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Bayesian Copulae Distributions, with Application to Operational Risk Management—Some Comments

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Abstract This paper points out mistakes in some results given in the paper "*Bayesian Copulae Distributions, with Application to Operational Risk Management*" by Luciana Dalla Valle, published in 2009 in volume 11, number 1 of "Methodology and Computing in Applied Probability". In particular, we explain why the inverse Wishart distribution is not a conjugate prior to the Gaussian copula.

Keywords Bayesian normal copula · Bayesian Student's t copula

AMS 2000 Subject Classification 62C10 · 62F15

1 Introduction

In this document we provide arguments supporting the claim that some results are incorrect in the paper Dalla Valle (2009) ("DV"). In particular, we show the claim that the inverse Wishart distribution is a conjugate prior to the Gaussian copula to be wrong.

We assume the reader to be familiar with

- The theory of copula functions. See Nelsen (2006) for an introduction to copulas and McNeil et al. (2005) for more details on elliptic distributions and elliptic copulas.
- Bayesian inference, in particular conjugate priors. See Lindley (1965) or Bernardo and Smith (1994) for an introduction.
- The original article DV. We will use the notation from DV throughout the paper.

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Section 2 gives a short introduction to copulas. Sections 3 and 4 lay out the mistakes in the Bayesian inference for Gaussian and t copulas, respectively. Section 5 describes a mistake in DV concerning the maximum likelihood estimator of the Gaussian copula. Section 6 concludes.

2 Copulas

Let $r \in \mathbb{N}$ denote the dimension of the random vectors under consideration.

Copulas are used to separate the dependence structure of a multivariate random vector from its margins. In fact, the joint distribution function

$$H(x_1, \ldots, x_r) = \mathbb{P}\left[X_1 \leq x_1, \ldots X_r \leq x_r\right]$$

of a random vector $\mathbf{X} \in \mathbb{R}^r$ can be written as

$$H(x_1, \ldots, x_r) = C\left(F_1(x_1), \ldots, F_r(x_r)\right) \quad \text{for all} \quad (x_1, \ldots, x_r)^T \in \mathbb{R}^r,$$

where $C : [0, 1]^r \to [0, 1]$ is a copula and F_i denotes the i-th marginal distribution functions. In case the F_i's are continuous, the copula is unique (by Sklar's Theorem, see Sklar 1959).

By turning this equation around, we can project any multivariate distribution to the unit square in order to recover the copula:

$$C(u_1, \ldots, u_r) = H\left(F_1^{-1}(u_1), \ldots, F_r^{-1}(u_r)\right),$$

where F_i^{-1} are the generalized inverse of F_i. In that way, we can obtain from every random vector of continuously distributed random variables the associated copula. For instance, if we project a multivariate normal distribution in $\mathbb{R}^r$ with mean $\mu \in \mathbb{R}^r$ and covariance matrix $\Sigma \in \mathbb{R}^{r\times r}$ to the unit cube $[0, 1]^r$ (by means of the equations above) we obtain a Gaussian copula.

3 A (Non-)Conjugate Prior to the Gaussian Copula

Let $\Sigma \in \mathbb{R}^{r\times r}$ be some correlation matrix, i.e. a symmetric positve definite (SPD) matrix with ones on the diagonal. It is easy to prove (and confirmed by Song 2000) that the multivariate normal $\mathcal{N}(\mathbf{0}, \Sigma)$ in $\mathbb{R}^r$ induces the Gaussian copula with density

$$c^{Ga}(\mathbf{u}|\Sigma) = \frac{1}{|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}\mathbf{x}^T\left(\Sigma^{-1} - I\right)\mathbf{x}\right),$$

where $|\cdot|$ denotes the determinant, $I \in \mathbb{R}^{r\times r}$ is the identity matrix, $x_i = \Phi^{-1}(u_i)$ for $i = 1, \ldots, r$ and $\Phi^{-1}(\cdot)$ is the inverse cdf of the standard normal distribution.

A copula is invariant under strictly increasing transformations of the margins: A positive scaling of the components of a multivariate normal changes the covariance structure but not the copula. Hence, the copula induced by the multivariate normal distribution $\mathcal{N}(\mu, \Sigma)$ (with Σ a covariance matrix, i.e., not necessarily a correlation matrix) depends only on the correlation matrix induced by Σ. Its copula density can be written as

$$c^{Ga}(\mathbf{u}|\Sigma) = \frac{1}{|\mathcal{P}(\Sigma)|^{1/2}} \exp\left(-\frac{1}{2}\mathbf{x}^T\left(\mathcal{P}(\Sigma)^{-1} - I\right)\mathbf{x}\right),$$

where $\mathcal{P} : \mathbb{R}^{r\times r} \to \mathbb{R}^{r\times r}$ is defined by

$$\mathcal{P}(\Sigma)_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}} \quad i, j = 1, \ldots, r.$$

Hence, the covariance matrix has to be projected to the corresponding correlation matrix in order to be plugged into the algebraic form of the copula density.

