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Mixed finite element methods for stationary incompressible magneto–hydrodynamics

Dominik Schötzau⋆

Department of Mathematics, University of Basel, 4051 Basel, Switzerland;
e-mail: schotzau@math.unibas.ch

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Summary. A new mixed variational formulation of the equations of stationary incompressible magneto–hydrodynamics is introduced and analyzed. The formulation is based on curl-conforming Sobolev spaces for the magnetic variables and is shown to be well-posed in (possibly non-convex) Lipschitz polyhedra. A finite element approximation is proposed where the hydrodynamic unknowns are discretized by standard inf-sup stable velocity-pressure space pairs and the magnetic ones by a mixed approach using Nédélec's elements of the first kind. An error analysis is carried out that shows that the proposed finite element approximation leads to quasi-optimal error bounds in the mesh-size.

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1 Introduction

Incompressible magneto-hydrodynamics (MHD) describes the flow of a viscous, incompressible and electrically conducting fluid. The governing equations form a multifield problem that arises in several applications such as, for example, liquid metals in magnetic pumps or aluminum electrolysis; we refer to [22, 24] for comprehensive accounts of the physical background of magneto-hydrodynamics. Several papers have been devoted to the design and the analysis of numerical schemes for the simulation of such fluids. We mention here [3] for an engineering approach to the numerical solution of transient incompressible MHD problems, with emphasis on long-term dissipativity of

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time integration. In [15], finite element methods (FEM) with suitable stabilization of the bilinear forms have been proposed and analyzed for linearized stationary problems in convection-dominated regimes. The recent work [17, 18] deals with a decoupled linear MHD problem involving electrically conducting and insulating regions.

The paper [19] gives a detailed existence theory and convergence analysis of finite element methods for nonlinear, fully coupled stationary incompressible MHD problems in domains $\Omega \subset \mathbb{R}^3$ that are either convex or whose boundaries are of class $C^{1,1}$. The functional setting is based on the standard Sobolev spaces $H^1(\Omega)$ and $L^2(\Omega)$. The hydrodynamic variables in the equations are discretized accordingly by inf-sup stable mixed elements and the magnetic ones by nodal, i.e., H^1-conforming, elements. The above assumptions on the smoothness of the boundary of the domain had to be made in order to assure sufficient regularity of the solution for the nodal FEM to converge. It has been known for some time, however, that in non-convex polyhedra Ω of engineering practice, the magnetic field may have regularity below $H^1(\Omega)^3$ and that nodal FEM discretizations, albeit stable, can converge to a magnetic field that misses certain singular (but physical) solution components induced by reentrant vertices or edges (for more details, see, e.g., [8] and the references cited therein). Consequently, in non-convex polyhedra Ω, setting the magnetic unknowns of the incompressible MHD equations in $H^1(\Omega)$ leads to a well-posed problem where the magnetic field cannot be correctly approximated.

This paper is devoted to the analysis and finite element approximation of nonlinear and fully coupled MHD problems in general Lipschitz polyhedra. To account for the possible low regularity of the magnetic field, we introduce a new mixed variational formulation for which the magnetic field belongs to the Sobolev space $H(\text{curl}; \Omega)$, as opposed to the $H^1(\Omega)$-based approaches that are employed in the works mentioned above. We first prove the existence of solutions to this formulation and show that the solutions are unique under standard smallness assumptions on the data (i.e., for small Reynolds numbers and small forcing terms). Although the proof of these results is carried out using well-known fixed point arguments developed for the stationary incompressible Navier-Stokes equations, see [16, 29] and the references therein, and follows [19] in some sense, it employs technical tools for the magnetic part of the equations that are substantially different than those in [19], owing to the novel functional-analytic framework. In particular, suitable Helmholtz decompositions of vector fields and the recent imbedding results from [2] will play a crucial role in our analysis. We then propose and analyze a new finite element approximation of the incompressible MHD equations, based on standard inf-sup stable velocity-pressure space pairs for the hydrodynamic variables and on a mixed formulation using Nédélec's elements of the first kind for the magnetic ones; see [25, 26] for the definition of these elements.

The well-posedness of the finite element formulation is established analogously to the analysis of the continuous problem, by using corresponding discrete Helmholtz decompositions and well-known results from the finite element theory for Maxwell's equations; see, e.g., the survey [21], the recent monograph [23] and the references therein. Finally, we present an error analysis that shows that the proposed finite element approximation leads to quasi-optimal error bounds in the mesh-size. In the note [28], the results of this paper have been announced and numerically confirmed for a linear MHD problem in two dimensions.

Throughout the paper, we use standard notation. For a Lipschitz polyhedron $\Omega \subset \mathbb{R}^3$, we denote by $L^p(\Omega)$, $1 \leq p \leq \infty$, the Lebesgue space of p-integrable functions, endowed with the norm $\|\cdot\|_{0,p}$. When $p = 2$, we simply write $\|\cdot\|_0$. The standard L^2-based Sobolev space with integer or fractional regularity exponent $s > 0$ is denoted by $H^s(\Omega)$. We write $\|\cdot\|_s$ for its norm. We define $H^1_0(\Omega)$ to be the subspace of $H^1(\Omega)$ of functions with zero trace on $\partial\Omega$. The dual space of $H^1_0(\Omega)$ is denoted by $H^{-1}(\Omega)$. Its norm is $\|\cdot\|_{-1}$. For a generic function space $X(\Omega)$ we write $X(\Omega)^3$ to denote vector fields whose components belong to $X(\Omega)$. This space is equipped with the usual product norm which we denote in the same way as the norm in $X(\Omega)$. We use $(\cdot,\cdot)$ for the inner product in $L^2(\Omega)^3$, and $\langle\cdot,\cdot\rangle$ for the duality pairing in $H^{-1}(\Omega)^3 \times H^1_0(\Omega)^3$. The spaces $H(\mathrm{curl};\Omega)$ and $H(\mathrm{div};\Omega)$ are the spaces of vector fields $\mathbf{c} \in L^2(\Omega)^3$ with $\mathrm{curl}\,\mathbf{c} \in L^2(\Omega)^3$ and $\mathrm{div}\,\mathbf{c} \in L^2(\Omega)$, respectively, endowed with the graph norms $\|\cdot\|_{\mathrm{curl}}$ and $\|\cdot\|_{\mathrm{div}}$. We denote by $H_0(\mathrm{curl};\Omega)$ and $H_0(\mathrm{div};\Omega)$ the subspaces of $H(\mathrm{curl};\Omega)$ and $H(\mathrm{div};\Omega)$ of fields with zero tangential trace and normal trace on $\partial\Omega$, respectively, and by $H(\mathrm{curl}^0;\Omega)$ and $H(\mathrm{div}^0;\Omega)$ the subspaces of $H(\mathrm{curl};\Omega)$ and $H(\mathrm{div};\Omega)$ of fields with zero rotation and divergence, respectively. We further set $H_0(\mathrm{div}^0;\Omega) = H_0(\mathrm{div};\Omega) \cap H(\mathrm{div}^0;\Omega)$.

The outline of the paper is as follows. In Section 2, we introduce the equations of incompressible magneto-hydrodynamics, propose a mixed variational formulation and prove existence as well as uniqueness, under the usual smallness assumptions on the data, of solutions. Section 3 is devoted to a mixed finite element discretization with emphasis on existence and stability results for the discrete formulation. Section 4 contains an error analysis of the proposed finite element method and shows that it leads to quasi-optimal error bounds in the mesh-size. In Section 5, we end our presentation with concluding remarks.

2 Stationary incompressible magneto–hydrodynamics

In this section, we first review the equations of incompressible magneto-hydrodynamics. We then introduce an auxiliary variational formulation based on solenoidal function spaces and establish existence and uniqueness results in general Lipschitz polyhedra. Finally, we propose and analyze a mixed formulation that incorporates the divergence constraints by the use of Lagrange multipliers.

2.1 The equations of incompressible magneto-hydrodynamics

Let Ω be a bounded Lipschitz polyhedron in $\mathbb{R}^3$. For simplicity, we assume that Ω is simply-connected, and that its boundary $\partial\Omega$ is connected. We consider the following incompressible MHD equations (see, e.g., [22,24] for comprehensive accounts of the physical background of magneto-hydrodynamics): find the velocity field $\mathbf{u}$, the pressure p, the magnetic field $\mathbf{b}$, and the scalar function r satisfying

$$-R_s^{-1}\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - S_c\,\mathrm{curl}\,\mathbf{b}\times\mathbf{b} = \mathbf{f} \qquad \text{in } \Omega, \tag{2.1}$$

$$R_m^{-1}\,S_c\;\mathrm{curl}(\mathrm{curl}\,\mathbf{b}) - S_c\,\mathrm{curl}(\mathbf{u}\times\mathbf{b}) - \nabla r = \mathbf{g} \qquad \text{in } \Omega, \tag{2.2}$$

$$\mathrm{div}\,\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{2.3}$$

$$\mathrm{div}\,\mathbf{b} = 0 \qquad \text{in } \Omega. \tag{2.4}$$

Here, R_s is the hydrodynamic Reynolds number, R_m the magnetic Reynolds number, S_c the coupling number, and $\mathbf{f} \in H^{-1}(\Omega)^3$ and $\mathbf{g} \in L^2(\Omega)^3$ are given source terms. We complete the above system with the homogeneous boundary conditions

$$\begin{aligned} \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega,\\ \mathbf{b}\cdot\mathbf{n} &= 0 && \text{on } \partial\Omega,\\ \mathbf{n}\times\mathrm{curl}\,\mathbf{b} &= \mathbf{0} && \text{on } \partial\Omega, \end{aligned} \tag{2.5}$$

where $\mathbf{n}$ denotes the outward normal unit vector on $\partial\Omega$. The scalar functions ρ and Γ will be required to have zero mean over Ω.

Remark 2.1 The scalar function r is the Lagrange multiplier associated to the constraint $\mathrm{div}\,\mathbf{b} = 0$; see, e.g., [10,32]. Formally, by taking the divergence of equation (2.2), we obtain $-\Delta r = \mathrm{div}\,\mathbf{g}$, see Remark 2.16 below. In particular, we have $r = 0$ if $\mathrm{div}\,\mathbf{g} = 0$. Thus, for $\mathbf{g} = \mathbf{0}$ the MHD problem (2.1)–(2.4) is the same as that considered in [19] or the linearized version thereof studied in [15].

Strongly related to boundary conditions in (2.5) is the following $L^2(\Omega)^3$–orthogonal Helmholtz decomposition

$$L^2(\Omega)^3 = H_0(\mathrm{div}^0;\Omega) \oplus \nabla H^1(\Omega), \tag{2.6}$$

valid under the above assumptions on the domain; see, e.g., [13, Section 4].

Remark 2.2 Our results are also valid for another frequently used set of boundary conditions for (2.1)–(2.4) given by

$$\mathbf{u} = \mathbf{0}, \qquad \mathbf{n} \times \mathbf{b} = \mathbf{0}, \qquad r = 0, \qquad \text{on } \partial\Omega,$$

see, e.g., [19] and the references cited therein. In this case, the corresponding decomposition is $L^2(\Omega)^3 = H(\text{div}^0; \Omega) \oplus \nabla H_0^1(\Omega)$; see [13, Section 4] or [23, Theorem 3.45].

2.2 Solenoidal function spaces

We first introduce an auxiliary variational formulation for (2.1)–(2.4) that is based on solenoidal function spaces. To this end, we define the space

$$\mathcal{H}(\Omega) := H(\text{curl}; \Omega) \cap H_0(\text{div}; \Omega),$$

and equip it with the norm

$$\|\mathbf{c}\|_{\mathcal{H}(\Omega)}^2 := \|\mathbf{c}\|_0^2 + \|\operatorname{curl} \mathbf{c}\|_0^2 + \|\operatorname{div} \mathbf{c}\|_0^2.$$

We then set

$$\mathbf{J} := H_0^1(\Omega)^3 \cap H(\text{div}^0; \Omega), \quad \mathbf{X} := \mathcal{H}(\Omega) \cap H(\text{div}^0; \Omega),$$

endowed with the norms $\mathbf{v} \mapsto \|\mathbf{v}\|_1$ and $\mathbf{c} \mapsto \|\mathbf{c}\|_{\text{curl}} = (\|\mathbf{c}\|_0^2 + \|\operatorname{curl} \mathbf{c}\|_0^2)^{\frac{1}{2}}$, respectively.

Next, we recall the following Poincaré-Friedrichs inequality in $\mathbf{X}$: there holds

$$\|\operatorname{curl} \mathbf{c}\|_0 \ \geq\ C\|\mathbf{c}\|_0 \qquad \forall \mathbf{c} \in \mathbf{X}, \tag{2.7}$$

with a constant $C > 0$ solely depending on the domain Ω; see, e.g., [13, Proposition 7.4] or [23, Corollary 3.51]. In view of (2.7), functions in $\mathbf{X}$ are uniquely defined by their rotation, and, on $\mathbf{X}$, the norm $\mathbf{c} \mapsto \|\operatorname{curl} \mathbf{c}\|_0$ is equivalent to the norm $\|\cdot\|_{\text{curl}}$.

