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Calculus of Variations

Schwarz operators of minimal surfaces spanning polygonal boundary curves

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Abstract This paper examines the Schwarz operator A and its relatives $\dot{A}$, $\bar{A}$ and $\bar{\dot{A}}$ that are assigned to a minimal surface X which maps consequtive arcs of the boundary of its parameter domain onto the straight lines which are determined by pairs P_j, P_{j+1} of two adjacent vertices of some simple closed polygon $\Gamma \subset \mathbb{R}^3$. In this case X possesses singularities in those boundary points which are mapped onto the vertices of the polygon Γ. Nevertheless it is shown that A and its closure $\bar{A}$ have essentially the same properties as the Schwarz operator assigned to a minimal surface which spans a smooth boundary contour. This result is used by the author to prove in [Jakob, Finiteness of the set of solutions of Plateau's problem for polygonal boundary curves. I.H.P. Analyse Non-lineaire (in press)] the finiteness of the number of immersed stable minimal surfaces which span an extreme simple closed polygon Γ, and in [Jakob, Local boundedness of the set of solutions of Plateau's problem for polygonal boundary curves (in press)] even the local boundedness of this number under sufficiently small perturbations of Γ.

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1 Introduction and main results

This paper is concerned with the Schwarz operator

$$A \equiv A^X := -\triangle + 2\,KE \tag{1}$$

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for a minimal surface X which maps consequtive arcs of the boundary of its parameter domain onto the straight lines that are determined by pairs P_j, P_{j+1} of two adjacent vertices of an arbitrarily fixed simple closed polygon $\Gamma \subset \mathbb{R}^3$ with $N+3$ vertices. Such a surface is given by a continuous $H^{1,2}$-mapping $X : \bar{B} \longrightarrow \mathbb{R}^3$ of the closure of the unit disc $B := \{w = (u, v) \in \mathbb{R}^2 \mid |w| < 1\}$ into $\mathbb{R}^3$ which is harmonic on B, satisfies

$$|X_u| = |X_v|, \quad \langle X_u, X_v\rangle = 0 \quad \text{on } B \tag{2}$$

and meets the boundary conditions $X(e^{i\theta}) \in \Gamma_j$ for $\theta \in [\tau_j, \tau_{j+1}]$, $j = 1, \ldots, N+3$, where Γ_j denotes the line $\{P_j + t\,(P_{j+1} - P_j) | t \in \mathbb{R}\}$ and where the τ_j are consequtive angles in $(0, 2\pi]$. We denote by $\tilde{\mathcal{M}}(\Gamma)$ the set of such surfaces. Furthermore K in (1) is the Gauss curvature of X and $E := |X_u|^2$. For minimal surfaces X bounded by some smooth contour Γ the behaviour of A^X is well known. The aim of this paper is to show that A^X respectively its closure $\overline{A^X}$ have essentially the same properties for minimal surfaces X with those "overshooting", piecewise linear boundary values, as explained above. The author is using this result in [7,8] for his proof of the boundedness of the number of immersed stable minimal surfaces spanning a simple closed polygon which is contained in a sufficiently small neighborhood of any fixed extreme simple closed polygon. The difficulty in studying A^X for a minimal surface X with overshooting, piecewise linear boundary constraints is caused by the fact that X is "singular" at the boundary points $e^{i\tau_j}$ which are mapped onto the corners P_j of Γ. Consequently the perturbing term KE of A^X is only of class $L^p(B)$ for some $p > 1$ on account of estimate (5) below. For some fixed $X \in \tilde{\mathcal{M}}(\Gamma)$ we shall consider $A \equiv A^X$ on

$$\text{Domain}(A) := \left\{\varphi \in C^2(B) \cap \mathring{H}^{1,2}(B) | A(\varphi) \in L^2(B)\right\}.$$

By $\dot{A}$ and $\dot{\triangle}$ we denote the minimal Schwarz and minimal Laplace operator on the domain $H^{2,2}(B) \cap C^2_0(B)$, respectively, where we set

$$C^2_0(B) := \left\{\varphi \in C^2(B) \cap C^0(\bar{B}) | \varphi|_{\partial B} \equiv 0\right\}.$$

Furthermore let $\overline{A}$, $\overline{\dot{A}}$ and $\overline{\dot{\triangle}}$ denote the $L^2(B)$-closures of A, $\dot{A}$ and $\dot{\triangle}$, respectively. Finally we consider the assigned quadratic form

$$J(\varphi) \equiv J^X(\varphi) := \int_B |\nabla\varphi|^2 + 2KE\,\varphi^2 \mathrm{d}w$$

which is defined for any $\varphi \in \mathring{H}^{1,2}(B)$ due to $KE \in L^p(B)$ for some $p > 1$. To study the spectra of A and $\bar{A}$ we investigate J on the function space

$$S\mathring{H}^{1,2}(B) := \left\{\varphi \in \mathring{H}^{1,2}(B) | \|\varphi\|_{L^2(B)} = 1\right\}.$$

Similarly we denote by $S(H^{2,2}(B) \cap \mathring{H}^{1,2}(B))$ and $S\text{Dom}(A)$ the intersections of the "$L^2(B)$-sphere" with the respective function spaces. Then we shall prove

Theorem 1 (i) *The spectra of A and $\bar{A}$ coincide. They are discrete and accumulate only at ∞; thus their eigenspaces are finite dimensional. Furthermore for their common smallest eigenvalue $\lambda_{\min} := \lambda_{\min}(A) = \lambda_{\min}(\bar{A})$ we have*

$$\lambda_{\min} = \inf_{S\text{Dom}(A)} J = \inf_{S\mathring{H}^{1,2}(B)} J = \inf_{S(H^{2,2}(B)\cap\mathring{H}^{1,2}(B))} J. \tag{3}$$

(ii) *For an eigenfunction* φ^* *in the eigenspace* $\mathrm{ES}_{\lambda_{\min}}(\bar{A})$ *there holds* $|\varphi^*| > 0 \quad \text{on } B$, *whence:*

$$\dim \mathrm{ES}_{\lambda_{\min}}(\bar{A}) = \dim \mathrm{ES}_{\lambda_{\min}}(A) = 1. \tag{4}$$

Especially an eigenfunction $\varphi^* \in \mathrm{ES}_{\lambda_{\min}}(A)$ *satisfies* $|\varphi^*| > 0 \quad \text{on } B$.

