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Comment on the Shiner–Davison–Landsberg Measure

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Abstract. The complexity measure from Shiner et al. [*Physical Review E* **59**, 1999, 1459–1464] (henceforth abbreviated as SDL-measure) has recently been the subject of a fierce debate. We discuss the properties and shortcomings of this measure, from the point of view of our recently constructed fundamental, statistical mechanics-based measures of complexity $C_s(\gamma, \beta)$ [Stoop et al., *J. Stat. Phys.* **114**, 2004, 1127–1137]. We show explicitly, what the shortcomings of the SDL-measure are: It is over-universal, and the implemented temperature dependence is trivial. We also show how the original SDL-approach can be modified to rule out these points of critique. Results of this modification are shown for the logistic parabola.

Key words: complexity, complexity measures, thermodynamics

1. Introduction

The question of how to measure complexity has continually attracted interest over the last decade, from the theoretical and the experimental points of view [1–21]. The Kolmogorov/Solomonoff algorithmic viewpoint [3, 4, 18] has been the most influential concept of complexity. The algorithmic complexity A of an object s is defined as the length of the shortest program P (in bits) that produces (prints) the object s

$$A(s) = \min_{C,P:C(P)=s} \log(\text{length}(P)), \tag{1}$$

where C is a computer. As there exists a universal computer, called the turing machine, which is able to simulate any other computer, $A(s)$ is a well-defined quantity.

The problem with this measure is that it violates basic conceptions of complexity. For example, random sequences are assigned the maximal complexity. However, computer-generated random sequences are generally the result of a simple random generator, which, obviously, has a finite algorithmic complexity. Moreover, using an intrinsic notion of complexity, truly random sequences appear no more complex than any pseudo-random sequences, even though the latter have a much shorter description length. In fact, finding the shortest description length is not that easy, as there are at least infinitely many programs to be checked. This is why the product of the algorithmic complexity with the difficulty of finding the right program might provide a more adequate approximation to the perceived, human, notion of complexity. Contrasting a plot of a two-dimensional embedded (pseudo-)random generator output against the output of the two-dimensional circle-standard map, illustrates the dilemma [8]: Whereas the random output is dull and void of structure, the output of the standard map appears as "interesting" and "complex." The reason for this inappropriateness as a natural perception of complexity, is that the algorithmic complexity has been devised as a measure of the complexity of objects generated by computers, or computer programs. In this context, the world appears to be rational (in the sense of rational numbers), allowing only a countable number of states to be distinguished. The real world, as e.g. generated by analog electronic circuits, however, is based on real numbers (even when rational numbers are measured), as is any biological or physical system. In fact, Gödel's Theorem [22] requires small digits in the measurements to be unpredictable *per se* (and not as a consequence of chaos theory), due to coupling to the rest of the world. Therefore, the need emerges to appropriately define measures of complexity for natural and physical systems.

2. The SDL-Measure Interpreted in Terms of the Fluctuation Spectrum

One recent approach to define complexity therefore builds on the requirement that the measure should be zero for truly random, as well as completely ordered, objects. Moreover, this complexity measure should be easy to evaluate; it should, in particular, not require hierarchical decomposition of the system. Starting from this position, Shiner et al. [1] defined their complexity measure as

$$\Gamma_{\alpha,\beta} = \Delta^{\alpha}(1 - \Delta)^{\beta}, \tag{2}$$

where $\Delta = S/S_{\max}$, with S being the Boltzmann–Gibbs–Shannon entropy. SDL interpreted Δ as the disorder, and $(1 - \Delta)$ as the order in the system. The rescaling by $S_{\max}$ maps measured order/disorder into the unit interval. However, as has been pointed out by several authors [21, 23], the SDL-measure has some important shortcomings. To elucidate the origin of these, and to point out ways to correct them, is the main content of our contribution.

3. A Recent Thermodynamic Formalism-Based Measure of Complexity

In order to achieve this, we contrast SDL with our previously proposed measure of complexity [2]. The latter is probably the most general statistical mechanics approach to complexity. For an observer-dependent variable ε, the fluctuation entropy spectrum $S(\varepsilon)$ is derived, using the thermodynamic formalism of dynamical systems [12]. This is achieved by a Legendre transform, applied to the free energy associated with the natural partition sum induced by the temporal evolution of the system. In more detail, the thermodynamic formalism departs from a partition function $Z(n, \beta, \nu)$, where n is the level or depth of the partition and β can be viewed as an inverse temperature. With $Z(n, \beta, \nu)$, a free energy

$$F(\beta) = \lim_{n\to\infty} \frac{1}{n} \log(Z(n, \beta, \nu)) \tag{3}$$

is associated, where in $F(\beta)$ we suppressed the dependence on the observable. β can be interpreted as an artificial temperature (that has no absolute zero, though). In the absence of phase transitions, an entropy function is obtained by means of the Legendre transform

$$S(\nu) = \nu\beta - F(\beta). \tag{4}$$

Requirements that apply to entropy functions are strict convexity with infinite derivatives at the two endpoints of the curve (in the absence of phase transition effects). From the large deviation entropy

