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A separable manifold failing to have the homotopy type of a CW-complex

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Abstract. We show that (a variation of) the Prüfer surface, which is an example of a separable non-metrizable 2-manifold, does not have the homotopy type of a CW-complex.

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1. Introduction. Our aim is to prove the following:

Theorem 1. *The Prüfer surface*[1]*, which is an example of a separable*[2] *non-metrizable manifold*[3]*, does not have the homotopy type of a CW-complex.*

This might sound like a dissonance in view of Milnor's Corollary 1 in [9], which states that *every separable manifold has the homotopy type of a (countable) CW-complex.*

Obviously Milnor's statement is formulated under the implicit proviso of metrizability. Indeed, the proof makes essential use of a metric, working as follows. Due to the results of J. H. C. Whitehead (see [17] and [18]), all that must be proved is that the given space X is dominated by a CW-complex. The first step is Hanner's theorem [5]: a space that is locally an ANR is an ANR (=absolute neighborhood retract). Then following Kuratowski [7], the space X is embedded via $\kappa : x \mapsto d(x, \cdot)$ where d is a bounded metric for X, into the Banach space of bounded continuous functions on X. The image happens to be closed in its convex

[1] Actually, the surface we consider is not the original Prüfer surface, but rather Calabi-Rosenlicht's slight modification of it. (This will be clarified by recalling the historical background in § 3.)
[2] A space is *separable* if it has a countable dense subset.
[3] Here this means a Hausdorff topological space which is locally Euclidean.

hull C, by an argument to be found in Wojdyslawski [19]. Since X is an ANR, there is an open neighborhood U in C which retracts to $\kappa(X)$. By transitivity of domination, it is enough to prove that U is dominated by a polyhedron P, which is constructed as the nerve of a suitable cover. The domination map $f : P \to U$ is defined by linearity (P being realized in $\mathbb{R}^V$ where V denotes the set of its vertices). The 'submission' map $g : U \to P$ is the barycentric map attached to a partition of unity (paracompactness is needed, but follows from metrizability). Lastly the homotopy $fg \simeq 1_U$ comes from local convexity considerations. For a detailed exposition see Palais [13]. (All this, being a powerful elaboration of the basic idea: embed the given space in an Euclidean space and triangulate an open tubular neighborhood of it.)

In particular Milnor's Corollary 1 does not apply to the manifold constructed (under the continuum hypothesis) by Rudin-Zenor [15], which is a hereditarily separable[4] non-metrizable manifold. The question of the contractibility of the Rudin-Zenor manifold then may appear as an interesting problem.

2. Proof of Theorem 1. Let us first give a picturesque description of the Prüfer surface P. We may think of P as the Euclidean plane from which a horizontal line has been removed, and then for each point of the line a small bridge is introduced in order to connect the upper to the lower half-plane (see Figure 1). A formal construction of P will be recalled in § 3. The main point-set theoretical information about P is its separability.

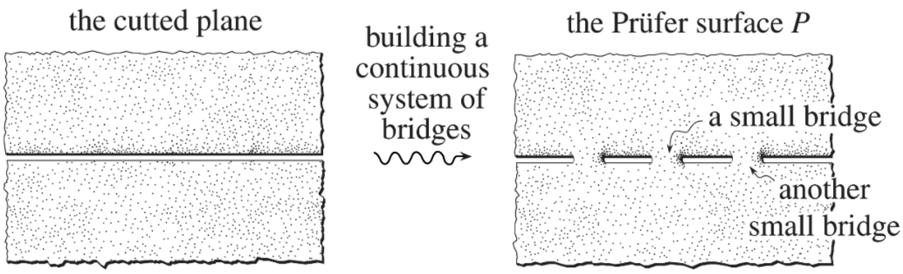

Figure 1

The fundamental group of P is easily identified, via van Kampen's theorem, as a free group on a continuum $\mathfrak{c}$=(cardinality of $\mathbb{R}$) of generators. (Details will be given in § 4.)

Theorem 1 is then best deduced[5] from the following:

[4]In the sense that each subspace is separable.
[5]This argument is due to the anonymous referee of a previous version of this paper. I would like to express him my gratitude for allowing me to reproduce it.