To prove this theorem we need some of Heinz' results (see [3,4]) about minimal surfaces with overshooting, piecewise linear boundary values. To this end we need some definitions:
Let Γ be some simple closed polygon in $\mathbb{R}^3$ with $N+3$ vertices ($N \in \mathbb{N}$)

$$(P_1, P_2, \ldots, P_{N+3}),$$

where we require the pairs of vectors $(P_{j+1} - P_j, P_j - P_{j-1})$ to be linear independent for $j = 1, \ldots, N+3$, with $P_0 := P_{N+3}$ and $P_{N+4} := P_1$. We consider the open bounded convex set T of N-tuples

$$(\tau_1, \tau_2, \ldots, \tau_N) =: \tau \in (0, \pi)^N,$$

which meet $0 < \tau_1 < \cdots < \tau_N < \pi$. Moreover we fix the three angles $\tau_{N+k} := \frac{\pi}{2}(1+k)$, $k = 1, 2, 3$. Now to any $\tau \in T$ we assign the set of surfaces

$$\tilde{\mathcal{U}}(\tau) := \left\{X \in C^0(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3) | X(e^{i\theta}) \in \Gamma_j \quad \text{for } \theta \in [\tau_j, \tau_{j+1}],\ 1 \le j \le N+3\right\},$$

where $\Gamma_j := \{P_j + t\,(P_{j+1} - P_j) | t \in \mathbb{R}\}$, $P_{N+4} := P_1$ and $\tau_{N+4} := \tau_1$. On account of Satz 1 in [3] one can define the map

$$\tilde{\psi}(\tau) := \text{unique minimizer of } \mathcal{D} \text{ within } \tilde{\mathcal{U}}(\tau),$$

where $\mathcal{D}$ denotes Dirichlet's integral. We will also use the notation $X(\,\cdot\,, \tau)$ for $\tilde{\psi}(\tau)$. From Satz 1 in [3] and Satz 1 in [4] we quote the following result:

Proposition 1 (i) *The surfaces* $\tilde{\psi}(\tau)$ *are harmonic on* $B \quad \forall \tau \in T$.

(ii) *The function* $\tilde{f} := \mathcal{D} \circ \tilde{\psi}$ *is of class* $C^\omega(T)$.

(iii) *A surface* $\tilde{\psi}(\tau)$ *is conformally parametrized on* B*, thus a minimal surface in* $\tilde{\mathcal{U}}(\tau)$*, if and only if* τ *is contained in* $K(\tilde{f})$*, the set of critical points of* $\tilde{f}$.

Point (i) of the above theorem and the Courant–Lebesgue Lemma imply (cf. [6, Chap. 4]) that

$$\begin{aligned} \tilde{\mathcal{M}}(\Gamma) &\equiv \{\text{set of minimal surfaces on } B\} \cap \bigcup_{\tau \in T} \tilde{\mathcal{U}}(\tau) \cap H^{1,2}(B, \mathbb{R}^3) \\ &= \left\{X \in \text{image}(\tilde{\psi}) | X \text{ is also conformally parametrized on } B\right\}. \end{aligned}$$

In the sequel we will only consider points $\tau \in K(\tilde{f})$, thus minimal surfaces $X(\,\cdot\,, \tau) \in \tilde{\mathcal{M}}(\Gamma)$, and will denote $A^\tau := -\triangle + 2\,(KE)^\tau$ and J^τ for the assigned Schwarz operators and quadratic forms. From [5], (3.3), resp. (34) in [7] we quote that there is some constant const.(τ), depending on τ and Γ only, such that

$$|(KE)^\tau(w)| \le \text{const.}(\tau) \sum_{k=1}^{N+3} |w - e^{i\tau_k}|^{-2+\alpha} \quad \forall w \in B, \tag{5}$$

for any $\tau \in K(\tilde{f})$ and some fixed $\alpha > 0$ that depends only on Γ. Moreover we are going to use the properties of the Green function (see [6, Proposition 6.1])

$$\tilde{G}(w,y) := \frac{1}{2\pi} \log\left(\frac{|1-\bar{w}y|}{|w-y|}\right), \tag{6}$$

which we consider on $(\bar{B} \times \bar{B}) \setminus \Lambda$ with $\Lambda := \{(w,y) \in \bar{B} \times \bar{B} | w = y\}$. In Proposition 6.2 in [6] the author proved that $\tilde{G}(\,\cdot\,, y)$ coincides with the weak $H^{1,s}(B)$-limit (for $s \in (1,2)$) and $L^p(B)$-limit [for $p \in (1,\infty)$] $G(\,\cdot\,,y)$ of some sequence $G^{\rho_j}(\,\cdot\,,y)$ of so-called mollified Green functions, for any $y \in B$ (see [6, (5.9), (5.10)]). Moreover we are going to use the assigned potential

