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Calculus of Variations

Unstable extremal surfaces of the "Shiffman functional" spanning rectifiable boundary curves

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Abstract In this paper we derive a sufficient condition for the existence of extremal surfaces of a parametric functional $\mathcal{J}$ with a dominant area term, which *do not* furnish global minima of $\mathcal{J}$ within the class $\mathcal{C}^*(\Gamma)$ of $H^{1,2}$-surfaces spanning an arbitrary closed rectifiable Jordan curve $\Gamma \subset \mathbb{R}^3$ that merely has to satisfy a chord-arc condition. The proof is based on the "*mountain pass* result" of (Jakob in Calc Var 21:401–427, 2004) which yields an unstable $\mathcal{J}$-extremal surface bounded by an arbitrary simple closed polygon and Heinz' "approximation method" in (Arch Rat Mech Anal 38:257–267, 1970). Hence, we give a precise proof of a partial result of the *mountain pass* theorem claimed by Shiffman in (Ann Math 45:543–576, 1944) who only outlined a very sketchy and partially incorrect proof.

1 Introduction and main result

Shiffman considered Plateau's problem for the two-dimensional parametric functional

$$\mathcal{J}(X) := \int_B F(X_u \wedge X_v) + k \mid X_u \wedge X_v \mid \mathrm{d}u\mathrm{d}v =: \mathcal{F}(X) + k\,\mathcal{A}(X),$$

on surfaces $X \in H^{1,2}(B,\mathbb{R}^3)$ of the type of the open disc $B := B_1^2(0) \subset \mathbb{R}^2$. The Lagrangian F is assumed to satisfy the following list of requirements (A*):

$$F \in C^0(\mathbb{R}^3) \cap C^2(\mathbb{R}^3 \setminus \{0\}), \tag{1}$$

$$F(tz) = t\,F(z) \quad \forall t \geq 0,\ \forall z \in \mathbb{R}^3, \tag{2}$$

$$m_1 \mid z \mid \leq F(z) \leq m_2 \mid z \mid \quad \forall z \in \mathbb{R}^3,\ 0 < m_1 \leq m_2, \tag{3}$$

$$F - \lambda \mid \cdot \mid \ \text{has to be convex on } \mathbb{R}^3, \text{ for some } \lambda > 0. \tag{4}$$

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As in [6,8] we will also consider integrands F that have to be only convex, i.e. that have to satisfy the list of requirements

$$(A) := \text{requirements (1)–(3) and convexity on } \mathbb{R}^3,$$

but eventually have to satisfy the additional requirement (R*):
The restriction of the function $g(z) := F(z) + F(-z)$ to the $\mathbb{S}^2$ is assumed to have three linearly independent critical points, i.e. there have to be at least three linearly independent unit vectors $a_1, a_2, a_3 \in \mathbb{S}^2$ such that $\nabla g(a_j) = r_j\, a_j^\top$, for some $r_j \in \mathbb{R}$, $j = 1, 2, 3$. Finally we assume as in [6] and [8] that

$$k > \max_{\mathbb{S}^2} F = m_2. \tag{5}$$

Thus $\mathcal{J}$ is a controlled perturbation of the area functional $\mathcal{A}$, where F depends only on the normal $X_u \wedge X_v$, but not on the position vector X itself. Moreover with respect to some closed rectifiable Jordan curve $\Gamma \subset \mathbb{R}^3$ we consider the Plateau class $\mathcal{C}^*(\Gamma)$ of surfaces $X \in H^{1,2}(B, \mathbb{R}^3)$ whose L^2-traces $X\,|_{\partial B}$ are continuous, monotonic mappings of $\mathbb{S}^1$ onto Γ satisfying a three-point condition:

$$X\,|_{\partial B}\,(\mathrm{e}^{\mathrm{i}\psi_k}) \overset{!}{=} P_k, \quad \psi_k := \frac{2\pi k}{3}, \quad k = 0, 1, 2, \tag{6}$$

where P_0, P_1, P_2 are three fixed points on Γ. Furthermore we topologize $\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3)$ by the $C^0(\bar{B}, \mathbb{R}^3)$-norm. Only assuming the requirements (A*) on the integrand F we are going to prove (see Definitions 4.2 and 4.3 in Sect. 4.2 and Definition 3.5 in [6])

Theorem 1.1 (Main result) *Let Γ be an arbitrary closed rectifiable Jordan curve in $\mathbb{R}^3$ satisfying a chord-arc condition* (57). *If there exist two different conformally parametrized surfaces $X_1 \neq X_2$ in $(\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3), \|\cdot\|_{C^0(\bar{B})})$ which are in a mountain pass situation w. r. to $\mathcal{J}$ with some elevation $e > 0$, then there exists a $\mathcal{J}$-extremal surface X^* in $\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3)$ with* $\mathcal{J}(X^*) \geq \max\{\mathcal{J}(X_1), \mathcal{J}(X_2)\} + \frac{e}{4} > \inf_{\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3)} \mathcal{J}$.

Following Shiffman we replace $\mathcal{J}$ by its *dominance* functional

$$\mathcal{I}(X) := \int_B F(X_u \wedge X_v) + \frac{k}{2}\,|\,DX\,|^2 \,\mathrm{d}u\mathrm{d}v = \mathcal{F}(X) + k\,\mathcal{D}(X).$$

Now a crucial tool which allows a derivation of the above theorem from the *mountain pass* result in [6] is the following compactness result of [8] for minimizers of $\mathcal{I}$, whose integrand F has to satisfy the requirements (A) and (R*), within boundary value classes $H^{1,2}_{\varphi}(B, \mathbb{R}^3)$, termed $\mathcal{I}$-surfaces (see Theorem 1.2 and Definition 1.1 in [8] for the notion "md"):

Theorem 1.2 *Let F be an integrand satisfying the requirements (A) and (R*). Let moreover $\{X^n\}$ be a sequence of $\mathcal{I}$-surfaces with $\mathcal{D}(X^n) \leq$ const., $\forall\, n \in \mathbb{N}$, and with equicontinuous and uniformly bounded boundary values. Then there exists a subsequence $\{X^{n_j}\}$ such that*

$$X^{n_j} \longrightarrow \bar{X} \quad \text{in } C^0(\bar{B}, \mathbb{R}^3) \quad \text{and} \quad X^{n_j} \rightharpoonup \bar{X} \quad \text{in } H^{1,2}(B, \mathbb{R}^3),$$

for a surface $\bar{X} \in H^{1,2}(B, \mathbb{R}^3) \cap C^0(\bar{B}, \mathbb{R}^3)$ with $\mathrm{md}((A\,\bar{X})_i) = 0$, $i = 1, 2, 3$.

Furthermore we show in Theorem 2.2 that such a limit surface $\bar{X}$ is an $\mathcal{I}$-surface again. It should be emphasized here that Shiffman asserted the wrong statement that the restriction of any even C^1-function to the $\mathbb{S}^2$ would possess three linearly independent critical points (see p. 552 in [12]), which would allow us to drop the unpleasant requirement (R*) from the very beginning. In fact the author constructed a counterexample of this assertion in Sect. 2 of [8] but proved on the other hand (in this section) that any integrand F that satisfies the requirements (A*) can be "approximated" by a family of Lagrangians $\{F_\epsilon\}_{\epsilon>0}$ meeting the conditions (A)+(R*) for sufficiently small ϵ, which we shall use in the final section of the present paper. In Sect. 4 we will follow the lines of Heinz's article [4] in which he proved an analogue of the mountain pass theorem 4.1 for the H-surface functional instead of $\mathcal{J}$ respectively $\mathcal{I}$ by approximating Γ by a sequence of simple closed polygons and applying his achievements of [3] and the "finite dimensional" mountain pass lemma.

2 $H^{1,2}_{\text{loc}}(B,\mathbb{R}^3)$-convergence and closedness of the set of $\mathcal{I}$-surfaces in $C^0(\bar{B},\mathbb{R}^3)$

In this section we give rigorous proofs of Theorems 10.2 and 10.3 in [12], pp. 558–561, where F is assumed to satisfy only the requirements (A). Throughout the paper we will use the notations $Z := X_u \wedge X_v$, $\delta Z := X_u \wedge \varphi_v + \varphi_u \wedge X_v$ and $\delta^2 Z := \varphi_u \wedge \varphi_v$ for any $X, \varphi \in H^{1,2}(B,\mathbb{R}^3)$,

$$\mathcal{R} := \mathcal{R}(X) := \{(u,v) \in B \mid (X_u \wedge X_v)(u,v) \neq 0\},$$
$$\mathcal{S} := \mathcal{S}(X) := B \setminus \mathcal{R}(X)$$

and $C_{r,\rho} := B_\rho(0) \setminus \overline{B_r(0)}$ for $r < \rho \in (0,1]$. Firstly we prove

Proposition 2.1 *Let $\{Y^n\}$ be a sequence in $H^{1,2}(B,\mathbb{R}^3)$ with $\mathcal{D}(Y^n) \leq \text{const.}$ and let $\{\delta_n\} \subset \mathbb{R}_{>0}$ be some sequence with $\delta_n \to 0$. Setting $r_n := r + \delta_n$ for each $r \in (0,1)$ we prove that*

$$m(r) := \liminf_{n\to\infty} \mathcal{D}_{C_{r,r_n}}(Y^n) = 0 \quad \text{for a.e. } r \in (0,1). \tag{7}$$

Proof We assume that there is some $\epsilon_0 > 0$ such that $P_\epsilon := \{r \in (0,1) \mid m(r) \geq \epsilon\}$ is non-empty for $\epsilon \in (0,\epsilon_0]$, otherwise we are done. We choose some $\epsilon \in (0,\epsilon_0]$ arbitrarily and a collection of finitely many different radii $r^1,\ldots,r^q$ in P_ϵ, where $q \leq \text{card}(P_\epsilon)$ is arbitrarily fixed (which means that we choose $q \in \mathbb{N}$ arbitrarily if P_ϵ should have infinitely many elements). Firstly due to $\delta_n \to 0$ there exists a number N_1 such that $C_{r^i,r^i_n} \cap C_{r^j,r^j_n} = \emptyset \ \ \forall i \neq j, \forall n > N_1$, which implies that

$$\sum_{i=1}^{q} \mathcal{D}_{C_{r^i,r^i_n}}(Y^n) \leq \mathcal{D}(Y^n) \leq \text{const.} =: M, \tag{8}$$

$\forall n > N_1$. Furthermore we can determine a number $N_2 \geq N_1$ such that $\mathcal{D}_{C_{r^i,r^i_n}}(Y^n) \geq \frac{m(r^i)}{2} \geq \frac{\epsilon}{2}\ \forall n > N_2$ and for $i = 1,\ldots,q$ simultaneously. Hence, together with (8) we see that $q\frac{\epsilon}{2} \leq M$, i.e. $q \leq \frac{2M}{\epsilon}$. This shows that $\text{card}(P_\epsilon) \leq \frac{2M}{\epsilon}$. Now every $r \in (0,1)$ with $m(r) > 0$ lies in some set $P_{\frac{1}{n}}$ for some $n > \frac{1}{m(r)}$, i.e. $\mathcal{B} := \{r \in (0,1) \mid m(r) > 0\} \subset$

$\bigcup_{n\in\mathbb{N}} P_{\frac{1}{n}}$ which is a countable set on account of $\mathrm{card}(P_{\frac{1}{n}}) \le 2Mn$, for $n > \frac{1}{\epsilon_0}$, and by $P_{\frac{1}{n}} \subset P_{\frac{1}{n'}}$ for $n \le n'$, thus in particular $\mathcal{L}^1(\mathcal{B}) = 0$. □

For the reader's convenience we recall here that we have by Proposition 3.3, Lemma 4.1 and (8) in [6]:

$$\begin{aligned}\delta^+\mathcal{I}(X,\varphi) &= \delta\mathcal{F}_{\mathcal{R}}(X,\varphi) + \delta^+\mathcal{F}_{\mathcal{S}}(X,\varphi) + k\,\delta\mathcal{D}(X,\varphi)\\ &= \int_{\mathcal{R}} \langle\nabla F(Z), \delta Z\rangle\,\mathrm{d}u\mathrm{d}v + \int_{\mathcal{S}} F(\delta Z)\,\mathrm{d}u\mathrm{d}v + k\int_B DX\cdot D\varphi\,\mathrm{d}u\mathrm{d}v \end{aligned} \tag{9}$$

for any $X, \varphi \in H^{1,2}(B,\mathbb{R}^3)$.

Theorem 2.1 *Let $\{X^n\}$ be a sequence of $\mathcal{I}$-surfaces with $\mathcal{D}(X^n) \le$ const., $\forall\, n \in \mathbb{N}$, and*

$$X^n \longrightarrow \bar{X} \quad \text{in} \quad C^0(\bar{B},\mathbb{R}^3)$$

for some $\bar{X} \in C^0(\bar{B},\mathbb{R}^3)$. Then there holds for every $r \in (0,1)$:

$$\| X^n - \bar{X} \|_{H^{1,2}(B_r(0))} \longrightarrow 0 \quad \text{for} \ \ n \to \infty. \tag{10}$$

Proof Without loss of generality we may assume that $\| \bar{X} - X^n \|_{C^0(\bar{B})} > 0\ \forall\, n \in \mathbb{N}$. We choose some $r \in (0,1)$ arbitrarily such that (7) holds for $Y^n := \bar{X} - X^n$ and $\delta_n := \| \bar{X} - X^n \|_{C^0(\bar{B})}$ and consider the sequence $\{r_n\}$ given by $r_n := r + \delta_n$ (as in (7)). Without loss of generality we may assume that $\{r_n\} \subset (r,1)\ \ \forall\, n \in \mathbb{N}$. By Lemma 2 of Sect. 2.5 in [7], p. 23, the $\mathcal{I}$-surfaces X^n are characterized by the variational inequality

$$\delta^+\mathcal{I}(X^n,\varphi) \ge 0 \quad \forall\, \varphi \in \mathring{H}^{1,2}(B,\mathbb{R}^3), \tag{11}$$

(see (9)) which we are going to test now by

$$\varphi^n(w) := \begin{cases} \bar{X}(w) - X^n(w) & : \ w \in B_r(0) \\ \frac{r_n - |w|}{r_n - r}(\bar{X}(w) - X^n(w)) & : \ w \in \overline{C_{r,r_n}} \\ 0 & : \ w \in C_{r_n,1}. \end{cases}$$

Knowing that $X^n, \bar{X} \in H^{1,2}(B,\mathbb{R}^3)$ one easily verifies that $\varphi^n \in \mathring{H}^{1,2}(B,\mathbb{R}^3)$, $\forall\, n \in \mathbb{N}$, on account of Lemma A 6.9 in [1], p. 254, and by the estimate

$$| D\varphi^n | \le \frac{r_n - |w|}{r_n - r} | D(\bar{X} - X^n) | + \frac{|\bar{X} - X^n|}{r_n - r} \le | D(\bar{X} - X^n) | + 1 \quad on \ \ C_{r,r_n}. \tag{12}$$

We will use the following abbreviations as in Sect. 4 of [6]:

$$Z^n := X^n_u \wedge X^n_v, \quad \delta Z^n := \varphi^n_u \wedge X^n_v + X^n_u \wedge \varphi^n_v, \quad \delta^2 Z^n := \varphi^n_u \wedge \varphi^n_v, \tag{13}$$

and we observe that

$$Z = Z^n + \delta Z^n + \delta^2 Z^n \quad on \ \ B_r(0). \tag{14}$$

Furthermore we define $\mathcal{R}^n := \mathcal{R}(X^n)$ and $\mathcal{S}^n := \mathcal{S}(X^n)$. Firstly we note:

$$\int_{B_\rho(0)} DX^n \cdot D(\bar{X} - X^n)\,\mathrm{d}u\mathrm{d}v = \mathcal{D}_{B_\rho(0)}(\bar{X}) - \mathcal{D}_{B_\rho(0)}(X^n) - \mathcal{D}_{B_\rho(0)}(\bar{X} - X^n)$$