Proposition 1. *If a separable topological space X is homotopy equivalent to a CW-complex K, then X is dominated by a countable subcomplex of K. In particular the fundamental group of X has to be countable .*

Proof. Let $f : X \to K$ be a homotopy equivalence and D be a countable dense subset of X. Then $f(D)$ is a countable subset of K. Since any point of a CW-complex is contained in a finite subcomplex, and the arbitrary union of subcomplexes is again a subcomplex, it follows that $f(D)$ is contained in a countable subcomplex L of K. Now using the inclusion $f(\overline{D}) \subset \overline{f(D)}$ (valid for any continuous map between arbitrary topological spaces), and the fact that subcomplexes are closed, one deduces that $f(X)$ is still contained in L. So, restricting g(=the homotopy inverse of f) to L gives a domination $d : L \to X$ with submission map $s : X \to L$ given by the co-restriction of f to L. (Indeed denoting by $i : L \hookrightarrow K$ the inclusion, we have $ds = gis = gf \simeq 1_X$.)

The last assertion follows from the fact that a countable CW-complex has a countable fundamental group. □

Note. Proposition 1 could as well be applied to the Hawaiian earrings to show they do not have the homotopy type of a CW-complex.

3. Construction of the Prüfer surface P. The following construction is due to Prüfer, first described in print by Radó [14]. However, our more geometric exposition is from [2], [12] (following an idea that can be traced back to R. L. Moore [11]). Historically, the Prüfer manifold emerged as an example of a non-triangulable surface[6], at about the same time as Radó [14] proved the triangulability of surfaces with countable base (in particular of Riemann surfaces).

We use $\mathbb{C}$ as model for the Euclidean plane. The idea is to consider the set P_0 formed by the points of the (open) upper half-plane $\mathcal{H} = \{z : \operatorname{Im}(z) > 0\}$ together with the set of all rays emanating from an arbitrary point of $\mathbb{R}$ and pointing into the upper half-plane. (Such rays are points of P_0.) We topologize P_0 with the usual topology on $\mathcal{H}$, and by taking as neighborhoods of a ray r (say emanating from $x \in \mathbb{R}$) the set of all rays with the same origin x deviating by at most ε radians from r, together with the points of $\mathcal{H}$ lying between the two extremal rays and at a (Euclidean) distance less than ε from x (compare left part of Figure 2).

The space P_0 turns out to be a surface-with-boundary. To see this one has only to check that a neighborhood N of a ray r is homeomorphic to the closed half-plane $\overline{\mathcal{H}}$. Intuitively one may argue as follows (compare Fig.3a)). One may first stretch N (which by construction is a sector swept out by focusing rays) to obtain parallel rays. Then the process of collapsing rays to their origins becomes one-to-one, and actually produces a boundary for the distorted sector. This vague idea can be made precise by writing down an explicit 'stretching'. One may choose

[6]Actually the first such example if we discard surfaces obtainable from Cantor's long ray.

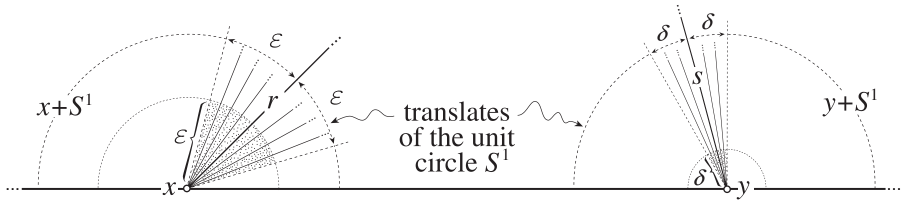

Figure 2. Some ray's neighborhoods

the map $\varphi : N \to \mathbb{C}$ taking a point z to $\varphi(z) = \sigma(z) + i|z - x|$ (where $\sigma(z)$ denotes the intersection of the line through x and z with $\lambda = \{z : \mathrm{Im}(z) = 1\}$, while taking a ray ρ to its intersection with λ (compare Fig.3.b)). Paying attention to follow how φ transforms typical neighborhoods shows that φ is a homeomorphism (once restricted to its image which is clearly homeomorphic to $\overline{\mathcal{H}}$).