DV continues on p. 102 with assuming an inverse-Wishart distribution with parameter $\alpha > r - 1$ and a SPD matrix $B \in \mathbb{R}^{r\times r}$ for the prior π for the underlying covariance matrix:

$$\pi(\Sigma) \propto |\Sigma|^{\alpha+\frac{r+1}{2}} \exp\left(-\operatorname{tr}\left(B\Sigma^{-1}\right)\right).$$

Using the algebraic form of the density of the Gaussian copula, we obtain for the posterior of Σ, given the observations $\{\mathbf{u}_t\}_t$:

$$\pi(\Sigma|\mathbf{x}) \propto |\mathcal{P}(\Sigma)|^{-M/2} \exp\left(-\frac{1}{2}\sum_{t=1}^{M} \mathbf{x}_t^T \left(\mathcal{P}(\Sigma)^{-1} - I\right)\mathbf{x}_t\right) |\Sigma|^{\alpha+\frac{r+1}{2}} \exp\left(-\operatorname{tr}\left(B\Sigma^{-1}\right)\right),$$

where, as before, $(\mathbf{x}_t)_i = \Phi^{-1}((\mathbf{u}_t)_i)$

This result does not correspond to the result shown in DV on page 102, where no projection $\mathcal{P}(\Sigma)$ is used. However, this projection has to be applied as

$$\frac{1}{|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}\mathbf{x}^T\left(\Sigma^{-1} - I\right)\mathbf{x}\right)$$

is *not* a copula density if Σ is a SPD matrix with diagonal entries not being equal to one (I.e., if Σ is not a correlation matrix).

Thus, the posterior $\pi(\Sigma|\mathbf{x})$ is not inverse Wishart distributed.

4 Bayesian Inference for the t Copula

A similar issue exists in Eq. 4 and Section 4.2 in DV. The t copula density implied by the multivariate t distribution with covariance matrix Σ has density

$$c^t(\mathbf{u}|\Sigma, \nu) = (\text{terms in } \nu)|\mathcal{P}(\Sigma)|^{-1/2}\left(1 + \frac{1}{\nu}\mathbf{x}^T\mathcal{P}(\Sigma)^{-1}\mathbf{x}\right)^{-\frac{\nu+r}{2}} \prod_{i=1}^{r}\left(1 + \frac{x_i^2}{\nu}\right)^{\frac{\nu+1}{2}},$$

where $x_i = t_\nu^{-1}(u_i)$ for all $i = 1, \ldots, r$.

Hence, for a covariance matrix Σ which is not a correlation matrix (i.e., has not only ones on the diagonal), we have, in general,

$$c^t(\mathbf{u}|\Sigma, \nu) \neq (\text{terms in } \nu)|\Sigma|^{-1/2}\left(1 + \frac{1}{\nu}\mathbf{x}^T\Sigma^{-1}\mathbf{x}\right)^{-\frac{\nu+r}{2}} \prod_{i=1}^{r}\left(1 + \frac{x_i^2}{\nu}\right)^{\frac{\nu+1}{2}}.$$

The deduction of the equations above can be done analogously as in Section 3.

5 Maximum Likelihood of the Gaussian Copula

Equation 6 in DV states that the maximum likelihood estimator of the correlation matrix of a Gaussian copula is equal to

$$\widehat{\Sigma}_{ML} = \frac{1}{M} \sum_{t=1}^{M} \mathbf{x}_t^T \mathbf{x}_t,$$

where the $\mathbf{x}_t$ are defined as in Section 3.

However this is wrong, see pp. 234–235 in McNeil et al. (2005). The maximum likelihood estimate $\widehat{\Sigma}_{ML}$ of a Gaussian copula cannot be calculated analytically. Note that the matrix above does not necessarily have ones on the diagonal either.

6 Conclusion

We described the reasons why the equations for the posterior distributions on pp. 102–103 as well as Eq. 6 in DV are wrong.

We believe that these mistakes cause the peculiarities of the risk measure figures given in Section 5 in DV. For instance, an expected shortfall at confidence level 95% of $8.28 \cdot 10^{38}$ seems unrealistically high.

It would probably be possible to correct the numerical results in DV. However, this would require a work much larger than the initial paper, as the corrected posterior density of the Gaussian copula is not analytically tractable because it is not from a known parametric family.

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