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# Scaled Limit and Rate of Convergence for the Largest Eigenvalue from the Generalized Cauchy Random Matrix Ensemble

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**Abstract** In this paper, we are interested in the asymptotic properties for the largest eigenvalue of the Hermitian random matrix ensemble, called the Generalized Cauchy ensemble $GCyE$, whose eigenvalues PDF is given by

$$\text{const} \cdot \prod_{1 \le j < k \le N} (x_j - x_k)^2 \prod_{j=1}^{N} (1 + i x_j)^{-s-N} (1 - i x_j)^{-\bar{s}-N} dx_j,$$

where $s$ is a complex number such that $\Re(s) > -1/2$ and where $N$ is the size of the matrix ensemble. Using results by Borodin and Olshanski (Commun. Math. Phys., 223(1):87–123, 2001), we first prove that for this ensemble, the law of the largest eigenvalue divided by $N$ converges to some probability distribution for all $s$ such that $\Re(s) > -1/2$. Using results by Forrester and Witte (Nagoya Math. J., 174:29–114, 2002) on the distribution of the largest eigenvalue for fixed $N$, we also express the limiting probability distribution in terms of some non-linear second order differential equation. Eventually, we show that the convergence of the probability distribution function of the re-scaled largest eigenvalue to the limiting one is at least of order $(1/N)$.

**Keywords** Random matrices · Generalized Cauchy Ensemble · Painlevé equations · Determinantal processes · Limit theorems

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## 1 Introduction and Results

Let $H(N)$ be the set of Hermitian matrices endowed with the measure

$$\mathrm{const}\cdot\det(1+X^2)^{-N}\prod_{1\le j<k\le N} dX_{jk}\prod_{i=1}^{N} dX_{ii},\quad X\in H(N), \tag{1.1}$$

where const is a normalizing constant, such that the total mass of $H(N)$ is equal to one. This measure is the analogue of the normalized Haar measure $\mu_N$ on the unitary group $U(N)$, if one relates $U(N)$ and $H(N)$ via the Cayley transform: $H(N)\ni X\mapsto U=\frac{X+i}{X-i}\in U(N)$. The measure (1.1) can be deformed to obtain the following two parameters probability measure:

$$\mathrm{const}\cdot\det((1+iX)^{-s-N})\det((1-iX)^{-\bar{s}-N})\prod_{1\le j<k\le N} dX_{jk}\prod_{i=1}^{N} dX_{ii}, \tag{1.2}$$

where $s$ is a complex parameter such that $\Re s>-1/2$ (otherwise the quantity involved in (1.2) does not integrate as is proved in [3]). Following Forrester and Witte [10] and [26], we call this measure the *generalized Cauchy measure* on $H(N)$. The name is chosen because if $s=0$ and $N=1$, (1.2) is nothing else than the density of a Cauchy random variable. This measure can be projected onto the space $\mathbb{R}^N/S(N)$, where $S(N)$ is the symmetric group of order $N$ and the quotient space is considered as the space of all (unordered) sets of eigenvalues of matrices in $H(N)$. This projection gives the eigenvalue density

$$\mathrm{const}\cdot\prod_{1\le j<k\le N}(x_j-x_k)^2\prod_{j=1}^{N} w_H(x_j)dx_j, \tag{1.3}$$

where $w_H(x_j)=(1+ix_j)^{-s-N}(1-ix_j)^{-\bar{s}-N}$, and where the $x_j$'s denote the eigenvalues, and as usual, the constant is chosen so that the total mass of $\mathbb{R}^N/S(N)$ is equal to one. $H(N)$, endowed with the generalized Cauchy measure, shall be called the *generalized Cauchy random matrix ensemble*, noted $GCyE$.

If one replaces the weight $w_H(x)$ in (1.3) by $w_2(x)=e^{-x^2}$, then one obtains the Hermite ensemble. Similarly, the choice $w_L(x)=x^a e^{-x}$ on $\mathbb{R}_+$ or $w_J(x)=(1-x)^\alpha(1+x)^\beta$ for $-1\le x\le 1$ leads to the Laguerre or Jacobi ensemble. The three classical weight functions $w_2$, $w_L$ and $w_J$ occur in the eigenvalue PDF for certain ensembles of Hermitian matrices based on matrices with independent Gaussian entries (see for example [9]). In [1], the defining property of a classical weight function in this context was identified as the following fact: If one writes the weight function $w(x)$ of an ensemble as $w(x)=e^{-2V(x)}$, with $2V'(x)=g(x)/f(x)$, $f(x)$ and $g(x)$ being polynomials in $x$, then the operator $n=f(d/dx)+(f'-g)/2$ increases the degree of the polynomials by one, and thus, $\deg f\le 2$, and $\deg g\le 1$. If $s\in(-1/2,\infty)$, this property actually also holds for the $GCyE$, and we obtain a fourth classical weight function (see also Witte and Forrester [26]). However, the construction of the matrix model for the $GCyE$ is different from the construction of the other three classical ensembles: A matrix model for the $GCyE$ will not have independent entries, but one can construct the ensemble via the Cayley transform. Indeed, following Borodin and Olshanski [3] (see also [9, 10, 26]) the measure (1.2) is, via the Cayley-transform, equivalent to the deformed normalized Haar measure $\mathrm{const}\cdot\det((1-U)^{\bar{s}})\det((1-U^*)^{s})\mu_N(dU)$, $U\in U(N)$. If we denote by $e^{i\theta_j}$, $j=1,\dots,N$, the eigenvalues of a unitary matrix with

$\theta_j \in [-\pi, \pi]$, the deformed Haar measure can, as in the Hermitian case, be projected to the eigenvalue probability measure to obtain the PDF

$$\text{const} \cdot \prod_{1 \le j < k \le N} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{N} w_U(\theta_j) d\theta_j, \tag{1.4}$$

where $w_U(\theta_j) = (1 - e^{i\theta_j})^{\overline{s}}(1 - e^{-i\theta_j})^s$, and $\theta_j \in [-\pi, \pi]$. This measure is defined on $\mathbb{S}^N/S(N)$, where $\mathbb{S}$ is the complex unit circle. Note, that this eigenvalue measure has a singularity at $\theta = 0$, if $s \neq 0$. Borodin and Olshanski [3] studied the measures (1.2), (1.3) and (1.4) in great detail due to their connections with representation theory of the infinite dimensional unitary group $U(\infty)$.

When $s \in (-1/2, \infty)$, (1.4) is nothing else than the eigenvalue distribution of the circular Jacobi unitary ensemble. This is a generalization of the Circular Unitary ensemble corresponding to the case $s = 0$. In fact, if $s = 1$, this corresponds to the CUE case with one eigenvalue fixed at one. More generally, for $s \in (-1/2, \infty)$ the singularity at one corresponds, in the log-gas picture, to a impurity with variable charge fixed at one, and mobile unit charges represented by the eigenvalues (see Witte and Forrester [26], and also [11]). It is the singularity at one that makes the study of this ensemble more difficult than the CUE. In the special case when $s = 0$, one can obtain the eigenvalues with PDF (1.3) from the eigenvalues of the circular unitary ensemble using a stereographic projection (see the book of Forrester [9], Chap. 2, Sect. 5 on the Cauchy ensemble). In fact, in this case, we get that (1.3) represents the Boltzmann factor for a one-component log-gas on the real line subject to the potential $2V(x) = N \log(1 + x^2)$. This corresponds to an external charge of strength $-N$ placed at the point $(0, 1)$ in the plane (this can also be generalized to arbitrary inverse temperature $\beta$ as given in the previous reference). Moreover, note that when $s \neq 0$, a construction of a random matrix ensemble with eigenvalue PDF (1.4) is given in [5].

In this paper, we are interested in the convergence and the asymptotic distribution of the re-scaled largest eigenvalue of a random matrix drawn from the generalized Cauchy ensemble, *for all admissible values of the parameter* $s$, namely $\Re(s) > -1/2$. Moreover, we will also address the problem of the rate of convergence in such a limit Theorem. In random matrix theory, the distribution of the largest eigenvalue as well as the problem of the convergence of the scaled largest eigenvalue, have received much attention (see e.g. [20–23]). Also the latter problem on the rate of convergence has been studied, especially in [12] and [6] for GUE and LUE matrices, and in [14] as well as in [8] for Wishart matrices. To deal with the law of the largest eigenvalue, there is a well established methodology (see [17]) for matrix ensembles with eigenvalue PDF of the form

$$\text{const} \cdot \prod_{1 \le j < k \le N} (x_j - x_k)^2 \prod_{j=1}^{N} w(x_j) dx_j, \tag{1.5}$$

where $w(x)$ is a weight function on $\mathbb{R}$. If one can define the set of monic orthogonal polynomials $\{p_n\}$ with respect to the weight function $w(x)$ on $\mathbb{R}$, then one defines the integral operator $K_N$ on $L_2(\mathbb{R})$, associated with the kernel $K_N(x, y) := \sum_{i=0}^{N-1} \frac{p_i(x)p_i(y)}{\|p_i\|^2} \sqrt{w(x)w(y)}$. Using this kernel, the formula to describe probabilities of the form

$$E(k, J) := \mathbb{P}[\text{there are exactly } k \text{ eigenvalues inside the interval } J],$$

where $J \subset \mathbb{R}$ and $k \in \mathbb{N}$, is (see [17]):

$$E(k, J) = \frac{(-1)^k}{k!} \frac{d^k}{dx^k} \det(I - xK_N)|_{x=1},$$

where the determinant is a Fredholm determinant and the operator $K_N$ is restricted to $J$. Note that if one takes $J = (t, \infty)$ for some $t \in \mathbb{R}$, then $E(0, (t, \infty))$ is simply the probability distribution of the largest eigenvalue, denoted from now on by $\lambda_1(N)$, of a $N \times N$ matrix in the respective ensemble. In their pioneering work [25], Tracy and Widom give a system of completely integrable differential equations to show how the probability $E(0, J)$ can be linked to solutions of certain Painlevé differential equations. Tracy and Widom apply their method to the finite Hermite, Laguerre and Jacobi ensembles. Moreover, one can also apply the method to scaling limits of random matrix ensembles, when the dimension $N$ goes to infinity. The sine kernel and its Painlevé-V representation for instance, as obtained by Jimbo, Miwa, Môri and Sato [13], arise if one takes the scaling limit in the bulk of the spectrum of the Gaussian Unitary Ensemble and of many other Hermitian matrix ensembles (see e.g. [15, 16, 18] and [19]). On the other hand, if one scales appropriately at the edge of the Gaussian Unitary Ensemble, one obtains an Airy kernel in the scaling limit with a Painlevé-II representation for the distribution of the largest eigenvalue (see [23]). Similar results have been obtained for the edge scalings of the Laguerre and Jacobi ensembles, where the Airy kernel has to be replaced by the Bessel kernel and the Painlevé-II equation by a Painlevé-V equation (see Tracy and Widom [24]). Soshnikov [22] gives an overview on scaling limit results for large random matrix ensembles.

For the eigenvalue measure (1.3), Borodin and Olshanski [3] give the kernel in the finite $N$ case, denoted by $K_N$ in the following (see Theorem 1.1), as well as a scaling limit of this kernel, when $N \to \infty$, denoted by $K_\infty$ (see (1.17)). Using the kernel $K_N$, one can set up the system of differential equations in the way of Tracy and Widom for the law $E(0, (t, \infty))$ of the largest eigenvalue $\lambda_1(N)$, for any $t \in \mathbb{R}$. In the case of a real parameter $s$, this has been done by Forrester and Witte in [26]. They obtain a characterization of the law of the largest eigenvalue in terms of a Painlevé-VI equation. More precisely, $(1 + t^2)$ times the logarithmic derivative of $E(0, (t, \infty))$ satisfies a Painlevé-VI equation. The same method suitably modified leads to a generalization of this result for complex $s$. However, the method of Tracy and Widom has the drawback that it only works for $s$ with $\Re s > 1/2$. Forrester and Witte propose in [10] an alternative method which makes use of $\tau$-function theory, to derive the Painlevé-VI characterization for $E(0, (t, \infty))$ for any $s$ such that $\Re s > -1/2$.

To sum up, for the generalized Cauchy ensemble, it is known that for finite $N$, $(1 + t^2)$ times the logarithmic derivative of $E(0, (t, \infty))$ satisfies a Painlevé-VI equation, for $t \in \mathbb{R}$. The orthogonal polynomials associated with the measure $w_H$ are known as well as the scaling limit of the associated kernel $K_N$, which we note $K_\infty$. One naturally expects $\lambda_1(N)$, appropriately scaled, to converge in law to the probability distribution $F_\infty(t) := \det(I - K_\infty)|_{L_2(t,\infty)}$, for $t > 0$ ($t \leq 0$ is not permissible in this particular case, as we will see in remark 1.6). We shall see below that this is indeed the case for all values of $s$ such that $\Re(s) > -1/2$. A natural question is: does $(1 + t^2)$ times the logarithmic derivative of $F_\infty(t)$ also satisfy some non-linear differential equation? And as previously mentioned, what is the rate of convergence to $F_\infty(t)$?

### Statement of the Main Results

We now state our main theorems. Our results are based on earlier work by Borodin and Olshanski [3] who obtained an explicit form for the orthogonal polynomials associated with the weight $w_H$ as well as the scaling limit for the associated kernel, and Forrester and Witte [10] who express, for fixed $N$ and for any complex number $s$, with $\Re(s) > -1/2$, the probability distribution of the largest eigenvalue $\lambda_1(N)$ in terms of some non-linear differential equation. For clarity and to fix the notations, we first state a Theorem of Borodin and Olshanski [3]. We refer the reader to the paper [3] for more information on the determinantal aspects. The discussion on the methods we use is postponed to the end of this Section.

