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The Eight-Vertex Model and Lattice Supersymmetry

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Abstract We show that the XYZ spin chain along the special line of couplings $J_xJ_y + J_xJ_z + J_yJ_z = 0$ possesses a hidden $\mathcal{N} = (2, 2)$ supersymmetry. This lattice supersymmetry is non-local and changes the number of sites. It extends to the full transfer matrix of the corresponding eight-vertex model. In particular, it is shown how to derive the supercharges from Baxter's Bethe ansatz. This analysis leads to new conjectures concerning the ground state for chains of odd length. We also discuss a correspondence between the spectrum of this XYZ chain and that of a manifestly supersymmetric staggered fermion chain.

Keywords Supersymmetry · Lattice models · Bethe ansatz

1 Introduction

The solution of the zero-field eight-vertex model by Baxter is a landmark in the theory of exactly solvable systems. The seminal papers [1–4, 6] present a variety of algebraic and analytic tools to compute the partition function, the eigenvalues and eigenvectors of its transfer matrix. The model is still under active study: in particular, Fabricius and McCoy showed that at the so-called root-of-unity points the spectrum of the transfer matrix possesses degeneracies not easily explained by its standard integrability alone [18, 26–28]. It is natural

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to attribute them to the presence of extended symmetries, possibly an elliptic generalisation of the sl_2-loop-algebra symmetries discovered for the six-vertex model [19, 25, 41].

The topic of this paper is the eight-vertex model at a particular root-of-unity point with an extended symmetry. We utilise the quantum XYZ spin chain, whose Hamiltonian commutes with the eight-vertex model transfer matrix [56]. More precisely, we study the Hamiltonian

$$H_N = -\frac{1}{2}\sum_{j=1}^{N}\big(J_x\sigma_j^x\sigma_{j+1}^x + J_y\sigma_j^y\sigma_{j+1}^y + J_z\sigma_j^z\sigma_{j+1}^z\big) + \frac{N(J_x+J_y+J_z)}{2} \tag{1a}$$

with periodic boundary conditions along the special line of couplings

$$J_xJ_y + J_xJ_z + J_yJ_z = 0. \tag{1b}$$

While it was already noticed in Baxter's original works that along the line of couplings (1) the ground state energy per site remains zero in the thermodynamic limit, its finite-size ground state was addressed much more recently. In [54, 55] Stroganov argued that (1) possesses exactly two zero-energy ground states for N odd. When $J_x = J_y$ (so that $\Delta \equiv J_z/J_x = -1/2$), the resulting critical XXZ chain has been extensively studied over the past decade [17, 20, 21, 31, 45–47, 61]. The components of the ground state possess some remarkable properties. They display a variety of relations with combinatorial quantities, such as the enumeration of alternating sign matrices and plane partitions. Many of the early conjectures, such as sum rules for the components, were then proved with the help of techniques such as the quantum Knizhnik-Zamolodchikov equation, and combinatorial tools [15, 22, 23, 49].

The fact that the ground state energy is exactly zero in the XXZ chain at $\Delta = -1/2$ for a finite odd number of sites was proved by exploiting a hidden supersymmetry [31, 61]. A ground-state energy of exactly zero is a common characteristic of theories with supersymmetry [59]. While quite a number of lattice models possess scaling limits described by field theories with supersymmetry [52], only a few are known where the supersymmetry is explicitly present on the lattice. Here, an unusual feature is that the supersymmetry operator changes the number of sites by one. While unusual, it is not unheard of; similar operators were for example studied in a spin chain arising from the integrable structures in four-dimensional gauge theory [11].

Remarkable properties of the zero-energy ground state persist along the entire line (1) [8, 9, 29, 43, 48]. Using the convenient parametrisation

$$J_x = 1+\zeta, \qquad J_y = 1-\zeta, \qquad J_z = \big(\zeta^2-1\big)/2, \tag{2}$$

computer results indicate that the ground state at odd N can be expressed as polynomials in ζ with positive integer coefficients. Non-linear recursion relations were observed for several components of the zero-energy ground states of (1) [8, 9, 43] (see also [50]). These relations are described by the tau-function hierarchies of the Painlevé VI equation [44]. Moreover, very simple expressions for one-point functions in the ground state such as the magnetisation were found by summing the expansion around the trivially solvable point $\zeta \to \infty$ [29].

Since the ground-state energy remains zero along the entire line (1), it is natural to expect that the supersymmetry persists off the critical point. This is true in the scaling limit, where the XYZ chain is described by the sine-Gordon field theory. This field theory possesses symmetries that can be related to the affine quantum group $U_q(\widehat{\mathrm{sl}}(2))$ with zero centre [12]. When $q^2 = -1$, the quantum group contains the $\mathcal{N} = (2,2)$ supersymmetry algebra. This is precisely the value of q corresponding to the scaling limit along the line (1). We thus refer to this line henceforth as the "supersymmetric" line.

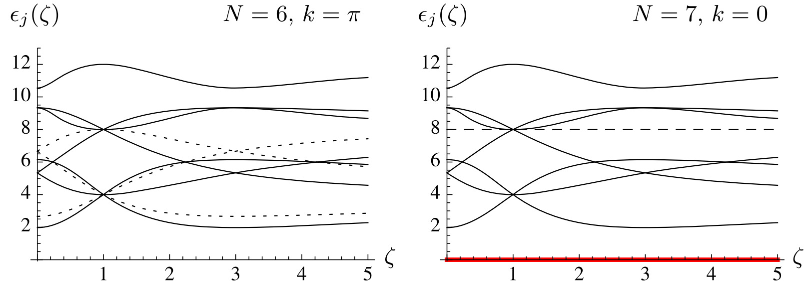


Fig. 1 Rescaled eigenvalues $\epsilon_j(\zeta) = 4E_j(\zeta)/(3+\zeta^2)$ for the XYZ Hamiltonian with $N = 6$ and $N = 7$ sites along the *supersymmetric line* as a function of ζ. The *plots* are restricted to the subsectors with momentum $k = \pi$ and 0 respectively. The *solid lines* correspond to exact common eigenvalues for the two chains. Moreover, the *plot for* $N = 7$ shows the existence of exact zero energy ground states

The purpose of this paper is to show that the supersymmetry remains exact on the lattice all along the supersymmetric line. The main focus in most (but not all [33]) earlier studies was the ground state of the system, which has exactly zero energy for any odd N. In this article, we study not only the ground states of this XYZ spin chain, but also consider its full spectrum. We show that for any value of ζ, in certain momentum sectors this model possesses a hidden exact symmetry which we call lattice supersymmetry, a lattice version of the well-known $\mathcal{N} = 2$ supersymmetry algebra [59]. These results provide a systematic generalisation of the XXZ results to the XYZ setting.

We illustrate this symmetry by explicitly diagonalising the XYZ Hamiltonian along the supersymmetric line. Figure 1 shows the example of chains with $N = 6$ and $N = 7$ sites: in particular momentum sectors, the corresponding Hamiltonians sites have exact common non-zero eigenvalues. This is a well-known feature of supersymmetric theories: states with $E > 0$ are doubly degenerate. Thus, in our case the common eigenvalues are candidates for supersymmetry doublets. This is a consequence of the existence of supercharges which change the number of sites N by one, and serve as intertwiners for the Hamiltonians of the corresponding chains. Moreover, two exact zero-energy ground states are found for odd N, and they are potential candidates for so-called supersymmetry singlets (who do not have a superpartner). In fact, we will show that in the sectors with momentum zero for odd N and momentum π for even N, the states organise themselves into quadruplets: an eigenvalue occurring first in the spectrum for $N-1$ sites, appears twice at N sites, and once at $N+1$ sites. The quadruplet structure hints at the existence of two copies of the supersymmetry algebra. Indeed by taking into account the symmetry under flipping all spins, we find an $\mathcal{N} = (2,2)$ supersymmetry algebra on the lattice.

The transfer matrix of the eight-vertex model commutes with the XYZ Hamiltonian, so the two have the same eigenstates. Thus one might expect that these properties carry over to the full model. As we will show, the existence of the $\mathcal{N} = (2,2)$ supersymmetry algebra is deeply rooted in the eight-vertex model, and can be derived from the Bethe ansatz. Moreover, the Bethe ansatz analysis leads to a novel characterisation of the ground states of the XYZ chain (1) for odd N.

The XYZ chain is closely connected to another model with lattice supersymmetry, introduced in [32] and thoroughly analysed in [31, 36, 37]. It describes spinless fermions on a chain with nearest-neighbour as well as the usual on-site exclusion. Connections between the two at the critical point are already known; energy levels coincide with those of the XXZ chain at $\Delta = -1/2$ with a momentum-dependent twist [31]. A similar correspondence also holds in the open chain with a magnetic field [61]. Moreover, the ground states of the fermion models display a variety of relations with the combinatorics described above [10]. Therefore, it is natural to ask if a similar equivalence holds off the critical point. Indeed, several connections between the zero-energy ground states for chains of odd length and the ground states of a staggered version of the fermion models were observed in [29, 30]. In this work, we provide more evidence for the connection of the two models. We observe the systematic existence of common eigenvalues in their spectra after proper identification of the relevant parameters, even in sectors where the supersymmetry is not fully realised.

The layout of this paper is the following. In Sect. 2, we introduce supercharges for the XYZ chain which change the number of sites and study their properties. In particular, we show that Hamiltonian can be obtained as a quadratic form of the supercharges in special momentum sectors. The combination with various other symmetries of the XYZ chain will lead to a second lattice supersymmetry, and thus explain the quadruplet structure in the spectra. In Sect. 3, we change our point of view and study the system using Bethe ansatz for the eight-vertex model. After recalling Baxter's original approach, we present a derivation of the supersymmetry from the Bethe equations, and show that it is a symmetry of the full transfer matrix along the supersymmetric line, connecting systems of different sizes. After this we proceed with a conjecture on the nature of the ground states for chains of odd length. The relation of the present model to the theory of fermions with nearest-neighbour exclusion is discussed in Sect. 4: we observe that the spectra of both theories have exact common eigenvalues. We suggest a mapping between the two models, based on the Bethe ansatz solution of the eight-vertex model. Finally, we present our conclusions and various open problems in Sect. 5. Some technical details are relegated to appendices.

Throughout the paper we report observations obtained from exact diagonalisation of small system sizes. Most of them show patterns which seem to hold for general N, and thus we formulate them as conjectures.

2 Supercharges and the XYZ Hamiltonian

In this section, we establish the relation between the XYZ chain (1) and lattice supersymmetry. We start with a review of the elementary symmetries of the XYZ chain and then proceed to the definition of the supercharges. The analysis of their properties under translation allows to show that they are nilpotent and write the Hamiltonian as quadratic forms of the supercharges. Next, we study the interplay with other symmetries such as parity and spin reversal, and establish the quadruplet structure of the eigenstates in special momentum sectors.

In the sequel, we will frequently deal with operators that change the number of sites of the chain. Hence, we will indicate by a subscript the length of the chain the corresponding operator acts on. For example, we denote by Q_N the supercharge acting on the Hilbert space for chains with N sites etc.

2.1 Elementary Symmetries

We start with a bit of notation. We denote by $\mathcal{H}_N = (\mathbb{C}^2)^{\otimes N}$ the usual Hilbert space for N spin-$1/2$ particles in a chain. We will use the standard orthonormal basis in which the operators σ_j^z are diagonal (σ^x, σ^y, σ^z are the usual Pauli matrices). Its basis vectors are labelled by configurations $\alpha = \alpha_1\alpha_2\cdots\alpha_N$ with $\alpha_j = +$ (spin up) or $-$ (spin down) for the j-th spin, and the σ_j^z operator acts according to

$$\sigma_j^z|\alpha_1\cdots\alpha_j\cdots\alpha_N\rangle = \alpha_j|\alpha_1\cdots\alpha_j\cdots\alpha_N\rangle. \tag{3}$$

Let us indicate here some elementary symmetries of the Hamiltonian (1), valid for any choice of J_x, J_y, J_z. First, it is invariant under translation. We will often use the translation operator T_N acting on basis states of $\mathcal{H}_N$ according to the usual rule

$$T_N|\alpha_1\cdots\alpha_{N-1}\alpha_N\rangle = |\alpha_N\alpha_1\cdots\alpha_{N-1}\rangle.$$

The translation invariance of the Hamiltonian implies that it commutes with this operator, and therefore they can be diagonalised simultaneously. Consider a state $|\psi\rangle$ such that $T_N|\psi\rangle = t_N|\psi\rangle$, then cyclicity implies $(T_N)^N = 1$. Therefore the eigenvalue t_N is an N-th root of unity. Writing $t_N = e^{\mathrm{i}k}$ we see that the momentum k has to be an integer multiple of $2\pi/N$. In this work, we focus mainly on momentum $k = 0$ for chains of odd length, and momentum $k = \pi$ for chains of even length.

Moreover, the Hamiltonian is invariant under reversal of the order of all spins. We thus define a parity operation P_N through

$$P_N|\alpha_1\alpha_2\cdots\alpha_{N-1}\alpha_N\rangle = |\alpha_N\alpha_{N-1}\cdots\alpha_2\alpha_1\rangle.$$

Obviously, we have $P_N^2 = 1$ and therefore eigenvalues ± 1.

Finally, notice that the Hamiltonian is a quadratic form of the Pauli matrices. Therefore it remains unchanged under global rotations by an angle π around any of the x-, y- or z-axis. Let us first consider the z-axis. The rotation is given by $\mathrm{i}^N S_N$ where

$$S_N = \sigma_1^z\sigma_2^z\cdots\sigma_N^z = \exp\left(\frac{\mathrm{i}\pi}{2}\sum_{j=1}^{N}(1-\sigma_j^z)\right). \tag{4}$$

The right-hand side makes evident that S_N has the eigenvalue ± 1 on configurations with an even/odd number of spins $-$.

Considering instead rotations by the angle π about the x-axis leads to the conclusion that the Hamiltonian commutes with the spin reversal operator R_N defined through

$$R_N = \sigma_1^x\sigma_2^x\cdots\sigma_N^x. \tag{5}$$

The operators S_N and R_N have the following (anti-)commutation relation:

$$S_N R_N = (-1)^N R_N S_N. \tag{6}$$

This implies in particular that for odd N the spin-reversal operator couples the sectors with even and odd number of spins down. Therefore any eigenvalue of H_N has even degeneracy and is at least doubly degenerate. This is reminiscent of *Kramers' degeneracy* in quantum mechanics [53]: all energy levels of a system with an odd number of spin-1/2 particles (fermions) are doubly degenerate as long as time-reversal symmetry is not broken. This suggests that the number of sites N can be viewed as a *fermion number*. In the next section, we give more evidence for this interpretation through an explicit construction of the supersymmetry algebra.

