

# Influence of a random telegraph process on the transport through a point contact

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**Abstract.** We describe the transport properties of a point contact under the influence of a classical two-level fluctuator. We employ a transfer matrix formalism allowing us to calculate arbitrary correlation functions of the stochastic process by mapping them on matrix products. The result is used to obtain the generating function of the full counting statistics of a classical point contact subject to a classical fluctuator, including extensions to a pair of two-level fluctuators as well as to a quantum point contact. We show that the noise in the quantum point contact is a sum of the (quantum) partitioning noise and the (classical) noise due to the two-level fluctuator. As a side result, we obtain the full counting statistics of a quantum point contact with time-dependent transmission probabilities.

## 1 Introduction

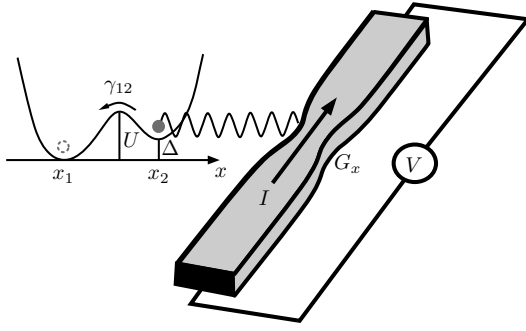
The impact of mobile impurities on transport through a quantum conductor attracted a lot of attention soon after it was realized that the conductance of a dirty coherent sample is sensitive to the position of a single impurity; this discovery formed the basis for the explanation of flicker noise as it appears due to the presence of bistable mobile impurities [1,2]. A detailed characterization of charge transport through a quantum conductor is provided by the full counting statistics (FCS) [3]. During the past decade, this description has been applied to numerous systems [4] and a first attempt to describe the influence of a two-level fluctuator on the FCS of quantum transport has been given in reference [5]. In the present paper, we calculate the full counting statistics of charge transport through a (classical or quantum) point contact coupled to a classical two-level fluctuator; we go beyond previous studies by considering the combined effects of the presence of one or many mobile impurities, as well as quantum-partitioning and the Fermi statistics on the FCS. Also, we reconsider carefully the case when partitioning is neglected and correct previous findings which are flawed when calculating the fourth or higher cumulants.

The influence of a fluctuating (uncontrollable) environment on a (controllable) device is a generic problem [6] and our work is related to other studies, e.g., the transport statistics through a quantum dot in the Coulomb-blockade regime [7,8] or the effect of a bistability on the transport through a quantum point contact [9,10]. Another example is the dephasing of qubits due to a classical [11] or

quantum [12] two-level system, or the studies on  $1/f$ -noise originating from telegraph noise due to classical [13,14] or quantum [11,15–17] two-level fluctuators. The two problems, full counting statistics of charge transport and dephasing of a quantum system (qubits) are related through the equivalence of fidelity [18] and full counting statistics [3], as has been pointed out recently [19].

In our analysis below, we describe the time evolution of the two-level fluctuator by rate equations which can be solved explicitly. We then study the full counting statistics of a wire with a conductance depending on the state of the fluctuator. The fluctuator induces noise in the transport current through the wire which we evaluate using a mapping of correlation functions on matrix products. Using this mapping, we are able to calculate the full counting statistics of a classical wire coupled to a two-level fluctuator. Furthermore, we discuss the situation of a second (independent) fluctuator and show that a non-linear interaction with the wire can lead to correlations in the noise even though the fluctuators evolve independent of each other. Finally, we apply our method to the case of a quantum wire which exhibits intrinsic partitioning (shot) noise [20,21]. We derive a formula which incorporates both classical- (due to the two-level fluctuator) and quantum- (due to the point contact) noise. Thereby, we give an explicit expression for the full counting statistics of a quantum point contact whose transmission probabilities change with time. As two-level fluctuators seem to be a major obstacle for achieving solid state implementations of qubits with long coherence times, being able to characterize the influence of a two-level fluctuator on transport through a point contact offers the possibility to

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**Fig. 1.** Sketch of the setup: a two-level system fluctuating incoherently between states  $x_{1,2}$  with rates  $\gamma_{12}$  and  $\gamma_{21}$  is coupled to a wire with a constriction. The conductance  $G$  of the wire changes according to the state of the fluctuator, inducing noise in the current flowing through the device.

learn about the fluctuating environment by measuring the full counting statistics through a nearby quantum point contact. The (partial) overlap of our results with previous work [7,9,11] will be discussed below.

## 2 Single two-level fluctuator

Consider a classical particle which can be trapped in an external potential at two positions denoted by  $x_1$  and  $x_2$ , cf. Figure 1. The potential is characterized by the energies  $E_{1,2}$  associated with the two valleys, which are shifted by the amount  $\Delta = E_2 - E_1$  with respect to each other, and the height of the barrier  $U$ . We assume the dynamics to be given by thermally activated hopping over the barrier [22]. The particle performs a random (Brownian) motion where the probabilities  $P_{1,2}$  to be in either valley obey the rate equations

$$\begin{aligned}\dot{P}_1(t) &= -\gamma_{21}P_1(t) + \gamma_{12}P_2(t), \\ \dot{P}_2(t) &= \gamma_{21}P_1(t) - \gamma_{12}P_2(t),\end{aligned}\quad (1)$$

where the rate to escape from  $x_1$  to  $x_2$  is given by  $\gamma_{21} = \gamma_{1 \rightarrow 2} \propto e^{-U/\vartheta}$  and the reverse process from  $x_2$  to  $x_1$  is governed by the rate  $\gamma_{12} = \gamma_{2 \rightarrow 1} \propto e^{(\Delta-U)/\vartheta}$ , with  $\vartheta$  the temperature. In equilibrium, the probabilities  $P_{1,2}^{\text{eq}}$  are such as to obey the balance equation  $dN_{21} = dN_{12}$  which equates the number of particles  $dN_{21} = \gamma_{21}P_1^{\text{eq}}dt$  going from  $x_1$  to  $x_2$  during the time  $dt$ , with the number of particles  $dN_{12} = \gamma_{12}P_2^{\text{eq}}dt$  going the opposite way. The equilibrium probabilities therefore satisfy the Gibbs weight

$$P_2^{\text{eq}}/P_1^{\text{eq}} = e^{-\Delta/\vartheta} \quad (2)$$

and together with the probability conservation  $P_1^{\text{eq}} + P_2^{\text{eq}} = 1$ , we obtain

$$P_1^{\text{eq}} = \gamma_{12}/\Gamma, \quad P_2^{\text{eq}} = \gamma_{21}/\Gamma, \quad (3)$$

where we have introduced the total rate  $\Gamma = \gamma_{12} + \gamma_{21}$ .