Borodin and Olshanski [3] give the correlation kernel for the determinantal point process defined by the measure (1.3). In fact, the monic orthogonal polynomial ensemble $\{p_m;\ m < \Re s + N - \frac{1}{2}\}$ on $\mathbb{R}$ associated with the weight $w_H(x)$, is defined by $p_0 \equiv 1$, and

$$p_m(x) = (x-i)^m {}_2F_1\left[-m,\ s+N-m,\ 2\Re s + 2N - 2m;\ \frac{2}{1+ix}\right], \tag{1.6}$$

where ${}_2F_1[a,\ b,\ c;\ x] = \sum_{n\geq 0} \frac{(a)_n(b)_n}{(c)_n n!} x^n$ is the Gauss hypergeometric function, and $(x)_n = x(x+1)\cdots(x+n-1)$. Using the Christoffel-Darboux formula and the theory of orthogonal polynomials, the following was proven by Borodin and Olshanski [3]:

**Theorem 1.1** *The $n$-point correlation function $(n \leq N)$ for the eigenvalue distribution* (1.3) *is given by*

$$\rho_n^{s,N}(x_1,\ldots,x_n) = \det\left(K_{s,N}(x_i,x_j)\right)_{i,j=1}^n,$$

*with the kernel $K_{s,N}(x,y)$ defined on $\mathbb{R}^2$ is given by*:

$$\begin{aligned} K_N(x,y) := K_{s,N}(x,y) &= \sum_{m=0}^{N-1} \frac{p_m(x)p_m(y)}{\|p_m\|^2}\sqrt{w_H(x)w_H(y)} \\ &= \frac{\phi(x)\psi(y)-\phi(y)\psi(x)}{x-y}, \end{aligned} \tag{1.7}$$

*with*

$$\phi(x) = \sqrt{Cw_H(x)}\,p_N(x), \tag{1.8}$$

*and*

$$\psi(x) = \sqrt{Cw_H(x)}\,p_{N-1}(x), \tag{1.9}$$

*where* $w_H(x) = (1+ix)^{-s-N}(1-ix)^{-\bar{s}-N} = (1+x^2)^{-\Re s - N} e^{2\Im s Arg(1+ix)}$ *and*

$$C := C_{N,s} = \frac{2^{2\Re s}}{\pi}\Gamma\begin{bmatrix} 2\Re s + N + 1, & s+1, & \bar{s}+1 \\ N, & 2\Re s + 1, & 2\Re s + 2 \end{bmatrix}. \tag{1.10}$$

*Here, we use the notation*:

$$\Gamma\begin{bmatrix} a, & b, & c, & \ldots \\ d, & e, & f, & \ldots \end{bmatrix} = \frac{\Gamma(a)\Gamma(b)\Gamma(c)\cdots}{\Gamma(d)\Gamma(e)\Gamma(f)\cdots}. \tag{1.11}$$

*Moreover, if $x = y$, the kernel is given by*:

$$K_N(x,x) = \phi'(x)\psi(x) - \phi(x)\psi'(x), \tag{1.12}$$

*using the Bernoulli-Hépital rule.*

Note that $p_N$ is well-defined (and in $L_2(w_H)$) only for $\Re s > 1/2$. However, it can be analytically continued to $\Re s > -1/2$ using the hypergeometric expression $p_N(x) = (x-i)^N {}_2F_1[-N,\ s,\ 2\Re s;\ 2/(1+ix)]$, except if $\Re s = 0$. Moreover, Borodin and Olshanski [3] give a way to get rid of the singularity at $\Re s = 0$. They introduce the polynomial

$$\begin{aligned}\tilde{p}_N(x) &= p_N(x) - \frac{iNs}{\Re s(2\Re s+1)} p_{N-1}(x)\\ &= (x-i)^N {}_2F_1\left[-N, s, 2\Re s + 1; \frac{2}{1+ix}\right].\end{aligned} \tag{1.13}$$

This polynomial makes sense for any $s \in \mathbb{C}$ with $\Re s > -1/2$ and one can define the kernel in Theorem 1.1 equivalently by:

$$K_N(x,y) = C\frac{\tilde{p}_N(x)p_{N-1}(y) - p_{N-1}(x)\tilde{p}_N(y)}{x-y}\sqrt{w_H(x)w_H(y)}. \tag{1.14}$$

We are interested in the distribution of the largest eigenvalue $\lambda_1(N)$ of a matrix in the $GCyE$. We have already seen that the probability that $\lambda_1(N)$ is smaller than $t$, is

$$E(0,(t,\infty)) = \det(I - K_N)|_{L_2(t,\infty)}, \tag{1.15}$$

for any $t \in \mathbb{R}$. Hence, we need to consider the operator $K_N$ with kernel $K_N(x,y)$ restricted to the interval $(t,\infty)$ to calculate the probability that no eigenvalue is in the interval $(t,\infty)$. This restriction is symmetric, with eigenvalues between 0 and 1. It is easy to see that $K_N$, restricted to any subinterval $J$ (or finite union of subintervals) of $\mathbb{R}$, has no eigenvalue equal to 1, since $E(0,(t,\infty)) > 0$ for any $t \in \mathbb{R}$. This is true because

$$P(\lambda_1(N) \le t) = \text{cst}\int_{(-\infty,t)^N} \prod (x_j - x_k)^2 \prod w_H(x_j)dx_1\cdots dx_N,$$

and the integrand is strictly positive. Moreover, restricting the correlation function $\rho_n^{s,N}$ of Theorem 1.1 to $J$ gives

$$\begin{aligned}\rho_n^{s,N}(x_1,\ldots,x_n)|_J &= \prod_{j=1}^n \chi_J(x_j)\rho_n^{s,N}(x_1,\ldots,x_n)\\ &= \prod_{j=1}^n \chi_J(x_j)\det(K_N(x_i,x_j))_{i,j=1}^n\\ &= \det(\chi_J(x_i)K_N(x_i,x_j)\chi_J(x_j))_{i,j=1}^n,\end{aligned} \tag{1.16}$$

where $\chi_J$ denotes the indicator function of the set $J$. Therefore, the restriction of $K_N$ to $J$, denoted by $K_{N,J}$, defines a determinantal process on $J$ with kernel $\chi_J(x)K_N(x,y)\chi_J(y) =: K_{N,J}(x,y)$.

Borodin and Olshanski [3] give a scaling limit for the kernel $K_N(x,y)$ given in Theorem 1.1. Namely, $\lim_{N\to\infty} NK_N(Nx,Ny) = K_\infty(x,y)$, for any $x,y \in \mathbb{R}^* = \mathbb{R}\backslash\{0\}$, where the kernel $K_\infty$ is defined by

$$K_\infty(x,y) = \frac{1}{2\pi} \frac{\Gamma(s+1)\Gamma(\overline{s}+1)}{\Gamma(2\Re s+1)\Gamma(2\Re s+2)} \frac{\tilde{P}(x)Q(y) - Q(x)\tilde{P}(y)}{x-y}, \tag{1.17}$$

if $x \neq y$, and,

$$K_\infty(x,x) = \frac{1}{2\pi} \frac{\Gamma(s+1)\Gamma(\overline{s}+1)}{\Gamma(2\Re s+1)\Gamma(2\Re s+2)} (\tilde{P}'(x)Q(x) - Q'(x)\tilde{P}(x)), \tag{1.18}$$

where

$$\tilde{P}(x) = |2/x|^{\Re s} e^{-i/x+\pi \Im s \,\mathrm{Sgn}(x)/2} {}_1F_1\left[s, 2\Re s+1; \frac{2i}{x}\right],$$

$$Q(x) = (2/x)|2/x|^{\Re s} e^{-i/x+\pi \Im s \,\mathrm{Sgn}(x)/2} {}_1F_1\left[s+1, 2\Re s+2; \frac{2i}{x}\right],$$

with

$${}_1F_1\left[r,q;x\right] = \sum_{n\geq 0} \frac{(r)_n}{(q)_n n!} x^n,$$

for any $r, q, x \in \mathbb{C}$.

*Remark 1.2* The kernel $K_\infty$ defines a determinantal point process (see [3] , Theorems IV and 6.1).

*Remark 1.3* If $s = 0$, the limiting kernel $K_\infty$ writes as

$$K_\infty(x_1,x_2) = \frac{1}{\pi} \frac{\sin(1/x_2 - 1/x_1)}{x_1 - x_2}.$$

Under the change of variable $y = \frac{1}{\pi x}$ and taking into account the corresponding change of the differential $dx$, $K_\infty$ translates to the famous sine kernel with correlation function

$$\rho_n(y_1,\dots,y_n) = \det\left(\frac{\sin(\pi(y_i - y_j))}{\pi(y_i - y_j)}\right)_{i,j=1}^n,$$

for any $n \in \mathbb{N}$ and $y_1,\dots,y_n \in \mathbb{R}$ (see Borodin and Olshanski [3]).

Before stating our main results, we need to introduce one more notation: we note $K_{[N]}(x,y)$ the kernel

$$K_{[N]}(x,y) := N K_N(Nx, Ny), \tag{1.19}$$

and $K_{[N]}$ the associated integral operator. We also recall the definition of the Fredholm determinant: if $K$ is an integral operator with kernel given by $K(x,y)$, then the $k$-correlation function $\rho_k$ is defined by:

$$\rho_k(x_1,\dots,x_k) = \det(K(x_i,x_j)_{1\leq i,j\leq k}).$$

The Fredholm determinant $F$, from $\mathbb{R}_+^*$ to $\mathbb{R}$, is then defined by

$$F(t) = 1 + \sum_{k\geq 1} \frac{(-1)^k}{k!} \int_{(t,\infty)^k} \rho_k(x_1, \ldots, x_k) dx_1 \cdots dx_k. \tag{1.20}$$

**Theorem 1.4** *For $s$ such that $\Re s > -1/2$ and $t > 0$, let $F_N$ be the Fredholm determinant associated with $K_{[N]}$, and let $F_\infty$ be the Fredholm determinant associated with $K_\infty$. Then, $F_N$ and $F_\infty$ are in $\mathcal{C}^3(\mathbb{R}_+^*, \mathbb{R})$, and for $p \in \{0, 1, 2, 3\}$, the $p$-th derivative of $F_N$ (with respect to $t$) converges pointwise to the $p$-th derivative of $F_\infty$.*

As an immediate consequence, one obtains the following convergence in law for the re-scaled largest eigenvalue:

**Corollary 1.5** *Given the set of $N \times N$ random Hermitian matrices $H(N)$ with the generalized Cauchy probability distribution (1.2), denote by $\lambda_1(N)$ the largest eigenvalue of such a randomly chosen matrix. Then, the law of $\lambda_1(N)/N$ converges to the distribution of the largest point of the determinantal process on $\mathbb{R}^*$ described by the limiting kernel $K_\infty(x, y)$ in the following sense*:

$$P\left[\frac{\lambda_1(N)}{N} \leq x_0\right] = \det(I - K_N)|_{L_2(Nx_0,\infty)} \longrightarrow \det(I - K_\infty)|_{L_2(x_0,\infty)}, \quad \textit{as } N \to \infty,$$

*for any $x_0 > 0$.*

*Remark 1.6* Note that in the case of finite $N$, the range of the largest eigenvalue is the whole real line, whereas in the limit case when $N \to \infty$, the range of the largest eigenvalue is $\mathbb{R}_+^*$. This is because an infinite number of points accumulate close to 0 (0 itself being excluded however). The accumulation of the points can be seen from the fact that due to the form of $K_\infty(x, x)$ (see (1.18)), $\lim_{\epsilon\to\infty} \int_\epsilon^\infty K_\infty(x, x)dx$ diverges.

Now, define

$$\theta_\infty(\tau) = \tau \frac{d \log \det(I - K_\infty)|_{L_2(\tau^{-1},\infty)}}{d\tau}, \quad \tau > 0. \tag{1.21}$$

Using the result of Forrester and Witte [10] for the distribution of the largest eigenvalue for fixed $N$ and Theorem 1.4, we are able to show:

**Theorem 1.7** *Let $s$ be such that $\Re s > -1/2$. Then the function $\theta_\infty$ given by (1.21) is well defined and is a solution to the Painlevé-V equation on $\mathbb{R}_+^*$*:

$$\begin{aligned} -\tau^2(\theta''(\tau))^2 &= \left[2(\tau\theta'(\tau) - \theta(\tau)) + (\theta'(\tau))^2 + i(\bar{s} - s)\theta'(\tau)\right]^2 \\ &\quad - (\theta'(\tau))^2(\theta'(\tau) - 2is)(\theta'(\tau) + 2i\bar{s}). \end{aligned} \tag{1.22}$$

*Remark 1.8* This implies in particular the result of Jimbo, Miwa, Môri and Sato [13] that the sine kernel, which is the special case of the $K_\infty$ kernel with parameter $s = 0$ (see remark 1.3), satisfies the Painlevé-V equation (1.22) with $s = 0$.

Eventually, following our initial motivation, we have the following result about the rate of convergence:

**Theorem 1.9** *For all $x_0 > 0$, and for $x > x_0$,*

$$\left| P\left[\frac{\lambda_1(N)}{N} \le x\right] - \det(I - K_\infty)|_{L_2(x,\infty)} \right| \le \frac{1}{N} C(x_0, s),$$

*where $C(x_0, s)$ is a constant depending only on $x_0$ and $s$.*

Now, we say a few words about the way we prove the above theorems. Our proofs, splitted into several technical lemmas, only use elementary methods; namely, they only involve checking pointwise convergence and domination in all the quantities involved in the Fredholm determinants of $K_{[N]}$ and $K_\infty$. We can then apply dominated convergence to show that the logarithmic derivative of the Fredholm determinant of $K_{[N]}$, as well as its derivatives, converge pointwise to the respective derivatives of the Fredholm determinant of $K_\infty$. This suffices to show that the Fredholm determinant of $K_\infty$ satisfies a Painlevé-V equation because we can write the rescaled finite $N$ Painlevé-VI equation of Forrester and Witte ([10, 11] and Theorem 2.1 below) as the sum of polynomial functions of the Fredholm determinant of $K_{[N]}$ and their first, second and third derivatives. Moreover, the various estimates and bounds we obtain for the different determinants and functions involved in our problem help us to obtain directly an estimate for the rate of convergence in Corollary 1.5 (that is Theorem 1.9).

Given Theorem 1.1 and the Painlevé VI characterization of Forrester and Witte [10], the results contained in Theorem 1.4 and Corollary 1.5 are very natural; but yet they have to be rigorously checked. As far as Theorem 1.7 is concerned, Borodin and Deift [2] obtain the same equation as (1.22) from the scaling limit of a Painlevé-VI equation characterizing a general 2F1-kernel similar to our kernel $K_N$ (Sect. 8 in [2]). They claim that it is natural to expect that the appropriately scaled logarithmic derivative of the Fredholm determinant of their 2F1-kernel solves this Painlevé-V equation. In fact, according to our Theorem 1.7, (1.21) corresponds to their limit, when $N \to \infty$, of the scaled solution of the Painlevé-VI equation and solves the Painlevé-V equation (1.22). Borodin and Deift's method is based on the combination of Riemann-Hilbert theory with the method of isomonodromic deformation of certain linear differential equations. The method is very powerful and general; however, we were not able to apply it in our situation; moreover, it seems that we would have to restrict ourselves to the values of $s$ such that $0 \le \Re(s) \le 1$. As we shall mention it later in the paper, all the ingredients seem to be there to apply the method of Tracy and Widom [25]; here again, we were not able to find our way: it seems to us (see next section) that with this method, we could obtain at best a second order non-linear differential equation for $\theta_\infty$, which is equivalent to (1.22), but for a restricted range of $s$: $\Re(s) > 1/2$, and thus excluding the case $s = 0$ of the sine kernel. On the other hand, our method to prove Theorem 1.7 heavily relies on the result of Forrester and Witte [10] for fixed $N$: hence we do not provide a general method to obtain Painlevé equations. However, it is an efficient approach to obtain some information about the rate of convergence in Corollary 1.5.