2.2 Supercharges

We proceed with the definition of the supercharges. It is useful to start by recalling the usual $\mathcal{N} = 2$ supersymmetry algebra [59]. It is built from two conjugate supercharges Q, $Q^\dagger$ that are nilpotent, i.e. their squares are zero: $Q^2 = (Q^\dagger)^2 = 0$. A further symmetry generator is the fermion number F. The supercharges obey the relations $[F, Q] = Q$ and $[F, Q^\dagger] = -Q^\dagger$. Hence Q increases the fermion number by one, while $Q^\dagger$ decreases it. The Hamiltonian is given as anticommutator $H = \{Q, Q^\dagger\}$. It conserves the fermion number and commutes with Q and $Q^\dagger$. The fact that H is of this form implies that its eigenvalues are non-negative. Any states with zero energy are automatically ground states, and annihilated by both supercharges. Thus they each are a supersymmetry singlet. Conversely, all states with positive energy $E>0$ form doublets of the supersymmetry algebra, with the fermion numbers of the two states in a doublet differing by one.

We here define the analogous supercharges on the lattice. We start with a generalisation of the work [61] and introduce operators q_j which map $\mathcal{H}_N$ to $\mathcal{H}_{N+1}$ through action on site j according to the following rules (the subscripts denote the positions of the corresponding spins):

$$q_j|\alpha_1\cdots\alpha_{j-1}\underset{j}{+}\alpha_{j+1}\cdots\alpha_N\rangle = 0, \tag{7a}$$

$$\begin{aligned}&q_j|\alpha_1\cdots\alpha_{j-1}\underset{j}{-}\alpha_{j+1}\cdots\alpha_N\rangle\\ &\quad=(-1)^{j-1}\big(|\alpha_1\cdots\alpha_{j-1}\underset{j}{+}\underset{j+1}{+}\alpha_{j+1}\cdots\alpha_N\rangle-\zeta|\alpha_1\cdots\alpha_{j-1}\underset{j}{-}\underset{j+1}{-}\alpha_{j+1}\cdots\alpha_N\rangle\big).\end{aligned} \tag{7b}$$

We see that while states with spins $+$ at site j are annihilated by q_j, the spins $-$ are transformed into pairs $++$ and $--$ with weights 1 and $-\zeta$ respectively. The pair creation implies a shift of the spin sequence $\alpha_{j+1}\cdots\alpha_N$ by one site to the right. This includes a "string" $(-1)^{j-1}$ which is crucial in the following. To respect the periodic boundary conditions here, we need to build eigenstates of the translation operator. We thus introduce an operator that creates a pair of like spins $++$ or $--$ on sites $N+1$ and 1, namely

$$q_0|\alpha_1\cdots\alpha_{N-1}\underset{N}{+}\rangle = 0, \tag{7c}$$

$$q_0|\alpha_1\cdots\alpha_{N-1}\underset{N}{-}\rangle = -\big(|\underset{1}{+}\alpha_1\cdots\alpha_{N-1}\underset{N+1}{+}\rangle-\zeta|\underset{1}{-}\alpha_1\cdots\alpha_{N-1}\underset{N+1}{-}\rangle\big). \tag{7d}$$

The operator q_0 acts always on the last site irrespectively of length of the chain with no string attached.

With the help of these "local" operators q_j we construct a supercharge Q_N. It increases the number of sites by one what supports the interpretation that N plays the role of the fermion number F. The construction goes as follows. Suppose that $|\psi\rangle$ is an eigenstate of the translation operator T_N with some eigenvalue t_N. Then we define $Q_N|\psi\rangle=0$ unless $t_N=(-1)^{N+1}$. If $t_N=(-1)^{N+1}$ however, we define

$$Q_N=\left(\frac{N}{N+1}\right)^{1/2}\sum_{j=0}^{N}q_j.$$

For $\zeta=0$ these operators decrease the number of spins $-$ by one but map a state of definite total magnetisation to another state of definite total magnetisation [61]. For finite ζ however, this is not the case because the two states on the right-hand side of (7) differ in magnetisation by two. This is related to the fact that the XYZ Hamiltonian can flip pairs $++$ and $--$ of adjacent spins, and thus conserves magnetisation only mod 2.

Because of q_0 the operator Q_N seems to distinguish the last site of the chain from the others. However, translation invariance removes this distinction, and we claim that Q_N is a well-defined mapping between the momentum spaces with $t_N=(-1)^{N+1}$. To show this it is useful to understand the transformation properties of q_j under translation. From their definition it is not difficult to show that

$$\begin{aligned}T_{N+1}q_jT_N^{-1}&=-q_{j+1},\quad j=0,\dots,N-1,\\ T_{N+1}q_N&=(-1)^Nq_0.\end{aligned} \tag{8}$$

Consider now an eigenvector $|\psi\rangle$ of the translation operator for the chain with N sites: $T_N|\psi\rangle = t_N|\psi\rangle$, $t_N = (-1)^{N+1}$. Upon action with Q_N we produce a vector $|\phi\rangle = Q_N|\psi\rangle$ belonging to $\mathcal{H}_{N+1}$. The application of T_{N+1} to this new vector leads to

$$T_{N+1}|\phi\rangle = \left(\sum_{j=1}^{N-1} T_{N+1}q_j T_N^{-1} + T_{N+1}q_N T_N^{-1} + T_{N+1}q_0 T_N^{-1}\right) T_N|\psi\rangle$$
$$= t_N\left(-\sum_{j=1}^{N} q_j + (-1)^N t_N^{-1} q_0\right)|\psi\rangle.$$

As $t_N = (-1)^{N+1}$ the new vector is an eigenvector of T_{N+1}, with eigenvalue $-t_N = t_{N+1}$, what proves our claim. Thus, we have $T_{N+1}Q_N T_N^{-1} = -Q_N$.

The fact that the operators Q_N are *bona fide* mappings between the momentum spaces of interest is crucial to the supersymmetric structure of the XYZ chain (1) that we describe now. First, the supercharges have "square zero" in the sense that

$$Q_{N+1}Q_N = 0. \tag{9}$$

Hence they can be thought of as fermionic. Second, *if restricted to the subsectors with translation eigenvalue* $t_N = (-1)^{N+1}$ the XYZ-Hamiltonian can be constructed from the supercharges and their Hermitian conjugates. The latter are defined in the usual way: if $|\psi\rangle$ is a vector in $\mathcal{H}_N$, and $|\phi\rangle$ in $\mathcal{H}_{N+1}$ then $\langle\psi|Q_N^\dagger|\phi\rangle = \langle\phi|Q_N|\psi\rangle^*$. With this definition, the Hamiltonian can be written as an "anticommutator"

$$H_N = Q_{N-1}Q_{N-1}^\dagger + Q_N^\dagger Q_N. \tag{10}$$

The proofs of (9) and (10) are elementary but cumbersome. We present the details in Appendices A.1 and A.2. Here, we study their consequences for the eigenvalue spectrum, and thus give an explanation of the common eigenvalues for systems of different size. We will see that all these properties are familiar from the theory of $\mathcal{N}=2$ supersymmetric quantum mechanics [59].

First of all, consider the eigenvalue equation $H_N|\psi\rangle = E|\psi\rangle$. Projecting back on $|\psi\rangle$ we find

$$\left\| Q_{N-1}^\dagger|\psi\rangle\right\|^2 + \left\| Q_N|\psi\rangle\right\|^2 = E\left\| |\psi\rangle\right\|^2. \tag{11}$$

It follows that the spectrum is non-negative: all $E \geq 0$. Let us first concentrate on strictly positive energies $E > 0$. For a chain with N sites these energies come in pairs in the sense that a given positive eigenvalue occurs in the spectrum at either $N+1$ or $N-1$ sites. This can be seen as follows. The structure of the Hamiltonian in (9) and (10) results in the commutation relation

$$H_{N+1}Q_N - Q_N H_N = 0. \tag{12}$$

Hence, if $|\psi\rangle$ is an eigenvector of H_N with eigenvalue E in the subspace with $t_N = (-1)^{N+1}$, then $Q_N|\psi\rangle$ is either zero or an eigenvector of H_{N+1} with the same eigenvalue E. Likewise $Q_{N-1}^\dagger|\psi\rangle$ is either zero or an eigenvector of H_{N-1} with the same eigenvalue E. However, one of the two vectors $Q_N|\psi\rangle$, $Q_{N-1}^\dagger|\psi\rangle$ must vanish.[1] Hence, every eigenstate with non-zero energy is part of a doublet

$$\big(|\psi\rangle, Q_N|\psi\rangle\big) \quad \text{and so} \quad \big(|\psi\rangle, Q_{N-1}^\dagger|\psi\rangle\big).$$

[1] To show this we write $H_N = H_N^{(1)} + H_N^{(2)}$ with $H_N^{(1)} = Q_N^\dagger Q_N$ and $H_N^{(2)} = Q_{N-1}Q_{N-1}^\dagger$. Then $H_N^{(1)}H_N^{(2)} = H_N^{(2)}H_N^{(1)} = 0$. Therefore, their respective eigenspaces associated with non-zero eigenvalues

Conversely, zero-energy states are unpaired (singlets). From (11) it follows that they must be solution to the two equations

$$Q_N|\psi\rangle = 0, \qquad Q^\dagger_{N-1}|\psi\rangle = 0. \tag{13}$$

It is well known that the solutions to these equations are related to the cohomology $\mathfrak{H}_N = \ker Q_N / \operatorname{im} Q_{N-1}$ of the operator Q_N. Every zero-energy eigenstate corresponds to a non-trivial (non-zero) element in $\mathfrak{H}_N$. This can be seen indirectly: suppose that there are two linearly independent zero-energy states $|\psi_1\rangle$ and $|\psi_2\rangle$, and assume that they are in the same cohomology class. This means that there is a state $|\phi\rangle$ for the chain with $N-1$ sites such that

$$|\psi_1\rangle = |\psi_2\rangle + Q_{N-1}|\phi\rangle.$$

We act with $Q^\dagger_{N-1}$ on both sides, and apply (13). This yields $Q^\dagger_{N-1}Q_{N-1}|\phi\rangle = 0$, and by reprojection on $|\phi\rangle$ to $\|Q_{N-1}|\phi\rangle\|^2 = 0$. This implies $Q_{N-1}|\phi\rangle = 0$ and therefore $|\psi_1\rangle = |\psi_2\rangle$—in contradiction to the assumption of linear independence. Hence every non-trivial cohomology class contains exactly one ground state. This has two consequences. First, every eigenstate with non-zero energy that is annihilated by Q_N can be written in the form $|\psi\rangle = Q_{N-1}|\phi\rangle$ where $|\phi\rangle$ is an eigenstate for the chain with $N-1$ sites. We will frequently use this property in the next section. Second, the number of zero-energy ground states is the number of distinct non-trivial elements in the cohomology. We formulate the following

Conjecture 1 *For odd N there are two non-trivial elements in* $\mathfrak{H}_N$, *whereas for even N there are none.*

We checked this conjecture up to $N = 11$ sites by evaluation of the row and column ranks of the rectangular matrices Q_N. Notice that for N odd it implies Stroganov's conjecture [55] on the existence of two zero energy ground states, provided that one can prove that they occur in the zero-momentum sector.

Before proceeding, let us make the following comment: both the nilpotency and the supersymmetric structure of the XYZ-Hamiltonian studied in this section are only valid in certain momentum sectors. The nature of their derivation, given in Appendices A.1 and A.2, reveals that this restriction comes from matching the periodic boundary conditions. Everything else follows from local relations. Thus, we conclude that the supersymmetry has to be present in the full problem for chains of infinite length.

2.3 Spin-Reversal Symmetry

In this and the next section we provide a detailed discussion of the relation between supersymmetry and the other symmetries of the Hamiltonian introduced in Sect. 2.1. Let us give a motivation for this. The relation $Q_{N+1}Q_N = 0$ implies that we cannot relate chains with N sites to $N+2$ sites by sole use of the supercharges defined in the previous sections. Yet, a detailed inspection of the spectra for small system sizes suggests such a connection. The most simple example is the eigenvalue $3+\zeta^2$, appearing once in the spectrum for $N = 2$

are orthogonal. If $H^{(1)}_N|\psi\rangle = E|\psi\rangle$ for some $E > 0$ then $H^{(2)}_N|\psi\rangle = 0$, by reprojection on $|\psi\rangle$ we find $\|Q^\dagger_{N-1}|\psi\rangle\|^2 = 0$, and therefore $Q^\dagger_{N-1}|\psi\rangle = 0$. It follows that $Q_N|\psi\rangle$ is non-zero because otherwise $|\psi\rangle$ would be a zero-energy state. Thus we found a doublet $(|\psi\rangle, Q_N|\psi\rangle)$. A similar argument applies to the eigenstates of $H^{(2)}_N$, and leads to pairs $(|\psi\rangle, Q^\dagger_{N-1}|\psi\rangle)$. Finally, as H_N commutes with $H^{(1)}_N$ and $H^{(2)}_N$ it follows that all non-zero eigenstates of our Hamiltonian organise in pairs in the sense stated above.

sites, twice for $N=3$ sites and once for $N=4$ sites. The inspection of other non-zero eigenvalues reveals similar patterns. This hints at a larger symmetry algebra which we analyse in this section.

The existence of these degeneracies can be explained through a remarkably simple observation. The supercharges Q_N introduced in Sect. 2.2 treat spins up and down in a very asymmetric way. However, the Hamiltonian commutes with the spin-reversal operator (5), as pointed out in Sect. 2.1. Therefore it seems natural to introduce the spin-reversed version of Q_N:

$$\widetilde{Q}_N = R_{N+1} Q_N R_N.$$

In order to understand the implications of this operator on the spectrum we need to work out the algebra generated Q_N, $\widetilde{Q}_N$ and their adjoints. The full list of relations reads

$$Q_N Q_{N-1} = \widetilde{Q}_N \widetilde{Q}_{N-1} = 0, \qquad Q_{N-1}^\dagger Q_N^\dagger = \widetilde{Q}_{N-1}^\dagger \widetilde{Q}_N^\dagger = 0, \tag{14a}$$

$$\widetilde{Q}_N^\dagger Q_N + Q_{N-1} \widetilde{Q}_{N-1}^\dagger = 0, \qquad Q_N^\dagger \widetilde{Q}_N + \widetilde{Q}_{N-1} Q_{N-1}^\dagger = 0, \tag{14b}$$

$$\widetilde{Q}_N Q_{N-1} + Q_N \widetilde{Q}_{N-1} = 0, \qquad Q_{N-1}^\dagger \widetilde{Q}_N^\dagger + \widetilde{Q}_{N-1}^\dagger Q_N^\dagger = 0. \tag{14c}$$

The Hamiltonian can be written as anticommutator of either set of supercharges

$$H_N = Q_N^\dagger Q_N + Q_{N-1} Q_{N-1}^\dagger = \widetilde{Q}_N^\dagger \widetilde{Q}_N + \widetilde{Q}_{N-1} \widetilde{Q}_{N-1}^\dagger. \tag{14d}$$

In fact, even any linear combination $\alpha Q_N + \beta \widetilde{Q}_N$ with $|\alpha|^2 + |\beta|^2 = 1$ is an admissible supercharge that will generate the Hamiltonian. We omit the proofs of (14b) and (14c) as they are tedious and very similar to the proof of nilpotency outlined in Appendix A.1.