$\forall\, \rho \in (0,1]$. Now combining this with (9), (11) and $F(0) = 0$ we arrive at:

$$0 \le \delta^+\mathcal{I}(X^n,\varphi^n) = \int\limits_{\mathcal{R}^n\cap B_r(0)} \langle \nabla F(Z^n), \delta Z^n\rangle \,\mathrm{d}u\mathrm{d}v + \int\limits_{\mathcal{R}^n\cap C_{r,r_n}} \langle \nabla F(Z^n), \delta Z^n\rangle \,\mathrm{d}u\mathrm{d}v + \int\limits_{\mathcal{S}^n\cap B_r(0)} F(\delta Z^n)\,\mathrm{d}u\mathrm{d}v + \int\limits_{\mathcal{S}^n\cap C_{r,r_n}} F(\delta Z^n)\,\mathrm{d}u\mathrm{d}v + k\,(\mathcal{D}_{B_r(0)}(\bar X) - \mathcal{D}_{B_r(0)}(X^n) - \mathcal{D}_{B_r(0)}(\bar X - X^n)) + k\int\limits_{C_{r,r_n}} DX^n\cdot D\varphi^n\,\mathrm{d}u\mathrm{d}v. \tag{15}$$

As in (9) and (11) of [6] we gain by (14), the convexity of $F \in C^2(\mathbb{R}^3\setminus\{0\})$, $\mid \nabla F \mid \le m_2$ on $\mathbb{R}^3 \setminus \{0\}$ and $\mid \delta^2 Z^n \mid \le \frac{1}{2} \mid D\varphi^n \mid^2$:

$$\mathcal{F}_{\mathcal{R}^n\cap B_r(0)}(\bar X) - \mathcal{F}_{\mathcal{R}^n\cap B_r(0)}(X^n) \ge \int\limits_{\mathcal{R}^n\cap B_r(0)} \langle \nabla F(Z^n), \delta Z^n\rangle\,\mathrm{d}u\mathrm{d}v - m_2\,\mathcal{D}_{\mathcal{R}^n\cap B_r(0)}(\varphi^n), \tag{16}$$

and together with $F \ge 0$ on $\mathbb{R}^3$ and $F(0) = 0$, using that $Z^n \equiv 0$ on $\mathcal{S}^n$:

$$\mathcal{F}_{\mathcal{S}^n\cap B_r(0)}(\bar X) - \mathcal{F}_{\mathcal{S}^n\cap B_r(0)}(X^n) \ge \int\limits_{\mathcal{S}^n\cap B_r(0)} F(\delta Z^n)\,\mathrm{d}u\mathrm{d}v - m_2\,\mathcal{D}_{\mathcal{S}^n\cap B_r(0)}(\varphi^n). \tag{17}$$

Now combining (16) and (17) with (15) and noting that $k > m_2$ we obtain:

$$\begin{aligned}
&\mathcal{I}_{B_r(0)}(\bar X) - \mathcal{I}_{B_r(0)}(X^n)\\
&\ge \int\limits_{\mathcal{R}^n\cap B_r(0)} \langle \nabla F(Z^n), \delta Z^n\rangle\,\mathrm{d}u\mathrm{d}v + \int\limits_{\mathcal{S}^n\cap B_r(0)} F(\delta Z^n)\,\mathrm{d}u\mathrm{d}v\\
&\quad - m_2\,\mathcal{D}_{B_r(0)}(\varphi^n) + k\,(\mathcal{D}_{B_r(0)}(\bar X) - \mathcal{D}_{B_r(0)}(X^n))\\
&\ge -\int\limits_{\mathcal{R}^n\cap C_{r,r_n}} \langle \nabla F(Z^n), \delta Z^n\rangle\,\mathrm{d}u\mathrm{d}v - \int\limits_{\mathcal{S}^n\cap C_{r,r_n}} F(\delta Z^n)\,\mathrm{d}u\mathrm{d}v\\
&\quad + (k - m_2)\,\mathcal{D}_{B_r(0)}(\varphi^n) - k\int\limits_{C_{r,r_n}} DX^n\cdot D\varphi^n\,\mathrm{d}u\mathrm{d}v\\
&\ge -\int\limits_{\mathcal{R}^n\cap C_{r,r_n}} \langle \nabla F(Z^n), \delta Z^n\rangle\,\mathrm{d}u\mathrm{d}v - \int\limits_{\mathcal{S}^n\cap C_{r,r_n}} F(\delta Z^n)\,\mathrm{d}u\mathrm{d}v\\
&\quad - k\int\limits_{C_{r,r_n}} DX^n\cdot D\varphi^n\,\mathrm{d}u\mathrm{d}v = -\delta^+\mathcal{I}_{C_{r,r_n}}(X^n,\varphi^n).
\end{aligned} \tag{18}$$

Next we gain by (12) and Cauchy–Schwarz inequality:

$$\mathcal{D}_{C_{r,r_n}}(\varphi^n) \le 2\,\mathcal{D}_{C_{r,r_n}}(\bar X - X^n) + 2\pi\,(r_n - r), \tag{19}$$

$\forall n \in \mathbb{N}$. Moreover by Proposition 2.1 and our choice of $r \in (0,1)$ we obtain an increasing sequence $\{n_j\} \subset \mathbb{N}$, depending on r, with

$$\mathcal{D}_{C_{r,r_{n_j}}}(\bar{X} - X^{n_j}) \longrightarrow 0 \qquad \textit{for} \quad j \to \infty. \tag{20}$$

Combining this with (19), $\mathcal{D}(X^{n_j}) \leq$ const. by hypothesis, $r_n \to r$ and Hölder's inequality we arrive at

$$\int_{C_{r,r_{n_j}}} DX^{n_j} \cdot D\varphi^{n_j} \, \mathrm{d}u\mathrm{d}v \leq \text{const.}\sqrt{\mathcal{D}_{C_{r,r_{n_j}}}(\varphi^{n_j})} \longrightarrow 0. \tag{21}$$

Moreover by (12) we estimate $\delta Z^{n_j} = \varphi_u^{n_j} \wedge X_v^{n_j} + X_u^{n_j} \wedge \varphi_v^{n_j}$ on $C_{r,r_{n_j}}$ by

$$|\, \delta Z^{n_j} \,| \leq 2 \,|\, DX^{n_j} \,||\, D\varphi^{n_j} \,| \leq 2 \,|\, DX^{n_j} \,| \,(|\, D(\bar{X} - X^{n_j}) \,| + 1),$$

which implies by Hölder's inequality, (20), $\mathcal{D}(X^{n_j}) \leq$ const. and $r_{n_j} \to r$:

$$\int_{C_{r,r_{n_j}}} |\, \delta Z^{n_j} \,| \, \mathrm{d}u\mathrm{d}v \leq \text{const.}\sqrt{\mathcal{D}_{C_{r,r_{n_j}}}(\bar{X} - X^{n_j})} + \text{const.}\sqrt{r_{n_j} - r} \longrightarrow 0. \tag{22}$$

Hence by $|\, \nabla F \,| \leq m_2$ on $\mathbb{R}^3 \setminus \{0\}$ and $F(z) \leq m_2 \,|\, z \,|$ $\forall z \in \mathbb{R}^3$ we obtain:

$$\left| \int_{\mathcal{R}^{n_j} \cap C_{r,r_{n_j}}} \langle \nabla F(Z^{n_j}), \delta Z^{n_j} \rangle \, \mathrm{d}u\mathrm{d}v \right| \leq m_2 \int_{C_{r,r_{n_j}}} |\, \delta Z^{n_j} \,| \, \mathrm{d}u\mathrm{d}v \longrightarrow 0, \tag{23}$$

$$\left| \int_{\mathcal{S}^{n_j} \cap C_{r,r_{n_j}}} F(\delta Z^{n_j}) \, \mathrm{d}u\mathrm{d}v \right| \leq m_2 \int_{C_{r,r_{n_j}}} |\, \delta Z^{n_j} \,| \, \mathrm{d}u\mathrm{d}v \longrightarrow 0. \tag{24}$$

Now combining (21), (23) and (24) with (18) we gain

$$\liminf_{j \to \infty} \,(\mathcal{I}_{B_r(0)}(\bar{X}) - \mathcal{I}_{B_r(0)}(X^{n_j})) \geq 0. \tag{25}$$

On the other hand we have $\| \, X^n \, \|_{H^{1,2}(B)} \leq$ const. by the requirements of the theorem, thus we obtain a weakly convergent subsequence $\{X^{n_i}\}$ of $\{X^{n_j}\}$:

$$X^{n_i} \rightharpoonup \bar{X} \quad \text{in} \;\; H^{1,2}(B, \mathbb{R}^3). \tag{26}$$

Hence, by the weak lower semicontinuity of $\mathcal{I}_{B_r(0)}$ (see [6], p. 403) and (25) we finally obtain

$$\limsup_{i \to \infty} \, \mathcal{I}_{B_r(0)}(X^{n_i}) \leq \limsup_{j \to \infty} \, \mathcal{I}_{B_r(0)}(X^{n_j}) \leq \mathcal{I}_{B_r(0)}(\bar{X}) \leq \liminf_{i \to \infty} \, \mathcal{I}_{B_r(0)}(X^{n_i}).$$

Due to this result and (26) we infer from Lemma 6 in Chap. 4 of [7]:

$$\mathcal{D}_{B_r(0)}(X^{n_i}) \longrightarrow \mathcal{D}_{B_r(0)}(\bar{X}) \quad \text{for} \;\; i \to \infty,$$

which again combined with (26) and the convergence of $\{X^n\}$ to $\bar{X}$ in $C^0(\bar{B}, \mathbb{R}^3)$ finally yields the assertion in (10) for the chosen radius $r \in (0,1)$ and the selected subsequence $\{X^{n_i}\}$. Now we suppose that there would exist some subsequence $\{X^{n_l}\}$ of the original sequence $\{X^n\}$ that satisfies

$$X^{n_l} \longrightarrow \tilde{X} \quad \text{in} \;\; H^{1,2}(B_r(0), \mathbb{R}^3) \tag{27}$$

for some different surface $\tilde{X} \neq \bar{X}$. Then we could apply Proposition 2.1 to $Y^l := \bar{X} - X^{n_l}$ and $\delta_l := \| \bar{X} - X^{n_l} \|_{C^0(\bar{B})}$ and could choose some radius $\tilde{r} \in (r, 1)$ such that (7) holds. Then by the above reasoning we would obtain a further subsequence $\{X^{n_m}\}$ of $\{X^{n_l}\}$ such that

$$X^{n_m} \longrightarrow \bar{X} \quad \text{in } H^{1,2}(B_{\tilde{r}}(0), \mathbb{R}^3),$$

thus especially in $H^{1,2}(B_r(0), \mathbb{R}^3)$ by $r < \tilde{r}$, which contradicts (27). Hence, we proved the assertion of the theorem for a.e. $r \in (0, 1)$, thus $\forall\, r \in (0, 1)$. □

A combination of this result with Lemma 2 of Sect. 2.5 in [7] yields

Theorem 2.2 *The limit surface $\bar{X}$ of Theorem 2.1 is an $\mathcal{I}$-surface again.*

Proof We choose some arbitrary $r \in (0, 1)$ and define $\mathcal{S}_r := \mathcal{S}(\bar{X}) \cap B_r$, $\mathcal{R}_r := \mathcal{R}(\bar{X}) \cap B_r$, $\mathcal{S}_r^n := \mathcal{S}(X^n) \cap B_r$, $\mathcal{R}_r^n := \mathcal{R}(X^n) \cap B_r$, with $B_r := B_r(0)$,

$$\sigma^n := \mathcal{S}_r^n \setminus \mathcal{S}_r = \mathcal{R}_r \setminus \mathcal{R}_r^n \quad \text{and} \quad \tau^n := \mathcal{S}_r \setminus \mathcal{S}_r^n = \mathcal{R}_r^n \setminus \mathcal{R}_r \tag{28}$$

and moreover $Z := \bar{X}_u \wedge \bar{X}_v$, $Z^n := X_u^n \wedge X_v^n$, $\delta Z := \bar{X}_u \wedge \varphi_v + \varphi_u \wedge \bar{X}_v$ and $\delta Z^n := X_u^n \wedge \varphi_v + \varphi_u \wedge X_v^n$ for some arbitrarily chosen $\varphi \in \mathring{H}^{1,2}(B_r(0), \mathbb{R}^3)$. The decisive step consists of the proof of

$$\delta^+ \mathcal{F}_{B_r(0)}(\bar{X}, \varphi) \geq \liminf_{n \to \infty} \delta^+ \mathcal{F}_{B_r(0)}(X^n, \varphi) \tag{29}$$

$\forall\, \varphi \in \mathring{H}^{1,2}(B_r(0), \mathbb{R}^3)$. Firstly we estimate:

$$| Z^n - Z | = | X_u^n \wedge X_v^n - \bar{X}_u \wedge \bar{X}_v | \leq (| DX^n | + | D\bar{X} |) \, | D(X^n - \bar{X}) | \, .$$

From this we infer by the Hölder inequality and (10):

$$\int\limits_{B_r(0)} | Z^n - Z | \; \mathrm{d}u\mathrm{d}v \leq 2 \left(\sqrt{\mathcal{D}_{B_r(0)}(X^n)} + \sqrt{\mathcal{D}_{B_r(0)}(\bar{X})} \right) \sqrt{\mathcal{D}_{B_r(0)}(X^n - \bar{X})} \longrightarrow 0. \tag{30}$$

Next we estimate:

$$| \delta Z^n - \delta Z | = | (X_u^n - \bar{X}_u) \wedge \varphi_v + \varphi_u \wedge (X_v^n - \bar{X}_v) | \leq 2 \; | D\varphi || D(X^n - \bar{X}) |,$$

which implies again by (10):

$$\int\limits_{B_r(0)} | \delta Z^n - \delta Z | \; \mathrm{d}u\mathrm{d}v \leq 4 \sqrt{\mathcal{D}_{B_r(0)}(\varphi) \; \mathcal{D}_{B_r(0)}(X^n - \bar{X})} \longrightarrow 0. \tag{31}$$

Next we split up the integrals on the sets $\mathcal{R}_r^n$ and $\mathcal{R}_r$ occuring in (29):

$$\begin{aligned}
&\int_{\mathcal{R}_r^n} \langle \nabla F(Z^n), \delta Z^n \rangle \, du dv - \int_{\mathcal{R}_r} \langle \nabla F(Z), \delta Z \rangle \, du dv \\
&= \int_{B_r(0)} \chi_{\mathcal{R}_r^n \cap \mathcal{R}_r} \langle \nabla F(Z^n), \delta Z^n \rangle + \chi_{\tau^n} \langle \nabla F(Z^n), \delta Z^n \rangle \\
&\quad - \chi_{\mathcal{R}_r \cap \mathcal{R}_r^n} \langle \nabla F(Z), \delta Z \rangle - \chi_{\sigma^n} \langle \nabla F(Z), \delta Z \rangle \, du dv \\
&= \int_{B_r(0)} \chi_{\mathcal{R}_r^n \cap \mathcal{R}_r} \langle \nabla F(Z^n), \delta Z^n - \delta Z \rangle \, du dv \\
&\quad + \int_{B_r(0)} \chi_{\mathcal{R}_r^n \cap \mathcal{R}_r} \langle \nabla F(Z^n) - \nabla F(Z), \delta Z \rangle \, du dv \\
&\quad - \int_{B_r(0)} \chi_{\sigma^n} \langle \nabla F(Z), \delta Z \rangle \, du dv + \int_{B_r(0)} \chi_{\tau^n} \langle \nabla F(Z^n), \delta Z^n \rangle \, du dv. \qquad (32)
\end{aligned}$$

For the first integral in (32) we have by $| \nabla F | \leq m_2$ on $\mathbb{R}^3 \setminus \{0\}$ and (31):

$$\left| \int_{B_r(0)} \chi_{\mathcal{R}_r^n \cap \mathcal{R}_r} \langle \nabla F(Z^n), \delta Z^n - \delta Z \rangle \, du dv \right| \leq m_2 \int_{B_r(0)} | \delta Z^n - \delta Z | \, du dv \longrightarrow 0. \qquad (33)$$

Now we are going to examine the second integral in (32). By (30) we obtain a subsequence $\{Z^{n_k}\}$ for which

$$Z^{n_k}(w) \longrightarrow Z(w) \quad \text{for a.e. } w \in B_r(0). \qquad (34)$$

We rename $\{n_k\}$ into $\{n\}$ again and shall consider this sequence henceforth. Now we choose some point $w \in B_r(0) \setminus \mathcal{N}$ arbitrarily, where $\mathcal{N} \subset B_r(0)$ is the subset of $\mathcal{L}^2$-measure zero on which (34) does not hold and δZ does not exist, and distinguish between the following two cases:

Case (1) There holds $w \in \mathcal{R}_r^{n_j} \cap \mathcal{R}_r$ for an increasing sequence $\{n_j\} \subset \mathbb{N}$. Then we obtain by (34) and the continuity of ∇F on $\mathbb{R}^3 \setminus \{0\}$:

$$\nabla F(Z^{n_j})(w) \longrightarrow \nabla F(Z)(w) \quad \text{for } j \to \infty.$$

As we have $\chi_{\mathcal{R}_r^n \cap \mathcal{R}_r}(w) = 0$ for $n \in \mathbb{N} \setminus \{n_j\}$ we can conclude:

$$\chi_{\mathcal{R}_r^n \cap \mathcal{R}_r}(w) \, (\nabla F(Z^n)(w) - \nabla F(Z)(w)) \, \delta Z(w) \longrightarrow 0 \quad \text{for } n \to \infty. \qquad (35)$$

Case (2) There exists some number $N \in \mathbb{N}$ such that $w \notin \mathcal{R}_r^n \cap \mathcal{R}_r$, i.e. $\chi_{\mathcal{R}_r^n \cap \mathcal{R}_r}(w) = 0$, $\forall n > N$. In this case we obtain (35) immediately.

