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On 3-dimensional asymptotically harmonic manifolds

Viktor Schroeder* and Hemangi Shah†

Abstract. Let (M, g) be a complete, simply connected Riemannian manifold of dimension 3 without conjugate points. We show that M is a hyperbolic manifold of constant sectional curvature $\frac{-h^2}{4}$, provided M is asymptotically harmonic of constant $h > 0$.

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1. Introduction. Let (M, g) be a complete, simply connected Riemannian manifold without conjugate points. Let SM be the unit tangent bundle of M. For $v \in SM$, let γ_v be the geodesic with $\gamma_v'(0) = v$ and $b_{v,t}(x) = \lim_{t\to\infty}(d(x, \gamma_v(t)) - t)$ the corresponding *Busemann function* for γ_v. The level sets ${b_v}^{-1}(t)$ are called *horospheres*.

A complete, simply connected Riemannian manifold without conjugate points is called *asymptotically harmonic* if the mean curvature of its horospheres is a universal constant, that is if its Busemann functions satisfy $\Delta b_v \equiv h$, $\forall v \in SM$, where h is a nonnegative constant. Then b_v is a smooth function on M for all v and all horospheres of M are smooth, simply connected hypersurfaces in M with constant mean curvature h.

For example, every simply connected, complete harmonic manifold without conjugate points is asymptotically harmonic.

For more details on this subject we refer to the discussion and to the references in [2]. Important result in this context are contained in [1], [3]. In [2] the following result was proved:

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Let M be a Hadamard manifold of dimension 3 whose sectional curvatures are bounded from above by a negative constant (i.e. $K \leq -a^2$ for some $a \neq 0$) and whose curvature tensor satisfies $\|\nabla R\| \leq C$ for a suitable constant C. If M is asymptotically harmonic, then M is symmetric and hence of constant sectional curvature.

We prove this result without any hypothesis on the curvature tensor.

Theorem 1.1. *Let (M, g) be a complete, simply connected Riemannian manifold of dimension 3 without conjugate points. If M is asymptotically harmonic of constant $h > 0$, then M is a manifold of constant sectional curvature $\frac{-h^2}{4}$.*

2. Proof of the Theorem. The first part of the proof (Lemma 2.1 to Lemma 2.3) is a modification of the results in [2]. Therefore we recall some notations which were already used in that paper. Our general assumption is that M is 3-dimensional, has no conjugate points and is asymptotically harmonic with constant $h > 0$. For $v \in SM$ and $x \in v^\perp$, let

$$u^+(v)(x) = \nabla_x \nabla b_{-v} \quad \text{and} \quad u^-(v)(x) = -\nabla_x \nabla b_v.$$

Thus $u^\pm(v) \in \mathrm{End}\,(v^\perp)$. With $\lambda_1(v), \lambda_2(v)$ we denote the eigenvalues of $u^+(v)$. The endomorphism fields $u^\pm$ satisfy the Riccati equation along the orbits of the geodesic flow $\varphi^t : SM \to SM$.

Thus if $u^\pm(t) := u^\pm(\varphi^t v)$ and $R(t) := R(\cdot, \gamma_v'(t))\gamma_v'(t) \in \mathrm{End}(\gamma_v'(t)^\perp)$, then

$$(u^\pm)' + (u^\pm)^2 + R = 0.$$

We define $V(v) = u^+(v) - u^-(v)$ and correspondingly $V(t) = V(\varphi^t(v))$ along $\gamma_v(t)$. We also define $X(v) = \frac{-1}{2}(u^+(v) + u^-(v))$ and $X(t) = X(\varphi^t(v))$. Then the Riccati equation for $u^\pm(t)$ yields

$$XV + VX = (u^-)^2 - (u^+)^2 = (u^+)' - (u^-)' = V'. \tag{1}$$

Lemma 2.1. *For fixed $v \in SM$ the map $t \mapsto \det V(\varphi^t v)$ is constant.*

