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ORIGINAL RESEARCH PAPER

The optimal dividend barrier in the Gamma–Omega model

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Abstract In the traditional actuarial risk model, if the surplus is negative, the company is ruined and has to go out of business. In this paper we distinguish between ruin (negative surplus) and bankruptcy (going out of business), where the probability of bankruptcy is a function of the level of negative surplus. The idea for this notion of bankruptcy comes from the observation that in some industries, companies can continue doing business even though they are technically ruined. Assuming that dividends can only be paid with a certain probability at each point of time, we derive closed-form formulas for the expected discounted dividends until bankruptcy under a barrier strategy. Subsequently, the optimal barrier is determined, and several explicit identities for the optimal value are found. The surplus process of the company is modeled by a Wiener process (Brownian motion).

1 Introduction

Classical risk theory had been synonymous with ruin theory. The central problem had been to calculate the (hopefully small) probability of ruin of a nonlife insurance

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company. However, it is unrealistic to assume that the surplus of a company can increase without bounds. Also, a main goal of a company should be the paying of dividends to its shareholders. The seminal paper of de Finetti [5] studies the criterion of maximizing expected discounted dividends until possible ruin of a company. If an optimal dividend strategy is applied, ruin is typically certain in the long run. Thus, de Finetti's idea marked a drastic departure from classical risk theory.

De Finetti's dividend problem has been an inspiration to substantial research in actuarial science. Two recent survey papers are Avanzi [4] and Albrecher and Thonhauser [3]. In general, the optimal dividend strategy can be complicated. However, in certain cases, it is a barrier strategy, and then the problem is reduced to finding the optimal barrier, a number.

It should be pointed out that dividends are a major topic of study in corporate finance. Indeed, within a few years of the publication of de Finetti [5], and apparently independent of it, the following related papers appeared: Shubik and Thompson [12], Miyasawa [10], and Takeuchi [13]. The companies modeled in these papers are not necessarily insurance companies; this is also the situation in our paper.

We make a distinction between ruin and bankruptcy. In the traditional actuarial model, if the surplus is negative, the company is ruined and has to go out of business. In particular, no dividends are paid after ruin. In this paper, a company with a negative surplus is assumed to be able to continue doing business as usual, until bankruptcy takes place. Thus, ruin is the situation when the surplus is negative, and bankruptcy means that the company goes out of business. We also assume that the probability of bankruptcy is a function of the level of negative surplus. The idea for this notion of bankruptcy comes from the observation that in some industries, companies can continue doing business even though they are technically ruined.

Motivated by Albrecher et al. [2], we assume that at each point of time, dividends can only be paid with a certain probability. As a consequence, the dividend payments under a barrier strategy constitute a discrete sequence of random variables, which has some practical appeal. The traditional continuous dividends can be retrieved from such a model as a limiting case.

This paper aims to obtain results that are intuitive, transparent or esthetical. For this reason, the surplus process of the company is modeled by a Wiener process (Brownian motion).

2 The model

As in Gerber and Shiu [9], the basic surplus process of a company is modelled by a Wiener process with expected increment $\mu > 0$ per unit time and variance σ^2 per unit time. However, the model is extended in two ways. First, if the surplus is negative, bankruptcy is not automatic. For a precise formulation, we introduce the bankruptcy rate function $\omega(x) \geq 0$, $x \leq 0$. This is a non-increasing (typically a decreasing) function; whenever the negative surplus is x, $\omega(x)dt$ is the probability of bankruptcy within dt time units. The second extension concerns the dividends to the shareholders of the company. The dividends can only be paid at certain random times and thus constitute a discrete sequence of random variables. As in Albrecher et al. [2], it is assumed that the waiting times between successive dates when dividends can be paid are independent random variables with a common exponential distribution of mean $1/\gamma$. In other words, at any time the probability that a dividend can be paid within dt time units is $\gamma\, dt$.

Remarks

(i) Without loss of generality, we assume that the bankruptcy rate function is positive for $x < 0$ and zero for $x > 0$. The case where it is positive for x below a critical level and zero above this level can be reduced to the case where the critical level is zero.

(ii) If $\omega(x)$ is infinite for $x \le x_0 < 0$ and $\omega(x) > 0$ for $x > x_0$, bankruptcy occurs at the latest when the surplus drops to x_0. In a sense, x_0 is the level of "certain bankruptcy". This concept differs from that of absolute ruin in Gerber [6].