Instead, we point out the striking analogies between (14) and the $\mathcal{N} = (2,2)$ supersymmetry algebra in two dimensional quantum field theory (see for example [35], Chap. 22). The latter consists of four supercharges $\mathsf{Q}_\pm, \overline{\mathsf{Q}}_\pm$, Hamiltonian H, momentum P and a fermion number F. The algebra is defined through the relations

$$\mathsf{Q}_\pm^2 = \overline{\mathsf{Q}}_\pm^2 = 0, \tag{15a}$$

$$\{\mathsf{Q}_\pm, \overline{\mathsf{Q}}_\mp\} = 0, \tag{15b}$$

$$\{\mathsf{Q}_+, \overline{\mathsf{Q}}_+\} = \Delta, \qquad \{\mathsf{Q}_-, \overline{\mathsf{Q}}_-\} = \Delta^*, \tag{15c}$$

$$\{\mathsf{Q}_+, \mathsf{Q}_-\} = \mathsf{H} + \mathsf{P}, \qquad \{\overline{\mathsf{Q}}_+, \overline{\mathsf{Q}}_-\} = \mathsf{H} - \mathsf{P}, \tag{15d}$$

$$[\mathsf{F}, \mathsf{Q}_\pm] = \pm \mathsf{Q}_\pm, \qquad [\mathsf{F}, \overline{\mathsf{Q}}_\pm] = \mp \overline{\mathsf{Q}}_\pm, \tag{15e}$$

together with the conjugation relations $\mathsf{Q}_\pm^\dagger = \mathsf{Q}_\mp$ and $\overline{\mathsf{Q}}_\pm^\dagger = \overline{\mathsf{Q}}_\mp$. The definition of H and P implies that they commute with all supercharges and the fermion number. The operators Δ and Δ^* are central elements. Non-zero values of the latter are usually an indication of topological sectors [60] which occur generically in non-compact spaces.

The similarity between the two algebraic structures (14) and (15) is certainly not a coincidence. In fact, as mentioned in the introduction, the scaling limit of the lattice model is described by the sine-Gordon field theory at the supersymmetric point. Precisely, the limit is $N \to \infty$, $\zeta \to 0$ with $g = \zeta N^{1/3}$ finite, and it yields the sine-Gordon theory with bare mass g and coupling $\beta = \sqrt{16\pi/3}$ in conventional units. It was shown in [12] that the sine-Gordon field theory possesses an affine quantum group symmetry $U_q(\widehat{\mathrm{sl}}(2))$, $q = \exp(-8\mathrm{i}\pi^2/\beta^2)$ with zero centre. For $\beta = \sqrt{16\pi/3}$ the latter is known to contain the $\mathcal{N} = (2,2)$ supersymmetry algebra [13], which indeed corresponds to our coupling.

Let us establish a dictionary between the lattice and field theory quantities. The first three lines of (14) and (15) are in one-to-one correspondence, provided that we identify the field theory charges with some linear combinations of the lattice supercharges, and replace anti-commutators by appropriate (and natural) expressions for the lattice operators. We see that there is no equivalent of the central charges $\varDelta$, $\varDelta^*$ in the lattice theory as the corresponding anticommutators are zero. We attribute this to the fact, that we have a finite, discrete and periodic space, and thus no topological/solitonic sector. The fourth line of (15) defines the field theory Hamiltonian and momentum. Whereas the former is related to the lattice XYZ Hamiltonian in a direct way, the identification between momenta in field and lattice theory needs to be supplemented by a shift by $(N+1)\pi \mod 2\pi$. The need for a shift becomes even more plausible when taking into account that the lattice equivalent of fermion number appears to be the number of sites of the spin chain. This is consistent with the Jordan-Wigner transformation appearing in the analysis of continuum limit of the XYZ chain [42].

It is known that all non-zero energy states of a theory possessing the $\mathcal{N}=(2,2)$ supersymmetry (15) are organised in quadruplets of the form $(|\psi\rangle; \mathsf{Q}_+|\psi\rangle, \overline{\mathsf{Q}}_-|\psi\rangle; \mathsf{Q}_+\overline{\mathsf{Q}}_-|\psi\rangle)$ with the state $|\psi\rangle$ being annihilated by Q_- and $\overline{\mathsf{Q}}_+$. The states in such a supermultiplet have all the same energy, and momentum, but differ in their fermion number. Given the similarities between the field theory and lattice algebra, it is natural to ask if there is a quadruplet structure in the lattice model. Indeed, we now show that the relations (14) imply that it exists, at least in the momentum sectors considered in this paper. From the last section, we already know that the eigenstates $|\psi\rangle$ of the lattice Hamiltonian H_N are part of doublets $(|\psi\rangle, Q_{N-1}^\dagger|\psi\rangle)$ or $(|\psi\rangle, Q_N|\psi\rangle)$. Without loss of generality, we focus on the second case $(|\psi\rangle, Q_N|\psi\rangle)$. Let us consider the vector $\widetilde{Q}_N|\psi\rangle$. There are two possibilities: it may either be non-zero or zero. First, suppose that $\widetilde{Q}_N|\psi\rangle$ is a non-zero vector. Clearly, it has the same energy as $Q_N|\psi\rangle$, and we might wonder if they coincide. In fact, we show that this cannot be the case, and that they are rather linearly independent. If there is linear dependence, then there must be non-zero numbers λ and μ such that $\lambda Q_N|\psi\rangle + \mu\widetilde{Q}_N|\psi\rangle = 0$. We show that $\lambda = 0$ by applying $Q_N^\dagger$ from the left. For the first term, we use $Q_N^\dagger Q_N|\psi\rangle = H_N|\psi\rangle = E|\psi\rangle$; for the second term, we make use of the anticommutation relation (14b) and write $Q_N^\dagger\widetilde{Q}_N|\psi\rangle = -\widetilde{Q}_{N-1}Q_{N-1}^\dagger|\psi\rangle = 0$. Thus, we are left with $-\lambda E|\psi\rangle = 0$ but because of $E>0$ we must have $\lambda = 0$. Likewise, one shows that $\mu = 0$. Thus, the two vectors are linearly independent. This implies in particular that in addition to $Q_{N-1}^\dagger|\psi\rangle = 0$ we have the equation $\widetilde{Q}_{N-1}^\dagger|\psi\rangle = 0$. Next, we increase once more the system size: consider the vector $Q_{N+1}\widetilde{Q}_N|\psi\rangle = -\widetilde{Q}_{N+1}Q_N|\psi\rangle$. This vector is non-zero as again can be shown by an indirect proof: if for example $Q_{N+1}\widetilde{Q}_N|\psi\rangle = 0$ then there would be some vector $|\phi\rangle$ for the chain with N sites such that $\widetilde{Q}_N|\psi\rangle = Q_N|\phi\rangle$. If we premultiply this relation by $Q_N^\dagger$ we find on the left-hand side $Q_N^\dagger\widetilde{Q}_N|\psi\rangle = -\widetilde{Q}_{N-1}Q_{N-1}^\dagger|\psi\rangle = 0$ because of (14b). Thus, the right-hand side becomes $Q_N^\dagger Q_N|\phi\rangle = 0$ what implies $Q_N|\phi\rangle = 0$. Yet, this is in contradiction to $\widetilde{Q}_N|\psi\rangle \neq 0$, proving our claim. We cannot apply more supercharges in order to increase the length of the chain because the state $Q_{N+1}\widetilde{Q}_N|\psi\rangle$ is annihilated by both Q_{N+2} (trivially), and $\widetilde{Q}_{N+2}$ (because of (14c)). Thus, we have constructed a quadruplet of one state at N sites, *two* states at $N+1$ sites, and one state $N+2$ sites, all of them having the same energy E with respect to the corresponding Hamiltonians:

$$\big(|\psi\rangle; Q_N|\psi\rangle, \widetilde{Q}_N|\psi\rangle; Q_{N+1}\widetilde{Q}_N|\psi\rangle\big).$$

The preceding construction assumes that $\widetilde{Q}_N|\psi\rangle$ is non-zero. Let us now consider the second case $\widetilde{Q}_N|\psi\rangle = 0$. As $E>0$ this can only be the case if there is a vector $|\phi\rangle$ for the chain with $N-1$ sites such that $|\psi\rangle = \widetilde{Q}_{N-1}|\phi\rangle$. Consider now the state $Q_{N-1}|\phi\rangle$. It cannot be zero: otherwise, we could write $0 = \widetilde{Q}_N(Q_{N-1}|\phi\rangle) = -Q_N(\widetilde{Q}_{N-1}|\phi\rangle) = -Q_N|\psi\rangle$ what contradicts our assumptions as we started from a doublet $(|\psi\rangle, Q_N|\psi\rangle)$. Moreover, the state $Q_{N-1}|\phi\rangle$ is linearly independent from $|\psi\rangle$, as follows from the same argument as above. We have $Q_N|\psi\rangle = Q_N\widetilde{Q}_{N-1}|\phi\rangle = -\widetilde{Q}_N Q_{N-1}|\phi\rangle$, and hence a similar quadruplet structure as before, this time however with one state at $N-1$ sites, *two* states at N sites, and one state $N+1$ sites:

$$\big(|\phi\rangle;\, Q_{N-1}|\phi\rangle, \widetilde{Q}_{N-1}|\phi\rangle;\, Q_N\widetilde{Q}_{N-1}|\phi\rangle\big), \quad \widetilde{Q}_{N-1}|\phi\rangle = |\psi\rangle. \tag{16}$$

Therefore, all states with non-zero energy must be part of a quadruplet. We see that our argument leads automatically to a degeneracy in the "middle" of such a quadruplet. In fact, there is a non-trivial conserved charge that maps between these two states. It is given by

$$C_N = \widetilde{Q}_N^\dagger Q_N = -Q_{N-1}\widetilde{Q}_{N-1}^\dagger.$$

The anticommutation relations imply that it commutes with the Hamiltonian and has square zero:

$$[H_N, C_N] = 0, \quad \text{and} \quad C_N^2 = 0.$$

Moreover, its Hermitian conjugate is the "spin-reversed" operator $C_N^\dagger = R_N C_N R_N$. They have the character of fermionic ladder operators. Indeed, let us consider the quadruplet (16) containing two states $|\psi\rangle = Q_{N-1}|\phi\rangle$ and $|\widetilde{\psi}\rangle = \widetilde{Q}_{N-1}|\phi\rangle$ at N sites. We find the following relations

$$C_N|\psi\rangle = 0, \qquad C_N|\widetilde{\psi}\rangle = -E|\psi\rangle,$$
$$C_N^\dagger|\widetilde{\psi}\rangle = 0, \qquad C_N^\dagger|\psi\rangle = -E|\widetilde{\psi}\rangle.$$

The other two states in the quadruplet, $|\phi\rangle$ and $Q_N\widetilde{Q}_{N-1}|\phi\rangle$, are annihilated by the corresponding operators C_{N-1} and C_{N+1} and their Hermitian conjugates. Thus, $(|\psi\rangle, |\widetilde{\psi}\rangle)$ can be thought of a doublet inside the quadruplet (16).

Even though C_N is a bilinear in the supercharges, it still can be thought of as fermionic in the following sense. The symmetry operator S_N defined in Sect. 2.1 anti-commutes with the fermion: $C_N S_N + S_N C_N = 0$, as can be shown using (6) and the fact that $S_{N+1}Q_N + Q_N S_N = 0$. If we now suppose that the state $|\psi\rangle$ is an eigenstate of S_N with eigenvalue $s = \pm 1$ then the anticommutation relation tells us that $|\widetilde{\psi}\rangle$ is also an eigenstate of S_N, however with eigenvalue $-s$. Therefore, we see that the fermionic operators C_N provide a mapping between the sectors with odd and even number of spins down. For chains of odd length, this connection is already established through the spin-reversal operator, as explained in Sect. 2.1. Namely, the states $(|\psi\rangle, |\widetilde{\psi}\rangle)$ can be mapped onto each other through spin reversal, as the state $|\phi\rangle$ is an eigenstate of the spin-reversal operator R_{N-1} when N is odd. For chains of even length, however, the spin-reversal operator fails to connect the two vectors, whereas the operator C_N does this independently of the number of sites.

2.4 Parity Symmetry

In the last part of this section, we analyse the relation between the supercharges and the parity operation. Using the definition of the "local" supercharges (7) we find the simple transformation laws

$$P_{N+1}q_j = (-1)^{N+1}q_{N-j+1}P_N, \qquad P_{N+1}q_0 = q_0 P_N T_N.$$

Let us now consider a state $|\psi\rangle$ for a chain of N sites with both definite translational behaviour $T_N|\psi\rangle = t_N|\psi\rangle$ and definite parity $P_N|\psi\rangle = p_N|\psi\rangle$, $p_N = \pm 1$. The parity operation reverses momentum, as can be seen from the relation $P_N T_N P_N = T_N^{-1}$. This implies that the translation eigenvalue t_N must be solution to $t_N^2 = 1$. Obviously, this is compatible with $t_N = (-1)^{N+1}$. Applying our rules, we find

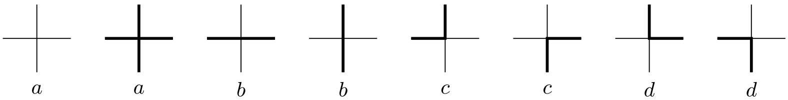


Fig. 2 The vertex configurations of the eight-vertex model (the *bold edges* have spin down). The associated weights a, b, c, d are invariant under spin reversal in the zero field case

$$P_{N+1}Q_N|\psi\rangle = (-1)^{N+1}p\left(\sum_{j=1}^{N} q_j + (-1)^{N+1}t_N\, q_0\right)|\psi\rangle$$
$$= (-1)^{N+1}p_N Q_N|\psi\rangle.$$

Thus, we find that (1) for N odd, $t_N = 1$ the action of Q_N preserves parity and (2) for N even, $t_N = -1$ the action of Q_N reverses parity.