Proof. Assume that $V(t)$ is invertible, then

$$\frac{d}{dt} \log \det V(t) = \mathrm{tr}\; V'(t)V^{-1}(t) = \mathrm{tr}\;(XV + VX)V^{-1}(t) = 2\;\mathrm{tr} X = 0.$$

The last step follows as M is asymptotically harmonic. Thus as long as $\det V(t) \neq 0$, it is constant along γ_v. Therefore $\det V(t)$ is constant along γ_v in any case. $\square$

Lemma 2.2. *Let $v \in SM$ be such that $V(v) = \mu\,\mathrm{Id}$, for some $\mu \in \mathbb{R}$, then $R(t) = \frac{-h^2}{4}\,\mathrm{Id}$, $\forall t$.*

Proof. Note that if $V(v) = \mu\,\mathrm{Id}$, then $V(\gamma_v'(t)) = h\,\mathrm{Id}$ for all t, as $\mathrm{tr}\, V \equiv 2h$ and by Lemma 2.1 the determinant of V is constant along $\gamma_v(t)$. Now by equation (1) $V' = XV + VX$. Hence, along γ_v, $V'(t) \equiv 0$. Thus $2hX = 0$ and since we assume $h > 0$ we have $X = 0$ along γ_v. Therefore, $u^+(t) = -u^-(t)$. But from the definition of V, $u^+(t) \equiv \frac{h}{2}\,\mathrm{Id}$ i.e u^+ is a scalar operator. By the Riccati equation $(u^+(t))^2 + R(t) = 0$, i.e. $R(t) = \frac{-h^2}{4}\,\mathrm{Id}$. □

Lemma 2.3. *For every point $p \in M$ there exists $v \in S_pM$ such that $R(x, v)v = \frac{-h^2}{4}\, x$, $\forall x \in v^\perp$. In particular,* $\mathrm{Ric}(v, v) = \frac{-h^2}{2}$.

Proof. Since TS^2 is nontrivial, an easy topological argument shows, that for every $p \in M$ there exists $v \in S_pM$ such that the two eigenvalues of $V(v)$ coincide. Thus $V(v) = \mu\,\mathrm{Id}$. The result now follows from Lemma 2.2. □

Lemma 2.4. *For all $v \in SM$ we have* $\mathrm{Ric}(v, v) \leq \frac{-h^2}{2}$.

Proof. The Riccati equation for $t \mapsto u^+(t)$ implies $(u^+)' + (u^+)^2 + R = 0$. Hence, $\mathrm{tr}(u^+)^2 + \mathrm{tr}\, R = 0$. Thus, $\mathrm{Ric}(v, v) = -({\lambda_1}^2(v) + {\lambda_2}^2(v))$. By hypothesis $\lambda_1(v) + \lambda_2(v) = h$, hence ${\lambda_1}^2(v) + {\lambda_2}^2(v) \geq \frac{h^2}{2}$. Consequently, $\mathrm{Ric}(v, v) \leq \frac{-h^2}{2}$. □

Lemma 2.5. *The sectional curvature K of M satisfies $K \leq -\frac{h^2}{4}$.*

Proof. Let $p \in M$, and let v be the vector in Lemma 2.3. Take $e_1 = v$, and let e_2 and e_3 be unit vectors orthogonal to e_1 so that $\{e_1, e_2, e_3\}$ forms an orthonormal basis of T_pM. Then $\{e_1\wedge e_2,\ e_1\wedge e_3,\ e_2\wedge e_3\}$ forms an orthonormal basis of $\Lambda^2 T_pM$. We want to show that the curvature operator, considered as map $R : \Lambda^2 T_pM \to \Lambda^2 T_pM$, $\langle R(X \wedge Y), V \wedge W\rangle = \langle R(X, Y)W, V\rangle$ is diagonal in this basis.