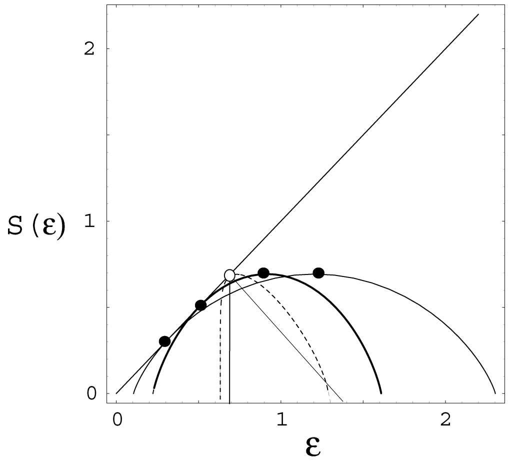

Figure 1. Fluctuation spectrum of different maps and specific entropy measures S_I and $S_{\max}$, respectively. Thick full lines, filled dots: Convex entropy functions $S(\varepsilon)$ obtained for two asymmetric tent maps of varying asymmetry. Dashed line, open dot: Numerical approximation of $S(\varepsilon)$ obtained for the fully developed parabola (partition level $n = 12$), which slowly converges towards the triangular function (thin full lines). In this case, S_I and $S_{\max}$ coincide. In the presence of first-order phase transitions, piecewise linear parts of the graph emerge, as is demonstrated by the parabola.

$S(\varepsilon)$, the complexity measure (for details see [2]) is calculated as

$$C_s(\gamma, \beta) = {\varepsilon_0}^{2\beta} \frac{\varepsilon_1}{\varepsilon_1 - \kappa} \int_{\mathrm{Supp}(\tilde{S})} (\tilde{S}(\tilde{\varepsilon})/\tilde{\varepsilon})^\gamma \, d\tilde{\varepsilon}. \tag{5}$$

Here, κ denotes a potential escape rate (which is nonzero only for repellors). The fluctuation spectrum generally has the convex form shown in Figure 1 for the asymmetric tent maps. In the presence of first-order phase transitions, straight-line parts emerge, as is shown by the example of the fully developed parabola. For the measure, the graph of $S(\varepsilon)$ *versus* ε has been rescaled by extracting the topological length scale ε_0 on both axes, which is indicated in (5) by the tildes.

In the context of the fluctuation entropy spectrum $S(\varepsilon)$, the SDL-complexity measure obtains a simple interpretation: S in their formula corresponds to the observable measure S_I in the fluctuation spectrum, defined by $S_I(\varepsilon) = \varepsilon$, whereas $S_{\max}$ corresponds to the topological entropy (the maximum of $S(\varepsilon)$). Their quantity is therefore proportional to the product of $S_I(S_{\max} - S_I)$. Geometrically, this measure is represented by the gray area of Figure 2.

4. Properties of the SDL-Measure

In particular, by construction the SDL-measure has the following unwanted properties: The measure is calculated from only two particularly significant points of the fluctuation spectrum: The natural measure S_I and the topological measure $S_{\max}$ (the latter sometimes also referred to as the balanced measure). The remaining shape of the spectrum $S(\varepsilon)$ remains without influence. This leads to an over-universality in the following sense: Dynamical systems that have entirely different properties may be attributed identical complexities. This feature is most dramatically illustrated by hyperbolic maps *versus* maps displaying phase-transition phenomena; in particular, intermittent maps that can

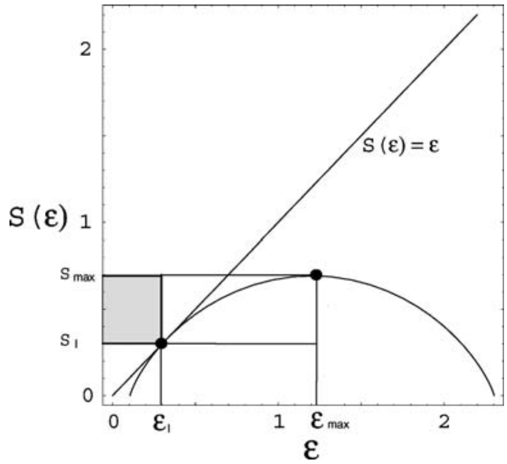

Figure 2. Geometric meaning of the SDL-complexity measure: The gray area has the size of $S_I(S_{\max} - S_I)$. For the fully developed parabola (see Figure 1), area zero would be obtained.