We studied the parity sectors by means of exact diagonalisation of the Hamiltonian up to $N = 11$ sites. For odd N we observed that the spectrum of the parity odd sector is contained in the parity even sector. Thus, we arrive at the following

Conjecture 2 *For odd N and zero momentum, the spectrum in the odd parity sector $p_N = -1$ is contained in the spectrum of the even parity sector $p_N = +1$.*

3 Supersymmetry and the Transfer Matrix of the Eight-Vertex Model

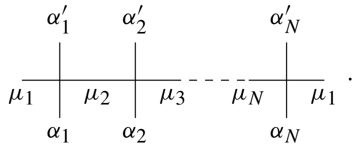
In this section, we consider the transfer matrix of the zero-field eight-vertex model. The main tool here is the Bethe ansatz for the transfer matrix established by Baxter [2–4]. We provide a derivation of the supersymmetry from this point of view, generalising the result of [31] from the critical point to the entire supersymmetric line.

After some basic definitions in Sect. 3.1, we provide a brief review of the Bethe ansatz for the eight-vertex model in the root-of-unity case in Sect. 3.2, in particular recalling the necessary change of basis of the Hilbert space. In Sect. 3.3 we establish the supersymmetry in the new basis. To make contact with the supercharges defined in our previous discussion, we have to transform back to the canonical spin basis what is discussed in Sect. 3.4. This leads to some new conjectures on the nature of the zero-energy ground states of the XYZ chain of odd length.

3.1 Basic Definitions

We start by recalling elementary facts about the eight-vertex model on the square lattice [5]. Each edge carries a classical $\mathbb{Z}_2$ "spin" variable $\pm$, corresponding to occupied/empty or spin up/down. The configurations are restricted in such a way that each vertex has an even number of spins down: the eight allowed vertex configurations are shown in Fig. 2. We associate a Boltzmann weight to each vertex, and the weight of a given lattice configuration is then simply the product over all the vertex weights. In the "zero-field" case, the weights are invariant under simultaneous reversal of all spins around a vertex. Thus, as shown in Fig. 2, there are four distinct weights, traditionally denoted by a, b, c, d. Suppose that the square lattice has say M rows and N columns, and is wrapped around a torus (periodic boundary conditions along the two directions). Then the model can conveniently be studied by the row-to-row transfer matrix $\boldsymbol{\mathcal{T}}_N$, whose matrix elements are defined as the sum over all configurations along a horizontal line, compatible with the spin values on the vertical edges:

$$\langle\alpha'|\mathcal{T}_N|\alpha\rangle = \sum_{\mu_1,\dots,\mu_N=\pm}$$



The invariance of the vertex weights under spin reversal implies that $[\mathcal{T}_N, R_N] = 0$. Moreover, the vertex rule implies that the transfer matrix conserves the number of down spins mod 2. Therefore we have $[\mathcal{T}_N, S_N] = 0$. Conservation of the total number of down spins is only possible in the six-vertex limit $d = 0$ (or $c = 0$).

To proceed we parametrise of the vertex weights in terms of Jacobi theta functions, following the definitions of [8, 9, 43] and [58]:

$$\begin{aligned}
a &= a(u) = \rho\vartheta_4\big(2\eta, q^2\big)\vartheta_4\big(u-\eta, q^2\big)\vartheta_1\big(u+\eta, q^2\big),\\
b &= b(u) = \rho\vartheta_4\big(2\eta, q^2\big)\vartheta_1\big(u-\eta, q^2\big)\vartheta_4\big(u+\eta, q^2\big),\\
c &= c(u) = \rho\vartheta_1\big(2\eta, q^2\big)\vartheta_4\big(u-\eta, q^2\big)\vartheta_4\big(u+\eta, q^2\big),\\
d &= d(u) = \rho\vartheta_1\big(2\eta, q^2\big)\vartheta_1\big(u-\eta, q^2\big)\vartheta_1\big(u+\eta, q^2\big).
\end{aligned}$$

Here u denotes the spectral parameter, η the so-called crossing parameter, and q the elliptic nome. Moreover, we choose the overall normalisation as $\rho = 2/\vartheta_2(0,q)\vartheta_4(0,q^2)$. This choice ensures that

$$h(u) = a(u) + b(u) = \vartheta_1(u, q), \tag{17}$$

a function which we shall use quite often (the right-hand side follows from standard identities for Jacobi theta functions [58]). With this parametrisation two transfer matrices with different spectral parameters u, u' commute:

$$\big[\mathcal{T}_N(u), \mathcal{T}_N\big(u'\big)\big] = 0. \tag{18}$$

This implies that the series expansion of the transfer matrix in the spectral parameter around any point yields a family of commuting operators. The most simple ones are the translation operator T_N and the XYZ-Hamiltonian

$$T_N = h(2\eta)^{-N}\mathcal{T}_N(\eta), \qquad H_N = a(\eta)/b'(\eta)\mathcal{T}_N(u)^{-1}\mathcal{T}'_N(u)|_{u=\eta}.$$

The Hamiltonian reduces exactly to our problem (1) if the crossing parameter is set to $\eta = \pi/3$. In this case, the variable ζ used to parametrise the supersymmetric line is related to the elliptic nome through

$$\zeta = \left(\frac{\vartheta_1(2\pi/3, q^2)}{\vartheta_4(2\pi/3, q^2)}\right)^2. \tag{19}$$

We are interested in using supersymmetry to study the eigenvalues and eigenvectors of the eight-vertex model transfer matrix. Because of (18) the eigenvectors do not depend on the spectral parameter, and so coincide with those of the XYZ Hamiltonian H_N, up to possible degeneracies. Such degeneracies do not seem to appear at generic values of the crossing parameter, but only at the special elliptic root of unity points $\eta = (m_1\pi + m_2\pi\tau)/L$, with m_1, m_2, L integers and $q = e^{\mathrm{i}\pi\tau}$, where additional symmetries are present [18]. In our case $\eta = \pi/3$ we have already shown that in the momentum sectors with $t_N = (-1)^{N+1}$ the eigenvectors organise into singlets, or quadruplets with the same value of E. Moreover, the eigenvectors in a given quadruplet each can be labelled by a distinct quantum number; two of them are for $N-1$ and $N+1$ sites, while we showed that the two at N sites have an even and odd numbers of spins down. Both the number of sites and the number mod 2 of spins down are preserved by the eight-vertex model transfer matrix, so barring any accidental degeneracies, these correspond to distinct eigenvectors of the transfer matrix as well. Since H_N is obtained from the logarithmic derivative of $\boldsymbol{\mathcal{T}}_N(u)$, the fact that their eigenvectors coincide makes it natural to hope that analogous structure occurs in the spectrum of the transfer matrix. We here show how at $\eta=\pi/3$ the supersymmetries described above indeed extend to the transfer matrix, and so give relations among the eigenvalues.

As an indication of the special properties occurring at $\eta=\pi/3$, we note that the zero-energy states $|\Psi^{\pm}\rangle$ of H_N for odd N, i.e. the supersymmetry singlets, have very simple transfer-matrix eigenvalues $\boldsymbol{\mathcal{T}}_N(u)|\Psi^{\pm}\rangle = \mathcal{T}_N(u)|\Psi^{\pm}\rangle$. They are given by [54, 55]

$$\mathcal{T}_N(u) = h(u)^N = \vartheta_1(u,q)^N.$$

The simplicity of this expression stresses the special nature of the two eigenstates. In the sequel we will see that the study of the transfer matrix eigenvalues leads naturally to a distinction of these states from the other eigenvectors.

3.2 Review of Baxter's Bethe Ansatz

Here we summarise the aspects of the coordinate-type Bethe ansatz [2–4] relevant to our derivation of the supersymmetry in the eight-vertex model at $\eta=\pi/3$.

Path Basis The transfer matrix of the eight-vertex model has no obvious particle-number conservation (such as conservation of the number of down-spins). This is a central difficulty when compared to the six-vertex model. In [3] Baxter developed a way to overcome this problem through the introduction of a basis upon which the transfer matrix acts in a way that resembles the six-vertex case.

For N sites the new basis vectors are labelled by a sequence of integers $\ell_1,\ell_2,\dots,\ell_N$, ℓ_{N+1} such that $|\ell_{j+1}-\ell_j|=1$, $j=1,\dots,N$. It is useful to think of a path starting at some height $\ell_1=\ell$ with the restriction that consecutive heights differ by ±1. In the following, we will therefore frequently call the corresponding set of vectors in $\mathcal{H}_N$ the "path basis". A down step or particle occurs at site j if $\ell_{j+1}-\ell_j=-1$, and an up step occurs otherwise. The path is completely characterised by ℓ and the positions $x_1,\dots,x_m$ of its m down steps. Hence for $x_k<j<x_{k+1}$ the local heights are given by

$$\ell_j = \ell + j - (2k+1). \tag{20}$$

The basis vectors are given as an N-fold tensor product of local vectors $|\Phi_{\ell,\ell'}\rangle$ in $\mathbb{C}^2$:

$$|\ell;x_1,\dots,x_m\rangle_N = \bigotimes_{j=1}^{N} |\Phi_{\ell_j,\ell_{j+1}}\rangle.$$

The factors are constructed from the local heights via

$$\begin{aligned} |\Phi_{\ell,\ell+1}\rangle &= \vartheta_1\big(s+(2\ell+1)\eta,q^2\big)|+\rangle + \vartheta_4\big(s+(2\ell+1)\eta,q^2\big)|-\rangle,\\ |\Phi_{\ell+1,\ell}\rangle &= \vartheta_1\big(t+(2\ell+1)\eta,q^2\big)|+\rangle + \vartheta_4\big(t+(2\ell+1)\eta,q^2\big)|-\rangle, \end{aligned} \tag{21}$$

where $|\pm\rangle$ are the local spin-1/2 basis vectors, and s and t arbitrary parameters such that the two vectors are linearly independent.

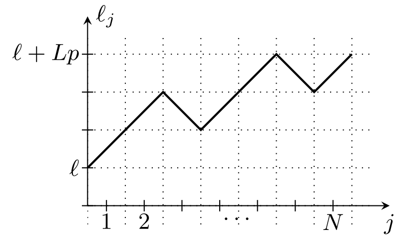


Fig. 3 A typical path for $L = 3$, two particles and $p = 1$

The transfer matrix maps these states onto themselves and conserves the number of particles, provided that the following condition is met [3]: for generic values of the crossing parameter η the initial and final height are identical $\ell_1 = \ell_{N+1}$, and therefore $N = 2m$. In the special case of elliptic roots of unity $\eta = (m_1\pi + m_2\pi\tau)/L$ however, this constraint can be relaxed to

$$N - 2m = Lp, \quad \text{for some } p \in \mathbb{Z} \tag{22}$$

because of the periodicity of the Jacobi theta functions involved in the construction of the vectors. The height difference between starting point and endpoint of the path is thus Lp as shown in Fig. 3. Moreover, because of the periodicity of the theta functions it is sufficient to restrict the initial height to $\ell_1 = 0, 1, \dots, L-1$ in this case.

It is instructive to compute the maximal dimension d_N of the subspace spanned by these vectors in the root-of unity-case by simple counting of the paths. There are $\binom{N}{m}$ arrangements of m particles, provided that (22) holds. Let us introduce an indicator function $\delta_L(n)$ which is 1 if $n = 0 \bmod L$, and 0 otherwise. We have the convenient representation

$$\delta_L(n) = \frac{1}{L}\sum_{j=0}^{L-1} e^{2\pi \mathrm{i} jn/L}.$$

We weight this by the number of arrangements and an additional factor L which takes into account the different choices for $\ell = 0, 1, \dots, L-1$. Summation over m yields

$$d_N = L\sum_{m=0}^{N}\delta_L(N-2m)\binom{N}{m} = 2^N\sum_{j=0}^{L-1}\left(\cos\left(\frac{2\pi j}{L}\right)\right)^N.$$

For the case of interest $L = 3$, we find

$$d_N = 2^N + 2(-1)^N.$$

We know that the dimension of the full Hilbert space $\mathcal{H}_N$ is 2^N. If we assume that all the vectors associated to paths are linearly independent then we conclude that for even N the path basis is redundant. For odd N however at least two vectors are missing, and thus the path basis does not span the entire Hilbert space. For small finite-size systems, it seems that exactly two vectors are missing, i.e. the existing $2^N - 2$ vectors are linearly independent. We shall assume the linear independence in the following, and will conjecture later that the two-dimensional complement of the path basis at N odd is spanned by the ground states of the XYZ Hamiltonian.

Eigenvectors and Bethe Equations The next step consists of decomposing the eigenvectors of the transfer matrix in terms of vectors in the path basis:

$$|\psi\rangle = \sum_{\ell=1}^{L}\omega^\ell \sum_{\{x_j\}}\psi(\ell; x_1, \dots, x_m)|\ell; x_1, \dots, x_m\rangle. \tag{23}$$

The summation over the positions of the particles is carried out in an ordered way $1 \leq x_1 < x_2 < \cdots < x_m \leq N$. Moreover ω is an L-th root of unity: $\omega^L = 1$. The wave functions $\psi(\ell; x_1, \ldots, x_m)$ are obtained through a Bethe-type ansatz. In order to describe it we need Baxter's "single particle" functions and "wave vectors" defined through

$$g_j(\ell, x) = e^{\mathrm{i}k_j x} \frac{h(w_{\ell+x-1} - \eta - u_j)}{h(w_{\ell+x-2})h(w_{\ell+x-1})}, \qquad e^{\mathrm{i}k_j} = \frac{h(u_j + \eta)}{h(u_j - \eta)}. \tag{24}$$

Here we used the function

$$w_\ell = (s+t)/2 - \pi/2 + 2\ell\eta \tag{25}$$

which is linear in ℓ, and contains the free parameters. The numbers $u_1, \ldots, u_m$ are the *Bethe roots* to be determined. With this notation the wave function is given in typical Bethe-ansatz form by

$$\begin{aligned}\psi(\ell|x_1, \ldots, x_m) \\ = \sum_\pi A_\pi g_{\pi(1)}(\ell, x_1) g_{\pi(2)}(\ell - 2, x_2) \cdots g_{\pi(m)}\big(\ell - 2(m-1), x_m\big).\end{aligned} \tag{26}$$

Here the sum runs over all permutations π of m objects. The $m!$ coefficients A_π satisfy the following relation: if τ is a transposition exchanging j and $j+1$, then we have for the permutation $\pi' = \pi \circ \tau$ the relation

$$\frac{A_{\pi'}}{A_\pi} = -\frac{h(u_{\pi(j+1)} - u_{\pi(j)} + 2\eta)}{h(u_{\pi(j)} - u_{\pi(j+1)} - 2\eta)}. \tag{27}$$

The left-hand side is commonly interpreted as the (bare) scattering matrix between two particles with "rapidities" $u_{\pi(j)}$ and $u_{\pi(j+1)}$. If they coincide then we find $A_{\pi'} = -A_\pi$, implying that the Bethe wave function vanishes.[2] This will be very important in our analysis.

