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A hierarchy of avalanche models on arbitrary topography

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Abstract We use the non-Cartesian, topography-based equations of mass and momentum balance for gravity driven frictional flows of Luca et al. (Math. Mod. Meth. Appl. Sci. 19, 127–171 (2009)) to motivate a study on various approximations of avalanche models for single-phase granular materials. By introducing scaling approximations we develop a hierarchy of model equations which differ by degrees in shallowness, basal curvature, peculiarity of constitutive formulation (non-Newtonian viscous fluids, Savage–Hutter model) and velocity profile parametrization. An interesting result is that differences due to the constitutive behaviour are largely eliminated by scaling approximations. Emphasis is on avalanche flows; however, most equations presented here can be used in the dynamics of other thin films on arbitrary surfaces.

1 Introduction

This paper is concerned with the derivation of a hierarchy of modelling equations for avalanches, debris flows or landslides down natural topographic terrains. Such models were derived in the past by (i) writing the balance laws of mass and linear momentum for an incompressible medium, (ii) postulating a stress deformation relation that characterizes the constitutive behaviour of the material in motion, (iii) formulating boundary conditions along the free surface and the base, (iv) non-dimensionalizing the equations, and then (v) using order estimates for the scales of the geometric and stress parameters that reduce the complexity of the equations (for a summarizing account with many modelling equations see [11,29]). The trend in the most recent papers is

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to write the governing equations in topography following curvilinear coordinates, since the topography often directly influences the dynamics. For channel flow, this has been done by Pudasaini et al. [26–28], provided the channel is mimicking the thalweg of an otherwise well defined cross sectional profile.

Bouchut and Westdickenberg [3], on the other hand, were the first to refer the governing equations (balance laws of mass and linear momentum and constitutive relations) to arbitrary basal topography. An equivalent, though somewhat restricted formulation of avalanching motion along arbitrary topography was also presented by De Toni and Scotton [7], who averaged the mass and momentum balance equations in the vertical direction. A set of equations of motion for the avalanche thickness and the depth-integrated velocity components parallel to the base was derived, and the Savage–Hutter [31] closure condition of Mohr–Coulomb-type was implemented. A topography following formulation has recently been given by Issler [14], the most distinguished feature being the use of Criminale–Ericksen–Filbey rheology with Mohr–Coulomb behaviour as introduced for snow avalanches by Norem et al. [21] (see the report by Harbitz [11]). A further formulation, essentially equivalent to that of [3], but using a different geometric formulation and combining it with the matrix notation of [3], was recently also given by Luca et al. [16]. These authors derived in full generality the depth-integrated mass and momentum balance equations and shallow avalanche equations for inviscid fluids and for Newtonian fluids subjected to viscous type basal friction. We emphasize that in the above mentioned papers the basal topography, across which the avalanche mass is moving, is fixed, i.e. time independent, and that no entrainment and deposition are envisaged. On the contrary, Bouchut et al. [4], and Tai and Kuo [32] accounted for such processes. Other papers, using non-Cartesian topography following coordinates for the flow of a fluid mass along a curved basal surface were not found.

This paper is a continuation of that by Luca et al. [16]. We assume fixed basal topography and ignore the entrainment processes. The three dimensional dimensionless governing equations of the flow of a fluid or debris along arbitrary topography are taken over from [16]. These equations hold for unspecified rheological behaviour and have been made dimensionless without introducing any geometric stretching, without making use of a shallowness property of the avalanching mass, and with utmost flexibility of the rheological behaviour of the avalanching material. It is the purpose of this paper to specify such assumptions and, in doing so, to deduce a hierarchy of avalanche models of differing complexity and varied physical description.

In Sects. 2 and 3 we give the topographic description and discuss the curvilinear coordinates that are used in the vicinity of an arbitrary surface. Section 4 commences with the presentation of the moving boundary value problem for avalanches—balances of mass and linear momentum, kinematic and stress boundary conditions—in intrinsic form, gives these statements in curvilinear coordinates introduced earlier, and presents depth-integrated versions of the dynamical equations, all in dimensionless form. So far the paper essentially summarizes results of [16] that are necessary for the developments in this paper. Section 5 introduces geometric and rheological scalings. We use the ratio of a typical thickness to a typical length as a small aspect ratio $\epsilon \ll 1$ and express all other scales in terms of it. The velocity profile is assumed to satisfy relations of Boussinesq type. For a profile expressed as a power law flow rule, with a power parameter and a slip parameter as measures for the shearing and sliding, explicit expressions for the “Boussinesq coefficients” in the depth-integrated convective acceleration terms are obtained. Rheological scalings concern also the components of the stress tensor in the curvilinear coordinate system. These are separately scaled for the normal and shear stresses on planes tangential to the basal surface and those on planes perpendicular to them. This allows a *classification of avalanche models* according to the rheological models used, which is introduced in Sect. 6. Essential in this classification are the relative weights of the basal shear stress, the depth-average of the shear stresses on planes parallel to the basal surface, and the normal and shear stresses on surfaces normal to the base. The analysis discloses four main classes of avalanche models which are defined in propositions. Roughly, the classes are:

(i) When the rheology is weakest, implying that sliding and shearing may occur but are small, only the constraint pressure of incompressibility survives; the rheological response is that of an inviscid fluid, except that non-uniformities in the velocity profile can be accounted for by those Boussinesq coefficients which deviate from unity.
(ii) For slightly larger resistance to shear, but still occurrence of sliding and shearing, only the basal shear stress is relevant, and may in this case be parameterized by a Coulomb or a viscous stress contribution or both. Thus, the rheological properties enter via the Boussinesq coefficients only.
(iii) For even more resistive rheology, in addition to the basal shear stress, the depth-averaged shear stress on planes parallel to the basal surface becomes significant. It is at this level where the rheological properties of the moving material are explicitly entering the formulation.
(iv) The most complex formulation emerges when the depth-integrated values of the stresses acting on planes perpendicular to the basal surface are sufficiently large to have to be considered. It is here, where the

normal and shear stresses parallel to the basal surface become significant. Such situations prevail when stress anisotropies or normal stress differences are important, or when the base-parallel distortions are large. All earth pressure models are of this kind [17,18,25,31], as are rheologies exhibiting normal stress effects.

All these facts are explained in detail in the sections following Sect. 6. In Sect. 7 we discuss bottom friction as parameterized by a Coulomb and viscous sliding law. In Sect. 8 it is shown how non-Newtonian viscous fluids fit the classes (ii) and (iii) introduced above. We quote rheological models from the literature fitting these classes. Finally we present a topography-adjusted version of the Savage–Hutter model which fits class (iv). We close with a summarizing discussion in Sect. 9.

In this paper the 2×2 matrices are denoted by upright capital boldface letters, e.g. $\mathbf{A}$, and the 2-column matrices are denoted by small boldface letters, e.g. $\mathbf{a}$. A similar notation, but with slanted letters, is used for vectors and tensors, e.g. $\boldsymbol{a}$, $\boldsymbol{A}$. The dyadic product of two column matrices $\mathbf{a}$ and $\mathbf{b}$ is defined as $\mathbf{a} \otimes \mathbf{b} \equiv \mathbf{a}\mathbf{b}^T$, where the superscript T denotes the transpose of a matrix. The symbol $\otimes$ also stands for the tensor product of two vectors. The inner product of the $m \times n$ matrices $\mathbf{A}$, $\mathbf{B}$ (we deal with the cases $m = n = 2$ and $m = 2, n = 1$) is defined as $\mathbf{A} \cdot \mathbf{B} = \operatorname{tr}(\mathbf{A}\mathbf{B}^T)$, where tr denotes the trace operator. Then, the Greek indices have the values 1 and 2, while the Latin indices range from 1 to 3. Finally, summation over repeated indices is understood.

2 Topography description

We suppose that the basal topography on which the avalanche flows is modelled by a regular surface $\mathcal{S}$ in the three-dimensional Euclidean space $\mathcal{E}$, such that, with respect to an orthogonal Cartesian coordinate system $Ox_1x_2x_3$, in which Ox_3 is the vertical direction, $\mathcal{S}$ has the representation

$$x_1 = x_1(\xi^1, \xi^2), \quad x_2 = x_2(\xi^1, \xi^2), \quad x_3 = b(x_1(\xi^1, \xi^2), x_2(\xi^1, \xi^2)). \tag{1}$$

The functions x_1, x_2, b, are assumed to be twice continuously differentiable on some open subset Δ_0 of $\mathbb{R}^2$. Moreover, the matrix

$$\mathbf{F} \equiv \left(\frac{\partial x_i}{\partial \xi^\alpha} \right)_{i,\alpha \in \{1,2\}} \tag{2}$$

is supposed to have a positive determinant ($\det \mathbf{F} > 0$). We denote by $\boldsymbol{\rho}$ the position vector of a point Q on $\mathcal{S}$, viz.,

$$\boldsymbol{\rho} = x_1(\xi^1, \xi^2)\, \boldsymbol{i}_1 + x_2(\xi^1, \xi^2)\, \boldsymbol{i}_2 + b(x_1(\xi^1, \xi^2), x_2(\xi^1, \xi^2))\, \boldsymbol{i}_3,$$

where $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ is the basis of the translation vector space $\mathcal{V}$ of $\mathcal{E}$ and associated to the Cartesian coordinate system $Ox_1x_2x_3$. Then, the vectors

$$\boldsymbol{\tau}_\alpha \equiv \frac{\partial \boldsymbol{\rho}}{\partial \xi^\alpha}, \quad \alpha \in \{1, 2\},$$

define the *natural basis* of the tangent space to $\mathcal{S}$ at Q. The reciprocal basis of the natural basis is denoted by $\{\boldsymbol{\tau}^1, \boldsymbol{\tau}^2\}$. A unit normal vector to $\mathcal{S}$ at Q is

$$\boldsymbol{n} \equiv \frac{\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2}{\|\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2\|},$$

where $\times$ stands for the cross product of two vectors in $\mathcal{V}$, and $\| \ \|$ represents the Euclidean norm of a vector in $\mathcal{V}$. We suppose that $\boldsymbol{n}$ points into the material body flowing on the topography, and denote by $-s_1, -s_2, c$ its components with respect to the Cartesian basis, i.e.

$$\boldsymbol{n} = -s_1 \boldsymbol{i}_1 - s_2 \boldsymbol{i}_2 + c \boldsymbol{i}_3, \quad c > 0.$$

(See [3] or [16] to deduce that c is positive.) We shall also use the notation

$$\mathbf{s} \equiv (s_1, s_2)^T.$$

The *coefficients of the first fundamental form* of the surface $\mathcal{S}$ are $\phi_{\alpha\beta} \equiv \boldsymbol{\tau}_\alpha \cdot \boldsymbol{\tau}_\beta$, and with $\phi^{\alpha\beta} \equiv \boldsymbol{\tau}^\alpha \cdot \boldsymbol{\tau}^\beta$ we have the matrix equalities, see Luca et al. [16],

$$(\phi_{\alpha\beta}) = \mathbf{M}_0^{-1}, \quad (\phi^{\alpha\beta}) = \mathbf{M}_0, \quad \mathbf{M}_0 \equiv \mathbf{F}^{-1}(\mathbf{I} - \mathbf{s} \otimes \mathbf{s})\mathbf{F}^{-T}, \tag{3.1–3}$$

where $\mathbf{I}$ is the 2×2 unit matrix.

Now, since the derivatives of $\boldsymbol{n}$ with respect to ξ^1, ξ^2 are vectors in the tangent space to $\mathcal{S}$ at Q, we have the representations

$$\frac{\partial \boldsymbol{n}}{\partial \xi^\beta} = -b_{\alpha\beta}\boldsymbol{\tau}^\alpha = -W^\alpha{}_\beta \boldsymbol{\tau}_\alpha, \quad \beta \in \{1, 2\},$$

defining the *coefficients* $b_{\alpha\beta}$ *of the second fundamental form of* $\mathcal{S}$, which we collect as entries of the (symmetric) matrix $\mathbf{H}$, and the entries $W^\alpha{}_\beta$ of the *Weingarten matrix*[1] $\mathbf{W}$, viz.,

$$\mathbf{H} \equiv (b_{\alpha\beta}), \quad \mathbf{W} \equiv (W^\alpha{}_\beta).$$

These matrices are related by

$$\mathbf{W} = \mathbf{M}_0 \mathbf{H}. \tag{4}$$

By the formulae

$$\mathcal{H} \equiv b_{\alpha\beta}\boldsymbol{\tau}^\alpha \otimes \boldsymbol{\tau}^\beta = W^\alpha{}_\beta \boldsymbol{\tau}_\alpha \otimes \boldsymbol{\tau}^\beta, \quad \Omega \equiv \tfrac{1}{2}\,\mathrm{tr}\,\mathcal{H} = \tfrac{1}{2}\,\mathrm{tr}\,\mathbf{W},$$

we introduce the *curvature tensor* $\mathcal{H}$ and the *mean curvature* Ω of $\mathcal{S}$ at Q.

3 Topography-based curvilinear coordinates

Next we recall the curvilinear coordinates used in Bouchut and Westdickenberg [3] and Luca et al. [16] to model avalanche flows on arbitrary topography. Thus, if $\boldsymbol{r}$ is the position vector of a point $P \in \mathcal{E}$ lying above the basal surface,

$$\boldsymbol{r} = x_1 \boldsymbol{i}_1 + x_2 \boldsymbol{i}_2 + x_3 \boldsymbol{i}_3 \equiv \boldsymbol{r}(x_1, x_2, x_3),$$

then relation

$$\boldsymbol{r}(x_1, x_2, x_3) = \boldsymbol{\rho}(\xi^1, \xi^2) + \xi\, \boldsymbol{n}(\xi^1, \xi^2) \equiv \boldsymbol{r}(\xi^1, \xi^2, \xi), \quad \xi > 0, \tag{5}$$

defines the change of coordinates

$$(x_1, x_2, x_3) \longleftrightarrow (\xi^1, \xi^2, \xi)$$

in a neighborhood of $\mathcal{S}$, provided $J \neq 0$, where J is the Jacobian of the transformation (5), given by

$$J = \frac{1}{c}\det \mathbf{B}, \quad \mathbf{B} \equiv \mathbf{F}(\mathbf{I} - \xi \mathbf{W}). \tag{6.1, 2}$$

Condition $J \neq 0$ is therefore equivalent to

$$\det(\mathbf{I} - \xi \mathbf{W}) = \det(\boldsymbol{I}_{\boldsymbol{\xi}} - \xi \mathcal{H}) \neq 0, \tag{7}$$

where $\boldsymbol{I}_{\boldsymbol{\xi}}$ is the restriction of the unit tensor $\boldsymbol{I}$ to the tangent space of $\mathcal{S}$ at $\boldsymbol{\xi} \equiv (\xi^1, \xi^2)$. In modelling avalanche flows, the curvature of the topography is taken sufficiently small, so we can assume that condition (7) holds in the domain occupied by the moving mass.

Then, the vectors

$$\boldsymbol{g}_\beta \equiv \frac{\partial \boldsymbol{r}}{\partial \xi^\beta} = (\delta^\alpha{}_\beta - \xi W^\alpha{}_\beta)\boldsymbol{\tau}_\alpha, \quad \beta \in \{1, 2\}, \quad \boldsymbol{g}_3 \equiv \frac{\partial \boldsymbol{r}}{\partial \xi} = \boldsymbol{n}, \tag{8}$$

[1] The matrices $\mathbf{H}$ and $\mathbf{W}$ are denoted by $\widetilde{\mathbf{H}}$ and $\widetilde{\mathbf{W}}$ in [16].

where $\delta^{\alpha}{}_{\beta}$ is the Kronecker symbol, form a basis of $\mathcal{V}$, which is called *the natural basis at the point* P. Note that $\boldsymbol{g}_1$, $\boldsymbol{g}_2$ are parallel to $\mathcal{S}$, and $\boldsymbol{g}_3$ is normal to $\mathcal{S}$, however, generally the basis $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$ is not orthonormal. The *covariant coefficients* $g_{ij} \equiv \boldsymbol{g}_i \cdot \boldsymbol{g}_j$ *of the metric tensor* are given by the block form matrix

$$(g_{ij}) = \begin{pmatrix} \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \quad \mathbf{M} \equiv \mathbf{B}^{-1}(\mathbf{I} - \mathbf{s} \otimes \mathbf{s})\mathbf{B}^{-T}. \tag{9.1, 2}$$

We notice that, in view of (3.3), (9.2), (6.2), we have

$$\mathbf{M}_0 = \mathbf{M}|_{\xi=0}, \quad \mathbf{M} = (\mathbf{I} - \xi\mathbf{W})^{-1}\mathbf{M}_0(\mathbf{I} - \xi\mathbf{W})^{-T}. \tag{10.1, 2}$$

Now, any vector $\boldsymbol{u} \in \mathcal{V}$ can be decomposed into a *tangential component* $\boldsymbol{u}_\tau$ and a *normal component* $\boldsymbol{u}_n$, as

$$\boldsymbol{u} = \underbrace{v^1\boldsymbol{g}_1 + v^2\boldsymbol{g}_2}+\underbrace{v^3\boldsymbol{g}_3}, \quad \boldsymbol{u}_\tau \equiv v^\beta \boldsymbol{g}_\beta = (\delta^{\alpha}{}_{\beta} - \xi W^{\alpha}{}_{\beta})\, v^\beta \boldsymbol{\tau}_\alpha, \quad \boldsymbol{u}_n \equiv v^3\boldsymbol{g}_3.$$

We have the relations

$$\boldsymbol{u}_n = (\boldsymbol{u}\cdot\boldsymbol{n})\,\boldsymbol{n}, \quad \boldsymbol{u}_\tau = \boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n},$$

implying the independence of the previous decomposition of the surface coordinates ξ^1, ξ^2.