be made to have identical values of S_I and $S_{\max}$, and hence complexity. Obviously, however, the intermittent maps are much more difficult to predict than hyperbolic ones, yet they are mapped on the identical family of complexity measures $\Gamma_{\alpha,\beta}$. Therefore, the detailed behavior of the system cannot be recovered by this approach, and using the exponents α, β is not a remedy for this situation. Instead, the information contained in the two points of the spectrum is re-distributed in a nontrivial way over the real axis.

5. Modification of the SDL-Measure

In order to obtain better measures based on the order–disorder approach, the trivial temperature dependence of the SDL-measure has to be replaced by a dependence that takes into account the whole spectrum. This can be achieved in the following way. First, the Stoop-complexity integrand

$$(\tilde{S}(\tilde{\varepsilon})/\tilde{\varepsilon})^{\gamma} \tag{6}$$

may be viewed as being related to complexities based on a SDL-like product integrand of the form

$$(1 - S(\varepsilon)/\varepsilon)^{\alpha}(S(\varepsilon)/\varepsilon)^{\beta}. \tag{7}$$

To achieve a more appropriate measure, we propose to integrate this function over all possible length scales. The result of this calculation for the logistic map is shown in Figure 3 for $\alpha = \beta = 1$. Note, however, that for this measure, the influence of the intermittent length scales will be reduced in comparison to our original complexity measure (5), something that we would prefer to avoid (for arguments, see [2]). As a more promising generalization of our measure, we could imagine considering independent exponentiation to the denominator of the integrand instead.

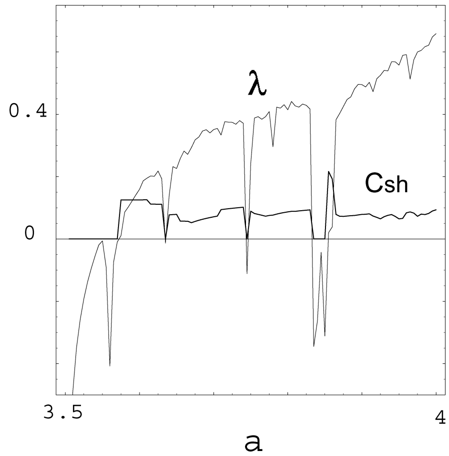

Figure 3. Modified SDL-complexity C_{Sh} and Lyapunov exponent λ for the logistic map, as a function of the order parameter a.

6. Results

This modification removes the criticized over-universality. A comparison of the SDL-complexity with the one obtained in the proposed way clearly elucidates notable differences between the two results (compare our Figure 3 with Figure 3 of Shiner et al. [1]): Whereas for the original measure we obtain an overall monotonously decreasing function, this is no longer the case with the modified measure. Furthermore, the original complexity measure yields nonzero complexity for completely ordered systems (period-doubling cascade cases or period-3 window), which seems counter-intuitive to us. The results obtained by the modified measure do not share this defect. They appear to fulfill the requirements asked for in the definition of a measure of complexity [1] much better: that it should be zero when either the system is completely ordered or random.

Our proposed approach is very general. In addition to one-dimensional systems, the evaluation can be performed for higher-dimensional systems as encountered in the description of electronic systems (usually dimensions 2–4), as well as for their time series. In the latter case, however, the approach requires substantially more efforts and care. It is hoped that by the evaluation of complexities along computational pathways, insight could be gained into the deeper nature of natural or artificial computation.

7. Conclusions

In this way, the SDL-measure can be modified to invalidate the most pertinent critiques. The nontrivial temperature dependence will remove the over-universality. In particular, the requirement by Binder and Perry [23], that at least some classes of systems known from hierarchical analysis should be discernible, is satisfied: Different dynamical systems will have distinct complexity measure families. In the comment by Crutchfield et al. [21], the authors criticized that any measure of complexity must be tied intrinsically to a process. In our modification, this is now indeed the case. Starting from the fundamental observations of complexity based on order and disorder, we have arrived at a measure that is no longer subject to the most pertinent critiques, and whose construction is entirely transparent. However, it remains to be seen whether a deeper significance can be attributed to the integrand of Equation (7), and, connected to this question, how useful the measure could be for practical applications.

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