J Risk Uncertainty (2008) 36:83–102
DOI 10.1007/s11166-007-9029-2

Prospect theory for continuous distributions

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Published online: 8 January 2008

Abstract We extend the original form of prospect theory by Kahneman and Tversky from finite lotteries to arbitrary probability distributions, using an approximation method based on weak-$\star$ convergence. The resulting formula is computationally easier than the corresponding formula for cumulative prospect theory and makes it possible to use prospect theory in future applications in economics and finance. Moreover, we suggest a method how to incorporate a crucial step of the "editing phase" into prospect theory and to remove in this way the discontinuity of the original model.

Keywords Prospect theory · Cumulative prospect theory · Continuity · Probability weighting · First-order stochastic dominance

JEL Classification D81

Since prospect theory (PT) has been introduced by Kahneman and Tversky (1979) as a descriptive model for decisions under risk, it has accommodated increasing empirical evidence, especially when compared with classical expected utility theory (EUT), which requires too strict assumptions regarding rationality from the decision makers. For an overview we refer to Schoemaker (1982) and Starmer (2000). Prospect theory adopts the basic framework from expected utility theory, but with additional psychological components based on the observations of the decision making process by real people.

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Prospect theory assumes that decision makers frame outcomes in terms of gains and losses, instead of the final wealth level that is used in expected utility theory. Accordingly, a value function v replaces the standard utility function. This value function has two parts, a concave part in the gain domain and a convex part in the loss domain, capturing the risk-averse tendency for gains and risk-seeking tendency for losses by many decision makers. Another important aspect is that probabilities are weighted by an S-shaped probability weighting function w, which is based on the observation that most people tend to overweight small probabilities and underweight large probabilities. Although the original formulation of prospect theory proposed by Kahneman and Tversky (1979) was only defined for lotteries with, at most, two non-zero outcomes, it can be generalized to n outcomes. Generalizations have been used by various authors, e.g., Fennema and Wakker (1997), Camerer and Ho (1994), Wakker (1989), Schneider and Lopes (1986). In this article, we study the original formulation for n outcomes by Karmakar (1978). The value of a lottery with outcomes x_i, each of probability p_i, is then given by

$$\mathrm{PT} = \frac{\sum_{i=1}^{n} w(p_i)v(x_i)}{\sum_{i=1}^{n} w(p_i)}. \tag{1}$$

In difference to Kahneman and Tversky (1979) the PT-value is here normalized by the sum of the weighted probabilities. We will see later why this normalization is necessary when studying a large or infinite number of outcomes.

Since Tversky and Kahneman (1992), the value function in PT is often chosen as

$$v(x) := \begin{cases} x^{\alpha}, & x \geq 0 \\ -\lambda(-x)^{\beta}, & x < 0, \end{cases}$$

where $\lambda \approx 2.25$ is called "loss-aversion" coefficient, and α, β describe the risk-attitudes for gains and losses. The choice for the weighting function given by Tversky and Kahneman (1992) is

$$w(p) := \frac{p^{\gamma}}{(p^{\gamma} + (1-p)^{\gamma})^{1/\gamma}} \tag{2}$$

with the parameter γ describing the amount of over- and underweighting.

The additional components of PT allow us to explain violations of some of the properties derived from EUT (in particular the Independence Axiom of von Neumann and Morgenstern), which have been frequently reported from experiments. However, PT is also criticized for having some undesirable characteristics, especially, the violation of first-order stochastic dominance and continuity. Another limitation of PT is that it can only be applied to discrete outcomes, but applications, e.g. in finance, require a theory for lotteries with non-discrete outcome distributions. A portfolio, for instance, might have any amount as return, not only a finite number of possible amounts. To solve these problems, a new theory, cumulative prospect theory (CPT) was proposed in Tversky and Kahneman (1992), where the *cumulative* probability distributions rather than the probabilities themselves are transformed by the probability weighting function (see also Wakker 1993).

CPT is often considered as an improvement over original PT, particularly because it does not violate stochastic dominance and it can be applied to continuous outcomes. However, the empirical comparisons of these two theories are still inconclusive: some data fit better with PT (Camerer and Ho 1994; Wu and Gonzalez 1996), some data fit better with CPT (Fennema and Wakker 1997). For example, in one recent study using a critical test, it has been found that the choices for gambles without a certainty effect are consistent with PT, but not CPT, whereas the choices for gambles with a certainty effect are consistent with both PT and CPT (Wu, Zhang and Abdellaoui 2005). Moreover, the studies that aimed at testing the key characteristics of the two theories even seem to suggest frequent contradictions with CPT. Various studies have reported systematic violations of properties of CPT such as ordinal independence, branch independence, event splitting effects and first-order stochastic dominance (Wu 1994; Birnbaum and McIntosh 1996; Birnbaum and Martin 2003; Birnbaum 2005; Humphrey 1995; Luce 1998; Starmer and Sugden 1993).

In particular, the violation of stochastic dominance is not necessarily a weakness for a descriptive decision theory because it has been observed that subjects frequently choose dominated lotteries especially when stochastic dominance is not transparent to them. In this respect, PT is even better than CPT in predicting such preference patterns (Tversky and Kahneman 1986; Birnbaum 2005). On the other hand, it seems that most people do not violate first-order stochastic dominance for lotteries with two outcomes, but this can be predicted by the original formulation (1) proposed by Karmakar (1978).[1]

Given the above evidence, it seems that CPT may be descriptively not as strong as PT. However, PT has some major disadvantages:

1. There is no generalization of PT to non-discrete outcome distributions.
2. It is not continuous, i.e., small changes in a lottery can produce large differences in its utility.

The purpose of this paper is two-fold: on the one hand, we want to generalize PT to non-discrete outcomes (Section 1), and on the other hand we want to show how by incorporating a central editing rule, the collecting of nearby outcomes (Kahneman and Tversky 1979), into PT, the theory can be made continuous (Section 2).