Hence, we gain (35) for a.e. $w \in B_r(0)$. Furthermore we see due to $| \nabla F | \leq m_2$ on $\mathbb{R}^3 \setminus \{0\}$:

$$| \chi_{\mathcal{R}_r^n \cap \mathcal{R}_r} (\nabla F(Z^n) - \nabla F(Z)) \, \delta Z | \leq 2 m_2 \, | \delta Z | \in L^1(B_r(0)),$$

$\forall n \in \mathbb{N}$. Therefore the Lebesgue convergence theorem finally implies that

$$\int_{B_r(0)} \chi_{\mathcal{R}_r^n \cap \mathcal{R}_r} (\nabla F(Z^n) - \nabla F(Z)) \, \delta Z \, du dv \longrightarrow 0. \qquad (36)$$

Now we examine the third integral in (32). We have $Z^n \equiv 0$ a.e. on $\sigma_n = \mathcal{S}_r^n \setminus \mathcal{S}_r$. Hence, we obtain by (30):

$$\int\limits_{B_r(0)} \chi_{\sigma^n} \mid Z \mid \, du dv = \int\limits_{B_r(0)} \chi_{\sigma^n} \mid Z - Z^n \mid \, du dv \longrightarrow 0.$$

Thus we gain an increasing sequence $\{n_k\}$ such that $\chi_{\sigma^{n_k}}(w) \mid Z(w) \mid \longrightarrow 0$ for a.e. $w \in B_r(0)$. Renaming $\{n_k\}$ into $\{n\}$ again and noticing that $\mid Z \mid > 0$ on $\sigma^n \subset \mathcal{R}_r$, $\forall\, n \in \mathbb{N}$, we arrive at $\chi_{\sigma^n}(w) \to 0$ for a.e. $w \in B_r(0)$, i.e.

$$\mathcal{L}^2(\sigma^n) \longrightarrow 0 \quad \text{for } n \to \infty. \tag{37}$$

As we know $\langle \nabla F(Z), \delta Z\rangle \in L^1(\mathcal{R}_r)$ due to $\mid \nabla F \mid \le m_2$ on $\mathbb{R}^3 \setminus \{0\}$ we infer from the absolute continuity of the Lebesgue integral that

$$\int\limits_{B_r(0)} \chi_{\sigma^n} \langle \nabla F(Z), \delta Z\rangle \, du dv \longrightarrow 0 \quad \text{for } n \to \infty. \tag{38}$$

Now the fourth integral in (32) has to be examined simultaneously with the remaining integrals on the sets $\mathcal{S}_r^n$ and $\mathcal{S}_r$ occuring in (29) respectively (9), which we also split up:

$$\begin{aligned}\int\limits_{\mathcal{S}_r^n} F(\delta Z^n)\, du dv - \int\limits_{\mathcal{S}_r} F(\delta Z)\, du dv &= \int\limits_{\mathcal{S}_r^n \cap \mathcal{S}_r} F(\delta Z^n)\, du dv + \int\limits_{\sigma^n} F(\delta Z^n)\, du dv \\ &\quad - \int\limits_{\mathcal{S}_r \cap \mathcal{S}_r^n} F(\delta Z)\, du dv - \int\limits_{\tau^n} F(\delta Z)\, du dv. \end{aligned} \tag{39}$$

Since F is Lipschitz continuous with Lip.-const.$= m_2$ by Lemma 3.2 in [6] we firstly obtain together with (31) that

$$\int\limits_{B_r(0)} \mid F(\delta Z^n) - F(\delta Z) \mid \, du dv \le m_2 \int\limits_{B_r(0)} \mid \delta Z^n - \delta Z \mid \, du dv \longrightarrow 0, \tag{40}$$

which estimates the difference of the first and third integral in (39) in particular. Now (40) yields a subsequence $\{\delta Z^{n_k}\}$ such that $F(\delta Z^{n_k})(w) \to F(\delta Z)(w)$ for a.e. $w \in B_r(0)$ and by Vitali's theorem we know that $\forall\, \epsilon > 0$ there exists some $\delta(\epsilon)$ such that

$$\int\limits_{E} F(\delta Z^{n_k})\, du dv < \epsilon, \quad \text{if } \mathcal{L}^2(E) < \delta(\epsilon) \tag{41}$$

uniformly $\forall\, k \in \mathbb{N}$. Again we rename $\{n_k\}$ into $\{n\}$. As (37) means that for any given $\delta > 0$ there is some $N(\delta)$ with $\mathcal{L}^2(\sigma^n) < \delta \;\; \forall\, n > N(\delta)$ we conclude together with (41) that

$$\int\limits_{\sigma^n} F(\delta Z^n)\, du dv \longrightarrow 0 \quad \text{for } n \to \infty. \tag{42}$$

Now there only remain the fourth integrals in (39) and (32). On $\tau^n = \mathcal{R}_r^n \setminus \mathcal{R}_r$ we obtain by the convexity of $F \in C^2(\mathbb{R}^3 \setminus \{0\})$ and its positive homogeneity:

$$\langle \nabla F(Z^n), \delta Z^n\rangle \le F(\delta Z^n) - F(Z^n) + \langle \nabla F(Z^n), Z^n\rangle = F(\delta Z^n). \tag{43}$$

Hence, we obtain together with (40):

$$\int_{\tau^n} \langle \nabla F(Z^n), \delta Z^n \rangle - F(\delta Z)\, \mathrm{d}u\mathrm{d}v \leq \int_{\tau^n} F(\delta Z^n) - F(\delta Z)\, \mathrm{d}u\mathrm{d}v \longrightarrow 0. \tag{44}$$

Now terming $\{n_j\} \subset \mathbb{N}$ the resulting increasing sequence, having selected subsequences several times after (29), and collecting (9), (32), (33), (36), (39), (40), (42) and (44) we finally conclude:

$$\begin{aligned} &\liminf_{n\to\infty} (\delta^+ \mathcal{F}_{B_r(0)}(X^n,\varphi) - \delta^+ \mathcal{F}_{B_r(0)}(\bar{X},\varphi)) \\ &\quad \leq \liminf_{j\to\infty} (\delta^+ \mathcal{F}_{B_r(0)}(X^{n_j},\varphi) - \delta^+ \mathcal{F}_{B_r(0)}(\bar{X},\varphi)) \\ &\quad = \liminf_{j\to\infty} \int_{\tau^{n_j}} \langle \nabla F(Z^{n_j}), \delta Z^{n_j} \rangle - F(\delta Z)\, \mathrm{d}u\mathrm{d}v \leq 0 \end{aligned}$$

$\forall\, \varphi \in \mathring{H}^{1,2}(B_r(0), \mathbb{R}^3)$, which proves (29). Moreover we obtain immediately by (10) (for the same sequence as in (29)):

$$\delta \mathcal{D}_{B_r(0)}(X^n,\varphi) = \int_{B_r(0)} DX^n \cdot D\varphi\, \mathrm{d}u\mathrm{d}v \longrightarrow \delta \mathcal{D}_{B_r(0)}(\bar{X},\varphi).$$

Hence, together with (29) and (9) we arrive at

$$\delta^+ \mathcal{I}_{B_r(0)}(\bar{X},\varphi) \geq \liminf_{n\to\infty}\, \delta^+ \mathcal{I}_{B_r(0)}(X^n,\varphi) \geq 0, \tag{45}$$

$\forall\, \varphi \in \mathring{H}^{1,2}(B_r(0), \mathbb{R}^3)$, where we used that the $\mathcal{I}$-surfaces X^n satisfy $\delta^+ \mathcal{I}_{B_r(0)}(X^n,\varphi) \geq 0 \;\; \forall\, \varphi \in \mathring{H}^{1,2}(B_r(0), \mathbb{R}^3)$ by Lemma 2 in Sect. 2.5 in [7] and $F(0) = 0$. Moreover for any $\varphi \in C_c^\infty(B, \mathbb{R}^3)$ we have $\mathrm{supp}(\varphi) \subset\subset B_r(0)$ for some $r \in (0,1)$ which satisfies (10), hence we gain by (45) and $F(0) = 0$:

$$\delta^+ \mathcal{I}(\bar{X},\varphi) \geq 0 \quad \forall\, \varphi \in C_c^\infty(B, \mathbb{R}^3). \tag{46}$$

Now we consider some arbitrarily fixed $\varphi \in \mathring{H}^{1,2}(B, \mathbb{R}^3)$ and some approximating sequence $\{\varphi^j\} \subset C_c^\infty(B, \mathbb{R}^3)$, i.e.

$$\varphi^j \longrightarrow \varphi \quad \text{in } \mathring{H}^{1,2}(B, \mathbb{R}^3). \tag{47}$$

We set $\delta Z^j := \bar{X}_u \wedge \varphi_v^j + \varphi_u^j \wedge \bar{X}_v$ and estimate $\mid \delta Z^j - \delta Z \mid \leq 2 \mid D\bar{X} \mid\mid D(\varphi^j - \varphi) \mid$, which implies by (47) $\int_B \mid \delta Z^j - \delta Z \mid \,\mathrm{d}u\mathrm{d}v \leq 4\sqrt{\mathcal{D}(\bar{X})\; \mathcal{D}(\varphi^j - \varphi)} \longrightarrow 0$. Therefore we obtain as in (33) and (40):

$$\left| \int_{\mathcal{R}} \langle \nabla F(Z), \delta Z^j - \delta Z \rangle\, \mathrm{d}u\mathrm{d}v \right| \leq m_2 \int_{\mathcal{R}} \mid \delta Z^j - \delta Z \mid \,\mathrm{d}u\mathrm{d}v \longrightarrow 0, \tag{48}$$

$$\left| \int_{\mathcal{S}} F(\delta Z^j) - F(\delta Z)\, \mathrm{d}u\mathrm{d}v \right| \leq m_2 \int_{\mathcal{S}} \mid \delta Z^j - \delta Z \mid \,\mathrm{d}u\mathrm{d}v \longrightarrow 0. \tag{49}$$

Moreover we have $\int_B D\bar{X} \cdot D\varphi^j\, \mathrm{d}u\mathrm{d}v \longrightarrow \int_B D\bar{X} \cdot D\varphi\, \mathrm{d}u\mathrm{d}v$ by (47). Hence, combining this with (48), (49) and (9) we finally infer from (46):

$$\delta^+\mathcal{I}(\bar{X},\varphi) = \lim_{j\to\infty} \delta^+\mathcal{I}(\bar{X},\varphi^j) \geq 0 \quad \forall\,\varphi \in \mathring{H}^{1,2}(B,\mathbb{R}^3),$$

which proves $\bar{X}$ to be an $\mathcal{I}$-surface by Lemma 2 in Sect. 2.5 in [7]. □

3 Continuity theorems for $\mathcal{A}$, $\mathcal{J}$ and $\mathcal{I}$

In this section we shall only quote Shiffman's "continuity theorems" 11.1 and 12.2 in [12] for the functionals $\mathcal{J}$ and $\mathcal{I}$ in application to sequences of $\mathcal{I}$-surfaces that converge in $C^0(\bar{B},\mathbb{R}^3)$, see Theorem 3.2 and Corollary 3.1 below. In fact these results can be easily derived from a deep "continuity theorem" for the area functional $\mathcal{A}$ applied to harmonic surfaces on ring regions $C_{\rho,1} = B_1(0) \setminus \overline{B_\rho(0)}$ with convergent boundary values in $(C^0 \cap BV)(\partial C_{\rho,1},\mathbb{R}^3)$ due to Morse and Tompkins in [9], which states precisely:

Theorem 3.1 *Let $\{\varphi_1^n\} \subset (C^0 \cap BV)(\partial B_1(0),\mathbb{R}^3)$ and $\{\varphi_\rho^n\} \subset (C^0 \cap BV)(\partial B_\rho(0),\mathbb{R}^3)$ be prescribed boundary values on $\partial C_{\rho,1} = \partial B_1(0) \cup \partial B_\rho(0)$ for some $\rho \in (0,1)$ such that*

$$\varphi_1^n \longrightarrow \varphi_1 \quad in\ \ C^0(\partial B_1(0),\mathbb{R}^3) \quad and \quad \mathcal{L}(\varphi_1^n) \longrightarrow \mathcal{L}(\varphi_1), \tag{50}$$

$$\varphi_\rho^n \longrightarrow \varphi_\rho \quad in\ \ C^0(\partial B_\rho(0),\mathbb{R}^3) \quad and \quad \mathcal{L}(\varphi_\rho^n) \longrightarrow \mathcal{L}(\varphi_\rho). \tag{51}$$

($\mathcal{L}$:=length) for some functions $\varphi_1 \in (C^0 \cap BV)(\partial B_1(0),\mathbb{R}^3)$ and $\varphi_\rho \in (C^0 \cap BV)(\partial B_\rho(0),\mathbb{R}^3)$. Then we prove for the harmonic extensions H^n respectively H of the boundary values $(\varphi_1^n,\varphi_\rho^n)$ respectively (φ_1,φ_ρ) on $\partial C_{\rho,1}$ that

$$\mathcal{A}_{C_{\rho,1}}(H^n) \longrightarrow \mathcal{A}_{C_{\rho,1}}(H) \quad for\ \ n \to \infty. \tag{52}$$

In the remaining part of this section the integrand F is assumed to satisfy only the requirements (A). We need the following estimate, Lemma 8.1 in [12], which is gained by "harmonic substitution".