Introducing the vector  $\mathbf{P}(t)$  with components  $P_{1,2}(t)$ , the rate equation (1) can be written as  $\dot{\mathbf{P}}(t) = -\mathbf{h}\mathbf{P}(t)$ ,

with the Fokker-Planck Hamiltonian [23] given by

$$\mathbf{h} = \begin{pmatrix} \gamma_{21} & -\gamma_{12} \\ -\gamma_{21} & \gamma_{12} \end{pmatrix}. \quad (4)$$

Note that the Hamiltonian  $\mathbf{h}$  is not Hermitian (left and right eigenvalues are not simply adjoint to each other). Nevertheless, its eigenvalues are real (and even positive). To make this point clear, we write the Hamiltonian  $\mathbf{h}$  in a new basis  $\mathbf{h}' = \mathbf{s}^{-1}\mathbf{h}\mathbf{s}$  using the transformation matrix  $\mathbf{s} = \text{diag}(\sqrt{P_1^{\text{eq}}}, \sqrt{P_2^{\text{eq}}})$ , such that

$$\mathbf{h}' = \begin{pmatrix} \gamma_{21} & -\Gamma\sqrt{P_1^{\text{eq}}P_2^{\text{eq}}} \\ -\Gamma\sqrt{P_1^{\text{eq}}P_2^{\text{eq}}} & \gamma_{12} \end{pmatrix}. \quad (5)$$

It is now visible that the matrix  $\mathbf{h}'$  (and therefore also the matrix  $\mathbf{h}$ ) has real (the matrix is symmetric) and positive (the determinant and the trace of the matrix are positive) eigenvalues. The evolution conserves the probability as  $\partial_t[P_1(t) + P_2(t)] = 0$ , as is evident from equation (1). This relation implies that  $(1, 1) \cdot \dot{\mathbf{P}}(t) = 0$  and it follows that  $\langle 0| = (1, 1)$  is a left eigenvector of the Hamiltonian  $\mathbf{h}$  to the eigenvalue 0. To every left eigenvector there exists a corresponding right eigenvector with the same eigenvalue which we will denote by  $|0\rangle$ . The right eigenvector to the eigenvalue 0 is given by the equilibrium distribution  $|0\rangle = \mathbf{P}^{\text{eq}}$ ,  $\mathbf{h}|0\rangle = 0$ . The second eigenvalue is given by  $\Gamma$  with the corresponding right [left] eigenvectors assuming the form  $|\Gamma\rangle = (1, -1)^T$  [ $\langle\Gamma| = (P_2^{\text{eq}}, -P_1^{\text{eq}})$ ]; note that the eigenvectors are normalized such that  $\langle a|b\rangle = \delta_{ab}$  and  $\sum_a |a\rangle\langle a| = \mathbb{1}_2$  with  $a, b \in \{0, \Gamma\}$ . Using the eigenbasis of  $\mathbf{h}$ , it is possible to compute the evolution operator  $\mathbf{P}(t > 0) = \exp(-\mathbf{h}t) = \sum_a e^{-at}|a\rangle\langle a|$ . The matrix element  $P_{mn}(t)$  denotes the conditional probability for the particle to be transferred from state  $n$  to  $m$  in the time  $t$ . The evolution only depends on the time difference as the Hamiltonian  $\mathbf{h}$  is time-independent. An explicit calculation yields the expression

$$\mathbf{P}(t) = \begin{pmatrix} P_1^{\text{eq}} + P_2^{\text{eq}}e^{-\Gamma t} & P_1^{\text{eq}}(1 - e^{-\Gamma t}) \\ P_2^{\text{eq}}(1 - e^{-\Gamma t}) & P_2^{\text{eq}} + P_1^{\text{eq}}e^{-\Gamma t} \end{pmatrix} \quad (6)$$

for the propagator during the time  $t$ . As the stochastic process is Markovian, the propagator (6) incorporates all the information needed in order to calculate general correlation functions of the stochastic process [24], see also below.

## 3 Classical wire

We consider a classical wire coupled to the two-level fluctuator. We assume the two-level system to be a charge impurity which interacts with the wire, e.g., via Coulomb forces. The net effect of the charge impurity is to change the conductance of the wire  $G_{1,2}$  depending on the position  $x(t) = x_{1,2}$  of the two-level system. The wire is biased by a constant voltage  $V$  such that the current is determined by  $I_{1,2} = VG_{1,2}$ . The current in the wire jumps

between  $I_1$  and  $I_2$  in a random way given by the dynamics of the two-level fluctuator which we assume to be in thermal equilibrium, i.e., in the state  $|0\rangle$ . The fluctuations of the two-level system induce current noise. In our discussion, any kind of back-action of the wire on the two-level fluctuator is neglected.