Let us first summarize our approach for extending PT to non-discrete outcome distributions: our central idea is here to use an approximation method. More precisely, we approximate non-discrete outcome distributions by finite lotteries. If we do this in Karmarkar's formulation (1), we can pass to the limit and obtain a well-defined expression for the non-discrete outcome distribution. In the simplest case of a continuous outcome distribution given

[1] Alternatively, it can also be achieved by editing rules, when using the formulation of Kahneman and Tversky (1979), but these editing rules are not always clearly defined.

by the probability density p and a probability weighting function w defined as in Eq. 2 we obtain with this method

$$PT(p) = \frac{\int v(x)p(x)^{\gamma}\,\mathrm{d}x}{\int p(x)^{\gamma}\,\mathrm{d}x}.$$

A more general formulation of this result can be found in Theorem 1 and Remark 1. Although the details to derive this formula are necessarily a bit technical, the general idea of this process can be illustrated by an analogy to histograms (compare Figure 1): a non-discrete outcome distribution (e.g. an absolutely continuous distribution as schematically depicted in the last picture of Figure 1) can be approximated by finer and finer histograms. Each histogram, however, can be represented by a finite lottery, where the outcomes are given by the position of the histogram bars, and the probabilities of these outcomes can be represented by the area of the bars. Approximating the distribution with finer and finer histograms, the number of bars and hence the number of outcomes increases and at the same time the area of the bars and hence the probability of the outcomes decreases. In the limit we arrive therefore at an integral formula for the PT-value of the non-discrete lottery where only the behavior of w close to zero plays a role—for this approach to work it is crucial that we have convergence. The question of convergence will turn out to be more than just mathematical hair-splitting: in fact, we will see that the method is only applicable if we use the formulation (1) of PT, i.e. that the normalization with the sum of the weighted probabilities is essential.

After extending PT to arbitrary outcome distribution, we try to solve the problem of continuity: a small change of a lottery (e.g. splitting a single outcome into two similar outcomes) changes the PT-value often substantially. This is the cause of several theoretical problems of PT that could only be resolved using cumulative prospect theory. Already Kahneman and Tversky (1979) suggest a so-called "editing phase" before the evaluation of the formula for the PT-value. In the editing phase in particular nearby outcomes are combined. We formalize this process and call the resulting modified theory smooth prospect theory (SPT). We show that this theory is in fact continuous. Its key idea is to make the outcomes "fuzzy", i.e. replacing an outcome at, say, x by an outcome-distribution around x. Nearby outcomes are then handled as something in between two separate outcomes and one combined outcome,

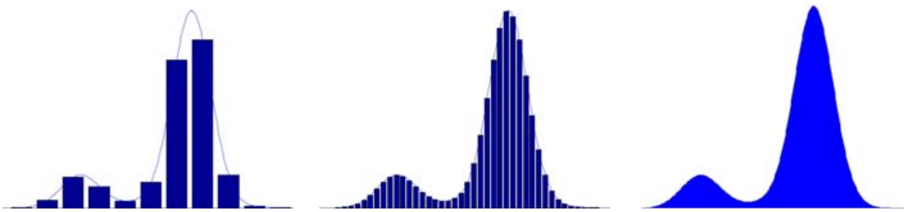

Fig. 1 Non-discrete outcome distributions are approximated by finer and finer histograms of uniform width

depending on their distance. If this is done in a mathematically sound way, it can be shown that this guarantees continuity. We show that other typical properties of PT are, nevertheless, still present, in particular violations of stochastic dominance.

In the final part, Section 3, we compare properties of the different variants of prospect theory with empirical evidence. Some mathematical background on continuity and the proofs of our results are provided in the Appendix.

1 Extending prospect theory to continuous probability distributions

In this section we derive a non-discrete variant of prospect theory. Afterwards we demonstrate why a naive extension of PT fails and why it is necessary to use the extension of PT as given in Eq. 1. For some refreshment on the mathematical background of the related concepts, we refer the reader to Appendix 1.

To extend PT to continuous distributions, we first consider a naive approach where we simply define the PT utility of a continuous distribution p on $\mathbb{R}$ by

$$\int v(x)w(p(x))\,\mathrm{d}x, \text{ or } \frac{\int v(x)w(p(x))\,\mathrm{d}x,}{\int w(p(x))\,\mathrm{d}x,}, \tag{3}$$

where v is the value function and w the weighting function as specified in Eq. 2.

Why do those "natural" attempts for extensions not work? First of all, the probability density $p(x)$ might take values larger than one, but w is only defined for values in [0,1]. Moreover, both formulas do not satisfy one of the most natural requirements for a decision theory: they are *not* invariant under changes of the coordinate system. If we "relabel" the monetary units from euros to cents, for example, it will lead to lower values of p and hence to a different over- or underweighting of the probability distribution:

Example 1 Let x be first given in euros and let p be an outcome distribution that attaches uniform probability for all outcomes between 0 and 1, i.e.

$$p(x) := \begin{cases} 1, & \text{for } x \in [0, 1], \\ 0, & \text{elsewhere.} \end{cases}$$

The PT-value therefore depends on $w(1) = 1$. For $v(x) = x$ we have PT $= 1/2$. Now let us consider the same lottery in cents. Then p becomes:

$$p(x) := \begin{cases} 1/100, & \text{for } x \in [0, 100], \\ 0, & \text{elsewhere.} \end{cases}$$

The PT-value in the first formulation depends now on $w(1/100)$ which is overweighted and therefore larger than $1/100$. For $v(x) = x/100$ (which gives the same value function as before after conversion into cents) we hence have $PT = 50 \cdot w(1/100) > 1/2$.

To show a contradiction to the second possible formulation, a slightly more complicated example would be needed, but the idea is the same.

Generally, the formulas do not give the same result if we replace $p(x)$ by $tp(tx)$ and $v(x)$ by $v(tx)$ for some $t > 0$, as a simple transformation shows. This leads to a direct dependence of decisions on the monetary unit in which the lotteries are phrased (as illustrated in the above example), which is certainly not a desired consequence, in particular since the independence is satisfied for the discrete case in PT, as well as for the general cases in CPT and EUT.