4 Model equations

The avalanche mass is treated as an incompressible single-phase continuum with uniform density ρ_0. Thus, the mass balance and momentum balance equations emerge as

$$\operatorname{div}\boldsymbol{u} = 0, \tag{11}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \operatorname{div}\left\{\boldsymbol{u} \otimes \boldsymbol{u} - \frac{1}{\rho_0}\boldsymbol{\sigma}\right\} = \boldsymbol{b}, \tag{12}$$

where div denotes the spatial divergence operator, t is the time, $\boldsymbol{u}$ is the velocity, $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\boldsymbol{b}$ is the body force per unit mass. At the basal topography the velocity $\boldsymbol{u}$ is supposed to satisfy

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \tag{13}$$

and the free surface of the flowing mass is assumed a material surface, given implicitly by

$$F(x_1, x_2, x_3, t) = 0, \tag{14}$$

with F satisfying the kinematic condition

$$\frac{\partial F}{\partial t} + \nabla F\cdot\boldsymbol{u} = 0 \quad \text{at} \quad F = 0, \tag{15}$$

where ∇ is the spatial gradient operator. Moreover, on the free surface, the *pressure* p, that enters the theory as independent variable in view of the incompressibility condition, is assumed[2]

$$p = 0 \quad \text{at} \quad F = 0. \tag{16}$$

In Luca et al. [16] Eqs. (11)–(16) have been written in the curvilinear coordinates described in Sect. 3. That is, they have been transformed into equations involving the contravariant components v^i, T^{ij}, $b^i\,(i, j = 1, 2, 3)$ of the velocity, stress tensor and body force, respectively:

$$\boldsymbol{u} = v^i\boldsymbol{g}_i, \quad \boldsymbol{\sigma} = T^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j, \quad \boldsymbol{b} = b^i\boldsymbol{g}_i.$$

In order to record these emerging equations, we introduce the *extra-stress tensor* $\boldsymbol{\sigma}_E$,

$$\boldsymbol{\sigma}_E \equiv p\boldsymbol{1} + \boldsymbol{\sigma} = T_E^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j.$$

[2] The motivation of using (16) instead of the traction-free surface boundary condition is given in Luca et al. [16].

Then, we collect the contravariant components of $\boldsymbol{u}$, $\boldsymbol{\sigma}_E$ and $\boldsymbol{b}$ as

$$\mathbf{v} \equiv \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \quad \mathrm{v} \equiv v^3, \quad \begin{pmatrix} \mathbf{P}_E & \mathbf{p}_E \\ \mathbf{p}_E^T & T_E^{33} \end{pmatrix} \equiv (T_E^{ij}), \quad \mathbf{b} \equiv \begin{pmatrix} b^1 \\ b^2 \end{pmatrix}, \quad \mathrm{b} \equiv b^3;$$

note that the 2×2 matrix $(T^{\alpha\beta})$ emerges as

$$(T^{\alpha\beta}) = -p\mathbf{M} + \mathbf{P}_E, \tag{17}$$

and define the (two-dimensional) Grad and Div operators by

$$\operatorname{Grad} f \equiv \left(\frac{\partial f}{\partial \xi^1}, \frac{\partial f}{\partial \xi^2}\right)^T, \quad \operatorname{Grad} \mathbf{w} \equiv \left(\frac{\partial w^\alpha}{\partial \xi^\beta}\right),$$
$$\operatorname{Div} \mathbf{w} \equiv \frac{\partial w^\alpha}{\partial \xi^\alpha}, \quad \operatorname{Div} \mathbf{P} \equiv \left(\frac{\partial P^{1\beta}}{\partial \xi^\beta}, \frac{\partial P^{2\beta}}{\partial \xi^\beta}\right)^T = \frac{\partial \mathbf{P}}{\partial \xi^\alpha}\mathbf{e}_\alpha,$$

where

$$f : \Delta_0 \to \mathbb{R}, \quad \mathbf{w} \equiv (w^1, w^2)^T : \Delta_0 \to \mathcal{M}_{2,1}, \quad \mathbf{P} \equiv (P^{\alpha\beta}) : \Delta_0 \to \mathcal{M}_2$$

are smooth fields, and $\mathbf{e}_1 \equiv (1, 0)^T$, $\mathbf{e}_2 \equiv (0, 1)^T$. In the above $\mathcal{M}_{2,1}$, $\mathcal{M}_2$ stand for the set of two-column matrices and the set of 2×2 matrices, respectively. Finally, assuming that ξ^1, ξ^2 are coordinates with dimension of length, and that the free surface equation (14) emerges as $\xi = h(\boldsymbol{\xi}, t)$ under transformation (5), we make use of non-dimensional variables according to the scalings

$$\begin{aligned} (x_1, x_2, x_3, t) &= L(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{t}/\sqrt{Lg}), \quad (\xi^1, \xi^2, \xi) = L(\hat{\xi}^1, \hat{\xi}^2, \hat{\xi}), \\ (b, h) &= L(\hat{b}, \hat{h}), \quad \mathbf{v} = \sqrt{Lg}\hat{\mathbf{v}}, \quad \mathrm{v} = \sqrt{Lg}\hat{\mathrm{v}}, \quad \boldsymbol{T} = \rho_0 g L \hat{\boldsymbol{T}}, \\ \mathbf{b} &= g\hat{\mathbf{b}}, \quad \mathrm{b} = g\hat{\mathrm{b}}, \end{aligned} \tag{18}$$

where L is a typical length tangent to the topography, $\boldsymbol{T} \equiv (T^{ij})$, and g is the constant gravitational acceleration. Dropping the hat and using the notations

$$\boldsymbol{\Gamma}(-p\,\mathbf{M}, \mathbf{0}) \equiv p\left\{\mathbf{B}^{-1}\frac{\partial \mathbf{B}}{\partial \xi^\alpha}\mathbf{M}\mathbf{e}_\alpha + \operatorname{tr}\left(\mathbf{W}(\mathbf{I} - \xi\mathbf{W})^{-1}\right)\mathbf{B}^{-1}\mathbf{s}\right\}, \tag{19}$$

$$\boldsymbol{\Gamma}(\mathbf{P}_E, \mathbf{p}_E) \equiv -\mathbf{B}^{-1}\frac{\partial \mathbf{B}}{\partial \xi^\alpha}\mathbf{P}_E\mathbf{e}_\alpha + 2\mathbf{B}^{-1}\mathbf{F}\mathbf{W}\mathbf{p}_E + \Gamma(\mathbf{P}_E)\mathbf{B}^{-1}\mathbf{s}, \tag{20}$$

$$\boldsymbol{\Gamma}(\mathbf{v}, \mathrm{v}) \equiv -\mathbf{B}^{-1}\frac{\partial \mathbf{B}}{\partial \xi^\alpha}(\mathbf{v} \otimes \mathbf{v})\mathbf{e}_\alpha + 2\mathrm{v}\mathbf{B}^{-1}\mathbf{F}\mathbf{W}\mathbf{v} + \Gamma(\mathbf{v})\mathbf{B}^{-1}\mathbf{s}, \tag{21}$$

$$\Gamma(\mathbf{P}_E) \equiv -\mathbf{B}^T\mathbf{F}^{-T}\mathbf{H}\cdot\mathbf{P}_E, \quad \Gamma(\mathbf{v}) \equiv -\mathbf{B}^T\mathbf{F}^{-T}\mathbf{H}\cdot(\mathbf{v} \otimes \mathbf{v}), \tag{22.1, 2}$$

we have obtained, see Luca et al. [16],

Proposition 1 *In the curvilinear coordinates* (5) *and in non-dimensional form, the mass balance equation* (11) *is given by*

$$\operatorname{Div}\{J\mathbf{v}\} + \frac{\partial}{\partial \xi}\{J\mathrm{v}\} = 0, \tag{23}$$

while the momentum balance Eq. (12) *turns into*

$$\begin{aligned} &\frac{\partial}{\partial t}\{J\mathbf{v}\} + \operatorname{Div}\{J(\mathbf{v} \otimes \mathbf{v} + p\,\mathbf{M} - \mathbf{P}_E)\} + \frac{\partial}{\partial \xi}\{J(\mathrm{v}\mathbf{v} - \mathbf{p}_E)\} \\ &\quad + J\boldsymbol{\Gamma}(-p\,\mathbf{M}, \mathbf{0}) + J\boldsymbol{\Gamma}(\mathbf{P}_E, \mathbf{p}_E) = J\mathbf{b} + J\boldsymbol{\Gamma}(\mathbf{v}, \mathrm{v}), \end{aligned} \tag{24}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\{J\mathrm{v}\} + \operatorname{Div}\{J(\mathrm{v}\mathbf{v} - \mathbf{p}_E)\} + \frac{\partial}{\partial \xi}\left\{J\left(\mathrm{v}^2 - T_E^{33}\right)\right\} \\ &\quad + J\frac{\partial p}{\partial \xi} + J\Gamma(\mathbf{P}_E) = J\mathrm{b} + J\Gamma(\mathbf{v}). \end{aligned} \tag{25}$$

Moreover, condition (13) *of the tangency of the velocity emerges as*

$$\mathrm{v} = 0 \quad at \quad \xi = 0, \tag{26}$$

the kinematic boundary condition (15) *appears as*

$$\frac{\partial h}{\partial t} + \mathrm{Grad}\, h \cdot \mathbf{v} = \mathrm{v} \quad at \quad \xi = h(\boldsymbol{\xi}, t), \tag{27}$$

and (16) *turns into*

$$p = 0 \quad at \quad \xi = h(\boldsymbol{\xi}, t). \tag{28}$$

Equation (24) is referred to as the *tangential* momentum balance equation, or the momentum balance equation *parallel* to the topography, while (25) is the *normal* momentum balance equation. The terms defined in (19)–(22) are due to the Christoffel coefficients.

The system (23)–(28), complemented by the closure relation for $\boldsymbol{\sigma}_E$, stands for the determination of the basic fields $p, \mathbf{v}, \mathrm{v}, h$. Due to its complexity, in avalanche modelling the idea is to deduce from (23)–(28) governing equations for the depth-average of the tangential velocity and for the thickness h. To this end the mass balance equation and the momentum balance equation parallel to the basal surface are averaged with respect to the thickness of the material body, usually taken following the vertical coordinate, see e.g. De Toni and Scotton [7]. In Bouchut and Westdickenberg [3] and Luca et al. [16], the integration is performed along the normal to the bed surface. Thus, the integration of (23) and (24) over ξ from 0 to $h(\boldsymbol{\xi}, t)$, with the account of boundary conditions (26) and (27) gives, see Luca et al. [16],

Proposition 2 *The depth-integrated* (*depth-averaged*) *mass balance equation and the depth-integrated* (*depth-averaged*) *tangential momentum balance equation are given, respectively, by*

$$\frac{\partial}{\partial t}\int_0^{h(\boldsymbol{\xi},t)} J\, d\xi + \mathrm{Div}\int_0^{h(\boldsymbol{\xi},t)} J\mathbf{v}\, d\xi = 0, \tag{29}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_0^{h(\boldsymbol{\xi},t)} J\mathbf{v}\, d\xi + \mathrm{Div}\int_0^{h(\boldsymbol{\xi},t)} J\{\mathbf{v}\otimes\mathbf{v} + p\,\mathbf{M} - \mathbf{P}_E\}\, d\xi + (J\,\mathbf{p}_E)|_{\xi=0} \\
&+ J|_{\xi=h}\,\{(-p\,\mathbf{M} + \mathbf{P}_E)\mathrm{Grad}\, h - \mathbf{p}_E\}_{\xi=h} \\
&+ \int_0^{h(\boldsymbol{\xi},t)} J\{\boldsymbol{\Gamma}(-p\,\mathbf{M}, \mathbf{0}) + \boldsymbol{\Gamma}(\mathbf{P}_E, \mathbf{p}_E)\}d\xi = \int_0^{h(\boldsymbol{\xi},t)} J\,\mathbf{b}\, d\xi + \int_0^{h(\boldsymbol{\xi},t)} J\,\boldsymbol{\Gamma}(\mathbf{v}, \mathrm{v})\, d\xi.
\end{aligned} \tag{30}$$

We now sketch the way by which we further proceed to derive the modelling equations for the averaged tangential velocity and for the avalanche thickness h. We recall that the intrinsic field equations and boundary conditions (11)–(16), when written in the curvilinear coordinates, emerge as

$$(23), \quad (24), \quad (25), \quad (26), \quad (27), \quad (28), \qquad \text{(I)}$$

and notice that, once the mass balance equation (23) and the boundary condition (26) are assumed, the kinematic boundary condition (27) is equivalent to Eq. (29). Therefore, the system of model equations (I) is equivalent to the system consisting of

$$(23), \quad (24), \quad (25), \quad (26), \quad (29), \quad (28). \qquad \text{(II)}$$

Next, we do not use system (I) or its equivalent form (II) to deduce the final modelling equations, but the system

$$(23), \quad (30), \quad (25), \quad (26), \quad (29), \quad (28), \qquad \text{(III)}$$

which is a consequence of (I), see also (II), and which we treat as follows.

The local mass balance equation (23) can be integrated by using the boundary condition (26) to obtain the normal velocity v,

$$\mathrm{v} = -\frac{1}{J}\int_0^{\xi} \mathrm{Div}\,(J\mathbf{v})\,ds, \tag{31}$$

and hence v in Eqs. (30) and (25) can be replaced by this expression. Then, using ordering approximations (for the avalanche depth, tangential velocity, stress components) in terms of an aspect ratio ϵ which accounts for the shallowness of the avalanche mass, the pressure p is obtained from the normal momentum balance equation (25) and boundary condition (28). The emerging expression of p is then substituted into the depth-integrated momentum balance equation (30). The remaining Eqs. (29) and (30) are next transformed into equations for the mean tangential velocity and avalanche depth h, by using a similar technique to that which is used in turbulence modelling. That is, the system of Eqs. (29) and (30) is closed not only with the constitutive relation for the stress tensor, but also with some flow rule relations. To keep the results as much general as possible, we first specify only the flow rules. The equations for the averaged tangential velocity and avalanche depth h deduced in this way are referred to as *shallow avalanche equations*, and they constitute the modelling equations for the flowing avalanche mass.

The advantage of using (III) is that the field variables in the final modelling equations are independent of the normal variable ξ. The method described here, usually called *depth-integration* or *depth-averaging* procedure, has also been used in Luca et al. [16] for the special cases of the inviscid fluid and of a viscous fluid with small viscosity and bottom viscous friction. In the next sections we continue the analysis performed in [16], first by assuming some kinematic and dynamic ordering approximations, and then, by specifying constitutive behaviours consistent with these approximations.

Before ending this section, we write the contravariant components D^{ij} of the strain rate tensor $\boldsymbol{D}$,

$$\boldsymbol{D} \equiv \tfrac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right),$$

in the block matrix

$$(D^{ij}) = \begin{pmatrix} \mathbf{D} & \mathbf{d} \\ \mathbf{d}^T & D^{33} \end{pmatrix}, \tag{32}$$

introduce the second invariant $II_{\boldsymbol{D}}$ of $\boldsymbol{D}$ by

$$II_{\boldsymbol{D}} \equiv \tfrac{1}{2}\,\mathrm{tr}\,\boldsymbol{D}^2,$$

and recall the following result, see Luca et al. [16].

Proposition 3 *The contravariant components of the strain rate tensor with respect to the natural basis* (8) *are given by*

$$\begin{aligned}\mathbf{D} = \frac{1}{2}\Bigg\{&\left[\mathrm{Grad}\,\mathbf{v} + \mathbf{B}^{-1}\left(\frac{\partial\mathbf{B}}{\partial\xi^{\alpha}}\,\mathbf{v}\otimes\mathbf{e}_{\alpha} + \mathbf{s}\otimes\mathbf{H}\mathbf{F}^{-1}\mathbf{B}\mathbf{v} - \mathrm{v}\mathbf{F}\mathbf{W}\right)\right]\mathbf{M}\\ &+\mathbf{M}\left[\mathrm{Grad}\,\mathbf{v} + \mathbf{B}^{-1}\left(\frac{\partial\mathbf{B}}{\partial\xi^{\alpha}}\,\mathbf{v}\otimes\mathbf{e}_{\alpha} + \mathbf{s}\otimes\mathbf{H}\mathbf{F}^{-1}\mathbf{B}\mathbf{v} - \mathrm{v}\mathbf{F}\mathbf{W}\right)\right]^T\Bigg\},\end{aligned} \tag{33}$$

$$\mathbf{d} = \tfrac{1}{2}\left\{\frac{\partial\mathbf{v}}{\partial\xi} + \mathbf{M}\,\mathrm{Grad}\,\mathrm{v}\right\}, \quad D^{33} = \frac{\partial\mathrm{v}}{\partial\xi}, \tag{34}$$

and the second invariant of the strain rate tensor has the form

$$II_{\boldsymbol{D}} = \tfrac{1}{2}\left\{\mathrm{tr}\,(\mathbf{D}\mathbf{M}^{-1})^2 + 2\mathbf{M}^{-1}\mathbf{d}\cdot\mathbf{d} + (D^{33})^2\right\}. \tag{35}$$

Note that, using (18) to non-dimensionalize (32)–(35), one formally obtains the same relations (32)–(35).

Next, the body force is assumed to be gravitational, so that, see [16],

$$\mathbf{b} = -c\,\mathbf{B}^{-1}\mathbf{s}, \quad \mathrm{b} = -c. \tag{36.1, 2}$$

Here the non-dimensional components are envisaged.

5 Geometric and rheological scalings

In this section we introduce the shallowness approximations (long-wave or lubrication approximations) that we need to exploit equations and boundary conditions (III), however without explicitly mentioning the constitutive relation for $\boldsymbol{\sigma}_E$. Doing so, we try to develop a general framework which is able to include various constitutive assumptions without too much effort. In Sect. 8 we shall specify constitutive relations that are in conformity with our approximations.