Lemma 3.1 *Let X be an $\mathcal{I}$-surface and $\Omega \subset B$ any open subset with a Lipschitz boundary. Then for the harmonic extension H of the boundary values $X\,|_{\partial\Omega}$ we have:*

$$\mathcal{F}_\Omega(X) \leq \mathcal{F}_\Omega(H) - k\,\mathcal{D}_\Omega(X-H).$$

Now Shiffman combined this estimate with Theorem 3.1 to achieve

Theorem 3.2 *Let $\{X^n\}$ be a sequence of $\mathcal{I}$-surfaces with $X^n\,|_{\partial B} \in (C^0 \cap BV)(\partial B,\mathbb{R}^3)$, $\mathcal{D}(X^n) \leq$ const. $\forall\, n \in \mathbb{N}$ and*

$$X^n \longrightarrow \bar{X} \quad in\ \ C^0(\bar{B},\mathbb{R}^3), \quad \mathcal{L}(X^n\,|_{\partial B}) \longrightarrow \mathcal{L}(\bar{X}\,|_{\partial B}) \tag{53}$$

for an $\mathcal{I}$-surface $\bar{X}$ with $\bar{X}\,|_{\partial B} \in (C^0 \cap BV)(\partial B,\mathbb{R}^3)$. Then there holds:

$$\mathcal{J}(X^n) \longrightarrow \mathcal{J}(\bar{X}) \quad for\ \ n \to \infty. \tag{54}$$

This theorem immediately implies Theorem 12.2 in [12]:

Corollary 3.1 *Let $\{X^n\}$ be a sequence of $\mathcal{I}$-surfaces as in Theorem 3.2 that are additionally (a.e.) conformally parametrized on B. Then firstly there holds*

$$\mathcal{I}(X^n) \longrightarrow \mathcal{I}(\bar{X}) \quad for\ \ n \to \infty, \tag{55}$$

where $\bar{X}$ is the limit $\mathcal{I}$-surface as in Theorem 3.2, and secondly $\bar{X}$ proves to be (a.e.) conformally parametrized on B.

Moreover we need an isoperimetric inequality for $\mathcal{A}_\Omega$ applied to harmonic surfaces on simply connected subdomains Ω of B whose boundary is a Jordan curve. This can easily be derived from the isoperimetric inequality for harmonic surfaces on B (see [2], pp. 134–138) by means of the homeomorphic extension of Riemann's mapping function $\phi : \Omega \xrightarrow{\cong} B$ onto $\bar{\Omega}$, whose existence can be guaranteed by requiring $\partial\Omega$ to be a Jordan curve, i.e. Ω to be a so-called Jordan domain (see [11], pp. 24–25).

Theorem 3.3 *Let Ω be a simply connected Jordan subdomain of B, $\varphi \in (C^0 \cap BV)(\partial\Omega, \mathbb{R}^3)$ and h the unique harmonic extension of φ onto $\bar{\Omega}$, then there holds:*

$$\mathcal{A}_\Omega(h) \le \frac{1}{4}\,\mathcal{L}(\varphi)^2.$$

Thus together with Lemma 3.1 and (3) one obtains finally (see [12], p. 557)

Corollary 3.2 *Let Ω be a simply connected Jordan subdomain of B whose boundary is additionally Lipschitzian and X an $\mathcal{I}$-surface with $X\,|_{\partial\Omega} \in (C^0 \cap BV)(\partial\Omega, \mathbb{R}^3)$, then there holds:*

$$\mathcal{A}_\Omega(X) \le \frac{m_2}{4m_1}\mathcal{L}(X\,|_{\partial\Omega})^2 \quad \text{and} \quad \mathcal{J}_\Omega(X) \le \left(1 + \frac{k}{m_1}\right)\frac{m_2}{4}\,\mathcal{L}(X\,|_{\partial\Omega})^2. \tag{56}$$

4 Combination with the results of [6] and [8]

In this section we combine all achievements of the preceding sections, of [6] and of [8] with a special continuity theorem, Proposition 4.4, which is shown similarly as Lemma 6 in [4], and a compactness result for boundary values, Proposition 4.5, in order to prove the following mountain pass result under the conditions (A) and (R*) on the integrand F (see Definition 4.2 and 4.3):

Theorem 4.1 *Let F be an integrand that satisfies the requirements (A)+(R*) and let Γ be an arbitrary closed rectifiable Jordan curve in $\mathbb{R}^3$ meeting a chord-arc condition (57). If there exist two different conformally parametrized surfaces $X_1 \ne X_2$ in $(\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3), \|\cdot\|_{C^0(\bar{B})})$ which are in a mountain pass situation w. r. to $\mathcal{J}$ with elevation $e \ge 0$, then there exists an unstable $\mathcal{J}$-extremal surface X^* in $\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3)$ with $\mathcal{J}(X^*) > \max\{\mathcal{J}(X_1), \mathcal{J}(X_2)\} + e$.*

In the first two subsections of this section it will suffice to impose only the requirements (A) on some arbitrarily fixed integrand F, but in Sects. 4.3 and 4.4 we will consider the integrand F that was fixed in the above theorem and thus has to meet additionally the requirement (R*).

4.1 Limit superior of continua

This subsection is devoted to the following notions of limits of sets:

Definition 4.1 *Let (Y,d) be some metric space. For any sequence of subsets $\{M^n\}_{n\in\mathbb{N}}$ of Y we define its limit inferior by*

$$\liminf_{n\in\mathbb{N}} M^n := \{y \in Y \mid \exists\, m_n \in M^n \text{ such that } d(m_n, y) \longrightarrow 0 \text{ for } n \to \infty\}$$

and its limit superior by

$$\limsup_{n\in\mathbb{N}} M^n := \{y \in Y \mid \exists \text{ some subseq. } \{M^{n_j}\} \text{ of } \{M^n\} \text{ and } m_j \in M^{n_j} \text{ such that } d(m_j, y) \longrightarrow 0 \text{ for } j \to \infty\}.$$

Firstly there holds the identity $\limsup_{n\in\mathbb{N}} M^n = \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{n\geq k} M^n}$, which is proved in [7], p. 86, and secondly we have (see [10], p. 388)

Proposition 4.1 *Let $\{M^n\}_{n\in\mathbb{N}}$ be some sequence of compact and connected subsets of a metric space (Y, d) such that $\overline{\bigcup_{n\in\mathbb{N}} M^n}$ is compact and $\liminf_{n\in\mathbb{N}} M^n \neq \emptyset$. Then $\limsup_{n\in\mathbb{N}} M^n$ is again compact and connected, i.e. a continuum.*

4.2 Mountain pass situation and instability

We consider some fixed simple closed polygon Γ with $N + 3$ vertices. As in Definition 7.4 in [6] we define the set of continua $\mathcal{P}_{(X_1,X_2)}$ containing a pair of surfaces X_1, X_2 in $(\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3), \|\cdot\|_{C^0(\bar{B})})$ and the set of continua $\wp_{(\tau_1,\tau_2)}$ containing a pair of points τ_1, τ_2 in the configuration space $T \subset (0, 2\pi)^N$. Using this we define similarly to Definition 7.7 and 7.5 in [6]:

Definition 4.2 (a) *Two different surfaces $X_1, X_2 \in (\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3), \|\cdot\|_{C^0(\bar{B})})$ are in a "mountain pass situation" with respect to $\mathcal{K} := \mathcal{J}, \mathcal{I}$ with elevation $e \geq 0$ if*

$$\sup_{\Sigma} \mathcal{K} > \max\{\mathcal{K}(X_1), \mathcal{K}(X_2)\} + e \quad \forall\, \Sigma \in \mathcal{P}_{(X_1,X_2)}.$$

(b) *A pair of different points $\tau_1, \tau_2 \in T \subset (0, 2\pi)^N$ is in a "mountain pass situation" with respect to $f^\Gamma = \mathcal{I} \circ \psi^\Gamma$ (see Definition 6.3 in [6]) if*

$$\max_P f^\Gamma > \max\{f^\Gamma(\tau_1), f^\Gamma(\tau_2)\} \quad \forall\, P \in \wp_{(\tau_1,\tau_2)}.$$

(c) *A set $P^* \in \wp_{(\tau_1,\tau_2)}$ with the property $\max_{P^*} f^\Gamma = \inf_{P\in\wp_{(\tau_1,\tau_2)}} \max_P f^\Gamma =: \beta(\tau_1, \tau_2)$ is called a minimizing connected set and we denote $P^*_\beta := \{\tau \in P^* \mid f^\Gamma(\tau) = \beta(\tau_1, \tau_2)\}$.*

Now similarly to the proof of Proposition 7.8 in [6] one can derive

Proposition 4.2 *If there exist two different conformally parametrized surfaces $X_1 \neq X_2$ in $(\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3), \|\cdot\|_{C^0(\bar{B})})$ that are in a mountain pass situation with respect to $\mathcal{J}$ with elevation $e \geq 0$, then the unique $\mathcal{I}$-surfaces X_l^* in the boundary value classes $H^{1,2}_{X_l|\partial B}(B, \mathbb{R}^3)$, $l = 1, 2$, are in a mountain pass situation with respect to $\mathcal{I}$ with elevation e.*

Definition 4.3 *We call a $\mathcal{J}$-extremal surface $X^* \in (\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3), \|\cdot\|_{C^0(\bar{B})})$ $\mathcal{K}$-unstable, for $\mathcal{K} = \mathcal{I}, \mathcal{J}$, if in every ϵ-ball $B_\epsilon(X^*) \cap \mathcal{C}^*(\Gamma)$ around X^* there is some surface $\tilde{X}$ such that $\mathcal{K}(\tilde{X}) < \mathcal{K}(X^*)$.*

4.3 Approximation of closed rectifiable Jordan curves by simple polygons

Firstly we need the following

Definition 4.4 (*i*) *Let* Γ *be an arbitrary closed rectifiable Jordan curve in* $\mathbb{R}^3$*. Then we term a simple closed polygon* $\tilde{\Gamma} \subset \mathbb{R}^3$ *a polygonal approximation of* Γ *if all vertices* $\tilde{A}_1, \dots, \tilde{A}_M$ $(M > 3)$ *of* $\tilde{\Gamma}$ *lie on* Γ *and if the arc on* Γ *between any two adjacent points* $\tilde{A}_m, \tilde{A}_{m+1}$*, which does not contain the remaining vertices of* $\tilde{\Gamma}$*, is indeed the shorter one* $\Gamma \mid_{(\tilde{A}_m, \tilde{A}_{m+1})}$ *connecting* $\tilde{A}_m$ *and* $\tilde{A}_{m+1}$*. We define its fineness by* $\triangle(\tilde{\Gamma}) := \max_{j=1,\dots,M} \mid \tilde{A}_j - \tilde{A}_{j-1} \mid$*, with* $\tilde{A}_0 := \tilde{A}_M$*.*

Definition 4.5 *A closed rectifiable Jordan curve* Γ *in* $\mathbb{R}^3$ *meets a chord-arc condition if there is a constant* C *such that*

$$\mathcal{L}(\Gamma \mid_{(P_1,P_2)}) \leq C \mid P_1 - P_2 \mid \quad \forall P_1, P_2 \in \Gamma, \tag{57}$$

where $\Gamma \mid_{(P_1,P_2)}$ *denotes the shorter arc on* Γ *connecting* P_1 *and* P_2*.*

Now we can state the following approximation lemma (see Lemma 5 in [4]):

Proposition 4.3 *Let* Γ *be an arbitrary closed rectifiable Jordan curve in* $\mathbb{R}^3$ *which satisfies a chord-arc condition* (57)*. Then there exists a sequence* $\{\Gamma^n\}$ *of polygonal approximations of* Γ *and homeomorphisms* $\varphi^n : \Gamma \xrightarrow{\cong} \Gamma^n$ *that keep the vertices of the* Γ^n *fixed and satisfy:*

$$\mathcal{L}(\Gamma^n) \longrightarrow \mathcal{L}(\Gamma), \quad \triangle(\Gamma^n) \longrightarrow 0, \quad \max_{P \in \Gamma} \mid P - \varphi^n(P) \mid \longrightarrow 0, \tag{58}$$

for $n \to \infty$*, and for any pair* $P_1, P_2 \in \Gamma$*:*

$$\mid \varphi^n(P_1) - \varphi^n(P_2) \mid \leq \mathcal{L}(\Gamma \mid_{(P_1,P_2)}) \quad \forall n \in \mathbb{N}. \tag{59}$$

Now let Γ be a fixed, closed rectifiable Jordan curve in $\mathbb{R}^3$ meeting a chord-arc condition (57) and $\{\Gamma^n\}$ a fixed sequence of polygonal approximations as in Prop. 4.3 with the vertices

$$\left(P_0^n, A_1^n, \dots, A_{l_n}^n; P_1^n; A_{l_n+1}^n, \dots, A_{m_n}^n; P_2^n; A_{m_n+1}^n, \dots, A_{N_n}^n\right), \tag{60}$$

where we may assume that the three points $\{P_k^n\}$ of the three-point-condition in $\mathcal{C}^*(\Gamma^n)$ satisfy $P_k^n \equiv P_k$, $k = 0, 1, 2$, (see (6)) and where $0 \leq l_n \leq m_n \leq N_n$ are fixed for each $n \in \mathbb{N}$. We consider some arbitrarily chosen $\mathcal{I}$-surface $X \in \mathcal{C}^*(\Gamma)$ and the sequence of boundary values $\varphi^n(X \mid_{\partial B}) : \mathbb{S}^1 \longrightarrow \Gamma^n$ which by their surjectivity give rise to a sequence of angles

$$0 = \psi_0 < \tau_1^n < \dots < \tau_{l_n}^n < \psi_1 < \dots < \tau_{m_n}^n < \psi_2 < \dots < \tau_{N_n}^n < 2\pi, \tag{61}$$

with $\psi_k = \frac{2k\pi}{3}$, for every $n \in \mathbb{N}$ such that

$$\varphi^n(X \mid_{\partial B})(e^{i\tau_j^n}) = A_j^n \quad \text{for } j = 1, \dots, N_n, \tag{62}$$

$$\text{respectively} \qquad \varphi^n(X \mid_{\partial B})(e^{i\psi_k}) \equiv P_k \quad \text{for } k = 0, 1, 2. \tag{63}$$

Hence, we obtain a sequence of tuples $\tau^n \in T^n \subset (0, 2\pi)^{N_n}$ (see Definition 6.1 in [6]) which yield the unique minimizers $X(\tau^n)$ of $\mathcal{I}$ in the sets $\mathcal{U}(\Gamma^n, \tau^n)$ (see (4), (5) and Definition 6.2, 6.3 in [6]). We are going to prove the crucial

Proposition 4.4 *If the integrand F of $\mathcal{F}$ satisfies the requirements* (A) *and* (R^*), *then there holds*

$$X(\tau^n) \longrightarrow X \quad \text{in } C^0(\bar{B},\mathbb{R}^3), \tag{64}$$

$$\mathcal{I}(X(\tau^n)) \longrightarrow \mathcal{I}(X) \quad \text{for } n \to \infty. \tag{65}$$

Proof We set $Z^n := \varphi^n(X\mid_{\partial B})$ and $\eta^n := Z^n - X\mid_{\partial B}$ and consider the harmonic extensions h respectively h^n of $X\mid_{\partial B}$ respectively η^n onto $\bar{B}$. By (59) and (57) we derive the estimate

$$\begin{aligned} &\mid \eta^n(\mathrm{e}^{i\alpha}) - \eta^n(\mathrm{e}^{i\beta}) \mid \leq \mid X(\mathrm{e}^{i\alpha}) - X(\mathrm{e}^{i\beta}) \mid + \mid Z^n(\mathrm{e}^{i\alpha}) - Z^n(\mathrm{e}^{i\beta}) \mid \\ &\quad \leq \mid X(\mathrm{e}^{i\alpha}) - X(\mathrm{e}^{i\beta}) \mid + \mathcal{L}(\Gamma\mid_{(X(\mathrm{e}^{i\alpha}),X(\mathrm{e}^{i\beta}))}) \leq (1+C) \mid X(\mathrm{e}^{i\alpha}) - X(\mathrm{e}^{i\beta}) \mid \end{aligned} \tag{66}$$