Can we rescue the "naive" approach 3 by replacing w by its derivative w'? It is easy to see that the same problems would arise: w' is (as w) only defined on [0, 1] and the consistency with respect to changes of monetary units would still be violated.

For these reasons we need to consider a more sophisticated approach based on an approximation method. The key idea is to approximate a continuous probability distribution by a sequence of finite lotteries. Finite lotteries can be represented by weighted sums of Dirac masses (see Appendix 1 for details). Therefore we can formulate our task as follows: we want to approximate the absolutely continuous probability measure p by a sequence of Dirac measures p_n. The usual PT utility for finitely many outcomes can be computed for p_n and we will study the limit $\lim_{n\to\infty} \mathrm{PT}(p_n)$. The hope is to find a limit functional that can be used to directly compute $PT(p)$. Unfortunately, not every possible approximation p_n of p will lead to the same limit functional, at least in the simple model considered in this section. This is caused by the highly discontinuous structure of the PT functional. However, we can select a "representative" approximation by formalizing the editing phase as collecting of nearby outcomes: we integrate the probability of all outcomes between, say, a and b into one event with an outcome of a. In this way, we transform the continuous distributions into a simple, discrete lottery where, e.g., the outcome a has a certain probability larger than zero. If we decompose the set of all possible outcomes into intervals of size $1/n$, we arrive at a lottery p_n. This process can be interpreted as the construction of a histogram from a probability distribution. When $n \to \infty$, this approximation becomes better and better, and our hope is that the associated PT-values of p_n will eventually converge to a PT-value that represents the continuous distribution given by p.

Mathematically spoken, we decompose $\mathbb{R}$ into intervals of equal size $1/n$ and replace on each interval p by a Dirac measure of corresponding weight. More precisely, we define

$$p_{z,n} := \int_{\frac{z}{n}}^{\frac{z+1}{n}} dp, \qquad \bar{p}_n := \sum_{z\in\mathbb{Z}} p_{z,n}\delta_{z/n}.$$

The measures $\bar{p}_n$ are still infinite sums of Diracs, but since p is a probability measure, it is easy to see that $\int_{[z/n,(z+1)/n)} dp \to 0$ for $|z| \to \infty$, thus we can neglect all, but finitely many intervals by making an arbitrarily small error. We call the resulting measure p_n. By a small lemma (McCann 1995), this approximates in fact p; in mathematical language: $p_n \overset{\star}{\rightharpoonup} p$.

Generally, we can use any decomposition of $\mathbb{R}$ into equally sized intervals $[x_i, x_{i+1})$ where the size $h_n := |x_i - x_{i+1}| \to 0$ and the union of these intervals covers all of $\mathbb{R}$ as $n \to \infty$. In fact, to keep the notation simpler, we will use this more general class of approximations from now on.

One reason why we have chosen the above approximation is that it is a *homogenous* approximation that allows for over- and underweighting of the probability in the limit. By "homogenous" we mean that the approximation does not depend directly on x. A non-homogenous approximation would mean, for instance, to choose in the above approximation the interval length h_n as a function of x. In the illustration of Figure 1 this corresponds to equally sized histogram bars in each approximation step. This is probably the most natural possible choice that we can make.[2]

We will demonstrate in the next section how incorporating the editing phase of prospect theory into our consideration makes it possible to derive this limit for arbitrary approximating sequences.

Theorem 1 *Let p be a probability distribution on $\mathbb{R}$ with exponential decay at infinity and let p_n be defined as above. Assume that $v \in C^1(\mathbb{R})$ has at most polynomial growth and that for the weighting function $w\colon [0,1] \to [0,1]$ there exists some $\alpha \in (0,1)$ and some finite number $C > 0$ such that*

$$\lim_{\varepsilon \to 0} \frac{w(\varepsilon)}{\varepsilon^\alpha} = C \tag{4}$$

for $\varepsilon \to 0$. Then the PT utility as formulated by Karmakar (1978)

$$PT(p_n) = \frac{\sum_z w(p_{n,z}) v(z/n)}{\sum_z w(p_{n,z})}$$

converges to

$$\lim_{n\to\infty} PT(p_n) = PT(p) := \frac{\int v(x) p(x)^\alpha \, dx}{\int p(x)^\alpha \, dx}. \tag{5}$$

The proof is given in Appendix 2. It can be easily generalized to the following case, which is important in many applications:

Remark 1 If p is a probability measure that can be written as a sum of finitely many weighted Dirac masses $\pi_i \delta_{x_i}$ and an absolutely continuous measure p_a, i.e., $p = p_a + \sum_{i=1}^n \pi_i \delta_{x_i}$, then we obtain the following limit:

$$\lim_{k\to\infty} PT(p_k) = PT(p) := \frac{\sum_{i=1}^n v(x_i)\pi_i^\alpha + \int v(x) p_a(x)^\alpha \, dx}{\sum_{i=1}^n \pi_i^\alpha + \int p_a(x)^\alpha \, dx}.$$

Remark 2 Condition (4) simply means that the weighting function close to zero is approximately p^α for some α. This is the case for most suggested

[2] A practical application can be found in Hens, Mayer and Rieger (2007) where historical data on stock returns is used to derive an (approximate) lottery describing their performance, which is in turn used to derive subjective PT utilities. The most natural method of forming a lottery is here to integrate the (discrete) events into a histogram that corresponds to a probability distribution. The results of this section make this approach possible.

weighting functions, in particular the one from Tversky and Kahneman (1992) where $\alpha < 1$. A weighting function with $\alpha = 1$ has been suggested in Rieger and Wang (2006). We could also consider the case $\alpha > 1$.

Remark 3 The assumption that p has exponential decay at infinity and that v has at most polynomial growth is needed in order to ensure that $\int p(x)^\alpha \, dx$ and $\int p(x)^\alpha v(x) \, dx$ both have finite values.[3] This problem is closely related to the variant of the St. Petersburg paradox that occurs in CPT and is interesting on its own. The curious reader may compare this with the results in Rieger and Wang (2006).