$$\mathcal{G}(\varphi)(w) := \int_B \tilde{G}(w,y)\,\varphi(y)\,\mathrm{d}y \quad \text{for } w \in \bar{B},$$

which is well defined for any $\varphi \in L^r(B)$, with $r > 1$, on account of $\tilde{G}(w,\,\cdot\,) \in L^p(B)$, $\forall p \in [1,\infty)$, $\forall w \in B$, and $\tilde{G}(w,\,\cdot\,) \equiv 0$ on B, $\forall w \in \partial B$, by Proposition 6.1 in [6]. Its most important features are Green's identity for any $\varphi \in H^{2,2}(B) \cap C_0^2(B)$ and $w \in B$:

$$-\varphi(w) = \int_B G(w,y)\,\dot{\triangle}\varphi(y)\,\mathrm{d}y \equiv \mathcal{G}(\dot{\triangle}\varphi)(w), \tag{7}$$

and the estimate

$$\|\mathcal{G}(\varphi)\|_{H^{2,2}(B)} \leq \text{const.}\,\|\varphi\|_{L^2(B)}, \tag{8}$$

for any $\varphi \in L^2(B)$. Now on account of the equality $\tilde{G}(\,\cdot\,,y) \equiv G(\,\cdot\,,y)$ one can combine properties of $\tilde{G}$ with the $L^p(B)$-estimate (5.11) in [6] for $G(\,\cdot\,,y)$ in order to prove the important assertion (3.11) in [5] (see [6, Proposition 7.1] for the proof), which states that for any $\varphi \in H^{2,2}(B) \cap C_0^2(B)$ and any $\tau \in K(\tilde{f})$ there holds the estimate

$$|(KE)^\tau \varphi(w)| \leq c(\tau,\alpha) \sum_{k=1}^{N+3} |w - e^{i\tau_k}|^{-1+\frac{\alpha}{2}}\,\|\triangle\varphi\|_{L^2(B)} \quad \forall w \in B. \tag{9}$$

By a well-known method (see e.g. [1, p. 108, Satz 2.23]) one proves that $H^{2,2}(B) \cap C_0^2(B)$ is densely contained in $H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ w. r. to the $H^{2,2}(B)$-norm. Hence, since the embedding $H^{2,2}(B) \hookrightarrow C^0(\bar{B})$ is continuous this implies

Proposition 2 *The estimate* (9) *holds for any* $\varphi \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ *and for any* $\tau \in K(\tilde{f})$.

Now a straight forward reasoning leads to (see [6, Proposition 7.3])

Proposition 3 *For any* $\varphi \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ *and any* $\tau \in K(\tilde{f})$ *there holds*:

$$\|2(KE)^\tau\,\varphi\|_{L^2(B)} \leq \frac{1}{2}\|\triangle\varphi\|_{L^2(B)} + c\,\|\varphi\|_{L^2(B)}, \tag{10}$$

for some constant $c = c(\tau)$ *that only depends on* τ.

We will abbreviate $A^\tau := A^{X(\,\cdot\,,\tau)}$ and $\dot{A}^\tau := \dot{A}^{X(\,\cdot\,,\tau)}$ in the sequel. From Proposition 3 we can derive firstly that $\mathrm{Dom}(\dot{A}^\tau) = H^{2,2}(B) \cap C_0^2(B)$ is contained in $\mathrm{Dom}(A^\tau)$, thus $\dot{A}^\tau \subset A^\tau$, and especially that A^τ is densely defined in $L^2(B)$, $\forall \tau \in K(\tilde{f})$. Moreover we have

Proposition 4 *A^τ is symmetric w. r. to $\langle \cdot, \cdot \rangle_{L^2(B)}$, i.e., $A^\tau \subset (A^\tau)^* \ \forall \tau \in K(\tilde{f})$.*

Proof We fix some $\tau \in K(\tilde{f})$. For any $\varphi \in \mathrm{Dom}(A^\tau)$ and $\psi \in C_c^\infty(B)$ we have $\nabla\varphi\,\psi \in C_c^1(B)$. Hence, by the divergence theorem we obtain

$$\langle A^\tau(\varphi), \psi \rangle_{L^2(B)} = \int_B \nabla\varphi \cdot \nabla\psi + 2\,(KE)^\tau \varphi\psi \,\mathrm{d}w =: \mathcal{L}^\tau(\varphi, \psi). \tag{11}$$

Now let $\psi \in \mathring{H}^{1,2}(B)$ be arbitrarily chosen and $\{\psi_j\} \subset C_c^\infty(B)$ with $\psi_j \longrightarrow \psi$ in $\mathring{H}^{1,2}(B)$. By Hölder's inequality and Sobolev's embedding theorem we achieve due to $1 - \frac{2}{2} = 0 > 0 - \frac{2}{q}, \forall q \in [1, \infty)$:

$$\begin{aligned}\|(KE)^\tau\,\varphi\,(\psi_j - \psi)\|_{L^1(B)} &\le \|(KE)^\tau\|_{L^{p^*}(B)}\|\varphi\|_{L^r(B)}\|\psi_j - \psi\|_{L^q(B)} \\ &\le \|(KE)^\tau\|_{L^{p^*}(B)}\|\varphi\|_{L^r(B)}\mathrm{const.}(q)\,\|\psi_j - \psi\|_{H^{1,2}(B)} \longrightarrow 0,\end{aligned}$$

for $j \to \infty$, with $\frac{1}{p^*} + \frac{1}{r} + \frac{1}{q} = 1$ and $p^* \in \left(1, \frac{2}{2-\alpha}\right)$. Hence, recalling that $A^\tau(\varphi) \in L^2(B)$ we gain (11) in the limit also for $\psi \in \mathring{H}^{1,2}(B)$, thus especially for any $\psi \in \mathrm{Dom}(A^\tau)$. Together with the symmetry of $\mathcal{L}^\tau(\cdot, \cdot)$ this yields for an arbitrary $\varphi \in \mathrm{Dom}(A^\tau)$:

$$\langle A^\tau(\varphi), \psi \rangle_{L^2(B)} = \mathcal{L}^\tau(\varphi, \psi) = \mathcal{L}^\tau(\psi, \varphi) = \langle \varphi, A^\tau(\psi) \rangle_{L^2(B)} \tag{12}$$