$\forall\,\alpha,\beta \in [0,2\pi]$. Now we combine this with Douglas' formula ([10], p. 277):

$$\begin{aligned} \mathcal{A}_0(\eta^n) &:= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\mid \eta^n(\mathrm{e}^{i\alpha}) - \eta^n(\mathrm{e}^{i\beta}) \mid^2}{4\sin^2(\frac{\alpha-\beta}{2})}\,\mathrm{d}\alpha\mathrm{d}\beta \\ &\leq \frac{(1+C)^2}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\mid X(\mathrm{e}^{i\alpha}) - X(\mathrm{e}^{i\beta}) \mid^2}{4\sin^2(\frac{\alpha-\beta}{2})}\,\mathrm{d}\alpha\mathrm{d}\beta \\ &= (1+C)^2\,\mathcal{A}_0(X\mid_{\partial B}) = (1+C)^2\,\mathcal{D}(h) \leq (1+C)^2\,\mathcal{D}(X). \end{aligned}$$

Hence, $(1+C)^2\,\frac{|X(\mathrm{e}^{i\alpha})-X(\mathrm{e}^{i\beta})|^2}{4\sin^2(\frac{\alpha-\beta}{2})}$ yields a Lebesgue dominating term for the integrands $\frac{|\eta^n(\mathrm{e}^{i\alpha})-\eta^n(\mathrm{e}^{i\beta})|^2}{4\sin^2(\frac{\alpha-\beta}{2})}$ on $[0,2\pi]^2$. Moreover we see by (58) that $\eta^n = \varphi^n(X\mid_{\partial B}) - X\mid_{\partial B} \longrightarrow 0$ in $C^0(\partial B,\mathbb{R}^3)$. Hence, we can infer by Lebesgue's convergence theorem:

$$\mathcal{D}(h^n) = \mathcal{A}_0(\eta^n) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\mid \eta^n(\mathrm{e}^{i\alpha}) - \eta^n(\mathrm{e}^{i\beta}) \mid^2}{4\sin^2(\frac{\alpha-\beta}{2})}\,\mathrm{d}\alpha\mathrm{d}\beta \longrightarrow 0. \tag{67}$$

Furthermore we consider the surfaces $X^n := X + h^n$ on $\bar{B}$. By (67) we have that $\mathcal{D}(X^n - X) = \mathcal{D}(h^n) \longrightarrow 0$, hence, together with $\mathcal{D}(X^n) \leq 2\,(\mathcal{D}(X) + \mathcal{D}(h^n)) \leq$const. Proposition 3.4 in [8] yields

$$\mid \mathcal{I}(X^n) - \mathcal{I}(X) \mid \leq \text{const.}\sqrt{\mathcal{D}(X^n - X)} \longrightarrow 0 \quad \text{for } n \to \infty. \tag{68}$$

Moreover we see $X^n\mid_{\partial B} = X\mid_{\partial B} + \eta^n = X\mid_{\partial B} + Z^n - X\mid_{\partial B} = \varphi^n(X\mid_{\partial B})$. Hence, since $\varphi^n(X\mid_{\partial B}) : \mathbb{S}^1 \longrightarrow \Gamma^n$ yields a weakly monotonic continuous map satisfying the Courant- and three-point-condition, (62) and (63), and by $h^n \in H^{1,2}(B,\mathbb{R}^3)$ we obtain that $X^n \in \mathcal{U}(\Gamma^n,\tau^n)$, $\forall\, n \in \mathbb{N}$ (see (4), (5) and Definition 6.2 in [6]). Thus we conclude for the unique minimizer $X(\tau^n)$ of $\mathcal{I}$ in $\mathcal{U}(\Gamma^n,\tau^n)$ $\mathcal{I}(X(\tau^n)) \leq \mathcal{I}(X^n)$, $\forall\, n \in \mathbb{N}$, implying together with (68):

$$\limsup_{n\to\infty} \mathcal{I}(X(\tau^n)) \leq \limsup_{n\to\infty} \mathcal{I}(X^n) = \lim_{n\to\infty} \mathcal{I}(X^n) = \mathcal{I}(X), \tag{69}$$

and especially

$$\mathcal{D}(X(\tau^n)) \leq \text{const.} \quad \forall\, n \in \mathbb{N}. \tag{70}$$

Moreover using that both $X(\tau^n), X^n \in \mathcal{U}(\Gamma^n, \tau^n)$ we gain by (58):

$$\begin{aligned} |\,(X(\tau^n) - X)\,|_{\partial B}| &\le |\,(X(\tau^n) - X^n)\,|_{\partial B}| + |\,(X^n - X)\,|_{\partial B}| \\ &\le \triangle(\Gamma^n) + |\,\eta^n\,| \longrightarrow 0 \qquad \text{in } C^0(\partial B). \end{aligned} \tag{71}$$

Now recalling that the $X(\tau^n)$ are $\mathcal{I}$-surfaces in particular (see Definitions 2.1 and 6.3 in [6]) and that F meets also (R^*) we infer by (70) and (71) that we may apply Theorems 1.2 and 2.2 which yield a subsequence $X(\tau^{n_j})$ with

$$X(\tau^{n_j}) \longrightarrow \bar{X} \quad \text{in } C^0(\bar{B}, \mathbb{R}^3), \tag{72}$$

for some $\mathcal{I}$-surface $\bar{X}$. Again by (71) we conclude that $\bar{X}\,|_{\partial B} = X\,|_{\partial B}$. Thus as we required X to be an $\mathcal{I}$-surface the uniqueness of $\mathcal{I}$-surfaces, by Theorem 4.3 in [6], yields $\bar{X} = X$. Hence, we gain the assertion (64) by (72) and the "principle of subsequences". Now combining this again with Theorem 1.2 we arrive at $X(\tau^{n_j}) \rightharpoonup X$ in $H^{1,2}(B, \mathbb{R}^3)$. Hence, by the weak lower semicontinuity of $\mathcal{I}$ and (69) we finally achieve:

$$\limsup_{j\to\infty} \mathcal{I}(X(\tau^{n_j})) \le \limsup_{n\to\infty} \mathcal{I}(X(\tau^n)) \le \mathcal{I}(X) \le \liminf_{j\to\infty} \mathcal{I}(X(\tau^{n_j})).$$

Thus we obtain the assertion (65) again by the "principle of subsequences". □

Finally we state a compactness result which is proved in [10], p. 208:

Proposition 4.5 *Let* Γ *and* $\{\Gamma^n\}$ *be as in Proposition 4.3 and* $X^n \in \mathcal{C}^*(\Gamma^n)$, $n \in \mathbb{N}$, *a sequence of surfaces with* $\mathcal{D}(X^n) \le const.$, $\forall\, n \in \mathbb{N}$, *satisfying the three-point-condition* $X^n(\mathrm{e}^{i\psi_k}) = P_k \in \Gamma \;\; \forall n \in \mathbb{N}$ (*see* (6) *and* (60)). *Then there exists a subsequence* $\{X^{n_k}\}$ *whose boundary values satisfy:*

$$X^{n_k}\,|_{\partial B} \longrightarrow \beta \quad in \;\; C^0(\partial B, \mathbb{R}^3),$$

where $\beta : \mathbb{S}^1 \longrightarrow \Gamma$ *is a continuous, weakly monotonic map onto* Γ, *with* $\beta(\mathrm{e}^{i\psi_k}) = P_k$.

4.4 Proof of Theorem 4.1

Firstly by Proposition 4.2 we obtain the existence of two $\mathcal{I}$-surfaces X_l^* in $H^{1,2}_{X_l|_{\partial B}}(B, \mathbb{R}^3)$, $l = 1, 2$, that satisfy in particular

$$\sup_{\Sigma} \mathcal{I} > \max_{l=1,2} \{\mathcal{I}(X_l^*)\} \quad \forall\, \Sigma \in \mathcal{P}_{(X_1^*, X_2^*)}. \tag{73}$$

Now let $\{\Gamma^n\}$ be a fixed sequence of polygonal approximations as in Proposition 4.3 whose vertices are given in (60) and $Z_l^n := \varphi^n(X_l^*\,|_{\partial B})$, for $l = 1, 2, n \in \mathbb{N}$. As explained in (61) and (62) we gain two sequences of tuples $\tau_l^n \in T^n \subset (0, 2\pi)^{N_n}$ with

$$Z_l^n(e^{i(\tau_l^n)_j}) = A_j^n, \quad l = 1, 2, \quad j = 1, \ldots, N_n, \quad \forall n \in \mathbb{N},$$

that yield the unique minimizers $X(\tau_l^n)$ of $\mathcal{I}$ in $\mathcal{U}(\Gamma^n, \tau_l^n)$ which satisfy by Proposition 4.4:

$$X(\tau_l^n) \longrightarrow X_l^* \qquad \text{in } C^0(\bar{B}, \mathbb{R}^3), \quad l = 1, 2, \tag{74}$$

$$\mathcal{I}(X(\tau_l^n)) \longrightarrow \mathcal{I}(X_l^*) \qquad \text{for } n \to \infty, \quad l = 1, 2, \tag{75}$$

where we used that F is required to meet also condition (R^*). Furthermore by Proposition 7.6 in [6] there exists a minimizing connected set $P^n \in \wp_{(\tau_1^n,\tau_2^n)}$ w. r. to the pair $\{\tau_l^n\}$ for every $n \in \mathbb{N}$, and we firstly prove:

$$\beta^n := \max_{P^n} f^{\Gamma^n} \leq \max\{\mathcal{I}(X(\tau_1^n)), \mathcal{I}(X(\tau_2^n)), C\,\mathcal{L}(\Gamma^n)^2\} \quad \forall\, n \in \mathbb{N}, \tag{76}$$

with $C := \left(1 + \frac{k}{m_1}\right)\frac{m_2}{4}$. For, if we assume that $\beta^n > \max_{l=1,2}\{\mathcal{I}(X(\tau_l^n))\} = \max_{l=1,2}\{f^{\Gamma^n}(\tau_l^n)\}$, for some $n \in \mathbb{N}$, then the pair $\{\tau_l^n\}$ is in a mountain pass situation w. r. to f^{Γ^n}, and the "finite dimensional" mountain pass lemma, Lemma 7.10 in [6], yields the existence of a critical point $\bar{\tau}^n \in P^n_{\beta^n}$ of f^{Γ^n}. Then by Theorem 6.17 in [6] the surface $X(\bar{\tau}^n) = \psi(\bar{\tau}^n)$ is a (a.e.) conformally parametrized $\mathcal{I}$-surface. Hence, in combination with $f^{\Gamma^n} = \mathcal{I} \circ \psi^{\Gamma^n}$ and the isoperimetric inequality for $\mathcal{J}$, Corollary 3.2, we gain:

$$\beta^n = \max_{P^n} f^{\Gamma^n} = f^{\Gamma^n}(\bar{\tau}^n) = \mathcal{I}(X(\bar{\tau}^n)) = \mathcal{J}(X(\bar{\tau}^n)) \leq C\,\mathcal{L}(\Gamma^n)^2,$$

with $C := \left(1 + \frac{k}{m_1}\right)\frac{m_2}{4}$, which proves (76). Combining (76) with (75) and (58) we obtain a convergent subsequence

$$\beta^{n_k} \longrightarrow d \quad \text{for some} \quad d \leq \max\{\mathcal{I}(X_1^*), \mathcal{I}(X_2^*), C\,\mathcal{L}(\Gamma)^2\}. \tag{77}$$

We rename $\{n_k\}$ into $\{n\}$ again and work with this subsequence henceforth. Now we consider the images $\Pi^n := \psi^{\Gamma^n}(P^n)$ which are compact and connected subsets of $(\mathcal{C}^*(\Gamma^n) \cap C^0(\bar{B}, \mathbb{R}^3), \|\cdot\|_{C^0(\bar{B})})$ on account of the continuity of ψ^{Γ^n} with respect to this topology on the target space, in particular, by Theorem 6.6 (i) in [6]. Now we are going to prove the relative compactness of the union $\bigcup_{n\in\mathbb{N}} \Pi^n$ (w. r. to $\|\cdot\|_{C^0(\bar{B})}$). To this end we firstly consider an arbitrary sequence $\{Y^k\} \subset \bigcup_{n\in\mathbb{N}} \Pi^n$. If $\{Y^k\}$ is contained in only finitely many Π^n then we can certainly select a convergent subsequence of $\{Y^k\}$ due to the compactness of the Π^n. Hence, we shall suppose the contrary, which means that we can select a subsequence $\{Y^{k_j}\}$ satisfying $Y^{k_j} \in \Pi^{n_j}$ $\forall j \in \mathbb{N}$, where $\{n_j\}$ is a monotonically increasing sequence in $\mathbb{N}$. In particular we have $Y^{k_j} \in \mathcal{C}^*(\Gamma^{n_j}) \cap C^0(\bar{B}, \mathbb{R}^3)$ $\forall j \in \mathbb{N}$. Furthermore as (77) implies $\mathcal{I}(Y) \leq \beta^n \leq$ const. $\forall\, Y \in \Pi^n$ and $\forall\, n \in \mathbb{N}$, we obtain especially

$$\mathcal{D}(Y) \leq const. \qquad \forall\, Y \in \bigcup_{n\in\mathbb{N}} \Pi^n. \tag{78}$$

Thus also noting (58) and (60) we may apply Proposition 4.5 yielding a further subsequence $\{Y^{k_l}\}$ with equicontinuous and uniformly bounded boundary values. Hence, due to (78) and since the sets $\Pi^n = \psi^{\Gamma^n}(P^n)$ consist of $\mathcal{I}$-surfaces we see that the Y^{k_l} meet all requirements of Theorem 1.2 which just guarantees the existence of a further convergent subsequence of $\{Y^{k_l}\}$ w. r. to $\|\cdot\|_{C^0(\bar{B})}$. Now together with a standard argument one also shows that every sequence $\{Y^k\} \subset \overline{\bigcup_{n\in\mathbb{N}} \Pi^n} \setminus \bigcup_{n\in\mathbb{N}} \Pi^n$ possesses a convergent subsequence, aswell, which yields the asserted compactness of $\overline{\bigcup_{n\in\mathbb{N}} \Pi^n}$. Moreover by $X(\tau_l^n) = \psi(\tau_l^n) \in \Pi^n$, for $l = 1, 2$, and recalling (74) we infer that

$$\{X_l^*\} \subset \liminf_{n\in\mathbb{N}} \Pi^n. \tag{79}$$

Hence, we see that the sequence $\{\Pi^n\}$ satisfies all requirements of Proposition 4.1 implying that $\Pi := \limsup_{n\in\mathbb{N}} \Pi^n$ is again compact and connected, i.e. a continuum.