### 3.1 Correlation functions – mapping on matrices

With our focus on the full counting statistics, we are interested in obtaining the moments of the charge  $Q = \int_0^t dt_1 I(t_1)$  transmitted through the point contact during the time  $t$ ; here,  $I(t_1)$  denotes  $I_{1,2}$  depending on the state  $x(t_1) = x_{1,2}$  of the two-level fluctuator at time  $t_1$ . As  $I(t_1) = VG(t_1)$ , we first concentrate on correlation functions of  $G(t)$ , where statistical averages over the stochastic process (1) will be denoted by  $\langle \cdot \rangle$ . In quantum mechanics, it is well-known that correlation functions can be evaluated either in the operator or in the path-integral formalism [25]. Likewise, we have the choice to apply a stochastic path-integral approach [7,26] or to use the operator formalism. Here, we stay with the operator formalism introduced in the previous section. Note that the propagator  $P_{mn}(t_2 - t_1)$ , cf. equation (6), denotes the conditional probability (the transfer matrix) to find the system in state  $x_m$  at time  $t_2$ , given that it resided in  $x_n$  at time  $t_1$ ,  $P_{mn}(t_2 - t_1) = \langle x(t_2) = x_m | x(t_1) = x_n \rangle$ ; i.e., within a path-integral formulation,  $P(t)$  involves already an integration over all possible paths between  $t_1$  and  $t_2$ .

The average conductance  $\langle G(t_1) \rangle$  is readily calculated. For a system residing in a stationary state given by  $\langle x(t_1) = x_n \rangle = P_n^{\text{eq}}$ , we obtain

$$G(t_1) = \sum_{n=1,2} G_n \langle x(t_1) = x_n \rangle = G_1 P_1^{\text{eq}} + G_2 P_2^{\text{eq}}. \quad (7)$$

The calculation of the conductance correlator  $\langle G(t_2)G(t_1) \rangle$  is more involved. We proceed slowly in order to motivate our general mapping between the calculation of correlation functions and the evaluation of matrix products of specific matrices. In order to calculate  $\langle G(t_2)G(t_1) \rangle$ , we assume first that  $t_2 > t_1$ ; classical correlators are symmetric so that the opposite ordering of times reduces to the same quantity. Using the fact that the stochastic process is Markovian, we expand the correlation function  $\langle G(t_2)G(t_1) \rangle = \sum_{m,n} G_m \langle x(t_2) = x_m | x(t_1) = x_n \rangle G_n \langle x(t_1) = x_n \rangle$  [24]; this expansion can be seen as a transfer-matrix expansion of the correlation function. We obtain the mapping for the correlator ( $t_2 > t_1$ )

$$\begin{aligned} \langle G(t_2)G(t_1) \rangle &= \sum_{mn} G_m P_{mn}(t_2 - t_1) G_n P_n^{\text{eq}} \\ &= \sum_{klmn} G_k \delta_{km} P_{ml}(t_2 - t_1) G_l \delta_{ln} P_n^{\text{eq}} \\ &= \langle 0 | G e^{-h(t_2 - t_1)} G | 0 \rangle \end{aligned} \quad (8)$$

with the diagonal matrix  $G_{mn} = G_n \delta_{mn}$ . Introducing the “interaction representation”

$$G_I(t) = e^{ht} G e^{-ht} \quad (9)$$

of the matrix  $G$ , the correlation function equation (8) can be further simplified to

$$\langle G(t_2)G(t_1) \rangle = \mathcal{T} \langle 0 | G_I(t_2) G_I(t_1) | 0 \rangle, \quad (10)$$

where the time-ordering operator  $\mathcal{T}$  has been included in order to relieve the restriction  $t_2 > t_1$ . It is easy to see that the above derivation is not restricted to the second order correlation function, but can be applied in the same way to higher order correlation functions. We thus arrive at the mapping

$$\langle G(t_N) \cdots G(t_1) \rangle = \mathcal{T} \langle 0 | G_I(t_N) \cdots G_I(t_1) | 0 \rangle, \quad (11)$$

where the left hand side is a correlation function for the classical stochastic process involving the two-level fluctuator and the right hand side is a matrix element involving the matrices  $G_I(t_n)$  and the vectors  $|0\rangle = P^{\text{eq}}$  and  $\langle 0| = (1, 1)$ .

### 3.2 Full counting statistics

We are now in the position to calculate the generating function

$$\chi(\lambda) = \langle e^{i\lambda \int_0^t dt' I(t')} \rangle \quad (12)$$

for the zero-frequency current-correlation functions (moments) of a classical point contact coupled to a two-level fluctuator. The moments are obtained as the Taylor coefficients  $\langle Q^n \rangle = (-i\partial_\lambda)^n \chi|_{\lambda=0}$ . Alternatively, the stochastic process can be characterized by irreducible cumulants which are given by the expansion coefficient of the logarithm of the characteristic function

$$\langle \langle Q^n \rangle \rangle = \left( \frac{d}{i d\lambda} \right)^n \log \chi(\lambda) \Big|_{\lambda=0}. \quad (13)$$

The characteristic function  $\chi(\lambda)$  can be recast in the form

$$\chi(\lambda) = \mathcal{T} \langle 0 | e^{i\lambda V \int_0^t dt' G_I(t')} | 0 \rangle \quad (14)$$

using the mapping equation (11). This equation can be rewritten using the well-known mapping between the “Schrödinger” and the “interaction” representation [23],  $e^{-(h+v)t} = e^{-ht} \mathcal{T} \exp[-\int_0^t dt' v_I(t')]$  with  $v_I(t) = e^{ht} v e^{-ht}$ . Here, we apply the relation in the opposite direction to arrive at an expression without the awkward time-ordering,

$$\chi(\lambda) = \langle 0 | e^{(-h+i\lambda V G)t} | 0 \rangle. \quad (15)$$

This formula was derived before by Bagrets and Nazarov using a stochastic path integral formulation of the problem (the Fokker-Planck Hamiltonian  $h$  and the counting field  $\lambda$  are denoted by  $\hat{L}$  and  $\lambda$  in their paper [7]). We believe that the present approach using transfer matrices and the mapping of the interaction picture onto the Schrödinger

picture is more transparent. Note though, that the counting field  $\lambda$  enters differently in their work compared to ours. Here,  $\lambda$  couples to the classical current  $I$  whereas Bagrets and Nazarov discuss the transport of individual (quantum) particles such that  $\lambda$  may only enter in the combination  $\exp(i\lambda)$  due to the quantization of charge.