The Bethe roots $u_1, \ldots, u_m$ remain to be determined. Baxter showed in [4] that if they solve the *Bethe equations*

$$\left(\frac{h(u_j + \eta)}{h(u_j - \eta)}\right)^N = -\omega^2 \prod_{k=1}^{m} \frac{h(u_j - u_k + 2\eta)}{h(u_j - u_k - 2\eta)}, \tag{28}$$

then (23) is an eigenvector of the transfer matrix $\boldsymbol{\mathcal{T}}_N(u)|\psi_N\rangle = \mathcal{T}_N(u)|\psi_N\rangle$. The corresponding eigenvalue can be obtained from the so-called $\mathcal{T}\mathcal{Q}$-equation

$$\mathcal{T}_N(u)\mathcal{Q}_N(u) = \omega\phi_N(u - \eta)\mathcal{Q}_N(u + 2\eta) + \omega^{-1}\phi_N(u + \eta)\mathcal{Q}_N(u - 2\eta), \tag{29}$$

[2] Let us suppose that two Bethe roots have the same value, say $u_{m-1} = u_m$. Now we modify the sum over permutations in (26) according to $\sum_\pi f_\pi = \sum_\pi f_{\tau\circ\pi}$ for some function f on the symmetric group S_m, where τ is an arbitrary permutation of m objects. We choose τ to be the transposition of $m-1$ and m. For any π define pre-images n_1, n_2 according to $\pi(n_1) = m-1$ and $\pi(n_2) = m$, then we find

$$\begin{aligned}\psi(\ell|x_1, \ldots, x_m) = \sum_\pi A_{\pi'} \prod_{j \neq n_1, n_2} g_{\pi(j)}\big(\ell - 2(j-1), x_j\big) \\ \times g_{\pi(n_2)}\big(\ell - 2(n_1 - 1), x_{n_1}\big) g_{\pi(n_1)}\big(\ell - 2(n_2 - 1), x_{n_2}\big).\end{aligned}$$

According to our assumption we have $A_{\pi'} = -A_\pi$. Moreover, from $u_{m-1} = u_m$ and (24), it is not difficult to see that $g_{m-1}(\ell, x) = g_m(\ell, x)$. Therefore we may write $g_{\pi(n_2)}(\ell - 2(n_1 - 1), x_{n_1}) g_{\pi(n_1)}(\ell - 2(n_2 - 1), x_{n_2}) = g_{\pi(n_1)}(\ell - 2(n_1 - 1), x_{n_1}) g_{\pi(n_2)}(\ell - 2(n_2 - 1), x_{n_2})$. Using these facts, we see that

$$\psi(\ell|x_1, \ldots, x_m) = -\psi(\ell|x_1, \ldots, x_m),$$

and therefore the wave function vanishes.

where $\phi_N(u) = h(u)^N = \vartheta_1(u,q)^N$, and

$$\mathcal{Q}_N(u) = \prod_{j=1}^{m} h(u-u_j)$$

is an elliptic polynomial with zeroes at the Bethe roots. In particular, setting $u=\eta$ we find that the eigenvalue t_N of the translation operator T_N is given in terms of the $\mathcal{Q}$-function as

$$t_N = \omega^{-1}\frac{\mathcal{Q}_N(-\eta)}{\mathcal{Q}_N(\eta)} = \omega^{-1}\prod_{j=1}^{m}\frac{h(u_j+\eta)}{h(u_j-\eta)} = \omega^{-1}\prod_{j=1}^{m} e^{\mathrm{i}k_j}, \tag{30}$$

where in the last step we used (24).

3.3 Derivation of the Supersymmetry from the Bethe Ansatz

We now use the Bethe ansatz to establish for the case $\eta=\pi/3$ the supersymmetry connecting systems with different numbers of sites N and $N\pm 1$.

We start by noting that from (22), the number of particles m in the path when $L=3$ must obey

$$N-2m = 3p$$

for some integer p. This relation is compatible with the simultaneous replacement $N\to N'=N-j$, $m\to m'=m+j$ and $p\to p'=p-j$ for some integer j. The supersymmetry charge Q_N studied in Sect. 2 increases the number of sites by one. However, in the context of the Bethe ansatz it turns out particularly convenient to consider an action like that of $Q_{N-1}^\dagger$: we choose $j=1$ and therefore *decrease* the length of the chain by one while adding a particle to the system. We discuss the relation with the supercharges studied previously in the next section.

Our strategy is to construct from a given solution $u_1,\dots,u_m$ of Bethe's equations at N sites a new solution $\tilde u_1,\dots,\tilde u_m,\tilde u_{m+1}$ at $N-1$ sites. We shall verify that a solution to this problem is simply given by $\tilde u_j = u_j$ for $j<m+1$ and $u_{m+1}=\pi$. Indeed, for the smaller system the first m Bethe equations with this choice become

$$\left(\frac{h(u_j+\eta)}{h(u_j-\eta)}\right)^{N-1} = -\omega^2\prod_{k=1}^{m}\frac{h(u_j-u_k+2\eta)}{h(u_j-u_k-2\eta)}\times\frac{h(u_j-\pi+2\eta)}{h(u_j-\pi-2\eta)}.$$

Using (28) and the antiperiodicity $h(u+\pi)=-h(u)$, this equation reduces to

$$\frac{h(u_j-\eta)}{h(u_j+\eta)} = \frac{h(u_j-\pi+2\eta)}{h(u_j+\pi-2\eta)}.$$

It holds for generic u_j if $\eta=\pi-2\eta \bmod \pi$, and thus in particular for the value $\eta=\pi/3$ we are interested in. However, we still have to check the $(m+1)$-th Bethe equation. We find

$$(-1)^{N+1} = \omega^2\prod_{j=1}^{m}\frac{h(u_j+\eta)}{h(u_j-\eta)}.$$

On the right-hand side we recognise the eigenvalue of the translation operator for the system with N sites (30). We conclude that the operation is possible only if $t_N=(-1)^{N+1}\omega^3$. But recall from the last section that for $\eta=\pi m/L$ the number ω is an L-th root of unity, in our case thus $\omega^3=1$, and therefore we find the symmetry in the momentum sector with

$$t_N = (-1)^{N+1}. \tag{31}$$

Consistency thus requires a restriction to the momentum sectors studied in Sect. 2.

As a side comment, let us notice that we could have started from the $\mathcal{T}\mathcal{Q}$-equation with arbitrary ω, deduced the Bethe equations from the requirement that $\mathcal{T}_N(u)$ is an entire function and *imposed* the lattice supersymmetry. Asking for consistency would have led us to $\omega^3 = 1$. In the six-vertex limit where the working is completely analogous, write $\omega = e^{i\phi}$ so that ϕ has the interpretation of a twist angle. We conclude that the twists leading to the symmetry here are $\phi = 0, \pm 2\pi/3$, as follows from the observations of [31]. These are precisely the values for which special simple eigenvalues of the transfer matrix, as well as relations to problems of enumerative combinatorics appear [17, 45, 46].

As a next step, we determine the relation between the corresponding eigenvalues of the transfer matrix from (29). For the $\mathcal{Q}$-function we find

$$\mathcal{Q}_{N-1}(u) = \prod_{k=1}^{m} h(u - u_j) \times h(u - \pi) = -h(u)\mathcal{Q}_N(u).$$

Using this relation, we deduce that

$$\mathcal{T}_N(u) + h(u)\mathcal{T}_{N-1}(u) = 0. \tag{32}$$

Setting $u = \eta$ we obtain a relation between the eigenvalues of the translation operators for both systems $t_{N-1} = -t_N = (-1)^N$. This is consistent with (31). For odd N we obtain thus the zero-momentum sector (invariant under translation), whereas for even N it is the π-momentum sector. This fits well the picture suggested by (24): the $(m+1)$-th particle with $u_{m+1} = \pi$ has momentum $k_{m+1} = \pi$, and therefore the eigenvalue of the translation operator is changed by a sign.

Relation Between Eigenvectors The preceding operation should manifest itself as an operation on the Hilbert space (or at least the special momentum sectors). In fact, we would like to introduce an operator $\hat{Q}_N : \mathcal{H}_N \to \mathcal{H}_{N+1}$ (not to be confused with Baxter's $\boldsymbol{Q}$-matrix) such that the eigenvectors of the transfer matrix that can be obtained from the Bethe ansatz are related according to

$$|\psi_{N-1}\rangle = \hat{Q}_{N-1}^{\dagger}|\psi_N\rangle.$$

Twofold application of $\hat{Q}_N^{\dagger}$ would lead to the injection of two particles with momentum π. However, in this case the Bethe wave function vanishes. Hence we can write on the subspace spanned by the path basis

$$\hat{Q}_{N-1}^{\dagger}\hat{Q}_N^{\dagger} = 0, \quad \text{or} \quad \hat{Q}_N\hat{Q}_{N-1} = 0.$$

We will now derive the explicit form of these operators, starting from the definition of the wave functions (26). To manipulate them, we need an explicit expression for the amplitudes A_π. In fact, their defining equation (27) can be solved up to a factor:

$$A_\pi = \operatorname{sgn}\pi \prod_{1 \le i < j \le m} h(u_{\pi(i)} - u_{\pi(j)} + 2\eta). \tag{33}$$

Let us consider the wave function (26) for $m+1$ particles, one of them having momentum $u_{m+1} = \pi$. It is a sum over permutations π of $\{1, 2, \dots, m+1\}$. For a start, let us consider in this sum only the permutations with $\pi(r) = m+1$ for some fixed $r = 1, \dots, m+1$. Any such permutation can be decomposed according to $\pi = \pi' \circ \pi''$ where $\pi'(m+1) = m+1$ and

$$\pi''(j) = \begin{cases} j, & j < r \\ m+1, & j = r \\ j-1, & j > r \end{cases}.$$

We have the signature $\operatorname{sgn}\pi'' = (-1)^{m+1-r}$ and thus $\operatorname{sgn}\pi = (-1)^{m+1-r}\operatorname{sgn}\pi'$. We use this in order to evaluate the Bethe amplitude (33) in terms of the permutation π'. After some algebra one finds

$$A_\pi = \operatorname{sgn}\pi' \prod_{1\le i<j\le m} h(u_{\pi'(i)} - u_{\pi'(j)} + 2\eta)\prod_{i=1}^{m} h(u_i - \eta) \prod_{j=r+1}^{m+1} e^{\mathrm{i}k_{\pi(j)}}.$$

We see that the only r-dependent term is the last product: a string of wave-vectors. Thus, we must understand how this affects the corresponding single-particle functions. For $j > r$ we notice the identity

$$e^{\mathrm{i}k_{\pi(j)}} g_{\pi(j)}(\ell - 2j + 2, x_j) = g_{\pi'(j-1)}\big(\ell - 2(j-1) + 2, x_j + 1\big),$$

which holds only because of $\eta = \pi/3$. This is already enough to simplify the wave function. As π' leaves $m+1$ unchanged, we can think of it as a permutation of only m objects. This can of course be done for any value of r separately. Collecting the different contributions, we find after some algebra a recursion relation for the wave functions

$$\begin{aligned}\psi(\ell; x_1,\dots,x_{m+1}) = \prod_{i=1}^{m} h(u_i-\eta)\sum_{r=1}^{m+1} g_{m+1}\big(\ell - 2(r-1), x_r\big)\\ \times \psi(\ell; x_1,\dots,x_{r-1}, x_{r+1}+1,\dots,x_{m+1}+1).\end{aligned}$$

The wave functions on the right hand side are the ones involving only the Bethe roots $u_1,\dots,u_m$, thus precisely the ones for the problem at N sites. However, notice that the positions of the particles are not arbitrary: for the r-th term we have $x_{r+1} + 1 - x_{r-1} \ge 2$. The picture becomes a little more transparent if we consider the induced operation on the basis states:[3]

$$\begin{aligned}\hat{Q}^\dagger_{N-1}|\ell; x_1,\dots,x_m\rangle_N = \mathcal{C}\sum_{r=1}^{m+1}\sum_{x=x_{r-1}+1}^{x_r-2} g_{m+1}\big(\ell-2(r-1), x\big)\\ \times |\ell; x_1,\dots,x_{r-1}, x, x_{r+1}-1,\dots,x_m-1\rangle_{N-1}\end{aligned} \qquad (34)$$

where we set $x_0 = 0$; $\mathcal{C}$ is a normalisation constant which can be chosen arbitrarily. We see that $\hat{Q}^\dagger_{N-1}$ inserts a new particle at position x between existing ones at x_{r-1} and x_r, provided that they are at least two sites apart (in order to guarantee $x_{r-1} < x < x_r - 1$). More precisely the operation $\hat{Q}^\dagger_{N-1}$ transforms locally two consecutive up-steps at $(x, x+1)$ to a single down-step at x, while shifting all particles on its right by one step to the left as illustrated in Fig. 4.

This is weighted by $\mathcal{C}g_{m+1}(\ell - 2r + 2, x_r = x)$. Notice that the local height at $x+1$ is given by $\ell_{x+1} = \ell + x - 2(r-1)$. Combining this with the definition of the single-particle wave function (24), we conclude that apart from the string the weight can be expressed in terms of the local height between the two up-segments alone (to see this recall that $h(u)$ is 2π-periodic and $w_{x+3} = w_x + 2\pi$). More explicitly, we make the convenient choice $\mathcal{C} = \prod_{\ell=1}^{3} h(w_\ell)$ and find the local weight

$$\mathcal{C}g_{m+1}(\ell - 2r + 2, x_r = x) = (-1)^x h(w_{\ell_{x+1}})^2. \qquad (35)$$

[3] Strictly speaking, $\hat{Q}^\dagger_{N-1}$ only acts on momentum states with $t_N = (-1)^{N+1}$. Clearly, these can be written as a superposition of path states. Thus, for simplicity, we present the action of the supercharge on the vectors in this decomposition.

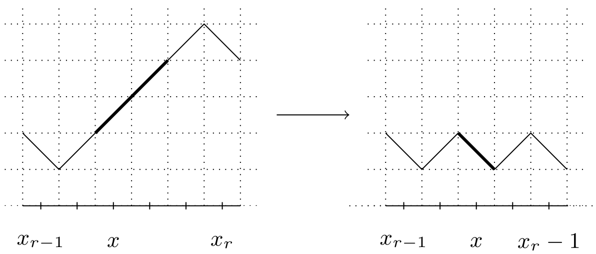


Fig. 4 Local action of $\hat{Q}_{N-1}^{\dagger}$: two steps up are transformed to a step down

This completes the definition of the supercharges acting on the path basis. From (32), we conclude that they act as intertwiners for the transfer matrices:

$$\mathcal{T}_N(u)\hat{Q}_{N-1} + h(u)\hat{Q}_{N-1}\mathcal{T}_{N-1}(u) = 0, \tag{36}$$

on the momentum sectors with $t_N = (-1)^{N+1}$. Of course, this relation only holds on the subspace spanned by the path basis.