The following imbedding results are crucial for our analysis.

Proposition 2.3 *We have the following imbedding properties:*

(1) The space $H^1(\Omega)^3$ is compactly imbedded into $L^q(\Omega)^3$ for any exponent $1 \leq q < 6$.
(2) There exists an exponent $s = s(\Omega) > \frac{1}{2}$ such that $\mathcal{H}(\Omega)$ is continuously imbedded into $H^s(\Omega)^3$.
(3) There exists a parameter $\delta_1 = \delta_1(\Omega) > 0$ such that $\mathcal{H}(\Omega)$ is compactly imbedded into $L^{3+\delta_1}(\Omega)^3$.

Proof. The first imbedding is well-known; see [16, Theorem I.1.3]. The second one follows from [2, Proposition 3.7]. To prove the third imbedding, let $s > \frac{1}{2}$ be such that $\mathcal{H}(\Omega) \hookrightarrow H^s(\Omega)^3$. We may assume that $s \in (\frac{1}{2}, 1]$. Then, since $H^s(\Omega)^3$ is compactly imbedded into $L^q(\Omega)^3$ for any $1 \leq q < \frac{6}{3-2s}$, see, for instance, [16, Theorem I.1.3 and Definition I.1.2], we can choose $q = 3 + \delta_1$ for a parameter $\delta_1 > 0$ in such a way that $\mathcal{H}(\Omega)$ is compactly imbedded into $L^q(\Omega)^3$. □

Next, we introduce the trilinear forms

$$
\begin{aligned}
c_0(\mathbf{w}; \mathbf{u}, \mathbf{v}) &:= \frac{1}{2} \int_\Omega (\mathbf{w} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} \, d\mathbf{x} - \frac{1}{2} \int_\Omega (\mathbf{w} \cdot \nabla)\mathbf{v} \cdot \mathbf{u} \, d\mathbf{x}, \\
c_1(\mathbf{d}; \mathbf{v}, \mathbf{b}) &:= S_c \int_\Omega (\operatorname{curl} \mathbf{b} \times \mathbf{d}) \cdot \mathbf{v} \, d\mathbf{x}, \\
c_2(\mathbf{d}; \mathbf{u}, \mathbf{c}) &:= S_c \int_\Omega (\mathbf{u} \times \mathbf{d}) \cdot \operatorname{curl} \mathbf{c} \, d\mathbf{x}.
\end{aligned}
$$

Remark 2.4 The form c_0 is the usual anti-symmetrized form for the nonlinear term of the Navier-Stokes operator; see, e.g., [29, Section II.3.2] for details.

The next two results show that the forms c_0, c_1 and c_2 are well-defined.

Lemma 2.5 *Let* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *in* $H^1(\Omega)^3$. *We have that*

$$
|c_0(\mathbf{w}; \mathbf{u}, \mathbf{v})| \leq C \, \|\mathbf{w}\|_{0,4} \, \|\mathbf{u}\|_1 \, \|\mathbf{v}\|_1 \; \leq \; C \|\mathbf{w}\|_1 \, \|\mathbf{u}\|_1 \, \|\mathbf{v}\|_1,
$$

for constants $C > 0$ *only depending on* Ω.

Proof. This follows from the compact imbedding $H^1(\Omega) \overset{c}{\hookrightarrow} L^4(\Omega)$ in Proposition 2.3 and Hölder's inequality; cf. [16, Lemma IV.2.1]. □

Lemma 2.6 *Let* $\mathbf{d} \in \mathbf{X}$, $\mathbf{u}, \mathbf{v} \in H^1(\Omega)^3$ *and* $\mathbf{b}, \mathbf{c} \in H(\operatorname{curl}; \Omega)$. *Let* $\delta_1 > 0$ *be such that* $\mathcal{H}(\Omega) \overset{c}{\hookrightarrow} L^{3+\delta_1}(\Omega)$ *according to Proposition 2.3. Then there exists a second parameter* $\delta_2 > 0$ *such that*

$$
\begin{aligned}
|c_1(\mathbf{d}; \mathbf{v}, \mathbf{b})| &\leq \; S_c \, \|\mathbf{d}\|_{0,3+\delta_1} \|\mathbf{v}\|_{0,6-\delta_2} \| \operatorname{curl} \mathbf{b}\|_0 \\
&\leq \; C \, S_c \, \|\mathbf{d}\|_{\mathrm{curl}} \|\mathbf{v}\|_1 \| \operatorname{curl} \mathbf{b}\|_0,
\end{aligned}
$$

and

$$
\begin{aligned}
|c_2(\mathbf{d}; \mathbf{u}, \mathbf{c})| &\leq \; S_c \, \|\mathbf{d}\|_{0,3+\delta_1} \|\mathbf{u}\|_{0,6-\delta_2} \| \operatorname{curl} \mathbf{c}\|_0 \\
&\leq \; C \, S_c \, \|\mathbf{d}\|_{\mathrm{curl}} \|\mathbf{u}\|_1 \| \operatorname{curl} \mathbf{c}\|_0,
\end{aligned}
$$

for constants $C > 0$ *only depending on* Ω.

Proof. In view of the imbedding properties in Proposition 2.3, we can choose $\delta_2 > 0$ such that $\frac{1}{3+\delta_1} + \frac{1}{6-\delta_2} = \frac{1}{2}$ and

$$\mathcal{H}(\Omega) \overset{c}{\hookrightarrow} L^{3+\delta_1}(\Omega)^3, \qquad H^1(\Omega)^3 \overset{c}{\hookrightarrow} L^{6-\delta_2}(\Omega)^3.$$

For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, we have $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$ and $\|\mathbf{x} \times \mathbf{y}\| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$. Therefore, by Hölder's inequality,

$$\begin{aligned} |c_1(\mathbf{d}; \mathbf{v}, \mathbf{b})| &\leq S_c \int_\Omega \|\mathbf{d}\| \, \|\mathbf{v}\| \, \|\operatorname{curl} \mathbf{b}\| \, d\mathbf{x} \\ &\leq S_c \, \|\mathbf{d}\|_{0,3+\delta_1} \|\mathbf{v}\|_{0,6-\delta_2} \|\operatorname{curl} \mathbf{b}\|_0. \end{aligned}$$

The above imbeddings and the fact that $\|\mathbf{d}\|_{\mathcal{H}(\Omega)} = \|\mathbf{d}\|_{\text{curl}}$ for $\mathbf{d} \in \mathbf{X}$ give the assertion for c_1. The proof for c_2 is analogous. □

Moreover, we recall the following skew-symmetry properties.

Lemma 2.7 *The following results hold:*

(1) Let $\mathbf{w}, \mathbf{v} \in H^1(\Omega)^3$. *Then we have*

$$c_0(\mathbf{w}; \mathbf{v}, \mathbf{v}) = 0.$$

(2) Let $\mathbf{d} \in \mathbf{X}$, $\mathbf{v} \in H^1(\Omega)^3$ *and* $\mathbf{c} \in H(\text{curl}; \Omega)$. *Then we have*

$$c_1(\mathbf{d}; \mathbf{v}, \mathbf{c}) + c_2(\mathbf{d}; \mathbf{v}, \mathbf{c}) = 0.$$

Proof. The first result is obvious. The second property follows from the fact that $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = -(\mathbf{z} \times \mathbf{y}) \cdot \mathbf{x}$ for vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$. □

By introducing the bilinear forms

$$\begin{aligned} a_s(\mathbf{u}, \mathbf{v}) &:= R_s^{-1} \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, d\mathbf{x}, \\ a_m(\mathbf{b}, \mathbf{c}) &:= R_m^{-1} S_c \int_\Omega \operatorname{curl} \mathbf{b} \cdot \operatorname{curl} \mathbf{c} \, d\mathbf{x}, \end{aligned}$$

we are ready to define an auxiliary variational formulation for (2.1)–(2.4) that is based on the solenoidal spaces $\mathbf{J}$ and $\mathbf{X}$.

Formulation 2.8 *Find* $(\mathbf{u}, \mathbf{b}) \in \mathbf{J} \times \mathbf{X}$ *such that*

$$\begin{aligned} a_s(\mathbf{u}, \mathbf{v}) + c_0(\mathbf{u}; \mathbf{u}, \mathbf{v}) - c_1(\mathbf{b}; \mathbf{v}, \mathbf{b}) &= \langle \mathbf{f}, \mathbf{v} \rangle, \\ a_m(\mathbf{b}, \mathbf{c}) - c_2(\mathbf{b}; \mathbf{u}, \mathbf{c}) &= (\mathbf{g}, \mathbf{c}) \end{aligned}$$

for all $(\mathbf{v}, \mathbf{c}) \in \mathbf{J} \times \mathbf{X}$.

Remark 2.9 Note that, due to the use of solenoidal function spaces, both the pressure p and the potential r have been eliminated from the equations and do not appear in the auxiliary variational problem in Formulation 2.8.

Remark 2.10 If we decompose the source term $\mathbf{g} \in L^2(\Omega)^3$ into $\mathbf{g} = \mathbf{g}_0 + \nabla\varphi$, with $\mathbf{g}_0 \in H_0(\mathrm{div}^0; \Omega)$ and $\varphi \in H^1(\Omega)$, according to (2.6), we have that $(\mathbf{g}, \mathbf{c}) = (\mathbf{g}_0, \mathbf{c})$ for all $\mathbf{c} \in \mathbf{X}$. Thus, the gradient part $\nabla\varphi$ in the decomposition of $\mathbf{g}$ does not appear in Formulation 2.8. A similar remark applies to the source term $\mathbf{f}$.

For the purpose of our analysis, it will be convenient to rewrite the variational problem in Formulation 2.8 in the compact form: find $(\mathbf{u}, \mathbf{b}) \in \mathbf{J} \times \mathbf{X}$ such that

$$\mathcal{A}(\mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}) + \mathcal{C}(\mathbf{u}, \mathbf{b}; \mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}) = \mathcal{L}(\mathbf{v}, \mathbf{c}) \tag{2.8}$$

for all $(\mathbf{v}, \mathbf{c}) \in \mathbf{J} \times \mathbf{X}$. Here,

$$\begin{aligned} \mathcal{A}(\mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}) &:= a_s(\mathbf{u}, \mathbf{v}) + a_m(\mathbf{b}, \mathbf{c}), \\ \mathcal{C}(\mathbf{w}, \mathbf{d}; \mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}) &:= c_0(\mathbf{w}; \mathbf{u}, \mathbf{v}) - c_1(\mathbf{d}; \mathbf{v}, \mathbf{b}) - c_2(\mathbf{d}; \mathbf{u}, \mathbf{c}), \\ \mathcal{L}(\mathbf{v}, \mathbf{c}) &:= \langle \mathbf{f}, \mathbf{v} \rangle + (\mathbf{g}, \mathbf{c}). \end{aligned}$$

Next, we equip the product space $H_0^1(\Omega)^3 \times H(\mathrm{curl}; \Omega)$ with the norm

$$|||\, (\mathbf{v}, \mathbf{c}) \,|||_A^2 := \|\mathbf{v}\|_1^2 + \|\mathbf{c}\|_{\mathrm{curl}}^2,$$

and set

$$\begin{aligned} |||\, \mathcal{L} \,|||_- &:= \sup_{(\mathbf{0},\mathbf{0}) \neq (\mathbf{v},\mathbf{c}) \in \mathbf{J} \times \mathbf{X}} \frac{\mathcal{L}(\mathbf{v}, \mathbf{c})}{|||\, (\mathbf{v}, \mathbf{c}) \,|||_A}, \\ |||\, \mathcal{L} \,|||_* &:= \left[\, \|\mathbf{f}\|_{-1}^2 + \|\mathbf{g}\|_0^2 \, \right]^{\frac{1}{2}}. \end{aligned}$$

Note that $|||\, \mathcal{L} \,|||_- \leq |||\, \mathcal{L} \,|||_*$.

We have the following properties for the forms $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{L}$. First, we note that

$$\mathcal{A}(\mathbf{v}, \mathbf{c}; \mathbf{v}, \mathbf{c}) \geq C_a \, \min\{R_s^{-1}, R_m^{-1} S_c\} \, |||\, (\mathbf{v}, \mathbf{c}) \,|||_A^2, \tag{2.9}$$

for all $(\mathbf{v}, \mathbf{c}) \in \mathbf{J} \times \mathbf{X}$, with a constant $C_a > 0$ only depending on Ω, and

$$\mathcal{C}(\mathbf{w}, \mathbf{d}; \mathbf{v}, \mathbf{c}; \mathbf{v}, \mathbf{c}) = 0, \tag{2.10}$$

for all $(\mathbf{w}, \mathbf{d}) \in H^1(\Omega)^3 \times \mathbf{X}$, $(\mathbf{v}, \mathbf{c}) \in H^1(\Omega)^3 \times H(\mathrm{curl}; \Omega)$. The coercivity of the form $\mathcal{A}$ follows from the Poincaré-Friedrichs inequality in $H_0^1(\Omega)$ and the one in (2.7). The skew-symmetry of the trilinear form $\mathcal{C}$ follows from Lemma 2.7.