The next scalings, which we first state and then motivate, use an aspect ratio $\epsilon \ll 1$ between a typical thickness normal to the topography and the typical length-scale L tangent to the topography, which has already been introduced in the non-dimensionalization procedure (18), as well as a constant $\gamma \in (0, 1)$. Thus, with the definition

$$\overline{f}\,(\boldsymbol{\xi}, t) \equiv \frac{1}{h(\boldsymbol{\xi}, t)} \int_0^{h(\boldsymbol{\xi},t)} f(\boldsymbol{\xi}, \xi, t)\, d\xi$$

for the *mean value* (along the depth) of a quantity f, we consider the following:

(a) Geometric approximation: the material layer is thin, i.e, $h = O(\epsilon)$.

(b) Flow rule approximations: the velocity $\boldsymbol{u}$ is such that $\mathbf{v} = O(1)$, and the following assumptions of Boussinesq type hold:

$$\int_0^{h(\boldsymbol{\xi},t)} \xi \mathbf{v}\, d\xi = \tfrac{1}{2} h^2 m_1 \overline{\mathbf{v}} + O(\epsilon^{2+\gamma}), \quad \int_0^{h(\boldsymbol{\xi},t)} \mathbf{v} \otimes \mathbf{v}\, d\xi = h\, m_2 \overline{\mathbf{v}} \otimes \overline{\mathbf{v}} + O(\epsilon^{2+\gamma}), \tag{37.1, 2}$$

$$\int_0^{h(\boldsymbol{\xi},t)} \xi \mathbf{v} \otimes \mathbf{v}\, d\xi = \tfrac{1}{2} h^2 m_3 \overline{\mathbf{v}} \otimes \overline{\mathbf{v}} + O(\epsilon^{2+\gamma}), \tag{37.3}$$

where m_1 to m_3 are supposed to be scalar functions of $\boldsymbol{\xi}, t$ of order $O(1)$. Moreover, we assume

$$\int_0^{h(\boldsymbol{\xi},t)} \mathrm{v} \mathbf{v}\, d\xi = \tfrac{1}{2} h^2 \beta\, \overline{\mathbf{v}} + O(\epsilon^{2+\gamma}), \tag{38}$$

where β is a scalar function of $\boldsymbol{\xi}, t$ of order $O(1)$. We refer to m_1, m_2, m_3 and β as *momentum correction factors* or *Boussinesq coefficients*.

(c) Dynamic approximations: corresponding to the motion of the avalanche mass, the stress tensor $\boldsymbol{\sigma}$ is such that

$$p = O(\epsilon), \quad \mathbf{P}_E = O(\epsilon), \quad \mathbf{p}_E = O(\epsilon^{\gamma}), \quad T_E^{33} = O(\epsilon^{1+\gamma}),$$

and on the free surface it satisfies

$$\mathbf{P}_E \mathrm{Grad}\, h - \mathbf{p}_E = O(\epsilon^{2+\gamma}) \quad \text{at} \quad \xi = h(\boldsymbol{\xi}, t). \tag{39}$$

We now explain our approximations. First, assumption (a) states that the avalanche body is spread out on the topography. It also implies $\xi = O(\epsilon)$, for $\xi \in [0, h]$.

Second, according to (b), the velocity components $\mathbf{v}$, parallel to the topography, are assumed significant. As a consequence, the normal velocity v emerges as $\mathrm{v} = O(\epsilon)$. Indeed, we have only to account for $\mathbf{v} = O(1)$ and $\xi = O(\epsilon)$ in (31), to see that $\mathrm{v} = O(\epsilon)$. We note that, with $J_0 \equiv J|_{\xi=0}$, (6) implies

$$J = J_0 \det(\mathbf{I} - \xi \mathbf{W}) = J_0(1 - 2\Omega\xi) + O(\epsilon^2), \tag{40}$$

and hence the normal velocity v emerges as

$$\mathrm{v} = -\frac{1}{J_0} \int_0^{\xi} \mathrm{Div}\,(J_0 \mathbf{v})\, ds + O(\epsilon^2) = -\frac{1}{J_0} \mathrm{Div} \left(J_0 \int_0^{\xi} \mathbf{v}\, ds \right) + O(\epsilon^2). \tag{41}$$

Then, for the approximations performed in the depth-integrated balance equations we need estimates for the integrals appearing on the left-hand side of (37) and (38). We motivate assumptions (37) and (38) as follows. The experimental data show that the tangential velocity is almost uniform along the avalanche depth, and that shearing may occur only near the bed surface. So, the "plug flow" and power law profiles seem to be good approximations of the real velocity. The flow rules (37) and (38) account for such profiles, as explained below.

(i) "Plug flow" is defined by the condition $\mathbf{v} = \mathbf{v}(\boldsymbol{\xi}, t)$, which gives $\mathbf{v} = \overline{\mathbf{v}}$. It is obvious that for this case the flow rule relations (37) are satisfied with $m_1 = m_2 = m_3 = 1$ and no negligible order terms. Next, from (41) we deduce

$$\mathrm{v} = -\frac{1}{J_0}\xi \mathrm{Div}\,(J_0\overline{\mathbf{v}}) + O(\epsilon^2),$$

and hence

$$\int_0^{h(\boldsymbol{\xi},t)} \mathrm{v}\mathbf{v}\, d\xi = -\frac{h^2}{2J_0}\mathrm{Div}\,(J_0\overline{\mathbf{v}})\,\overline{\mathbf{v}} + O(\epsilon^3), \tag{42}$$

implying that (38) is satisfied with $\beta = -\frac{1}{J_0}\mathrm{Div}\,(J_0\overline{\mathbf{v}})$.

(ii) The power law velocity profile is defined as, see Hutter et al. [13],

$$\mathbf{v} = \mathbf{v}_h - \left(1 - \frac{\xi}{h}\right)^{n+1}(\mathbf{v}_h - \mathbf{v}_0), \quad \mathbf{v}_0 = \tilde{\chi}\mathbf{v}_h, \quad n > 0, \quad \tilde{\chi} \in [0, 1], \tag{43}$$

where $\mathbf{v}_0 \equiv \mathbf{v}|_{\xi=0}$, $\mathbf{v}_h \equiv \mathbf{v}|_{\xi=h}$, n is a constant, and $\tilde{\chi}$ a function of $\boldsymbol{\xi}, t$. It is clear that for $\tilde{\chi} = 1$ the power law flow (43) degenerates into "plug flow", while $\tilde{\chi} = 0$ implies $\mathbf{v}_0 = \mathbf{0}$. In order to see that the profile (43) satisfies (37) and (38), it is advantageous to represent $\mathbf{v}$ as

$$\begin{aligned} \mathbf{v} &= \overline{\mathbf{v}} + \frac{1}{n+1}\left\{1 - (n+2)\left(1 - \frac{\xi}{h}\right)^{n+1}\right\}(\overline{\mathbf{v}} - \mathbf{v}_0), \\ \mathbf{v}_0 &= \chi\overline{\mathbf{v}}, \quad \chi \equiv \frac{(n+2)\tilde{\chi}}{n+1+\tilde{\chi}} \in [0, 1]. \end{aligned} \tag{44}$$

Then it can be checked that Eqs. (37) are satisfied with

$$\begin{aligned} m_1 &= 1 + \frac{1-\chi}{n+3}, \quad m_2 = 1 + \frac{(1-\chi)^2}{2n+3}, \\ m_3 &= 1 + \frac{2(1-\chi)}{n+3} + \frac{3(1-\chi)^2}{(n+3)(2n+3)}, \end{aligned} \tag{45}$$

and (38) holds with

$$\begin{aligned} \beta &= -\frac{2}{J_0 h^2}\int_0^{h(\boldsymbol{\xi},t)} (1 + \alpha m_4)\,\mathrm{Div}\{J_0(\xi + \alpha\, m_5 h)\overline{\mathbf{v}}\}\, d\xi, \\ \alpha &\equiv 1 - \chi, \quad m_4 \equiv \frac{1}{n+1}\left\{1 - (n+2)\left(1 - \frac{\xi}{h}\right)^{n+1}\right\}, \\ m_5 &\equiv \frac{1}{n+1}\left[\left(1 - \frac{\xi}{h}\right)^{n+2} - \left(1 - \frac{\xi}{h}\right)\right]. \end{aligned} \tag{46}$$

For the profile (44) there are no negligible terms in (37) and (38). From (45) we can see that $1 \le m_1, m_2, m_3 < 2$. Then, if $1 - \chi = O(\epsilon^\gamma)$, implying $\overline{\mathbf{v}} - \mathbf{v}_0 = O(\epsilon^\gamma)$, we have

$$m_1 = m_3 = 1 + O(\epsilon^\gamma), \quad \alpha = O(\epsilon^\gamma),$$

such that the parameters m_1, m_3 can be replaced by 1 in (37.1, 3) and (38) emerges as

$$\int_0^{h(\boldsymbol{\xi},t)} \mathrm{v}\mathbf{v}\, d\xi = -\frac{h^2}{2J_0}\,\mathrm{Div}\,(J_0\overline{\mathbf{v}})\,\overline{\mathbf{v}} + O(\epsilon^{2+\gamma}). \tag{47}$$

When sliding of the avalanche mass is significant, $1-\chi = O(\epsilon^\gamma)$ is the most likely case, since we expect $\mathbf{v}_0$ to be only slightly different from $\overline{\mathbf{v}}$. If, moreover, $1-\chi = O(\epsilon)$, that is $\overline{\mathbf{v}} - \mathbf{v}_0 = O(\epsilon)$, then

$$m_1 = m_3 = 1 + O(\epsilon), \quad m_2 = 1 + O(\epsilon^2), \quad \alpha = O(\epsilon),$$

implying that the parameters m_1, m_2, m_3 can be replaced by 1 in (37), where now the neglected terms are of order $O(\epsilon^3)$ instead of $O(\epsilon^{2+\gamma})$, and that (38) turns into (42).

It is worthwhile to mention that the Boussinesq coefficients m_1 to m_3 and β in (37) and (38) must in general be interpreted as material dependent parameters, since the velocity profile of the avalanche is influenced by the rheological properties of the flowing mass. For instance, if an ideal fluid is envisaged, then it is reasonable to assume "plug flow" profile, viz. $\chi = 1$ in (44), while for a viscous fluid with large viscosity, a power-law velocity profile with $\mathbf{v}_0 = \mathbf{0}$, viz. $\chi = 0$ in (44), is a more reasonable assumption. Obviously, the emerging coefficients m_1 to m_3 and β corresponding to these cases do not coincide. Moreover, in Luca et al. [16], for a Newtonian fluid the "sliding" coefficient χ has been expressed in terms of the dynamic viscosity, see also Sect. 8.1.2 in this paper, implying explicit dependence of m_1 to m_3 and β on viscosity.

In view of (41), assumption (38) actually refers to $\mathbf{v}$. Since in the general case (i.e. no "plug flow", no power law velocity profile) it is not clear how we can postulate β. In a first trial in the numerical computations, we suggest the particular form (47) of assumption (38).

For further use we note that with relation (37.1) the mean tangential velocity field $\overline{\boldsymbol{u}}_\tau$ emerges as

$$\overline{\boldsymbol{u}}_\tau = \left\{\delta^\alpha{}_\beta - \tfrac{1}{2} h\, m_1 W^\alpha{}_\beta\right\} \overline{v}^\beta \boldsymbol{\tau}_\alpha + O(\epsilon^{1+\gamma}), \tag{48}$$

implying explicit dependence of the tangential velocity vector on the curvature of the basal topography.

Finally, assumption (c) states that the pressure is of the order of the hydrostatic pressure, and that the normal stresses parallel to the base T_E^{11}, T_E^{22}, as well as the shear stresses T_E^{12}, $\mathbf{p}_E$ are small, while the dissipative normal stress T_E^{33} is insignificant. Moreover, noting that the so called traction-free surface boundary condition

$$\boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{0} \quad \text{at } \xi = h(\boldsymbol{\xi}, t),$$

where $\boldsymbol{n}$ is a unit normal vector to the free surface, yields in curvilinear coordinates

$$(-p\,\mathbf{M} + \mathbf{P}_E)\mathrm{Grad}\, h = \mathbf{p}_E, \quad \mathbf{p}_E \cdot \mathrm{Grad}\, h = -p + T_E^{33} \quad \text{at} \quad \xi = h(\boldsymbol{\xi}, t),$$

see Luca et al. [16], assumptions (28) and (39), and $\mathbf{p}_E = O(\epsilon^\gamma)$ and $T_E^{33} = O(\epsilon^{1+\gamma})$ show that the traction-free surface boundary condition is approximately satisfied.

In the next section we shall use the assumptions (a)–(c) in order to derive the equations governing the mean motion of the avalanche, and then to distinguish various particular forms of these equations, depending on the approximating order of the stress components.

6 Classification of avalanche models

Under the scalings introduced in the previous section we are able to deduce the shallow avalanche equations, that we state in the following proposition, the proof of which is relegated to the Appendix.

Proposition 4 *Under assumptions* (a)–(c) *the mean pressure field in a free surface incompressible fluid is given by*

$$\overline{p} = \tfrac{1}{2} h (c + a m_3) + O(\epsilon^{1+\gamma}), \quad a \equiv \mathbf{H}\overline{\mathbf{v}} \cdot \overline{\mathbf{v}}, \tag{49}$$

and the governing equations for the determination of the mean velocity components $\overline{\mathbf{v}}$ *and of the free surface height* h *are*

$$\frac{\partial}{\partial t}\{J_0 h(1-\Omega h)\}+\mathrm{Div}\{J_0 h(1-\Omega h\, m_1)\overline{\mathbf{v}}\}=O(\epsilon^{2+\gamma}), \tag{50}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\{J_0 h(1-\Omega h\, m_1)\overline{\mathbf{v}}\}+\mathrm{Div}\Big\{J_0 h\big[(m_2-\Omega h\, m_3)\overline{\mathbf{v}}\otimes\overline{\mathbf{v}}+\overline{p}\,\mathbf{M}_0-\overline{\mathbf{P}}_E\big]\Big\}\\
&\quad -J_0 h\left\{\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial \xi^\alpha}\overline{\mathbf{P}}_E\,\mathbf{e}_\alpha-2\mathbf{W}\overline{\mathbf{p}}_E+(\mathbf{H}\cdot\overline{\mathbf{P}}_E)\mathbf{F}^{-1}\mathbf{s}\right\}\\
&=-J_0\,\mathbf{p}_E|_{\xi=0}-J_0 h\Big\{(c+a\,m_2-\tfrac{1}{2}\tilde{a}h\,m_3)\mathbf{I}+\overline{p}\,\mathbf{W}\Big\}\mathbf{F}^{-1}\mathbf{s}\\
&\quad -\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial \xi^\alpha}\{J_0 h\,[(m_2-\Omega h\, m_3)\overline{\mathbf{v}}\otimes\overline{\mathbf{v}}+\overline{p}\,\mathbf{M}_0]\}\mathbf{e}_\alpha\\
&\quad +\tfrac{1}{2}\,J_0 h^2 m_3\mathbf{F}^{-1}\frac{\partial}{\partial \xi^\alpha}(\mathbf{F}\mathbf{W}\mathbf{F}^{-1})\mathbf{F}(\overline{\mathbf{v}}\otimes\overline{\mathbf{v}})\mathbf{e}_\alpha+J_0 h^2\beta\mathbf{W}\overline{\mathbf{v}}+O(\epsilon^{2+\gamma}),
\end{aligned} \tag{51}$$

where $\tilde{a}\equiv\mathbf{H}\overline{\mathbf{v}}\cdot\mathbf{W}\overline{\mathbf{v}}=\mathbf{H}\overline{\mathbf{v}}\cdot\mathbf{M}_0\mathbf{H}\overline{\mathbf{v}}\geq 0$.

In (51) it is understood that $\overline{p}$ is replaced with $h(c+a\,m_3)/2$, see (49). For further use we also note that, if $\mathbf{p}_E=O(\epsilon)$ and $T_E^{33}=O(\epsilon^2)$, then at $\xi=0$ the pressure p is given by

$$p\,|_{\xi=0}=h(c+a\,m_2)+O(\epsilon^2), \tag{52}$$

see (A.2) and the remark thereafter, as well as (37.2).

We note that, for the "plug flow" of the avalanche mass and for the power law velocity profile (44), in which $1-\chi=O(\epsilon^\gamma)$, β can be substituted in (51) with

$$-\frac{1}{J_0}\mathrm{Div}\,(J_0\overline{\mathbf{v}}),$$

and that multiplying (51) from the left with $\mathbf{F}$ we deduce

$$\begin{aligned}
&\frac{\partial}{\partial t}\{J_0 h(1-\Omega h\, m_1)\mathbf{F}\overline{\mathbf{v}}\}+\mathrm{Div}\Big\{J_0 h\mathbf{F}\big[(m_2-\Omega h\, m_3)\overline{\mathbf{v}}\otimes\overline{\mathbf{v}}+\overline{p}\,\mathbf{M}_0-\overline{\mathbf{P}}_E\big]\Big\}\\
&\quad +J_0 h\left\{2\mathbf{F}\mathbf{W}\overline{\mathbf{p}}_E-(\mathbf{H}\cdot\overline{\mathbf{P}}_E)\mathbf{s}\right\}\\
&=-J_0\mathbf{F}\,\mathbf{p}_E|_{\xi=0}-J_0 h\Big\{(c+a\,m_2-\tfrac{1}{2}\tilde{a}h\,m_3)\mathbf{I}+\overline{p}\,\mathbf{F}\mathbf{W}\mathbf{F}^{-1}\Big\}\mathbf{s}\\
&\quad +\tfrac{1}{2}\,J_0 h^2 m_3\frac{\partial}{\partial \xi^\alpha}(\mathbf{F}\mathbf{W}\mathbf{F}^{-1})\mathbf{F}(\overline{\mathbf{v}}\otimes\overline{\mathbf{v}})\mathbf{e}_\alpha+J_0 h^2\beta\mathbf{F}\mathbf{W}\overline{\mathbf{v}}+O(\epsilon^{2+\gamma}),
\end{aligned}$$

which is equivalent to (51), however, more useful for computations.