We finish this section by pointing out that—as in the case of the XYZ chain—the notion of "particle" is somewhat arbitrary. One could as well have chosen the steps up as particles. From the local vectors (21) we see that this corresponds essentially to exchanging the parameters s and t. This would lead to a second supersymmetry operation with the same local weights (35), which transforms locally two consecutive steps down to a single step up, and thus resembling strongly the case studied in Sect. 2. We will discuss their connections in the next section.

3.4 Supercharges in the Spin Representation and the XYZ Ground States

Having found an operator $\hat{Q}_N^{\dagger}$ (and thus $\hat{Q}_N$) defined through its action on states of the path basis it seems natural to ask how it acts on simple spin states, i.e. momentum states built from a spin configuration. Generically, the path states are rather complicated superpositions thereof, and hence we have to find the transformation relating the two bases. In order to work it out, we must address the question of incompleteness of the path basis for chains of odd length, pointed out in Sect. 3.2. In fact, for odd N we have to *define* the action of $\hat{Q}_{N-1}^{\dagger}$ on the missing two states which we denote by $|\Psi_{\pm}\rangle$.

Let us first state a simple observation: as the $|\Psi_{\pm}\rangle$ are not in the path basis, they cannot be obtained through the action of $\hat{Q}_N^{\dagger}$ on any state in the Hilbert space $\mathcal{H}_{N+1}$ for the chain with $N+1$ sites. Second, we extend the definition of $\hat{Q}_{N-1}^{\dagger}$ in the most natural way: $\hat{Q}_{N-1}^{\dagger}|\Psi_{\pm}\rangle = 0$. Of course, the same reasoning applies to the operator $\hat{Q}_N$ itself, and thus we have

$$\hat{Q}_N|\Psi_{\pm}\rangle = 0, \quad \text{and} \quad |\Psi_{\pm}\rangle \neq \hat{Q}_{N-1}|\phi\rangle \quad \text{for all } |\phi\rangle \in \mathcal{H}_{N-1}. \tag{37}$$

Notice that this provides a consistent extension of the nilpotency property $\hat{Q}_{N+1}\hat{Q}_N = 0$. In a more mathematical parlance, we extend thus the definition of $\hat{Q}_N$ in such a way that the missing states are closed, but not exact with respect to the operators $\hat{Q}_N$.

Given this extension, the next steps are to construct the relation between the path basis and the spin basis, and to understand the action of the operator $\hat{Q}_N$ on simple spin states. We have not found a systematic construction for generic N. It seems that this is related to quite non-trivial identities between Jacobi theta functions, as suggested by the most simple example $\hat{Q}_2$. We work out this special case in Appendix B. It suggests the following

Conjecture 3 *For any N the operator $\hat{Q}_N$ is a linear combination of the supercharges Q_N and $\widetilde{Q}_N$, defined in Sect. 2, with coefficients depending on the free parameters s and t. The latter can be fine-tuned in order to make one of the coefficients vanish. As a generalisation of (36) we have the intertwining relation*

$$\boldsymbol{\mathcal{T}}_N(u)Q_{N-1} + h(u)Q_{N-1}\boldsymbol{\mathcal{T}}_{N-1}(u) = 0$$

(*and a similar relation for $\widetilde{Q}_N$*) *on the momentum sectors with $t_N = (-1)^{N+1}$.*

It is easy to check that this equation is compatible with the commutation relations between the supercharges and the XYZ Hamiltonian (12). More importantly however, this conjecture—if true—has some interesting consequences for the ground states of the XYZ spin chain along the supersymmetric line. In fact, combining it with (37) we conclude that the two missing states $|\Psi_\pm\rangle$ correspond to non-trivial elements in the cohomology of the supercharge Q_N (or $\widetilde{Q}_N$), discussed previously. Therefore they are perfect candidates for the ground states of the XYZ Hamiltonian for chain with odd N. Indeed, we verified this conjecture up to $N = 9$ sites by checking that the ground states obtained through exact diagonalisation of the Hamiltonian are indeed orthogonal to all states in the path basis provided that we impose the relation between ζ and the elliptic nome q given in (19). Thus we are led to

Conjecture 4 *For odd N the subspace of vectors that are orthogonal to the path basis is two-dimensional. It is spanned by the two zero-energy ground states of the XYZ-Hamiltonian* (1) *which are invariant under translation.*

This is perhaps the most surprising outcome of our analysis because it suggests that for $\eta = \pi/3$ the ground states at odd N cannot directly be obtained from Baxter's Bethe ansatz. Notice that this observation is different from the widely discussed question of the completeness of the Bethe ansatz (see e.g. [7]) as here the Bethe ansatz (as it stands) does not apply to the missing states.

These two states are thus eigenstates of the eight-vertex transfer matrix with eigenvalue $\mathcal{T}_N(u) = \vartheta_1(u,q)^N$ as was conjectured by Stroganov [54, 55]. The conjecture was extended to the inhomogeneous eight-vertex model, defined by allowing on any site j a shift of the spectral parameter $u \to u - u_j$. The conjectured eigenvalue is [48]

$$\mathcal{T}_N(u) = \vartheta_1(u-u_1,q)\vartheta_1(u-u_2,q)\cdots\vartheta_1(u-u_N,q). \tag{38}$$

The simple product structure of this eigenvalue suggests that there is a local mechanism leading to its existence. Here we seek to extend our Conjecture 4 to the inhomogeneous setting. The construction of the path basis parallels the homogeneous case, with a slight modification of the local vectors (21). For site j they become

$$|\Phi^{(j)}_{\ell,\ell+1}\rangle = \vartheta_1\big(s+(2\ell+1)\eta+u_j,q^2\big)|+\rangle + \vartheta_4\big(s+(2\ell+1)\eta+u_j,q^2\big)|-\rangle,$$
$$|\Phi^{(j)}_{\ell+1,\ell}\rangle = \vartheta_1\big(t+(2\ell+1)\eta-u_j,q^2\big)|+\rangle + \vartheta_4\big(t+(2\ell+1)\eta-u_j,q^2\big)|-\rangle,$$

where $|\pm\rangle$ are the local spin-1/2 basis vectors. Using this, we checked numerically for $N = 3$, 5 and 7 sites, and random choices for the spectral parameters u_j the following

Conjecture 5 *For odd N the subspace of vectors that are orthogonal to the inhomogeneous path basis is two-dimensional. It is spanned by the two eigenstates of the inhomogeneous transfer matrix with eigenvalue* (38).

4 Relation to Lattice Fermions with Hard-Core Exclusion

In this section, we present new observations about the connection between the XYZ model along the supersymmetric line and the staggered supersymmetric fermion chains with nearest-neighbour exclusion considered in [29, 30] (see also [38]). We provide a first step to construct a mapping between the models, based on the path description of the states in Baxter's Bethe ansatz for the eight-vertex model. Such a mapping was relatively straightforward to obtain for the XXZ case [31, 61], because in both cases there is a conserved $U(1)$ symmetry, and have closely related Bethe equations. Here there remains a $U(1)$ symmetry in the fermion model, but there is no such manifest symmetry in the XYZ chain. Nevertheless, there appears to be a close relation between the spectra in the two cases. Moreover, we explain how there is evidence of the hard-fermion structure in the path basis of the eight-vertex model by exploiting the mod 3-periodicity of the heights in the path description.

4.1 Conjectures Relating the Spectra

Let us recall the model of [31, 32], describing spinless fermions on a periodic one-dimensional lattice with $N_{(f)}$ sites. The fermions are subject to the constraint that no two adjacent sites are both occupied. Defining the ordinary fermion creation and annihilation operators to be $c_j^\dagger$ and c_j with $\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$ and $\{c_i, c_j^\dagger\} = \delta_{ij}$, the constraint amounts to restricting the usual fermionic Hilbert space to states annihilated by $n_j n_{j+1}$, where $n_j = c_j^\dagger c_j$ is the fermion number operator. Fermions respecting this constraint are annihilated and created by the operators $d_j = (1 - n_{j-1})c_j(1 - n_{j+1})$ and $d_j^\dagger = (1 - n_{j-1})c_j^\dagger(1 - n_{j+1})$. The model has explicit $\mathcal{N} = 2$ supersymmetry: it is built from a supercharge

$$Q_{(f)} = \sum_{j=1}^{N_{(f)}} \lambda_j d_j^\dagger,$$

where the λ_j are non-zero real coupling constants (possible phases may be removed by simple gauge transformations). The Hamiltonian is given as anticommutator $H_{(f)} = \{Q_{(f)}, Q_{(f)}^\dagger\}$. For periodic boundary conditions $d_j = d_{j+N_{(f)}}$ on the fermions (known as *Ramond boundary conditions*), the Hamiltonian is

$$H_{(f)} = \sum_{j=1}^{N_{(f)}} \lambda_j \lambda_{j+1} \big(d_{j+1}^\dagger d_j + d_j^\dagger d_{j+1}\big) + \sum_{j=1}^{N_{(f)}} \lambda_j^2 (1 - n_{j-1})(1 - n_{j+1}).$$

This thus includes a hopping term, a chemical potential, and a next-to-nearest neighbour repulsion. Notice that unlike in the XYZ chain where the magnetisation is not conserved, this Hamiltonian conserves fermion number for any values of the λ_j.

Following [29, 30] we now consider the case where length of the fermion chain is a multiple of three, and the coupling constants are staggered with period three. Then the problem is invariant under translation by three sites: if $T_{(f)}$ is the translation operator on the fermion chain then we have $[H_{(f)}, T_{(f)}^3] = 0$. In this case, one can show that for $N_{(f)} = 3m$ the model has exactly two zero-energy ground states with m fermions in the "momentum sector" where $T_{(f)}^3 \equiv (-1)^{m+1}$. The precise form of these ground states depends on the values of λ_1, λ_2, λ_3. The most general case is analysed in [14]. Here we describe the choice $\lambda_1 = y$, $\lambda_2 = 1$, $\lambda_3 = y$ for some real y. This was the case studied in [30], where we conjectured that after the change of variable

$$\zeta^2 = 1 + 8y^2 \tag{39}$$

the two zero-energy ground states of the fermion chain at $N_{(f)}=3m$, and the two zero-energy ground states of the XYZ spin chain with $N=2m+1$ sites share some components which are polynomials in ζ and are related to a tau-function hierarchy associated with the Painlevé VI equation [8, 9, 43].

It is natural to ask if the relation between the two models is deeper. Indeed, if we rewrite the spectrum of the fermion chain in terms of the variable ζ by using (39) then a number of eigenvalues in the spectra of the XYZ Hamiltonian at $N=2m$ and $N=2m+1$ coincide *exactly* with eigenvalues of the fermion chain Hamiltonian $4H_{(f)}$ (the factor 4 is just an issue of normalisation). This statement can be sharpened by analysing different momentum sectors. As an example, we provide the characteristic polynomial $\det(E-H_N)$ for the XYZ Hamiltonian with $N=4$ sites in the subsector with momentum $k=\pi$:

$$\big(E-(\zeta^2+3)\big)\big(E-(\zeta^2+2\zeta+5)\big)\big(E-(\zeta^2-2\zeta+5)\big)\big(E-2(\zeta^2+1)\big). \tag{40}$$

The characteristic polynomial $\det(\epsilon-H_{(f)})$ for the fermion model at $N_{(f)}=6$ sites in the subsector with $m=2$ particles and $T^3_{(f)}=-1$ is given by:

$$\epsilon^2\big(\epsilon-(1+4y^2)\big)\big(\epsilon-(1+2y^2)\big)\big(\epsilon^2-(3+4y^2)\epsilon+2(1+2y^2+2y^4)\big). \tag{41}$$

If we set $\epsilon=E/4$ and use the change of variables (39), then (41) coincides with (40) up to the factor E^2 and an unimportant global numerical factor. Hence we see that upon the change of variables, the spectra coincide with the exception that the zero-energy states are absent in the XYZ spectrum. This coincidence of the XYZ spectrum at $N=2m$ on the sector with momentum π, and the fermion model at $N_{(f)}=3m$ on the sectors with $T^3_{(f)}=(-1)^{m+1}$ appears to be systematic for small m, but different multiplicities of various eigenvalues occur for $m\geq 4$. Studying the spectra up to $m=6$, we are led to the following conjecture:

Conjecture 6 *The spectrum of the XYZ Hamiltonian H_N for $N=2m$ sites in the sector with momentum π coincides with the spectrum of the staggered fermion chain $4H_{(f)}$ with $N_{(f)}=3m$ sites in the sector where $T^3_{(f)}=(-1)^{m+1}$ if variables are changed according to* (39), *with two exceptions*: (i) *the eigenvalue $E=0$ is missing in the XYZ spectrum and* (ii) *the two models lead to different multiplicities of the eigenvalues.*

This conjecture identifies sectors of the two models where their supersymmetries are exactly realised. Yet, it appears that the connection is even deeper. We analysed the relations between the models for antiperiodic or *Neveu-Schwarz boundary conditions* $d_{j+N_{(f)}}=-d_j$ on the fermions. In this case, the supersymmetry of the fermion model is broken. The spectrum needs no longer be positive, and indeed the ground state has negative energy. These boundary conditions are equivalent to a twist in the Hamiltonian, leading to the term $-\lambda_{N_{(f)}}\lambda_1(d_1^\dagger d_{N_{(f)}}+d_{N_{(f)}}^\dagger d_1)$. This sector is unlikely to share properties with the momentum sectors discussed so far in this paper because they have explicit unbroken supersymmetry. We found however coincidence with the spectrum of the XYZ chain of even length $N=2m$ and *zero* momentum. We illustrate it once again by showing the explicit characteristic polynomials for $m=2$. The XYZ Hamiltonian for $N=4$ sites, and momentum $k=0$ has the characteristic polynomial

$$\begin{aligned}&(E-4)\big(E-(\zeta-1)^2\big)\big(E-(\zeta+1)^2\big)\\&\quad\times\big(E^3-3E^2(\zeta^2+1)+2E(\zeta^2+3)^2+8(\zeta^2-1)^2\big).\end{aligned}$$

The characteristic polynomial of the fermion model at $N_{(f)}=6$ with Neveu-Schwarz boundary conditions restricted to the sector $T^3_{(f)}=1$ is given by

$$(\epsilon-1)\big(\epsilon^3-3\epsilon^2(2y^2+1)+2\epsilon(2y^2+1)^2+8y^4\big)\big(\epsilon^2-\epsilon(4y^2+1)+4y^4\big).$$

Again, if we set $\epsilon = E/4$ and perform the change of variables (39) we find that the two polynomials coincide up to some unimportant numerical factor. Studying small systems up to $m = 6$ we are led to the

Conjecture 7 *The spectrum of the XYZ Hamiltonian for $N = 2m$ sites in the sector with zero momentum coincides with the spectrum of $4H_{(f)}$ for the twisted staggered fermion chain with $N_{(f)} = 3m$ sites in the sector where $T^3_{(f)} = (-1)^m$ after changing variables according to* (39). *The multiplicities of the eigenvalues in the two models are different.*

4.2 A Mapping to Hard-Particle Configurations

Conjectures 6 and 7, relating the spectra of the XYZ chain and the staggered fermion model, raise naturally the question if there is a mapping between the models, at least in some subsectors. At the XXZ point $\zeta = 0$ such a mapping was discussed in [31]: the fermion chain with $N_{(f)}$ sites and m fermions is equivalent to the twisted spin chain with $N = N_{(f)} - m$ sites and m spins down, the twist being the eigenvalue of the translation operator in the fermion model. If we represent an occupied site on the fermion chain by $\bullet$ and an empty site by $\circ$ then the correspondence between fermions and spins is given by

$$\rangle\!\!-\!\bullet\!-\!\!\langle \quad \leftrightarrow \quad - \qquad \text{and} \qquad \rangle\!\!-\!\!\langle \quad \leftrightarrow \quad +$$

This mapping has no direct generalisation to the off-critical case. The reason is the absence of conservation of the number of down spins in the general XYZ chain as opposed to the particle conservation in the staggered fermion chain. However, the path basis was designed to implement particle conservation (the number of down steps). Thus, we focus on the path states, and try to conceive a mapping between them and the fermion model.