This approach to extend prospect theory to continuous distributions has not been used before to our knowledge. One property of this approach might at first glance contradict a core ingredient of PT, namely the S-shape of the weighting function. In fact, the probability density $p(x)$ is only weighted as $p(x)^\alpha$, so we have a strictly concave "probability weighting function" and no S-shape. However, this is not really the case: the normalization ensures that large probabilities are still underweighted. A drawback of this result is that the probability weighting function can no longer be freely fitted to individual behavior. Instead there is only one parameter (here written as α) left that can be adjusted. One can look at this phenomenon from two sides: on the one hand this demonstrates that the freedom in the choice of w in the original formulation of prospect theory is misleading, since it disappears when we study lotteries with many or even infinitely many outcomes. On the other hand, fewer degrees of freedom mean fewer difficult decisions regarding the choices of the functional form of w, which can also be seen as a conceptual advantage.

Let us now explain why it is essential to use the formulation of n-outcome lotteries by Karmarkar and not the formulation $\widetilde{PT}(p) = \sum w(p_n)v(x_n)$, as introduced implicitly in Schneider and Lopes (1986). We will show the following (at first glance slightly surprising) result:

Theorem 2 *Let p be a continuous probability distribution on* $\mathbb{R}$ *with expected utility*

$$\mathrm{EU}(p) := \int v(x)p(x) \, dx \neq 0$$

and let p_n *be defined as above. Assume* $v \in C^1(\mathbb{R})$*. Moreover, assume that for the weighting function* $w\colon [0,1] \to [0,1]$ *there exists some* $\alpha \in (0,1]$*. Then the* $\widetilde{PT}$ *utility of Schneider and Lopes* (1986)

$$\widetilde{PT}(p_n) = \sum_z w(p_{n,z})v(z/n)$$

[3]The assumption can be weakened, e.g. to $\int p(x)^\alpha \, dx < +\infty$ if v is bounded.

converges to

$$\lim_{n\to\infty} \widetilde{PT}(p_n) = \begin{cases} \infty, & \text{if } \alpha < 1, \\ C \cdot EU(p), & \text{if } \alpha = 1, \end{cases}$$

We remark that this theorem also holds when $EU(p)$ is infinite. The proof of the theorem is given in Appendix 2.

Remark 4 The condition $EU(p) \neq 0$ is a technical condition that is only used in the case $\alpha < 1$. Since the expected utility is only meaningful up to an affine transformation, this can be assumed without loss of generality.

Theorem 2 highlights the difficulty of probability weighting in this formulation of the n-outcome prospect theory: in the approximation process, the single probabilities become smaller and smaller, hence (if $\alpha < 1$) the overweighting becomes stronger and stronger and finally leads to an infinite utility. In the case $\alpha = 1$, however, the relative difference between the overweighting becomes smaller and smaller as the single probabilities become small, hence in the limit the overweighting does not play a role any more and we arrive simply at a variant of the expected utility.

One might wonder at this point whether it is really a problem that the PT-values diverge to infinity in the approximation process (if $\alpha < 1$), since they have per se no real meaning: what matters only, are preference relations expressed by the *differences* between the PT-values of lotteries. There are two reasons why this argument is not convincing:

- First, it would imply that we can make a lottery with positive value arbitrarily attractive if only we decompose its events into more and more single events. Applied to the problem of continuous lotteries that would mean that by approximating them differently fine, we could induce any preference relation between them.
- Second, we could not compare two continuous lotteries directly, since their PT-values were both infinite. Instead, we would always have to compare their approximations and hope that the preferences expressed by them converge. This would render any practical application as impossible.[4]

For all of these reasons it is therefore necessary to rely on the formulation 1 and Theorem 1.

[4]Since we obtain directly infinity as the limit, rather than an integral formulation with infinite value, it is also not possible to consider the difference of two lotteries by writing their values under the same integral. This method would allow, e.g., in expected utility theory, where we have such an integral formulation, to define preferences over pairs of some lotteries that each have infinite utility.

2 Continuity in prospect theory

The original prospect theory lacks some properties which would be quite natural to assume. We have already discussed stochastic dominance. Now we turn our attention to another problem: the discontinuity of PT. In the classical framework of PT, there is a process called the "editing phase" which "filters" this (and other) problems. The person is assumed to process first the presented lotteries, in particular by collecting outcomes with identical (or nearly identical) values to a single event. As an example for this editing process consider the following two lotteries:

probability	0.8	0.1	0.1
outcome	0€	9.99€	10€

probability	0.8	0.2
outcome	0€	10€

Without an editing phase, PT could value the first higher than the second, due to the strong overweighting of the low probability 0.1. In the editing phase, however, the first lottery would be converted into a lottery similar to the second one, by simply collecting the very similar payoffs of 9.99 and 10€.

In this example we see two effects of the editing phase: on the one hand, it avoids certain stochastic dominance violations, based on event splitting. On the other hand, it avoids a discontinuity of the theory: without this editing phase, a sequence p_n of lotteries of the first type that converges to the second lottery does not satisfy the continuity condition $PT(p_n) \to PT(p)$. However, the editing phase is, as Kahneman and Tversky (1979) admit, not very well defined.

In this section we present a modification of PT that incorporates this basic idea of the editing phase. At the end of this section we compare certain properties of this modified version, called smooth prospect theory (SPT), with PT and CPT to discuss its usefulness.

Our idea is to model the editing phase by making our evaluation on the payoffs a little bit "fuzzy" (in a well-defined way). We introduce a parameter $\varepsilon > 0$ and assume that outcomes that differ only by less than ε are more and more considered to be the same. In a first step we therefore transform $p := \sum_{i=1}^{n} p_i \delta_{x_i}$ into an absolute continuous probability distribution p_ε by the formula

$$p_\varepsilon(x) := \frac{1}{2\varepsilon} \sum_{i=1}^{n} p_i \chi_{[x_i-\varepsilon, x_i+\varepsilon]}(x),$$

where $\chi_{[a,b]}$, the *indicator function* of the interval $[a, b]$, is given by

$$\chi_{[a,b]}(x) := \begin{cases} 1\ , & \text{when } a \le x \le b \\ 0\ , & \text{elsewhere.} \end{cases}$$

In Figure 2 we give an illustration for such a transformation, based on a lottery similar to the one from our initial example. The idea is closely related to the concept of "kernel estimates" and to "mollifiers" in analysis.