The shallow avalanche equations (49)–(51) simplify under various stronger assumptions on the degrees of scaling of the extra-stress components, and for slightly curved basal surfaces. We next sequentially refer to such situations. To this end, in the bulk matter we choose several alternative dynamic assumptions replacing (c), for instance by

(c$^{\mathrm{ii}}$) $p=O(\epsilon)$, $\mathbf{P}_E=O(\epsilon^{1+\gamma})$, $\mathbf{p}_E=O(\epsilon^\gamma)$, $T_E^{33}=O(\epsilon^{1+\gamma})$,
(c$^{\mathrm{ii}}$) $p=O(\epsilon)$, $\mathbf{P}_E=O(\epsilon^{1+\gamma})$, $\mathbf{p}_E=O(\epsilon)$, $T_E^{33}=O(\epsilon^{1+\gamma})$,
(c$^{\mathrm{iii}}$) $p=O(\epsilon)$, $\mathbf{P}_E=O(\epsilon^{1+\gamma})$, $\mathbf{p}_E=O(\epsilon^{1+\gamma})$, $T_E^{33}=O(\epsilon^{1+\gamma})$,
(c$^{\mathrm{iv}}$) $p=O(\epsilon)$, $\mathbf{P}_E=O(\epsilon^{1+\gamma})$, $\mathbf{p}_E=O(\epsilon^{2+\gamma})$, $T_E^{33}=O(\epsilon^{1+\gamma})$,
(c$^{\mathrm{v}}$) $p=O(\epsilon)$, $\mathbf{P}_E=O(\epsilon)$, $\mathbf{p}_E=O(\epsilon^{1+\gamma})$, $T_E^{33}=O(\epsilon^{1+\gamma})$.

In (c$^{\mathrm{i}}$)–(c$^{\mathrm{v}}$) it is understood that (39) holds. Now, with $\mathbf{P}_E=O(\epsilon^{1+\gamma})$, the terms in (51) containing the products $h\,\overline{\mathbf{P}}_E$ are of order $O(\epsilon^{2+\gamma})$, and hence negligible. Similarly, if $\mathbf{p}_E$ is of order $O(\epsilon^{1+\gamma})$, the term in (51) containing the mean shear stress $\overline{\mathbf{p}}_E$ is negligible. For $\mathbf{p}_E=O(\epsilon^{2+\gamma})$, the basal shear stress is also insignificant. We therefore obtain

Table 1 Classification of avalanche models in terms of the scalings of the stress components

	Stresses $\mathbf{P}_E$	Shear stresses $\mathbf{p}_E$	Normal stress T_E^{33}	Non-negligible stress components in (51)
(c)	$O(\epsilon)$	$O(\epsilon^{\gamma})$	$O(\epsilon^{1+\gamma})$	$\overline{\mathbf{P}}_E, \overline{\mathbf{p}}_E, \mathbf{p}_E\|_{\xi=0}$
(c$^{\text{i}}$)	$O(\epsilon^{1+\gamma})$	$O(\epsilon^{\gamma})$	$O(\epsilon^{1+\gamma})$	$\overline{\mathbf{p}}_E, \mathbf{p}_E\|_{\xi=0}$
(c$^{\text{ii}}$)	$O(\epsilon^{1+\gamma})$	$O(\epsilon)$	$O(\epsilon^{1+\gamma})$	$\overline{\mathbf{p}}_E, \mathbf{p}_E\|_{\xi=0}$
(c$^{\text{iii}}$)	$O(\epsilon^{1+\gamma})$	$O(\epsilon^{1+\gamma})$	$O(\epsilon^{1+\gamma})$	$\mathbf{p}_E\|_{\xi=0}$
(c$^{\text{iv}}$)	$O(\epsilon^{1+\gamma})$	$O(\epsilon^{2+\gamma})$	$O(\epsilon^{1+\gamma})$	–
(c$^{\text{v}}$)	$O(\epsilon)$	$O(\epsilon^{1+\gamma})$	$O(\epsilon^{1+\gamma})$	$\overline{\mathbf{P}}_E, \mathbf{p}_E\|_{\xi=0}$

Proposition 5 (1) *Under assumptions* (a), (b), (c$^{\text{i}}$) *or* (a), (b), (c$^{\text{ii}}$), *the basic equations for the unknown fields* $\overline{\mathbf{v}}$ *and h are* (50) *and* (51), *in which the terms containing* $\overline{\mathbf{P}}_E$ *are omitted.*

(2) *Under assumptions* (a), (b), (c$^{\text{iii}}$), *the basic equations for the unknown fields* $\overline{\mathbf{v}}$ *and h are* (50) *and* (51), *in which the terms containing* $\overline{\mathbf{P}}_E, \overline{\mathbf{p}}_E$ *are omitted.*

(3) *Under assumptions* (a), (b), (c$^{\text{iv}}$), *the avalanche mass flows down as if it were the ideal fluid, the corresponding governing equations being* (50) *and* (51), *in which all the extra stress components are neglected.*

(4) *Under assumptions* (a), (b), (c$^{\text{v}}$), *the basic equations for the unknown fields* $\overline{\mathbf{v}}$ *and h are* (50) *and* (51), *in which the term containing* $\overline{\mathbf{p}}_E$ *is omitted.*

Table 1 schematically shows the various avalanche classes derived in Propositions 4 and 5 in terms of the scalings of the stress components. We distinguish four main classes: (i) one corresponding to the case for which all the stress components in the avalanche equations are negligible; (ii) one in which only the basal shear stress $\mathbf{p}_E|_{\xi=0}$ enters the governing equations; (iii) one in which $\mathbf{p}_E|_{\xi=0}$ and $\overline{\mathbf{p}}_E$ are significant, however, $\overline{\mathbf{P}}_E$ can be neglected, and (iv) one in which $\overline{\mathbf{P}}_E$ is non-negligible.

Luca et al. [16] derived the avalanche equations for the ideal fluid and for a Newtonian fluid with small viscosity and viscous sliding law. These models fit the classes (i) and (ii), respectively. In Sect. 8 below we deal with situations that suit to classes (iii) and (iv).

The next propositions show modelling equations for avalanche flows on slightly curved topographies.

Proposition 6 *Under assumptions* (a)–(c), *if* $\mathcal{H} = O(\epsilon^{\gamma'})$, $\gamma' \in (0, 1)$, *then the mean pressure field in a free surface incompressible fluid is given by*

$$\overline{p} = \tfrac{1}{2}ch + O(\epsilon^{1+\hat{\gamma}}), \quad \hat{\gamma} \equiv \min\{\gamma, \gamma'\}, \tag{53}$$

and the governing equations for the determination of the mean velocity components $\overline{\mathbf{v}}$ *and of the free surface height h are*

$$\frac{\partial}{\partial t}\{J_0 h\} + \text{Div}\{J_0 h\,\overline{\mathbf{v}}\} = O(\epsilon^{2+\hat{\gamma}}), \tag{54}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\{J_0 h\overline{\mathbf{v}}\}+\text{Div}\big\{J_0 h\big[m_2\overline{\mathbf{v}}\otimes\overline{\mathbf{v}}+\overline{p}\,\mathbf{M}_0-\overline{\mathbf{P}}_E\big]\big\}-J_0 h\left\{\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial\xi^\alpha}\overline{\mathbf{P}}_E\,\mathbf{e}_\alpha-2\mathbf{W}\overline{\mathbf{p}}_E\right\}\\&\quad=-J_0\,\mathbf{p}_E|_{\xi=0}-J_0 h(c+a\,m_2)\mathbf{F}^{-1}\mathbf{s}-\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial\xi^\alpha}\{J_0 h\,[m_2\overline{\mathbf{v}}\otimes\overline{\mathbf{v}}+\overline{p}\,\mathbf{M}_0]\}\mathbf{e}_\alpha+O(\epsilon^{2+\hat{\gamma}}).\end{aligned} \tag{55}$$

The statements of Proposition 6 follow immediately by neglecting the terms of order $O(\epsilon^{2+\gamma'})$ in (49)–(51). Note that relations (53)–(55) do not contain β and the parameters m_1, m_3. Equation (55) simplifies even more under the following circumstances.

Proposition 7 *Under assumptions* (a), (b), (c$^{\text{ii}}$) *and* $\mathcal{H} = O(\epsilon^{\gamma'})$, *or* (a), (b), (c$^{\text{i}}$) *and* $\mathcal{H} = O(\epsilon)$, *equations* (53), (54) *hold, and* (55) *turns into*

$$\begin{aligned}&\frac{\partial}{\partial t}\{J_0 h\overline{\mathbf{v}}\}+\text{Div}\,\{J_0 h\,[m_2\overline{\mathbf{v}}\otimes\overline{\mathbf{v}}+\overline{p}\,\mathbf{M}_0]\}\\&\quad=-J_0\,\mathbf{p}_E|_{\xi=0}-J_0 h(c+a\,m_2)\mathbf{F}^{-1}\mathbf{s}-\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial\xi^\alpha}\{J_0 h\,[m_2\overline{\mathbf{v}}\otimes\overline{\mathbf{v}}+\overline{p}\mathbf{M}_0]\}\mathbf{e}_\alpha+O(\epsilon^{2+\hat{\gamma}}),\end{aligned} \tag{56}$$

or, equivalently,

$$\frac{\partial}{\partial t}\{J_0 h \mathbf{F}\overline{\mathbf{v}}\} + \operatorname{Div}\{J_0 h \mathbf{F}[m_2 \overline{\mathbf{v}} \otimes \overline{\mathbf{v}} + \overline{p}\,\mathbf{M}_0]\} = -J_0 \mathbf{F}\,\mathbf{p}_E|_{\xi=0} - J_0 h(c + a\,m_2)\mathbf{s} + O(\epsilon^{2+\hat{\gamma}}).$$

Equation (56) shows that the extra stress components of the avalanche mass enter the model only through the basal shear stress, and that the curvature of the topography is accounted for by the scalar a. Note that in Eqs. (55), (56) a term which could have been neglected is still present. Indeed,

$$\operatorname{Div}\{J_0 h \overline{p}\,\mathbf{M}_0\} = \tfrac{1}{2}\,c\operatorname{Div}(J_0 h^2 \mathbf{M}_0) + \tfrac{1}{2} J_0 h^2 \mathbf{M}_0 \operatorname{Grad} c,$$

and $\operatorname{Grad} c = O(\mathcal{H})$, see Luca et al. [16]. This term (involving $\operatorname{Grad} c$) can be eliminated by proceeding as in [16]. For instance, it can be shown that equation (56) emerges as

$$\begin{aligned} J_0 h &\left\{\frac{\partial \overline{\mathbf{v}}}{\partial t} + m_2(\operatorname{Grad}\overline{\mathbf{v}})\,\overline{\mathbf{v}} + (\overline{\mathbf{v}} \otimes \overline{\mathbf{v}})\operatorname{Grad} m_2 + c\mathbf{M}_0 \operatorname{Grad} h\right\} + (m_2 - 1)\operatorname{Div}(J_0 h \overline{\mathbf{v}}) \\ &= -J_0\,\mathbf{p}_E|_{\xi=0} - J_0 h(c + a\,m_2)\mathbf{F}^{-1}\mathbf{s} - J_0 h\,m_2 \mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial \xi^\alpha}(\overline{\mathbf{v}} \otimes \overline{\mathbf{v}})\mathbf{e}_\alpha + O(\epsilon^{2+\hat{\gamma}}). \end{aligned} \tag{57}$$

However, unlike (56), Eq. (57) is not in conservative form. For the "plug flow" of an ideal fluid, for which $m_2 = 1$ and $\mathbf{p}_E = \mathbf{0}$, Eq. (57) has been derived in Luca et al. [16] assuming $\mathcal{H} = O(\epsilon)$.

7 Bottom Coulomb versus viscous friction

In applied rheology of avalanching debris with or without an interstitial fluid, but treated as a single constituent "fluid" material, the basal shear stress $\mathbf{p}_E|_{\xi=0}$ must be prescribed in terms of a phenomenological relation, expressing the local kinematic and/or dynamic state. Two different situations may be envisaged, either no-slip or, when sliding prevails, a stress related sliding law. If the material is a slurry with a dense concentration of small particles, smaller than the roughness scale of the basal surface, then no-slip is likely. This also means large stresses, corresponding to $\mathbf{p}_E|_{\xi=0} = O(\epsilon^\gamma)$ or $\mathbf{p}_E|_{\xi=0} = O(\epsilon)$. In this case $\mathbf{p}_E|_{\xi=0}$ can be determined from the constitutive relation of the material in the bulk. On the other hand, if the roughness scale of the basal surface is smaller than the typical diameter of the material particles, it is expected that the avalanche mass slips on the surface. This situation requires the postulation of a sliding law, in which the friction force (i.e. $-(\boldsymbol{\sigma n} - (\boldsymbol{\sigma n}\cdot\boldsymbol{n})\boldsymbol{n})$ evaluated at the basal surface, where $\boldsymbol{n}$ is the unit normal vector pointing into the avalanche body) can be given as a constitutive quantity expressing the interaction of the avalanche mass with the basal surface. At dense concentrations of the particles this friction is (i) due to the *rubbing* of the particles on the surface, corresponding to, e.g., Coulomb friction, $-\boldsymbol{\tau}_{\text{Coulomb}}$, and (ii) due to *collisions*, corresponding, e.g., to viscous friction, $-\boldsymbol{\tau}_{\text{viscous}}$, with[3]

$$\boldsymbol{\tau}_{\text{Coulomb}} = (\tan\delta)(-\boldsymbol{\sigma n}\cdot\boldsymbol{n})_+ \operatorname{sgn}\boldsymbol{u}, \quad \boldsymbol{\tau}_{\text{viscous}} = \rho_0 C(\|\boldsymbol{u}\|)\boldsymbol{u}. \tag{58.1, 2}$$

In the above $\boldsymbol{u}$ is the basal velocity, which is assumed tangent to the sliding surface,

$$\operatorname{sgn}\boldsymbol{u} \equiv \begin{cases} \frac{1}{\|\boldsymbol{u}\|}\boldsymbol{u}, & \text{if } \boldsymbol{u} \neq \boldsymbol{0}, \\ \text{any tangent vector } \boldsymbol{m} \text{ to } \mathcal{S}, \quad \|\boldsymbol{m}\| \leq 1, & \text{if } \boldsymbol{u} = \boldsymbol{0}, \end{cases}$$

δ is the *basal angle of friction*, $\tan\delta > 0$, the index $+$ stands for the positive part of a quantity, i.e. $f_+ \equiv \max\{0, f\}$, and $C > 0$ is the *drag coefficient*, in the linear case constant, more often, however, taken to be linear in $\|\boldsymbol{u}\|$. In this latter case, $C(\|\boldsymbol{u}\|) = \tilde{c}\|\boldsymbol{u}\|$, with dimensionless $\tilde{c} \lessapprox 2.5 \times 10^{-3} (= O(\epsilon^{1+\gamma})$ if $\epsilon \approx 10^{-2}$, $\gamma \approx \frac{1}{2}$). The distribution of the shear traction into a dry and a viscous component depends on the mean free path between the particles: in a slurry $\|\boldsymbol{\tau}_{\text{Coulomb}}\| \ll \|\boldsymbol{\tau}_{\text{viscous}}\|$, whilst in a very dense debris flow $\|\boldsymbol{\tau}_{\text{Coulomb}}\| \gg \|\boldsymbol{\tau}_{\text{viscous}}\|$. All these cases are conveniently accounted for by writing

$$\boldsymbol{\sigma n} - (\boldsymbol{\sigma n}\cdot\boldsymbol{n})\boldsymbol{n} = \phi\boldsymbol{\tau}_{\text{Coulomb}} + (1-\phi)\boldsymbol{\tau}_{\text{viscous}} \quad \text{at } \xi = 0, \tag{59}$$

with $\phi \in [0, 1]$, see e.g. Voellmy [33], Gray and Tai [9], Pudasaini and Hutter [29, p. 145]. With such a parametrization, δ can be determined from a static heap test, and C from a flow test at dilute concentration.

[3] Here physical (i.e. dimensional) quantities are envisaged.