Let us consider a typical path starting at height $\ell_1 = \ell$ and terminating at some $\ell_{N+1} = \ell + 3p$ for fixed integer p. As before, let m be the number of decreasing steps. Recall that adjacent heights obey $\ell_{j+1} - \ell_j = \pm 1$. However, notice that at $\eta = \pi/3$ we may shift any local height variable by a multiple of three without changing the corresponding state, as can be seen from the vectors (21). Thus, instead of a decreasing step $\ell_{j+1} = \ell_j - 1$ we can modify the path locally according to $\ell_{j+1} = \ell_j + 2$ as shown in Fig. 5(a). This motivates the following construction: given a path we replace each decreasing step by a step of two units up, and then continue with usual, appropriately shifted steps up. This procedure yields a new, monotone increasing path from height ℓ to height $\ell + N + m + 1$ as illustrated in Fig. 6(a). Next, we associate to the each of the two types of steps particle configurations along the vertical axis according to the rules display in Fig. 5(b). This is quite reminiscent of the correspondence in the critical case. Thus, we obtain from a path a particle configuration with $N_{(f)} = N + m$ sites and m particles (see Fig. 6(a) for illustration), with the hard-core rule that particles cannot be adjacent to each other, just like in the fermionic case. Notice in particular that because of the condition $N = 2m + 3p$, the length of the particle chain $N_{(f)} = 3(m + p)$ is always a multiple of three.

For fixed ℓ, the position of the particles $\bullet$ on the vertical axis are given as

$$y_j = x_j + \ell + j, \quad j = 1, \ldots, m.$$

We fix the origin at $\ell = y = 0$, and consider periodic boundary conditions, so the y_j's are considered mod $N_{(f)}$.

The mapping between paths and particle configurations is not one-to-one. We illustrate this by analysing the effect of a translation on the path configuration by one step to the

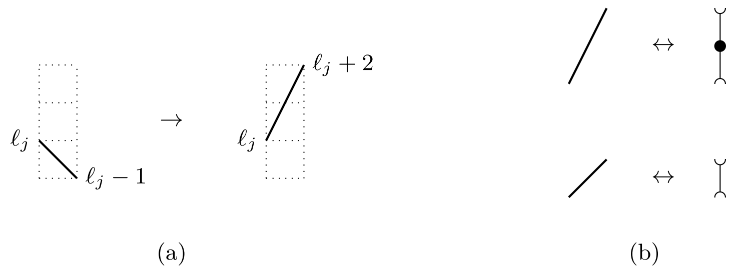


Fig. 5 (**a**) Local modification of the path. (**b**) Correspondence between path steps and particle configurations

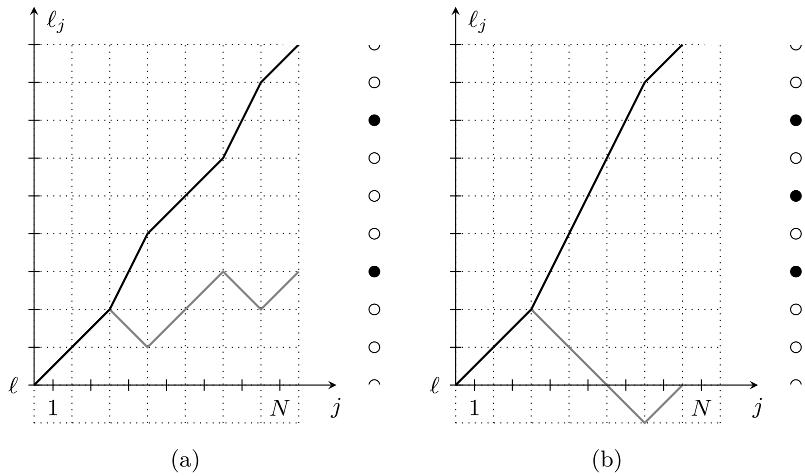


Fig. 6 (**a**) Mapping from path configuration to a particle configuration with hard-core exclusion. (**b**) Insertion of a particle through local operation of the supercharges for the XYZ chain

right. The last step of the path is simply removed, and glued to the first one. However, we would like to respect the rule that the initial height of the new path is 0, 1 or 2. Thus, a vertical shift of all heights by ± 3 units might be necessary, and lead to a different particle configuration. There are multiple cases. (i) If the last step of the given path goes up, and $\ell = 1, 2$ the translation has no effect on the configuration of hard-particles. (ii) However, if $\ell = 0$ the path has to be shifted by three units, and thus the particle positions are cyclically translated by 3: $y_j \rightarrow y_j + 3 \bmod N_{(f)}$. (iii) If the last step of the initial path is decreasing, then a translation of the path leaves the particle configuration unchanged if $\ell = 2$. (iv) For $\ell = 0, 1$ however, the positions in the particle configuration constructed from the translated path are shifted according to $y_j \rightarrow y_j + 3 \bmod N_{(f)}$. For given N and paths with m steps down, the number of hard-particle configurations obtained through the mapping is obtained by counting the paths corresponding to cases (ii) and (iv). This yields

$$\binom{N-1}{m} + 2\binom{N-1}{m-1} = \frac{N_{(f)}}{N_{(f)} - m}\binom{N_{(f)} - m}{m},$$

where we used $N_{(f)} = N + m$. A little combinatorics shows that this is the number of possible hard-particle configurations for $N_{(f)}$ sites and m particles. Also, we see that translation of the path configurations is related to translation of the hard-particle configurations along the vertical axis by three steps.

Next, we would like to understand the nature of these particles by examining the local action of the XYZ supercharges, explained in Sect. 3.3, on the particle configurations. Recall that the action on the path states corresponds locally to transforming two up steps into a single down step. Given our rules identifying paths with particle configurations, it is not difficult to see that this corresponds to insertion of a particle while respecting the nearest-neighbour exclusion rule (see Fig. 6(b)). Recall that this comes with a weight $(-1)^x h(w_{\ell_{x+1}})^2$ in terms of the positions for the path. The corresponding position of insertion in the hard-particle state is $y = x + \ell + j$ where j is the number of particles on sites $0, 1, \ldots, y-2$. Hence, the weight becomes $(-1)^{j+y-\ell} h(w_y)^2$ as follows from (25) and the periodicity of $h(u)$. The factor $(-1)^j$ in the string suggests that the hard particles are indeed fermions. Furthermore it is tempting to use this in order to identify the coupling constants λ_y. This requires taking into account a systematic identification of the hard-particle states in terms of the path states (what is delicate as the proposed mapping is not one-to-one), changes of normalisations through the supersymmetry operation, and finally the restriction to the special momentum spaces for both models. While we are not in a position to carry out this program, we nevertheless put forward the following

Conjecture 8 *The coupling constants of the corresponding fermion model are given by*

$$\lambda_y = \left|\vartheta_1(w_y, q)\right|^{3/2},$$

where w_y is the linear function defined in (25).

We see that these coupling constants depend only on the combination $s+t$. The evidence for this conjecture is that this parametrisation of the coupling constants uniformises a family of elliptic curves appearing in the coordinate direct coordinate Bethe ansatz for the fermion chain [14]. In particular, it implies that upon appropriate rescaling the eigenvalues of the fermion chain do not depend on $s+t$.

5 Conclusion

We have studied the XYZ chain and the eight-vertex model along the supersymmetric line, and showed that it possesses an $\mathcal{N} = (2,2)$ supersymmetry on the lattice. A consequence is that chains of different length have common positive energy levels in certain momentum sectors, which are organised into supersymmetry quadruplets. Moreover, we presented a derivation of the supersymmetry by means of the Bethe ansatz for the eight-vertex model, and showed that the supercharges perform simple local operations on the path basis. This analysis led us to a novel characterisation for the ground states of the XYZ chain with odd length. Finally, we reported some observations that the XYZ chain along the supersymmetric line and the staggered supersymmetric fermion chains with nearest-neighbour exclusion have exact common eigenvalues in certain subsectors.

There are many open questions and extensions. To us, it seems most interesting to clarify further the nature of the ground states for the chains of odd length. We hope that the supersymmetry will be helpful, for instance to prove that there are exactly two zero-energy ground states. A central tool in supersymmetric theories is the Witten index $\mathrm{tr}(-1)^F$ [59]: it provides a lower bound on the number of zero-energy states. Indeed, it would be interesting to define this quantity or find at least a suitable analogue for the present theory. As the fermion number coincides with the number of sites, the formal generalisation leads to a trace which runs over an infinite collection of Hilbert spaces, what one would have to make sense of. Similar considerations apply to the index $\mathrm{tr}((-1)^F F e^{-\beta H})$ defined in [16]. A possible way to resolve these problems might be to establish a more complete mapping between the XYZ chain and the staggered fermion chain. For the latter, there exists a standard procedure to find the Witten index, and determine the exact number of ground states using cohomology arguments (see e.g. [39]). Further insights into the structural properties of the ground states will certainly be obtained by considering the inhomogeneous eight-vertex model, as was the case in the trigonometric limit [20–23]. Almost all developments in this work considered the homogeneous version, and the supersymmetry appears to be intimately related to translation invariance. It would be interesting to see if (and how) this symmetry persists in the inhomogeneous case.

Finally, let us point out that the supersymmetry presented in this article is a particular feature of the $\eta = \pi/3$ model. It is natural to ask for an extension to general roots-of-unity points such as $\eta = \pi/(k+2)$ with $k = 1, 2, 3, \ldots$. The case $k = 2$ was already addressed in [31] from the point of view of fermions with generalised exclusion rules. This allowed the identification of a supersymmetric point for the Fateev-Zamolodchikov integrable spin-1 chain. For more general trigonometric models, the points $\eta = \pi/(k+2)$ were identified as the combinatorial points for fused spin-$k/2$ models [62], as anticipated in [24] (see also [51]). Indeed, using the works [40, 57] we can show that these coincide precisely with the cases where a lattice supersymmetry is present. This generalisation will be addressed in a forthcoming publication [34].

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Appendix A: Properties of the Supercharges

In this appendix we present some technical details about the properties of the supercharges introduced in Sect. 2. In the first part, we prove the nilpotency property, and in the second part, we show that their anticommutator generates the XYZ Hamiltonian.

A.1 Nilpotency

Let us prove that the operators Q_N "have square zero" in the sense that

$$Q_{N+1} Q_N = 0.$$

To this end, we need a set of anticommutation rules for the local operators q_j defined in the main text (7). We have the rule

$$q_i q_j + q_{j+1} q_i = 0, \quad 1 \le i < j \le N. \tag{42}$$

This can be shown along the lines of [61], and therefore we only sketch the proof for another relation involving q_0. Let us first consider $q_0 q_j$. We find that its action non-zero only on states having spins $-$ at position j and N. We find

$$q_0 q_j|\cdots \underset{j}{-}\cdots\underset{N}{-}\rangle = (-1)^j\big(|+\cdots\underset{j+1}{+}+\cdots\underset{N+2}{+}+\rangle - \zeta|+\cdots\underset{j+1}{-}-\cdots\underset{N+2}{+}\rangle$$
$$-\zeta|-\cdots\underset{j+1}{+}+\cdots\underset{N}{-}\rangle + \zeta^2|-\cdots\underset{j+1}{-}-\cdots\underset{N}{-}\rangle\big).$$

Reversing the order of the q's, we have to take into account the shift and therefore consider $q_{j+1}q_0$. Its action yields

$$q_{j+1}q_0|\cdots\underset{j}{-}\cdots\underset{N}{-}\rangle = (-1)^{j+1}\big(|+\cdots\underset{j+1}{+}+\cdots\underset{N}{+}\rangle - \zeta|+\cdots\underset{j+1}{-}-\cdots\underset{N}{+}\rangle$$
$$-\zeta|1\cdots\underset{j+1}{+}+\cdots\underset{N}{-}\rangle + \zeta^2|-\cdots\underset{j+1}{-}-\cdots\underset{N}{-}\rangle\big).$$

We see that this result coincides with the previous one, except for a minus sign. Combining these two equations, we find therefore

$$q_0 q_j + q_{j+1}q_0 = 0, \quad j = 1,\ldots,N-1 \tag{43}$$

when acting on $\mathcal{H}_N$.