## 2 Proof of Theorems 1.4, and 1.7

In this section, we split the proofs of Theorems 1.4, and 1.7 into several technical Lemmas. The notations are those introduced in Sect. 1. Throughout this paper, the notation $C(a_0, a_1, \dots, a_n)$ stands for a positive constant which only depends on the parameters $a_0, a_1, \dots, a_n$, and whose value may change from line to line (we shall not be interested in explicit values for the different constants). We first bring in an ODE that $\theta_\infty$ should satisfy; then we prove several technical lemmas about the convergence of the correlation functions and the derivatives of the kernel $K_{[N]}$. We shall use these lemmas to show that $\theta_\infty(t)$ is indeed well defined (i.e. $F_\infty(t)$ is non-zero for any $t > 0$) and to prove Theorems 1.4 and 1.7.

## 2.1 Scaling Limits

We now state a result by Forrester and Witte [10] which will play an important role in the proof of Theorem 1.7.

**Theorem 2.1** *For* $\Re(s) > -1/2$, *define*

$$\begin{aligned}\sigma(t) &= (1+t^2)\frac{d}{dt}\log\det(I-K_N)|_{L_2(t,\infty)}\\ &= (1+t^2)\frac{d}{dt}\log P[\textit{there is no eigenvalue inside } (t,\infty)].\end{aligned}$$

*Then, for* $t \in \mathbb{R}$, $\sigma(t)$ *satisfies the equation*:

$$\begin{aligned}&(1+t^2)(\sigma'')^2 + 4(1+t^2)(\sigma')^3 - 8t(\sigma')^2\sigma\\ &\quad + 4\sigma^2(\sigma' - (\Re s)^2) + 8(t(\Re s^2) - \Re s\Im s - N\Im s)\sigma\sigma'\\ &\quad + 4(2t\Im s(N+\Re s) - (\Im s)^2 - t^2(\Re s)^2 + N(2\Re s + N))(\sigma')^2 = 0. \qquad (2.1)\end{aligned}$$

*Remark 2.2* The ODE (2.1) is equivalent to the master Painlevé equation (SD-I) of Cosgrove and Scoufis [7]. Cosgrove and Scoufis, show that the solution of this equation can be expressed in terms of the solution of a Painlevé-VI equation using a Bäcklund transform. In the case of $s$ real, this transformation is described in Forrester and Witte [26].

*Remark 2.3* One can also attempt to use the general method introduced by Tracy and Widom in [25] for kernels of the form (1.7) to prove the above Theorem. Their method establishes a system of PDE's, the so called Jimbo-Miwa-Môri-Sato equations, which can be reduced to a Painlevé-type equation. The PDE's consist of a set of universal equations and a set of equations depending on the specific form of the following recurrence differential equation for $\phi$ and $\psi$:

$$\begin{aligned}m(x)\phi'(x) &= A(x)\phi(x) + B(x)\psi(x),\\ m(x)\psi'(x) &= -C(x)\phi(x) - A(x)\psi(x),\end{aligned} \qquad (2.2)$$

where $A, B, C$ and $m$ are polynomials in $x$. Doing the calculations for the case of the determinantal process with kernel $K_N$, one obtains that for $\phi$ and $\psi$ given in Theorem 1.1, the recurrence equations (2.2) hold with:

$$\begin{aligned}m(x) &= 1 + x^2,\\ A(x) &= -x\Re s + \Im s\left(1 + \frac{N}{\Re s}\right),\\ B(x) &= \frac{|s|^2}{\Re s^2}N\frac{2\Re s + N}{2\Re s + 1},\\ C(x) &= 2\Re s + 1.\end{aligned}$$

Note that this only makes sense if $\Re s \neq 0$. One can then show that for $t \in \mathbb{R}$, and $\Re(s) > 1/2$, (2.1) holds. In the case of $s \in (1/2, \infty) \subset \mathbb{R}$, this Theorem was obtained in this way by Forrester and Witte in [26] (Proposition 4). We would like to shortly explain the reason why we were able to make this method work only for $\Re(s) > 1/2$. Indeed, the method of Tracy and Widom has originally been developed for finite intervals (or unions of finite intervals). If one applies the method to the case of a semi-infinite interval $(t, \infty)$, one has to consider an interval $(t, a)$, where $a > t$. Then, one writes down the PDE's of Tracy and Widom for that interval and takes the limit in all the equations as $a \to \infty$. Note that the variables in these PDE's are the end-points $t$ and $a$ of the interval. It is clear, that one has to be careful about the convergence of the quantities involved in these equations, when $a \to \infty$. In particular, one needs in our case that the term $(1 + a^2)Q(a)R(t, a)$, where $R(x, y)$ is the kernel of the resolvent operator $K_{N,J}(1 - K_{N,J})^{-1}$, and $Q(x) = (I - K_{N,J})^{-1}\phi(x)$, which is of order $a^{1-2\Re s}$, tends to zero, when $a \to \infty$. This implies the restriction $\Re s > 1/2$. One might encounter the same type of obstacle in an attempt to prove Theorem 1.7 with this method (we will give the corresponding recurrence equations for $\phi$ and $\psi$ in the case of $K_\infty$ in Remark 2.17).

We now show that when $N \to \infty$, the ODE (2.1) converges to a $\sigma$-version of the Painlevé-V equation. This limiting equation is also given in Borodin and Deift [2] (Proposition 8.14). Borodin and Deift obtain this equation as a scaling limit of a Painlevé-VI equation characterizing their 2F1-kernel. However, their 2F1-kernel is different from our kernel $K_N$.

Set for $\tau > 0$,

$$\theta(\tau) := \theta_N(\tau) := \tau \frac{d \log \det(1 - K_N)|_{L_2(N\tau^{-1}, \infty)}}{d\tau}, \tag{2.3}$$

where $R(x, y)$ is the kernel of the resolvent operator $K_{N,J}(1 - K_{N,J})^{-1}$. Then,

$$\theta(\tau) = \tau\left(-\frac{N}{\tau^2}\right) R\left(\frac{N}{\tau}, \frac{N}{\tau}\right) = -\frac{N}{\tau}\left[\frac{\sigma(\frac{N}{\tau})}{1 + \frac{N^2}{\tau^2}}\right].$$

It follows that

$$\sigma\left(\frac{N}{\tau}\right) = -\theta(\tau)\left(\frac{\tau}{N} + \frac{N}{\tau}\right),$$

$$\sigma'\left(\frac{N}{\tau}\right) = \frac{\tau^2}{N^2}(\tau\theta'(\tau) + \theta(\tau)) + (\tau\theta'(\tau) - \theta(\tau)),$$

$$\sigma''\left(\frac{N}{\tau}\right) = -\frac{\tau^3}{N^3}[4\tau\theta'(\tau) + 2\theta(\tau) + \tau^2\theta''(\tau)] - \frac{\tau^3}{N}\theta''(\tau).$$

Now, put this into the ODE (2.1) with $t = \frac{N}{\tau}$. After dividing by $N^2$, we obtain:

$$\begin{aligned}
&\left(\frac{1}{\tau^2}\right)^2 (\tau^3\theta''(\tau))^2 + 4\left(\frac{1}{\tau^2}\right)(\tau\theta'(\tau) - \theta(\tau))^3 + \frac{8}{\tau}(\tau\theta'(\tau) - \theta(\tau))^2\frac{\theta(\tau)}{\tau} \\
&+ 4\left(\frac{\theta(\tau)}{\tau}\right)^2(\tau\theta'(\tau) - \theta(\tau) - (\Re s)^2) - 8\left(\frac{(\Re s)^2}{\tau} - \Im s\right)\frac{\theta(\tau)}{\tau}(\tau\theta'(\tau) - \theta(\tau)) \\
&+ 4\left[2\frac{\Im s}{\tau} - \frac{(\Re s)^2}{\tau^2} + 1\right](\tau\theta'(\tau) - \theta(\tau))^2 = O(N^{-1}).
\end{aligned}$$

This gives

$$-\tau^2(\theta''(\tau))^2 = 4\left\{(\theta'(\tau))^2(\tau\theta'(\tau)-\theta(\tau)-(\Re s)^2)+2\Im s\,\theta'(\tau)(\tau\theta'(\tau)-\theta(\tau)) + (\tau\theta'(\tau)-\theta(\tau))^2\right\} + O(N^{-1}).$$

Now if one neglects the terms of order $O(N^{-1})$, it is easy to see that this is precisely equation (1.22). But this is also exactly the $\sigma$-form of the Painlevé-V equation in Borodin and Deift [2], Proposition 8.14.

Hence, $\theta_N(\tau)(=\theta(\tau))$ satisfies a differential equation which tends to the $\sigma$-Painlevé-V equation and we have the following Proposition:

**Proposition 2.4** *The ODE (2.1) with the change of variable $t = N/\tau$, $\tau > 0$, is solved by $\theta_N(\tau)$, and is of the form*

$$\sum_{k=0}^{m} N^{-k}\frac{P_k(\tau,\theta_N(\tau),\theta'_N(\tau),\theta''_N(\tau))}{\tau^q} = 0,$$

*where $m$ and $q$ are universal integers and the $P_k$'s are polynomials which are independent of $N$. Moreover, $P_0(\tau,\theta_N(\tau),\theta'_N(\tau),\theta''_N(\tau))\tau^{-q}$ corresponds to the $\sigma$-form of the Painlevé-V equation (1.22).*

*Remark 2.5* We note that $\theta_N(\tau)$, given by (2.3), is a solution of the ODE (2.1), with $t = N/\tau$. Moreover, we know that $\lim_{N\to\infty} NK_N(x,y) = K_\infty(x,y)$, for any $x, y \in \mathbb{R}^*$. Hence it is natural to guess that $\theta_\infty(\tau)$ should satisfy the ODE (1.22).

## 2.2 Some Technical Lemmas

For clarity, we decompose the proof of our Theorems into several Lemmas about the convergence of correlation functions and the derivatives of the kernel $K_{[N]}$.

**Lemma 2.6** *Let $K$ be a function in $C^2((\mathbb{R}_+^*)^2,\mathbb{R})$, such that for all $k \in \mathbb{N}$, and $x_1, x_2, \dots, x_k > 0$, the matrix $K(x_i,x_j)_{1\le i,j\le k}$ is symmetric and positive. Define the $k$-correlation function $\rho_k$ by:*

$$\rho_k(x_1,\dots,x_k) = \det(K(x_i,x_j)_{1\le i,j\le k}),$$

*and suppose that for $(p,q) \in \{(i,j);\ i,j\in\mathbb{N}_0,\ i+j\le 2\}$, for some $\alpha > 1/2$, and for all $x_0 > 0$, one has the upper bound*

$$\left|\frac{\partial^{p+q}}{\partial x^p \partial y^q}K(x,y)\right| \le \frac{C(x_0)}{(xy)^\alpha}, \tag{2.4}$$

*if $x, y \ge x_0$. Then, $\rho_k$ is in $C^2((\mathbb{R}_+^*)^k,\mathbb{R})$ for all $k$, and for all $x_0 > 0$, $x_1,\dots,x_k \ge x_0$, one has:*

$$\left|\frac{\partial^p}{\partial x_j^p}\rho_k(x_1,\dots,x_k)\right| \le \frac{(C(x_0))^k}{(x_1\cdots x_k)^{2\alpha}}, \tag{2.5}$$

*if $p \in \{0,1,2\}$ and $j \in \{1,\dots,k\}$. Moreover,*

$$\frac{\partial^p}{\partial x_j^p}\rho_k(x_1,\dots,x_k) = 0 \tag{2.6}$$

*if $p \in \{0,1\}$, $j \in \{1,\dots,k\}$ and if there exists $j' \ne j$ such that $x_j = x_{j'}$.*

*Proof* Fix $k \in \mathbb{N}$. The fact that $\rho_k$ is in $C^2$ is an immediate consequence of the fact that $K$ is in $C^2$. For $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_k$ fixed, the function:

$$t \mapsto \rho_k(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_k)$$

is positive by the positivity of $K$, and equal to zero if $t = x_{j'}$ for some $j' \in \{1, \dots, j-1, j+1, \dots, k\}$. Therefore, $t = x_{j'}$ is a local minimum of this function and one deduces the equality (2.6). We now turn to the proof of (2.5). By symmetry of $\rho_k$, we only need to show the case $j = 1$. We isolate the terms containing $x_1$ in the determinant defining $\rho_k$ to obtain:

$$\begin{aligned}\rho_k(x_1, \dots, x_k) &= K(x_1, x_1) \det(K(x_{l+1}, x_{m+1})_{1 \le l,m \le k-1}) \\ &\quad + \sum_{2 \le i,j \le k} (-1)^{i+j-1} K(x_i, x_1) K(x_1, x_j) \\ &\quad \times \det(K(x_{l+1+\mathbb{1}_{l \ge i-1}}, x_{m+1+\mathbb{1}_{m \ge j-1}})_{1 \le l,m \le k-2}),\end{aligned}$$

where we take the convention that an empty sum is equal to 0 and an empty determinant is equal to 1. One deduces:

$$\begin{aligned}\frac{\partial}{\partial x_1} \rho_k(x_1, \dots, x_k) &= (K'_1 + K'_2)(x_1, x_1) \det(K(x_{l+1}, x_{m+1})_{1 \le l,m \le k-1}) \\ &\quad + \sum_{2 \le i,j \le k} (-1)^{i+j-1} (K'_2(x_i, x_1) K(x_1, x_j) + K(x_i, x_1) K'_1(x_1, x_j)) \\ &\quad \times \det(K(x_{l+1+\mathbb{1}_{l \ge i-1}}, x_{m+1+\mathbb{1}_{m \ge j-1}})_{1 \le l,m \le k-2}),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial x_1^2} \rho_k(x_1, \dots, x_k) &= (K''_{1,1} + 2K''_{1,2} + K''_{2,2})(x_1, x_1) \det(K(x_{l+1}, x_{m+1})_{1 \le l,m \le k-1}) \\ &\quad + \sum_{2 \le i,j \le k} (-1)^{i+j-1} (K''_2(x_i, x_1) K(x_1, x_j) + 2K'_2(x_i, x_1) K'_1(x_1, x_j) \\ &\quad + K(x_i, x_1) K''_1(x_1, x_j)) \det(K(x_{l+1+\mathbb{1}_{l \ge i-1}}, x_{m+1+\mathbb{1}_{m \ge j-1}})_{1 \le l,m \le k-2}),\end{aligned}$$

where for $p, q \in \{1, 2\}$, $K'_p$ denotes the derivative of $K$ with respect to the $p$-th variable, and $K''_{p,q}$ denotes the second derivative of $K$ with respect to the $p$-th and the $q$-th variable. By the positivity of $K$, there exists, for all $r \in \mathbb{N}$ and $y_1, \dots, y_r, z_1, \dots, z_r > 0$, vectors $e_1, \dots, e_r, f_1, \dots, f_r$ of an Euclidean space $E$ equipped with its usual scalar product $(.|.)$, such that $(e_i | f_j) = K(y_i, z_j)$ for all $i, j \in \{1, \dots, r\}$. Now, we can define a scalar product on the $r$-th exterior power of $E$ by setting

$$(u_1 \wedge \dots \wedge u_r | v_1 \wedge \dots \wedge v_r) = \det((u_i | v_j)_{1 \le i,j \le r}),$$

for all $u_1, \dots, u_r, v_1, \dots, v_r \in E$. Note that this scalar product is nothing else than a Gram determinant and we have the upper bound

$$|\det((e_i | f_j)_{1 \le i,j \le r})| \le \prod_{i=1}^{r} \|e_i\|_E \prod_{i=1}^{r} \|f_i\|_E,$$