Because of Lemma 2.5 and Lemma 2.6, we further have the continuity properties

$$|\mathcal{A}(\mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c})| \leq \max\{R_s^{-1}, R_m^{-1} S_c\} \, |||\, (\mathbf{u}, \mathbf{b}) \,|||_A |||\, (\mathbf{v}, \mathbf{c}) \,|||_A, \tag{2.11}$$

for all $(\mathbf{u}, \mathbf{b}), (\mathbf{v}, \mathbf{c}) \in H^1(\Omega)^3 \times H(\mathrm{curl}; \Omega)$, and

(2.12)
$$|\mathcal{C}(\mathbf{w},\mathbf{d};\mathbf{u},\mathbf{b};\mathbf{v},\mathbf{c})| \leq C_c \max\{1, S_c\} \,|||\, (\mathbf{w},\mathbf{d}) \,|||_A |||\, (\mathbf{u},\mathbf{b}) \,|||_A |||\, (\mathbf{v},\mathbf{c}) \,|||_A,$$

for all $(\mathbf{w},\mathbf{d}) \in H^1(\Omega)^3 \times \mathbf{X}$ and $(\mathbf{u},\mathbf{b})$, $(\mathbf{v},\mathbf{c}) \in H^1(\Omega)^3 \times H(\text{curl};\Omega)$. The constant $C_c > 0$ solely depends on Ω. Finally, we have

$$|\mathcal{L}(\mathbf{v},\mathbf{c})| \leq |||\, \mathcal{L} \,|||_{-} |||\, (\mathbf{v},\mathbf{c}) \,|||_A, \tag{2.13}$$

for all $(\mathbf{v},\mathbf{c}) \in \mathbf{J} \times \mathbf{X}$, and

$$|\mathcal{L}(\mathbf{v},\mathbf{c})| \leq |||\, \mathcal{L} \,|||_{*} |||\, (\mathbf{v},\mathbf{c}) \,|||_A, \tag{2.14}$$

for all $(\mathbf{v},\mathbf{c}) \in H^1(\Omega)^3 \times H(\text{curl};\Omega)$.

2.3 Existence and uniqueness of solutions

We address the existence and uniqueness of solutions to Formulation 2.8 by applying the abstract theory developed in [16, Section IV.1] for a class of non-linear problems (that includes the stationary incompressible Navier-Stokes equations).

To this end, we first recall the results from [16, Theorem IV.1.2 and Theorem IV.1.3] in a form which is convenient for our analysis.

Theorem 2.11 *Let V be a separable Hilbert space with norm $\|\cdot\|_V$, l a linear functional in the dual space V', and $(u,v,w) \mapsto a(u;v,w)$ a trilinear mapping $V \times V \times V \to \mathbb{R}$ satisfying the following hypotheses:*

- *There exists a constant $\alpha > 0$ such that*
$$|a(u;v,w)| \leq \alpha \|u\|_V \|v\|_V \|w\|_V \qquad \forall u,v,w \in V.$$
- *There exists a constant $\beta > 0$ such that*
$$a(u;v,v) \geq \beta \|v\|_V^2 \qquad \forall u,v \in V.$$
- *The mapping $u \mapsto a(u;u,v)$ is sequentially weakly continuous on V. That is, if $u_m \to u$ weakly in V for $m \to \infty$, then*
$$a(u_m;u_m,v) \to a(u;u,v), \qquad m \to \infty,$$
for all $v \in V$.

Then the problem: find $u \in V$ such that
$$a(u;u,v) = l(v) \qquad \forall v \in V,$$
has at least one solution $u \in V$. Any solution $u \in V$ satisfies the stability bound $\|u\|_V \leq \beta^{-1}\|l\|_{V'}$. Furthermore, if
$$\alpha\beta^{-2}\|l\|_{V'} < 1,$$
the above problem has a unique solution $u \in V$.

Let us now show the following existence result for Formulation 2.8.

Theorem 2.12 *For* $\mathbf{f} \in H^{-1}(\Omega)^3$ *and* $\mathbf{g} \in L^2(\Omega)^3$*, there exists at least one solution* $(\mathbf{u}, \mathbf{b})$ *in* $\mathbf{J} \times \mathbf{X}$ *of the variational problem in Formulation 2.8. Furthermore, we have the stability bound*

$$|||\,(\mathbf{u}, \mathbf{b})\,|||_A \leq \frac{|||\,\mathcal{L}\,|||_-}{C_a \min\{R_s^{-1}, R_m^{-1} S_c\}}$$

for any solution $(\mathbf{u}, \mathbf{b}) \in \mathbf{J} \times \mathbf{X}$.

Proof. We shall apply Theorem 2.11 for the formulation (2.8). To this end, we proceed in several steps.

Step 1 First, we show that the space $\mathbf{J} \times \mathbf{X}$ is separable. To do so, we note that the space $\mathbf{J}$ is a closed subspace of $H^1(\Omega)^3$ and thus clearly separable. Moreover, the space $\mathcal{H}(\Omega)$ is isomorphic to a closed subspace of $L^2(\Omega)^7$. This can be seen by defining the mapping $T : \mathcal{H}(\Omega) \to L^2(\Omega)^7$ by $T(\mathbf{c}) = (\mathbf{c}, \operatorname{curl} \mathbf{c}, \operatorname{div} \mathbf{c})$. Obviously, $\|T(\mathbf{c})\|_0 = \|\mathbf{c}\|_{\mathcal{H}(\Omega)}$. Thus, $\mathcal{H}(\Omega)$ and its closed subspace $\mathbf{X}$ are separable. The product space $\mathbf{J} \times \mathbf{X}$ is therefore separable.

Step 2 Next, we show that the mapping

$$(\mathbf{u}, \mathbf{b}) \mapsto \mathcal{A}(\mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}) + \mathcal{C}(\mathbf{u}, \mathbf{b}; \mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}) \tag{2.15}$$

is sequentially weakly continuous on $\mathbf{J} \times \mathbf{X}$.

To this end, let $(\mathbf{u}_m, \mathbf{b}_m)_{m\in\mathbb{N}}$ be a sequence in $\mathbf{J} \times \mathbf{X}$ that weakly converges to $(\mathbf{u}, \mathbf{b}) \in \mathbf{J} \times \mathbf{X}$. Obviously, $\lim_{m\to\infty} \mathcal{A}(\mathbf{u}_m, \mathbf{b}_m; \mathbf{v}, \mathbf{c}) = \mathcal{A}(\mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c})$ due to the continuity property (2.11) of the form $\mathcal{A}$. Furthermore, due to the compact imbeddings in Proposition 2.3, we have that $\mathbf{u}_m \to \mathbf{u}$ strongly in $L^4(\Omega)^3$ and $L^{6-\delta_2}(\Omega)^3$ and $\mathbf{b}_m \to \mathbf{b}$ strongly in $L^{3+\delta_1}(\Omega)^3$ for the parameters $\delta_1 > 0$ and $\delta_2 > 0$ from Proposition 2.3 and Lemma 2.6. Linearity in the first two arguments of $\mathcal{C}$ then gives

$$\begin{aligned}
&|\mathcal{C}(\mathbf{u}_m, \mathbf{b}_m; \mathbf{u}_m, \mathbf{b}_m; \mathbf{v}, \mathbf{c}) - \mathcal{C}(\mathbf{u}, \mathbf{b}; \mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c})| \\
&\quad \leq \quad |\mathcal{C}(\mathbf{u}_m - \mathbf{u}, \mathbf{b}_m - \mathbf{b}; \mathbf{u}_m, \mathbf{b}_m; \mathbf{v}, \mathbf{c})| + |\mathcal{C}(\mathbf{u}, \mathbf{b}; \mathbf{u}_m - \mathbf{u}, \mathbf{b}_m - \mathbf{b}; \mathbf{v}, \mathbf{c})|.
\end{aligned}$$

Thanks to Lemma 2.5, Lemma 2.6, and Proposition 2.3, we have

$$\begin{aligned}
|c_0(\mathbf{u}_m - \mathbf{u}; \mathbf{u}_m; \mathbf{v})| &\leq C\|\mathbf{u} - \mathbf{u}_m\|_{0,4}\|\mathbf{u}_m\|_1\|\mathbf{v}\|_1, \\
|c_1(\mathbf{b}_m - \mathbf{b}; \mathbf{v}, \mathbf{b}_m)| &\leq C S_c\|\mathbf{b}_m - \mathbf{b}\|_{0,3+\delta_1}\|\mathbf{v}\|_1\|\operatorname{curl} \mathbf{b}_m\|_0, \\
|c_2(\mathbf{b}_m - \mathbf{b}; \mathbf{u}_m, \mathbf{c})| &\leq C S_c\|\mathbf{b}_m - \mathbf{b}\|_{0,3+\delta_1}\|\mathbf{u}_m\|_1\|\operatorname{curl} \mathbf{c}\|_0.
\end{aligned}$$

Hence, using the boundedness of $(\mathbf{u}_m, \mathbf{b}_m)$ in $\mathbf{J} \times \mathbf{X}$ shows that

$$\lim_{m\to\infty} |\mathcal{C}(\mathbf{u}_m - \mathbf{u}, \mathbf{b}_m - \mathbf{b}; \mathbf{u}_m, \mathbf{b}_m; \mathbf{v}, \mathbf{c})| = 0.$$

It remains to see that $|\mathcal{C}(\mathbf{u}, \mathbf{b}; \mathbf{u}_m - \mathbf{u}, \mathbf{b}_m - \mathbf{b}; \mathbf{v}, \mathbf{c})|$ converges to zero. To do so, we first note that

$$\lim_{m\to\infty} |c_0(\mathbf{u}; \mathbf{u}_m - \mathbf{u}, \mathbf{v})| = 0,$$
$$\lim_{m\to\infty} |c_1(\mathbf{b}; \mathbf{v}, \mathbf{b}_m - \mathbf{b})| = 0,$$

since $\mathbf{w} \mapsto c_0(\mathbf{u}; \mathbf{w}, \mathbf{v})$ and $\mathbf{c} \mapsto c_1(\mathbf{b}; \mathbf{v}, \mathbf{c})$ are continuous functionals on $\mathbf{J}$ and $\mathbf{X}$, respectively, due to Lemma 2.5 and Lemma 2.6. Furthermore,

$$|c_2(\mathbf{b}; \mathbf{u}_m - \mathbf{u}, \mathbf{c})| \leq S_c \|\mathbf{b}\|_{\mathrm{curl}} \|\mathbf{u}_m - \mathbf{u}\|_{0,6-\delta_2} \| \operatorname{curl} \mathbf{c}\|_0.$$

Thus,

$$\lim_{m\to\infty} |\mathcal{C}(\mathbf{u}, \mathbf{b}; \mathbf{u}_m - \mathbf{u}, \mathbf{b}_m - \mathbf{b}; \mathbf{v}, \mathbf{c})| = 0.$$

This shows that the mapping in (2.15) is sequentially weakly continuous.

Step 3 The coercivity and continuity properties in (2.9)–(2.13) and the results in Step 1 and Step 2 verify the hypotheses in Theorem 2.11 for the formulation in (2.8). Referring to Theorem 2.11 thus proves the assertion. □

For small data, we obtain the following uniqueness result from Theorem 2.11.

Theorem 2.13 *Assume that*

$$\frac{C_c \max\{1, S_c\} \,|\!|\!| \mathcal{L} |\!|\!|_{-}}{C_a^2 \min\{R_s^{-2}, R_m^{-2} S_c^2\}} < 1. \tag{2.16}$$

Then the variational problem in Formulation 2.8 has a unique solution $(\mathbf{u}, \mathbf{b}) \in \mathbf{J} \times \mathbf{X}$.

Remark 2.14 The results in Theorem 2.12 and Theorem 2.13 hold true verbatim for the boundary conditions in Remark 2.2 if we replace $\mathcal{H}(\Omega)$ and $\mathbf{X}$ by

$$\mathcal{H}(\Omega) = H_0(\mathrm{curl}; \Omega) \cap H(\mathrm{div}; \Omega), \qquad \mathbf{X} = \mathcal{H}(\Omega) \cap H(\mathrm{div}^0; \Omega),$$

respectively.

2.4 Mixed variational formulation

We are now ready to define a mixed variational formulation for the MHD equations (2.1)–(2.4). To do so, we first introduce the spaces

$$\mathbf{V} := H_0^1(\Omega)^3, \qquad Q := L^2(\Omega)/\mathbb{R},$$
$$\mathbf{C} := H(\text{curl}; \Omega), \qquad S := H^1(\Omega)/\mathbb{R},$$

and equip them with the norms $\| \cdot \|_1$, $\| \cdot \|_0$, $\| \cdot \|_{\text{curl}}$ and $\| \cdot \|_1$, respectively.