Furthermore by the definition of Π for any $X \in \Pi$ there exists a subsequence $\{\Pi^{n_k}\}$ and $\mathcal{I}$-surfaces $X^k \in \Pi^{n_k} \subset \mathcal{C}^*(\Gamma^{n_k}) \cap C^0(\bar{B}, \mathbb{R}^3)$ with

$$X^k \longrightarrow X \quad \text{in } C^0(\bar{B}, \mathbb{R}^3). \tag{80}$$

Now recalling (78) Theorem 2.2 yields that X has to be an $\mathcal{I}$-surface again which lies in $\mathcal{C}^*(\Gamma)$ on account of Proposition 4.5 (see again (60)). Hence, Π is a continuum consisting of $\mathcal{I}$-surfaces in $\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3)$ and containing the pair $\{X_l^*\}$ due to (79), which implies $\Pi \in \mathcal{P}_{(X_1^*, X_2^*)}$ in particular, thus

$$\sup_{\Pi} \mathcal{I} > \max_{l=1,2} \{\mathcal{I}(X_l^*)\} \tag{81}$$

on account of (73). Next we prove that

$$\beta := \sup_{\Pi} \mathcal{I} \leq d. \tag{82}$$

If this would be wrong then there would have to exist some surface $X \in \Pi$ with $\mathcal{I}(X) > d$. By the definition of Π we infer the existence of some sequence $\{X^k\}$ as in (80) which implies together with (78) $\| X^k \|_{H^{1,2}(B)} \leq$ const. Hence, we obtain some subsequence $X^j \in \Pi^{n_j}$ with $X^j \rightharpoonup X$ in $H^{1,2}(B, \mathbb{R}^3)$, which yields by the weak lower semicontinuity of $\mathcal{I}$ and (77):

$$d < \mathcal{I}(X) \leq \liminf_{j \to \infty} \mathcal{I}(X^j) \leq \liminf_{j \to \infty} \beta^{n_j} = \lim_{n \to \infty} \beta^n = d,$$

which is a contradiction. Hence, combining (82) with (75), (77), (81) and $f^{\Gamma^n} = \mathcal{I} \circ \psi^{\Gamma^n}$ we conclude that there exists some $n_0 \in \mathbb{N}$ such that

$$\beta^n > \max_{l=1,2} \{\mathcal{I}(X(\tau_l^n))\} = \max_{l=1,2} \{f^{\Gamma^n}(\tau_l^n)\} \qquad \forall n > n_0. \tag{83}$$

As below (76) this yields by Lemma 7.10 in [6] a critical point $\bar{\tau}^n \in P^n_{\beta^n}$ of f^{Γ^n} and by Theorem 6.17 in [6] a conformally parametrized $\mathcal{I}$-surface $X(\bar{\tau}^n) \in \Pi^n$ satisfying

$$\beta^n = \mathcal{I}(X(\bar{\tau}^n)) \qquad \forall n > n_0. \tag{84}$$

Now as below (78) we firstly infer by (78) (and (60)) that we may apply Proposition 4.5 yielding a subsequence $\{X(\bar{\tau}^{n_k})\}$ with converging boundary values in $C^0(\partial B, \mathbb{R}^3)$, which enables us to apply Theorem 1.2 to the $\mathcal{I}$-surfaces $X(\bar{\tau}^{n_k})$ guaranteeing the existence of a further convergent subsequence:

$$X(\bar{\tau}^{n_j}) \longrightarrow \bar{X} \quad \text{in } C^0(\bar{B}, \mathbb{R}^3). \tag{85}$$

Hence, since $X(\bar{\tau}^{n_j}) \in \Pi^{n_j}$ we obtain $\bar{X} \in \Pi$ by the definition of Π, which implies in particular that $\bar{X}$ has to be again an $\mathcal{I}$-surface lying in $\mathcal{C}^*(\Gamma)$. Since we additionally know that the $\mathcal{I}$-surfaces $X(\bar{\tau}^{n_j})$ are conformally parametrized and that

$$\mathcal{L}(X(\bar{\tau}^{n_j}) \mid_{\partial B}) = \mathcal{L}(\Gamma^{n_j}) \longrightarrow \mathcal{L}(\Gamma) = \mathcal{L}(\bar{X} \mid_{\partial B}) \quad \text{for } j \to \infty$$

on account of the weak monotonicity of the boundary values and (58), we infer from Corollary 3.1 that

$$\mathcal{I}(X(\bar{\tau}^{n_j})) \longrightarrow \mathcal{I}(\bar{X}) \quad \text{for } j \to \infty \tag{86}$$

and that $\bar{X}$ is also conformally parametrized on B, hence in particular a $\mathcal{J}$-extremal surface by Lemma 3.6 in [6]. Now combining (77), (82), (84) and (86) with the fact that $\bar{X} \in \Pi$ we arrive at:

$$\beta \le d \longleftarrow \beta^{n_j} = \mathcal{I}(X(\bar{\tau}^{n_j})) \longrightarrow \mathcal{I}(\bar{X}) \le \sup_{\Pi} \mathcal{I} = \beta \quad \text{for } j \to \infty, \tag{87}$$

which implies at once:

$$\mathcal{I}(\bar{X}) = d = \beta, \tag{88}$$

i.e. $\bar{X}$ "sits on the top of Π". This gives rise to consider the set

$$\Pi^* := \{X \in \Pi \mid \mathcal{I}(X) = \beta,\ X \text{ is conform. param. on } B\}\ (\neq \emptyset). \tag{89}$$

Furthermore (81) guarantees that $\Pi \setminus \Pi^* \neq \emptyset$. Now we prove that Π^* is closed. To this end we consider a sequence $\{Y^j\} \subset (\Pi^*, \|\cdot\|_{C^0(\bar{B})})$ satisfying

$$Y^j \longrightarrow Y \quad \text{in } C^0(\bar{B}, \mathbb{R}^3).$$

First of all we see that $Y \in \Pi$, as Π is closed. As all Y^j are conformally parametrized $\mathcal{I}$-surfaces in $\mathcal{C}^*(\Gamma)$, satisfying $\mathcal{L}(Y^j\mid_{\partial B}) \equiv \mathcal{L}(\Gamma)$ and $\mathcal{D}(Y^j) \le \frac{\beta}{k}\ \forall j \in \mathbb{N}$ by (89) we see due to Corollary 3.1 that firstly $\beta \equiv \mathcal{I}(Y^j) \longrightarrow \mathcal{I}(Y)$, thus $\mathcal{I}(Y) = \beta$, and secondly that Y is again conformally parametrized on B. Hence, in fact we confirm that $Y \in \Pi^*$. Using this we can conclude that the boundary $\partial\Pi^*$ of Π^* in Π is non-empty, i.e. there exists at least one point $X^* \in \Pi^*$ which satisfies $B_\epsilon(X^*) \cap (\Pi \setminus \Pi^*) \neq \emptyset$ $\forall \epsilon > 0$. Otherwise Π^* would be an open and closed subset of the connected set Π, in contradiction to the fact that both $\Pi \setminus \Pi^*$ and Π^* are non-empty. We choose such a boundary point X^* and show firstly that X^* is $\mathcal{I}$-unstable. To this end we consider the (non-empty) intersection $B_\epsilon(X^*) \cap (\Pi \setminus \Pi^*)$ for an arbitrarily fixed $\epsilon > 0$. If there were a surface $\tilde{X}$ in $B_\epsilon(X^*) \cap (\Pi \setminus \Pi^*)$ with $\mathcal{I}(\tilde{X}) < \beta = \mathcal{I}(X^*)$, then we would be done. Hence, we have to consider the case in which $\mathcal{I}(Y) \ge \beta \quad \forall Y \in B_\epsilon(X^*) \cap (\Pi \setminus \Pi^*)$, but then we have

$$\beta \le \mathcal{I}(Y) \le \sup_{\Pi} \mathcal{I} = \beta, \quad \text{i.e.} \quad \mathcal{I}(Y) = \beta \quad \forall Y \in B_\epsilon(X^*) \cap (\Pi \setminus \Pi^*). \tag{90}$$

Now we fix some $Y \in B_\epsilon(X^*) \cap (\Pi \setminus \Pi^*)$ and choose another ball $B_\delta(Y) \subset B_\epsilon(X^*)$ around Y for a sufficiently small $\delta > 0$. Again we only have to consider the case in which

$$\mathcal{I}(Z) \ge \beta = \mathcal{I}(Y) \quad \forall Z \in B_\delta(Y) \cap \mathcal{C}^*(\Gamma), \tag{91}$$

otherwise we would be done. Now we choose an arbitrary family $\phi_\epsilon : \bar{B} \overset{\cong}{\longrightarrow} \bar{B}$ of inner variations of "medium type", i.e. of the class $\mathcal{V}$, as defined in Definition 6.7 in [6], which do not affect the three points $\{e^{i\psi_k}\}$ of the three-point-condition. Then the inner variations $Y \circ \phi_\epsilon$ still satisfy $Y \circ \phi_\epsilon \in B_\delta(Y) \cap \mathcal{C}^*(\Gamma)$, for $|\epsilon| \le \epsilon_0$ sufficiently small. Hence, we infer by (91):

$$\mathcal{F}(Y) + k\,\mathcal{D}(Y) = \mathcal{I}(Y) \le \mathcal{I}(Y \circ \phi_\epsilon) = \mathcal{F}(Y \circ \phi_\epsilon) + k\,\mathcal{D}(Y \circ \phi_\epsilon) \quad \forall\ |\epsilon| \le \epsilon_0.$$

Together with the invariance of the parametric functional $\mathcal{F}$ w. r. to orientation preserving reparametrizations of $\bar{B}$ we arrive at $\mathcal{D}(Y) \le \mathcal{D}(Y \circ \phi_\epsilon)$, $\forall\ |\epsilon| \le \epsilon_0$, yielding $\partial\mathcal{D}(Y, \lambda) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathcal{D}(Y \circ \phi_\epsilon)\mid_{\epsilon=0} = 0$, with $\lambda := \frac{\mathrm{d}}{\mathrm{d}\epsilon}\phi_\epsilon \mid_{\epsilon=0}$ (see Proposition 6.10 in [6]). Moreover an arbitrary family $\{\phi_\epsilon\} \in \mathcal{V}$ can be "renormed" by a uniquely determined family of Moebius transformations $\{K_\epsilon\} \subset \mathrm{Aut}(B)$, which means that $\tilde{\phi}_\epsilon := \phi_\epsilon \circ K_\epsilon$

satisfies $\tilde{\phi}_\epsilon(\mathrm{e}^{i\psi_k}) \equiv \mathrm{e}^{i\psi_k}$ and again $\{\tilde{\phi}_\epsilon\} \in \mathcal{V}$ (see Remark 6.11 in [6] and p. 71 in [7]). Since $\mathcal{D}$ is invariant with respect to conformal reparametrizations of $\bar{B}$ we infer for an arbitrary family $\{\phi_\epsilon\} \in \mathcal{V}$:

$$\partial\mathcal{D}(Y,\lambda) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathcal{D}(Y\circ\phi_\epsilon)\mid_{\epsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\mathcal{D}(Y\circ\tilde{\phi}_\epsilon)\mid_{\epsilon=0} = \partial\mathcal{D}(Y,\tilde{\lambda}) = 0,$$

with $\lambda := \frac{\mathrm{d}}{\mathrm{d}\epsilon}\phi_\epsilon \mid_{\epsilon=0}$ and $\tilde{\lambda} := \frac{\mathrm{d}}{\mathrm{d}\epsilon}\tilde{\phi}_\epsilon \mid_{\epsilon=0}$. Now by Lemma 6.18 and Proposition 6.19 in [6] we conclude from this that Y is conformally parametrized on B. Thus together with (90) we conclude $Y \in \Pi^*$, in contradiction to our choice $Y \in B_\epsilon(X^*) \cap (\Pi \setminus \Pi^*)$. Thus in fact there has to be a surface $\tilde{X} \in B_\epsilon(X^*) \cap (\Pi \setminus \Pi^*) \subset B_\epsilon(X^*) \cap \mathcal{C}^*(\Gamma)$ with $\mathcal{I}(\tilde{X}) < \mathcal{I}(X^*)$. Now using $\mathcal{J} \le \mathcal{I}$ and that X^* is conformally parametrized we conclude from this:

$$\mathcal{J}(\tilde{X}) \le \mathcal{I}(\tilde{X}) < \mathcal{I}(X^*) = \mathcal{J}(X^*),$$

which proves the $\mathcal{J}$-instability of the $\mathcal{J}$-extremal surface $X^* \in \mathcal{C}^*(\Gamma) \cap C^0(\bar{B},\mathbb{R}^3)$. Moreover by our assumption that X_1 and X_2 are two different conformally parametrized surfaces in $\mathcal{C}^*(\Gamma) \cap C^0(\bar{B},\mathbb{R}^3)$ that are in a mountain pass situation w. r. to $\mathcal{J}$ with elevation $e \ge 0$ we obtain as in the proof of Proposition 7.8 in [6] that for the continuum $\Pi \in \mathcal{P}_{(X_1^*,X_2^*)}$ there has to exist some $\Sigma^* \in \mathcal{P}_{(X_1,X_2)}$ with $\sup_\Pi \mathcal{I} \ge \sup_{\Sigma^*} \mathcal{I}$. Thus again using that $\mathcal{J} \le \mathcal{I}$ and that X_1 and X_2 are in a mountain pass situation w. r. to $\mathcal{J}$ with elevation e we finally obtain by $\Sigma^* \in \mathcal{P}_{(X_1,X_2)}$:

$$\mathcal{J}(X^*) = \mathcal{I}(X^*) = \beta = \sup_\Pi \mathcal{I} \ge \sup_{\Sigma^*} \mathcal{I} \ge \sup_{\Sigma^*} \mathcal{J} > \max\{\mathcal{J}(X_1), \mathcal{J}(X_2)\} + e.$$

5 Proof of the main result

In this final section we drop the condition (R*) on the integrand F but require F to meet (A*) instead of only (A), i.e. that $F - \lambda \mid \cdot \mid$ is convex, for some fixed $\lambda > 0$, and consider an approximating sequence of integrands $\{F_\epsilon\}$ for F in the sense of Proposition 2.1 in [8], satisfying the requirements (A) and (R*). We will denote $\mathcal{F}_\epsilon(X) := \int_B F_\epsilon(X_u \wedge X_v)\,\mathrm{d}u\mathrm{d}v$, $\mathcal{J}_\epsilon := \mathcal{F}_\epsilon + k\,\mathcal{A}$ and $\mathcal{I}_\epsilon := \mathcal{F}_\epsilon + k\,\mathcal{D}$. Firstly we need the following crucial compactness result which we can derive from an idea due to Hildebrandt in [5], Theorems 4.1 and 4.2, on account of our isoperimetric inequality (56) for $\mathcal{I}_\epsilon$-surfaces and the imposed chord-arc-condition (57) on Γ.

Theorem 5.1 *An arbitrary family $\{X_\epsilon\}_{\epsilon>0}$ of $\mathcal{I}_\epsilon$- (respectively $\mathcal{J}_\epsilon$-) extremal surfaces in $\mathcal{C}^*(\Gamma)$ is equicontinuous on $\bar{B}$.*

Proof Let $w_0 \in B_2(0)$ be an arbitrary point and set $S_r(w_0) := B \cap B_r(w_0)$, for any $r > 0$, $C'_r(w_0) \cup C_r(w_0) := (B_r(w_0) \cap \partial B) \cup (\partial B_r(w_0) \cap \bar{B}) = \partial S_r(w_0)$, $\{\zeta_r^1(w_0), \zeta_r^2(w_0)\} := \partial B_r(w_0) \cap \partial B$, $\gamma_\epsilon(r) := \gamma_\epsilon(r,w_0) := trace(X_\epsilon \mid_{C_r(w_0)})$, $\gamma'_\epsilon(r) := \gamma'_\epsilon(r,w_0) := trace(X_\epsilon \mid_{C'_r(w_0)})$. Noting that the F_ϵ all share the same growth constants m_1 and m_2 we firstly infer from the isoperimetric inequality (56) for $\mathcal{I}_\epsilon$-surfaces, the conformality of the X_ϵ on B and $\{X_\epsilon\} \subset \mathcal{C}^*(\Gamma)$:

$$\mathcal{D}(X_\epsilon) \le \frac{m_2}{4m_1}\mathcal{L}(\Gamma)^2 \quad \forall\, \epsilon > 0. \tag{92}$$