$\forall \psi \in \mathrm{Dom}(A^\tau)$, which shows indeed $\mathrm{Dom}(A^\tau) \subset \mathrm{Dom}((A^\tau)^*)$ and $(A^\tau)^*(\varphi) = A^\tau(\varphi)$, just as asserted. □

From this and the symmetry of $\dot{A}^\tau$ and $\dot{\triangle}$ on $H^{2,2}(B) \cap C_0^2(B)$ one can easily derive that $A^\tau, \dot{A}^\tau$ and $\dot{\triangle}$ are closable in $L^2(B)$, $\forall \tau \in K(\tilde{f})$. Now we can prove

Proposition 5 *There holds* $\mathrm{Dom}(\bar{\dot{\triangle}}) = \mathrm{Dom}(\overline{\dot{A}^\tau}) = H^{2,2}(B) \cap \mathring{H}^{1,2}(B)\ \forall \tau \in K(\tilde{f})$.

Proof We fix some $\tau \in K(\tilde{f})$ arbitrarily and choose some $\varphi \in \mathrm{Dom}(\bar{\dot{\triangle}})$. Thus there is a sequence $\{\varphi_m\} \subset H^{2,2}(B) \cap C_0^2(B) = \mathrm{Dom}(\dot{\triangle})$ such that

$$\varphi_m \longrightarrow \varphi \ \text{ and } \ \dot{\triangle}\varphi_m \longrightarrow \bar{\dot{\triangle}}(\varphi) \quad \text{in } L^2(B). \tag{13}$$

By (10) we see that

$$\|2(KE)^\tau(\varphi_n - \varphi_m)\|_{L^2(B)} \le \frac{1}{2}\|\dot{\triangle}\varphi_n - \dot{\triangle}\varphi_m\|_{L^2(B)} + c\,\|\varphi_n - \varphi_m\|_{L^2(B)}, \tag{14}$$

thus that $\{2(KE)^\tau\varphi_m\}$ is a Cauchy sequence in $L^2(B)$. Now from (13) we can deduce the pointwise convergence

$$(KE)^\tau\varphi_{m_k}(w) \longrightarrow (KE)^\tau\varphi(w) \quad \text{for a.e. } w \in B, \tag{15}$$

for some suitable sequence $\{m_k\}$, which shows that $(KE)^\tau\,\varphi_m \longrightarrow (KE)^\tau\,\varphi$ in $L^2(B)$ and therefore again with (13):

$$\dot{A}^\tau(\varphi_m) = -\dot{\triangle}\varphi_m + 2(KE)^\tau\varphi_m \longrightarrow -\bar{\dot{\triangle}}(\varphi) + 2(KE)^\tau\varphi = \overline{\dot{A}^\tau}(\varphi)$$

in $L^2(B)$, which proves that $\varphi \in \mathrm{Dom}(\overline{\dot{A}^\tau})$.

Now let some $\varphi \in \mathrm{Dom}(\overline{\dot{A}^\tau})$ be given arbitrarily, which means that there exists a sequence $\{\varphi_m\} \subset H^{2,2}(B) \cap C_0^2(B)$ satisfying

$$\varphi_m \longrightarrow \varphi \ \text{ and } \ \dot{A}^\tau(\varphi_m) \longrightarrow \overline{\dot{A}^\tau}(\varphi) \quad \text{in } L^2(B). \tag{16}$$

For some arbitrary $\psi \in H^{2,2}(B) \cap C_0^2(B)$ we have by (10):

$$\|\dot{A}^\tau(\psi)\|_{L^2(B)} \geq \|\dot{\Delta}\psi\|_{L^2(B)} - \|2(KE)^\tau\psi\|_{L^2(B)} \geq \frac{1}{2}\|\dot{\Delta}\psi\|_{L^2(B)} - c\|\psi\|_{L^2(B)},$$

and therefore $\|\dot{\Delta}\psi\|_{L^2(B)} \leq 2\,\|\dot{A}^\tau(\psi)\|_{L^2(B)} + 2c\|\psi\|_{L^2(B)}$. Combining this with (16) we conclude that $\{\dot{\Delta}\varphi_m\}$ is a Cauchy sequence in $L^2(B)$, and therefore also $\{2\,(KE)^\tau\varphi_m\} = \{\dot{\Delta}\varphi_m + \dot{A}^\tau(\varphi_m)\}$ due to the second convergence in (16). Now due to the first convergence in (16) we conclude again (15) and thus $(KE)^\tau\,\varphi_m \longrightarrow (KE)^\tau\,\varphi$ in $L^2(B)$ and therefore again with the second convergence in (16):

$$\dot{\Delta}\varphi_m = -\dot{A}^\tau(\varphi_m) + 2(KE)^\tau\varphi_m \longrightarrow -\overline{\dot{A}^\tau}(\varphi) + 2(KE)^\tau\varphi = \bar{\dot{\Delta}}\varphi$$