$\|.\|_E$ being the norm associated to $(.|.)$. This last bound is equivalent to

$$|\det(K(y_i, z_j)_{1\leq i,j\leq r})| \leq \sqrt{\prod_{i=1}^{r} K(y_i, y_i) \prod_{i=1}^{r} K(z_i, z_i)}. \tag{2.7}$$

Now, let $x_0 > 0$ and $x_1, \ldots, x_k \geq x_0$. The bound (2.4) given in the statement of the Lemma and the inequality (2.7) imply

$$|\det(K(x_{l+1}, x_{m+1})_{1\leq l,m\leq k-1})| \leq \frac{(C(x_0))^{k-1}}{(x_2 x_3 \cdots x_k)^{2\alpha}}$$

and

$$\begin{aligned}
&|\det(K(x_{l+1+\mathbb{1}_{l\geq i-1}}, x_{m+1+\mathbb{1}_{m\geq j-1}})_{1\leq l,m\leq k-2})| \\
&\leq \frac{(C(x_0))^{k-2}}{(x_2 x_3 \cdots x_{i-1} x_{i+1} \cdots x_k)^{\alpha} (x_2 x_3 \cdots x_{j-1} x_{j+1} \cdots x_k)^{\alpha}} \\
&= \frac{(C(x_0))^{k-2} (x_i x_j)^{\alpha}}{(x_2 \cdots x_k)^{2\alpha}}.
\end{aligned}$$

Hence, each term involved in the expressions of $\rho_k$ and its two first derivatives with respect to $x_1$ is smaller than $4(C(x_0))^k/(x_1 \cdots x_k)^{2\alpha}$ and therefore, the absolute values of $\rho_k$ an its derivatives are bounded by $4((k-1)^2+1)(C(x_0))^k/(x_1 \cdots x_k)^{2\alpha} \leq 4^k (C(x_0))^k/(x_1 \cdots x_k)^{2\alpha}$, implying the bound (2.5). □

*Remark 2.7* In the above proof, the value of $C(x_0)$ does not change. It is thus possible to take $C(x_0)$ in the inequality (2.5) to be equal to 4 times the value of $C(x_0)$ in (2.4).

We now have to prove that the re-scaled kernel $K_{[N]}$ satisfies the hypothesis of Lemma 2.6, and that its partial derivatives converge pointwise to the partial derivatives of $K_\infty$. In the following, we introduce the notation

$$F_{n,h,a}(x) = {}_2F_1\left[-n, h, a; 2/(1+ix)\right],$$

for $(n, h, a) \in \mathbb{N} \times \mathbb{C} \times \mathbb{R}_+^*$.

**Lemma 2.8** *Let $\epsilon \in \{0, 1\}$, $h \in \mathbb{C}$, $a \in \mathbb{R}_+^*$. For $N \in \mathbb{N}$, we set $n := N - \epsilon$. Then, $x \mapsto F_{n,h,a}(Nx)$ and $x \mapsto {}_1F_1[h, a; 2i/x]$ are in $C^\infty(\mathbb{R}^*)$, and for all $p \in \mathbb{N}$ and $x \in \mathbb{R}^*$:*

$$\frac{d^p}{dx^p}(F_{n,h,a}(Nx)) \underset{N\to\infty}{\longrightarrow} \frac{d^p}{dx^p}({}_1F_1[h, a; 2i/x]).$$

*Moreover, for all $x_0 > 0$ and for all $x \in \mathbb{R}$ such that $|x| \geq x_0$, one has the bound*

$$\left|\frac{d^p}{dx^p}(F_{n,h,a}(Nx))\right| \leq \frac{C(x_0, h, a, p)}{|x|^{p+\mathbb{1}_{p>0}}}.$$

*Proof* One has

$$F_{n,h,a}(Nx) = \sum_{k=0}^{\infty} \frac{(-n)_k (h)_k}{(a)_k k!} \left(\frac{2}{1+Nix}\right)^k,$$

where only a finite number of the summands are different from zero. This implies that the function is $C^\infty$ on $\mathbb{R}^*$, and

$$\frac{d^p}{dx^p}(F_{n,h,a}(Nx)) = \sum_{k=0}^{\infty} \frac{(-n)_k (h)_k}{(a)_k k!} (k)_p \left(\frac{2}{1+Nix}\right)^{k+p} \left(-\frac{iN}{2}\right)^p.$$

The term of order $k$ in this sum is dominated by (note that $a > 0$)

$$\frac{(|h|)_k}{(a)_k k!} (k)_p \frac{2^k}{|x|^{k+p}},$$

and for fixed $x$, tends to

$$\frac{(h)_k}{(a)_k k!} (k)_p \frac{(2i)^k (-1)^p}{x^{k+p}},$$

when $N \to \infty$. One deduces, that for $|x| \geq x_0 > 0$:

$$\begin{aligned}
\left|\frac{d^p}{dx^p}(F_{n,h,a}(Nx))\right| &\leq \sum_{k=0}^{\infty} \frac{(|h|)_k}{(a)_k k!} (k)_p \frac{2^k}{|x|^{k+p}} \\
&\leq \mathbb{1}_{p=0} + \frac{1}{|x|^{p+1}} \sum_{k=1}^{\infty} \frac{(|h|)_k}{(a)_k k!} (k)_p \frac{2^k}{x_0^{k-1}} \\
&\leq \frac{C(x_0, h, a, p)}{|x|^{p+\mathbb{1}_{p>0}}}
\end{aligned}$$

which is the desired bound. Now, by dominated convergence, one has

$$\frac{d^p}{dx^p}(F_{n,h,a}(Nx)) \underset{N\to\infty}{\longrightarrow} \sum_{k=0}^{\infty} \frac{(h)_k}{(a)_k k!} (k)_p \frac{(2i)^k (-1)^p}{x^{k+p}}.$$

Hence, Lemma 2.8 is proved if we show that $x \mapsto {}_1F_1[h, a; 2i/x]$ is $C^\infty$ on $\mathbb{R}^*$, and that

$$\frac{d^p}{dx^p}({}_1F_1[h, a; 2i/x]) = \sum_{k=0}^{\infty} \frac{(h)_k}{(a)_k k!} (k)_p \frac{(2i)^k (-1)^p}{x^{k+p}}. \tag{2.8}$$

But the sum in (2.8) is obtained by taking the derivative of order $p$ of each term of the sum defining ${}_1F_1$. Therefore, we are done, since this term by term derivation is justified by the domination of the right hand side of (2.8) by $C(x_0, h, a, p)/|x|^{p+\mathbb{1}_{p>0}}$ on $\mathbb{R}\backslash(-x_0, x_0)$. □

**Lemma 2.9** *Fix $s$ such that $\Re s > -\frac{1}{2}$. Define the functions $\tilde{P}_N$ and $Q_N$ by*

$$\tilde{P}_N(x) = 2^{\Re s} \left(\frac{\Gamma(2\Re s + N + 1)}{N\Gamma(N)}\right)^{1/2} \tilde{p}_N(Nx)\sqrt{w_H(Nx)},$$

$$Q_N(x) = 2^{\Re s+1} \left(\frac{N\Gamma(2\Re s + N + 1)}{\Gamma(N)}\right)^{1/2} p_{N-1}(Nx)\sqrt{w_H(Nx)},$$

*where* $\tilde{p}_N$, $p_{N-1}$ *and* $w_H$ *are given in Theorem* 1.1 *and the remark below that Theorem. Then,* $\tilde{P}_N$ *and* $Q_N$ *are* $C^\infty$ *on* $\mathbb{R}$, $\tilde{P}$ *and* $Q$, *defined below* (1.17), *are* $C^\infty$ *on* $\mathbb{R}^*$, *and for all* $x \in \mathbb{R}^*$, $p \in \mathbb{N}_0$,

$$(\mathrm{Sgn}(x))^N \tilde{P}_N^{(p)}(x) \underset{N\to\infty}{\longrightarrow} \tilde{P}^{(p)}(x),$$

$$(\mathrm{Sgn}(x))^N Q_N^{(p)}(x) \underset{N\to\infty}{\longrightarrow} Q^{(p)}(x).$$

*Moreover, for all* $p \in \mathbb{N}_0$, $x_0 > 0$, *one has the following bounds*:

$$\left|\tilde{P}_N^{(p)}(x)\right| \leq \frac{C(x_0, s, p)}{|x|^{p+\Re s}},$$

*and*

$$\left|Q_N^{(p)}(x)\right| \leq \frac{C(x_0, s, p)}{|x|^{p+1+\Re s}},$$

*for all* $|x| \geq x_0$.

*Proof* We define

$$\Phi_N(x) = D(N,n,s)(Nx-i)^n F_{n,h,a}(Nx)(1+iNx)^{(-s-N)/2}(1-iNx)^{(-\overline{s}-N)/2},$$

where

$$D(N,n,s) = 2^{\Re s+(N-n)}\left(\frac{\Gamma(2\Re s+N+1)}{N\Gamma(N)}\right)^{1/2} N^{N-n},$$

and $N-n \in \{0,1\}$ (see Lemma 2.8). Then, if $(n,h,a) = (N,s,2\Re s+1)$, $\Phi_N(x) = \tilde{P}_N(x)$ and if $(n,h,a) = (N-1,s+1,2\Re s+2)$, $\Phi_N(x) = Q_N(x)$. Moreover, note that $\Phi_N$ is a product of $C^\infty$ functions on $\mathbb{R}$.

Now, for $\delta \in \{-1,1\}$:

$$\log(1+\delta iNx) = \log(1-\delta i/Nx) + \log(N|x|) + i\frac{\pi}{2}\delta\,\mathrm{Sgn}(x),$$

because both sides of the equality have an imaginary part in $(-\pi,\pi)$ and their exponentials are equal. Hence,

$$\begin{aligned}&\left(\frac{-s+N}{2}-(N-n)\right)\log(1+iNx) + \frac{-\overline{s}-N}{2}\log(1-iNx)\\ &= \left(\frac{-s+N}{2}-(N-n)\right)\log(1-i/Nx) + \frac{-\overline{s}-N}{2}\log(1+i/Nx)\\ &\quad - (\Re s+(N-n))\log(N|x|) + ni\pi\,\mathrm{Sgn}(x)/2 + \pi\Im s\,\mathrm{Sgn}(x)/2.\end{aligned}$$

This implies:

$$\begin{aligned}\Phi_N(x) &= D(N,n,s)(-i)^n(1+iNx)^{(-s+N)/2-(N-n)}(1-iNx)^{(-\overline{s}-N)/2}F_{n,h,a}(Nx)\\ &= D(N,n,s)(-i)^n(N|x|)^{-\Re s-(N-n)}e^{ni\pi\,\mathrm{Sgn}(x)/2}e^{\pi\Im s\,\mathrm{Sgn}(x)/2}\\ &\quad\times(1-i/Nx)^{(N-s)/2-(N-n)}(1+i/Nx)^{(-\overline{s}-N)/2}F_{n,h,a}(Nx)\end{aligned}$$

$$= D(N,n,s)(\mathrm{Sgn}(x))^n (2N)^{-\Re s-(N-n)}(2/|x|)^{\Re s+N-n} e^{\pi \Im s\,\mathrm{Sgn}(x)/2}$$
$$\times (1-i/Nx)^{(N-s)/2-(N-n)}(1+i/Nx)^{(-\bar{s}-N)/2} F_{n,h,a}(Nx)$$
$$= D'(N,s)(\mathrm{Sgn}(x))^N e^{\pi \Im s\,\mathrm{Sgn}(x)/2}(2/x)^{N-n}(2/|x|)^{\Re s}$$
$$\times (1-i/Nx)^{(N-s)/2-(N-n)}(1+i/Nx)^{(-\bar{s}-N)/2} F_{n,h,a}(Nx), \tag{2.9}$$

where for $s$ fixed,

$$D'(N,s) = D(N,n,s)(2N)^{-\Re s-(N-n)} = \left(\frac{\Gamma(2\Re s+N+1)}{N^{2\Re s+1}\Gamma(N)}\right)^{1/2}. \tag{2.10}$$

This tends to 1 when $N$ goes to infinity. In particular $D'(N,s)$ can be bounded by some $C(s)$, not depending on $N$. We investigate all the terms in (2.9) separately in the following.