Next, we introduce the forms $b_s : Q \times \mathbf{V} \to \mathbb{R}$ and $b_m : S \times \mathbf{C} \to \mathbb{R}$ given by

$$b_s(q, \mathbf{v}) := -\int_\Omega q \operatorname{div} \mathbf{v} \, d\mathbf{x}, \qquad b_m(s, \mathbf{c}) := -\int_\Omega \nabla s \cdot \mathbf{c} \, d\mathbf{x}.$$

These forms are continuous and satisfy the following inf-sup conditions

$$\begin{aligned} &\inf_{0 \neq q \in Q} \sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{V}} \frac{b_s(q, \mathbf{v})}{\|\mathbf{v}\|_1 \|q\|_0} \geq \Gamma_s > 0, \\ &\inf_{0 \neq s \in S} \sup_{\mathbf{0} \neq \mathbf{c} \in \mathbf{C}} \frac{b_m(s, \mathbf{c})}{\|\mathbf{c}\|_{\text{curl}} \|s\|_1} \geq \Gamma_m > 0, \end{aligned} \tag{2.17}$$

respectively, with constants Γ_s and Γ_m only depending on Ω. The inf-sup condition for the form b_s can be found in, e.g., [16, Section I.5.1]. For the form b_m, it can be easily seen by noting that, for $s \in S$ arbitrary, we have $\nabla s \in \mathbf{c}$ and thus by the Poincaré-Friedrichs inequality in $H^1(\Omega)/\mathbb{R}$:

$$\sup_{\mathbf{0} \neq \mathbf{c} \in \mathbf{C}} \frac{b_m(s, \mathbf{c})}{\|\mathbf{c}\|_{\text{curl}} \|s\|_1} \geq \frac{b_m(s, -\nabla s)}{\|\nabla s\|_{\text{curl}} \|s\|_1} = \frac{\|\nabla s\|_0^2}{\|\nabla s\|_0 \|s\|_1} \geq \Gamma_m > 0,$$

see also [25, Theorem 9] and [21, Section 5.4].

We define the following mixed variational formulation for (2.1)–(2.4).

Formulation 2.15 *Find* $(\mathbf{u}, p, \mathbf{b}, r) \in \mathbf{J} \times Q \times \mathbf{X} \times S$ *such that*

$$\begin{aligned} a_s(\mathbf{u}, \mathbf{v}) + c_0(\mathbf{u}; \mathbf{u}, \mathbf{v}) - c_1(\mathbf{b}; \mathbf{v}, \mathbf{b}) + b_s(p, \mathbf{v}) &= \langle \mathbf{f}, \mathbf{v} \rangle, \\ a_m(\mathbf{b}, \mathbf{c}) - c_2(\mathbf{b}; \mathbf{u}, \mathbf{c}) + b_m(r, \mathbf{c}) &= (\mathbf{g}, \mathbf{c}) \end{aligned}$$

for all $(\mathbf{v}, \mathbf{c}) \in \mathbf{V} \times \mathbf{C}$.

Remark 2.16 By decomposing the data $\mathbf{g}$ into $\mathbf{g} = \mathbf{g}_0 + \nabla\varphi$, with $\mathbf{g}_0 \in H_0(\text{div}^0; \Omega)$ and $\varphi \in S$, according to (2.6), we have that the multiplier r solves the problem $-(\nabla r, \nabla s) = (\nabla\varphi, \nabla s)$ for all $s \in S$. This can be seen by choosing test functions $\mathbf{c} = \nabla s$ in the second equation of the problem in Formulation 2.15. In particular, for a divergence–free function $\mathbf{g} \in H_0(\text{div}^0; \Omega)$, we have that $r = 0$.

It will be convenient to introduce the global form

$$\mathcal{B}(q, s; \mathbf{v}, \mathbf{c}) := b_s(q, \mathbf{v}) + b_m(s, \mathbf{c}),$$

and to define the kernel

$$Z := \{(\mathbf{v}, \mathbf{c}) \in \mathbf{V} \times \mathbf{C} : \mathcal{B}(q, s; \mathbf{v}, \mathbf{c}) = 0 \ \forall \, (q, s) \in Q \times S\}.$$

Notice that a function $(\mathbf{v}, \mathbf{c}) \in \mathbf{V} \times \mathbf{C}$ belongs to $\mathbf{J} \times \mathbf{X}$ if and only if $(\mathbf{v}, \mathbf{c}) \in Z$. This follows from standard properties of the divergence operator, see [16, Section I.2.2], and from the Helmholtz decomposition in (2.6). Thus, we have

$$Z = \mathbf{J} \times \mathbf{X}, \tag{2.18}$$

and can rewrite the variational problem in Formulation 2.15 in the compact form: find $(\mathbf{u}, p, \mathbf{b}, r) \in \mathbf{V} \times Q \times \mathbf{C} \times S$ such that

$$\begin{aligned} \mathcal{A}(\mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}) + \mathcal{C}(\mathbf{u}, \mathbf{b}; \mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}) + \mathcal{B}(p, r; \mathbf{v}, \mathbf{c}) &= \mathcal{L}(\mathbf{v}, \mathbf{c}), \\ \mathcal{B}(q, s; \mathbf{u}, \mathbf{b}) &= 0 \end{aligned} \tag{2.19}$$

for all $(\mathbf{v}, q, \mathbf{c}, s) \in \mathbf{V} \times Q \times \mathbf{C} \times S$.

By endowing the space $Q \times S$ with the product norm

$$\| (q, s) \|_B^2 := \|q\|_0^2 + \|s\|_1^2,$$

we have, in addition to the properties in (2.9)–(2.14),

$$|\mathcal{B}(q, s; \mathbf{v}, \mathbf{c})| \leq C_b \| (q, s) \|_B \| (\mathbf{v}, \mathbf{c}) \|_A, \tag{2.20}$$

for all $(\mathbf{v}, q, \mathbf{c}, s) \in \mathbf{V} \times Q \times \mathbf{C} \times S$, with a continuity constant $C_b > 0$. Furthermore, we have the following inf-sup condition for the form $\mathcal{B}$.

Lemma 2.17 *There is a constant $\Gamma > 0$ solely depending on Ω such that*

$$\sup_{(\mathbf{0},\mathbf{0})\neq(\mathbf{v},\mathbf{c})\in\mathbf{V}\times\mathbf{C}} \frac{\mathcal{B}(q, s; \mathbf{v}, \mathbf{c})}{\| (\mathbf{v}, \mathbf{c}) \|_A} \geq \Gamma \| (q, s) \|_B$$

for all $(q, s) \in Q \times S$.

Proof. Let $(q, s) \in Q \times S$. Thanks to (2.17), there exist functions $\mathbf{v} \in \mathbf{V}$ and $\mathbf{c} \in \mathbf{C}$ such that

$$b_s(q, \mathbf{v}) \geq \|q\|_0^2, \qquad b_m(s, \mathbf{c}) \geq \|s\|_1^2,$$

and

$$\|\mathbf{v}\|_1 \leq \Gamma_s^{-1} \|q\|_0, \qquad \|\mathbf{c}\|_{\mathrm{curl}} \leq \Gamma_m^{-1} \|s\|_1.$$

Therefore, we obtain

$$\mathcal{B}(q, s; \mathbf{v}, \mathbf{c}) \geq \|q\|_0^2 + \|s\|_1^2 = \| (q, s) \|_B^2,$$

and

$$\begin{aligned} \| (\mathbf{v}, \mathbf{c}) \|_A^2 &\leq \max\{\Gamma_s^{-2}, \Gamma_m^{-2}\}\big[\|q\|_0^2 + \|s\|_1^2\big] \\ &\leq \max\{\Gamma_s^{-2}, \Gamma_m^{-2}\} \| (q, s) \|_B^2. \end{aligned}$$

The assertion follows. □

Corollary 2.18 *For* $\mathbf{f} \in H^{-1}(\Omega)^3$ *and* $\mathbf{g} \in L^2(\Omega)^3$*, there exists at least one solution* $(\mathbf{u}, p, \mathbf{b}, r)$ *in* $\mathbf{J} \times Q \times \mathbf{X} \times S$ *of the mixed problem in Formulation 2.15. We have the stability bounds*

$$\|\!| (\mathbf{u}, \mathbf{b}) \|\!|_A \le \frac{\|\!| \mathcal{L} \|\!|_-}{C_a \min\{R_s^{-1}, R_m^{-1} S_c\}},$$

and

$$\|\!| (p, r) \|\!|_B \le \Gamma^{-1} \Big[\|\!| \mathcal{L} \|\!|_* + \max\{R_s^{-1}, R_m^{-1} S_c\} \|\!| (\mathbf{u}, b) \|\!|_A + C_c \max\{1, S_c\} \|\!| (\mathbf{u}, b) \|\!|_A^2 \Big],$$

for any solution $(\mathbf{u}, p, \mathbf{b}, r) \in \mathbf{J} \times Q \times \mathbf{X} \times S$.

Moreover, under assumption (2.16), the problem in Formulation 2.15 has a unique solution.

Remark 2.19 In convex or smooth domains Ω, existence and uniqueness results of this type have been proven in [19, Theorem 4.7] for a variational formulation of the MHD equations (2.1)–(2.4) that is based on the standard Sobolev space $H^1(\Omega)^3$ for the magnetic field $\mathbf{b}$.

Proof. The proof of Corollary 2.18 follows from the theory of saddle point problems; see [5, Section II.1] or [16, Section I.4.1 and Section IV.I]. Let $(\mathbf{u}, \mathbf{b}) \in \mathbf{J} \times \mathbf{X}$ be the solution of the auxiliary problem in Formulation 2.8, in accordance to the results in Theorem 2.12 and Theorem 2.13. Due to the inf-sup condition in Lemma 2.17 and the continuity properties of $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{L}$, it is possible to uniquely solve the following problem for the multiplier (p, r): find $(p, r) \in Q \times S$ such that

$$\mathcal{B}(p, r; \mathbf{v}, \mathbf{c}) = \mathcal{L}(\mathbf{v}, \mathbf{c}) - \mathcal{A}(\mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}) - \mathcal{C}(\mathbf{u}; \mathbf{b}; \mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c})$$

for all $(\mathbf{v}, \mathbf{c}) \in (\mathbf{V} \times \mathbf{C})/(\mathbf{J} \times \mathbf{X})$; see, e.g., [16, Section I.4.1 and Theorem IV.1.4]. The function $(\mathbf{u}, p, \mathbf{b}, r)$ then solves (2.19) and thus is a solution of Formulation 2.15.

Furthermore, we obtain from Lemma 2.17

$$\begin{aligned} \Gamma \|\!| (p, r) \|\!|_B & \\ &\le \sup_{(\mathbf{0},\mathbf{0}) \neq (\mathbf{v},\mathbf{c}) \in \mathbf{V} \times \mathbf{C}} \frac{\mathcal{B}(p, r; \mathbf{v}, \mathbf{c})}{\|\!| (\mathbf{v}, \mathbf{c}) \|\!|_A} \\ &\le \sup_{(\mathbf{0},\mathbf{0}) \neq (\mathbf{v},\mathbf{c}) \in \mathbf{V} \times \mathbf{C}} \frac{\mathcal{L}(\mathbf{v}, \mathbf{c}) - \mathcal{A}(\mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}) - \mathcal{C}(\mathbf{u}; \mathbf{b}; \mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c})}{\|\!| (\mathbf{v}, \mathbf{c}) \|\!|_A}, \end{aligned}$$

from where the stability bound for (p, r) follows with the continuity of the forms $\mathcal{L}$, $\mathcal{A}$ and $\mathcal{C}$. □

Remark 2.20 For the boundary conditions in Remark 2.2, the spaces $\mathbf{C}$ and S in Formulation 2.15 have to be replaced by $\mathbf{C} = H_0(\text{curl}; \Omega)$ and $S = H_0^1(\Omega)$, respectively. Corollary 2.18 holds then true verbatim.

3 Finite element discretization

In this section, we introduce a mixed finite element approximation of the MHD equations in (2.1)–(2.4). The approximation is based on Nédélec's first family of elements for the discretization of the magnetic field. We then establish the well-posedness of the finite element formulation.

3.1 Meshes and finite element spaces

Throughout, we consider regular and quasi-uniform meshes $\mathcal{T}_h$ of mesh-size h that partition Ω into tetrahedra $\{K\}$. Let $\mathcal{P}_k(K)$ be the space of polynomials of total degree at most $k \geq 0$ on K and $\widetilde{\mathcal{P}}_k(K)$ the space of homogeneous polynomials of degree k on K. The space $\mathcal{D}_k(K)$ denotes the polynomials $\mathbf{p}$ in $\widetilde{\mathcal{P}}_k(K)^3$ that satisfy $\mathbf{p}(\mathbf{x}) \cdot \mathbf{x} = 0$ on K. For $k \geq 1$, we define the space

$$\mathcal{N}_k(K) = \mathcal{P}_{k-1}(K)^3 \oplus \mathcal{D}_k(K).$$

Note that $\mathcal{N}_k(K) \subset \mathcal{P}_k(K)^3$.