These relations are useful in order to prove that the supercharges are nilpotent in the sense stated above. In a first step, we observe that the (42) and (43) can be used to reduce the product of the supercharges to

$$Q_{N+1}Q_N = \left(\frac{N}{N+2}\right)^{1/2}\left(\sum_{j=0}^{N}(q_{j+1}q_j + q_j^2) + q_0 q_N + q_{N+1}q_0\right).$$

Let us examine the different terms in this sum. The individual terms are non-vanishing only if they act on the following states in $\mathcal{H}_N$:

$$\begin{aligned}(q_{j+1}q_j + q_j^2)|\cdots\underset{j}{-}\cdots\rangle &= \zeta\big(|\cdots\underset{j}{-}+\underset{j+2}{+}\cdots\rangle - |\cdots\underset{j}{+}+\underset{j+2}{-}\cdots\rangle\big), \quad j = 1,\ldots,N,\\ (q_1q_0 + q_0^2)|\cdots\underset{N}{-}\rangle &= \zeta\big(|\underset{1}{+}+\cdots\underset{N+2}{-}\rangle - |\underset{1}{+}-\cdots\underset{N+2}{+}\rangle\big),\\ (q_0q_N + q_{N+1}q_0)|\cdots\underset{N}{-}\rangle &= (-1)^{N+1}\zeta\big(|\underset{1}{+}\cdots\underset{N+1}{-}+\rangle - |\underset{1}{-}\cdots\underset{N+1}{+}+\rangle\big).\end{aligned} \tag{44}$$

As all the expressions are proportional to ζ we see that in the XXZ limit $\zeta = 0$ the relation $Q_{N+1}Q_N = 0$ is immediate. Actually, it would not even be necessary to impose the restriction to certain momentum spaces in this case. However, for general $\zeta \neq 0$ the relation only survives on the special momentum sectors. Intuitively, this can be seen as follows: we see that the operations defined in (44) insert pairs $++$ to left and right of a spin $-$, thus we expect that the summation of these on a periodic chain will lead to telescopic cancellations. This will however only work in a momentum sector compatible with the sign in the third expression in (44) which coincides with the eigenvalue of the translation operator. More concretely, let us consider the case of $Q_{N+1}Q_N$ on a momentum state $|\psi_\alpha\rangle$ built from spin configuration $\alpha = \alpha_1\alpha_2\cdots\alpha_N$, that is

$$|\psi_\alpha\rangle = \sum_{j=0}^{N} t_N^j T_N^{-j}|\alpha\rangle, \quad t_N = (-1)^{N+1}.$$

We leave aside the issue of normalisation. Let us suppose that α has m spins $-$ and denote their positions by $x_1,\ldots,x_m$. Using the rules defined in (44) we can write

$$C\zeta^{-1}Q_{N+1}Q_N|\psi_\alpha\rangle = \sum_{j=0}^{N-1} t_N^j\big(T_N^{-j}|\alpha\rangle\otimes|++\rangle - |++\rangle\otimes T_N^{-j}|\alpha\rangle\big)$$

$$+(-1)^{N+1}\sum_{\ell=1}^{m} t_N^{x_\ell}\big(|+\rangle\otimes T_N^{-x_\ell}|\alpha\rangle\otimes|+\rangle - T_N^{1-x_\ell}|\alpha\rangle\otimes|{+}{+}\rangle\big)$$
$$+\sum_{\ell=1}^{m} t_N^{x_\ell}\big(|{+}{+}\rangle\otimes T_N^{-x_\ell}|\alpha\rangle - |+\rangle\otimes T_N^{1-x_\ell}|\alpha\rangle\otimes|+\rangle\big),$$

where $C=\sqrt{(N+2)/N}$. Because of $t_N=(-1)^{N+1}$ this can be written in terms of the positions $y_1,\dots,y_{N-m}$ of the individual spins $+$ in α. We find a simplified expression:

$$C\zeta^{-1}Q_{N+1}Q_N|\psi_\alpha\rangle = (-1)^N\sum_{\ell=1}^{N-m} t_N^{y_\ell}\big(|+\rangle\otimes T_N^{-y_\ell}|\alpha\rangle\otimes|+\rangle - T_N^{1-y_\ell}|\alpha\rangle\otimes|{+}{+}\rangle\big)$$
$$-\sum_{\ell=1}^{N-m} t_N^{y_\ell}\big(|{+}{+}\rangle\otimes T_N^{-y_\ell}|\alpha\rangle - |+\rangle\otimes T_N^{1-y_\ell}|\alpha\rangle\otimes|+\rangle\big).$$

Now observe that if the configuration α has a spin $+$ at position y_i we can write

$$|+\rangle\otimes T_N^{-y_i}|\alpha\rangle = T_N^{1-y_i}|\alpha\rangle\otimes|+\rangle.$$

Using this in the preceding formula we conclude that all terms cancel mutually. This proves the statement $Q_{N+1}Q_N=0$.

A.2 The Hamiltonian as an Anticommutator

In this appendix we show in detail that if we restrict the Hamiltonian to subsectors where the eigenvalue of the translation operator is $t_N=(-1)^{N+1}$ then it can be written as "anti-commutator"

$$H_N = Q_N^\dagger Q_N + Q_{N-1}Q_{N-1}^\dagger. \tag{45}$$

First, it is useful to introduce the projector on the momentum spaces that we are interested in. It is given by

$$\Pi_N = \frac{1}{N}\sum_{j=0}^{N-1} t_N^j T_N^{-j} = \frac{1}{N}\sum_{j=0}^{N-1}(-1)^{(N+1)j}T_N^{-j}.$$

From the definition of the supercharges (7) and the translation properties established in (8), we conclude that $Q_N\Pi_N=\Pi_{N+1}Q_N=Q_N$. Using this, we conclude that

$$Q_N^\dagger Q_N = N(N+1)\Pi_N q_1^\dagger \Pi_{N+1} q_1 \Pi_N,$$
$$Q_{N-1}Q_{N-1}^\dagger = N(N-1)\Pi_N q_1 \Pi_{N-1} q_1^\dagger \Pi_N.$$

Second, using the definition of the projector, and again the rules (8), we find that

$$(N+1)\Pi_N q_1^\dagger \Pi_{N+1} q_1 \Pi_N = \Pi_N\left(\sum_{j=0}^{N} q_j^\dagger q_1\right)\Pi_N,$$
$$(N-1)\Pi_N q_1 \Pi_{N-1} q_1^\dagger \Pi_N = \Pi_N\left(\sum_{j=1}^{N-1} q_1 q_j^\dagger\right)\Pi_N.$$

We reduce in a third step the sum of these expressions through an application of the following anticommutation relations

$$q_i q_j^\dagger + q_{j+1}^\dagger q_i = 0, \quad 1 \le i < j-1 \le N-1,$$
$$q_0 q_j^\dagger + q_{j+1}^\dagger q_0 = 0, \quad 2 \le i \le N-1.$$

These can be derived in a similar way as the relations (42) and (43). After some algebra, we are left with

$$\Pi_N H_N \Pi_N^{-1} = N \Pi_N \big(q_1^\dagger q_1 + q_1 q_1^\dagger + q_2^\dagger q_1 + q_0^\dagger q_1\big) \Pi_N^{-1}.$$

The remaining quadratic terms can be expressed through simple spin operators. We find

$$q_j q_j^\dagger = \frac{1}{4}\big((1+\sigma_j^z)(1+\sigma_{j+1}^z) + \zeta^2(1-\sigma_j^z)(1-\sigma_{j+1}^z)\big) - \zeta\big(\sigma_j^+\sigma_{j+1}^+ + \sigma_j^-\sigma_{j+1}^-\big),$$
$$q_j^\dagger q_j = \frac{1}{2}(1+\zeta^2)(1-\sigma_j^z),$$
$$q_{j+1}^\dagger q_j = -\frac{\zeta^2}{4}(1-\sigma_j^z)(1-\sigma_{j+1}^z) - \sigma_j^+\sigma_{j+1}^-,$$

and $q_0^\dagger q_1 = T_N^{-1} q_1^\dagger q_2 T_N$. Using these relations and again translation invariance, we conclude that

$$\Pi_N H_N \Pi_N^{-1} = -N \Pi_N \big(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+ + \zeta\big(\sigma_1^+\sigma_2^+ + \sigma_1^-\sigma_2^-\big)\big)\Pi_N - N\Pi_N\left(\left(\frac{\zeta^2-1}{4}\right)\sigma_1^z\sigma_2^z - \frac{3+\zeta^2}{4}\right)\Pi_N.$$

The expression on the right-hand side is nothing but the restriction of the XYZ-Hamiltonian (1), (2) to the momentum sectors with $t_N = (-1)^{N+1}$, what proves the statement.

Appendix B: Reduction from $N = 3$ to $N = 2$ Sites

In this appendix we show that the supercharge in the path basis can be written as linear superposition of the supercharges defined in Sect. 2.2 for the most simple case of three and two sites.

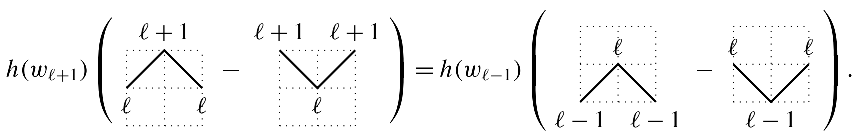
There is a single π-momentum state for $N = 2$ sites. In the canonical basis it is given by (we do not normalise the states):

$$|\phi\rangle = |{+}{-}\rangle - |{-}{+}\rangle.$$

For $N = 3$ sites, there are four possible states which are invariant under translation

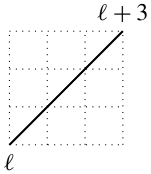
$$|\psi_1\rangle = |{+}{+}{+}\rangle, \qquad |\psi_2\rangle = |{-}{+}{+}\rangle + |{+}{-}{+}\rangle + |{+}{+}{-}\rangle,$$
$$|\psi_3\rangle = |{+}{-}{-}\rangle + |{-}{+}{-}\rangle + |{-}{-}{+}\rangle, \qquad |\psi_4\rangle = |{-}{-}{-}\rangle.$$

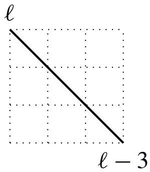
Let us now turn to the path basis. For even N it is redundant. Indeed, for $N = 2$ there are $\nu(2) = 6$ admissible paths but the Hilbert space has dimension $d = 2^N = 4$. Indeed, one verifies that the different states are related through the identity



In some sense, the relation is trivial here because the difference of the path states on both sides is proportional to the singlet state $|\phi\rangle$, and equality of the proportionality factors on both sides is readily verified.

For $N=3$ there are two states in the path basis which are invariant under translation. Up to factors, they are given by

$|\chi_1\rangle = \sum_{\ell=0}^{2}$ 

, $|\chi_2\rangle = \sum_{\ell=0}^{2}$ 

.

As mentioned before, the path basis for odd N is incomplete. In this concrete example, we see that the two states that are missing have to be invariant under translation. The requirement that they are orthogonal to all path states determines them up to linear combinations and normalisations. We find it convenient to choose

$$|\chi_3\rangle = \zeta|\psi_1\rangle + |\psi_3\rangle, \qquad |\chi_4\rangle = \zeta|\psi_4\rangle + |\psi_2\rangle, \tag{46}$$

where we used the coordinate basis.

In order to find the action on $\hat{Q}_2^\dagger$ on the vectors $|\psi_j\rangle$ we write simply decompose the path basis and the two missing states for three sites in terms of the spin basis according to $|\chi_i\rangle = \sum_{j=1}^{4} A_{ij}|\psi_j\rangle$. Hence

$$\hat{Q}_2^\dagger|\chi_i\rangle = \sum_{j=1}^{4} A_{ij}\,\hat{Q}_2^\dagger|\psi_j\rangle = b_i|\phi\rangle,$$

where the b_i are constants (in this example the map is necessarily of rank 1). Hence

$$\hat{Q}_2^\dagger|\psi_i\rangle = \sum_{j=1}^{4}\left(A^{-1}\right)_{ij} b_j|\phi\rangle. \tag{47}$$

Thus we have to determine the matrix A involved in the change of basis and the vector b. For the former it is convenient to abbreviate $f_j(x) = \prod_{k=0}^{2}\vartheta_j(x+2\pi k/3, q^2)$. Then we find that

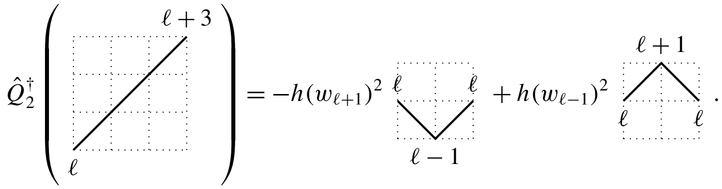
$$A = \begin{pmatrix} 3f_1(s) & -\zeta f_4(s) & -\zeta f_1(s) & 3f_4(s) \\ 3f_1(t) & -\zeta f_4(t) & -\zeta f_1(t) & 3f_4(t) \\ \zeta & 0 & 1 & 0 \\ 0 & 1 & 0 & \zeta \end{pmatrix},$$

where we used the theta function identity

$$\begin{aligned} &\vartheta_1(u,q^2)\left(\vartheta_4\left(u-\frac{\pi}{3},q^2\right)\vartheta_4\left(u+\frac{\pi}{3},q^2\right) + \zeta\,\vartheta_1\left(u-\frac{\pi}{3},q^2\right)\vartheta_1\left(u+\frac{\pi}{3},q^2\right)\right) \\ &\quad = \vartheta_4(u,q^2)\left(\vartheta_4\left(u-\frac{\pi}{3},q^2\right)\vartheta_1\left(u+\frac{\pi}{3},q^2\right) + \vartheta_1\left(u-\frac{\pi}{3},q^2\right)\vartheta_4\left(u+\frac{\pi}{3},q^2\right)\right) \end{aligned}$$

(notice that using the definition of ζ this turns out to be an identity involving products of *five* theta functions and therefore does not simply follow from Riemann's identity).

Next, let us determine the b_j. The only path state which is not annihilated by $\hat{Q}_2^\dagger$ is $|\chi_1\rangle$. The application of the local transformation rules defined in (34) and (35) leads to



Finally the summation over $\ell = 0, 1, 2$ then yields $b_1 = h((s-t)/2)\sum_{\ell=0}^{2} h(w_\ell)^3$. From the rules for the action of $\hat{Q}_2^\dagger$ it is evident that $b_2 = 0$. Not evident however are the values of b_3 and b_4. We follow the proposal made in the main text: the supercharges annihilate the two missing states at odd length. Hence we set $b_3 = b_4 = 0$. Then it is a simple matter to find the action of $\hat{Q}_2^\dagger$ on the spin states. After having computed the inverse A^{-1} of the coordinate transformation we find from (47)

$$\hat{Q}_2^\dagger = \text{const.} \times \big(f_4(t) Q_2^\dagger + f_1(t)\widetilde{Q}_2^\dagger\big),$$

with the functions $f_j(t)$ defined above. The overall factor is a function of s, t and q. The action of the operators $Q_2^\dagger$ and $\widetilde{Q}_2^\dagger$ on the spin states is

$$Q_2^\dagger|\psi_1\rangle = -\sqrt{\frac{3}{2}}|\phi\rangle, \qquad Q_2^\dagger|\psi_2\rangle = 0, \qquad Q_2^\dagger|\psi_3\rangle = \zeta\sqrt{\frac{3}{2}}|\phi\rangle, \qquad Q_2^\dagger|\psi_4\rangle = 0,$$

$$\widetilde{Q}_2^\dagger|\psi_1\rangle = 0, \qquad \widetilde{Q}_2^\dagger|\psi_2\rangle = -\zeta\sqrt{\frac{3}{2}}|\phi\rangle, \qquad \widetilde{Q}_2^\dagger|\psi_3\rangle = 0, \qquad \widetilde{Q}_2^\dagger|\psi_4\rangle = \sqrt{\frac{3}{2}}|\phi\rangle.$$

Thus, we see that $\widetilde{Q}_2^\dagger = R_2 Q_2^\dagger R_3$ where R_N is the spin-reversal operator introduced in Sect. 2.1. Find with thus the Hermitian conjugates of the supercharges constructed in Sect. 2.

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