For further convenience, we subtract the average charge  $\langle Q \rangle = V\langle G \rangle t$  in order to obtain the reduced full counting statistics

$$\hat{\chi}(\lambda) = \chi(\lambda)e^{-i\lambda\langle Q \rangle} = \langle 0 | e^{-\hat{h}t} | 0 \rangle, \quad (16)$$

with the matrix  $\hat{h} = \mathbf{h} - i\lambda V(\mathbf{G} - \langle G \rangle)$ . Apart from the average charge (which is zero for  $\hat{\chi}$ ), both  $\log \chi$  and  $\log \hat{\chi}$  generate the same cumulants. The explicit calculation of the characteristic function of the full counting statistics  $\hat{\chi}$  involves the eigenvalues

$$\hat{h}_{\pm} = \frac{\Gamma}{2} \left[ 1 + i\lambda \Delta g \Delta P^{\text{eq}} \pm \sqrt{1 + 2i\lambda \Delta g \Delta P^{\text{eq}} - \lambda^2 (\Delta g)^2} \right]$$

of the matrix  $\hat{h}$ , where we have introduced the difference in the equilibrium population  $\Delta P^{\text{eq}} = P_2^{\text{eq}} - P_1^{\text{eq}}$  and the difference in the (dimensionless) conductance  $\Delta g = V(G_2 - G_1)/\Gamma$ .

### 3.3 Asymptotic long-time limit

For long times  $\Gamma t \gg 1$ , the matrix exponential (16) is dominated by the eigenvalue  $\hat{h}_-$  of  $\hat{h}$  with the smallest real part [7,27,28]. We obtain an explicit expression for the generating function

$$\log \hat{\chi}_{\gg}(\lambda) = -\hat{h}_- t. \quad (17)$$

All cumulants become linear in  $t$ , due to the fact that the autocorrelation time in the system is given by  $\Gamma^{-1}$  and every state decays to the equilibrium state after this time. The fluctuations for  $\Gamma t \gg 1$  can be seen as a sum of independent stochastic processes and the cumulant generating function  $\log \hat{\chi}$  becomes extensive in  $t$ . Interestingly, it is possible to obtain an explicit relation for the cumulants ( $n \geq 2$ )

$$\begin{aligned} \langle\langle Q^n \rangle\rangle_{\gg} &= n! V^n (G_1 - G_2)^n \Gamma^{1-n} t \\ &\times \sum_{k=1}^{n-1} N_{n-1,k} (-1)^{k+1} (P_1^{\text{eq}})^k (P_2^{\text{eq}})^{n-k}, \end{aligned} \quad (18)$$

with the Narayana numbers  $N_{n,m} = \binom{n}{m} \binom{n}{m-1} / n$  [29]. The cumulants in (18) grow factorially in magnitude with  $n$  and oscillate as a function of  $\Delta P^{\text{eq}}$  which are generic features of high-order cumulants [30]. The first couple of cumulants assume the form

$$\begin{aligned} \langle\langle Q^2 \rangle\rangle_{\gg} &= 2P_1^{\text{eq}} P_2^{\text{eq}} \frac{V^2 (G_1 - G_2)^2 t}{\Gamma}, \\ \langle\langle Q^3 \rangle\rangle_{\gg} &= 6P_1^{\text{eq}} P_2^{\text{eq}} (P_2^{\text{eq}} - P_1^{\text{eq}}) \frac{V^3 (G_1 - G_2)^3 t}{\Gamma^2}. \end{aligned} \quad (19)$$

Note that all the odd cumulants vanish if the process is symmetric  $\gamma_{12} = \gamma_{21}$ , cf. equation (3). The cumulants in (18) and (19) agree with the result in reference [5]. However, starting with the 4-th cumulant a discrepancy arises; e.g., for the 4-th cumulant we obtain

$$\begin{aligned} \langle\langle Q^4 \rangle\rangle_{\gg} &= \\ &= 24P_1^{\text{eq}} P_2^{\text{eq}} [(P_2^{\text{eq}} - P_1^{\text{eq}})^2 - P_1^{\text{eq}} P_2^{\text{eq}}] \frac{V^4 (G_1 - G_2)^4 t}{\Gamma^3}, \end{aligned}$$

which is different from the result [5]

$$\langle\langle Q^4 \rangle\rangle_{\gg} = 24P_1^{\text{eq}} P_2^{\text{eq}} (P_2^{\text{eq}} - P_1^{\text{eq}})^2 \frac{V^4 (G_1 - G_2)^4 t}{\Gamma^3}.$$

The latter result is incorrect as it misses terms due to the implicit assumption in reference [5] that the reduced conductance correlators  $\langle\langle G(t_n) \cdots G(t_1) \rangle\rangle$  may only depend on the largest time difference  $t_n - t_1$ . Even though this hypothesis is correct for correlators up to  $n = 3$ , it fails for higher-order correlators.

### 3.4 Short times

For short times,  $t \ll \Gamma^{-1}$  there is no evolution of the two-level system and we can set  $ht = 0$ . The full counting statistics reads

$$\begin{aligned} \hat{\chi}_{\ll}(\lambda) &= \langle 0 | e^{i\lambda V(\mathbf{G} - \langle G \rangle)t} | 0 \rangle \\ &= P_2^{\text{eq}} e^{i\lambda V(G_2 - G_1)P_1^{\text{eq}} t} + P_1^{\text{eq}} e^{i\lambda V(G_1 - G_2)P_2^{\text{eq}} t}, \end{aligned} \quad (20)$$

with the cumulants ( $n \geq 2$ ) given by

$$\begin{aligned} \langle\langle Q^n \rangle\rangle_{\ll} &= V^n (G_1 - G_2)^n t^n \\ &\times \sum_{k=1}^{n-1} E_{n-1,k-1} (-1)^{k+1} (P_1^{\text{eq}})^k (P_2^{\text{eq}})^{n-k}, \end{aligned} \quad (21)$$

where the Eulerian numbers  $E_{n,m}$  are defined through  $E_{n,m} = \sum_{k=0}^m (-1)^k \binom{n+1}{k} (m+1-k)^n$  [29]. In the short time limit, the cumulants grow like  $\langle\langle Q^n \rangle\rangle \propto t^n$ , i.e., higher order cumulants are suppressed at short times. The first couple of cumulants are explicitly given by

$$\begin{aligned} \langle\langle Q^2 \rangle\rangle_{\ll} &= P_1^{\text{eq}} P_2^{\text{eq}} V^2 (G_1 - G_2)^2 t^2, \\ \langle\langle Q^3 \rangle\rangle_{\ll} &= P_1^{\text{eq}} P_2^{\text{eq}} (P_2^{\text{eq}} - P_1^{\text{eq}}) V^3 (G_1 - G_2)^3 t^3. \end{aligned} \quad (22)$$