in $L^2(B)$, i.e., that $\varphi \in \mathrm{Dom}(\bar{\dot{\Delta}})$.
Finally we have to prove that $\mathrm{Dom}(\bar{\dot{\Delta}}) = H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$. Firstly let $\varphi \in \mathrm{Dom}(\bar{\dot{\Delta}})$ be chosen arbitrarily, thus there exists a sequence $\{\varphi_m\} \subset H^{2,2}(B) \cap C_0^2(B) = \mathrm{Dom}(\dot{\Delta})$ satisfying (13). By (7), inequality (8) and (13) we achieve:

$$\|\varphi_m\|_{H^{2,2}(B)} = \|\mathcal{G}(\dot{\Delta}\varphi_m)\|_{H^{2,2}(B)} \leq \text{const.}\|\dot{\Delta}\varphi_m\|_{L^2(B)} \leq \text{const.} \tag{17}$$

$\forall m \in \mathbb{N}$. Hence, together with the compactness of the embedding $H^{2,2}(B) \hookrightarrow L^2(B)$ and (13) we achieve the existence of a subsequence $\{\varphi_{m_k}\}$ such that $\varphi_{m_k} \rightharpoonup \varphi$ weakly in $H^{2,2}(B)$. This shows indeed $\varphi \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ due to $\mathring{H}^{1,2}(B) \supset \mathrm{Dom}(\dot{\Delta})$. Finally the inclusion $H^{2,2}(B) \cap \mathring{H}^{1,2}(B) \subset \mathrm{Dom}(\bar{\dot{\Delta}})$ follows immediately from the fact that $H^{2,2}(B) \cap C_0^2(B)$ is densely contained in $H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ w. r. to the $H^{2,2}(B)$-norm. $\square$

Now we are going to prove the essential self-adjointness of A^τ. By means of the continuity of $\mathcal{G} : L^2(B) \longrightarrow H^{2,2}(B)$ and (7) one can prove as in [10], p. 59, that $\dot{\Delta}$ is essentially self-adjoint w. r. to $\langle\cdot,\cdot\rangle_{L^2(B)}$, i.e., $\bar{\dot{\Delta}} = (\dot{\Delta})^*$. Together with estimate (10), for $\tau \in K(\tilde{f})$, and the obvious symmetry of $(KE)^\tau$ we infer from Theorem 4.4 in [9, p. 288]:

Proposition 6 $\dot{A}^\tau = -\dot{\Delta} + 2\,(KE)^\tau$ *is essentially self-adjoint w. r. to* $\langle\cdot,\cdot\rangle_{L^2(B)}$, *i.e.,* $\overline{\dot{A}^\tau} = (\dot{A}^\tau)^*$, $\forall\tau \in K(\tilde{f})$.

Now combining Proposition 4 with the fact that $\mathrm{Dom}(A^\tau)$ is densely contained in $L^2(B)$ w. r. to $\|\cdot\|_{L^2(B)}$ we can derive by twice applying Theorem 5.29 in [9, p. 168]:

Proposition 7 $(A^\tau)^*$ *is densely defined in* $L^2(B)$ *and closed,* $(A^\tau)^{**} = \bar{A}^\tau$ *and* $(A^\tau)^* = \overline{(A^\tau)^*} = ((A^\tau)^*)^{**}$ $\forall\tau \in K(\tilde{f})$.

Summarizing we obtain

Proposition 8 $(\dot{A}^\tau)^* = \overline{\dot{A}^\tau} = \bar{A}^\tau = (A^\tau)^*$ *are self-adjoint operators with domain* $H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$, $\forall\tau \in K(\tilde{f})$.

Proof We fix some $\tau \in K(\tilde{f})$. Firstly there holds by Proposition 4: $\dot{A}^\tau \subset A^\tau \subset (A^\tau)^*$. Combining this with Propositions 6 and 7 we achieve:

$$\overline{(\dot{A}^\tau)^*} = \overline{\dot{A}^\tau} \subset \bar{A}^\tau \subset \overline{(A^\tau)^*} = ((A^\tau)^*)^{**} = ((A^\tau)^{**})^* = (\bar{A}^\tau)^* \subset (\overline{\dot{A}^\tau})^*.$$

Hence, also noting that $\overline{(A^\tau)^*} = (A^\tau)^*$ by Proposition 7, we can conclude that $\overline{\dot{A}^\tau} = \bar{A}^\tau = (A^\tau)^*$ are self-adjoint operators with domain $H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ by Proposition 6. Furthermore applying Theorem 5.29 in [9, p. 168], to the densely defined and closable operator $\dot{A}^\tau$ we obtain that $\left(\dot{A}^\tau\right)^*$ is densely defined in $L^2(B)$, closed, i.e., $\left(\dot{A}^\tau\right)^* = \overline{\left(\dot{A}^\tau\right)^*}$, and $\left(\dot{A}^\tau\right)^{**} = \overline{\dot{A}^\tau}$. Now applying it to the densely defined and closed operator $\left(\dot{A}^\tau\right)^*$ again we gain that $\left(\left(\dot{A}^\tau\right)^*\right)^{**} = \overline{\left(\dot{A}^\tau\right)^*}$. Hence, we achieve together with Proposition 6 that

$$\overline{\dot{A}^\tau} = \left(\overline{\dot{A}^\tau}\right)^* = \left(\left(\dot{A}^\tau\right)^{**}\right)^* = \left(\left(\dot{A}^\tau\right)^*\right)^{**} = \overline{\left(\dot{A}^\tau\right)^*} = \left(\dot{A}^\tau\right)^*.$$