To approximate the unknowns $(\mathbf{u}, p)$ in Formulation 2.15, we use standard finite element spaces

$$\mathbf{V}_h \subset \mathbf{V}, \qquad Q_h \subset Q, \tag{3.1}$$

that are based on the meshes $\mathcal{T}_h$. We assume that the pair $\mathbf{V}_h \times Q_h$ satisfies the discrete inf-sup condition

$$\inf_{0 \neq q \in Q_h} \sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{V}_h} \frac{b_s(q, \mathbf{v})}{\|\mathbf{v}\|_1 \|q\|_0} \geq \gamma_s > 0, \tag{3.2}$$

with an inf-sup constant γ_s that is independent of the mesh-size h.

A wide variety of spaces $\mathbf{V}_h$ and Q_h fulfilling the inf-sup condition in (3.2) have been proposed in the literature; we refer to [5, Chapter IV], [16, Chapter II] and the references cited therein.

To approximate the unknowns $(\mathbf{b}, r)$ in Formulation 2.15, we use Nédélec's first family of spaces, see [25,26], given by

$$\mathbf{C}_h = \{\, \mathbf{c} \in \mathbf{C} \mid \mathbf{u}|_K \in \mathcal{N}_k(K),\ K \in \mathcal{T}_h \,\}, \tag{3.3}$$

combined with the standard H^1-conforming space

$$S_h = \{\, s \in S \mid s|_K \in \mathcal{P}_k(K),\ K \in \mathcal{T}_h \,\}, \tag{3.4}$$

for an approximation order $k \geq 1$. Since $\nabla S_h \subset \mathbf{C}_h$, it can be seen as on the continuous level that $\mathbf{C}_h \times S_h$ satisfies the inf-sup condition

$$\inf_{0 \neq s \in S_h} \sup_{\mathbf{0} \neq \mathbf{c} \in \mathbf{C}_h} \frac{b_m(s, \mathbf{c})}{\|\mathbf{c}\|_{\text{curl}} \|s\|_1} \geq \gamma_m > 0, \tag{3.5}$$

with γ_m only depending on Ω; see also [25, Theorem 9] and [21, Section 5.4]. In fact, we have $\gamma_m = \Gamma_m$, with Γ_m being the constant in (2.17).

Remark 3.1 It is also possible to base the space $\mathbf{C}_h$ on Nédélec's elements of the second type; see [27] for the definition of these elements. The space S_h then has to be suitably adjusted; see [27] or [23, Section 8.2].

3.2 Discretely solenoidal function spaces

As on the continuous level, we first eliminate the multipliers by introducing the following spaces of discretely divergence-free functions:

$$\mathbf{J}_h := \{ \mathbf{v} \in \mathbf{V}_h \mid b_s(q, \mathbf{v}) = 0 \ \forall q \in Q_h \},$$
$$\mathbf{X}_h := \{ \mathbf{c} \in \mathbf{C}_h \mid b_m(s, \mathbf{c}) = 0 \ \forall s \in S_h \}.$$

Note that $\mathbf{J}_h \not\subset \mathbf{J}$ and $\mathbf{X}_h \not\subset \mathbf{X}$ which complicates the analysis of the discrete problem and especially the treatment of the coupling forms c_1 and c_2.

One of our main tools will be the following discrete Helmholtz decomposition:

$$\mathbf{C}_h = \mathbf{X}_h \oplus \nabla S_h, \tag{3.6}$$

the decomposition being orthogonal in $L^2(\Omega)^3$; see [25–27] or [23, Section 7.2.1]. Furthermore, the space $\mathbf{X}_h$ is known to satisfy the following discrete Poincaré-Friedrichs inequality:

$$\|\operatorname{curl} \mathbf{c}\|_0 \geq C \|\mathbf{c}\|_0 \qquad \forall \mathbf{c} \in \mathbf{X}_h, \tag{3.7}$$

with a constant $C > 0$ independent of the mesh-size h; cf., e.g., [21, Theorem 4.7].

First, we show that the trilinear forms c_0, c_1 and c_2 are well-defined on the discrete level; in view of Lemma 2.5, this is evident for the form c_0 since $\mathbf{V}_h \subset \mathbf{V}$. To study the forms c_1 and c_2, we set

$$\mathbf{X}(h) := \mathbf{X} + \mathbf{X}_h,$$

equipped with the norm $\| \cdot \|_{\text{curl}}$.

Proposition 3.2 *For* $\mathbf{d} \in \mathbf{X}(h)$*,* $\mathbf{v} \in \mathbf{V}_h$ *and* $\mathbf{b} \in \mathbf{C}$*, we have*

$$|c_1(\mathbf{d}; \mathbf{v}, \mathbf{b})| \leq C\, S_c\, \|\mathbf{d}\|_{\text{curl}} \|\mathbf{v}\|_1 \|\operatorname{curl} \mathbf{b}\|_0.$$

Moreover, for $\mathbf{d} \in \mathbf{X}(h)$*,* $\mathbf{u} \in \mathbf{V}$ *and* $\mathbf{c} \in \mathbf{C}_h$*, we have*

$$|c_2(\mathbf{d}; \mathbf{u}, \mathbf{c})| \leq C\, S_c\, \|\mathbf{d}\|_{\text{curl}} \|\mathbf{u}\|_1 \|\operatorname{curl} \mathbf{c}\|_0.$$

The constants $C > 0$ *are independent of the mesh-size* h*.*

Proof. We proceed in several steps.

Step 1 We start by noting that there is a linear mapping $\mathbf{H} : \mathbf{X_h} \to \mathbf{X}$ such that $\operatorname{curl} \mathbf{d} = \operatorname{curl}(\mathbf{H}\mathbf{d})$ for all $\mathbf{d} \in \mathbf{X}_h$. Since functions in $\mathbf{X}$ are uniquely defined by their rotations, see [21, Section 4] and the Poincaré-Friedrichs inequality in (2.7), this mapping is well-defined. We then have

$$\|\mathbf{d} - \mathbf{H}\mathbf{d}\|_0 \leq C h^{\frac{1}{2}+\sigma} \|\operatorname{curl} \mathbf{d}\|_0 \qquad \forall\, \mathbf{d} \in \mathbf{X}_h, \tag{3.8}$$

for an exponent $\sigma > 0$ depending solely on the domain Ω. This can be easily seen by adapting the arguments in, e.g., [21, Section 4] to our boundary conditions.

Step 2 We recall the following inverse estimate from [6, Theorem 3.2.6]. On a quasi-uniform mesh there holds

$$\|\varphi\|_{0,q} \leq C h^{3(\frac{1}{q}-\frac{1}{p})} \|\varphi\|_{0,p}, \qquad 1 \leq p \leq q \leq \infty, \tag{3.9}$$

for all piecewise polynomial functions φ, with a constant $C > 0$ independent of the mesh-size h.

Step 3 We prove the assertion for the form c_1. For $\mathbf{d} \in \mathbf{X}$, the claim follows from Lemma 2.6 since $\mathbf{V}_h \subset \mathbf{V}$. Thus, we may assume that $\mathbf{d} \in \mathbf{X}_h$. By the triangle inequality, Lemma 2.6, the definition of $\mathbf{H}$, and the Poincaré-Friedrichs inequality (2.7), we first note that

$$\begin{aligned} |c_1(\mathbf{d}; \mathbf{v}, \mathbf{b})| &\leq |c_1(\mathbf{d} - \mathbf{H}\mathbf{d}; \mathbf{v}, \mathbf{b})| + |c_1(\mathbf{H}\mathbf{d}; \mathbf{v}, \mathbf{b})| \\ &\leq |c_1(\mathbf{d} - \mathbf{H}\mathbf{d}; \mathbf{v}, \mathbf{b})| + C S_c \|\mathbf{H}\mathbf{d}\|_{\text{curl}} \|\mathbf{v}\|_1 \|\operatorname{curl} \mathbf{b}\|_0 \\ &= |c_1(\mathbf{d} - \mathbf{H}\mathbf{d}; \mathbf{v}, \mathbf{b})| + C S_c \|\operatorname{curl} \mathbf{d}\|_0 \|\mathbf{v}\|_1 \|\operatorname{curl} \mathbf{b}\|_0, \end{aligned}$$

so that it remains to estimate the term $|c_1(\mathbf{d} - \mathbf{H}\mathbf{d}; \mathbf{v}, \mathbf{b})|$. To do so, fix $0 < \sigma' \leq \sigma$ and choose $p \in [1, 6)$ such that

$$\frac{3}{p} = \frac{1}{2} + \sigma'. \tag{3.10}$$

We obtain

$$\begin{aligned} |c_1(\mathbf{d}-\mathbf{Hd};\mathbf{v},\mathbf{b})| &\leq S_c\|\mathbf{v}\|_{0,\infty}\|\mathbf{d}-\mathbf{Hd}\|_0\|\operatorname{curl}\mathbf{b}\|_0 \\ &\leq C S_c\, h^{-\frac{1}{2}-\sigma'}\, h^{\frac{1}{2}+\sigma}\,\|\mathbf{v}\|_{0,p}\|\operatorname{curl}\mathbf{d}\|_0\|\operatorname{curl}\mathbf{b}\|_0 \\ &\leq C S_c\,\|\mathbf{v}\|_1\|\operatorname{curl}\mathbf{d}\|_0\|\operatorname{curl}\mathbf{b}\|_0. \end{aligned}$$

Here, we have used Hölder's inequality, the inverse estimate (3.9) from Step 2 (with $q=\infty$ and the exponent p in (3.10)), the approximation result from (3.8) from Step 1, and the first imbedding from Proposition 2.3. This proves the assertion for c_1.

Step 4 For c_2, we proceed as before and may assume that $\mathbf{d}\in\mathbf{X}_h$. For $0<\sigma'\leq\sigma$ we choose parameters $p_1, p_2\geq 1$ such that

$$\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{2}, \qquad \frac{1}{p_1}-\frac{1}{2}=-\frac{1}{6}-\frac{\sigma'}{3}, \qquad p_2\in[1,6).$$

As before, we only need to bound the term $c_2(\mathbf{d}-\mathbf{Hd};\mathbf{u},\mathbf{c})$. We have

$$\begin{aligned} |c_2(\mathbf{d}-\mathbf{Hd};\mathbf{u},\mathbf{c})| &\leq S_c\,\|\mathbf{d}-\mathbf{Hd}\|_0\|\mathbf{u}\|_{0,p_2}\|\operatorname{curl}\mathbf{c}\|_{0,p_1} \\ &\leq C S_c\, h^{\frac{1}{2}+\sigma}h^{-\frac{1}{2}-\sigma'}\|\operatorname{curl}\mathbf{d}\|_0\|\mathbf{u}\|_{0,p_2}\|\operatorname{curl}\mathbf{c}\|_0 \\ &\leq C S_c\|\operatorname{curl}\mathbf{d}\|_0\|\mathbf{u}\|_1\|\operatorname{curl}\mathbf{c}\|_0. \end{aligned}$$

Similarly to Step 3, we used Hölder's inequality, the approximation result (3.8) from Step 1, the inverse estimate (3.9) from Step 2 (with $q=p_1$ and $p=2$), and the first imbedding from Proposition 2.3. Further, since $|c_2(\mathbf{Hd};\mathbf{u},\mathbf{c})|$ can be bounded by Lemma 2.6, the assertion for c_2 follows. □

As a consequence of Proposition 3.2, the trilinear form $\mathcal{C}$ is continuous.

Corollary 3.1 *Let* $\mathbf{w},\mathbf{u}\in\mathbf{V}$, $\mathbf{d}\in\mathbf{X}(h)$, $\mathbf{b}\in\mathbf{C}$, *and* $(\mathbf{v},\mathbf{c})\in\mathbf{V}_h\times\mathbf{C}_h$. *Then there holds*

$$|\mathcal{C}(\mathbf{w},\mathbf{d};\mathbf{u},\mathbf{b};\mathbf{v},\mathbf{c})|\leq C_c\max\{1,S_c\}|||\,(\mathbf{w},\mathbf{d})\,|||_A|||\,(\mathbf{u},\mathbf{b})\,|||_A|||\,(\mathbf{v},\mathbf{c})\,|||_A$$

for a constant $C_c>0$ *independent of the mesh-size* h.

Furthermore, we have that $\mathcal{C}(\mathbf{w},\mathbf{d};\mathbf{v},\mathbf{c};\mathbf{v},\mathbf{c})=0$.

Note that, for simplicity, we have used the same notation for the continuity constant C_c as in (2.12). The discrete version of the auxiliary problem in Formulation 2.8 is then as follows.