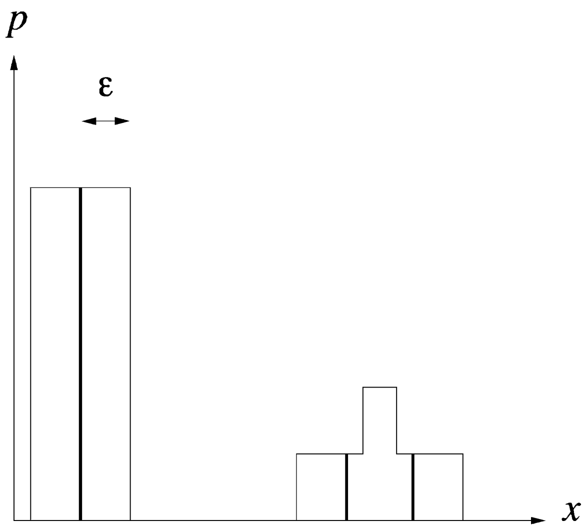

Fig. 2 A lottery before (*thick lines*) and after (*thin lines*) the "editing phase" as described by the smooth prospect theory model

Given this transformed probability, we now need to define its subjective utility in a way that we recover the classical PT when $\varepsilon \to 0$. To this aim, we need to define the *smooth prospect theory* (SPT) of the lottery p by

$$SPT_\varepsilon(p) := \frac{\int w\left(\sum_{i=1}^n p_i \chi_{[x_i-\varepsilon, x_i+\varepsilon]}(x)\right) v(x)\, dx}{\int w\left(\sum_{i=1}^n p_i \chi_{[x_i-\varepsilon, x_i+\varepsilon]}(x)\right)\, dx} \tag{6}$$

or more generally for arbitrary probability measures p by

$$SPT_\varepsilon(p) := \frac{\int w\left(\int_{x-\varepsilon}^{x+\varepsilon} dp\right) v(x)\, dx}{\int w\left(\int_{x-\varepsilon}^{x+\varepsilon} dp\right)\, dx}.$$

In the following, we will occasionally omit the index ε. Then ε is an arbitrary fixed positive number. The following proposition (which is proved in Appendix 2) shows that our definition is meaningful, i.e. invariant under rescaling of the monetary unit, and that it coincides with PT for $\varepsilon \to 0$.

Proposition 1 *Let* $SPT_\varepsilon(p)$ *be given by* Eq. 6. *Then* $\lim_{\varepsilon\to 0} SPT_\varepsilon(p) = PT(p)$. *Moreover,* SPT_ε *is invariant under affine rescaling.*

The main purpose of incorporating the editing phase into the mathematical formalism was to avoid the discontinuity of the original theory. The next theorem shows that *SPT* is in fact continuous:

Theorem 3 *Let* p^k *and* p *be probability measures with* $p^k \overset{\star}{\rightharpoonup} p$, *then* $SPT(p^k) \to SPT(p)$, *i.e., smooth prospect theory is continuous.*

We prove this result in Appendix 2.

SPT still allows for violations of stochastic dominance.

Remark 5 Although SPT is continuous, it can still violate the stochastic dominance principle if $\varepsilon > 0$ is chosen small enough.

In fact, one can show that for every fixed $\varepsilon > 0$, the stochastic dominance principle can be violated for some lotteries. Hence the "collecting" of similar outcomes by itself is not a sufficient explanation for the avoidance of dominated lotteries. The proof of this is relatively easy: one just needs to construct a lottery with outcomes being apart at least ε and so low probabilities that the overweighting still leads to a stochastic dominance violation similar to the initial example. However, the number of cases with stochastic dominance violations decreases, when ε increases.

One main advantage of incorporating the editing phase into the functional form is that it allows us to obtain the non-discrete generalization of PT that we have already derived in the previous section via *arbitrary approximations*. In fact, we can study the same limit for SPT that we have studied for the discrete form of PT in the previous section, where now at the same time we let ε go to zero. We will see that the resulting limit is again the non-discrete version of PT defined above. The importance of this result is that we do not need any more restrictions on the sequence of approximating measures. This underlines that the limit functional (5) is indeed the natural generalization of discrete PT:

Theorem 4 *Let p be an absolutely continuous probability measure*[5] *and $p(x)$ its probability density on $\mathbb{R}$ with at least exponential decay at infinity. Let p^k be a sequence of probability measures with $p^k \overset{\star}{\rightharpoonup} p$. Assume that $v \in C^1(\mathbb{R})$ has at most polynomial growth and that for the weighting function $w\colon [0,1] \to [0,1]$ there exists some $\alpha \in (0,1)$ and some $C > 0$ such that $\lim_{\delta \to 0} w(\delta)\delta^{-\alpha} = C$. Then, for all sequences $k(\varepsilon) \to \infty$ that converge sufficiently slowly as $\varepsilon \to 0$, the SPT utility of p^k converges to $PT(p)$, i.e.:*

$$\lim_{\varepsilon \to 0} SPT_\varepsilon(p^{k(\varepsilon)}) = PT(p) = \frac{\int v(x)p(x)^\alpha \, dx}{\int p(x)^\alpha \, dx}.$$

The proof is based on the previous convergence results, see Appendix 2.

In the following section we discuss the properties of the variants of prospect theory and compare them with experimental findings.

3 A comparison of the PT-family

Prospect theory and cumulative prospect theory have been accepted as the most competitive alternative theories of expected utility theory to describe decision under risks. Although CPT is often considered to have mathematically

[5]This result can be generalized in the spirit of Remark 1.

more elegant properties, empirical evidence sometimes suggests that the original PT may capture certain psychological processes that cannot be predicted by CPT.