Thus the Courant–Lebesgue Lemma yields the equicontinuity of the boundary values $\{X_\epsilon \mid_{\partial B}\}$. Using this we prove now that there is some $R > 0$ independent of

w_0 and ϵ such that $\gamma'_\epsilon(r)$ coincides with the shorter arc on Γ connecting $X_\epsilon(\zeta_r^1(w_0))$ and $X_\epsilon(\zeta_r^2(w_0))$ for any $r \le R$, where we note that if $C'_r(w_0)$ is empty, for some w_0 and $r > 0$, then the corresponding empty arcs $\gamma'_\epsilon(r)$ tautologically satisfy this condition. For if this were not true, then for every $R > 0$ there would have to exist some point $w_R \in B_2(0)$ and some $\epsilon(R) > 0$ such that $\gamma'_{\epsilon(R)}(R)$ was the longer arc on Γ connecting $X_{\epsilon(R)}(\zeta_R^1(w_R))$ and $X_{\epsilon(R)}(\zeta_R^2(w_R))$. Due to $\{w_R\} \subset B_2(0)$ we would obtain a null-sequence $\{R_j\}$ such that $w_{R_j} \longrightarrow w^*$ for some point $w^* \in \overline{B_2(0)}$. Thus there exists some index N such that $C'_{R_j}(w_{R_j})$ would contain at most one of the points $\{e^{i\psi_k}\}$, $k = 0, 1, 2$, of the three-point condition for $j > N$, which implies that $\Gamma \setminus \gamma'_{\epsilon_j}(R_j)$ would contain at least two points, say P_1, P_2, of the three-point condition for $j > N$, where we denote $\epsilon_j := \epsilon(R_j)$. Since we may apply the chord-arc condition to the shorter arcs $\Gamma \setminus \gamma'_{\epsilon_j}(R_j)$ we can conclude from the equicontinuity of $\{X_{\epsilon_j}\ |_{\partial B}\}$ and $|\ \zeta_{R_j}^1(w_{R_j}) - \zeta_{R_j}^2(w_{R_j})\ | < 2\,R_j \longrightarrow 0$:

$$\mathcal{L}(\Gamma \setminus \gamma'_{\epsilon_j}(R_j)) \le C\ |\ X_{\epsilon_j}(\zeta_{R_j}^1(w_{R_j})) - X_{\epsilon_j}(\zeta_{R_j}^2(w_{R_j}))\ | \longrightarrow 0 \tag{93}$$

for $j \to \infty$. On the other hand we know that $\mathcal{L}(\Gamma \setminus \gamma'_{\epsilon_j}(R_j)) \ge |\ P_1 - P_2\ |\ \forall j > N$, which contradicts (93) and proves our claim. Hence, applying the chord-arc condition to $\gamma'_\epsilon(r)$ we achieve:

$$\mathcal{L}(\gamma'_\epsilon(r)) \le C\ |\ X_\epsilon(\zeta_r^1(w_0)) - X_\epsilon(\zeta_r^2(w_0))\ | \tag{94}$$

$\forall r \le R$, any $w_0 \in B_2(0)$ and any $\epsilon > 0$. As we have trivially $|\ X_\epsilon(\zeta_r^1(w_0)) - X_\epsilon(\zeta_r^2(w_0))\ | \le \mathcal{L}(\gamma_\epsilon(r))$ we arrive at

$$\mathcal{L}(\gamma'_\epsilon(r)) \le C\,\mathcal{L}(\gamma_\epsilon(r)) \tag{95}$$

$\forall r \le R$ and any $\epsilon > 0$, where $w_0 \in B_2(0)$ is arbitrarily chosen. We note that if $C'_r(w_0)$ is empty and therefore $\mathcal{L}(\gamma'_\epsilon(r)) \equiv 0$, then (95) is satisfied trivially. Now we combine this estimate with the isoperimetric inequality (56) applied to the conformally parametrized $\mathcal{I}_\epsilon$-surfaces X_ϵ on $S_r(w_0)$, which yields for $\phi_\epsilon(r) := \phi_\epsilon(r, w_0) := \mathcal{D}_{S_r(w_0)}(X_\epsilon)$:

$$\phi_\epsilon(r) = \mathcal{A}_{S_r(w_0)}(X_\epsilon) \le \frac{m_2}{4m_1}(\mathcal{L}(\gamma'_\epsilon(r)) + \mathcal{L}(\gamma_\epsilon(r)))^2 \le \frac{m_2}{4m_1}(C+1)^2\,\mathcal{L}(\gamma_\epsilon(r))^2, \tag{96}$$

$\forall r \le R$ and any $\epsilon > 0$. Now introducing polar coordinates about the point w_0 one easily achieves

$$\frac{\mathrm{d}}{\mathrm{d}r}\phi_\epsilon(r) \ge \int_{\theta^1(r)}^{\theta^2(r)} \frac{1}{r}\ |\ (X_\epsilon)_\theta(r, \theta)\ |^2\ \mathrm{d}\theta,$$

for a.e. $r > 0$. From this one derives as in the proof of Theorem 4.1 in [5]:

$$\mathcal{L}(\gamma_\epsilon(r))^2 \le 4\pi r\,\frac{\mathrm{d}}{\mathrm{d}r}\phi_\epsilon(r),$$

for a.e. $r > 0$ and any $\epsilon > 0$. Hence, in combination with (96) we achieve the differential inequality $\phi_\epsilon(r) \le \frac{r}{\mu}\,\frac{\mathrm{d}}{\mathrm{d}r}\phi_\epsilon(r)$, for a.e. $r \in (0, R)$ and any $\epsilon > 0$, with $\mu := \frac{m_1}{m_2\pi\,(C+1)^2}$.

Now by a well-known lemma, Lemma 4.2 in [5], and (92) we achieve a uniform "Dirichlet growth condition" for the X_ϵ:

$$\mathcal{D}_{S_r(w_0)}(X_\epsilon) = \phi_\epsilon(r) \le \phi_\epsilon(R)\Big(\frac{r}{R}\Big)^\mu \le \frac{m_2}{4m_1}\mathcal{L}(\Gamma)^2\Big(\frac{r}{R}\Big)^\mu,$$

$\forall r \le R$, any $\epsilon > 0$ and any $w_0 \in B_2(0)$. Thus by a well-known reasoning due to Morrey, as stated in Lemma 2.3 in [5], one can derive from this estimate a uniform bound of the "Hölder-quotients" of the X_ϵ on $\bar{B}$, which only depends on m_1, m_2, C and $\mathcal{L}(\Gamma)$, i.e.

$$\mid X_\epsilon(w) - X_\epsilon(w') \mid \le const.(m_1, m_2, C, \mathcal{L}(\Gamma)) \mid w - w' \mid^{\frac{\mu}{2}} \quad \forall w, w' \in \bar{B}$$

and for any $\epsilon > 0$, which proves the assertion of the theorem. □

Now let Ω be an arbitrary subdomain of B and $(\mathcal{F}_\epsilon)_\Omega(X) := \int_\Omega F_\epsilon(X_u \wedge X_v)\,\mathrm{d}u\mathrm{d}v$. We can easily derive from (9) in [8]:

Proposition 5.1 *There holds for any $X \in H^{1,2}(B, \mathbb{R}^3)$, for $\epsilon \searrow 0$:*

$$\mid (\mathcal{F}_\epsilon)_\Omega(X) - \mathcal{F}_\Omega(X) \mid \le \sup_{\mathbb{R}^3\setminus\{0\}} \mid \nabla F_\epsilon - \nabla F \mid \mathcal{A}_\Omega(X) \longrightarrow 0. \tag{97}$$

Proof By $F_\epsilon(0) = F(0) = 0$ and $\langle \nabla F_\epsilon(z), z\rangle = F_\epsilon(z)$ respectively $\langle \nabla F(z), z\rangle = F(z)$, for $z \ne 0$, we obtain, abbreviating $Z := X_u \wedge X_v$ and $\mathcal{R} := \mathcal{R}(X)$:

$$\begin{aligned}\mid (\mathcal{F}_\epsilon)_\Omega(X) - \mathcal{F}_\Omega(X) \mid &\le \int_\Omega \mid F_\epsilon(Z) - F(Z) \mid \,\mathrm{d}u\mathrm{d}v \\ &= \int_{\Omega\cap\mathcal{R}} \mid F_\epsilon(Z) - F(Z) \mid \,\mathrm{d}u\mathrm{d}v = \int_{\Omega\cap\mathcal{R}} \mid \langle \nabla F_\epsilon(Z) - \nabla F(Z), Z\rangle \mid \,\mathrm{d}u\mathrm{d}v \\ &\le \sup_{\mathbb{R}^3\setminus\{0\}} \mid \nabla F_\epsilon - \nabla F \mid \mathcal{A}_\Omega(X) \longrightarrow 0, \quad \text{for } \epsilon \searrow 0,\end{aligned}$$

due to property (9) of $\{F_\epsilon\}$ in [8]. □

Now we can prove (see Definition 4.2)

Proposition 5.2 *If $X_1, X_2 \in \mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3)$ are in a mountain pass situation with respect to $\mathcal{J}$ with some elevation $e > 0$, then there exists some $\bar{\epsilon} > 0$ such that X_1, X_2 are in a mountain pass situation with respect to $\mathcal{J}_\epsilon$ with elevation $\frac{e}{4}$, $\forall \epsilon < \bar{\epsilon}$.*

Proof We denote $m := \max_{l=1,2}\{\mathcal{J}(X_l)\}$ and $M := m + e$. Firstly we infer from the above proposition the existence of some $\bar{\epsilon} > 0$ such that

$$\mid \mathcal{J}_\epsilon(X_l) - \mathcal{J}(X_l) \mid = \mid \mathcal{F}_\epsilon(X_l) - \mathcal{F}(X_l) \mid < \frac{e}{4} \tag{98}$$

$\forall \epsilon < \bar{\epsilon}$ and for $l = 1, 2$. Now we choose some $\Sigma \in \mathcal{P}_{(X_1, X_2)}$ arbitrarily. By our requirement there has to exist some surface $X \in \Sigma$ with $\mathcal{J}(X) = M + \delta$ for some $\delta > 0$, which yields in particular $\mathcal{A}(X) \le \frac{M+\delta}{k}$. Hence, denoting $\rho(\epsilon) := \sup_{\mathbb{R}^3\setminus\{0\}} \mid \nabla F_\epsilon - \nabla F \mid$ we obtain together with the above proposition

$$\begin{aligned}\mathcal{J}_\epsilon(X) \ge \mathcal{J}(X) - \mid \mathcal{J}_\epsilon(X) - \mathcal{J}(X) \mid &\ge M + \delta - \rho(\epsilon)\frac{M+\delta}{k} \\ &= \Big(1 - \frac{\rho(\epsilon)}{k}\Big)(M+\delta) > \Big(1 - \frac{\rho(\epsilon)}{k}\Big)M,\end{aligned} \tag{99}$$

for sufficiently small ϵ. Hence, by choosing $\bar{\epsilon}$ that small such that $\rho(\epsilon) < \frac{k}{2}\Big(1 - \frac{m}{M}\Big)$, $\forall\, \epsilon < \bar{\epsilon}$, we obtain by (99):

$$\sup_{\Sigma} \mathcal{J}_\epsilon \geq \mathcal{J}_\epsilon(X) > \Big(1 - \frac{\rho(\epsilon)}{k}\Big) M > \frac{M+m}{2} = m + \frac{\mathrm{e}}{2},$$

and therefore together with (98):

$$\sup_{\Sigma} \mathcal{J}_\epsilon > \Big(m + \frac{e}{4}\Big) + \frac{\mathrm{e}}{4} > \max_{l=1,2}\{\mathcal{J}_\epsilon(X_l)\} + \frac{e}{4},$$

$\forall\, \epsilon < \bar{\epsilon}$ and for any $\Sigma \in \mathcal{P}_{(X_1,X_2)}$, which proves our assertion. □

Hence, combining the above proposition with the requirements of our desired main result, Theorem 1.1, we achieve

Corollary 5.1 *There exists some sequence $\{X^*_{\epsilon_n}\}$ of unstable $\mathcal{J}_{\epsilon_n}$-extremal surfaces with $\mathcal{J}_{\epsilon_n}(X^*_{\epsilon_n}) > \max\{\mathcal{J}_{\epsilon_n}(X_1), \mathcal{J}_{\epsilon_n}(X_2)\} + \frac{e}{4}$, for some null-sequence $\{\epsilon_n\}$, and some limit surface X^* in $\mathcal{C}^*(\Gamma) \cap C^0(\bar{B}, \mathbb{R}^3)$ such that*

$$X^*_{\epsilon_n} \longrightarrow X^* \quad in \;\; C^0(\bar{B}, \mathbb{R}^3) \quad and\ weakly\ in \;\; H^{1,2}(B, \mathbb{R}^3). \tag{100}$$

Proof By Proposition 5.2 we can apply Theorem 4.1 to X_1, X_2 and F_ϵ, for $\epsilon < \bar{\epsilon}$, yielding the existence of some unstable $\mathcal{J}_\epsilon$-extremal surface $\{X^*_\epsilon\}$ with $\mathcal{J}_\epsilon(X^*_\epsilon) > \max\{\mathcal{J}_\epsilon(X_1), \mathcal{J}_\epsilon(X_2)\} + \frac{e}{4}$. Moreover by Theorem 5.1 the family $\{X^*_\epsilon\}_{\epsilon<\bar{\epsilon}}$ is equicontinuous on $\bar{B}$ and by (92) together with a suitable Poincaré inequality we also know that $\| X^*_\epsilon \|_{H^{1,2}(B)} \leq$ const. Thus together with Rellich's embedding theorem, Riesz' selection theorem and the proof of Arzela-Ascoli's theorem we achieve our assertion. □

Now combining this with (56) and Proposition 5.1 we can apply the ideas of the proof of Theorem 2.1 in order to show

Theorem 5.2 *There holds also for every $r \in (0, 1)$:*

$$\| X^*_{\epsilon_n} - X^* \|_{H^{1,2}(B_r(0))} \longrightarrow 0 \quad for \;\; n \to \infty. \tag{101}$$

In particular, X^* turns out to be conformally parametrized a.e. on B.

Proof We denote $(\mathcal{I}_\epsilon)_\Omega := (\mathcal{F}_\epsilon)_\Omega + k\,\mathcal{D}_\Omega$ for any domain $\Omega \subset B$. Firstly we infer from Proposition 5.1:

$$| (\mathcal{I}_\epsilon)_\Omega(X^*) - \mathcal{I}_\Omega(X^*) | = | (\mathcal{F}_\epsilon)_\Omega(X^*) - \mathcal{F}_\Omega(X^*) | \longrightarrow 0, \tag{102}$$

and together with inequality (56) and property (9) of $\{F_\epsilon\}$ in [8] even:

$$\begin{aligned} | (\mathcal{I}_\epsilon)_\Omega(X^*_\epsilon) - \mathcal{I}_\Omega(X^*_\epsilon) | &= | (\mathcal{F}_\epsilon)_\Omega(X^*_\epsilon) - \mathcal{F}_\Omega(X^*_\epsilon) | \\ &\leq \rho(\epsilon)\, \mathcal{A}_B(X^*_\epsilon) \leq \rho(\epsilon)\, \frac{m_2}{4\, m_1} \mathcal{L}(\Gamma)^2 \longrightarrow 0 \quad \text{for } \epsilon \searrow 0, \end{aligned} \tag{103}$$

with $\rho(\epsilon) := \sup_{\mathbb{R}^3 \setminus \{0\}} | \nabla F_\epsilon - \nabla F |$. We choose some $r \in (0, 1)$ arbitrarily such that (7) holds for $Y^n := X^* - X^*_{\epsilon_n}$ and set $\delta_n := \| X^* - X^*_{\epsilon_n} \|_{C^0(\bar{B})}$, consider the sequence $\{r_n\}$

given by $r_n := r + \delta_n$ and set $B_r := B_r(0)$. Firstly as in (20) we select a subsequence $\{\epsilon_{n_j}\}$, depending on r, such that

$$\mathcal{D}_{C_{r,r_{n_j}}}(X^* - X^*_{\epsilon_{n_j}}) \longrightarrow 0 \quad \text{for } j \to \infty. \tag{104}$$

Now we can follow the lines of the proof of Theorem 2.1 in order to show:

$$\limsup_{j\to\infty} (\mathcal{I}_{\epsilon_{n_j}})_{B_r}(X^*_{\epsilon_{n_j}}) \le \lim_{j\to\infty} (\mathcal{I}_{\epsilon_{n_j}})_{B_r}(X^*) = \mathcal{I}_{B_r}(X^*). \tag{105}$$