### 3.5 Arbitrary times

Expanding the matrix exponential in equation (16) in its eigenbasis, the generator for the full counting statistics reads

$$\hat{\chi} = \frac{\hat{h}_+ e^{-\hat{h}_- t} - \hat{h}_- e^{-\hat{h}_+ t}}{\hat{h}_+ - \hat{h}_-} \quad (23)$$

for arbitrary times. This result has been first derived in reference [11] in the context of dephasing of a qubit due to

the interaction with a classical two-level fluctuator, where  $\lambda$  denotes the interaction of the qubit with the fluctuator. Here, we are interested in the transport properties of a point contact characterized by cumulants which are given by the Taylor expansion of  $\log \chi$  around  $\lambda = 0$ ; the relation between these two problems is a consequence of the generic equivalence between full counting statistics and fidelity, see reference [19]. The first couple of cumulants are given by

$$\begin{aligned} \langle\langle Q^2 \rangle\rangle &= 2P_1^{\text{eq}}P_2^{\text{eq}} \frac{V^2(G_1 - G_2)^2[(\Gamma t - 1) + e^{-\Gamma t}]}{\Gamma^2}, \\ \langle\langle Q^3 \rangle\rangle &= 6P_1^{\text{eq}}P_2^{\text{eq}}(P_2^{\text{eq}} - P_1^{\text{eq}}) \\ &\quad \times \frac{V^3(G_1 - G_2)^3[(\Gamma t - 2) + (\Gamma t + 2)e^{-\Gamma t}]}{\Gamma^3}. \end{aligned} \quad (24)$$

### 3.6 Symmetric fluctuator

A special situation is given when the two-level fluctuator is symmetric,  $\Delta = 0$ , i.e.,  $P_1^{\text{eq}} = P_2^{\text{eq}} = 1/2$ . Then the characteristic function assumes the simple form

$$\log \chi_{\gg}(\lambda) = \frac{\Gamma t}{2} \left[ \sqrt{1 - \lambda^2 V^2 (G_1 - G_2)^2 / \Gamma^2} - 1 \right] \quad (25)$$

for long times. In the short time limit, the generating function

$$\chi_{\ll}(\lambda) = \cos[\lambda V (G_1 - G_2) t / 2] \quad (26)$$

becomes periodic. In both cases, due to the symmetry of the states  $x_1$  and  $x_2$ , only the even cumulants are nonvanishing.

## 4 A pair of two-level fluctuators

Needless to say, the mapping of Section 3.1 is not restricted to a single two-level fluctuator. It can be generalized to an arbitrary number of states whose dynamics is governed by classical rate equations described by a Fokker-Planck Hamiltonian  $\mathbf{h}$ . To illustrate this concept, the example of two classical, uncorrelated two-level fluctuators coupled to a wire is discussed in the following. We restrict ourselves to the case where the dynamics of the two-level systems is completely independent of each other such that  $\mathbf{h} = \mathbf{h}^\alpha + \mathbf{h}^\beta$ ; here and in the following, we denote quantities involving only the first (second) fluctuator with a superscript  $\alpha(\beta)$ , e.g.,  $\mathbf{h}^\alpha = \mathbf{h}^\alpha \otimes \mathbb{1}^\beta$ . Written explicitly in the basis  $\{|1\rangle^\alpha \otimes |1\rangle^\beta, |2\rangle^\alpha \otimes |1\rangle^\beta, |1\rangle^\alpha \otimes |2\rangle^\beta, |2\rangle^\alpha \otimes |2\rangle^\beta\}$ , the Fokker-Planck Hamiltonian reads

$$\mathbf{h} = \begin{pmatrix} \gamma_{21}^\alpha + \gamma_{21}^\beta & -\gamma_{12}^\alpha & -\gamma_{12}^\beta & 0 \\ -\gamma_{21}^\alpha & \gamma_{12}^\alpha + \gamma_{21}^\beta & 0 & -\gamma_{12}^\beta \\ -\gamma_{21}^\beta & 0 & \gamma_{21}^\alpha + \gamma_{12}^\beta & -\gamma_{12}^\alpha \\ 0 & -\gamma_{21}^\beta & -\gamma_{21}^\alpha & \gamma_{12}^\alpha + \gamma_{12}^\beta \end{pmatrix}. \quad (27)$$

One is tempted to think that the independent dynamics of the two subsystems would lead to independent statistics, such that the characteristic function of the full counting statistics is given by the product of the individual characteristic functions. Indeed, this is the generic case for two-level fluctuators coupling to a qubit, where the combined effect leads to  $1/f$  noise [11,13,15–17]. However, here, this argument is only valid when the interaction with the wire is “linear” such that the effects of the individual subsystems simply add up, in formula  $\mathbf{G} = \mathbf{G}^\alpha + \mathbf{G}^\beta$ . Having the model of Figure 1 in mind, this assumption is incorrect as a (quantum) point contact does not react linearly on changes in the gate potential and, therefore, the noise of individual fluctuators does not simply add up. In the following, we first treat the (simple) case of linear interaction and then comment on the correlation which arises in the general case by applying perturbation theory in the nonlinearity.