□

Now we are going to prove Theorem 1. As in (11) we will use the bilinear form

$$\mathcal{L}^\tau(\varphi, \psi) := \int_B \nabla\varphi \cdot \nabla\psi + 2\,(KE)^\tau \varphi\psi \,\mathrm{d}w,$$

for $\varphi, \psi \in \mathring{H}^{1,2}(B)$ assigned to some $\tau \in K(\tilde{f})$, thus especially $J^\tau(\varphi) \equiv \mathcal{L}^\tau(\varphi, \varphi)$. In the sequel we fix some $\tau \in K(\tilde{f})$, thus some minimal surface $X(\,\cdot\,, \tau) \in \tilde{\mathcal{M}}(\Gamma)$, and $p^* \in \left(1, \frac{2}{2-\alpha}\right)$ arbitrarily and abbreviate $A := A^\tau$, $\mathcal{L} := \mathcal{L}^\tau$ and $J := J^\tau$. The final tool of the proof of Theorem 1 is

Proposition 9 *There exists some constant $C(p^*)$ such that:*

$$J(\varphi) \geq \frac{1}{2} \int_B |\nabla\varphi|^2 \,\mathrm{d}w - C(p^*)\|KE\|_{L^{p^*}(B)} \quad \forall \varphi \in S\mathring{H}^{1,2}(B). \tag{18}$$

Proof We consider the continuous embeddings $\mathring{H}^{1,2}(B) \hookrightarrow L^q(B) \hookrightarrow L^2(B)$, for any $q \geq 2$, where the first one is compact due to Sobolev's embedding theorem. Hence, we may apply Ehrling's interpolation lemma, yielding

$$\|\varphi\|_{L^q(B)} \leq \epsilon \|\varphi\|_{\mathring{H}^{1,2}(B)} + C(q, \epsilon) \quad \forall \varphi \in S\mathring{H}^{1,2}(B),$$

for any $\epsilon > 0$ and any $q \geq 2$, where we used the requirement $\|\varphi\|_{L^2(B)} = 1$. Hence, together with Hölder's, Cauchy–Schwarz' and Poincaré's inequalities we achieve for any $\epsilon > 0$:

$$\begin{aligned} \|KE\,\varphi^2\|_{L^1(B)} &\leq \|KE\|_{L^{p^*}(B)} \|\varphi\|^2_{L^{2p'}(B)} \leq \|KE\|_{L^{p^*}(B)} (\epsilon \|\varphi\|_{\mathring{H}^{1,2}(B)} + C(p', \epsilon))^2 \\ &\leq \|KE\|_{L^{p^*}(B)} 2\,(\epsilon^2 (C_P + 1) \int_B |\nabla\varphi|^2 \,\mathrm{d}w + C(p', \epsilon)^2), \end{aligned}$$

with $\frac{1}{p^*} + \frac{1}{p'} = 1$, and therefore by the definition of J:

$$J(\varphi) \geq (1 - 4\,\|KE\|_{L^{p^*}(B)}\,(C_P + 1)\,\epsilon^2) \int_B |\nabla\varphi|^2 \,\mathrm{d}w - 4\,\|KE\|_{L^{p^*}(B)}\, C(p', \epsilon)^2,$$

for any $\varphi \in S\mathring{H}^{1,2}(B)$, which yields our assertion by a suitable choice of ϵ. □

In order to prove Theorem 1 we shall apply Courant's technique for obtaining eigenvalues and eigenfunctions of A by minimizing the quadratic form J on $S\mathring{H}^{1,2}(B)$ with respect to subsidiary conditions. We shall only sketch the necessary steps.

Proof of Theorem 1 Firstly the above proposition guarantees the existence of $\inf_{S\mathring{H}^{1,2}(B)} J$. Hence, we may consider some sequence $\{\varphi_j\} \subset S\mathring{H}^{1,2}(B)$ such that $J(\varphi_j) \searrow \inf_{S\mathring{H}^{1,2}(B)} J$, and again using (18) we conclude together with Poincaré's inequality that $\|\varphi_j\|_{H^{1,2}(B)} \leq$ const. Thus we can extract some subsequence $\{\varphi_{j_k}\}$ such that

$$\varphi_{j_k} \rightharpoonup \varphi^* \quad \text{weakly in } H^{1,2}(B),$$

for some $\varphi^* \in \mathring{H}^{1,2}(B)$. Since this implies $\varphi_{j_k} \longrightarrow \varphi^*$ in $L^q(B)$, for any $q \geq 1$, we infer $\varphi^* \in S\mathring{H}^{1,2}(B)$. Furthermore this implies:

$$\|KE\,(\varphi_{j_k}^2 - (\varphi^*)^2)\|_{L^1(B)} \leq \|KE\|_{L^{p^*}(B)} \|\varphi_{j_k}^2 - (\varphi^*)^2\|_{L^{p'}(B)} \longrightarrow 0,$$

with $\frac{1}{p^*} + \frac{1}{p'} = 1$. Hence, J inherits the weak lower semicontinuity of the Dirichlet integral:

$$\begin{aligned} J(\varphi^*) &= \int_B |\nabla\varphi^*|^2 + 2\,(KE)^\tau (\varphi^*)^2 \,\mathrm{d}w \qquad (19)\\ &\leq \liminf_{k\to\infty} \int_B |\nabla\varphi_{j_k}|^2 \,\mathrm{d}w + 2 \lim_{k\to\infty} \int_B KE\,\varphi_{j_k}^2 \,\mathrm{d}w = \liminf_{k\to\infty} J(\varphi_{j_k}) = \inf_{S\mathring{H}^{1,2}(B)} J, \end{aligned}$$