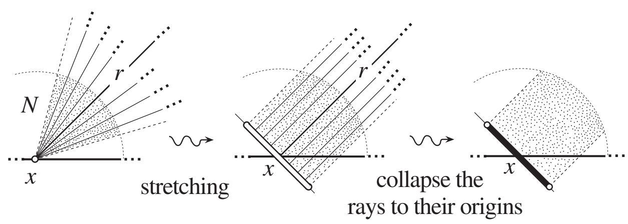

Figure 3a

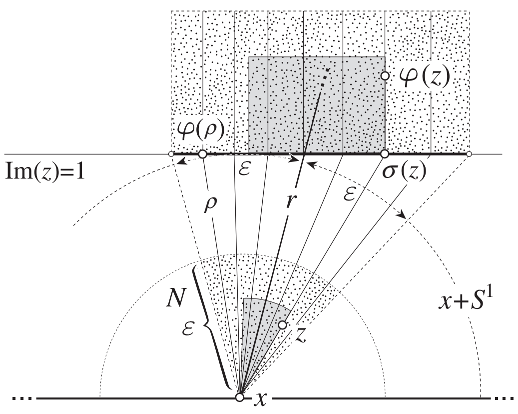

Figure 3b

Note. This approach involving rays brings the Prüfer surface in close analogy to projective geometry, but some readers may find it confusing and prefer the extremely useful exposition in [6].

Observe that P_0 has a continuum $\mathfrak{c}$ of boundary components, each homeomorphic to the real line $\mathbb{R}$. (The Hausdorffness of P_0 is easily verified.)

Now given W a manifold-with-boundary, there are two obvious ways to produce a manifold M (without boundary). A first method is by *collaring*: we set $M = W \cup_{\mathrm{id}_{\partial W}} (\partial W \times [0, 1))$. A second option is by *doubling*: $M = W \cup_{\mathrm{id}_{\partial W}} W$ (two copies of W are glued along their boundaries).

For $W = P_0$, the process of collaring leads to the 'original' Prüfer surface (the one described in [14]). In this case there is in M an uncountable family of pairwise disjoint open sets, so that M fails to be separable.

The second option leads to the surface P we are interested in, since it is separable. We also call it a Prüfer surface, even though it seems to appear explicitly only in the paper by Calabi-Rosenlicht [3].

Proposition 2. *The Prüfer surface P obtained by doubling P_0, is a connected (Hausdorff) 2-manifold which is separable, but contains an uncountable discrete subspace, and therefore is non-metrizable.*

Proof. Observe that the rational points $\mathbb{Q} + i\mathbb{Q}_{>0}$ give a countable dense subset of P_0, and so P is clearly separable. Further we note that the family of all rays $(r_x)_{x\in\mathbb{R}}$, say orthogonal to $\mathbb{R}$, is an uncountable discrete subspace of P, since given any ray r_x one can find an open neighborhood of it cutting out only this single ray from the whole family. It follows that P is not hereditarily separable, so not second countable, and therefore non-metrizable. (As is well-known, metrizability and second countability are equivalent for connected manifolds (see [16] or [12]). Actually, since P is separable, its non-metrizability is more economically deduced from the fact that metrizable plus separable imply second countable.) □

At this stage one could already observe the following:

Corollary 1. *The Prüfer surface P (and more generally any non-metrizable manifold) is not homeomorphic to a CW-complex.*

Proof. This follows from the paracompactness of CW-complexes established by Miyazaki [10], and the equivalence between the concepts of paracompactness and metrizability, when spaces are restricted to be manifolds (see again [16] or [12]). □

4. The fundamental group of the Prüfer surface. The following information on the fundamental group of P completes the proof of Theorem 1.

Proposition 3. *$\pi_1(P)$ is a free group on a continuum $\mathfrak{c}$ of generators.*