Introducing the reference conductance  $G_0 = G_0^\alpha + G_0^\beta$  as well as the induced changes  $\Delta G^x = G_1^x - G_0^x$  due to the fluctuator  $x = \alpha, \beta$ , the conductance matrix for linear interaction is given by

$$\mathbf{G} = G_0 \mathbb{1}_4 + \text{diag}(0, \Delta G^\alpha, \Delta G^\beta, \Delta G^\alpha + \Delta G^\beta); \quad (28)$$

the increase of conductance in the case when both fluctuators are in state  $x_2$  is the sum of the respective increases when only one of the fluctuators is in state  $x_2$ , that is to say, the effects of the two fluctuators simply add up. In this case, the characteristic function assumes the form  $\chi(\lambda) = \chi^\alpha(\lambda)\chi^\beta(\lambda)$  and the cumulants  $\langle\langle Q^n \rangle\rangle$  become a sum of cumulants generated by the two individual subsystems.

In the case of a general (diagonal) matrix  $\mathbf{G}$ , the solution of the problem involves the determination of the roots of a polynomial of fourth degree and the characteristic function does not separate, even though the time evolution of the two fluctuators is completely independent of each other. To be more explicit, we want to show how a small perturbation  $\Delta G \ll \Delta G^\alpha + \Delta G^\beta$  in  $\mathbf{G}_{44}$  destroying the linearity (additivity) leads to correlations which can be arbitrary large for long times. We define  $\chi^{\text{corr}}(\lambda) = \chi(\lambda)/\chi^\alpha(\lambda)\chi^\beta(\lambda)$  as the part of the characteristic function which describes the correlation between the action of the individual subsystems. In the long-time limit, to first order in  $\Delta G$ , we can apply standard perturbation theory to find the correction to the lowest eigenvalue of equation (27). The cumulant generating function for the correlation is given by

$$\log \chi_{\gg}^{\text{corr}}(\lambda) = i\lambda V \Delta G t \prod_{x=\alpha,\beta} \frac{(\Gamma^x + i\lambda V \Delta G^x) P_2^{\text{eq},x} - \hat{h}_-^x}{\hat{h}_+^x - \hat{h}_-^x}. \quad (29)$$

The average transmitted charge changes according to

$$\Delta \langle Q \rangle = P_2^{\text{eq},\alpha} P_2^{\text{eq},\beta} V \Delta G t, \quad (30)$$

with  $P_2^{\text{eq},\alpha}P_2^{\text{eq},\beta}$  the probability to be in the state  $|2\rangle^\alpha \otimes |2\rangle^\beta$  and  $V\Delta G$  the change in the current. Less trivial, the correlation contribution to the noise

$$\Delta\langle\langle Q^2\rangle\rangle = 4V\Delta\langle Q\rangle \left[ \frac{P_1^{\text{eq},\alpha}}{\Gamma^\alpha}\Delta G^\alpha + \frac{P_1^{\text{eq},\beta}}{\Gamma^\beta}\Delta G^\beta \right] \quad (31)$$

depends both on  $\Delta G^\alpha$  and  $\Delta G^\beta$ .

## 5 Quantum wire

Considering a quantum rather than a classical wire, additional noise appears due to the probabilistic nature of the charge transport (due to partitioning) even in the absence of a fluctuating environment. Given a quantum wire with  $N$  channels characterized by their transmission eigenvalues  $T^\gamma$ ,  $\gamma = 1, \dots, N$ , and biased by a voltage potential  $V$ , the characteristic function of the full counting statistics is given by [3]

$$\log \chi^q(\lambda) = \frac{qVt}{2\pi\hbar} \sum_\gamma \log[1 + (e^{iq\lambda} - 1)T^\gamma], \quad (32)$$

with  $q$  the charge of the electron; this result is valid in the asymptotic limit  $qVt/\hbar \gg 1$ , for low-temperatures  $\vartheta \ll qV$ , and with the proviso that the energy-dependence of the transmission eigenvalues is negligible in the energy interval set by the voltage. The quantum nature of the fermions leads to the noise

$$\langle\langle Q^2\rangle\rangle^q = q\langle Q\rangle^q \frac{\sum_\gamma T^\gamma(1 - T^\gamma)}{\sum_\gamma T^\gamma} \quad (33)$$

which disappears provided that all the channels are either closed  $T^\gamma = 0$  or completely open  $T^\gamma = 1$ . Note that the noise in equation (33) is sub-Poissonian, i.e., the Fano factor  $F = \langle\langle Q^2\rangle\rangle^q/q\langle Q\rangle^q$  is less than 1.

Here, we are interested in the case where the quantum wire is capacitively coupled to a two-level fluctuator such that the transmission eigenvalues  $T_\gamma$  change over time; note that we neglect a possible energy dependence of the transmission eigenvalues, which corresponds to the fact that we assume that the scattering center does not produce any time delay due to the scattering event. The characteristic function  $\log \chi^q(\lambda) = \det \mathbf{Q}$  is given by the determinant of the matrix [31]

$$\mathbf{Q}_{k\gamma, k'\gamma'} = \langle \phi_{k\gamma}(t) | e^{i\lambda q \mathcal{Q}_t} | \phi_{k'\gamma'}(t) \rangle \quad (34)$$

with the counting operator  $\mathcal{Q}_t = \int_I dx |x\rangle\langle x|$  integrated over the interval  $I = [0, v_F t]$  and

$$\phi_{k\gamma}(x, t) = \begin{cases} e^{ik(x - v_F t)} + r_\gamma(t + x/v_F) e^{-ik(x + v_F t)} & x < 0 \\ \tau_\gamma(t - x/v_F) e^{ik(x - v_F t)} & x > 0 \end{cases} \quad (35)$$

the single-particle solution of the time-dependent Schrödinger equation involving a scattering center at  $x = 0$  with time-dependent transmission [reflection] amplitude