Let $G$ be the function defined by:

$$G(y) := (1-iy/N)^{(N-s)/2-(N-n)}(1+iy/N)^{(-\bar{s}-N)/2}.$$

This function is $C^\infty$ on $\mathbb{R}$ and one has:

$$G^{(p)}(y) = G(y)\sum_{q=0}^{p} C(p,q)(i/N)^q(-i/N)^{p-q}(-(N-s)/2+N-n)_q$$
$$\times ((N+\bar{s})/2)_{p-q}(1-iy/N)^{-q}(1+iy/N)^{-(p-q)}.$$

For $s$, $y$, $p$ and $N-n\in\{0,1\}$ fixed, the last sum is dominated by some constant $C(s,p)$ only depending on $s$ and $p$ and tends to $(-i)^p$, as $N\to\infty$. Moreover, $G(y)$ tends to $e^{-iy}$, and

$$G(y) = \left(\frac{1-iy/N}{1+iy/N}\right)^{(N-i\Im s)/2}(1-iy/N)^{-(N-n)}(1+y^2/N^2)^{-\Re s/2}.$$

A simple computation, yields the following:

$$|G(y)| \le C(s)\left(1+\frac{y^2}{N^2}\right)^{-\Re s/2} \le C(s)(1+y^2)^{1/4}.$$

This implies that $G^{(p)}(y)$ tends to $(-i)^p e^{-iy}$ when $N$ goes to infinity, and that

$$\left|G^{(p)}(y)\right| \le C(s,p)(1+y^2)^{1/4}.$$

Now, for all $f$ in $C^\infty(\mathbb{R})$, the function $g$ defined by $x\mapsto f(1/x)$ is in $C^\infty(\mathbb{R}^*)$, and there exist universal integers $(\mu_{p,k})_{p\in\mathbb{N}_0,0\le k\le p}$, such that $\mu_{p,0}=0$ for all $p\ge 1$, and for $p\in\mathbb{N}_0$,

$$g^{(p)}(x) = \sum_{k=0}^{p}\frac{\mu_{p,k}}{x^{p+k}} f^{(k)}(1/x).$$

Applying this formula to the functions $G$ and $y\to e^{-iy}$, one obtains the following pointwise convergence (for $x\neq 0$):

$$\frac{d^p}{dx^p}\left[(1-i/Nx)^{(N-s)/2-(N-n)}(1+i/Nx)^{(-\bar{s}-N)/2}\right] \underset{N\to\infty}{\longrightarrow} \frac{d^p}{dx^p}(e^{-i/x}) \tag{2.11}$$

with, for $|x| \geq x_0 > 0$,

$$\left| \frac{d^p}{dx^p} \left[ (1 - i/Nx)^{(N-s)/2-(N-n)} (1 + i/Nx)^{(-\bar{s}-N)/2} \right] \right| \leq \frac{C(x_0, s, p)}{|x|^{p+\mathbb{1}_{p>0}}}. \tag{2.12}$$

Recall that by Lemma 2.8, one has the convergence

$$\frac{d^p}{dx^p}(F_{n,h,a}(Nx)) \underset{N\to\infty}{\longrightarrow} \frac{d^p}{dx^p}({}_1F_1[h, a; 2i/x]), \tag{2.13}$$

and the bound

$$\left| \frac{d^p}{dx^p}(F_{n,h,a}(Nx)) \right| \leq \frac{C(x_0, h, a, p)}{|x|^{p+\mathbb{1}_{p>0}}} \leq \frac{C(x_0, s, p)}{|x|^{p+\mathbb{1}_{p>0}}}, \tag{2.14}$$

since $(h, a)$ only depends on $s$ in the relevant cases (see the beginning of the proof). Moreover,

$$\left| \frac{d^p}{dx^p} \left[ (2/x)^{N-n} (2/|x|)^{\Re s} \right] \right| \leq \frac{C(s, p)}{|x|^{\Re s+(N-n)+p}}. \tag{2.15}$$

We can now give the derivatives of $\Phi_N$, using (2.9). One has for $p \geq 0$:

$$\begin{aligned}
(\mathrm{Sgn}(x))^N \frac{d^p}{dx^p}(\Phi_N(x)) &= D'(N, s) e^{\pi \Im s \,\mathrm{Sgn}(x)/2} \\
&\quad \times \sum_{q_1+q_2+q_3=p} \frac{p!}{q_1! q_2! q_3!} \frac{d^{q_1}}{dx^{q_1}} \left[ (2/x)^{N-n} (2/|x|)^{\Re s} \right] \\
&\quad \times \frac{d^{q_2}}{dx^{q_2}} \left[ (1 - i/Nx)^{(N-s)/2-(N-n)} (1 + i/Nx)^{(-N-\bar{s})/2} \right] \\
&\quad \times \frac{d^{q_3}}{dx^{q_3}} \left[ F_{n,h,a}(Nx) \right].
\end{aligned}$$

By (2.10), (2.11) and (2.13), whenever $s$, $x$ and $N - n \in \{0, 1\}$ are fixed, this expression tends to

$$\begin{aligned}
&e^{\pi \Im s \,\mathrm{Sgn}(x)/2} \sum_{q_1+q_2+q_3=p} \frac{p!}{q_1! q_2! q_3!} \frac{d^{q_1}}{dx^{q_1}} \left[ (2/x)^{N-n} (2/|x|)^{\Re s} \right] \\
&\quad \times \frac{d^{q_2}}{dx^{q_2}} \left[ e^{-i/x} \right] \frac{d^{q_3}}{dx^{q_3}} \left( {}_1F_1[h, a; 2i/x] \right),
\end{aligned}$$

for $N \to \infty$. But this is precisely the $p$-th derivative of $\tilde{P}$ at $x$ if $\Phi_N = \tilde{P}_N$, and the $p$-th derivative of $Q$ at $x$ if $\Phi_N = Q_N$. Moreover, for $|x| \geq x_0 > 0$, one easily obtains the bound

$$\left| \frac{d^p}{dx^p}(\Phi_N(x)) \right| \leq \frac{C(x_0, s, p)}{|x|^{\Re s+(N-n)+p}},$$

using (2.10), (2.12), (2.14) and (2.15). This completes the proof of the Lemma. □

**Lemma 2.10** *Let $f$ and $g$ be two functions which are $C^\infty$ from $\mathbb{R}^*$ to $\mathbb{R}$. We define the function $\phi$ from $(\mathbb{R}^*)^2$ to $\mathbb{R}$ by*

$$\phi(x, y) := \frac{f(x)g(y) - g(x)f(y)}{x - y},$$

*for* $x \neq y$, *and*

$$\phi(x,x) := f'(x)g(x) - g'(x)f(x).$$

*Then,* $\phi$ *is* $C^\infty$ *on* $(\mathbb{R}^*)^2$ *and for all* $p, q \in \mathbb{N}_0$:

(a) *If* $x \neq y$:

$$\frac{\partial^{p+q}\phi}{\partial x^p \partial y^q} = \sum_{k=0}^{p}\sum_{l=0}^{q} C_p^k C_q^l \frac{f^{(k)}(x)g^{(l)}(y) - g^{(k)}(x)f^{(l)}(y)}{(x-y)^{p+q-k-l+1}}(-1)^{p-k}(p+q-k-l)!.$$

(b) *If* $x$ *and* $y$ *have same sign*:

$$\frac{\partial^{p+q}\phi}{\partial x^p \partial y^q} = \sum_{k=0}^{q} C_q^k \Bigg[ g^{(q-k)}(y) \int_0^1 f^{(k+p+1)}(y+\theta(x-y))\theta^p(1-\theta)^k d\theta \\ - f^{(q-k)}(y) \int_0^1 g^{(k+p+1)}(y+\theta(x-y))\theta^p(1-\theta)^k d\theta \Bigg].$$

*Proof* (a) By induction, one proves that for all $p, q \in \mathbb{N}_0$, and for $x, y \in \mathbb{R}$ distincts and different from zero, it is possible to take, in a neighborhood of $(x, y)$, $p$ derivatives of $\phi$ with respect to $x$ and $q$ derivatives of $\phi$ with respect to $y$, in any order, with a result equal to the expression given in the statement of the Lemma. This implies the existence and the continuity of all partial derivatives of $\phi$ in $(\mathbb{R}^*)^2 \backslash \{(x,x), x \in \mathbb{R}^*\}$. Therefore, $\phi$ is $C^\infty$ in this open subset of $(\mathbb{R}^*)^2$.

(b) With the same method as in (a), we obtain that $\phi$ is $C^\infty$ on $(\mathbb{R}^*_-)^2 \cup (\mathbb{R}^*_+)^2$. The only technical issues are the continuity and the derivation under the integral. These can easily be justified by the boundedness of the derivatives of $f$ and $g$ in any compact set of $\mathbb{R}^*$. □

**Proposition 2.11** *Let* $x, y \in \mathbb{R}^*$ *and let* $\Re s > -1/2$. *Then* $K_{[N]}$ *and* $K_\infty$ *are* $C^\infty$ *in* $(\mathbb{R}^*)^2$ *and for all* $p, q \in \mathbb{N}_0$,

$$(\mathrm{Sgn}(xy))^N \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_{[N]}(x,y) \underset{N\to\infty}{\longrightarrow} \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_\infty(x,y).$$

*Moreover, for any* $x_0 > 0$, *and* $|x|, |y| \geq x_0 > 0$:

$$\left| \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_{[N]}(x,y) \right| \leq \frac{C(x_0, s, p, q)}{|x|^{\Re s+p+1}|y|^{\Re s+q+1}}.$$

Note that the pointwise convergence in the case $p = q = 0$ corresponds to the convergence result for the kernels given by Borodin and Olshanski [3].

*Proof* One has

$$(\mathrm{Sgn}(xy))^N K_{[N]}(x,y) \\ = \frac{1}{2\pi} \frac{\Gamma(s+1)\Gamma(\overline{s}+1)}{\Gamma(2\Re s+1)\Gamma(2\Re s+2)} \\ \times \frac{(\mathrm{Sgn}(x))^N \tilde{P}_N(x)(\mathrm{Sgn}(y))^N Q_N(y) - (\mathrm{Sgn}(y))^N \tilde{P}_N(y)(\mathrm{Sgn}(x))^N Q_N(x)}{x-y}$$

for $x \neq y$, and

$$K_{[N]}(x,x) = \frac{1}{2\pi} \frac{\Gamma(s+1)\Gamma(\overline{s}+1)}{\Gamma(2\Re s+1)\Gamma(2\Re s+2)} (\tilde{P}_N'(x) Q_N(x) - Q_N'(x)\tilde{P}_N(x)),$$

with $\tilde{P}_N$ and $Q_N$ defined in Lemma 2.9. Recall the definition of $K_\infty$ in (1.17) and (1.18). Now, $\tilde{P}_N$, $Q_N$, $\tilde{P}$ and $Q$ are in $C^\infty(\mathbb{R}^*)$ (see Lemma 2.9) and hence, by Lemma 2.10, $K_{[N]}$ and $K_\infty$ are in $C^\infty((\mathbb{R}^*)^2)$.

Moreover, by Lemma 2.9, the derivatives of $x \mapsto \mathrm{Sgn}^N(x)\tilde{P}_N(x)$ and $x \mapsto \mathrm{Sgn}^N(x) \times Q_N(x)$ converge pointwise to the corresponding derivatives of $\tilde{P}$ and $Q$. Considering, for $x \neq y$, the expression (a) of Lemma 2.10, and for $x = y$, the expression (b), one easily deduces the pointwise convergence of the derivatives of $(x,y) \mapsto (\mathrm{Sgn}(xy))^N K_{[N]}(x,y)$ towards the corresponding derivatives of $K_\infty$.

Finally, the bounds given in the statement of the Lemma can be obtained from the bounds of the derivatives of $\tilde{P}_N$ and $Q_N$, given in Lemma 2.9, and by applying the formula (a) of Lemma 2.10 if $xy < 0$ or $\max(|x|,|y|) > 2\min(|x|,|y|)$ (which implies $|x-y| \geq \max(|x|,|y|)/2$), or the formula (b) if $xy > 0$ and $\max(|x|,|y|) \leq 2\min(|x|,|y|)$. □

Summarizing, we have:

**Proposition 2.12** *Let $s$ be such that $\Re s > -\frac{1}{2}$. Then, the restriction to $\mathbb{R}_+^*$ of the scaled kernel $K_{[N]}$ and the kernel $K_\infty$ satisfy the conditions of Lemma 2.6. Moreover, for all $p, q \in \mathbb{N}_0$, the partial derivatives*

$$\mathrm{Sgn}(xy)^N \frac{\partial^{p+q}}{\partial x^p \partial y^q} K_{[N]}(x,y)$$

*converge pointwise to the corresponding partial derivatives of $K_\infty(x,y)$.*

*Proof* This follows immediately from Proposition 2.11 and the fact that these kernels are real symmetric and positive because they are kernels of determinantal processes on the real line (see remark 1.2 for the kernel $K_\infty$). □

The next step is to analyze the convergence of the Fredholm determinant of $K_{N,J}$ and its derivatives to the corresponding derivatives of the Fredholm determinant of $K_{\infty,J}$, for $J = (t,\infty)$, $t > 0$.