From Lemma 2.3 we see $R(e_2, e_1)e_1 = \frac{-h^2}{4}\, e_2$, $R(e_3, e_1)e_1 = \frac{-h^2}{4}\, e_3$. Thus $K(e_1, e_2) = K(e_1, e_3) = \frac{-h^2}{4}$ and $K(e_2, e_3) \leq \frac{-h^2}{4}$ as $\mathrm{Ric}(e_3, e_3) \leq \frac{-h^2}{2}$, where $K(v, w)$ denotes the sectional curvature of the plane spanned by v and w. We will prove below that

$$\langle R(e_1, e_3)e_3, e_2\rangle = 0 \quad\text{and}\quad \langle R(e_1, e_2)e_2, e_3\rangle = 0. \tag{2}$$

Assuming this for a moment, it follows that $R(e_1 \wedge e_3) \perp \mathrm{span}\{e_1 \wedge e_2,\ e_2 \wedge e_3\}$ and $R(e_1 \wedge e_2) \perp \mathrm{span}\{e_1 \wedge e_3, e_2 \wedge e_3\}$. Hence,

$$R(e_1 \wedge e_2) = \frac{-h^2}{4}\, e_1 \wedge e_2 \text{ and } R(e_1 \wedge e_3) = \frac{-h^2}{4}\, e_1 \wedge e_3.$$

Since $e_1 \wedge e_2$ and $e_1 \wedge e_3$ are eigenvectors of R, also $e_2 \wedge e_3$ is an eigenvector and we obtain

$$R(e_2 \wedge e_3) = K(e_2, e_3)\; e_2 \wedge e_3.$$

Thus the curvature operator is diagonal in the basis $\{e_1 \wedge e_2,\ e_1 \wedge e_3,\ e_2 \wedge e_3\}$ and all eigenvalues are $\leq \frac{-h^2}{4}$, which proves the result.

It remains to show (2). Consider for $t \in (-\varepsilon, \varepsilon)$ the vectors $v_t = \cos t e_1 + \sin t e_2$. Then,

$$\begin{aligned} f(t) := \mathrm{Ric}(v_t, v_t) &= K(v_t, e_3) + K(v_t, -e_1 \sin t + e_2 \cos t) \\ = K(e_1, e_2) + \sin^2 t\, K(e_2, e_3) &+ \cos^2 t\, K(e_1, e_3) + \sin 2t\, \langle R(e_1, e_3)e_3, e_2\rangle\,. \end{aligned}$$

By Lemma 2.4 $f(0) = \mathrm{Ric}(v,v) = \frac{-h^2}{2}$ is maximal and hence $f'(0) = 0$. This implies the first equation in (2). If we replace e_2 by e_3 in the above computation we obtain the second equation. □

Finally we come to the

Proof of Theorem 1.1. Lemma 2.5 implies that $K_M \leq \frac{-h^2}{4}$. By standard comparison geometry we obtain $\lambda_1(v) \geq \frac{h}{2}$ and $\lambda_2(v) \geq \frac{h}{2}$. Now $\lambda_1 + \lambda_2 = h$ implies that $\lambda_1 = \lambda_2 = \frac{h}{2}$. Hence, $u^+(v)$ is a scalar operator and therefore $R(x,v)v = \frac{-h^2}{4}\, x$, $\forall v$ and $\forall x \in v^\perp$. Thus, $K_M \equiv \frac{-h^2}{4}$.

3. Final Remark. We expect that the result holds also in the case $h = 0$, i.e. if all horospheres are minimal. Our argument, however, uses $h > 0$ essentially in the proof of Lemma 2.2.

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VIKTOR SCHROEDER, Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich.
e-mail: `vschroed@math.unizh.ch`

HEMANGI SHAH, School of Mathematics, Tata Institute of Fundamental Research Dr. Homi Bhabha Road, Mumbai - 400005, India.
e-mail: `hema@math.tifr.res.in`

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