$\tau_\gamma(t)$  [ $r_\gamma(t)$ ]; note that we have suppressed the transverse part of the wave function belonging to the channel index  $\gamma$ . Equation (35) is valid in the linear-spectrum approximation, where  $v_F$  is the Fermi velocity and  $0 \leq k \leq k_F$ <sup>1</sup>. As the matrix  $\mathbf{Q}$  is block-diagonal in  $\gamma$  and constitutes a Toeplitz matrix with respect to the index  $k$ , its determinant can be shown (using the technique discussed in Ref. [31]) to have the form

$$\log \chi^q(\lambda) = \frac{qV}{2\pi\hbar} \int_0^t dt' \sum_\gamma \log[1 + (e^{iq\lambda} - 1)T_\gamma(t')] \quad (36)$$

for long times  $qVt/\hbar \gg 1$ ; the expression equation (36) reduces to equation (32) when the transmission probability does not change in time. We now add the classical two-level fluctuator to the system: depending on the state  $x = x_{1,2}$  of the nearby two-level fluctuator, the quantum wire is described by one of the two sets of transmission eigenvalues  $T_{1,2}^\gamma$ . The total generating function  $\chi^{\text{qtl}}(\lambda)$  is an average over the individual contributions

$$\begin{aligned} \chi^{\text{qtl}}(\lambda) &= \langle \chi^q(\lambda) \rangle \\ &= \left\langle e^{(qV/2\pi\hbar) \int_0^t dt' \sum_\gamma \log[1 + (e^{iq\lambda} - 1)T^\gamma(t')]} \right\rangle, \end{aligned} \quad (37)$$

where the average  $\langle \cdot \rangle$  is over the stochastic process of the two-level fluctuator. Equation (37) can be calculated explicitly with the method outlined in Section 3.1. Indeed, equation (12) goes over to equation (37) via replacing  $i\lambda I(t)$  with  $(qV/2\pi\hbar) \sum_\gamma \log[1 + (e^{iq\lambda} - 1)T^\gamma(t)]$ . In this mapping, the conductances  $G_n$  are changed to  $(q/2\pi i\hbar\lambda) \sum_\gamma \log[1 + (e^{iq\lambda} - 1)T_n^\gamma]$  with  $T_n^\gamma$  the transparency of the channel  $\gamma$  when the two-level fluctuator is in the state  $n$ . Inserting

$$i\lambda\Delta g \rightarrow \mu = \frac{qV}{2\pi\hbar\Gamma} \sum_\gamma \log \left[ \frac{1 + (e^{iq\lambda} - 1)T_2^\gamma}{1 + (e^{iq\lambda} - 1)T_1^\gamma} \right] \quad (38)$$

into equation (17) and using equation (16), we obtain

$$\begin{aligned} \log \chi_{\gg}^{\text{qtl}}(\lambda) &= \sum_{n,\gamma} P_n^{\text{eq}} \log \chi_n^q(\lambda) \\ &= \frac{\Gamma t}{2} \left[ 1 + \mu \Delta P^{\text{eq}} - \sqrt{1 + 2\mu \Delta P^{\text{eq}} + \mu^2} \right]; \end{aligned} \quad (39)$$

here,  $\log \chi_n^q(\lambda) = (qVt/2\pi\hbar) \sum_\gamma \log[1 + (e^{iq\lambda} - 1)T_n^\gamma]$  is the characteristic function for the FCS, equation (32), dependent on the system's state ( $n = 1, 2$ ) and  $\gamma$  runs over the channel index. The expression (39) coincides with the result obtained by Jordan and Sukhorukov who considered the case of rare transitions ( $\hbar\Gamma \ll qV$ ), see reference [9]. In their work, Jordan and Sukhorukov describe the transport of a conserved charge in a classical bistable system

<sup>1</sup> Note that the time-dependence of the scattering matrix promotes the incoming state to higher energies. In order that the linear spectrum approximation is still valid the time dependence of  $T(t)$  should not be too abrupt.

where large charge fluctuations drive transitions between stable states with different transport characteristics. The generic fluctuations present in the system's stable states then combine with the fluctuations of the bistable charge to generate the transport statistics of the bistable system. While describing a very general situation, the specific analysis in reference [9] is limited to those cases where the charge-switching events are rare on the time scale of the underlying fluctuations in the stable states. The translation to our model may not be obvious from the start and is done by choosing the process of charge partitioning for generating the underlying fluctuations (with different transmission rates  $T_n^\gamma$ ,  $n = 1, 2$  for the two stable states), i.e., the generators  $\log \chi_n^q(\lambda)$  correspond to the long time generators  $H_n t$  in reference [9]. The non-linearity leading to switching between stable states generates the transition probabilities  $\Gamma_{1,2}$ ; the latter are determined by an instanton trajectory and have to be small in the case of reference [9]. In our model these rates are given as the basic input parameters  $\gamma_{ij}$  defining our two-level fluctuator. The condition of rare charge switching events corresponds to the requirement that  $\hbar\Gamma \ll qV$ . In this limit equation (39) reduces to the result of the classical point contact, equation (17), with the conductances given by

$$G_{1,2} = \frac{q^2}{2\pi\hbar} \sum_{\gamma} T_{1,2}^{\gamma}. \quad (40)$$

We observe that the statistics is dominated by the fluctuations of the impurity and the results derived in the previous sections remain valid in the case of a quantum wire when the classical conductance is replaced by the Landauer formula (40).

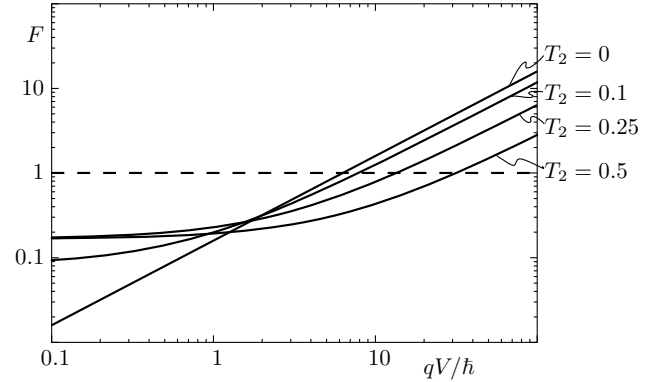
However, our result equation (39) is also valid in the opposite regime  $\hbar\Gamma \gg qV$ , i.e., when the two-level fluctuator noise acts on a timescale which is fast compared to the partitioning noise. Then only the first term in (39) contributes and the cumulant generating function is the average of the expressions (32) for the quantum point contact, to be taken over the positions  $x_{1,2}$  with weights given by the probabilities  $P_{1,2}^{\text{eq}}$ . Having access to both regimes, it is possible to study the crossover from classical noise (due to the two-level fluctuator) to quantum-partitioning noise (due to the quantum point contact). To this end, we calculate the long-time asymptotics of the first two moments of (37),