**Lemma 2.13** *Let $F$ be a function defined from $(\mathbb{R}_+^*)^{k+1}$ to $\mathbb{R}$, for some $k \in \mathbb{N}$. We suppose that $F$ is in $C^1$, and that there exists, for some $\alpha > 1$ and for all $x_0 > 0$, a bound of the form*

$$|F(t,x_1,x_2,\dots,x_k)| + \left|\frac{\partial}{\partial t} F(t,x_1,x_2,\dots,x_k)\right| \leq \frac{C(x_0)}{(x_1\cdots x_k)^\alpha},$$

*for all $t, x_1, \dots, x_k \geq x_0$. Then, the integrals involved in the definitions of the following two functions from $\mathbb{R}_+^*$ to $\mathbb{R}$ are absolutely convergent:*

$$H_0 : t \mapsto \int_{(t,\infty)^k} F(t,x_1,\dots,x_k) dx_1 \cdots dx_k,$$

*and*

$$H_1 : t \mapsto \int_{(t,\infty)^k} \frac{\partial}{\partial t} F(t,x_1,\dots,x_k) dx_1 \cdots dx_k$$

$$-\sum_{l=1}^{k}\int_{(t,\infty)^{k-1}} F(t,x_1,\dots,x_{l-1},t,x_{l+1},\dots,x_k)dx_1\cdots dx_{l-1}dx_{l+1}\cdots dx_k.$$

*Moreover, the first derivative of $H_0$ is continuous and equal to $H_1$.*

*Proof* Due to the bound given in the Lemma, it is clear that all the integrals in the definition of $H_0$ and $H_1$ are absolutely convergent. Therefore, for $0 < t < t'$, we can use Fubini's Theorem in order to compute the integral

$$\int_t^{t'} H_1(u)du.$$

Straightforward computations show that this integral is equal to $H_0(t') - H_0(t)$. Hence, if we prove that $H_1$ is continuous, we are done. Now, let $t > x_0 > 0$. For $t' > x_0$, one has

$$\begin{aligned}
&|H_1(t') - H_1(t)|\\
&\quad\le \int_{(x_0,\infty)^k}\left|\frac{\partial}{\partial t'}F(t',x_1,\dots,x_k)\mathbb{1}_{\{x_1,\dots,x_k>t'\}}\right.\\
&\qquad\left.-\frac{\partial}{\partial t}F(t,x_1,\dots,x_k)\mathbb{1}_{\{x_1,\dots,x_k>t\}}\right|dx_1\cdots dx_k\\
&\qquad+\sum_{l=1}^{k}\int_{(x_0,\infty)^{k-1}}\big|F(t',x_1,\dots,x_{l-1},t',x_{l+1},\dots,x_k)\mathbb{1}_{\{x_1,\dots,x_{l-1},x_{l+1},\dots x_k>t'\}}\\
&\qquad-F(t,x_1,\dots,x_{l-1},t,x_{l+1},\dots,x_k)\mathbb{1}_{\{x_1,\dots,x_{l-1},x_{l+1},\dots x_k>t\}}\big|\,dx_1\cdots dx_{l-1}dx_{l+1}\cdots dx_k.
\end{aligned}$$

All the terms inside the integrals converge to zero almost everywhere when $t' \to t$ (more precisely, whenever the minimum of the $x_j$'s is different from $t$). Hence, by dominated convergence, $|H_1(t') - H_1(t)|$ tends to zero when $t' \to t$. □

**Lemma 2.14** *Let $K$ be a function satisfying the conditions of Lemma 2.6. Then, using the notation of that Lemma,*

$$\sum_{k\ge1}\frac{1}{k!}\int_{(t,\infty)^k}\rho_k(x_1,\dots,x_k)dx_1\cdots dx_k < \infty$$

*for all $t > 0$. Moreover, the Fredholm determinant $F$, from $\mathbb{R}_+^*$ to $\mathbb{R}$, defined in (1.20) is in $C^3$, and its derivatives are given by*

$$F'(t) = \sum_{k\ge0}\frac{(-1)^k}{k!}\int_{(t,\infty)^k}\rho_{k+1}(t,x_1,\dots,x_k)dx_1\cdots dx_k,$$

$$F''(t) = \sum_{k\ge0}\frac{(-1)^k}{k!}\int_{(t,\infty)^k}\frac{\partial}{\partial t}\rho_{k+1}(t,x_1,\dots,x_k)dx_1\cdots dx_k,$$

$$F'''(t) = \sum_{k\ge0}\frac{(-1)^k}{k!}\int_{(t,\infty)^k}\frac{\partial^2}{\partial t^2}\rho_{k+1}(t,x_1,\dots,x_k)dx_1\cdots dx_k,$$

*where all the sums and the integrals above are absolutely convergent.*

*Proof* For $k \geq 1$, we define $F_k$ by

$$F_k(t) = \frac{(-1)^k}{k!} \int_{(t,\infty)^k} \rho_k(x_1, \dots, x_k) dx_1 \cdots dx_k.$$

The integral is finite because of the bounds given in Lemma 2.6. By the same bounds, one can apply Lemma 2.13 three times, to obtain that $F_k$ is in $C^3$, with the derivatives given by

$$F_k'(t) = \frac{(-1)^{k-1}}{(k-1)!} \int_{(t,\infty)^{k-1}} \rho_k(t, x_1, \dots, x_{k-1}) dx_1 \cdots dx_{k-1},$$

$$F_k''(t) = \frac{(-1)^{k-1}}{(k-1)!} \int_{(t,\infty)^{k-1}} \frac{\partial}{\partial t} \rho_k(t, x_1, \dots, x_{k-1}) dx_1 \cdots dx_{k-1},$$

$$F_k'''(t) = \frac{(-1)^{k-1}}{(k-1)!} \int_{(t,\infty)^{k-1}} \frac{\partial^2}{\partial t^2} \rho_k(t, x_1, \dots, x_{k-1}) dx_1 \cdots dx_{k-1},$$

where again all the integrals are absolutely convergent by Lemma 2.6. Note that we use (2.6) to calculate the derivatives above. Moreover, for $p \in \{0, 1, 2, 3\}$, (2.5) gives the following bound for any $x_0 > 0$:

$$\sup_{t \geq x_0} |F_k^{(p)}(t)| \leq \frac{(C(x_0))^k}{(k-1)!}.$$

Using dominated convergence, we have that the sum

$$\sum_{k \geq 1} F_k(t)$$

is absolutely convergent, and that its $p$-th derivative, $p \in \{0, 1, 2, 3\}$ with respect to $t$ is continuous and given by the absolutely convergent sum

$$\sum_{k \geq 1} F_k^{(p)}(t).$$ □

### 2.3 $\theta_\infty$ is Well Defined

In order to prove that $\theta_\infty$ is well defined, we need to prove that $F_\infty(t)$ never vanishes for $t > 0$ (recall from Remark 1.6 that the range of the largest eigenvalue is $\mathbb{R}_+^*$). We note that $F_\infty(t)$ is the Fredholm determinant of the restriction of the operator $K_\infty$ to the space $L^2((t, \infty))$, which can also be seen as the operator on $L^2((t_0, \infty))$ with kernel $(x, y) \to K_\infty(x, y) \mathbb{1}_{x,y>t}$, for some $t_0$ such that $t > t_0 > 0$. This operator is positive, and we recall that it is a trace class operator, since:

$$\int_{(t,\infty)} K_\infty(x, x)\, dx < \infty.$$

Therefore, the Fredholm determinant of this operator is given by the convergent product of $1 - \lambda_j$, where $(\lambda_j)_{j \in \mathbb{N}}$ is the decreasing sequence of its (positive) eigenvalues, with multiplicity. This implies that the determinant is zero if and only if 1 is an eigenvalue of the operator: hence, we only need to prove that this is not the case. Indeed, if 1 is an eigenvalue, there exists $f \neq 0$ in $L^2((t_0, \infty))$ such that for almost all $x \in (t_0, \infty)$:

$$f(x) = \mathbb{1}_{x>t} \int_t^\infty K_\infty(x, y)\, f(y)\, dy.$$

Hence $f(x) = 0$ for almost every $x \leq t$, and

$$f = p_{(t,\infty)} K_{\infty,(t_0,\infty)} f$$

in $L^2((t_0,\infty))$, where $K_{\infty,(t_0,\infty)}$ is the operator on this space, with kernel $K_\infty$, and $p_{(t,\infty)}$ is the projection on the space of functions supported by $(t,\infty)$. Now, if we denote $g := K_{\infty,(t_0,\infty)} f$,

$$\|g\|^2_{L^2((t_0,\infty))} = \int_{t_0}^{\infty} \int_{t_0}^{\infty} \int_{t_0}^{\infty} K_\infty(x,y)\, K_\infty(x,z)\, f(y)\, f(z)\, dx\, dy\, dz.$$

By dominated convergence, one can check that $\|g\|^2_{L^2((t_0,\infty))}$ is the limit of

$$\int_{t_0}^{\infty} \int_{t_0}^{\infty} \int_{t_0}^{\infty} K_{[N]}(x,y)\, K_{[N]}(x,z)\, f(y)\, f(z)\, dx\, dy\, dz$$

when $N$ goes to infinity. This expression is equal to $\| p_{(t_0,\infty)} K_{[N]} \tilde{f} \|^2_{L^2(\mathbb{R})}$, and hence, smaller than or equal to $\| K_{[N]} \tilde{f} \|^2_{L^2(\mathbb{R})}$, where the operators $p_{(t_0,\infty)}$ and $K_{[N]}$ act on $L^2(\mathbb{R})$, and where $\tilde{f}$ is equal to $f$ on $(t_0,\infty)$ and equal to zero on $(-\infty, t_0]$. Now, $K_{[N]}$ (as $K_N$) is an orthogonal projector on $L^2(\mathbb{R})$ (with an $N$-dimensional image), hence, $\|K_{[N]}\tilde{f}\|_{L^2(\mathbb{R})} \leq \|\tilde{f}\|_{L^2(\mathbb{R})}$. This implies:

$$\|g\|_{L^2((t_0,\infty))} \leq \|f\|_{L^2((t_0,\infty))}.$$

Now, with obvious notation:

$$\begin{aligned} \|g\|^2_{L^2((t_0,\infty))} &= \|p_{(t,\infty)} g\|^2_{L^2((t_0,\infty))} + \|p_{(t_0,t]} g\|^2_{L^2((t_0,\infty))} \\ &= \|f\|^2_{L^2((t_0,\infty))} + \|p_{(t_0,t]} g\|^2_{L^2((t_0,\infty))} \end{aligned}$$

since $f = p_{(t,\infty)} g$. By comparing the last two equations, one deduces that

$$\|p_{(t_0,t]} g\|^2_{L^2((t_0,\infty))} = 0,$$

which implies that $g$ is supported by $(t,\infty)$, and

$$f = p_{(t,\infty)} g = g = K_{\infty,(t_0,\infty)} f.$$

Hence, $K_{\infty,(t_0,\infty)} f$ (equal to $f$), takes the value zero a.e. on the interval $(t_0, t)$. Since $f$ is different from zero, one easily deduces a contradiction from the following Lemma:

**Lemma 2.15** *Let $f$ be a function in $L^2((t,\infty))$ for some $t > 0$. Then the function $g$ from $\mathbb{R}_+^*$ to $\mathbb{R}$, defined by*:

$$g(x) = \int_t^{\infty} K_\infty(x,y)\, f(y)\, dy$$

*is analytic on* $\{z \in \mathbb{C};\ \Re(z) > 0\}$.

*Proof* It is sufficient to prove that for all $x_0$ such that $0 < x_0 < t/2$, $g$ can be extended to a holomorphic function on the set $H_{x_0} := \{x \in \mathbb{C}; \Re(x) > x_0\}$. Let $(\epsilon, h, a)$ be equal to

$(0, s, 2\Re(s)+1)$ or $(1, s+1, 2\Re(s)+2)$, and $\Phi$ equal to $\tilde{P}$ in the first case, $Q$ in the second case. One has for $x \in \mathbb{R}_+^*$:

$$\Phi(x) = \left(\frac{2}{x}\right)^{\Re(s)+\epsilon} e^{-i/x} e^{\pi \Im(s)/2} {}_1F_1[h, a; 2i/x].$$

$\Phi$ can easily be extended to $H_{x_0}$: for the first factor, one can use the standard extension of the logarithm (defined on $\mathbb{C}\backslash\mathbb{R}_-$), and the last factor is a hypergeometric series which is uniformly convergent on $H_{x_0}$. Moreover, it is easy to check (by using dominated convergence for the hypergeometric factor), that this extension of $\Phi$ is holomorphic with derivative:

$$\begin{aligned}\Phi'(x) = {} & e^{\pi \Im s/2} \left(\frac{2}{x}\right)^{\Re s+\epsilon} e^{-i/x} \\ & \times \left[\frac{-(\Re s+\epsilon)}{x} {}_1F_1[h, a; 2i/x] + \frac{i}{x^2} {}_1F_1[h, a; 2i/x] \right. \\ & \left. - \sum_{k=0}^{\infty} \frac{(h)_k (2i)^k k}{(a)_k k!} \left(\frac{1}{x}\right)^{k+1}\right].\end{aligned}$$

With these formulae, one deduces the following bounds, available on the whole set $H_{x_0}$:

$$|\Phi(x)| \leq \frac{C(x_0, s)}{|x|^{\Re(s)+\epsilon}},$$

$$|\Phi'(x)| \leq \frac{C(x_0, s)}{|x|^{\Re(s)+\epsilon+1}}.$$

Now, let us fix $y \in (t, \infty)$. Recall that for $x \in \mathbb{R}_+^* \backslash \{y\}$:

$$K_\infty(x, y) = \frac{1}{2\pi} \frac{\Gamma(s+1)\Gamma(\overline{s}+1)}{\Gamma(2\Re s+1)\Gamma(2\Re s+2)} \frac{\tilde{P}(x)Q(y) - Q(x)\tilde{P}(y)}{x-y}. \tag{2.16}$$

This formula is meaningful for all $x \in H_{x_0} \backslash \{y\}$ and gives an analytic continuation of $x \mapsto K_\infty(x, y)$ to this set. Now, for $x > x_0$, one also has the formula:

$$K_\infty(x, y) = \frac{1}{2\pi} \frac{\Gamma(s+1)\Gamma(\overline{s}+1)}{\Gamma(2\Re s+1)\Gamma(2\Re s+2)} \mathbb{E}\left[\tilde{P}'(Z)Q(y) - Q'(Z)\tilde{P}(y)\right],$$

where $Z$ is a uniform random variable on the segment $[x, y]$. By the bounds obtained for $\Phi$ and $\Phi'$, one deduces that the continuation of $x \mapsto K_\infty(x, y)$ to the set $H_{x_0} \backslash \{y\}$ is bounded in the neighborhood of $y$, and hence can be extended to $H_{x_0}$. By construction, this extension coincides with $K_\infty(x, y)$ for $x \in (x_0, \infty)\backslash\{y\}$, and in fact it coincides on the whole interval $(x_0, \infty)$, since $K_\infty(x, y)$ tends to $K_\infty(y, y)$ when $x$ is real and tends to $y$. In other words, we have constructed an extension of $x \mapsto K_\infty(x, y)$ which is holomorphic on $H_{x_0}$. Now, let us take $x \in H_{x_0}$ such that $|x-y| \geq y/2$, which implies that $|x-y| \geq C(|x|+y)$ for a universal constant $C$. By using this inequality and the bounds on $\tilde{P}$ and $Q$, one obtains:

$$|K_\infty(x, y)| \leq \frac{C(s, x_0)}{|xy|^{\Re(s)+1}}.$$

By taking the derivative of the equation (2.16), one obtains the bound (again for $x \in H_{x_0}$ and $|x - y| \geq y/2$):

$$\left| \frac{\partial}{\partial x} K_\infty(x, y) \right| \leq \frac{C(s, x_0)}{|x|^{\Re(s)+2} y^{\Re(s)+1}}.$$

Now, the maximum principle implies that the condition $|x - y| \geq y/2$ can be removed in the last two bounds. By using these bounds, Cauchy-Schwarz inequality and dominated convergence, one deduces that the function:

$$x \mapsto \int_t^\infty K_\infty(x, y) f(y) \, dy$$

is well defined on the set $H_{x_0}$, and admits a derivative, given by the formula:

$$x \mapsto \int_t^\infty \left( \frac{\partial}{\partial x} K_\infty(x, y) \right) f(y) \, dy. \qquad \square$$

## 2.4 Proof of Theorem 1.4

Note that by Proposition 2.12, $K_{[N]}$ and $K_\infty$ satisfy the conditions of Lemma 2.6. For $k, N \in \mathbb{N}$, let $\rho_{k,N}$ be the $k$-correlation function associated with $K_{[N]}$ and $\rho_{k,\infty}$ the $k$-correlation function associated with $K_\infty$. By Lemma 2.14, $F_N$ is well defined for $N \in \mathbb{N} \cup \{\infty\}$, and $C^3$. The explicit expressions of $F_N$ and $F_\infty$ and their derivatives are given in Lemma 2.14 by replacing $\rho_k$ by $\rho_{k,N}$ and $\rho_{k,\infty}$ respectively. Now, for $k \geq 1$, all the partial derivatives of any order of $\rho_{k,N}$ converge pointwise to the corresponding derivatives of $\rho_{k,\infty}$ when $N$ goes to infinity. This is due to the explicit expression of $\rho_{k,N}$ as a determinant and the convergence given by Proposition 2.12. Moreover, by that same Proposition, there exists $\alpha > 1/2$ only depending on $s$ such that

$$\left| \frac{\partial^p}{\partial x_1^p} \rho_{k,N}(x_1, \ldots, x_k) \right| \leq \frac{C(x_0, s)^k}{(x_1 \cdots x_k)^{2\alpha}},$$

for $p \in \{0, 1, 2\}$, and for all $x_1, \ldots, x_k \geq x_0 > 0$. In particular, this bound is uniform with respect to $N$, and it is now easy to deduce the pointwise convergence of the derivatives of $F_N$ (up to order 3), by dominated convergence.