Formulation 3.3 *Find* $(\mathbf{u}_h,\mathbf{b}_h)\in\mathbf{J}_h\times\mathbf{X}_h$ *such that*

$$\begin{aligned} a_s(\mathbf{u}_h,\mathbf{v})+c_0(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v})-c_1(\mathbf{b}_h;\mathbf{v},\mathbf{b}_h) &= \langle\mathbf{f},\mathbf{v}\rangle, \\ a_m(\mathbf{b}_h,\mathbf{c})-c_2(\mathbf{b}_h;\mathbf{u}_h,\mathbf{c}) &= (\mathbf{g},\mathbf{c}) \end{aligned}$$

for all $(\mathbf{v},\mathbf{c})\in\mathbf{J}_h\times\mathbf{X}_h$.

The compact form of the variational problem in Formulation 3.3 is: find $(\mathbf{u}_h, \mathbf{b}_h) \in \mathbf{J}_h \times \mathbf{X}_h$ such that

$$\mathcal{A}(\mathbf{u}_h, \mathbf{b}_h; \mathbf{v}, \mathbf{c}) + \mathcal{C}(\mathbf{u}_h, \mathbf{b}_h; \mathbf{u}_h, \mathbf{b}_h; \mathbf{v}, \mathbf{c}) = \mathcal{L}(\mathbf{v}, \mathbf{c}) \tag{3.11}$$

for all $(\mathbf{v}, \mathbf{c}) \in \mathbf{J}_h \times \mathbf{X}_h$.

In view of (2.11) and Corollary 3.1, the forms $\mathcal{A}$ and $\mathcal{C}$ are continuous on the discrete level. Furthermore, we have

$$\mathcal{A}(\mathbf{v}, \mathbf{c}; \mathbf{v}, \mathbf{c}) \geq C_a \min\{R_s^{-1}, R_m^{-1} S_c\} \, ||| (\mathbf{v}, \mathbf{c}) |||_A^2, \tag{3.12}$$

for all $(\mathbf{v}, \mathbf{c}) \in \mathbf{J}_h \times \mathbf{X}_h$, with a constant $C_a > 0$ independent of the mesh-size h, again denoted as in (2.9) for simplicity. The coercivity with respect to $\mathbf{c}$ follows from the discrete Poincaré-Friedrichs inequality in (3.7). Finally, we need the discrete counterpart of $||| \mathcal{L} |||_-$ given by

$$||| \mathcal{L} |||_h := \sup_{(\mathbf{0},\mathbf{0}) \neq (\mathbf{v},\mathbf{c}) \in \mathbf{J}_h \times \mathbf{X}_h} \frac{\mathcal{L}(\mathbf{v}, \mathbf{c})}{||| (\mathbf{v}, \mathbf{c}) |||_A}.$$

Again, $||| \mathcal{L} |||_h \leq ||| \mathcal{L} |||_*$.

From Theorem 2.11, we obtain the following result.

Theorem 3.4 *For* $\mathbf{f} \in H^{-1}(\Omega)^3$ *and* $\mathbf{g} \in L^2(\Omega)^3$, *there exists at least one solution* $(\mathbf{u}_h, \mathbf{b}_h)$ *in* $\mathbf{J}_h \times \mathbf{X}_h$ *of the problem in Formulation 3.3. We have the stability bound*

$$||| (\mathbf{u}_h, \mathbf{b}_h) |||_A \leq \frac{||| \mathcal{L} |||_h}{C_a \min\{R_s^{-1}, R_m^{-1} S_c\}}$$

for any solution $(\mathbf{u}_h, \mathbf{b}_h) \in \mathbf{J}_h \times \mathbf{X}_h$. *Furthermore, for small data with*

$$\frac{C_c \max\{1, S_c\} \, ||| \mathcal{L} |||_h}{C_a^2 \min\{R_s^{-2}, R_m^{-2} S_c^2\}} < 1, \tag{3.13}$$

the problem in Formulation 3.3 has a unique solution $(\mathbf{u}_h, \mathbf{b}_h) \in \mathbf{J}_h \times \mathbf{X}_h$.

3.3 Mixed finite element approximation

With the discrete spaces from Section 3.1, the finite element approximation of the weak problem in Formulation 2.15 reads as follows.

Formulation 3.5 *Find* $(\mathbf{u}_h, p_h, \mathbf{b}_h, r_h) \in \mathbf{J}_h \times Q_h \times \mathbf{X}_h \times S_h$ *such that*

$$\begin{aligned} a_s(\mathbf{u}_h, \mathbf{v}) + c_0(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}) - c_1(\mathbf{b}_h; \mathbf{v}, \mathbf{b}_h) + b_s(p_h, \mathbf{v}) &= \langle \mathbf{f}, \mathbf{v} \rangle, \\ a_m(\mathbf{b}_h, \mathbf{c}) - c_2(\mathbf{b}_h; \mathbf{u}_h, \mathbf{c}) + b_m(r_h, \mathbf{c}) &= (\mathbf{g}, \mathbf{c}) \end{aligned}$$

for all $(\mathbf{v}, \mathbf{c}) \in \mathbf{V}_h \times \mathbf{C}_h$.

Remark 3.6 Writing $\mathbf{g} = \mathbf{g}_0 + \nabla\phi$ as in Remark 2.16, it can be seen that r_h solves $-(\nabla r_h, \nabla s) = (\nabla\phi, \nabla s)$, $s \in S_h$. Thus, for a solenoidal source term $\mathbf{g} \in H_0(\mathrm{div}^0; \Omega)$, we have $r_h = 0$.

By introducing the discrete kernel

$$Z_h := \{(\mathbf{v}, \mathbf{c}) \in \mathbf{V}_h \times \mathbf{C}_h : \mathcal{B}(q, s; \mathbf{v}, \mathbf{c}) = 0 \quad \forall\, (q, s) \in Q_h \times S_h\},$$

we have

$$Z_h = \mathbf{J}_h \times \mathbf{X}_h. \tag{3.14}$$

Therefore, we can write the variational problem in Formulation 3.5 in the form: find $(\mathbf{u}_h, p_h, \mathbf{b}_h, r_h) \in \mathbf{V}_h \times Q_h \times \mathbf{C}_h \times S_h$ such that

(3.15)
$$\begin{aligned} \mathcal{A}(\mathbf{u}_h, \mathbf{b}_h; \mathbf{v}, \mathbf{c}) + \mathcal{C}(\mathbf{u}_h, \mathbf{b}_h; \mathbf{u}_h, \mathbf{b}_h; \mathbf{v}, \mathbf{c}) + \mathcal{B}(p_h, r_h; \mathbf{v}, \mathbf{c}) &= \mathcal{L}(\mathbf{v}, \mathbf{c}), \\ \mathcal{B}(q, s; \mathbf{u}_h, \mathbf{b}_h) &= 0 \end{aligned}$$

for all $(\mathbf{v}, q, \mathbf{c}, s) \in \mathbf{V}_h \times Q_h \times \mathbf{C}_h \times S_h$.

The discrete inf-sup conditions for b_s and b_m in (3.2) and (3.5), respectively, give the discrete analogue of Lemma 2.17.

Lemma 3.7 *There is a constant $\gamma > 0$ only depending on γ_s and γ_m in (3.2) and (3.5) such that*

$$\sup_{(\mathbf{v},\mathbf{c}) \in \mathbf{V}_h \times \mathbf{C}_h} \frac{\mathcal{B}(q, s; \mathbf{v}, \mathbf{c})}{|\!|\!| (\mathbf{v}, \mathbf{c}) |\!|\!|_A} \geq \gamma |\!|\!| (q, s) |\!|\!|_B,$$

for all $(q, s) \in Q_h \times S_h$. In particular, γ is independent of the mesh-size h.

From Theorem 2.11, we thus obtain the discrete counterpart of Corollary 2.18.

Corollary 3.2 *For $\mathbf{f} \in H^{-1}(\Omega)^3$ and $\mathbf{g} \in L^2(\Omega)^3$, there exists at least one solution $(\mathbf{u}_h, p_h, \mathbf{b}_h, r_h)$ in $\mathbf{J}_h \times Q_h \times \mathbf{X}_h \times S_h$ of the discrete problem in Formulation 3.5. We have the stability bounds*

$$|\!|\!| (\mathbf{u}_h, \mathbf{b}_h) |\!|\!|_A \leq \frac{|\!|\!| \mathcal{L} |\!|\!|_h}{C_a \min\{R_s^{-1}, R_m^{-1} S_c\}},$$

and

$$\begin{aligned} |\!|\!| (p_h, r_h) |\!|\!|_B \leq \gamma^{-1} \Big[& |\!|\!| \mathcal{L} |\!|\!|_* + \max\{R_s^{-1}, R_m^{-1} S_c\} |\!|\!| (\mathbf{u}_h, b_h) |\!|\!|_A \\ & + C_c \max\{1, S_c\} |\!|\!| (\mathbf{u}_h, b_h) |\!|\!|_A^2 \Big], \end{aligned}$$

for any solution $(\mathbf{u}_h, p_h, \mathbf{b}_h, r_h) \in \mathbf{J}_h \times Q_h \times \mathbf{X}_h \times S_h$.

Under assumption (3.13), the discrete problem in Formulation 3.5 has a unique solution $(\mathbf{u}_h, p_h, \mathbf{b}_h, r_h)$ *in* $\mathbf{J}_h \times Q_h \times \mathbf{X}_h \times S_h$.

Remark 3.8 The above results can be easily adapted to the boundary conditions in Remark 2.2. In this case, we take

$$\mathbf{C}_h = \{ \mathbf{c} \in H_0(\text{curl}; \Omega) \mid \mathbf{u}|_K \in \mathcal{N}_k(K), \ K \in \mathcal{T}_h \}$$

and set

$$S_h = \{ s \in H_0^1(\Omega) \mid s|_K \in \mathcal{P}_k(K), \ K \in \mathcal{T}_h \},$$

according to Remark 2.20.

3.4 Solution methods

Let us briefly discuss some solution methods to numerically find the approximation $(\mathbf{u}_h, p_h, \mathbf{b}_h, r_h)$ of the problem in Formulation 3.5; we also refer to the discussions and numerical comparisons in [19, Section 7] and [14, Chapter VII].

First, we consider the classical coupled Picard iteration that reads as follows: given $(\mathbf{u}_h^\ell, p_h^\ell, \mathbf{b}_h^\ell, r_h^\ell) \in \mathbf{J}_h \times Q_h \times \mathbf{X}_h \times S_h$, determine the next iterate by solving the following linearized MHD problem: find $(\mathbf{u}_h^{\ell+1}, p_h^{\ell+1}, \mathbf{b}_h^{\ell+1}, r_h^{\ell+1}) \in \mathbf{J}_h \times Q_h \times \mathbf{X}_h \times S_h$ such that

$$\begin{aligned} a_s(\mathbf{u}_h^{\ell+1}, \mathbf{v}) + c_0(\mathbf{u}_h^\ell; \mathbf{u}_h^{\ell+1}, \mathbf{v}) - c_1(\mathbf{b}_h^\ell; \mathbf{v}, \mathbf{b}_h^{\ell+1}) + b_s(p_h^{\ell+1}, \mathbf{v}) &= \langle \mathbf{f}, \mathbf{v} \rangle, \\ a_m(\mathbf{b}_h^{\ell+1}, \mathbf{c}) - c_2(\mathbf{b}_h^\ell; \mathbf{u}_h^{\ell+1}, \mathbf{c}) + b_m(r_h^{\ell+1}, \mathbf{c}) &= (\mathbf{g}, \mathbf{c}) \end{aligned}$$

for all $(\mathbf{v}, q, \mathbf{c}, s) \in \mathbf{V}_h \times Q_h \times \mathbf{C}_h \times S_h$. Under assumption (3.13) on the data, the iterates $\{(\mathbf{u}_h^\ell, p_h^\ell, \mathbf{b}_h^\ell, r_h^\ell)\}_\ell$ converge to the solution $(\mathbf{u}_h, p_h, \mathbf{b}_h, r_h)$ of the problem in Formulation 3.5 for any initial guess $(\mathbf{u}_h^0, p_h^0, \mathbf{b}_h^0, r_h^0) \in \mathbf{J}_h \times Q_h \times \mathbf{X}_h \times S_h$. The initial guess $(\mathbf{u}_h^0, p_h^0, \mathbf{b}_h^0, r_h^0)$ can be found, for example, by solving the above problem with data $\mathbf{u}_h^{-1} = \mathbf{b}_h^{-1} = \mathbf{0}$.

The coupled Picard iteration requires the solution of a fully coupled mixed system in each iteration step which is quite costly. On the other hand, a wide variety of efficient solution techniques for linear systems of this form are available nowadays. We mention here only the very efficient and robust saddle point solvers for incompressible flow problems in [12,11,31] and the references therein. These approaches, however, need to be extended to MHD problems of the above form by combining them with efficient $H(\text{curl}; \Omega)$-based solution techniques for Maxwell's equations; see, e.g., [4,20,30] and the references therein. We also point out in passing that, in typical incompressible MHD applications, only the hydrodynamic Reynolds number R_s

is large, say in the range 10^2–10^5, while the magnetic one is moderate, say $R_m \approx 10^{-1}$ and $S_c \approx 1$. Therefore, the structure of the linearized MHD problem above is closely related to the Oseen problem of incompressible fluid flow.