Using the scalings (18) we have $\boldsymbol{u} = \sqrt{Lg}\,\hat{\boldsymbol{u}}$, $\boldsymbol{\tau}_{\text{viscous}} = \rho_0 g L\,\hat{\boldsymbol{\tau}}_{\text{viscous}}$, and hence the non-dimensional form of (58.2) is

$$\hat{\boldsymbol{\tau}}_{\text{viscous}} = \hat{C}(\|\hat{\boldsymbol{u}}\|)\hat{\boldsymbol{u}}, \quad \hat{C}(\|\hat{\boldsymbol{u}}\|) \equiv C(\sqrt{Lg}\|\hat{\boldsymbol{u}}\|)/\sqrt{Lg}. \tag{60.1, 2}$$

Next the hat will be dropped in (60.1). Thus, since

$$\boldsymbol{\sigma n}\cdot\boldsymbol{n} = T^{33}, \quad \boldsymbol{\sigma n} - (\boldsymbol{\sigma n}\cdot\boldsymbol{n})\boldsymbol{n} = T^{13}\boldsymbol{g}_1 + T^{23}\boldsymbol{g}_2, \quad \|\boldsymbol{u}\|^2 = g_{ij}v^i v^j,$$

using $\mathrm{v} = 0$ at $\xi = 0$ and expression (9) for the coefficients g_{ij} of the metric tensor, in the curvilinear coordinates used in this paper the non-dimensional form of condition (59) emerges as

$$\mathbf{p}_E|_{\xi=0} = \phi(\tan\delta)\left\{(p - T_E^{33})\big|_{\xi=0}\right\}_+ \operatorname{sgn}\mathbf{v}_0 + (1-\phi)C\left(\sqrt{\mathbf{M}_0^{-1}\mathbf{v}_0\cdot\mathbf{v}_0}\right)\mathbf{v}_0, \tag{61}$$

where $\mathbf{v}_0 \equiv \mathbf{v}|_{\xi=0}$, and for a two-column $\mathbf{x}$ the multivalued function $\operatorname{sgn}\mathbf{x}$ is defined as

$$\operatorname{sgn}\mathbf{x} \equiv \begin{cases} \frac{1}{\sqrt{\mathbf{M}_0^{-1}\mathbf{x}\cdot\mathbf{x}}}\mathbf{x}, & \text{if } \mathbf{x} \neq \mathbf{0}, \\ \text{any 2-column } \mathbf{m},\ \mathbf{M}_0^{-1}\mathbf{m}\cdot\mathbf{m} \leq 1, & \text{if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Therefore, with

$$\mathbf{p}_E = O(\epsilon^{1+\gamma}), \quad p = O(\epsilon), \quad T_E^{33} = O(\epsilon^2), \quad \tan\delta = O(\epsilon^\gamma), \quad C = O(\epsilon^{1+\gamma}),$$

condition (61) makes sense, and therefore the bottom friction condition (59) can be used to express the basal shear stress $\mathbf{p}_E|_{\xi=0}$ in the shallow avalanche equation (51). Using (52), condition (61) takes the form

$$\mathbf{p}_E|_{\xi=0} = \phi h(\tan\delta)\{c + a\,m_2\}_+ \operatorname{sgn}\mathbf{v}_0 + (1-\phi)C\left(\sqrt{\mathbf{M}_0^{-1}\mathbf{v}_0\cdot\mathbf{v}_0}\right)\mathbf{v}_0 + O(\epsilon^{2+\gamma}).$$

However, since the mean velocity is searched for, the condition above must be written in terms of $\overline{\mathbf{v}}$. This can be achieved, e.g., by assuming that $\mathbf{v}_0$ is collinear to $\overline{\mathbf{v}}$, that is, $\mathbf{v}_0 = \chi\overline{\mathbf{v}}$, $\chi > 0$. Thus, we obtain

$$\mathbf{p}_E|_{\xi=0} = \phi h(\tan\delta)\{c + a\,m_2\}_+ \operatorname{sgn}\overline{\mathbf{v}} + (1-\phi)\chi C\left(\chi\sqrt{\mathbf{M}_0^{-1}\overline{\mathbf{v}}\cdot\overline{\mathbf{v}}}\right)\overline{\mathbf{v}} + O(\epsilon^{2+\gamma}). \tag{62}$$

In (62) the negligible terms $O(\epsilon^{2+\gamma})$ are due to (52). Thus, for $\phi = 0$, there are no negligible terms in (62).

Now, referring to item (ii) of Proposition 5 we can state

Proposition 8 *Assuming* $\mathbf{P}_E = O(\epsilon)$, $\mathbf{p}_E = O(\epsilon^{1+\gamma})$, $T_E^{33} = O(\epsilon^2)$, (39) *and the bottom friction condition* (62) *in which* $\tan\delta = O(\epsilon^\gamma)$, $C = O(\epsilon^{1+\gamma})$, *the shallow avalanche equations are* (49)–(51)*, in which the term containing* $\overline{\mathbf{p}}_E$ *is omitted, and the basal shear stress is given by* (62)*. If, moreover,* $\mathbf{P}_E = O(\epsilon^{1+\gamma})$*, the terms involving* $\overline{\mathbf{P}}_E$ *can also be neglected. For this latter case, there are the parameters* m_1 *to* m_3, β, χ, *the friction angle* δ *and the drag coefficient* C *that describe the motion of the avalanche. All these parameters, which are phenomenological coefficients, are materially dependent.*

8 Viscous debris and avalanche models

The equations stated in Propositions 4 to 8 must be complemented by closure relations for the stress tensor fitting the dynamical assumptions (c) or variants during the avalanche motion. Next we first deal with non-Newtonian viscous fluids, and then we focus on the Savage–Hutter avalanche approach.

8.1 Non-Newtonian viscous fluids

We consider the (non-dimensional) constitutive equation

$$\boldsymbol{\sigma}_E = 2\eta(\dot{\gamma})\boldsymbol{D}, \quad \dot{\gamma} \equiv 2\sqrt{II_{\boldsymbol{D}}}, \tag{63}$$

defining a *non-Newtonian viscous fluid*; $\eta(\dot{\gamma})$ is the *effective* (or *apparent*) *viscosity*, which is a function of the *shear rate* (or *stretching*) $\dot{\gamma}$. In the curvilinear coordinates (5), relation (63) emerges as

$$\mathbf{P}_E = 2\eta(\dot{\gamma})\mathbf{D}, \quad \mathbf{p}_E = 2\eta(\dot{\gamma})\,\mathbf{d}, \quad T_E^{33} = 2\eta(\dot{\gamma})D^{33}, \tag{64.1–3}$$

and in view of (33) and (34) we have

$$\mathbf{D} = O(1), \quad \mathbf{d} = O(\epsilon^{-1}), \quad D^{33} = O(1), \tag{65}$$

whence we deduce

$$\mathbf{P}_E = O(\eta), \quad \mathbf{p}_E = O(\eta/\epsilon), \quad T_E^{33} = O(\eta). \tag{66}$$

With the scalings (18) the non-dimensional effective viscosity η appears as the inverse of a Reynolds number Re, viz.,

$$\eta = \frac{1}{Re}, \quad Re \equiv \frac{\rho_0 L\sqrt{Lg}}{\eta_{\text{dim}}},$$

where η_{dim} is the physical effective viscosity. For snow and debris avalanches we have $\epsilon \approx 10^{-2}$, $\gamma \approx \frac{1}{2}$ (see [29, p. 188]), while typical Reynolds numbers range from 10^3 to 10^7, implying typical viscosities η of order $O(\epsilon^{1+\gamma})$ or less. In view of (66), we therefore have situations in which various dynamic assumptions are fulfilled. However, unless $\eta = O(\epsilon^{3+\gamma})$, the dynamic restriction (39) still remains, now in the form

$$\mathbf{p}_E = O(\epsilon^{2+\gamma}) \quad \text{at} \quad \xi = h(\boldsymbol{\xi}, t). \tag{67}$$

As a consequence, various simplified versions of equations presented in Proposition 4 are available, see Proposition 5. Thus, if $\eta = O(\epsilon^{1+\gamma})$ or $\eta = O(\epsilon^2)$, item (i) of Proposition 5 holds, while if $\eta = O(\epsilon^{2+\gamma})$, one applies item (ii) of Proposition 5. More specifically, we have

Proposition 9 *The avalanche equations corresponding to the non-Newtonian fluid fit class* (i) *if* $\eta = O(\epsilon^{3+\gamma})$ *or less. Moreover, assuming* (67)*, these equations also fit class* (ii) *if* $\eta = O(\epsilon^{2+\gamma})$ *or* $\eta = O(\epsilon^3)$*, and class* (iii) *if* $\eta = O(\epsilon^{1+\gamma})$ *or* $\eta = O(\epsilon^2)$.

Similar considerations can be made when referring to Proposition 7. Table 2 concisely expresses the statements of Proposition 9.

Next we first deal with the case $\eta = O(\epsilon^{1+\gamma})$ or $\eta = O(\epsilon^2)$, for which we need to relate the shear stresses $\mathbf{p}_E|_{\xi=0}$ and $\overline{\mathbf{p}}_E$ to the basic unknown fields $\overline{\mathbf{v}}$ and h by means of the closure relation (64.2). We shall do this in the next subsection by adopting the no-slip boundary condition. The case of small viscosity, $\eta = O(\epsilon^{2+\gamma})$, when only $\mathbf{p}_E|_{\xi=0}$ is needed, shall be dealt with in Sect. 8.1.2 by considering non-zero basal velocity.

Table 2 Classification of non-Newtonian avalanche models in terms of the scaling of viscosity

Non-dimensional viscosity η	Non-negligible stress components in (51)
$O(\epsilon^{3+\gamma})$	–
$O(\epsilon^{2+\gamma})$, $O(\epsilon^3)$	$\mathbf{p}_E\|_{\xi=0}$
$O(\epsilon^{1+\gamma})$, $O(\epsilon^2)$	$\overline{\mathbf{p}}_E$, $\mathbf{p}_E\|_{\xi=0}$

8.1.1 Non-Newtonian viscous fluids with significant viscosity

Now, let us suppose that $\eta = O(\epsilon^{1+\gamma})$ or $\eta = O(\epsilon^2)$. For concreteness we shall refer to $\eta = O(\epsilon^{1+\gamma})$. Since the viscosity is not too small, the sliding basal velocity is expected to be small, and so its influence on the mean flow is not significant. Condition $\boldsymbol{u}|_{\xi=0} = \boldsymbol{0}$, which we next assume, can then be considered a good approximation. Moreover, in order to express $\overline{\mathbf{p}}_E$ in terms of $\overline{\mathbf{v}}$ and h, we suppose that the velocity components $\mathbf{v}$ are given by (43), in which $\tilde{\chi} = 0$, that is

$$\mathbf{v} = \left\{1 - \left(1 - \frac{\xi}{h}\right)^{n+1}\right\} \mathbf{v}_h = \frac{n+2}{n+1}\left\{1 - \left(1 - \frac{\xi}{h}\right)^{n+1}\right\} \overline{\mathbf{v}}. \tag{68}$$

The parameters m_1 to m_3, β are then given by (45) and (46), in which $\chi = 0$. Assuming (68), it is understood that $h \neq 0$. Of course, the thickness of the real avalanche mass can have zero values. So, we expect the solution h of the emerging modelling equations to be "small" at those $(\boldsymbol{\xi}, t)$ where the thickness of the real avalanche is zero. By the power law profile (68), in which $n \geq 3$ is expected, we postulate velocity components that are almost uniform through the depth, except for small values of ξ. We also note that the no-slip boundary condition is often used in thin layer theories dealing with non-Newtonian viscous fluids, see e.g. Ng and Mei [20], Berezin and Spodareva [2], Perazzo and Gratton [23], Huang and Garcia [12]. Finally, it is worthwhile to mention that power law velocity profiles (68) have been obtained as exact solutions of the approximating equations describing the stationary motion on an inclined plane of a power law fluid with the so-called *power law index* $\lambda = 1/n$, see e.g. Berezin and Spodareva [2], Perazzo and Gratton [23].

Given the velocity field (68) and the constitutive Eq. (64), we can determine the shear stress $\mathbf{p}_E$. We have

$$\mathbf{p}_E = \eta(\dot{\gamma})\frac{n+2}{h}\left(1 - \frac{\xi}{h}\right)^n \overline{\mathbf{v}} + \eta(\dot{\gamma})\mathbf{M}\mathrm{Grad}\, \mathrm{v}, \tag{69}$$

and therefore, with $\eta = O(\epsilon^{1+\gamma})$ and $\mathrm{v} = O(\epsilon)$, we deduce

$$\mathbf{p}_E = \eta(\dot{\gamma})\frac{n+2}{h}\left(1 - \frac{\xi}{h}\right)^n \overline{\mathbf{v}} + O(\epsilon^{2+\gamma}), \tag{70}$$

which in particular shows that condition (67) is satisfied. We now proceed to determine $\mathbf{p}_E$ at $\xi = 0$. For $\mathrm{v} = 0$ at $\xi = 0$, from (69) we obtain

$$\mathbf{p}_E|_{\xi=0} = \eta(\dot{\gamma}_0)\frac{n+2}{h}\overline{\mathbf{v}}, \quad \dot{\gamma}_0 \equiv \dot{\gamma}|_{\xi=0}, \tag{71}$$

in which we have to determine $\dot{\gamma}_0$. To this end, using condition $\mathbf{v} = \mathbf{0}$ at $\xi = 0$, from (31) we deduce

$$\frac{\partial \mathrm{v}}{\partial \xi} = 0 \quad \text{at } \xi = 0,$$

and so (33) and (34) imply

$$\mathbf{D} = \mathbf{0}, \quad \mathbf{d} = \frac{1}{2}\frac{\partial \mathbf{v}}{\partial \xi}, \quad D^{33} = 0 \quad \text{at } \xi = 0, \tag{72}$$

whence

$$II_{\boldsymbol{D}} = \tfrac{1}{4}\mathbf{M}^{-1}\frac{\partial \mathbf{v}}{\partial \xi} \cdot \frac{\partial \mathbf{v}}{\partial \xi} \quad \text{at } \xi = 0,$$

see formula (35) expressing the second invariant of $\boldsymbol{D}$. Using (68), we obtain

$$II_{\boldsymbol{D}}|_{\xi=0} = \frac{(n+2)^2}{4h^2}\mathbf{M}_0^{-1}\overline{\mathbf{v}} \cdot \overline{\mathbf{v}},$$

and therefore

$$\dot{\gamma}_0 = 2\sqrt{II_{\boldsymbol{D}}|_{\xi=0}} = \frac{n+2}{h}\sqrt{\mathbf{M}_0^{-1}\overline{\mathbf{v}} \cdot \overline{\mathbf{v}}}.$$

Thus, we arrive at

Proposition 10 *Assume the motion of the non-Newtonian fluid* (63) *be such that* $\boldsymbol{u}|_{\xi=0} = \boldsymbol{0}$ *and* (68) *hold. Then we have:*

(i) *The basal shear stress* $\mathbf{p}_E|_{\xi=0}$ *is given by the exact formula*

$$\mathbf{p}_E|_{\xi=0} = \eta(\dot{\gamma}_0)\frac{n+2}{h}\overline{\mathbf{v}}, \quad \dot{\gamma}_0 = \frac{n+2}{h}\sqrt{\mathbf{M}_0^{-1}\overline{\mathbf{v}}\cdot\overline{\mathbf{v}}}\,.$$

(ii) *If* $\eta = O(\epsilon^{1+\gamma})$, *condition* (67) *is fulfilled and the mean shear stress* $\overline{\mathbf{p}}_E$ *is given by*

$$\overline{\mathbf{p}}_E = \frac{n+2}{h}\,\overline{\eta(\dot{\gamma})\left(1-\frac{\xi}{h}\right)^n}\,\overline{\mathbf{v}} + O(\epsilon^{2+\gamma}).$$

Once the the viscosity function η is known, the mean shear stress $\overline{\mathbf{p}}_E$ in the proposition above can be further expressed in terms of $\overline{\mathbf{v}}$, h by accounting for the power law velocity profile (68) and normal velocity (31) in expression (35) of $II_{\boldsymbol{D}}$.

8.1.2 Non-Newtonian viscous fluids with small viscosity

Now we consider $\eta = O(\epsilon^{2+\gamma})$. By Proposition 9 we have only to make precise the bottom shear stress. Since now the shear stress in the normal direction is not so large, we expect a non-zero basal velocity, and therefore a friction law can be used to express the basal shear stress. For instance, assuming (67) and $\eta = O(\epsilon^{2+\gamma})$, the statements of Proposition 8 hold, and hence one possible choice for $\mathbf{p}_E|_{\xi=0}$ is to assume the bottom friction condition (62). Note that the emerging shallow avalanche equations contain parameters $m_1, m_2, m_3, \beta, \chi$ and the drag coefficient C, however, the viscosity η does not enter these equations explicitly.

For the case $\phi = 0$ in (62), that is viscous friction is dominant, and for a drag coefficient given by $C(\|\boldsymbol{u}\|) = \tilde{c}\,\|\boldsymbol{u}\|$, $\tilde{c} = O(\epsilon^{1+\gamma})$, that is

$$\mathbf{p}_E|_{\xi=0} = \tilde{c}\,\chi^2\sqrt{\mathbf{M}_0^{-1}\overline{\mathbf{v}}\cdot\overline{\mathbf{v}}}\,\overline{\mathbf{v}}, \quad \tilde{c} = O(\epsilon^{1+\gamma}), \tag{73}$$

we indicate a possible way to determine m_1, m_2, m_3, β and χ in the modelling equations. The idea is to assume the power-law velocity profile (44) and to use the constitutive Eq. (64.2) in the bulk fluid. First we note that

$$\mathbf{p}_E = \eta(\dot{\gamma})\left\{\frac{\partial\mathbf{v}}{\partial\xi} + \mathbf{M}\mathrm{Grad}\,\mathrm{v}\right\} = \eta(\dot{\gamma})\frac{n+2}{h}\left(1-\frac{\xi}{h}\right)^n(1-\chi)\overline{\mathbf{v}} + O(\epsilon^{3+\gamma}), \tag{74}$$

and hence condition (67) is fulfilled. Then, the parameters m_1 to m_3 are given by (45), β is shown in (46), and it remains to obtain χ. To this end we shall equate expression (73) for the basal shear stress and

$$\mathbf{p}_E|_{\xi=0} = \eta(\dot{\gamma}_0)\left.\frac{\partial\mathbf{v}}{\partial\xi}\right|_{\xi=0} = \eta(\dot{\gamma}_0)\frac{n+2}{h}(1-\chi)\overline{\mathbf{v}},$$

which is deduced from (74) by accounting for $\mathrm{v} = 0$ at $\xi = 0$. Therefore, we have

$$\eta(\dot{\gamma}_0)\frac{n+2}{h}(1-\chi) = \tilde{c}\,\chi^2\sqrt{\mathbf{M}_0^{-1}\overline{\mathbf{v}}\cdot\overline{\mathbf{v}}}, \tag{75}$$

which, once the viscosity function η is known and $\dot{\gamma}_0$ is expressed in terms of $\overline{\mathbf{v}}$, h, constitutes the equation for the determination of χ. For a Newtonian fluid the viscosity η is constant, and Eq. (75) has a unique solution $\chi \in (0, 1)$, as it can easily be seen. The case of the Newtonian fluid has been treated in Luca et al. [16]. For an arbitrary viscosity function η one can only state that, if there exists a positive solution of (75), then it belongs to $(0, 1)$. Indeed, if such a solution exists, it satisfies

$$\chi = \frac{-1+\sqrt{1+4\tilde{m}}}{2\tilde{m}}, \quad \tilde{m} \equiv \frac{\tilde{c}h}{\eta(\dot{\gamma}_0)(n+2)}\sqrt{\mathbf{M}_0^{-1}\overline{\mathbf{v}}\cdot\overline{\mathbf{v}}},$$

which gives $\chi \in (0, 1)$. It is, however, clear that, for a non-constant viscosity function η, it may happen that Eq. (75) possesses no solution. In other words, postulating a power law velocity profile and also a viscous friction law will lead to a contradiction. We also mention that we have conducted a similar analysis by postulating the power law velocity profile (43) and the classical bottom Coulomb friction condition, i.e. (62) with $\phi = 1$. We have obtained a non-realistic result, and that is why we have assumed a viscous friction in this paper.