Proof. (After M. Baillif). For all $x \in \mathbb{R}$, let U_x be the open neighborhood of x in P_0 depicted in Figure 2 where r is chosen orthogonal to $\mathbb{R}$ and $\varepsilon = \frac{\pi}{2}$. Let then B_x be U_x taken together with its symmetrical copy U_x^σ, so $B_x = U_x \cup U_x^\sigma$ is an open set of P (we can think of it as a 'bridge' linking the upper to the lower half-plane). For all $x \in \mathbb{R} - \{0\} = \mathbb{R}^*$, put $O_x = \mathcal{H} \cup \mathcal{H}^\sigma \cup B_0 \cup B_x$. The collection $(O_x)_{x \in \mathbb{R}^*}$ forms an open cover of P, which satisfies the hypotheses of van Kampen's theorem, since $O_x \cap O_y$ is arcwise-connected. Furthermore $\bigcap_{x \in \mathbb{R}^*} O_x = \mathcal{H} \cup \mathcal{H}^\sigma \cup B_0$ is homeomorphic to the union of $\mathbb{C} - \mathbb{R}$ with an open interval from $\mathbb{R}$ (by the same kind of argument as the one cartooned in Fig.3.a)), and so is simply connected. Moreover each member O_x of this cover is homeomorphic to the union of $\mathbb{C} - \mathbb{R}$ with two disjoint (real) intervals. Hence it has the homotopy type of the circle S^1. The result follows by van Kampen's theorem. □

Note. It is well-known that the fundamental group of any metrizable open surface is a free group ([1], [4], [8]). Whether this freeness holds true behind the horizon of metrizability seems to be a difficult question.

5. The case of non-Hausdorff manifolds. We conclude by making some simple observations concerning complications arising in the relation between manifolds and CW-complexes, in the case that the Hausdorff separation axiom is relaxed from the definition of a manifold. Then already one of the simplest examples of a 'manifold', the so-called *line with two origins* (obtained from two copies of $\mathbb{R}$ by identifying corresponding points outside the origin, see Figure 4) fails to have the homotopy type of a CW-complex (and this in spite of the fact that it is well-behaved from the point of view of second countability). In fact, we even have a worse situation:

Proposition 4. *The line with two origins R does not have the homotopy type of any Hausdorff topological space.*

Proof. We need two preliminary remarks.

- First remember that there is a general Hausdorffization process applicable to any space X, which leads to a Hausdorff space X_{Haus} with a map $X \to X_{\text{Haus}}$. This is obtained by factorizing the given space by the smallest Hausdorff equivalence relation. It has the property that any continuous map from X to a Hausdorff space H factors through X_{Haus}.
- Second by Mayer-Vietoris it is easy to check that the first homology group $H_1(R, \mathbb{Z})$ is infinite cyclic.

We are now ready to prove Proposition 4. Assume there is a homotopy equivalence $f : R \to H$ for some Hausdorff space H. Then f factors through R_{Haus}, which is nothing but the usual real line $\mathbb{R}$. But this being contractible, it follows by functoriality that the morphism $H_1(f)$ is zero, in contradiction to the non-vanishing of $H_1(R, \mathbb{Z})$. □

Finally, let us consider a variant of the line with two origins obtained by identifying in two copies of $\mathbb{R}$ corresponding points outside some closed interval (see Figure 5). The resulting quotient space is again a non-Hausdorff manifold which is easily seen to be homotopy equivalent to the circle S^1.

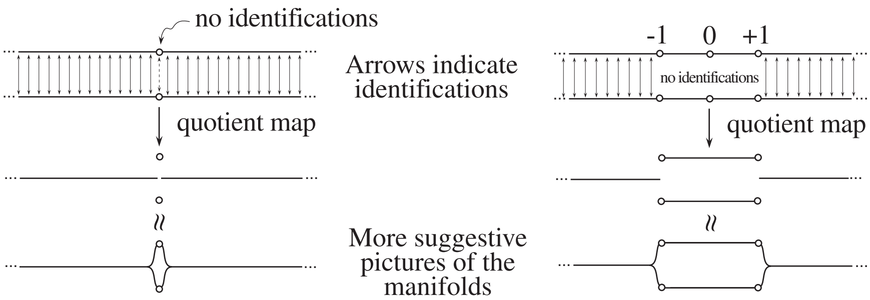

FIGURE 4. The line with two origins

FIGURE 5. Another non-Hausdorff manifold

So, it is not so much relaxing the Hausdorff axiom, that leads us outside the class $\mathcal{W}$ of spaces having the homotopy type of a CW-complex. Rather it is the strange geometric behavior of 'extremely narrow bifurcations' inherent to some non-Hausdorff manifolds, which appears as something alien to the combinatorial nature of CW-complexes.

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