The application of Newton's method gives rise to similar linear MHD problems in each iteration step. The convergence of the iterates is generally faster, but only guaranteed for initial guesses that are sufficiently close to the exact solution; cf. [19, 14]. Finally, we mention the decoupled Picard iteration that reads as follows: given $(\mathbf{u}_h^\ell, p_h^\ell, \mathbf{b}_h^\ell, r_h^\ell) \in \mathbf{J}_h \times Q_h \times \mathbf{X}_h \times S_h$, find the next iterate $(\mathbf{u}_h^{\ell+1}, p_h^{\ell+1}, \mathbf{b}_h^{\ell+1}, r_h^{\ell+1}) \in \mathbf{J}_h \times Q_h \times \mathbf{X}_h \times S_h$ by solving:

$$a_s(\mathbf{u}_h^{\ell+1}, \mathbf{v}) + b_s(p_h^{\ell+1}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle - c_0(\mathbf{u}_h^\ell; \mathbf{u}_h^\ell, \mathbf{v}) + c_1(\mathbf{b}_h^\ell; \mathbf{v}, \mathbf{b}_h^\ell),$$
$$a_m(\mathbf{b}_h^{\ell+1}, \mathbf{c}) + b_m(r_h^{\ell+1}, \mathbf{c}) = (\mathbf{g}, \mathbf{c}) + c_2(\mathbf{b}_h^\ell; \mathbf{u}_h^\ell, \mathbf{c})$$

for all $(\mathbf{v}, q, \mathbf{c}, s) \in \mathbf{V}_h \times Q_h \times \mathbf{C}_h \times S_h$. However, this procedure only converges for initial guesses that are sufficiently close to the exact solution and is instable for higher Reynolds numbers; cf., e.g., [19, Proposition 7.2] or [14, Section VII.5]. On the other hand, the equations completely decouple into a standard Stokes problem for the iterate $(\mathbf{u}_h^{\ell+1}, p_h^{\ell+1})$ and a standard Maxwell problem for $(\mathbf{b}_h^{\ell+1}, r_h^{\ell+1})$ for which efficient solvers are available, as discussed above. Improved iterative procedures that decouple into an Oseen-type problem and a Maxwell problem were recently developed in [14, Section VII.5].

Remark 3.9 If the linearized MHD problems above are convection-dominated, it might be necessary for their efficient solution to include suitable stabilization terms in the bilinear forms in order to avoid numerical instabilities. This can be done by employing the techniques recently developed in [15]. However, as our analysis is mainly concerned with the incorporation of the divergence constraint $\operatorname{div} \mathbf{b} = 0$ via the mixed approach, this point is not further investigated in the present paper and remains to be addressed elsewhere.

4 Error analysis

In this section, we show that the mixed finite element method proposed in Formulation 3.5 yields quasi-optimal error bounds in the mesh-size h. We then discuss the convergence rates that are obtained under standard smoothness assumptions.

4.1 Error bounds

We begin by addressing the error in $(\mathbf{u}, \mathbf{b})$. We assume that the data is sufficiently small so that both the continuous and the discrete problem are uniquely solvable.

Theorem 4.1 *Assume that*

$$\frac{C_c \max\{1, S_c\} \,|\!|\!|\, \mathcal{L} \,|\!|\!|_*}{C_a^2 \min\{R_s^{-2}, R_m^{-2} S_c^2\}} < \frac{1}{2}. \tag{4.1}$$

Let $(\mathbf{u}, p, \mathbf{b}, r) \in \mathbf{J} \times Q \times \mathbf{X} \times S$ *be the solution of the problem in Formulation 2.15 and let* $(\mathbf{u}_h, p_h, \mathbf{b}_h, r_h) \in \mathbf{J}_h \times Q_h \times \mathbf{X}_h \times S_h$ *be its approximation given in Formulation 3.5. Then we have that*

$$\begin{aligned} |\!|\!|\, (\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h) \,|\!|\!|_A \leq C &\inf_{(\mathbf{v},\mathbf{c}) \in \mathbf{V}_h \times \mathbf{C}_h} |\!|\!|\, (\mathbf{u} - \mathbf{v}, \mathbf{b} - \mathbf{c}) \,|\!|\!|_A \\ &+ C \inf_{(q,s) \in Q_h \times S_h} |\!|\!|\, (p - q, r - s) \,|\!|\!|_B, \end{aligned}$$

with a constant $C > 0$ *that is independent of the mesh-size* h.

Proof. We proceed in several steps.

Step 1 We first note that the error satisfies

$$\begin{aligned} &\mathcal{A}(\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h; \mathbf{v}, \mathbf{c}) + \mathcal{B}(p - p_h, r - r_h; \mathbf{v}, \mathbf{c}) \\ &\quad + \mathcal{C}(\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h; \mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}) \\ &\quad + \mathcal{C}(\mathbf{u}_h, \mathbf{b}_h; \mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h; \mathbf{v}, \mathbf{c}) = 0 \end{aligned} \tag{4.2}$$

for all $(\mathbf{v}, \mathbf{c}) \in \mathbf{V}_h \times \mathbf{C}_h$. This can be easily seen by subtracting the discrete formulation (3.15) from the continuous one in (2.19) and by using the trilinearity of the form $\mathcal{C}$. Note that all the terms are well-defined due to Corollary 3.1.

Step 2 Let $(\mathbf{v}, \mathbf{c})$ be in the discrete kernel Z_h. Recall that $Z_h = \mathbf{J}_h \times \mathbf{X}_h$, according to (3.14). Using the orthogonality property from Step 1, we have

$$\begin{aligned} &\mathcal{A}(\mathbf{v} - \mathbf{u}_h, \mathbf{c} - \mathbf{b}_h; \mathbf{v} - \mathbf{u}_h, \mathbf{c} - \mathbf{b}_h) \\ &\quad + \mathcal{C}(\mathbf{v} - \mathbf{u}_h, \mathbf{c} - \mathbf{b}_h; \mathbf{u}, \mathbf{b}; \mathbf{v} - \mathbf{u}_h, \mathbf{c} - \mathbf{b}_h) \\ &\quad + \mathcal{C}(\mathbf{u}_h, \mathbf{b}_h; \mathbf{v} - \mathbf{u}_h, \mathbf{c} - \mathbf{b}_h; \mathbf{v} - \mathbf{u}_h, \mathbf{c} - \mathbf{b}_h) \\ &= \mathcal{A}(\mathbf{v} - \mathbf{u}, \mathbf{c} - \mathbf{b}; \mathbf{v} - \mathbf{u}_h, \mathbf{c} - \mathbf{b}_h) \\ &\quad + \mathcal{C}(\mathbf{v} - \mathbf{u}, \mathbf{c} - \mathbf{b}; \mathbf{u}, \mathbf{b}; \mathbf{v} - \mathbf{u}_h, \mathbf{c} - \mathbf{b}_h) \\ &\quad + \mathcal{C}(\mathbf{u}_h, \mathbf{b}_h; \mathbf{v} - \mathbf{u}, \mathbf{c} - \mathbf{b}; \mathbf{v} - \mathbf{u}_h, \mathbf{c} - \mathbf{b}_h) \\ &\quad - \mathcal{B}(p - p_h, r - r_h; \mathbf{v} - \mathbf{u}_h, \mathbf{c} - \mathbf{b}_h). \end{aligned} \tag{4.3}$$

Again, all the terms in (4.3) are well-defined. We note that

$$\mathcal{C}(\mathbf{u}_h, \mathbf{b}_h; \mathbf{v} - \mathbf{u}_h, \mathbf{c} - \mathbf{b}_h; \mathbf{v} - \mathbf{u}_h, \mathbf{c} - \mathbf{b}_h) = 0,$$

in view of the skew-symmetry of the form $\mathcal{C}$ in Corollary 3.1. Since $(\mathbf{v} - \mathbf{u}_h, \mathbf{c} - \mathbf{b}_h)$ belongs to the kernel Z_h, we have

$$\mathcal{B}(p - p_h, r - r_h; \mathbf{v} - \mathbf{u}_h, \mathbf{c} - \mathbf{b}_h) = \mathcal{B}(p - q, r - s; \mathbf{v} - \mathbf{u}_h, \mathbf{c} - \mathbf{b}_h)$$

for any $(q, s) \in Q_h \times S_h$.

Using the continuity properties of the forms $\mathcal{A}$, $\mathcal{C}$, $\mathcal{B}$, and the stability bounds for $|||\,(\mathbf{u},\mathbf{b})\,|||_A$ and $|||\,(\mathbf{u}_h,\mathbf{b}_h)\,|||_A$ in Corollary 2.18 and Corollary 3.2, respectively, the right-hand side of (4.3) can be bounded from above by

$$\begin{aligned}
\text{r.h.s.} \leq\ & |||\,(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\,|||_A \Big[\max\{R_s^{-1}, R_m^{-1}S_c\} |||\,(\mathbf{u}-\mathbf{v},\mathbf{b}-\mathbf{c})\,|||_A \\
&+ C_c \max\{1, S_c\} |||\,(\mathbf{u}-\mathbf{v},\mathbf{b}-\mathbf{c})\,|||_A |||\,(\mathbf{u},\mathbf{b})\,|||_A \\
&+ C_c \max\{1, S_c\} |||\,(\mathbf{u}-\mathbf{v},\mathbf{b}-\mathbf{c})\,|||_A |||\,(\mathbf{u}_h,\mathbf{b}_h)\,|||_A \\
&+ C_b |||\,(p-q, r-s)\,|||_B \Big] \\
\leq\ & C |||\,(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\,|||_A \Big[|||\,(\mathbf{u}-\mathbf{v},\mathbf{b}-\mathbf{c})\,|||_A + |||\,(p-q,r-s)\,|||_B \Big].
\end{aligned}$$

Next, using the coercivity property (3.12) of the form $\mathcal{A}$ on the kernel Z_h, continuity of $\mathcal{C}$ in Corollary 3.1, the stability bound for $|||\,(\mathbf{u},\mathbf{b})\,|||_A$ in Corollary 2.18, and assumption (4.1) allows us to bound the left-hand side of (4.3) from below by

$$\begin{aligned}
\text{l.h.s.} \geq\ & C_a \min\{R_s^{-1}, R_m^{-1}S_c\} |||\,(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\,|||_A^2 \\
&- C_c \max\{1, S_c\} |||\,(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\,|||_A^2 |||\,(\mathbf{u},\mathbf{b})\,|||_A \\
\geq\ & \frac{1}{2} C_a \min\{R_s^{-1}, R_m^{-1}S_c\} |||\,(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\,|||_A^2.
\end{aligned}$$

Combining these bounds yields

$$|||\,(\mathbf{v}-\mathbf{u}_h,\mathbf{c}-\mathbf{b}_h)\,|||_A \leq C |||\,(\mathbf{u}-\mathbf{v},\mathbf{b}-\mathbf{c})\,|||_A + C |||\,(p-q,r-s)\,|||_B.$$

Therefore, by the triangle inequality,

$$|||\,(\mathbf{u}-\mathbf{u}_h,\mathbf{b}-\mathbf{b}_h)\,|||_A \leq C |||\,(\mathbf{u}-\mathbf{v},\mathbf{b}-\mathbf{c})\,|||_A + C |||\,(p-q,r-s)\,|||_B,$$

for all $(\mathbf{v},\mathbf{c}) \in Z_h$, $(q,s) \in Q_h \times S_h$.

Step 3 Let now $(\mathbf{v},\mathbf{c}) \in \mathbf{V}_h \times \mathbf{C}_h$ be arbitrary. Let $(\mathbf{w},\mathbf{d}) \in \mathbf{V}_h \times \mathbf{C}_h$ be a solution of

$$\mathcal{B}(\mathsf{q},\mathsf{s};\mathbf{w},\mathbf{d}) = \mathcal{B}(\mathsf{q},\mathsf{s};\mathbf{u}-\mathbf{v},\mathbf{b}-\mathbf{c}) \qquad \forall(\mathsf{q},\mathsf{s}) \in Q_h \times S_h.$$

From the inf-sup condition in Lemma 3.7 and the continuity of the form $\mathcal{B}$, there exists a solution to this problem that satisfies

$$|||\,(\mathbf{w},\mathbf{d})\,|||_A \leq C |||\,(\mathbf{u}-\mathbf{v},\mathbf{b}-\mathbf{c})\,|||_A; \tag{4.4}$$

see [5, Section II.2.2] for details. Then, $(\mathbf{w}+\mathbf{v}, \mathbf{d}+\mathbf{b}) \in Z_h$ can be inserted in the bound of Step 2. With the triangle inequality, we obtain

$$\begin{aligned} \|\!|\!| (\mathbf{u}-\mathbf{u}_h, \mathbf{b}-\mathbf{b}_h) \|\!|\!|_A &\leq C \|\!|\!| (\mathbf{u}-\mathbf{v}, \mathbf{b}-\mathbf{c}) \|\!|\!|_A \\ &\quad + C \|\!|\!| (\mathbf{w}, \mathbf{d}) \|\!|\!|_A + C \|\!|\!| (p-q, r-s) \|\!|\!|_B. \end{aligned}$$

Referring to (4.4) finishes the proof. □

Next, we bound the error in (p, r).