Now, note that, if $\tilde{c} = O(\epsilon^{1+2\gamma})$, then (75) shows that $1-\chi = O(\epsilon^{\gamma})$, and if $\tilde{c} = O(\epsilon^{2+\gamma})$, then $1-\chi = O(\epsilon)$. Moreover, for this latter case, from (74) we deduce $\mathbf{p}_E = O(\epsilon^{2+\gamma})$, and therefore the negligible terms in the linear momentum balance equation (51) are of order $O(\epsilon^3)$ (see the proof of Proposition 4 in the Appendix). Using these remarks and recalling the comments following (46), as well as the results of this subsection, we deduce

Proposition 11 *For the non-Newtonian fluid* (63) *with* $\eta = O(\epsilon^{2+\gamma})$*, we have:*

(i) *Postulating* (67) *and the basal Coulomb/viscous friction law* (59)*, the corresponding shallow avalanche equations are* (49)–(51)*, in which the terms containing* $\overline{\mathbf{P}}_E$ *and* $\overline{\mathbf{p}}_E$ *are omitted, the basal shear stress is* (62)*, and the parameters* m_1, m_2, m_3, β *and* χ *must be a priori prescribed.*

(ii) *For only viscous friction with the linear drag coefficient* $\tilde{c} = O(\epsilon^{1+\gamma})$*, and for the velocity profile* (43)*, if equation* (75) *has a solution in* $(0, 1)$*, say* χ_0*, the parameters* m_1 *to* m_3 *and* β *are given by* (45) *and* (46)*, in which* $\chi = \chi_0$*. Moreover, if* $\tilde{c} = O(\epsilon^{1+2\gamma})$*, the modelling equations* (49)–(51) *emerge as*

$$
\begin{aligned}
&\overline{p} = \tfrac{1}{2}h(c+a) + O(\epsilon^{1+\gamma}),\\
&\frac{\partial}{\partial t}\{J_0 h(1-\Omega h)\} + \mathrm{Div}\{J_0 h(1-\Omega h)\,\overline{\mathbf{v}}\} = O(\epsilon^{2+\gamma}),\\
&\frac{\partial}{\partial t}\{J_0 h(1-\Omega h)\overline{\mathbf{v}}\} + \mathrm{Div}\{J_0 h[(m_2-\Omega h)\overline{\mathbf{v}}\otimes\overline{\mathbf{v}} + \overline{p}\,\mathbf{M}_0]\}\\
&\quad = -J_0\tilde{c}\,\chi_0^2\sqrt{\mathbf{M}_0^{-1}\overline{\mathbf{v}}\cdot\overline{\mathbf{v}}}\;\overline{\mathbf{v}} - J_0 h\big\{(c + a\,m_2 - \tfrac{1}{2}\tilde{a}h)\mathbf{I} + \overline{p}\,\mathbf{W}\big\}\,\mathbf{F}^{-1}\mathbf{s}\\
&\qquad - \mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial \xi^{\alpha}}\,\{J_0 h\,[(m_2-\Omega h)\overline{\mathbf{v}}\otimes\overline{\mathbf{v}} + \overline{p}\,\mathbf{M}_0]\}\mathbf{e}_{\alpha}\\
&\qquad + \tfrac{1}{2}\,J_0 h^2\mathbf{F}^{-1}\frac{\partial}{\partial \xi^{\alpha}}(\mathbf{F}\mathbf{W}\mathbf{F}^{-1})\mathbf{F}(\overline{\mathbf{v}}\otimes\overline{\mathbf{v}})\mathbf{e}_{\alpha} - h^2\,\mathrm{Div}\,(J_0\overline{\mathbf{v}})\,\mathbf{W}\overline{\mathbf{v}} + O(\epsilon^{2+\gamma}).
\end{aligned}
$$

If all the more $\tilde{c} = O(\epsilon^{2+\gamma})$*, then the negligible term in the expression of the mean pressure is of order* $O(\epsilon^2)$ *instead of* $O(\epsilon^{1+\gamma})$*, in the mass and momentum balance equations above the neglected terms are of order* $O(\epsilon^3)$ *instead of* $O(\epsilon^{2+\gamma})$*, and* m_2 *can be replaced by* 1.

For the case stated in item (ii) in the proposition above, due to the complexity of Eq. (75) for non-constant viscosity, in numerical computations it is perhaps more suitable to follow item (i), and to calibrate the coefficients m_1, m_2, m_3, β, as well as the sliding parameter χ, so that the theoretical results agree sufficiently well with the experimental data. However, by doing so, the viscosity does not appear explicitly in the final formulae.

8.1.3 Examples

It is clear that the results stated in Proposition 9 concern a wide diversity of non-Newtonian viscous fluids, in particular (i) *shear thinning* fluids,[4] for which the maximum viscosity, i.e. η_{dim} at $\dot{\gamma} = 0$, is sufficiently small, and (ii) *shear thickening* fluids,[4] for which the viscosity remains bounded and sufficiently small when the shear rate increases. Note that in this subsection $\dot{\gamma}$ denotes the dimensional shear rate.

Debris flows are often described by non-Newtonian/Newtonian stress-stretching relations, with or without *yield stress* τ_0, such as the *Herschel–Bulkley fluid* law, viz.,

$$
\eta_{\mathrm{dim}}(\dot{\gamma}) = k\,\dot{\gamma}^{\lambda-1} + \frac{\tau_0}{\dot{\gamma}}, \qquad \lambda > 0, \quad k > 0, \quad \tau_0 \geq 0, \tag{76}
$$

which combines the *power law fluid* ($\tau_0 = 0$), with the *Bingham fluid* ($\lambda = 1$, $\tau_0 \neq 0$). For the cases (i) $\lambda \in (0, 1)$ and (ii) $\lambda = 1$, $\tau_0 \neq 0$, relation (76) refers to shear thinning fluids, while $\lambda > 1$ corresponds to shear thickening fluids. The common fluids are shear thinning, so that next we shall consider such fluids, as described by (76)

[4] A fluid shows *shear thinning* (or *pseudoplastic*)/*shear thickening* (or *dilatant*) behaviour, if its viscosity decreases/increases with increasing shear rate. Accordingly, a fluid is called shear thinning/thickening if it has only shear thinning/thickening behaviour.

with ($\tau_0 \neq 0$) or without ($\tau_0 = 0$) yield stress. Thus, for shear thinning fluids the viscosity (76) becomes very large for small to zero shear rates, and therefore it cannot fulfill the ordering assumptions on η in Proposition 9, for any k and τ_0. However, model (76) is not realistic, since an infinite viscosity at low stretching is physically not possible. Realistic is, however, that at small shear rates, the physical viscosity, with or without yield stress, has a non-zero finite value, that is possibly large.

To overcome this inconvenience for the Bingham fluid, Papanastasiou [22] proposed a regularized version with a parameter m of dimension "time" (s), viz.,

$$\eta_{\text{dim}}(\dot{\gamma}) = k + \frac{\tau_0}{\dot{\gamma}}\,(1 - e^{-m\,\dot{\gamma}}), \quad m > 0, \tag{77}$$

which gives

$$\lim_{\dot{\gamma}\to 0} \eta_{\text{dim}}(\dot{\gamma}) = k + \tau_0 m \neq \infty.$$

For the case $\lambda \in (0, 1)$, with or without yield stress, we propose the following regularized version of (76)

$$\eta_{\text{dim}}(\dot{\gamma}) = k\,\dot{\gamma}^{\lambda-1}\left(1 - e^{-l\,\dot{\gamma}^{1-\lambda}}\right) + \frac{\tau_0}{\dot{\gamma}}\,(1 - e^{-m\dot{\gamma}}), \qquad l > 0, \quad m > 0, \tag{78}$$

which implies

$$\begin{gathered} \lim_{\dot{\gamma}\to 0} \eta_{\text{dim}}(\dot{\gamma}) = kl + \tau_0 m \neq \infty, \quad \eta_{\text{dim}}(\dot{\gamma}) \approx k\dot{\gamma}^{\lambda-1} \quad \text{for} \quad \dot{\gamma} \to \infty, \\ \eta'_{\text{dim}}(\dot{\gamma}) < 0, \quad \lim_{\dot{\gamma}\to 0} \eta'_{\text{dim}}(\dot{\gamma}) = -\infty, \end{gathered} \tag{79}$$

showing that the mentioned singularity at zero stretching in (76) is removed, and that the basic features of the Herschel–Bulkley model are preserved.[5] The regularizing parameters l ($s^{1-\lambda}$) and m (s) can be chosen sufficiently large, if "infinite" viscosity at low shear stress is to be mimicked.

Now we estimate numerical values for the maximum non-dimensional viscosity

$$\eta_{\max} \equiv \frac{kl + \tau_0 m}{\rho_0 L \sqrt{Lg}},$$

where l is replaced by 1 if the Bingham fluid is envisaged, by using typical values of ρ_0, L for an avalanche flow, viz., $\rho_0 = 2,000$ Kg m^{-3}, $L = 100$ m, and taking $g = 10$ m s^{-1}. For various debris flows and slurries, Wang et al. [34, p. 39, p. 117] quote values of the Bingham viscosity, i.e. k in (76) with $\lambda = 1$, from 4×10^{-2} to 2×10^{-1} Pa s, while for the yield stress they mention values between 4 and 50 Pa. So, for viscosity k and yield stress τ_0 small, in the sense that $kl + \tau_0 m \leq 10^2$ Pa s, we have

$$\eta_{\max} \leq 0.16 \times 10^{-4}\,\text{Pa s}.$$

With $\epsilon = 10^{-2}$ this yields $\eta_{\max} = O(\epsilon^{2+\gamma})$.

Consider now the case with strong viscosity/yield stress. Huang and Garcia [12] quote $\tau_0 = 200$–300 Pa for a mud flow, and Coussot and Piau [5] give numerical values for the parameters k and τ_0 of the model (76) describing a water-kaoline mixture in the range [10, 300] Pa s$^{\lambda}$ and [20, 500] Pa, respectively. So, if we choose for a realistic range

$$kl + \tau_0 m \in [10^3,\ 10^4]\,\text{Pa s},$$

we obtain

$$\eta_{\max} \in [0.16 \times 10^{-3},\ 0.16 \times 10^{-2}].$$

[5] Mendes and Dutra [19] have introduced

$$\eta_{\text{dim}}(\dot{\gamma}) = \left(k\,\dot{\gamma}^{\lambda-1} + \frac{\tau_0}{\dot{\gamma}}\right)(1 - e^{-m\dot{\gamma}}), \quad m > 0$$

as the regularized version of (76). Unlike (76) and (78) this viscosity shows shear thickening behaviour at zero shear rates, since $\lim_{\dot{\gamma}\to 0} \eta'_{\text{dim}}(\dot{\gamma}) = +\infty$.

With the same scale for ϵ, i.e. $\epsilon = 10^{-2}$, this implies $\eta_{\max} = O(\epsilon^{1+\gamma})$.

We deduce that the Herschel–Bulkely model can be used within the theory developed in this paper, if one uses a regularized version which has bounded viscosity, for instance (78).

However, like the Herschel–Bulkley model (76), the regularized version (78) predicts dramatical changes of the viscosity by slightly changing the shearing. Indeed, at zero shear rates the derivative of η_{dim} approaches $-\infty$, see (79). A more suitable viscosity function with or without yield stress, overcoming this non-realistic feature and which has a power-law behaviour for sufficiently large shear rates, is

$$\eta_{\text{dim}}(\dot\gamma) = \eta_1\, e^{-t_1\,\dot\gamma} + \frac{2}{\pi}\,\eta_2\left(\frac{\dot\gamma}{\dot\gamma_r}\right)^{\lambda-1}\arctan\left(t_2\left(\frac{\dot\gamma}{\dot\gamma_r}\right)^{\beta}\right) + \frac{\tau_0}{\dot\gamma}\,(1 - e^{-m\,\dot\gamma}), \tag{80}$$

with constant viscosities η_1, η_2 (Pa s), $\dot\gamma_r$ a stretching (s^{-1}), τ_0 the yield stress (Pa), t_1, m times (s), λ, β, t_2 non-dimensional constants, for which numbers must be given, such that

$$\begin{aligned} &\eta_1 > 0, \quad \eta_2 \ge 0, \quad \tau_0 \ge 0, \quad \lambda \in (0, 1),\\ &t_1 \ge 0 \quad t_2, m > 0, \quad \beta + \lambda - 2 > 0. \end{aligned} \tag{81}$$

Conditions (81) guarantee

$$\lim_{\dot\gamma\to 0}\eta_{\text{dim}}(\dot\gamma) = \eta_1 + \tau_0\, m \ne \infty, \quad \lim_{\dot\gamma\to 0}\eta'_{\text{dim}}(\dot\gamma) = -\eta_1 t_1 - \tfrac{1}{2}\tau_0 m^2 = \text{ finite},$$

$$\eta_{\text{dim}}(\dot\gamma) \approx \eta_2\left(\frac{\dot\gamma}{\dot\gamma_r}\right)^{\lambda-1} \quad \text{for} \quad \dot\gamma \to \infty,$$

and that shear thinning behaviour prevails at least for large stretching. Model (80) includes the Bingham–Papanastasiou fluid (77) ($t_1 = 0, \eta_2 = 0, \tau_0 \ne 0$) and the model introduced by Zhu et al. [35] ($\eta_2 = 0$) as an extension of the De Kee and Turcotte [6] proposal. The viscosity model (80) seems to be new. Now, using again $\rho_0 = 2{,}000$ Kg m^{-3}, $L = 100$ m, $g = 10$ m s^{-1}, we estimate the maximum non-dimensional viscosity

$$\eta_{\max} \equiv \frac{\eta_1 + \tau_0 m}{\rho_0 L\sqrt{Lg}}.$$

For a water-glycerol mixture, Ancey [1] quotes numerical values (Pa s) of the Newtonian viscosity in the range 10^{-3} to 10^0, while Zhu et al. [35] deduced, e.g. $\eta_1 = 0.15$ Pa s for their carbopol dispersion. So, we conjecture $\eta_1 \in [10^{-3},\ 10^0]$ Pa s in (80). Then, for small yield stress, in the sense that $\tau_0 m \le 10^2$ Pa s, we thus have

$$\eta_1 + \tau_0 m \in [10^{-3},\ 10^2]\,\text{Pa s},$$

and hence

$$\eta_{\max} \in [0.16 \times 10^{-6},\ 0.16 \times 10^{-4}].$$

With $\epsilon = 10^{-2}$ this yields $\eta_{\max} = O(\epsilon^{2+\gamma})$.

We refer now to the case with strong yield stress. Zhu et al. [35] used $\tau_0 = 3$ Pa and $m = 2{,}000$ s for a carbopol dispersion. So, if we choose for a realistic range

$$\eta_1 + \tau_0 m \in [10^3,\ 10^4]\,\text{Pa s},$$

we obtain

$$\eta_{\max} \in [0.16 \times 10^{-3},\ 0.16 \times 10^{-2}],$$

which implies $\eta_{\max} = O(\epsilon^{1+\gamma})$.

We conclude that the proposed modified Herschel–Bulkley viscosity (80) is a possible candidate to model a larger class of realistic rheological properties of thin flows within the theory developed in this paper. It could be criticized as being too demanding for engineers, since many parameters must be experimentally determined. However, many parameters of the list (81) are introduced for regularization purposes only and therefore need not be selected accurately, so that rough estimates should generally suffice. Moreover, the parameters make the model more flexible, and thus useful in computations when numerical data have to be fitted to the experimental data.

There are also non-Newtonian fluids which exhibit plateaux both at small and large strain rates. These may be described by the *Cross model*

$$\frac{\eta-\eta_\infty}{\eta_0-\eta_\infty}=\frac{1}{1+k\,\dot{\gamma}^{1-\lambda}},\quad k>0,\quad \lambda>0,\quad \eta_0>\eta_\infty,$$

with (physical) viscosities[6] η_0 and η_∞. These laws are characterized by λ and k (Pa s$^\lambda$), and may have shear thinning ($\lambda \in (0,1)$) or shear thickening ($\lambda > 1$) behaviour. The maximum viscosity is η_0 and thus they pose no difficulties for Proposition 9.

To summarize: we have shown that our thin film approximations with $\eta = O(\epsilon^{1+\gamma})$ or $\eta = O(\epsilon^{2+\gamma})$ are realistic avalanche models for a number of non-Newtonian fluids arising both in the geophysical and chemical context.