To this end we simply have to replace F by F_{ϵ_n}, $\bar{X}$ by X^* and X^n by $X^*_{\epsilon_n}$, thus now using the notations $\mathcal{R}^n := \mathcal{R}(X^*_{\epsilon_n})$, Z^n and δZ^n with the analogous meanings. We test the variational equalities

$$\delta\mathcal{I}_{\epsilon_n}(X^*_{\epsilon_n}, \varphi) = 0 \quad \forall\, \varphi \in \mathring{H}^{1,2}(B, \mathbb{R}^3)$$

(see (9) and p. 407 in [6]) of the $\mathcal{I}_{\epsilon_n}$-extremal surfaces $X^*_{\epsilon_n}$ by the functions

$$\varphi^n(w) := \begin{cases} X^*(w) - X^*_{\epsilon_n}(w) & : \quad w \in B_r(0) \\ \frac{r_n - |w|}{r_n - r}(X^*(w) - X^*_{\epsilon_n}(w)) & : \quad w \in \overline{C_{r,r_n}} \\ 0 & : \quad w \in C_{r_n,1}. \end{cases}$$

Since the F_ϵ share the same properties (A) with the replaced F, especially the same growth constants m_1 and m_2, we obtain as in (18), (21) and (23) on account of (104) for the subsequence $\{\epsilon_{n_j}\}$:

$$\begin{aligned} &(\mathcal{I}_{\epsilon_{n_j}})_{B_r}(X^*) - (\mathcal{I}_{\epsilon_{n_j}})_{B_r}(X^*_{\epsilon_{n_j}}) \\ &\ge - \int\limits_{\mathcal{R}^{n_j} \cap C_{r,r_{n_j}}} \langle \nabla F_{\epsilon_{n_j}}(Z^{n_j}), \delta Z^{n_j} \rangle \, \mathrm{d}u\mathrm{d}v - k \int\limits_{C_{r,r_{n_j}}} DX^*_{\epsilon_{n_j}} \cdot D\varphi^{n_j} \, \mathrm{d}u\mathrm{d}v \\ &= -\delta(\mathcal{I}_{\epsilon_{n_j}})_{C_{r,r_{n_j}}}(X^*_{\epsilon_{n_j}}, \varphi^{n_j}) \longrightarrow 0, \end{aligned}$$

for $j \to \infty$, i.e.

$$\liminf_{j\to\infty} ((\mathcal{I}_{\epsilon_{n_j}})_{B_r}(X^*) - (\mathcal{I}_{\epsilon_{n_j}})_{B_r}(X^*_{\epsilon_{n_j}})) \ge 0,$$

which yields (105) taking also (102) into account. As we also know that $X^*_{\epsilon_n} \rightharpoonup X^*$ in $H^{1,2}(B, \mathbb{R}^3)$ by (100) we obtain by the weak lower semicontinuity of $\mathcal{I}_{B_r}$, (103) and (105):

$$\begin{aligned} \limsup_{j\to\infty} (\mathcal{I}_{\epsilon_{n_j}})_{B_r}(X^*_{\epsilon_{n_j}}) \le \mathcal{I}_{B_r}(X^*) &\le \liminf_{j\to\infty} \mathcal{I}_{B_r}(X^*_{\epsilon_{n_j}}) \\ &= \liminf_{j\to\infty} (\mathcal{I}_{\epsilon_{n_j}})_{B_r}(X^*_{\epsilon_{n_j}}), \end{aligned}$$

and therefore again combined with (103):

$$\mathcal{I}_{B_r}(X^*) = \lim_{j\to\infty} (\mathcal{I}_{\epsilon_{n_j}})_{B_r}(X^*_{\epsilon_{n_j}}) = \lim_{j\to\infty} \mathcal{I}_{B_r}(X^*_{\epsilon_{n_j}}).$$

Now in combination with the weak $H^{1,2}(B)$-convergence in (100) we infer by Lemma 6 on p. 43 in [7] that

$$\mathcal{D}_{B_r}(X^*_{\epsilon_{n_j}}) \longrightarrow \mathcal{D}_{B_r}(X^*) \quad \text{for } j \to \infty, \tag{106}$$

thus again combined with the weak $H^{1,2}(B)$-convergence and the $C^0(\bar{B})$-convergence in (100) we arrive at the assertion (101) for the chosen radius $r \in (0,1)$ and the considered subsequence $\{X^*_{\epsilon_{n_j}}\}$. Now (101) follows exactly as in the ending of the proof of Theorem 2.1, which implies in particular that X^* inherits its conformality from the $X^*_{\epsilon_n}$ a.e. on $B_r(0)$ for any $r < 1$, thus a.e. on B. □

Now we follow the lines of the proof of Theorem 2.2 to achieve

Theorem 5.3 *The limit surface X^* of Corollary 5.1 is an $\mathcal{I}$-surface again, thus a $\mathcal{J}$-extremal surface.*

Proof We replace $\bar{X}$ by X^* and X^n by $X^*_{\epsilon_n}$ in the proof of Theorem 2.2, choose some arbitrary $r \in (0,1)$, define the sets $\mathcal{S}_r$, $\mathcal{R}_r$, $\mathcal{S}^n_r$, $\mathcal{R}^n_r$, σ^n and τ^n analogously to (28) and use the abbreviations Z, Z^n, δZ and δZ^n with the analogous meanings. Taking into account the conformality of the involved surfaces the desired inequality (29) becomes now

$$\delta\mathcal{F}_{B_r}(X^*,\varphi) \geq \liminf_{n\to\infty} \delta(\mathcal{F}_{\epsilon_n})_{B_r}(X^*_{\epsilon_n},\varphi) \tag{107}$$

$\forall\, \varphi \in \mathring{H}^{1,2}(B_r(0),\mathbb{R}^3)$, with $B_r := B_r(0)$, and (32) turns into

$$\begin{aligned}
&\int_{\mathcal{R}^n_r} \langle \nabla F_{\epsilon_n}(Z^n), \delta Z^n\rangle \, \mathrm{d}u\mathrm{d}v - \int_{\mathcal{R}_r} \langle \nabla F(Z), \delta Z\rangle \, \mathrm{d}u\mathrm{d}v \\
&\quad = \int_{B_r} \chi_{\mathcal{R}^n_r\cap\mathcal{R}_r} \langle \nabla F_{\epsilon_n}(Z^n), \delta Z^n - \delta Z\rangle \, \mathrm{d}u\mathrm{d}v \\
&\qquad + \int_{B_r} \chi_{\mathcal{R}^n_r\cap\mathcal{R}_r} \langle \nabla F_{\epsilon_n}(Z^n) - \nabla F(Z), \delta Z\rangle \, \mathrm{d}u\mathrm{d}v \\
&\qquad - \int_{B_r} \chi_{\sigma^n} \langle \nabla F(Z), \delta Z\rangle \, \mathrm{d}u\mathrm{d}v + \int_{B_r} \chi_{\tau^n} \langle \nabla F_{\epsilon_n}(Z^n), \delta Z^n\rangle \, \mathrm{d}u\mathrm{d}v.
\end{aligned} \tag{108}$$

Since the F_ϵ have the same growth constants as F we obtain from the above theorem the analogue of (33). Furthermore due to (101) we obtain a subsequence $\{Z^{n_k}\}$ for which

$$Z^{n_k}(w) \longrightarrow Z(w) \quad \text{for a.e. } w \in B_r(0). \tag{109}$$

We rename $\{n_k\}$ into $\{n\}$ again and shall consider this sequence henceforth. Now we choose some point $w \in B_r(0) \setminus \mathcal{N}$ arbitrarily, where $\mathcal{N} \subset B_r(0)$ is defined as in the proof of Theorem 2.2, and only have to refine the argument for the first case in which we suppose to hold $w \in \mathcal{R}^{n_j}_r \cap \mathcal{R}_r$ for an increasing sequence $\{n_j\} \subset \mathbb{N}$. Then we obtain by (109), the continuity of ∇F on $\mathbb{R}^3 \setminus \{0\}$ and (9) in [8]:

$$\begin{aligned}
&|\, \nabla F_{\epsilon_{n_j}}(Z^{n_j})(w) - \nabla F(Z)(w) \,| \\
&\quad \leq |\, \nabla F_{\epsilon_{n_j}}(Z^{n_j})(w) - \nabla F(Z^{n_j})(w) \,| + |\, \nabla F(Z^{n_j})(w) - \nabla F(Z)(w) \,| \\
&\quad \leq \sup_{\mathbb{R}^3\setminus\{0\}} |\, \nabla F_{\epsilon_{n_j}} - \nabla F \,| + |\, \nabla F(Z^{n_j})(w) - \nabla F(Z)(w) \,| \longrightarrow 0 \quad \text{for } j \to \infty.
\end{aligned}$$

As we have $\chi_{\mathcal{R}^n_r\cap\mathcal{R}_r}(w) = 0$ for $n \in \mathbb{N} \setminus \{n_j\}$ we can conclude:

$$\chi_{\mathcal{R}^n_r\cap\mathcal{R}_r}(w)\, (\nabla F_{\epsilon_n}(Z^n)(w) - \nabla F(Z)(w))\, \delta Z(w) \longrightarrow 0 \quad \text{for } n \to \infty, \tag{110}$$

and in Case (2), i.e. if there exists some number $N \in \mathbb{N}$ such that $w \notin \mathcal{R}_r^n \cap \mathcal{R}_r, \forall n > N$, we obtain (110) immediately. Hence, we gain (110) for a.e. $w \in B_r(0)$ and by $|\nabla F_{\epsilon_n}|$, $|\nabla F| \leq m_2$ on $\mathbb{R}^3 \setminus \{0\}$ Lebesgue's convergence theorem finally implies

$$\int_{B_r} \chi_{\mathcal{R}_r^n \cap \mathcal{R}_r} (\nabla F_{\epsilon_n}(Z^n) - \nabla F(Z))\, \delta Z \, \mathrm{d}u\mathrm{d}v \longrightarrow 0. \tag{111}$$

Moreover we gain here (38) without any changes on account of (101). Finally we achieve as in (43) by the properties (A) of the F_{ϵ_n} for any $n \in \mathbb{N}$:

$$\langle \nabla F_{\epsilon_n}(Z^n), \delta Z^n \rangle \leq F_{\epsilon_n}(\delta Z^n). \tag{112}$$

Hence, noting that $\delta Z = 0$ on $\tau^n = \mathcal{S}_r \setminus \mathcal{S}_r^n$ we obtain by $F_{\epsilon_n}(0) = 0$, (112), the Lipschitz continuity of the F_{ϵ_n} with Lip.-const.$= m_2$ and (101):

$$\begin{aligned}\int_{\tau^n} \langle \nabla F_{\epsilon_n}(Z^n), \delta Z^n \rangle \, \mathrm{d}u\mathrm{d}v &\leq \int_{\tau^n} F_{\epsilon_n}(\delta Z^n) - F_{\epsilon_n}(\delta Z)\, \mathrm{d}u\mathrm{d}v \\ &\leq m_2 \int_{B_r} |\, \delta Z^n - \delta Z \,| \; \mathrm{d}u\mathrm{d}v \longrightarrow 0.\end{aligned}$$

Hence, combining this with (108), (111), the analogous convergences of (33) and (38) and recalling that we have selected subsequences twice, we obtain for a subsequence $\{n_j\}$:

$$\begin{aligned}&\liminf_{n\to\infty} (\delta(\mathcal{F}_{\epsilon_n})_{B_r}(X^*_{\epsilon_n}, \varphi) - \delta\mathcal{F}_{B_r}(X^*, \varphi)) \\ &\quad \leq \liminf_{j\to\infty} (\delta(\mathcal{F}_{\epsilon_{n_j}})_{B_r}(X^*_{\epsilon_{n_j}}, \varphi) - \delta\mathcal{F}_{B_r}(X^*, \varphi)) \\ &\quad = \liminf_{j\to\infty} \left(\int_{\mathcal{R}_r^{n_j}} \langle \nabla F_{\epsilon_{n_j}}(Z^{n_j}), \delta Z^{n_j} \rangle \, \mathrm{d}u\mathrm{d}v - \int_{\mathcal{R}_r} \langle \nabla F(Z), \delta Z \rangle \, \mathrm{d}u\mathrm{d}v \right) \leq 0\end{aligned}$$

$\forall \varphi \in \mathring{H}^{1,2}(B_r(0), \mathbb{R}^3)$, which proves (107). Together with (101) we gain

$$\delta\mathcal{I}_{B_r}(X^*, \varphi) \geq \liminf_{n\to\infty} \; \delta(\mathcal{I}_{\epsilon_n})_{B_r}(X^*_{\epsilon_n}, \varphi) = 0$$

$\forall \varphi \in \mathring{H}^{1,2}(B_r(0), \mathbb{R}^3)$, where we used that the $X^*_{\epsilon_n}$ are $\mathcal{I}_{\epsilon_n}$-extremal surfaces. Now one can follow the ending of the proof of Theorem 2.2 to achieve even

$$\delta\mathcal{I}(X^*, \varphi) \geq 0 \quad \forall \varphi \in \mathring{H}^{1,2}(B, \mathbb{R}^3),$$

which characterizes X^* to be an $\mathcal{I}$-surface by Lemma 2 in Section 2.5 in [7]. □

Now following the lines of Shiffman's proof of the "continuity theorem" 11.1 in [12], i.e. of Theorem 3.2, we show

Corollary 5.2 *There holds also*

$$\lim_{n\to\infty} \mathcal{J}(X^*_{\epsilon_n}) = \mathcal{J}(X^*) = \lim_{n\to\infty} \mathcal{J}_{\epsilon_n}(X^*_{\epsilon_n}). \tag{113}$$

Proof On account of Corollary 5.1, Theorem 5.2, $\mathcal{L}(X^*_{\epsilon_n} \mid_{\partial B}) \equiv \mathcal{L}(\Gamma) = \mathcal{L}(X^* \mid_{\partial B})$, $m_1 \mid z \mid \leq F_\epsilon(z) \leq m_2 \mid z \mid \ \forall z \in \mathbb{R}^3$ uniformly in $\epsilon > 0$ and (92) one can see that all estimates in the proof of the "continuity theorem" 11.1 in [12], i.e. of Theorem 3.2, especially the estimate on p. 548 in [12], remain valid uniformly in n, if we replace F by F_{ϵ_n}, $\mathcal{J}$ by $\mathcal{J}_{\epsilon_n}$, X^n by $X^*_{\epsilon_n}$ and $\bar{X}$ by X^*. Hence, we conclude firstly only for a subsequence $\{\epsilon_{n_j}\}$ that for any $\rho > 0$ there exists some $N(\rho) \in \mathbb{N}$ with

$$\mid \mathcal{J}_{\epsilon_{n_j}}(X^*_{\epsilon_{n_j}}) - \mathcal{J}_{\epsilon_{n_j}}(X^*) \mid < \left(1 + 3m_2 + \frac{2m_2 + m_1}{m_1} k\right) \rho,$$

if $j > N(\rho)$. Thus together with (102) and (103) for $\Omega := B$ we obtain:

$$\lim_{j \to \infty} \mathcal{J}(X^*_{\epsilon_{n_j}}) = \mathcal{J}(X^*) = \lim_{j \to \infty} \mathcal{J}_{\epsilon_{n_j}}(X^*_{\epsilon_{n_j}}).$$

Now the principle of subsequences yields assertion (113). □

Hence, combining (113) with Corollary 5.1 and (97) we arrive at:

$$\mathcal{J}(X^*) \longleftarrow \mathcal{J}_{\epsilon_n}(X^*_{\epsilon_n}) > \max_{l=1,2}\{\mathcal{J}_{\epsilon_n}(X_l)\} + \frac{\mathrm{e}}{4} \longrightarrow \max_{l=1,2}\{\mathcal{J}(X_l)\} + \frac{e}{4},$$

which finally proves the last assertion of the main result, Theorem 1.1, about the $\mathcal{J}$-(respectively $\mathcal{I}$-) extremal surface X^*.

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