Theorem 4.2 *Assume (4.1). Let* $(\mathbf{u}, p, \mathbf{b}, r) \in \mathbf{J} \times Q \times \mathbf{X} \times S$ *be the solution of the problem Formulation 2.15 and* $(\mathbf{u}_h, p_h, \mathbf{b}_h, r_h) \in \mathbf{J}_h \times Q_h \times \mathbf{X}_h \times S_h$ *its approximation given in Formulation 3.5. Then we have that*

$$\begin{aligned} \|\!|\!| (p-p_h, r-r_h) \|\!|\!|_B &\leq C \inf_{(\mathbf{v},\mathbf{c}) \in \mathbf{V}_h \times \mathbf{C}_h} \|\!|\!| (\mathbf{u}-\mathbf{v}, \mathbf{b}-\mathbf{c}) \|\!|\!|_A \\ &\quad + C \inf_{(q,s) \in Q_h \times S_h} \|\!|\!| (p-q, r-s) \|\!|\!|_B, \end{aligned}$$

with a constant $C > 0$ *that is independent of the mesh-size* h.

Proof. Fix $(q, r) \in Q_h \times S_h$. From the inf-sup condition in Lemma 3.7, we have

$$\gamma \|\!|\!| (p_h-q, r_h-s) \|\!|\!|_B \leq \sup_{(\mathbf{v},\mathbf{c}) \in \mathbf{V}_h \times \mathbf{C}_h} \frac{\mathcal{B}(p_h-q, r_h-s; \mathbf{v}, \mathbf{c})}{\|\!|\!| (\mathbf{v}, \mathbf{c}) \|\!|\!|_A} \leq T_1 + T_2,$$

with

$$\begin{aligned} T_1 &= \sup_{(\mathbf{v},\mathbf{c}) \in \mathbf{V}_h \times \mathbf{C}_h} \frac{\mathcal{B}(p-q, r-s; \mathbf{v}, \mathbf{c})}{\|\!|\!| (\mathbf{v}, \mathbf{c}) \|\!|\!|_A}, \\ T_2 &= \sup_{(\mathbf{v},\mathbf{c}) \in \mathbf{V}_h \times \mathbf{C}_h} \frac{\mathcal{B}(p_h-p, r_h-r; \mathbf{v}, \mathbf{c})}{\|\!|\!| (\mathbf{v}, \mathbf{c}) \|\!|\!|_A}. \end{aligned}$$

Obviously, using the continuity of $\mathcal{B}$ in (2.20),

$$T_1 \leq C_b \|\!|\!| (p-q, r-s) \|\!|\!|_B.$$

In view of (4.2), we further have

$$\begin{aligned} \mathcal{B}(p_h-p, r_h-r; \mathbf{v}, \mathbf{c}) &= \mathcal{A}(\mathbf{u}-\mathbf{u}_h, \mathbf{b}-\mathbf{b}_h; \mathbf{v}, \mathbf{c}) \\ &\quad + \mathcal{C}(\mathbf{u}_h, \mathbf{b}_h; \mathbf{u}-\mathbf{u}_h, \mathbf{b}-\mathbf{b}_h; \mathbf{v}, \mathbf{c}) \\ &\quad + \mathcal{C}(\mathbf{u}-\mathbf{u}_h, \mathbf{b}-\mathbf{b}_h; \mathbf{u}, \mathbf{b}; \mathbf{v}, \mathbf{c}). \end{aligned}$$

Therefore, we obtain with the continuity of the forms $\mathcal{A}$ and $\mathcal{C}$

$$\begin{aligned} T_1 \leq \|\!|\!| (\mathbf{u}-\mathbf{u}_h, \mathbf{b}-\mathbf{b}_h) \|\!|\!|_A \Big[&\max\{R_s^{-1}, R_m^{-1} S_c\} \\ &+ C_c \max\{1, S_c\} \|\!|\!| (\mathbf{u}, \mathbf{b}) \|\!|\!|_A + C_c \max\{1, S_c\} \|\!|\!| (\mathbf{u}_h, \mathbf{b}_h) \|\!|\!|_A \Big]. \end{aligned}$$

Taking into account the stability bounds for $(\mathbf{u}, b)$ and $(\mathbf{u}_h, \mathbf{b}_h)$ in Corollary 2.18 and Corollary 3.2, respectively, gives

$$|||\, (p_h - q, r_h - s)\, |||_B \le C|||\, (\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h)\, |||_A + C|||\, (p - q, r - s)\, |||_B.$$

The triangle inequality and the bound in Theorem 4.1 complete the proof. □

Let now $\{\mathbf{V}_h\}_h$, $\{Q_h\}_h$, $\{\mathbf{C}_h\}_h$, and $\{S_h\}_h$ be sequences of finite element spaces, assuming that

$$\inf_{(\mathbf{v},\mathbf{c})\in \mathbf{V}_h\times \mathbf{C}_h} |||\, (\mathbf{u} - \mathbf{v}, \mathbf{b} - \mathbf{c})\, |||_A \to 0, \qquad h \to 0,$$
$$\inf_{(q,s)\in Q_h\times S_h} |||\, (p - q, r - s)\, |||_B \to 0, \qquad h \to 0.$$

Note that due to the density of $C^\infty(\overline{\Omega})$ functions in the continuous spaces, these assumptions are justified; see [23, Theorem 3.26] for the density of $C^\infty(\overline{\Omega})^3$ in $H(\mathrm{curl}; \Omega)$.

We then immediately have the following result.

Corollary 4.1 *Assume (4.1). Let* $(\mathbf{u}, p, \mathbf{b}, r) \in \mathbf{J} \times Q \times \mathbf{X} \times S$ *be the solution of the problem in Formulation 2.15 and* $\{(\mathbf{u}_h, p_h, \mathbf{b}_h, r_h)\}_h$ *its approximations obtained with the above sequence of spaces. Then we have*

$$\lim_{h\to 0} |||\, (\mathbf{u} - \mathbf{u}_h, \mathbf{b} - \mathbf{b}_h)\, |||_A = 0, \qquad \lim_{h\to 0} |||\, (p - p_h, r - r_h)\, |||_B = 0.$$

Remark 4.3 The result in Corollary 4.1 ensures the convergence of the finite element approximations in non-convex polyhedra as $h \to 0$. It is known that such a convergence result cannot hold if the magnetic field is approximated by nodal (i.e. H^1-conforming) elements. This is due to the fact that in non-convex polyhedra the strongest magnetic singularities do not lie in $H^1(\Omega)^3$; see [8]. In convex or smooth domains, on the other hand, quasi-optimal error bounds for nodal FEM discretizations of the MHD equations (2.1)–(2.4) have been established in [19, Theorem 6.4].

4.2 Convergence rates

Let us finish this section by discussing the convergence rates that are obtained from Theorem 4.1 and Theorem 4.2.

First, we assume that the following standard approximation property holds for the velocity-pressure space pair $\mathbf{V}_h \times Q_h$:

(4.5) $$\inf_{\mathbf{v}\in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}\|_1 + \inf_{q\in Q_h} \|p - q\|_0 \le Ch^{\min\{s,k\}}\big[\, \|\mathbf{u}\|_{s+1} + \|p\|_s \,\big],$$

for $(\mathbf{u}, p) \in H^{s+1}(\Omega)^3 \times H^s(\Omega)$, an exponent $s > \frac{1}{2}$, and an approximation order $k \ge 1$.

Then, for $k \geq 1$, we have for the pair $\mathbf{C}_h \times S_h$:

(4.6)
$$\inf_{\mathbf{c}\in\mathbf{C}_h} \|\mathbf{b}-\mathbf{c}\|_{\mathrm{curl}} + \inf_{s\in S_h} \|r-s\|_1 \leq Ch^{\min\{s,k\}}\big[\, \|\mathbf{b}\|_s + \|\operatorname{curl}\mathbf{b}\|_s + \|r\|_{s+1}\,\big],$$

provided that $\mathbf{b} \in H^s(\Omega)^3$, $\operatorname{curl}\mathbf{b} \in H^s(\Omega)^3$, and $r \in H^{1+s}(\Omega)$, for an exponent $s > \frac{1}{2}$. The approximation property for $\mathbf{b}$ in the norm $\|\cdot\|_{\mathrm{curl}}$ can be found in, e.g., [1, Proposition 5.6], whereas the one for r follows from standard approximation theory for H^1-conforming spaces.

Using (4.5) and (4.6), we conclude that the results in Theorem 4.1 and Theorem 4.2 yield the following convergence rates.

Corollary 4.2 *Assume (4.1) and let* $\mathbf{V}_h \times Q_h$ *satisfy (4.5). Let the exact solution* $(\mathbf{u}, p, \mathbf{b}, r)$ *to the MHD equations (2.1)–(2.4) satisfy*

$$\mathbf{u} \in H^{s+1}(\Omega)^3, \qquad p \in H^s(\Omega), \tag{4.7}$$

and

$$\mathbf{b} \in H^s(\Omega)^3, \qquad \operatorname{curl}\mathbf{b} \in H^s(\Omega)^3, \qquad r \in H^{1+s}(\Omega), \tag{4.8}$$

for a regularity exponent $s > \frac{1}{2}$*. Let* $(\mathbf{u}_h, p_h, \mathbf{b}_h, r_h) \in \mathbf{J}_h \times Q_h \times \mathbf{X}_h \times S_h$ *be the finite element approximation in Formulation 3.5. Then we have the error bound*

$$\begin{aligned} &|||\, (\mathbf{u}-\mathbf{u}_h, \mathbf{b}-\mathbf{b}_h)\, |||_A + |||\, (p-p_h, r-r_h)\, |||_B \\ &\quad \leq Ch^{\min\{s,k\}}\Big[\|\mathbf{u}\|_{s+1} + \|p\|_s + \|\mathbf{b}\|_s + \|\operatorname{curl}\mathbf{b}\|_s + \|r\|_{s+1}\Big], \end{aligned}$$

with a constant $C > 0$ *independent of the mesh-size* h.

Remark 4.4 We point out that, for the error bound in Corollary 4.2 to hold, it is not necessary that the magnetic field $\mathbf{b}$ belongs to $H^1(\Omega)^3$. However, a precise characterization of the regularity properties of the complete MHD system in (2.1)–(2.4) does not seem to be available in the literature and is beyond the scope of the present paper. The regularity assumptions for $(\mathbf{u}, p, \mathbf{b})$ in (4.7) and (4.8) are realistic for linear MHD problems in polyhedra as they can be derived from corresponding results for linearized Navier-Stokes and Maxwell problems, respectively; see, e.g., [9,2,7,8] and the references therein. We also recall that for a divergence-free source term $\mathbf{g}$ (often encountered in practice) we have $r = 0$ and the assumption $r \in H^{s+1}(\Omega)$ is trivially satisfied.

Remark 4.5 The numerical tests in the recent note [28] have confirmed the convergence rates in Corollary 4.2 for a two-dimensional linear MHD problem in a non-convex polygon where nodal FEM is known to fail to converge to the exact magnetic field. There, the approximation was based on quadrilateral meshes using standard $\mathcal{Q}_2 - \mathcal{Q}_0 - \mathcal{Q}_1$ elements for the approximation of the velocity $\mathbf{u}$, the pressure p and the multiplier r, respectively, combined with lowest-order Nédélec's elements for the magnetic field $\mathbf{b}$ (in two dimensions these elements correspond to rotated Raviart-Thomas elements).

5 Conclusions

In this paper, we have presented a new existence and uniqueness theory for the equations of incompressible magneto-hydrodynamics, based on a variational setting that employs the space $H(\text{curl}; \Omega)$ for the magnetic field. Our results are valid in general Lipschitz polyhedra with possible reentrant edges or vertices; these are domains for which nodal FEM for the magnetic field is known to fail to converge to the exact solution. A new mixed finite element approximation has then been proposed using Nédélec's first family of elements for the magnetic field. Our analysis shows that the method is stable and quasi-optimal in the mesh-size. The first numerical tests in [28] demonstrate the ability of the proposed method to resolve highly singular solutions whose magnetic components have regularity below $H^1(\Omega)$.

As our finite element schemes have essentially been devised for the discretization of the elliptic operators underlying the incompressible MHD equations in polyhedra, they may suffer from the usual numerical instabilities in highly transport-dominated situations and have to be stabilized with techniques similar to those proposed in [15]. This remains to be addressed elsewhere.

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