8.2 Savage–Hutter approach

In this section we implement the Savage–Hutter closure model. Thus, we assume that the avalanching mass slides on the basal surface, experiencing the classical Coulomb friction law, i.e. (59) with $\phi = 1$, and that (39) and

$$\mathbf{P}_E=O(\epsilon),\quad \mathbf{p}_E=O(\epsilon^{1+\gamma}),\quad T_E^{33}=O(\epsilon^2),\quad \tan\delta=O(\epsilon^\gamma) \tag{82.1–4}$$

hold. Consequently, with $\mathbf{v}_0=\chi\overline{\mathbf{v}}$, we can appeal to Proposition 8, and hence the significant stresses in (51) are $\overline{\mathbf{P}}_E$ and $\mathbf{p}_E|_{\xi=0}$, with $\mathbf{p}_E|_{\xi=0}$ as given by (62) in which $\phi = 1$. Therefore, it only remains to give a closure relation for $\overline{\mathbf{P}}_E$. We shall do this by following the Savage–Hutter approach.

First, we define the *mean surface stretching* $\boldsymbol{D}_{\mathbb{S}}\overline{\boldsymbol{u}}_\tau$ by

$$\boldsymbol{D}_{\mathbb{S}}\overline{\boldsymbol{u}}_\tau \equiv \tfrac{1}{2}\left(\nabla_{\mathbb{S}}\overline{\boldsymbol{u}}_\tau+\nabla_{\mathbb{S}}\overline{\boldsymbol{u}}_\tau^{\,T}\right),$$

where the differential operator $\nabla_{\mathbb{S}}$ is the surface gradient, see Appendix B. Then, let $\{\boldsymbol{w}_1,\boldsymbol{w}_2\}$ be an orthonormal basis of the tangent space to the surface $\mathbb{S}$ consisting of eigenvectors of the mean surface stretching (we assume $\boldsymbol{D}_{\mathbb{S}}\overline{\boldsymbol{u}}_\tau \neq \boldsymbol{0}$). With respect to the basis $\{\boldsymbol{w}_1,\boldsymbol{w}_2\}$ we have the representation $\overline{\boldsymbol{u}}_\tau = u^\alpha \boldsymbol{w}_\alpha$. Now, we define another orthonormal basis of the tangent space to $\mathbb{S}$ by taking

$$\boldsymbol{f}_1=\boldsymbol{w}_1,\quad \boldsymbol{f}_2=\boldsymbol{w}_2\quad \text{if}\quad |u^1|\geq|u^2|,$$

and

$$\boldsymbol{f}_1=\boldsymbol{w}_2,\quad \boldsymbol{f}_2=\boldsymbol{w}_1\quad \text{if}\quad |u^1|<|u^2|.$$

We denote by $\mathbf{C}\equiv(C^\alpha{}_\beta)$ the change of basis matrix relating the bases $\{\boldsymbol{\tau}_1,\boldsymbol{\tau}_2\}$ and $\{\boldsymbol{f}_1,\boldsymbol{f}_2\}$, viz.,

$$\boldsymbol{f}_\beta=C^\alpha{}_\beta\boldsymbol{\tau}_\alpha,\quad \beta\in\{1,2\}. \tag{83}$$

Finally, recalling (17), in view of (82.1), $p = O(\epsilon)$ and $\mathbf{M} = \mathbf{M}_0 + O(\epsilon)$, we have

$$(T^{\alpha\beta})=-p\,\mathbf{M}_0+\mathbf{P}_E+O(\epsilon^2)=O(\epsilon),$$

and using (8) and (82.2, 3), the (symmetric) stress tensor $\boldsymbol{\sigma}$ emerges as

$$\begin{aligned}\boldsymbol{\sigma}&=T^{ij}\boldsymbol{g}_i\otimes\boldsymbol{g}_j=T^{\alpha\beta}(\delta^\gamma{}_\alpha-\xi W^\gamma{}_\alpha)(\delta^\sigma{}_\beta-\xi W^\sigma{}_\beta)\,\boldsymbol{\tau}_\gamma\otimes\boldsymbol{\tau}_\sigma\\&\quad+T^{\alpha3}(\boldsymbol{g}_\alpha\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{g}_\alpha)+T^{33}\boldsymbol{n}\otimes\boldsymbol{n}=T^{\gamma\sigma}\boldsymbol{\tau}_\gamma\otimes\boldsymbol{\tau}_\sigma-p\,\boldsymbol{n}\otimes\boldsymbol{n}+O(\epsilon^{1+\gamma}).\end{aligned}$$

In the above, $(T^{\alpha\beta})=-p\,\mathbf{M}_0+\mathbf{P}_E$ is understood, and thus we have

$$\overline{\boldsymbol{\sigma}}=\overline{T}^{\gamma\sigma}\boldsymbol{\tau}_\gamma\otimes\boldsymbol{\tau}_\sigma-\overline{p}\,\boldsymbol{n}\otimes\boldsymbol{n}+O(\epsilon^{1+\gamma}),\quad (\overline{T}^{\alpha\beta})=-\overline{p}\,\mathbf{M}_0+\overline{\mathbf{P}}_E. \tag{84.1, 2}$$

[6] For $\lambda \in (0,1)$ we have $\eta_0=\eta(0)$ and $\eta_\infty=\lim_{\dot\gamma\to\infty}\eta(\dot\gamma)$, but for $\lambda>1$ the notation should be modified, since $\eta_0=\lim_{\dot\gamma\to\infty}\eta(\dot\gamma)$ and $\eta_\infty=\lim_{\dot\gamma\to0}\eta(\dot\gamma)$.

Expressions (84.1) and (83) show that the mean stress $\overline{\boldsymbol{\sigma}}$ can also be written in terms of the basis $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{n}\}$ as

$$\overline{\boldsymbol{\sigma}} = \widetilde{P}^{\alpha\beta} \boldsymbol{f}_\alpha \otimes \boldsymbol{f}_\beta - \overline{p}\, \boldsymbol{n} \otimes \boldsymbol{n} + O(\epsilon^{1+\gamma}),$$

with the 2×2 matrices $(\overline{T}^{\alpha\beta})$ and $(\widetilde{P}^{\alpha\beta}) \equiv \widetilde{\mathbf{P}}$ related by

$$(\overline{T}^{\alpha\beta}) = \mathbf{C}\widetilde{\mathbf{P}}\mathbf{C}^T. \tag{85}$$

Combining (84.2) and (85) we obtain

$$\overline{\mathbf{P}}_E = \overline{p}\,\mathbf{M}_0 + \mathbf{C}\,\widetilde{\mathbf{P}}\mathbf{C}^T. \tag{86}$$

We now make the following ad hoc “constitutive” assumption of Savage–Hutter type:

$$\widetilde{\mathbf{P}} = -\overline{p} \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \tag{87}$$

where the *earth pressure coefficients* k_1, k_2 are given in terms of

$$\begin{aligned}
k^1_{act} &\equiv \tfrac{2}{\cos^2\varphi} \left\{1 - \sqrt{1 - \sec^2\delta\, \cos^2\varphi}\right\} - 1, \\
k^1_{pass} &\equiv \tfrac{2}{\cos^2\varphi} \left\{1 + \sqrt{1 - \sec^2\delta\, \cos^2\varphi}\right\} - 1, \\
k^2_{act} &\equiv \tfrac{1}{2} \left\{k^1_{act} + 1 - \sqrt{(k^1_{act} - 1)^2 + 4\tan^2\delta}\right\}, \\
k^2_{pass} &\equiv \tfrac{1}{2} \left\{k^1_{pass} + 1 + \sqrt{(k^1_{pass} - 1)^2 + 4\tan^2\delta}\right\}
\end{aligned}$$

as follows:

$$k_1 = \begin{cases} k^1_{act}, & \text{if} \quad \lambda_1 \geq 0 \\ k^1_{pass}, & \text{if} \quad \lambda_1 < 0 \end{cases}, \qquad k_2 = \begin{cases} k^2_{act}, & \text{if} \quad \lambda_2 \geq 0 \\ k^2_{pass}, & \text{if} \quad \lambda_2 < 0 \end{cases}. \tag{88}$$

In the above φ is the *internal angle of friction*, $\varphi > \delta$, and λ_α denotes the eigenvalue of the mean surface stretching corresponding to $\boldsymbol{f}_\alpha$, $\alpha = 1, 2$. Inserting (87) into (86), we obtain

Proposition 12 *The shallow avalanche equations of Savage–Hutter type are (49)–(51), in which the mean shear stress* $\overline{\mathbf{p}}_E$ *is omitted, the basal shear stress* $\mathbf{p}_E|_{\xi=0}$ *is given by the Coulomb friction law (62) with* $\phi = 1$, $\tan\delta = O(\epsilon^\gamma)$*, and the mean stress* $\overline{\mathbf{P}}_E$ *is given by*

$$\overline{\mathbf{P}}_E = \overline{p}\,\mathbf{M}_0 - \overline{p}\,\mathbf{C} \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \mathbf{C}^T,$$

with the matrix $\mathbf{C}$ *and the earth pressure coefficients* k_1, k_2 *defined in (83) and (88), respectively.*

We finally note that, representing $\boldsymbol{D}_{\mathcal{S}}\overline{\boldsymbol{u}}_\tau$ as

$$\boldsymbol{D}_{\mathcal{S}}\overline{\boldsymbol{u}}_\tau = \mathcal{D}^\alpha{}_\beta \boldsymbol{\tau}_\alpha \otimes \boldsymbol{\tau}^\beta,$$

the matrix $(\mathcal{D}^\alpha{}_\beta)$ is given by (B.4) in Appendix B, in which $\mathbf{u}$ is taken as

$$\mathbf{u} = (\mathbf{I} - \tfrac{1}{2} h m_1 \mathbf{W})\overline{\mathbf{v}},$$

since $\overline{\boldsymbol{u}}_\tau$ emerges as (48). The matrix $(\mathcal{D}^\alpha{}_\beta)$ is needed to determine the eigenvalues and the eigenvectors of $\boldsymbol{D}_{\mathcal{S}}\overline{\boldsymbol{u}}_\tau$.

9 Conclusions

In this paper we have presented a general derivation of model equations for thin film flows of a finite mass of a material that can be described by the dynamical equations of balances of mass and linear momentum of a one constituent continuum.

One essential idea is to formulate the governing equations in topography adjusted coordinates across which the avalanche mass is moving. This is prerequisite to introduce the shallowness approximation in an objective form. Gravity driven flows are generally thin, and this shallowness ought to be measured by an aspect ratio involving the thickness to length ratio *relative to the given topography*. Most numerical models of landslides and debris movements measure thicknesses vertically and extents horizontally, and thus may violate shallowness orderings in steep topographies.

A second important idea pursued in this paper is the recognition that gravity driven flows of debris may occur in various different forms; they may thus be described by different rheological postulates, and stresses that develop inside the moving mass may largely differ in magnitude from one event to another. So, depending on the numerical values of the stress components some of these components may be ignored in a thin-layer model. As a consequence, the resulting avalanche model may have a simplified rheological complexity, perhaps so simple that the rheological information contained in the original model is completely or partly lost. This then naturally leads to the question whether avalanche models that are characterized by the shallowness property can be classified according to the rheological properties of the underlying moving material. The answer to this question was given in this paper and it sheds light on some controversial but unsolved discussions regarding the correct avalanche model.

Indeed, within the context of shallow geometries we have shown that, depending on the order of magnitudes of the stress components on an element with surfaces perpendicular and tangential to the basal topography, four classes of avalanche models can be defined.[7]

(i) When resistance to shearing and sliding may occur, but is small, only the constraint pressure due to incompressibility survives (see Proposition 5). The material responds in a shallow avalanche theory as if it were an incompressible inviscid fluid, no matter what rheological postulate one may have formulated for the material behaviour initially. There is no basal shear stress, and material properties that differ from an incompressible ideal fluid are only evidenced in the Boussinesq coefficients characterizing the non-uniformity of the tangential velocity components through depth. We know of no model in practice that would be so simple.

(ii) For slightly larger resistance to shearing, only the basal shear stress is relevant, and hence, again with basal sliding, the frictional properties of the contact surface are important. These may be formulated by a dry Coulomb-type friction law and/or a viscous sliding law—depending on whether the contact properties are due to rubbing of particles over the surface or due to particles bouncing at the wall leading to viscous friction or both, see Proposition 8. In this case the rheological properties of the fluid comprising the avalanching mass enter via the Boussinesq coefficients, the friction angle and/or the drag coefficient only. This complexity of avalanche models has been very popular in the past. The famous Voellmy snow avalanche model [33] and its extensions Perla et al. [24], Salm [30] are of this type as are all early Russian snow avalanche models [8,11,29]. These models experienced recently a revival in the computational avalanche literature by Gray et al. [10] and Bouchut and collaborators [3,4], however, with justification of their use only by Bouchut et al.

(iii) If the resistance against shearing is even larger, as may be the case for a dense slurry with particle diameters that are smaller than the roughness scales at the basal surface, then the rheological properties of the fluid play a more significant role in the model, by also entering it via the depth-averaged value of the shear stresses on planes parallel to the basal surface. With the no slip condition at the basal surface, the depth-averaged shear stresses and the basal shear stresses may then be computed with an assumed velocity profile (having no slip) and a given stress stretching relation, which we have explicitly shown for a non-Newtonian fluid. Dense clay suspensions in water belong to this class. Furthermore, creeping flow of a non-linearly viscous fluid is also a typical example for this. Geophysical applications are the motion of cold ice in ice sheets or of soil in rock glaciers, however, in ice sheets the temperature variation cannot be ignored as was done in our case, and the temperature at the base must be below melting to avoid sliding. The temperature dependence can be mimicked by choosing a representative depth dependence for it in the expression for the apparent viscosity.

(iv) When the depth-integrated normal and shear stresses acting on planes perpendicular to the basal surface are large, then the most complex avalanche model ensues in which the rheological properties of the fluid within the avalanching mass play a role. Such situations prevail, when stress anisotropies or normal stress differences are important or when the base-parallel distortions are large. The former arise, e.g., in all earth

[7] See also Sect. 1.

pressure models in which the normal stresses parallel to the basal surface are expressed in terms of the normal stress perpendicular to it with an earth pressure coefficient that is either invariant against rotations perpendicular to the basal surface [15,17,18] or direction dependent according to the sign of the stretching (Savage–Hutter, see Sect. 8.2). In the formulations of Iverson [15] and McDougall and Hungr [17,18] the shear stress on planes perpendicular to the basal surface is also parameterized with an earth pressure coefficient.

The results make clear that the exact form of the avalanche equations depends on the relative weights of the stresses versus the aspect ratio of the shallowness approximation. Class (i) avalanches are unlikely, but class (ii) avalanches are among the most often applied models. They enjoy the property of sensitivity to the rheological properties of the fluid comprising the moving mass via the Boussinesq coefficients mainly, and they teach us that all attention must be focussed at the parametrization of the basal friction law. Within this class, it is almost vain to argue about the rheology (constitutive equation) of the fluid. Within the class (iii), however, it is exactly reverse if the no slip condition is assumed; here, all information comes from the constitutive properties of the material comprising the moving avalanche. The most complex situation prevails for class (iv) models. Here, generally both the rheological properties of the moving mass as well as its interaction with the basal surface via the sliding law are important. This enlarged complexity must be considered in all those cases when the dynamics of the avalanche must be modelled from its inception at the initial position at rest through the catastrophic motion to (and including) its deposit. A convincing demonstration for this is given in Fig 10.35 of [29] that shows the computed avalanche geometry for an ideal fluid and its comparison with experimental results. They do not agree at all. The reason is as follows: as a granular avalanche moves into a deposit, large basal-parallel normal stresses are active, which via the earth pressure coefficients give rise to relatively large anisotropic stress effects which are dominant in this final stretch of motion.

Appendix A: Proof of Proposition 4

Let us first prove (49). To this end we use the normal momentum balance (25), which with the aid of (36.2) and of the ordering approximations in Sect. 5, turns into

$$\underbrace{\frac{\partial}{\partial t}(J\mathrm{v})}_{O(\epsilon)} + \underbrace{\mathrm{Div}\,(J\,\mathrm{v}\mathbf{v} - \mathbf{p}_E)}_{O(\epsilon^\gamma)} + \underbrace{\frac{\partial}{\partial \xi}\left(J\,(\mathrm{v}^2 - T_E^{33})\right)}_{O(\epsilon^\gamma)} + \underbrace{J\,\frac{\partial p}{\partial \xi}}_{O(1)} + \underbrace{J\Gamma(\mathbf{P}_E)}_{O(\epsilon)} = \underbrace{-Jc + J\Gamma(\mathbf{v})}_{O(1)},$$

and hence[8]

$$\frac{\partial p}{\partial \xi} = -c + \Gamma(\mathbf{v}) + O(\epsilon^\gamma). \tag{A.1}$$

Definitions (22.2) and (6.2) of $\Gamma(\mathbf{v})$ and $\mathbf{B}$, respectively, give

$$\Gamma(\mathbf{v}) = -\mathbf{H}\cdot(\mathbf{v}\otimes\mathbf{v}) + O(\epsilon),$$

and thus, integrating (A.1) from ξ to h and accounting for (28), we deduce

$$p = c(h-\xi) + \mathbf{H}\cdot\int_{\xi}^{h(\boldsymbol{\xi},t)} \mathbf{v}\otimes\mathbf{v}\,ds + O(\epsilon^{1+\gamma}). \tag{A.2}$$

If $\mathbf{p}_E = O(\epsilon)$, $T_E^{33} = O(\epsilon^2)$, in the above the term $O(\epsilon^{1+\gamma})$ can be replaced by $O(\epsilon^2)$, since now the neglected terms in (A.1) are $O(\epsilon)$. Then, noticing that in view of (37.3) we have

$$\int_0^{h(\boldsymbol{\xi},t)}\left(\int_{\xi}^{h(\boldsymbol{\xi},t)} \mathbf{v}(s)\otimes\mathbf{v}(s)\,ds\right)d\xi = \int_0^{h(\boldsymbol{\xi},t)}\left(\int_0^{s} \mathbf{v}(s)\otimes\mathbf{v}(s)\,d\xi\right)ds$$

$$= \int_0^{h(\boldsymbol{\xi},t)} s\mathbf{v}(s)\otimes\mathbf{v}(s)\,ds = \tfrac{1}{2}h^2 m_3\bar{\mathbf{v}}\otimes\bar{\mathbf{v}} + O(\epsilon^{2+\gamma}),$$

[8] In the next approximation procedure the pressure p appears under the integral from 0 to $h(\boldsymbol{\xi},t)$, where the terms of order $O(\epsilon^{1+\gamma})$ will be neglected, and thus the determination of p up to terms of order $O(\epsilon^{1+\gamma})$ suffices.

the mean value of p is shown to be given by (49). For further use we introduce

$$p^\star \equiv c(h-\xi) + \mathbf{H}\cdot \int_{\xi}^{h(\boldsymbol{\xi},t)} \mathbf{v}\otimes\mathbf{v}\, ds,$$

such that

$$p = p^\star + O(\epsilon^{1+\gamma}), \quad \overline{p} = \overline{p^\star} + O(\epsilon^{1+\gamma}), \quad \overline{p^\star} = \tfrac{1}{2}\, h\,(c + a\, m_3).$$

We now refer to the depth-integrated mass balance equation (29). Here we have only to use expression (40) of J and assumption (37.1) in (29) to deduce (50).