In Table 1, we compare several mathematical properties of PT, SPT and CPT to the empirical evidence (see also Wu, Zhang and Abdellaoui 2005). We see that the formulation of PT by Karmakar (1978) has certain advantages over the formulation by Schneider and Lopes (1986), in particular violations of internality and (in the two outcome cases) of stochastic dominance are avoided. Moreover, we have seen that the formulation by Karmakar (1978) can be extended to continuous outcomes, whereas the non-normalized version cannot (compare Theorem 1 and Theorem 2). SPT outperforms PT in that it does not violate continuity and predicts that closer outcomes are less overweighted than very distinct outcomes, which is psychologically very plausible. Compared to other PT theories, the rank-dependent property of CPT is more consistent with empirical evidence. However, it fails to predict that distinct outcomes receive more weights than aggregated outcomes, which has been found in experiments.

Table 1 Comparisons of prospect theory (PT), smooth prospect theory (SPT), and cumulative prospect theory (CPT)

Properties	PT[a]	PT[b]	SPT	CPT	Empirical evidence[c]
Violation of independence axiom	*Yes*	*Yes*	*Yes*	*Yes*	Yes
Explanation of Allais paradox	*Yes*	*Yes*	*Yes*	*Yes*	Yes
Violation of stochastic dominance for lotteries with two outcomes	*No*	Yes	*No*	*No*	No
Violation of stochastic dominance (three or more outcomes)	*Yes*	*Yes*	*Yes*	No	Yes
Violation of internality[d]	*No*	Yes	*No*	*No*	No
Inverse S-shaped weighting function which implies lower- and upper-subadditivity	*Yes*	*Yes*	*Yes*	*Yes*	Yes
Distinctive outcomes receive more weights (support theory)	*Yes*	*Yes*	*Yes*	No	Yes
Among distinctive outcomes, closer outcomes get less weights	No	No	Yes	No	Plausible
Violation of continuity	Yes	Yes	No	No	—
Can be applied for non-discrete distributions	*Yes*	No	*Yes*	*Yes*	—

Properties correlating to experimental evidence are italicized.

[a] As in Karmakar (1978).

[b] As implicitly used in Schneider and Lopes (1986).

[c] A collection of the experimental evidence regarding these properties can be found by Wu, Zhang and Abdellaoui (2005). We summarize their report in the form of this table and extend it slightly. For references on the particular findings compare, e.g., Karmakar (1979), Starmer and Sugden (1993), Wu (1994), Birnbaum and McIntosh (1996), Humphrey (1995), Luce (1998), Birnbaum and Martin (2003), Birnbaum (2005).

[d] This means, that the certainty equivalent of a lottery can be larger than any outcome of the lottery. As an example consider $v(x) = x$ and $w(0.25) = 0.4$, then the certainty equivalent of the lottery (.25, 100; .25, 90; .25, 80; .25, 0) (expressed in values) in prospect theory as defined in Schneider and Lopes (1986) is 108, greater than any of the outcomes. For further discussions see also Gneezy, List and Wu (2006).

CPT also fails to predict the violation of stochastic dominance documented in empirical findings (Tversky and Kahneman 1986; Birnbaum 2005).

Another aspect that sometimes plays a role is of a pragmatic nature: In some applications, in particular in areas such as behavioral finance when one has to work with huge amounts of data points (e.g., historical stock returns), an advantage of PT (or SPT) is the reduced computation load as compared to CPT, because an ordering of the probabilities by their outcomes is not necessary. This application was previously not possible, since no continuous model for PT had been available.

4 Conclusions

We have extended prospect theory for the evaluation of non-discrete outcome distributions, while still preserving the positive features of PT. This extension is therefore more consistent with some of the patterns observed in experiments than CPT, although it does not require more fitting parameters. The continuous limit is even to some extent independent of the precise choice of the weighting function which brings a substantial simplification over CPT, however, at the same time limits the flexibility of the theory.

The discontinuities of PT can be removed in a natural way by incorporating a central idea of the "editing phase" by Kahneman and Tversky (1979) into the functional. The resulting modified theory, smooth prospect theory (SPT) therefore combines many of the advantages of PT and CPT in one model.

With these improvements for the classical prospect theory, in particular the extension to non-discrete lotteries, we built a foundation for applications of PT in financial economics and other areas, where up to now the only possibility was to use the conceptually different and numerically and analytically harder CPT.

Acknowledgements We thank Thorsten Hens for many discussions and his steady support for our projects and János Mayer for his constructive contributions. We also thank Pavlo Blavatskyy for his remarks and the anonymous referee for his valuable suggestions. Financial support by the National Centre of Competence in Research "Financial Valuation and Risk Management" (NCCR FINRISK), Project 3, "Evolution and Foundations of Financial Markets", and by the University Priority Program "Finance and Financial Markets" of the University of Zürich is gratefully acknowledged.

Appendix

Appendix 1: Probability measures and continuity

We provide some information for non-specialists on a couple of mathematical concepts that we have applied in the previous sections. We apologize to the cognoscenti for making a complicated subject seem easy, while at the same time trying not to be too imprecise.

Let $N \in \mathbb{N}$. We recall that a *probability measure* p on $\mathbb{R}^N$ is a non-negative measure with $||p|| := \int_{\mathbb{R}^N} dp = 1$. A probability measure is *absolutely continuous* if there exists an integrable function p_a such that we can write

$$\int_{\mathbb{R}^N} f(x)dp(x) = \int_{\mathbb{R}^N} f(x)p_a(x)dx$$

for every continuous function f. A *Dirac mass* δ_{x_0} is defined by $\int_{\mathbb{R}^N} f(x)\delta_{x_0}(x) = f(x_0)$ for every continuous function f. In particular, we can write any probability distribution p with only finitely many outcomes $x_1, \ldots x_n$ of corresponding probabilities $p_1, \ldots, p_n$ as a sum of Dirac weighted masses

$$p = \sum_{i=1}^{n} p_i \delta_{x_i}.$$

This formulation enables us to handle the two typical situations of discrete lotteries (finitely many outcomes) and continuous outcome distributions (e.g., normally distributed outcomes) simultaneously.

