CHRISTOPHE COURBAGE

ON BIVARIATE RISK PREMIA

ABSTRACT. This note examines the conditions under which the bivariate risk premium for one risk may be negative even if both risks are positively correlated, using a mean variance setting. The link between the bivariate risk premium and the partial bivariate risk premia is also investigated.

KEY WORDS: Multivariate risk aversion, Bivariate risk premium, Partial bivariate risk premium, Correlated risks, Cross derivatives

1. INTRODUCTION

For the last two decades, the literature on risk aversion measures with multiple risks has been both extensive and successful (see Doherty et al., 1981; Kihlstrom et al., 1981; Ross, 1981; Doherty et al., 1987; Pratt et al., 1987; Gollier et al., 1996 and others). These works deal with aversion to one risk when another risk is present and consider only unidimensional utility functions. Doherty et al. (1981, 1987) examine the link between partial risk premium (which removes only part of the risk) and Arrow (1971)–Pratt (1964)'s risk premium. Moreover, these authors show, in a mean-variance setting, that a negative correlation between risks is a necessary condition under which the risk premium for one risk is negative, while another background risk is present.

However, for numerous economic problems, adequate modeling requires multivariate utility functions (taking into consideration arguments other than final wealth). Such problems call for generalizing the well-known univariate risk premium concept to the multidimensional case (see Kihlstrom et al., 1974; Duncan, 1977; Karni, 1979; Ambarish et al., 1987; Demers et al., 1991).

The conditions ensuring a negative bivariate risk premium for one risk in the presence of another in a bidimensional environment (when each argument of the utility function depends on a single source of risk) are different from a unidimensional environment (where the utility function takes one argument that depends on two

Theory and Decision **50:** 29–34, 2001.

sources of risk). In this note we examine some key properties of bivariate risk premia in a mean-variance setting. Firstly, we show that the partial bivariate risk premium can be negative, even when the two risks are positively correlated. Secondly, we investigate the link between the bivariate risk premium and the partial bivariate risk premia.

2. THE BIVARIATE RISK PREMIUM

Let $u(w)$ denote a real-valued von Neumann-Morgenstern utility function where w is a 2-dimensional vector. Assume that u is strictly increasing in each component of w, concave and twice continuously differentiable. Let $\tilde{z}$ be a random vector in R^2 such as $E(\tilde{z}) = 0$ and denote by $V = [\sigma_{ij}]$ the positive semi-definite variance-covariance matrix of $\tilde{z}$.

Following Kihlstrom et al. (1974), we define the bivariate risk premium $\pi(w, \tilde{z})$ arbitrarily on the first argument:

$$E[u(w_1 + \tilde{z}_1, w_2 + \tilde{z}_2)] = u(w_1 - \pi, w_2) \tag{1}$$

Assuming small risks,[1] it is viable to approximate each side of equation (1) by a Taylor series expansion of u around w (Duncan (1977); Karni (1979)), giving:

$$\pi(w, \tilde{z}) = -\frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_{ij} \frac{u_{ij}(w)}{u_1(w)} \tag{2}$$

where u_1 and u_{ij} denote the first and second partial derivatives of u with respect to its ith and jth arguments, respectively.

3. THE PARTIAL BIVARIATE RISK PREMIUM

The partial multivariate risk premium, as defined by Chalfant et al. (1993), is the amount an individual would pay to avoid one component of the multivariate risk in the presence of others.

Define the partial bivariate risk premium π_1 for risk $\tilde{z}_1$ by:

$$E[u(w_1 + \tilde{z}_1, w_2 + \tilde{z}_2)] = E[u(w_1 - \pi_1, w_2 + \tilde{z}_2)] \tag{3}$$

For small risks, following Chalfant et al. (1993), a Taylor approximation of expression (3) yields:[2]

$$\pi_1 = -\frac{1}{2}\sigma_{11}\frac{u_{11}}{u_1} - \sigma_{12}\frac{u_{12}}{u_1} \tag{4}$$

Contrary to the univariate case, a negative correlation between the risks is no longer a necessary condition for π_1 to be negative.

From Equation (4), we obtain the following result.

PROPOSITION 1. *If $u_{12} > 0$, a positive correlation between the risks is a necessary condition to have π_1 negative.*

The interpretation is the following:

In the univariate case, if risks are positively correlated, elimination of the first risk reduces total risk, so the individual will be willing to pay to get rid of it. However, if the risks are negatively correlated, elimination of the first risk increases the total risk, and the individual can be incited not to eliminate it.

The analysis is less straightforward in a bivariate case as it is necessary to consider the sign of u_{12} in addition to the magnitude and sign of the risks. In this case, the value of u_{12} is of importance. Indeed, when u_{12} equals zero, the change in the risky situation for good one does not affect the marginal utility of good two. Since marginal utility does not change, the higher derivatives of u with respect to the second argument do not change either, and the perception of the second risk is unaltered. Of course, when u_{12} is different from zero, the elimination of the first risk modifies all the derivatives with respect to the second argument, and the perception of the second risk is altered.

Hence, if the endowment in terms of two goods is positively correlated, a consumer would not want to reduce the risk of the endowment of the first good if it is strongly complementary with the second good, whose risk cannot be reduced.

4. LINK BETWEEN THE PREMIA

An interesting relation between the bivariate risk premium and the partial bivariate risk premia can be defined as follows.