## 2.5 Proof of Theorem 1.7

Theorem 1.7 follows immediately from Proposition 2.4 and the following Proposition:

**Proposition 2.16** *Let $s$ be such that $\Re s > -1/2$, and $F_N$, $N \in \mathbb{N}$, and $F_\infty$ be as in Theorem 1.4. Then, for $N \in \mathbb{N} \cup \{\infty\}$, the function $\theta_N$ from $\mathbb{R}_+^*$ to $\mathbb{R}$, defined by*

$$\theta_N(\tau) = \tau \frac{d}{d\tau} \log(F_N(\tau^{-1})),$$

*is well defined and $C^2$. Moreover, for $p \in \{0, 1, 2\}$, the derivatives $\theta_N^{(p)}$ converge pointwise to $\theta_\infty^{(p)}$ (defined by (1.21)).*

*Proof* Recall that for $t > 0$, $F_N(t)$ is the probability that a random matrix of dimension $N$, following the generalized Cauchy weight (1.2), has no eigenvalue in $(Nt, \infty)$. Therefore, $F_N(t) > 0$, for any $t > 0$. Similarly, $F_\infty(t)$ is the probability that the limiting determinantal process has no point in $(t, \infty)$, which is also different from zero for any $t > 0$, as we proved in Sect. 2.3. Therefore, for all $N \in \mathbb{N} \cup \{\infty\}$, $\theta_N$ is well-defined and

$$\theta_N(\tau) = -\frac{F_N'(\tau^{-1})}{\tau F_N(\tau^{-1})}.$$

Since $F_N$ is in $C^3$, $\theta_N$ is in $C^2$, for all $N \in \mathbb{N} \cup \{\infty\}$, and one can give explicit expressions for $\theta_N$ and for its first two derivatives (see Lemma 2.14). It is now easy to deduce from these explicit expressions and the pointwise convergence of the first three derivatives of $F_N$ assured by Theorem 1.4, the pointwise convergence for the first two derivatives of $\theta_N$, when $N \in \mathbb{N}$ goes to infinity. □

*Remark 2.17* Note that most probably, it is also possible to derive the fact that the kernel $K_\infty$ gives rise to a solution of the Painlevé-V equation (1.22) directly by the methods of Tracy and Widom [25] in an analogous way then the one used to obtain the Painlevé-VI equation (2.1) in the finite $N$ case. In fact, the recurrence equations (2.2) in the infinite case are:

$$x^2 P'(x) = \left(-x\Re s + \frac{\Im s}{\Re s}\right) P(x) + \frac{|s|^2}{\Re s^2} \frac{1}{2\Re s + 1} Q(x),$$
$$x^2 Q'(x) = -(2\Re s + 1)\, P(x) - \left(-x\Re s + \frac{\Im s}{\Re s}\right) Q(x),$$

where $P$ and $Q$ are as in the definition of $K_\infty$ in (1.17) and (1.18). However, this method will has several drawbacks, as already mentioned in the introduction.

## 3 The Convergence Rate: Proof of Theorem 1.9

We first need the rate of convergence for the scaled kernel $K_{[N]}(x, y) = N K_N(Nx, Ny)$:

**Lemma 3.1** *Let $x, y > x_0 > 0$. Then there exists a constant $C(x_0, s) > 0$ only depending on $x_0$ and $s \in \mathbb{C}$ ($\Re s > -1/2$), such that*

$$\left|K_{[N]}(x, y) - K_\infty(x, y)\right| \le \frac{1}{N} \frac{C(x_0, s)}{(xy)^{\Re s + 1}}.$$

In the following proof, $C(a, b, \dots)$ denotes a strictly positive constant only depending on $a, b, \dots$ which may change from line to line.

*Proof* Let $x, y > x_0$, $x \neq y$. Then, setting $C(s) = |\frac{1}{2\pi} \frac{\Gamma(s+1)\Gamma(\bar{s}+1)}{\Gamma(2\Re s+1)\Gamma(2\Re s+2)}|$, and using the notations from Lemma 2.9, we have

$$\begin{aligned}&\left|K_{[N]}(x, y) - K_\infty(x, y)\right| \\ &\quad = C(s) \left|\frac{1}{x-y}\right| \left|\tilde{P}_N(x) Q_N(y) - \tilde{P}_N(y) Q_N(x) - (\tilde{P}(x) Q(y) - \tilde{P}(y) Q(x))\right|\end{aligned}$$

$$\leq C(s)\left|\frac{1}{x-y}\right|\left\{\left|\tilde{P}_N(x)Q_N(y)-\tilde{P}(x)Q(y)\right|+\left|\tilde{P}_N(y)Q_N(x)-\tilde{P}(y)Q(x)\right|\right\}$$

$$\leq C(s)\left|\frac{1}{x-y}\right|\left\{\left|\tilde{P}_N(x)-\tilde{P}(x)\right||Q_N(y)|+|Q_N(y)-Q(y)|\left|\tilde{P}(x)\right|\right.$$

$$\left.+\left|\tilde{P}_N(y)-\tilde{P}(y)\right||Q_N(x)|+|Q_N(x)-Q(x)|\left|\tilde{P}(y)\right|\right\}. \tag{3.1}$$

Similarly, if $x, y > x_0$, it is easy to check (by using the fundamental Theorem of calculus) that

$$\left|K_{[N]}(x,y)-K_\infty(x,y)\right|$$

$$\leq C(s)\mathbb{E}\left[\left|\tilde{P}'_N(Z)-\tilde{P}'(Z)\right||Q_N(x)|+|Q_N(x)-Q(x)|\left|\tilde{P}'(Z)\right|\right.$$

$$\left.+\left|\tilde{P}_N(x)-\tilde{P}(x)\right|\left|Q'_N(Z)\right|+\left|Q'_N(Z)-Q'(Z)\right|\left|\tilde{P}(x)\right|\right], \tag{3.2}$$

where $Z$ is a uniform random variable in the interval $[x, y]$.

By using (3.1) if $\max(x, y) \geq 2\min(x, y)$ and (3.2) if $\max(x, y) < 2\min(x, y)$, one deduces that the Lemma is proved, if we show that for $p \in \{0, 1\}$,

$$\left|\tilde{P}_N^{(p)}(x)-\tilde{P}^{(p)}(x)\right| \leq \frac{1}{N}\frac{C(x_0,s,p)}{x^{p+\Re s}}, \tag{3.3}$$

and

$$\left|Q_N^{(p)}(x)-Q^{(p)}(x)\right| \leq \frac{1}{N}\frac{C(x_0,s,p)}{x^{p+1+\Re s}}. \tag{3.4}$$

Recall from (2.9), the following function (note that $x > x_0 > 0$):

$$\Phi_N(x) = D'(N,s)e^{\pi\Im s/2}\left(\frac{2}{x}\right)^{N-n}\left(\frac{2}{x}\right)^{\Re s}$$

$$\times\left(1-\frac{i}{Nx}\right)^{(N-s)/2-(N-n)}\left(1+\frac{i}{Nx}\right)^{-(\bar{s}+N)/2}F_{n,h,a}(Nx),$$

and let us define similarly:

$$\Phi(x) = e^{\pi\Im s/2}\left(\frac{2}{x}\right)^{N-n}\left(\frac{2}{x}\right)^{\Re s}e^{-i/x}{}_1F_1\left[h,a;2i/x\right],$$

where $(n, h, a) = (N, s, 2\Re s + 1)$ and $\Phi_N(x) = \tilde{P}_N(x)$, or $(n, h, a) = (N - 1, s + 1, 2\Re s + 2)$ and $\Phi_N(x) = Q_N(x)$, for $N \in \mathbb{N}^*$ (recall that $N - n = 0$ in the first case and $N - n = 1$ in the second case). It suffices to show that for $p \in \{0, 1\}$, $|\Phi_N^{(p)}(x) - \Phi^{(p)}(x)| \leq \frac{C(x_0,s,p)}{Nx^{\Re(s)+1+p}}$ to deduce (3.3) and (3.4). Let us first investigate the case $p = 0$:

$$|\Phi_N(x)-\Phi(x)|$$

$$\leq e^{\pi\Im s/2}\left(\frac{2}{x}\right)^{\Re s+(N-n)}\left\{\left|D'(N,s)-1\right|\left|(1-i/(Nx))^{(N-s)/2-(N-n)}\right.\right.$$

$$\times\ (1+i/(Nx))^{-(N+\overline{s})/2}\, F_{n,h,a}(Nx)\big|$$
$$+\big|(1-i/(Nx))^{(N-s)/2-(N-n)}\,(1+i/(Nx))^{-(N+\overline{s})/2}-e^{-i/x}\big|$$
$$\times\ \big|F_{n,h,a}(Nx)\big|+\big|e^{-i/x}\big|\,\big|F_{n,h,a}(Nx)-{}_1F_1\left[h,a;2i/x\right]\big|\big\}. \tag{3.5}$$

We show that the bracket $\{.\}$ is bounded uniformly by $\frac{1}{N}C(x_0,s)$. In the following, we look at the three summands in the bracket separately. For the first one, we have by (2.12) and (2.14) that

$$\big|(1-i/(Nx))^{(N-s)/2-(N-n)}\,(1+i/(Nx))^{-(N+\overline{s})/2}\,F_{n,h,a}(Nx)\big|\le C(x_0,s).$$

Moreover, it is easy to check (for example, by using Stirling formula) that

$$\left|\frac{\Gamma(2\Re s+N+1)}{N^{2\Re s+1}\Gamma(N)}-1\right|\le\frac{1}{N}C(s).$$

Now, if some sequence $a_N>0$ converges to $a>0$ in the order $1/N$ as $N\to\infty$, $\sqrt{a_N}\to\sqrt{a}$, in the order $1/N$ as well, for $N\to\infty$. Hence,

$$\big|D'(N,s)-1\big|=\left|\left(\frac{\Gamma(2\Re s+N+1)}{N^{2\Re s+1}\Gamma(N)}\right)^{1/2}-1\right|\le\frac{1}{N}C(s).$$

Thus, the first term in the bracket $\{.\}$ of (3.5) is bounded by $C(x_0,s)/N$. Let us look at the second term:

$$\big|F_{n,h,a}(Nx)\big|\le C(x_0,s),$$

again according to (2.14). Moreover,

$$\begin{aligned}&\big|(1-i/(Nx))^{(N-s)/2-(N-n)}(1+i/Nx)^{-(N+\overline{s})/2}-e^{-i/x}\big|\\&\quad\le\big|(1-i/(Nx))^{(N-s)/2}(1+i/(Nx))^{-(N+\overline{s})/2}-e^{-i/x}\big|\,\big|(1-i/(Nx))^{-(N-n)}\big|\\&\qquad+\big|e^{-i/x}\big|\,\big|(1-i/(Nx))^{-(N-n)}-1\big|.\end{aligned}\tag{3.6}$$

It is clear, that the second term in the sum is bounded by $C(x_0)/N$. For the first term, the second factor is bounded by $C(x_0)$, whereas for the first factor, we have the following:

$$\begin{aligned}&\left|\left(\frac{1-i/(Nx)}{1+i/(Nx)}\right)^{N/2}\left(\frac{1-i/(Nx)}{1+i/(Nx)}\right)^{-i\Im s/2}\left(1+1/(Nx)^2\right)^{-\Re s/2}-e^{-i/x}\right|\\&\quad\le\left|\left(\frac{1-i/(Nx)}{1+i/(Nx)}\right)^{N/2}-e^{-i/x}\right|\left|\left(\frac{1-i/(Nx)}{1+i/(Nx)}\right)^{-i\Im s/2}\right|\left|\left(1+1/(Nx)^2\right)^{-\Re s/2}\right|\\&\qquad+\left|e^{-i/x}\right|\left|\left(\frac{1-i/(Nx)}{1+i/(Nx)}\right)^{-i\Im s/2}-1\right|\left|\left(1+1/(Nx)^2\right)^{-\Re s/2}\right|\\&\qquad+\left|e^{-i/x}\right|\left|\left(1+1/(Nx)^2\right)^{-\Re s/2}-1\right|.\end{aligned}\tag{3.7}$$

We investigate all terms in this sum separately: $|(1+1/(Nx)^2)^{-\Re s/2}-1|$ can be bounded by $C(x_0,s)/N$ using binomial series, and

$$\left|\left(\frac{1-i/(Nx)}{1+i/(Nx)}\right)^{-i\Im s/2}\right| = |\exp\{-\Im s \operatorname{Arg}(1+i/Nx)\}| \leq C(x_0,s).$$

Furthermore,

$$\begin{aligned}
&\left|\left(\frac{1-i/(Nx)}{1+i/(Nx)}\right)^{-i\Im s/2}-1\right| \\
&\quad = |\exp\{-\Im s \operatorname{Arg}(1+i/(Nx))\}-1| \\
&\quad = |\exp\{-\Im s \operatorname{Arctan}(1/(Nx))\}-1| \\
&\quad \leq \left|\sum_{k=0}^{\infty}\frac{\left(-\Im s\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}(1/(Nx))^{2n+1}\right)^k}{k!}-1\right| \\
&\quad \leq \frac{1}{N}C(x_0,s).
\end{aligned}$$

Here, we use the fact that the Taylor series for the arctangent is absolutely convergent if $0<1/(Nx)<1$, which is true for $N$ large enough. Now, by considering the series of the complex logarithm of $1\pm i/(Nx)$ (absolutely convergent for $N$ large enough), one can show that

$$\left|(1\pm i/(Nx))^{\mp N/2}-e^{-i/(2x)}\right| \leq \frac{1}{N}C(x_0).$$

The remaining terms in the sum (3.7) are clearly bounded by $C(x_0,s)$ and hence, the second term in the sum (3.5) converges to zero in the order $1/N$.