$$\langle Q \rangle_{\gg}^{\text{qt1}} = V(P_1^{\text{eq}}G_1 + P_2^{\text{eq}}G_2)t \quad (41)$$

for the average charge and

$$\begin{aligned} \langle \langle Q^2 \rangle \rangle_{\gg}^{\text{qt1}} &= qVt \frac{q^2}{2\pi\hbar} \sum_{n,\gamma} P_n^{\text{eq}} T_n^\gamma (1 - T_n^\gamma) \\ &+ 2P_1^{\text{eq}}P_2^{\text{eq}} \frac{V^2(G_1 - G_2)^2 t}{\Gamma} \end{aligned} \quad (42)$$

for the noise. The noise is simply given by the sum of the quantum partitioning noise (first term) and the noise due to the dynamics of the impurity (second term). Note



**Fig. 2.** Plot of the Fano factor  $F$  as a function of the bias voltage  $V$  for a single mode wire with  $P_1^{\text{eq}} = P_2^{\text{eq}} = 1/2$ ,  $T_1 = 1$ , and  $T_2 = 0, 0.1, 0.3$ , and  $0.5$ . The Fano factor starts of at a value  $T_2(1 - T_2)/(1 + T_2) < 1$  for small voltages. In the opposite regime, it is given by  $[(1 - T_2)^2/(1 + T_2)]qV/2\pi\hbar\Gamma$  which becomes superpoissonian for  $V$  large enough.

the crossover of the noise (42) from sub-Poissonian  $F \leq 1$  for a fast fluctuator with  $\hbar\Gamma \gg qV$  (the first term in (42) dominates) to super-Poissonian  $F \geq 1$  when the fluctuator is slow  $\hbar\Gamma \ll qV$  (the second term in (42) dominates), provided that  $G_1 \neq G_2$ , see Figure 2.

## 6 Experimental test

It is difficult to experimentally confirm the crossover from sub- to super-Poissonian noise as described in the previous section as the quantum-partitioning noise is typically small and thus the noise of the classical two-level fluctuator will dominate. As a promising setup, we envision coupling a quantum point contact in GaAs/AlGaAs to a double dot, e.g., in an InAs nanowire, which serves as tunable two-level fluctuator. Such a system was studied recently, see reference [32].

To observe the crossover, both the noise  $S_{\text{tl}}(0)$  from the classical two-level fluctuator and the quantum-partitioning noise  $S_{\text{qp}}(0)$  (at zero frequency) have to dominate over the thermal Nyquist-Johnson noise which is given by

$$S_{\text{NJ}}(0) \approx \frac{2q^2 k_B T}{\pi\hbar} \simeq 10^{-28} \text{ A}^2 \text{ s} \quad (43)$$

at 50 mK. The quantum-partitioning noise [first term in Eq. (42)] can be estimated as

$$S_{\text{qp}}(0) \approx \frac{q^3 V}{8\pi\hbar} \simeq V[\text{mV}] 10^{-27} \text{ A}^2 \text{ s} \quad (44)$$

when the system is tuned in the middle of a conductance step with  $T \approx 1/2$ . Note that for bias voltages  $V \geq 0.1$  mV the quantum-partitioning noise is larger than the thermal noise floor.

Tunneling of an electron between the two quantum dots with a rate  $\Gamma$  leads to a change in the conductance.

In reference [32], this change was of the order of  $0.1 q^2/\hbar$ . However, in order to being able to observe the crossover the capacitive coupling of the double dot to the quantum point contact should be reduced to a level such that  $G_1 - G_2 \approx 0.001 q^2/\hbar$ . This provides us with the estimate (using  $P_1^{\text{eq}} = P_2^{\text{eq}} = 1/2$ )

$$S_{\text{tl}}(0) = \frac{V^2(G_1 - G_2)^2}{\Gamma} \simeq \frac{V^2 [\text{mV}^2]}{\Gamma [\text{MHz}]} 10^{-26} \text{ A}^2 \text{ s} \quad (45)$$

for the zero-frequency noise due to the two-level fluctuator (second term in Eq. (42)). At a bias voltage  $V \simeq 1$  mV with a rate  $\Gamma \simeq 100$  MHz, the noise due to the fluctuator is given by  $S_{\text{tl}}(0) \simeq 10^{-28} \text{ A}^2 \text{ s}$  dominated by the quantum-partitioning noise with  $S_{\text{tl}}(0) \simeq 10^{-27} \text{ A}^2 \text{ s}$ . Note that in experiments rates of the order of 10–100 kHz have been observed [32,33]. We expect that rates in the 100 MHz regime to be realistic due to the exponential dependence of the tunneling rate on the potential barrier. Increasing the bias voltage  $V$  to values of a few mV the noise due to the classical two-level fluctuator starts to dominate and the crossover as depicted in Figure 2 can be observed.

## 7 Conclusion

We have determined the influence of a thermally driven two-level fluctuator on a point contact through the calculation of the full counting statistics of transported charge. Both, classical and quantum point contacts have been considered and extensions to multiple fluctuators have been discussed. In our analysis, we have made use of a mapping between correlation functions of classical stochastic processes and simple time-ordered matrix products. For the case of a quantum point contact, we have shown that the partitioning noise and the noise due the two-level fluctuator add up and the noise crosses over from sub- to super-Poissonian depending on the applied voltage bias. To extend the present formalism to the case of current correlators at finite frequencies, which provides additional insights into the dynamics of the two-level fluctuator, is an interesting problem for future studies.

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