Let π_2 be the partial bivariate risk premium (specified in the second argument) for risk $\tilde{z}_2$:

$$E[u(w_1+\tilde{z}_1, w_2+\tilde{z}_2)] = E[u(w_1+\tilde{z}_1, w_2-\pi_2)] \tag{5}$$

For small risks, a Taylor series expansion around (w_1, w_2) of the left-hand side of Equation (5) gives:

$$\begin{aligned} E[u(w_1+\tilde{z}_1, w_2+\tilde{z}_2)] = {} & u(w_1, w_2) + \frac{1}{2}\sigma_{11}u_{11}(w_1, w_2) \\ & + \frac{1}{2}\sigma_{22}u_{22}(w_1, w_2) \\ & + \sigma_{12}u_{12}(w_1, w_2) + o(\mathrm{tr}V) \end{aligned}$$

where $\mathrm{tr}V = \sigma_{11} + \sigma_{22}$. Now, expanding the right-hand side of Equation (5) around (w_1, w_2), we have:

$$\begin{aligned} E[u(w_1+\tilde{z}_1, w_2-\pi_2)] = {} & u(w_1, w_2) - \pi_2 u_2(w_1, w_2) \\ & + \frac{1}{2}\sigma_{11}u_{11}(w_1, w_2) + o(\sigma_{11}). \end{aligned}$$

According to Chalfant et al. (1993), we ignore terms that contain π_2^2 since those are of the same order as the remainders. Now by assumption that the risks are small, we can ignore the remainder terms. Approximating Equation (5) using the truncated Taylor expansions thus gives an approximation for the partial bivariate risk premium:

$$\pi_2 = -\frac{1}{2}\sigma_{22}\frac{u_{22}}{u_2} - \sigma_{12}\frac{u_{12}}{u_2} \tag{6}$$

So, from Equations (4), (6) and (2), the following proposition is straightforward.

PROPOSITION 2. $\pi = \pi_1 + \pi_2(u_2/u_1) + \sigma_{12}(u_{12}/u_1)$.

If both risks are independent or if $u_{12} = 0$ then $\pi = \pi_1 + \pi_2(u_2/u_1)$, using good 1 as numeraire. The premium an individual is willing to pay to remove all risk is therefore equal to the sum of the premia to remove each risk separately.

A simple illustration of this result when $\sigma_{12} = 0$ or $u_{12} = 0$ is that a monopoly insurer will extract the same rent if the aggregate risk in the economy is either equally or unequally distributed. For large, independent risks $\pi > \pi_1 + \pi_2$, when the individual has a unidimensional utility and risk aversion is risk vulnerable (Eeckhoudt et al. (1998); see Gollier et al. (1996) for a definition of risk vulnerability).

ACKNOWLEDGMENTS

I am grateful to Louis Eeckhoudt and Henri Loubergé for their very helpful comments.

NOTES

1. According to Karni (1979), small risks are such that $\Pr\{(z_1, z_2) \in B\} = 1$, where B is a two-dimensional ball of center (0,0) with radius ε, which is arbitrarily close to zero.
2. Equation (4) is the two-dimensional case of the partial multivariate risk premium approximation defined by Chalfant et al. (1993).

REFERENCES

Ambarish, R. and Kallberg, J.G. (1987), Multivariate risk premiums, *Theory and Decision* 22: 77–96.

Arrow, K.J. (1971), *Essays in the theory of risk bearing* Chicago: Markham.

Chalfant, J.A. and Finkelshtain, I. (1993), Portfolio choices in the presence of other risks, *Management Science* 39(8): 925–936.

Demers, F. and Demers, M. (1991), Multivariate risk aversion and uninsurable risks: theory and applications, *The Geneva Papers on Risk and Insurance Theory* 16(1): 7–43.

Doherty, N.A., Loubergé, H. and Schlesinger, H. (1987), Additive and multiplicative risk premiums with multiple sources of risk, *Scandinavian Actuarial Journal:* 41–49.

Doherty, N.A. and Schlesinger, H. (1981), A note on risk premium with random initial wealth, *Insurance: Mathematics and Economics* 52: 183–186.

Duncan, G.T. (1977), A matrix measure of multivariate local risk aversion, *Econometrica* 45: 895–903.

Eeckhoudt, L. and Gollier, C. (1988), Which shape for the cost curve of risk; Working Paper, Faculté Universitaires Catholiques de Mons, Belgium.

Gollier, C. and Pratt, J.W. (1996), Risk vulnerability and the tempering effect of background risk, *Econometrica* 64: 1109–1124.

Karni, E. (1979), Multivariate risk aversion, *Econometrica* 47: 1391–1401.
Kihlstrom, R.E. and Mirman, L.J. (1974), Risk aversion with many commodities, *Journal of Economic Theory* 8: 361–388.
Kihlstrom, R.E., Romer, D. and Williams, S. (1981), Risk aversion with random initial wealth, *Econometrica* 49: 911–921.
Pratt, J.W. (1964), Risk aversion in the small and in the large, *Econometrica* 32: 122–134.
Pratt, J.W. and Zeckhauser, R. (1987), Proper risk aversion, *Econometrica* 55: 143–154.
Ross, S. (1981), Some stronger measures of risk aversion in the small and in the large with applications, *Econometrica* 49: 621–638.

Address for correspondence: Christophe Courbage, University of Geneva, Department of Political Economy, 102 Bd Carl-Vogt, CH-1211 Genève 4, Switzerland
Phone: (41)-22-7058274; Fax: (41)-22-7058293;
E-mail: courbage@ecopo.unige.ch