Of course, a probability measure can be much more complicated. Of practical relevance is the case where it is a sum of an absolutely continuous measure and weighted Dirac masses. In this case we have discrete and continuous parts.

A central tool of this article is the approximation of measures by other measures. To be able to approximate measures, we need to have a notion of convergence. In other words: we want to define when a measure p is approximated by a sequence of measures $(p_n)_{n\in\mathbb{N}}$. In order to motivate the mathematical definition, let us consider first the naive approach: we define an approximation by requiring that $p_n(x)$ converges to $p(x)$, i.e. that the probability $p_n(x)$ of every outcome x converges to $p(x)$. This seemingly natural approach fails for two reasons: first, we are dealing with probability measures for which it is difficult to define $p(x)$ in a reasonable way. (p is not simply a function.) Second, we would exclude that $p_n = \delta_{1/n}$ approximates $p = \delta_0$, thus the convergence property would be too strong.

There is a better approach: We could say that p_n converges to p if every expected utility of p_n converges to the expected utility of p. This would imply that, in the limit, every rational person would be indifferent between p and p_n. This idea motivates the mathematical concept of weak-$\star$-convergence:

Definition 1 (Weak-$\star$-convergence of probability measures) We say that a sequence (p_n) of probability measures on $\mathbb{R}^N$ converges weak-$\star$ to a probability measure p if for all bounded continuous functions f

$$\int_{\mathbb{R}^N} f(x)dp_n(x) \to \int_{\mathbb{R}^N} f(x)dp(x)$$

holds. We write this as $p_n \overset{\star}{\rightharpoonup} p$. The function f is sometimes called a *test function*.

To see the correspondence to the intuitive approach sketched above, consider $f(x)$ as a utility function.

Finally, we recall the concept of continuity. The word "continuous" unfortunately has two quite different meanings in the English language. Since this may lead to some confusion in this article, we briefly explain the two concepts: First, *continuous* means *non-discrete*. We have already used this notion when talking about measures (or lotteries). As an example, think on a normal distribution in contrast to a lottery with finitely many outcomes. Second, *continuous* means *not discontinuous*. We say that a function F is *continuous* in this sense, if for all sequences x_n converging to x, we have $F(x_n) \to F(x)$. This second type of continuity is an important concept in every model. Roughly spoken, we want a model to be continuously depending on its parameters, since parameters can usually only be measured with a certain amount of precision. This is the case even in a mathematically well-sounded area like physics, and even more so in behavioral decision theory where the precision of experiments is obviously limited. Whereas PT is discontinuous, EUT, CPT, SPT are continuous, i.e., if $p_n \overset{\star}{\rightharpoonup} p$ then, e.g., $SPT(p_n) \to SPT(p)$.

Appendix 2: Mathematical proofs

Proof of Theorem 1 We first assume for simplicity that the support of p is bounded and that the union of the intervals $[x_i, x_{i+1}]$ covers supp p for all n. We define $h_n := |x_i - x_{i+1}|$. (Remember that, since the decomposition is assumed to be homogenous, h_n does not depend on i.) Since p is absolute continuous, $p_{i,n} := \int_{x_i}^{x_{i+1}} p(x)\,dx$ converges to zero as $n \to \infty$. Hence we can use Eq. 4 to prove

$$
\begin{aligned}
\lim_{n\to\infty} PT(p_n) &= \lim_{n\to\infty} \frac{\sum_i w(p_{i,n})v(x_i)}{\sum_{i=1}^n w(p_{i,n})} \\
&= \lim_{n\to\infty} \frac{\sum_i C(p_{i,n})^\alpha v(x_i)}{\sum_i C(p_{i,n})^\alpha} \\
&= \lim_{n\to\infty} \frac{h_n^\alpha \sum_i \left(\fint_{x_i}^{x} x_{i+1}\, p(x)\,dx \right)^\alpha v(x_i)}{h_n^\alpha \sum_i \left(\fint_{x_i}^{x} x_{i+1}\, p(x)\,dx \right)^\alpha}.
\end{aligned}
$$

In the next step, we transform the integrals into averages of which we can finally take the limit $n \to \infty$ (since p is continuous):

$$
\begin{aligned}
\lim_{n\to\infty} PT(p_n) &= \lim_{n\to\infty} \frac{\sum_i \left(\fint_{x_i}^{x_{i+1}} p(x)\,dx \right)^\alpha v(x_i)}{\sum_i \left(\fint_{x_i}^{x_{i+1}} p(x)\,dx \right)^\alpha} \\
&= \frac{\int v(x)p(x)^\alpha\,dx}{\int p(x)^\alpha}.
\end{aligned}
$$

This concludes the proof of Theorem 1. □

Proof of Theorem 2 Following the same ideas as in the proof of Theorem 1, we compute:

$$\lim_{n\to\infty} \widetilde{PT}(p_n) = \lim_{n\to\infty} \sum_i w(p_{i,n}) v(x_i)$$
$$= \lim_{n\to\infty} \sum_i C (p_{i,n})^\alpha v(x_i). \tag{7}$$

In the case $\alpha = 1$, we derive from this

$$\lim_{n\to\infty} \widetilde{PT}(p_n) = \lim_{n\to\infty} \sum_i C \left(\int_{x_i}^{x_{i+1}} v(\xi)\, dp + \int_{x_i}^{x_{i+1}} (v(x_i) - v(\xi)) p(x)\, dx \right).$$

Since $|v(x_i) - v(\xi)| \leq |v'(x_i)||x_i - \xi| + O(|x_i - \xi|^2) \to 0$ as $n \to \infty$, for every converging sequence of x_i, we get

$$\lim_{n\to\infty} \widetilde{PT}(p_n) = C \lim_{n\to\infty} \sum_i \int_{x_i}^{x_{i+1}} v(\xi) p(x)\, dx$$
$$= C \cdot EU(p).$$