thus $J(\varphi^*) = \inf_{S\mathring{H}^{1,2}(B)} J$. Now we construct recursively a filtration of subspaces $\mathring{H}^{1,2}(B) =: U_1 \supset U_2 \supset U_3 \cdots$ of $\mathring{H}^{1,2}(B)$ by

$$U_i := \big\{\eta \in \mathring{H}^{1,2}(B) | \langle \eta, \varphi_j^* \rangle_{L^2(B)} = 0, \;\; j = 1, \ldots, i-1\big\}, \tag{20}$$

for $i \geq 2$, and $SU_i := U_i \cap S\mathring{H}^{1,2}(B)$, where we set $\varphi_1^* := \varphi^*$ and the $\varphi_i^* \in SU_i$ have to minimize J:

$$J(\varphi_i^*) \overset{!}{=} \inf_{SU_i} J =: \lambda_i. \tag{21}$$

We obtain those minimizers φ_i^*, $i \geq 2$, exactly by the same procedure which yielded φ^* above since the U_i's are closed w. r. to weak $H^{1,2}(B)$-convergence and non-trivial, otherwise there would hold $\mathrm{Span}(\varphi_1^*, \ldots, \varphi_{i-1}^*)^\perp = \{0\}$ [$\perp$ w. r. to $\langle\cdot,\cdot\rangle_{L^2(B)}$ in $\mathring{H}^{1,2}(B)$] which contradicts $\dim \mathring{H}^{1,2}(B) = \infty$ due to the projection theorem. By construction of our filtration the sequence $\{\lambda_i\}$ is increasing. Furthermore $\{\infty\}$ is its only point of accumulation since if there was a bounded subsequence $\{\lambda_{i_k}\}$ then we would conclude by (21), (18) and Poincaré's inequality that $\|\varphi_{i_k}^*\|_{H^{1,2}(B)} \leq$const. $\forall k \in \mathbb{N}$. Hence, since the embedding $H^{1,2}(B) \hookrightarrow L^2(B)$ is compact, $\{\varphi_{i_k}^*\}$ would possess a Cauchy-subsequence w. r. to $\|\cdot\|_{L^2(B)}$, which contradicts the fact that

$$\langle \varphi_i^* - \varphi_j^*, \varphi_i^* - \varphi_j^* \rangle_{L^2(B)} = \|\varphi_i^*\|_{L^2(B)}^2 - 2\,\langle \varphi_i^*, \varphi_j^* \rangle_{L^2(B)} + \|\varphi_j^*\|_{L^2(B)}^2 = 2 - 2\delta_{ij}$$

$\forall i, j \in \mathbb{N}$, by (20) and $\varphi_i^* \in SU_i$. Now we are going to prove that the φ_i^* and λ_i are indeed eigenfunctions and eigenvalues of A and $\bar{A}$. For some fixed i we consider an arbitrary $\psi \in U_i$ and the function

$$f_i(\epsilon) := J(\varphi_i^* + \epsilon\psi) - \lambda_i \|\varphi_i^* + \epsilon\psi\|_{L^2(B)}^2 \quad \text{on } [-\epsilon_0, \epsilon_0],$$

for $\epsilon_0 > 0$ that small such that $\|\varphi_i^* + \epsilon\psi\|_{L^2(B)} > 0\ \forall\epsilon \in [-\epsilon_0, \epsilon_0]$. Then we obtain for any $\psi \in U_i$ and any $i \in \mathbb{N}$, abbreviating $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{L^2(B)}$:

$$0 = \frac{\mathrm{d}}{\mathrm{d}\epsilon} f_i(\epsilon)|_{\epsilon=0} = 2\,(\mathcal{L}(\varphi_i^*, \psi) - \lambda_i\,\langle\varphi_i^*, \psi\rangle).$$

Next a standard reasoning yields $\mathcal{L}(\varphi_i^*, \psi) = \lambda_i\,\langle\varphi_i^*, \psi\rangle$ even for any $\psi \in \mathring{H}^{1,2}(B)$, i.e.,

$$A(\varphi_i^*) = \lambda_i\,\varphi_i^* \quad \text{weakly on } B \tag{22}$$

$\forall i \in \mathbb{N}$. Now we know that our coefficients $2(KE)^\tau - \lambda_i$ are of class $C^\infty(B)$ for any $\tau \in K(\tilde{f})$ (see [7, (35)]). Thus the L^2-regularity theory, Theorem 8.13 in [2], yields that $\varphi_i^* \in C^\infty(B)\ \forall i \in \mathbb{N}$. Hence, if we test (22) with an arbitrary $\psi \in C_c^\infty(B)$ and apply the divergence theorem to $\nabla\varphi_i^*\,\psi \in C_c^\infty(B)$, then we obtain:

$$\langle A(\varphi_i^*), \psi\rangle = \mathcal{L}(\varphi_i^*, \psi) = \lambda_i\,\langle\varphi_i^*, \psi\rangle.$$