We investigate the third term in (3.5): Clearly, $|e^{-i/x}|=1$. The second factor in the third term requires somewhat more work:

$$\begin{aligned}
&\left|F_{n,h,a}(Nx)-{}_1F_1[h,a;2i/x]\right| \\
&\quad = \left|\sum_{k=0}^{\infty}\frac{(-n)_k(h)_k 2^k}{(a)_k k!}\left(\frac{1}{1+iNx}\right)^k-\sum_{k=0}^{\infty}\frac{(h)_k(2i)^k}{(a)_k k!}\left(\frac{1}{x}\right)^k\right| \\
&\quad \leq \sum_{k=1}^{\infty}\frac{(|h|)_k 2^k}{(a)_k k!}\left|(-n)_k\left(\frac{1}{i-Nx}\right)^k-\left(\frac{1}{x}\right)^k\right|,
\end{aligned}$$

where the last inequality is true because of the absolute convergence of both sums. Now,

$$\begin{aligned}
&\left|(-n)_k\left(\frac{1}{i-Nx}\right)^k-\left(\frac{1}{x}\right)^k\right| \\
&\quad \leq \frac{1}{x_0^k}\left|1-\frac{(-n)_k}{((i/x)-N)^k}\right|
\end{aligned}$$

$$= \frac{1}{x_0^k} \left| 1 - \prod_{l=N-n}^{N-n+k-1} \frac{l-N}{(i/x)-N} \right|$$

$$= \frac{1}{x_0^k} \left| 1 - \prod_{l=N-n}^{N-n+k-1} \frac{(N-l)_+}{N-(i/x)} \right|.$$

Since all the factors in the last product have a module smaller than 1, it is possible to deduce:

$$\left| (-n)_k \left( \frac{1}{i-Nx} \right)^k - \left( \frac{1}{x} \right)^k \right|$$

$$\leq \frac{1}{x_0^k} \sum_{l=N-n}^{N-n+k-1} \left| 1 - \frac{(N-l)_+}{N-(i/x)} \right|$$

$$\leq \frac{1}{x_0^k} \sum_{l=N-n}^{N-n+k-1} \frac{l+1/x}{N}$$

$$\leq \frac{1}{x_0^k} \frac{k^2 + k/x_0}{N}.$$

This bound implies easily that:

$$\left| F_{n,h,a}(Nx) - {}_1F_1[h,a;2i/x] \right| \leq \frac{C(s,x_0)}{N},$$

and we can deduce:

$$|\Phi_N(x) - \Phi(x)| \leq \frac{1}{N} \frac{C(x_0,s)}{x^{\Re s+(N-n)}}.$$

Therefore, (3.3) and (3.4) are proved for $p=0$.

It remains to prove that

$$|\Phi'_N(x) - \Phi'(x)| \leq \frac{1}{N} \frac{C(x_0,s)}{x^{\Re s+(N-n)+1}},$$

to show (3.3) and (3.4) for $p=1$. But this is immediate using the same methods as above and the fact that we can write

$$\begin{aligned}
\Phi'_N(x) = D'(N,s) e^{\pi \Im s/2} \left( \frac{2}{x} \right)^{\Re s+(N-n)} & (1-i/(Nx))^{(N-s)/2-(N-n)} \\
& \times (1+i/(Nx))^{-(\overline{s}+N)/2} \left[ \frac{-(\Re s+(N-n))}{x} F_{n,h,a}(Nx) \right. \\
& + \frac{i}{x^2} \left\{ \left( \frac{1-s/N}{2} - \frac{N-n}{N} \right) \frac{1}{1-i/(Nx)} + \frac{1+\overline{s}/N}{2} \frac{1}{1+i/(Nx)} \right\} F_{n,h,a}(Nx) \\
& \left. + \sum_{k=0}^{\infty} \frac{(-n)_k (h)_k k 2^{k+1}}{(a)_k k!} \left( -\frac{iN}{2} \right) \left( \frac{1}{1+iNx} \right)^{k+1} \right],
\end{aligned}$$

and

$$\Phi'(x) = e^{\pi \Im s/2} \left(\frac{2}{x}\right)^{\Re s + (N-n)} e^{-i/x} \left[ \frac{-(\Re s + (N-n))}{x} {}_1F_1[h, a; 2i/x] \right.$$
$$\left. + \frac{i}{x^2} {}_1F_1[h, a; 2i/x] - \sum_{k=0}^{\infty} \frac{(h)_k (2i)^k k}{(a)_k k!} \left(\frac{1}{x}\right)^{k+1} \right].$$

This ends the proof. □

Now we prove Theorem 1.9. Let us first prove the following result: for all $n \in \mathbb{N}^*$, and for all symmetric and positive $n \times n$ matrices $A$ and $B$ such that $\sup_{1 \le i,j \le n} |A_{i,j}| \le \alpha$, $\sup_{1 \le i,j \le n} |B_{i,j}| \le \alpha$ and $\sup_{1 \le i,j \le n} |A_{i,j} - B_{i,j}| \le \beta$ for some $\alpha, \beta > 0$, one has

$$|\det(B) - \det(A)| \le \beta n^2 \alpha^{n-1}. \tag{3.8}$$

Indeed, the following formula holds:

$$\det(B) - \det(A) = \int_0^1 d\lambda \text{ Diff} \det[A + \lambda(B - A)].(B - A),$$

where for $C := A + \lambda(B - A)$, Diff $\det[C].(B - A)$ denotes the image of the matrix $B - A$ by the differential of the determinant, taken at point $C$. Now, $C$ is symmetric, positive, and $|C_{i,j}| \le \alpha$ for all indices $i, j$, since $C$ is a barycenter of $A$ and $B$, with positive coefficients. Moreover, the derivative of $C$ with respect to the coefficient of indices $i, j$ is (up to a possible change of sign) the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing the line $i$ and the column $j$ of $C$. By using the same arguments as in the proof of inequality (2.7), one can easily deduce that this derivative is bounded by $\alpha^{n-1}$. Hence:

$$|\det(B) - \det(A)| \le \int_0^1 d\lambda \, \alpha^{n-1} \sum_{1 \le i,j \le n} |B_{i,j} - A_{i,j}|$$

which implies (3.8). Now, we can compare the determinants of $(K_{[N]}(x_i, x_j))_{i,j=1}^n$ and $(K_\infty(x_i, x_j))_{i,j=1}^n$ for $x_1, \dots, x_n > x_0$ by applying (3.8) to:

$$A_{i,j} = (x_i x_j)^{\Re(s)+1} K_{[N]}(x_i, x_j),$$
$$B_{i,j} = (x_i x_j)^{\Re(s)+1} K_\infty(x_i, x_j),$$
$$\alpha = C(x_0, s), \qquad \beta = C(x_0, s)/N.$$

Here, we use the bounds for $K_{[N]}$, $K_\infty$ and their difference given in Proposition 2.11 and in Lemma 3.1. We obtain:

$$\left|\det(K_{[N]}(x_i, x_j)_{i,j=1}^n) - \det(K_\infty(x_i, x_j)_{i,j=1}^n)\right|$$
$$\le \frac{1}{(x_1 \cdots x_n)^{2\Re(s)+2}} \frac{n^2}{N} (C(x_0, s))^n.$$

This implies

$$\begin{aligned}
&\left| P\left[\frac{\lambda_1(N)}{N} \le x\right] - \det(I - K_\infty)|_{L_2(t,\infty)} \right| \\
&\le \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(x,\infty)^n} \left|\det(K_{[N]}(x_i, x_j)_{i,j=1}^n) - \det(K_\infty(x_i, x_j)_{i,j=1}^n)\right| dx_1 \cdots dx_n \\
&\le \sum_{n=1}^{\infty} \frac{1}{n!} \frac{n^2}{N} \left( \int_{(x,\infty)} \frac{C(x_0, s)}{y^{2\Re s+2}} dy \right)^n \\
&\le \frac{1}{N} \sum_{n=1}^{\infty} \frac{n}{(n-1)!} \left( \int_{(x_0,\infty)} \frac{C(x_0, s)}{y^{2\Re s+2}} dy \right)^n \\
&\le C(x_0, s)/N,
\end{aligned}$$

since the last sum is convergent and depends only on $x_0$ and $s$.

## 4 Concluding Remark About $U(N)$

With the notations and results from Secti. 2.1, we know that the distribution of $\lambda_1(N)$, the largest eigenvalue of a matrix in $H(N)$ under the distribution (1.2), can be written as

$$P[\lambda_1(N) \le a] = \exp\left( -\int_a^\infty \frac{\sigma(t)}{1+t^2} dt \right). \tag{4.1}$$

Using the Cayley transform $H(N) \ni X \mapsto U = \frac{X+i}{X-i} \in U(N)$, we can map the generalized Cauchy measure from $H(N)$ to the measure (1.4) on $U(N)$. The inverse of the Cayley transform writes as

$$\theta \longmapsto i \frac{e^{i\theta}+1}{e^{i\theta}-1} = \cot\left(\frac{\theta}{2}\right),$$

for $\theta \in [-\pi, \pi]$. $\theta = 0$ is mapped to $\infty$ by definition. Using this application, (4.1) turns into:

$$P[\theta_1(N) \ge y] = \exp\left( -\frac{1}{2} \int_0^y d\phi\, \sigma\left(\cot\left(\frac{\phi}{2}\right)\right)\right), \tag{4.2}$$

for $y = 2\operatorname{arccot}(a)$, $y \in [0, 2\pi]$, and $e^{i\theta_1(N)} = \frac{\lambda_1(N)+i}{\lambda_1(N)-i}$. $\theta_1(N)$ being here in $[0, 2\pi]$ (and not in $[-\pi, \pi]$!). In other words, the distribution of the largest eigenvalue on the real line of a random matrix $H \in H(N)$ with measure (1.2), maps to the distribution of the eigenvalue with smallest angle of a random matrix $U \in U(N)$ satisfying the law (1.4). Here, smallest angle has to be understood as the eigenvalue which is closest to 1 looking counterclockwise on the circle from the point 1.

According to [4], the eigenvalues $\{e^{i\theta_1}, \dots, e^{i\theta_N}\}$, (recall that $\theta_i \in [-\pi, \pi]$) of a random unitary matrix $U$, satisfying the law (1.4), also determine a determinantal point process with correlation kernel

$$K_N^U(e^{i\alpha}, e^{i\beta}) = d_N(s)\sqrt{w_U(\alpha)w_U(\beta)} \times \frac{e^{iN\frac{\alpha-\beta}{2}} Q_N^s(e^{-i\alpha}) Q_N^{\overline{s}}(e^{i\beta}) - e^{-iN\frac{\alpha-\beta}{2}} Q_N^{\overline{s}}(e^{i\alpha}) Q_N^s(e^{-i\beta})}{e^{i\frac{\alpha-\beta}{2}} - e^{-i\frac{\alpha-\beta}{2}}}, \tag{4.3}$$

where $d_N(s) = \frac{1}{2\pi}\frac{(\overline{s}+1)_N(s+1)_N}{(2\Re s+1)_N N!}\frac{\Gamma(1+s)\Gamma(1+\overline{s})}{\Gamma(1+2\Re s)}$, $Q_N^s(x) = {}_2F_1[s, -n, -n-\overline{s}; x]$ and $w_U$ is the weight defined after (1.4). If $N \to \infty$, the rescaled correlation kernel $\frac{1}{N}K_N^U(e^{i\alpha/N}, e^{i\beta/N})$ converges to

$$K^U(\alpha, \beta) = e(s)|\alpha\beta|^{\Re s} e^{-\frac{\pi}{2}\Im s(\mathrm{Sgn}(\alpha)+\mathrm{Sgn}(\beta))} \times \frac{e^{i\frac{\alpha-\beta}{2}} Q^s(-i\alpha) Q^{\overline{s}}(i\beta) - e^{-i\frac{\alpha-\beta}{2}} Q^{\overline{s}}(i\alpha) Q^s(-i\beta)}{\alpha - \beta}, \tag{4.4}$$

where $e(s) = \frac{1}{2\pi i}\frac{\Gamma(s+1)\Gamma(\overline{s}+1)}{\Gamma(2\Re s+1)^2}$, and $Q^s(x) = {}_1F_1[s, 2\Re s+1; x]$ (again according to [4]). In [4], it is also shown that the kernel $K^U$ coincides up to multiplication by a constant with the limiting kernel $K_\infty$ from (1.17) if one changes the variables in (4.4) to $\alpha = \frac{2}{x}$ and $\beta = \frac{2}{y}$, $x, y \in \mathbb{R}^*$. This not surprising because a scaling $x \mapsto Nx$ for the eigenvalues in the Hermitian case corresponds to a scaling $\alpha \mapsto \frac{\alpha}{N}$ for the eigenvalues in the unitary case as can be seen from the elementary fact that for $x \in \mathbb{R}^*$, and $N \in \mathbb{N}$, one has

$$\frac{Nx+i}{Nx-i} = e^{\frac{2i}{Nx}+O(N^{-2})}. \tag{4.5}$$

*Remark 4.1* Note that because of the $O(N^{-2})$ term in the argument of (4.5), it is not possible to give an identity involving the kernel $K_N$ of Theorem 1.1 and the kernel (4.3).

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