We now consider the case $\alpha < 1$. From estimate 7 we obtain

$$\lim_{n\to\infty} \widetilde{PT}(p_n) = \lim_{n\to\infty} \sum_i C \frac{p_{i,n}}{(p_{i,n})^{1-\alpha}} v(x_i).$$

We estimate $(p_{i,n})^{1-\alpha} \leq (s_n)^{1-\alpha}$ with $s_n := \sup_i p_{i,n}$. Since $p_{i,n} \to 0$ as $n \to \infty$, we have $s_n \to 0$ as $n \to \infty$. Therefore we arrive at

$$\lim_{n\to\infty} \widetilde{PT}(p_n) \geq C \lim_{n\to\infty} s_n^{\alpha-1} \sum_i p_{i,n} v(x_i)$$
$$= C \lim_{n\to\infty} s_n^{\alpha-1} EU(p),$$

which is infinite, since $EU(p) \neq 0$.

Let us now check the case when supp p is unbounded. We replace p with its restriction to the interval $[-m, +m]$ and call this restriction p^m. Fixing m and using what we have just proved for measures with bounded support, we see that in the case $\alpha = 1$

$$\lim_{n\to\infty} \widetilde{PT}\left(p_n^m\right) = C \lim_{n\to\infty} \sum_i \int_{x_i}^{x_{i+1}} v(\xi) p^m(x)\, dx$$
$$= C \cdot \int_{-m}^{+m} v(x) p^m(x)\, dx.$$

Since $\int_{-m}^{+m} v(x) p^m(x)\, dx \to EU(p)$ as $m \to \infty$ (where we allow for infinity as value of $EU(p)$), the general case follows.

A similar consideration proves the result for $\alpha < 1$ with supp p unbounded. □

Proof of Proposition 1 If $\varepsilon > 0$ is smaller than $\min_{i,j} |x_i - x_j|$, then Eq. 6 simplifies to

$$SPT_\varepsilon(p) = \frac{\sum_{i=1}^n \int_{x_i-\varepsilon}^{x_i+\varepsilon} w(p_i)v(x)\,dx}{\sum_{i=1}^n \int_{x_i-\varepsilon}^{x_i+\varepsilon} w(p_i)\,dx} = \frac{\sum_{i=1}^n w(p_i)2\varepsilon \fint_{x_i-\varepsilon}^{x_i+\varepsilon} v(x)\,dx}{\sum_{i=1}^n w(p_i)2\varepsilon}.$$

Since $v \in C^1$, we obtain

$$\lim_{\varepsilon\to 0} SPT_\varepsilon(p) = \frac{\sum_{i=1}^n w(p_i)v(x_i)}{\sum_{i=1}^n w(p_i)}.$$

A straightforward computation finally shows that *SPT* is invariant under affine rescaling of the monetary units. □

Proof of Theorem 3 Since $p^k \overset{\star}{\rightharpoonup} p$, we have for $x \in \mathbb{R}$ either that $\int_{x-\varepsilon}^{x+\varepsilon} dp^k \to \int_{x-\varepsilon}^{x+\varepsilon} dp$ or that $p(\{x-\varepsilon\} \cup \{x+\varepsilon\}) > 0$. We claim that the latter case can only happen at most for countably many $x \in \mathbb{R}$:

We denote the set of all x for which $p(\{x\}) > 0$ by S. Since p is a probability measure and therefore in particular bounded and non-negative, we have

$$1 = \int_{\mathbb{R}} dp \geq \sum_{x\in S} p(\{x\}).$$

This implies that for every $\delta > 0$ there can be only finitely many $x \in S$ such that $p(\{x\}) \geq \delta$. We can therefore enumerate all $x \in S$ by sorting them with respect to $p(\{x\})$ in descending order. Therefore S is countable and accordingly $p(\{x-\varepsilon\} \cup \{x+\varepsilon\}) > 0$ can only be the case for countably many $x \in \mathbb{R}$. This implies that $\int_{x-\varepsilon}^{x+\varepsilon} dp^k \to \int_{x-\varepsilon}^{x+\varepsilon} dp$ for a.e. $x \in \mathbb{R}$, since a countable set on $\mathbb{R}$ has measure zero. Since w is continuous and bounded, we obtain $w\left(\int_{x-\varepsilon}^{x+\varepsilon} dp^k\right) \to w\left(\int_{x-\varepsilon}^{x+\varepsilon} dp\right)$ and therefore

$$\int w\left(\int_{x-\varepsilon}^{x+\varepsilon} dp^k\right) v(x)\,dx \to \int w\left(\int_{x-\varepsilon}^{x+\varepsilon} dp\right) v(x)\,dx.$$

Since $\int w\left(\int_{x-\varepsilon}^{x+\varepsilon} dp^k\right) dx$ is uniformly positive, the convergence carries over to the quotient and we have proved the continuity of SPT. □

Proof of Theorem 4 Let $p^k \overset{\star}{\rightharpoonup} p$, then we know for all $\varepsilon > 0$ that $SPT_\varepsilon(p^k) \to SPT_\varepsilon(p)$ as $k \to \infty$, and also that $SPT_\varepsilon(p) \to PT(p)$ as $\varepsilon \to 0$. We construct a diagonal sequence $(\varepsilon, k(\varepsilon))$ with the desired property as follows:

For every $\varepsilon > 0$ choose k such that $|SPT_\varepsilon(p^k) - SPT_\varepsilon(p)| < |SPT_\varepsilon(p) - PT(p)|$. Then

$$|SPT_\varepsilon(p^{k(\varepsilon)}) - PT(p)| \leq |SPT_\varepsilon(p^{k(\varepsilon)}) - SPT_\varepsilon(p)| + |SPT_\varepsilon(p) - PT(p)| < 2|SPT_\varepsilon(p) - PT(p)| \;\to\; 0 \text{ as } \varepsilon \to 0.$$

□

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