Finally, we consider (30) in order to derive (51). The calculations will be performed by neglecting the terms of order $O(\epsilon^{1+\gamma})$ under the integral sign.[9] Since the integral under the time derivative in (30) has been already evaluated when deducing (50), we pass to the second term on the left-hand side of Eq. (30). For $p = p^\star + O(\epsilon^{1+\gamma})$, $p^\star = O(\epsilon)$, $\mathbf{P}_E = O(\epsilon)$ and $\mathbf{M} = \mathbf{M}_0 + O(\epsilon)$, with the aid of (40) we have

$$J\,(\mathbf{v}\otimes\mathbf{v} + p\,\mathbf{M} - \mathbf{P}_E) = J_0(1-2\Omega\xi)\mathbf{v}\otimes\mathbf{v} + J_0 p^\star \mathbf{M}_0 - J_0\mathbf{P}_E + O(\epsilon^{1+\gamma}),$$

which, combined with (37.2, 3), implies

$$\int_0^{h(\boldsymbol{\xi},t)} J\,(\mathbf{v}\otimes\mathbf{v} + p\,\mathbf{M} - \mathbf{P}_E)\, d\xi = J_0 h\left\{(m_2 - \Omega h m_3)\overline{\mathbf{v}}\otimes\overline{\mathbf{v}} + \overline{p^\star}\,\mathbf{M}_0 - \overline{\mathbf{P}}_E\right\} + O(\epsilon^{2+\gamma}). \tag{A.3}$$

We pass to the next two terms on the left-hand side of Eq. (30): $(J\,\mathbf{p}_E)|_{\xi=0}$ can be written as

$$(J\,\mathbf{p}_E)|_{\xi=0} = J_0\;\mathbf{p}_E|_{\xi=0}\,, \tag{A.4}$$

and the term evaluated at $\xi = h$ is $O(\epsilon^{2+\gamma})$, in view of assumptions (28) and (39).

Then, the third (and last) integral on the left-hand side of (30) will be evaluated using (19), (20), (22.1), (6.2) and (10.2), $p = p^\star + O(\epsilon^{1+\gamma})$, $p^\star = O(\epsilon)$, $\mathbf{P}_E = O(\epsilon)$, $\mathbf{p}_E = O(\epsilon^\gamma)$, as follows:

$$\Gamma(-p\,\mathbf{M}, \mathbf{0}) = p^\star\left\{\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial\xi^\alpha}\,\mathbf{M}_0\mathbf{e}_\alpha + 2\Omega\mathbf{F}^{-1}\mathbf{s}\right\} + O(\epsilon^{1+\gamma}),$$
$$\Gamma(\mathbf{P}_E, \mathbf{p}_E) = -\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial\xi^\alpha}\,\mathbf{P}_E\mathbf{e}_\alpha + 2\mathbf{W}\mathbf{p}_E - (\mathbf{H}\cdot\mathbf{P}_E)\mathbf{F}^{-1}\mathbf{s} + O(\epsilon^{1+\gamma}).$$

Thus, with $J = J_0 + O(\epsilon)$, we obtain

$$\int_0^{h(\boldsymbol{\xi},t)} J\{\Gamma(-p\,\mathbf{M},\mathbf{0}) + \Gamma(\mathbf{P}_E,\mathbf{p}_E)\}\, d\xi = J_0 h\,\overline{p^\star}\,\mathbf{F}^{-1}\left\{\frac{\partial\mathbf{F}}{\partial\xi^\alpha}\,\mathbf{M}_0\mathbf{e}_\alpha + 2\Omega\mathbf{s}\right\}$$
$$- J_0 h\left\{\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial\xi^\alpha}\,\overline{\mathbf{P}}_E\,\mathbf{e}_\alpha - 2\mathbf{W}\overline{\mathbf{p}}_E + (\mathbf{H}\cdot\overline{\mathbf{P}}_E)\mathbf{F}^{-1}\mathbf{s}\right\} + O(\epsilon^{2+\gamma}). \tag{A.5}$$

Now we pass to the terms on the right-hand side of Eq. (30). First, using (36), (40) and

$$\mathbf{B}^{-1} = (\mathbf{I} - \xi\mathbf{W})^{-1}\mathbf{F}^{-1} = (\mathbf{I} + \xi\mathbf{W})\,\mathbf{F}^{-1} + O(\epsilon^2), \tag{A.6}$$

we have

$$J\mathbf{b} = -J_0 c\,\{(1-2\Omega\xi)\mathbf{I} + \xi\mathbf{W}\}\,\mathbf{F}^{-1}\mathbf{s} + O(\epsilon^2),$$

[9] The only neglected terms of order $O(\epsilon^{1+\gamma})$ are due to p, and $\mathbf{p}_E$ in $\boldsymbol{\Gamma}(\mathbf{P}_E, \mathbf{p}_E)$; all the other neglected terms are $O(\epsilon^2)$.

whence

$$\int_0^{h(\boldsymbol{\xi},t)} J\,\mathbf{b}\,d\xi = -J_0 hc\left\{(1-\Omega h)\mathbf{I}+\tfrac{1}{2}h\mathbf{W}\right\}\mathbf{F}^{-1}\mathbf{s}+O(\epsilon^3). \tag{A.7}$$

Next, we use (40), (A.6) and $\mathrm{v} = O(\epsilon)$ to obtain

$$\begin{aligned}
J\Gamma(\mathbf{v})\mathbf{B}^{-1}\mathbf{s} &= -J_0\Big\{(1-2\Omega\xi)(\mathbf{H}\cdot(\mathbf{v}\otimes\mathbf{v}))\mathbf{I}+\xi(\mathbf{H}\cdot(\mathbf{v}\otimes\mathbf{v}))\mathbf{W}\\
&\qquad -\xi(\mathbf{W}^T\mathbf{H}\cdot(\mathbf{v}\otimes\mathbf{v}))\mathbf{I}\Big\}\mathbf{F}^{-1}\mathbf{s}+O(\epsilon^2),\\
J\mathrm{v}\mathbf{B}^{-1}\mathbf{F}\mathbf{W} &= J_0\mathrm{v}\mathbf{W}+O(\epsilon^2),\\
J\mathbf{B}^{-1}\frac{\partial\mathbf{B}}{\partial\xi^\alpha} &= J_0\left\{(1-2\Omega\xi)\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial\xi^\alpha}-\xi\left(\mathbf{F}^{-1}\frac{\partial}{\partial\xi^\alpha}(\mathbf{F}\mathbf{W})-\mathbf{W}\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial\xi^\alpha}\right)\right\}+O(\epsilon^2)\\
&= J_0\left\{(1-2\Omega\xi)\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial\xi^\alpha}-\xi\mathbf{F}^{-1}\frac{\partial}{\partial\xi^\alpha}(\mathbf{F}\mathbf{W}\mathbf{F}^{-1})\mathbf{F}\right\}+O(\epsilon^2).
\end{aligned}$$

Then, by appeal to (37) and (38), with $a\equiv\mathbf{H}\overline{\mathbf{v}}\cdot\overline{\mathbf{v}}$, $\tilde{a}\equiv\mathbf{H}\overline{\mathbf{v}}\cdot\mathbf{W}\overline{\mathbf{v}}$ we arrive at

$$\begin{aligned}
\int_0^{h(\boldsymbol{\xi},t)} J\,\Gamma(\mathbf{v},\mathrm{v})\,d\xi &= -\,J_0h(m_2-\Omega h\,m_3)\mathbf{F}^{-1}\frac{\partial\mathbf{F}}{\partial\xi^\alpha}(\overline{\mathbf{v}}\otimes\overline{\mathbf{v}})\,\mathbf{e}_\alpha\\
&\quad+\frac{1}{2}J_0h^2m_3\mathbf{F}^{-1}\frac{\partial}{\partial\xi^\alpha}(\mathbf{F}\mathbf{W}\mathbf{F}^{-1})\mathbf{F}\,(\overline{\mathbf{v}}\otimes\overline{\mathbf{v}})\,\mathbf{e}_\alpha+J_0h^2\beta\mathbf{W}\overline{\mathbf{v}}\\
&\quad-J_0h\left\{(a\,m_2-a\Omega h\,m_3-\tfrac{1}{2}\tilde{a}h\,m_3)\mathbf{I}+\tfrac{1}{2}ah\,m_3\mathbf{W}\right\}\mathbf{F}^{-1}\mathbf{s}+O(\epsilon^{2+\gamma}).
\end{aligned}\tag{A.8}$$

Finally, collecting (A.3)–(A.5), (A.7), (A.8) in (30), we obtain the desired result (51). □

Appendix B: The surface gradient in curvilinear coordinates

Let $\boldsymbol{u}$ be a tangent vector field to a surface $\mathcal{S}$ given by the parametrization $x_k = x_k(\xi^1,\xi^2)$, $k = 1, 2, 3$. Representing $\boldsymbol{u}$ with respect to the natural basis $\{\boldsymbol{\tau}_1,\boldsymbol{\tau}_2\}$ of the tangent space to $\mathcal{S}$ as

$$\boldsymbol{u} = u^\alpha\boldsymbol{\tau}_\alpha,$$

the *surface gradient* $\nabla_{\mathcal{S}}\boldsymbol{u}$ is the second order tensor on the tangent space to $\mathcal{S}$ defined as

$$\nabla_{\mathcal{S}}\boldsymbol{u}\equiv u^\alpha{}_{;\beta}\boldsymbol{\tau}_\alpha\otimes\boldsymbol{\tau}^\beta,$$

where

$$u^\alpha{}_{;\beta}\equiv\frac{\partial u^\alpha}{\partial\xi^\beta}+\Gamma^\alpha_{\gamma\beta}u^\gamma,\quad \Gamma^\alpha_{\gamma\beta}\equiv\phi^{\alpha\sigma}\frac{\partial^2x_i}{\partial\xi^\beta\partial\xi^\gamma}\frac{\partial x_i}{\partial\xi^\sigma},\quad \phi^{\alpha\sigma}\equiv\boldsymbol{\tau}^\alpha\cdot\boldsymbol{\tau}^\sigma, \tag{B.1.1–3}$$

with $\{\boldsymbol{\tau}^1,\boldsymbol{\tau}^2\}$ the reciprocal basis of $\{\boldsymbol{\tau}_1,\boldsymbol{\tau}_2\}$. The symmetric part of the surface gradient $\nabla_{\mathcal{S}}\boldsymbol{u}$ is

$$\boldsymbol{D}_{\mathcal{S}}\boldsymbol{u}\equiv\tfrac{1}{2}\left(\nabla_{\mathcal{S}}\boldsymbol{u}+\nabla_{\mathcal{S}}\boldsymbol{u}^T\right).$$

Our aim is to determine the matrix $(\mathcal{D}^\alpha{}_\beta)$ in the representation

$$\boldsymbol{D}_{\mathcal{S}}\boldsymbol{u}=\mathcal{D}^\alpha{}_\beta\boldsymbol{\tau}_\alpha\otimes\boldsymbol{\tau}^\beta,$$

for the case of parametrization (1) of the surface $\mathcal{S}$, see Sect. 2. To this end we note that

$$\mathcal{D}^\alpha{}_\beta=\tfrac{1}{2}\left\{u^\alpha{}_{;\beta}+\phi^{\alpha\sigma}u^\gamma{}_{;\sigma}\,\phi_{\gamma\beta}\right\},$$

which implies the matrix equality

$$(\mathcal{D}^{\alpha}{}_{\beta}) = \frac{1}{2}\left\{(u^{\alpha}{}_{;\beta}) + \mathbf{M}_0 (u^{\alpha}{}_{;\beta})^T \mathbf{M}_0^{-1}\right\}, \tag{B.2}$$

see (3), and therefore we have to determine $(u^{\alpha}{}_{;\beta})$.

Thus, we first notice that (B.1.1) gives the matrix equality

$$(u^{\alpha}{}_{;\beta}) = \left(\frac{\partial u^{\alpha}}{\partial \xi^{\beta}}\right) + \left(\Gamma^{\alpha}_{\gamma\beta}\, u^{\gamma}\right) = \operatorname{Grad}\mathbf{u} + u^{\gamma}\overset{(\gamma)}{\mathbf{\Gamma}}, \tag{B.3}$$

where the 2-column $\mathbf{u}$ and the 2×2 matrix $\overset{(\gamma)}{\mathbf{\Gamma}}$ are introduced by

$$\mathbf{u} \equiv (u^1, u^2)^T, \quad \overset{(\gamma)}{\mathbf{\Gamma}} \equiv \left(\Gamma^{\alpha}_{\gamma\beta}\right)_{\alpha,\beta\in\{1,2\}}.$$

To deduce the components $\Gamma^{\alpha}_{\gamma\beta}$ of $\overset{(\gamma)}{\mathbf{\Gamma}}$ we use (B.1.2), (1), (2) and (3.2). We therefore have

$$\Gamma^{\alpha}_{\gamma\beta} = \phi^{\alpha\sigma}\sum_{i=1}^{2}\frac{\partial F_{i\beta}}{\partial \xi^{\gamma}}F_{i\sigma} + \phi^{\alpha\sigma}\frac{\partial}{\partial \xi^{\gamma}}\left(\frac{\partial b}{\partial \xi^{\beta}}\right)\frac{\partial b}{\partial \xi^{\sigma}},$$

whence

$$\overset{(\gamma)}{\mathbf{\Gamma}} = \mathbf{M}_0\mathbf{F}^T\frac{\partial \mathbf{F}}{\partial \xi^{\gamma}} + \mathbf{M}_0 \operatorname{Grad} b \otimes \frac{\partial}{\partial \xi^{\gamma}}(\operatorname{Grad} b).$$

Now, with

$$\operatorname{Grad} b = \frac{1}{c}\,\mathbf{F}^T\mathbf{s}, \quad \operatorname{Grad} c = -c\mathbf{H}\mathbf{F}^{-1}\mathbf{s}, \quad \operatorname{Grad}\mathbf{s} = \mathbf{F}\mathbf{W},$$

see Luca et al. [16] (we recall that the matrix $\mathbf{H}$ is denoted by $\widetilde{\mathbf{H}}$ in [16]), after a routine calculation involving (3.3), (10.3), we obtain

$$\mathbf{M}_0 \operatorname{Grad} b = c\mathbf{F}^{-1}\mathbf{s}, \quad \frac{\partial}{\partial \xi^{\gamma}}(\operatorname{Grad} b) = \frac{1}{c}\,\mathbf{H}\mathbf{e}_{\gamma} + \frac{1}{c}\frac{\partial \mathbf{F}^T}{\partial \xi^{\gamma}}\,\mathbf{s},$$

and hence

$$\overset{(\gamma)}{\mathbf{\Gamma}} = \mathbf{F}^{-1}\left(\frac{\partial \mathbf{F}}{\partial \xi^{\gamma}} + (\mathbf{s}\otimes\mathbf{e}_{\gamma})\,\mathbf{H}\right).$$

Next, substituting the above expression of $\overset{(\gamma)}{\mathbf{\Gamma}}$ into (B.3) and noticing that $\partial F_{i\sigma}/\partial\xi^{\gamma} = \partial F_{i\gamma}/\partial\xi^{\sigma}$ implies

$$u^{\gamma}\frac{\partial \mathbf{F}}{\partial \xi^{\gamma}} = \frac{\partial \mathbf{F}}{\partial \xi^{\gamma}}\,\mathbf{u}\otimes\mathbf{e}_{\gamma},$$

we arrive at

$$(u^{\alpha}{}_{;\beta}) = \operatorname{Grad}\mathbf{u} + \mathbf{F}^{-1}\left(\frac{\partial \mathbf{F}}{\partial \xi^{\gamma}}\,\mathbf{u}\otimes\mathbf{e}_{\gamma} + (\mathbf{s}\otimes\mathbf{u})\mathbf{H}\right).$$

Finally, substituting this expression into (B.2) we obtain the desired result, viz.,

$$\begin{aligned}(\mathcal{D}^{\alpha}{}_{\beta}) = \tfrac{1}{2}\Bigg\{&\operatorname{Grad}\mathbf{u} + \mathbf{F}^{-1}\left(\frac{\partial \mathbf{F}}{\partial \xi^{\gamma}}\,\mathbf{u}\otimes\mathbf{e}_{\gamma} + (\mathbf{s}\otimes\mathbf{u})\mathbf{H}\right)\\ &+ \mathbf{M}_0\left[\operatorname{Grad}\mathbf{u} + \mathbf{F}^{-1}\left(\frac{\partial \mathbf{F}}{\partial \xi^{\gamma}}\,\mathbf{u}\otimes\mathbf{e}_{\gamma} + (\mathbf{s}\otimes\mathbf{u})\mathbf{H}\right)\right]^T\mathbf{M}_0^{-1}\Bigg\}.\end{aligned} \tag{B.4}$$

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