Thus the fundamental lemma of the calculus of variations yields the Eq. 22 even in the classical sense on B. In particular we see that $\varphi_i^* \in \mathrm{Dom}(A)$, thus indeed the φ_i^*'s and the λ_i's are eigenfunctions and eigenvalues of A and therefore also of $\bar{A}\ \forall i \in \mathbb{N}$. Next a standard reasoning yields $\|\psi\|^2_{L^2(B)} = \sum_{j=1}^\infty \langle\varphi_j^*, \psi\rangle^2$ for any $\psi \in \mathring{H}^{1,2}(B)$. Now we suppose that $\lambda \notin \{\lambda_i\}$ is a further eigenvalue of $\bar{A}$ and $\phi \in ES_\lambda(\bar{A})$ a corresponding eigenfunction. Since $\phi \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B) = \mathrm{Dom}(\bar{A})$ by Theorem 8 we have $\nabla\phi\,\psi \in \mathring{H}^{1,1}(B)$ for any $\psi \in C_c^\infty(B)$. Hence, applying the divergence theorem to $\nabla\phi\,\psi$ we obtain

$$\mathcal{L}(\phi, \psi) = \langle\bar{A}(\phi), \psi\rangle = \lambda\,\langle\phi, \psi\rangle, \tag{23}$$

and we achieve this equality also for any $\psi \in \mathring{H}^{1,2}(B)$ exactly as in the proof of Proposition 4 by approximation. Now testing this weak equation with $\psi := \varphi_i^*$ for an arbitrary $i \in \mathbb{N}$ we conclude together with (22):

$$\lambda\,\langle\phi, \varphi_i^*\rangle = \mathcal{L}(\phi, \varphi_i^*) = \mathcal{L}(\varphi_i^*, \phi) = \lambda_i\,\langle\varphi_i^*, \phi\rangle,$$

hence, $0 = (\lambda - \lambda_i)\ \langle\varphi_i^*, \phi\rangle, \forall i \in \mathbb{N}$, which would imply that all the coordinates $\langle\varphi_i^*, \phi\rangle$ of ϕ would vanish and therefore $0 = \sum_{j=1}^\infty \langle\varphi_j^*, \phi\rangle^2 = \|\phi\|^2_{L^2(B)}$. But ϕ is an eigenfunction. Hence, we have proved so far $\{\lambda_i\} = Spec(\bar{A}) \supset Spec(A) \supset \{\lambda_i\}$ and therefore also $\{\lambda_i\} = Spec(A)$. Finally we infer from $\mathrm{Dom}(A) \subset \mathrm{Dom}(\bar{A}) = H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$, $\varphi^* \in S\mathrm{Dom}(A)$ and (19):

$$\inf_{S\mathring{H}^{1,2}(B)} J \le \inf_{S(H^{2,2}(B)\cap\mathring{H}^{1,2}(B))} J \le \inf_{S\mathrm{Dom}(A)} J \le J(\varphi^*) = \inf_{S\mathring{H}^{1,2}(B)} J,$$

which together with $\inf_{S\mathring{H}^{1,2}(B)} J = \lambda_1 = \lambda_{\min}(A) = \lambda_{\min}(\bar{A})$ completes also the proof of (3). The second part of the theorem now follows along usual lines by employing Harnack's inequality. Let $\varphi^* \in ES_{\lambda_{\min}}(\bar{A}) \subset H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ with $\|\varphi^*\|_{L^2(B)} = 1$ be given arbitrarily. We assume the existence of some point $w_0 \in B$ with $\varphi^*(w_0) = 0$. Firstly we note that $|\varphi^*| \in \mathring{H}^{1,2}(B)$ and that $\int_B |\nabla|\varphi^*||^2\,\mathrm{d}w = \int_B |\nabla\varphi^*|^2\,\mathrm{d}w$. Moreover applying (23) to $\phi := \varphi^*$ and $\psi := \varphi^*$ we obtain by (3):

$$J(|\varphi^*|) = J(\varphi^*) = \langle\bar{A}(\varphi^*), \varphi^*\rangle_{L^2(B)} = \lambda_{\min}\langle\varphi^*, \varphi^*\rangle_{L^2(B)} = \lambda_{\min} = \inf_{S\mathring{H}^{1,2}(B)} J.$$

Hence, exactly as we achieved (22) we conclude now due to $|\varphi^*| \in \mathring{H}^{1,2}(B)$:

$$A(|\varphi^*|) = \lambda_{\min}\, |\varphi^*| \quad \text{weakly on } B.$$

Now we may apply Harnack's inequality, Theorem 8.20 in [2], to $|\varphi^*| \geq 0$ on any disc $B_{4R}(w_0) \subset\subset B$, yielding $\sup_{B_R(w_0)} |\varphi^*| \leq \text{const.} \inf_{B_R(w_0)} |\varphi^*|$. Hence, from $\varphi^*(w_0) = 0$ we can conclude now that $\varphi^* \equiv 0$ on $B_R(w_0)$ and thus that $\varphi^* \equiv 0$ on B by a successive use of Harnack's inequality, which contradicts our assumption $\|\varphi^*\|_{L^2(B)} = 1$. Thus we have proved indeed for an arbitrary eigenfunction $\varphi^* \in ES_{\lambda_{\min}}(\bar{A})$ that $\varphi^* > 0$ or < 0 on B. Now we assume that $\dim ES_{\lambda_{\min}}(\bar{A}) > 1$. On account of the projection theorem we could choose two $\mathrm{L}^2(B)$-orthogonal eigenfunctions $\varphi^*, \bar{\varphi}^*$ in $ES_{\lambda_{\min}}(\bar{A})$, i.e., with $\langle \varphi^*, \bar{\varphi}^* \rangle_{L^2(B)} = 0$, in contradiction to $\langle \varphi^*, \bar{\varphi}^* \rangle_{L^2(B)} > 0$ or < 0. As we have $\{0\} \neq ES_{\lambda_{\min}}(A) \subset ES_{\lambda_{\min}}(\bar{A})$ we arrive at (4). □

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