3 Barrier strategies

A barrier dividend strategy is given by a parameter $b \ge 0$. If at a potential dividend-payment time the surplus is above b, the excess is paid as a dividend. Dividends are discounted at a constant force of interest $\delta > 0$. Let $V(x; b)$ denote the expectation of the discounted dividends until bankruptcy, considered as a function of the initial surplus x and subject to the barrier strategy with parameter b. Here, x is any real number, not necessarily positive. The function $V(x; b)$ is characterized by the system of differential equations

$$\frac{\sigma^2}{2}V''(x;b) + \mu V'(x;b) - [\delta + \omega(x)]V(x;b) = 0, \quad x<0, \tag{1}$$

$$\frac{\sigma^2}{2}V''(x;b) + \mu V'(x;b) - \delta V(x;b) = 0, \quad 0<x<b, \tag{2}$$

$$\frac{\sigma^2}{2}V''(x;b) + \mu V'(x;b) - \delta V(x;b) + \gamma[x - b - V(x;b) + V(b;b)] = 0, \quad x > b, \tag{3}$$

together with the requirements that $V(x; b)$ and $V'(x; b)$ are continuous functions of x, that $V(-\infty; b) = 0$, and that $V(x; b)$ is linearly bounded for $x \to \infty$.

Equations 1–3 can be derived from the fact that the expected instantaneous total return (over a time interval of length dt) must be the sum of the expected instantaneous change of value and the expected instantaneous dividend. For example, if $x > b$, this is the condition that

$$V(x;b)\delta dt = \{E[dV] - [V(x;b) - V(b;b)]\gamma dt\} + (x-b)\gamma dt,$$

where $E[dV] = \frac{\sigma^2}{2}V''(x;b)dt + \mu V'(x;b)dt$. From this, (3) follows. Similarly, if $x < 0$, the condition is that

$$V(x;b)\delta dt = E[dV] - V(x;b)\omega(x)dt,$$

from which (1) follows.

Remark Because $V(x; b)$ and $V'(x; b)$ are continuous functions, it follows from (1) that $V''(x; b)$ is discontinuous whenever the monotone function $\omega(x)$ has a jump.

4 Alternative interpretations for *V*(*x*; *b*)

In Eq. 1, the force of interest δ and the bankruptcy rate $\omega(x)$ play the same mathematical role, because only their sum matters. Based on this observation and to make the point, we introduce two alternative "extreme" models. They both yield the same function $V(x; b)$ as an expectation. Let

$$v(x) = \begin{cases} \delta + \omega(x) & if x<0, \\ \delta & if x > 0. \end{cases}$$

Then Eqs. 1–3 can be written as

$$\frac{\sigma^2}{2}V''(x;b) + \mu V'(x;b) - v(x)V(x;b) = 0, \quad x<b, \tag{4}$$

$$\frac{\sigma^2}{2}V''(x;b) + \mu V'(x;b) - v(x)V(x;b) + \gamma[x - b - V(x;b) + V(b;b)] = 0, \quad x > b. \tag{5}$$

In the first alternative model, the rate of bankruptcy is zero (that is, there is no bankruptcy in this model) and the function $v(x)$ plays the role of a surplus-dependent force of interest, because $\omega(x)$ has been added to δ. In some sense it reflects the seriousness of the financial situation of the company. In the second alternative model, the force of interest is zero and bankruptcy takes place according to the modified bankruptcy rate function $v(x)$, the sum of the original bankruptcy rate $\omega(x)$ and a constant termination rate δ. That is, δdt is the probability that the company will become bankrupt, due to external events or circumstances, within *dt* time units. (In a Lévy process framework, this is often referred as an "exponential killing" of a process.) In both models, $V(x; b)$ satisfies Eqs. 4 and 5 and is therefore the same function. However note that in each model, $V(x; b)$ is the expectation of an underlying random variable, and that these random variables are not the same.

In Sect. 3, we applied the fact that the expected instantaneous total return (over an interval of length *dt*) is the sum of the expected instantaneous change of value and the expected instantaneous dividend. It is instructive to compare this decomposition in the two models. In the first alternative model, we have

$$V(x;b)\nu(x)dt = E[dV] + 0, \quad x<b,$$
$$V(x;b)\nu(x)dt = \{E[dV] - [V(x;b) - V(b;b)]\gamma dt\} + (x-b)\gamma dt, \quad x > b.$$

In the second alternative model, the expected total return is zero. Hence, the decomposition is

$$0 = E[dV] - V(x;b)\nu(x)dt, \quad x<b,$$
$$0 = \{E[dV] - V(x;b)\nu(x)dt - [V(x;b) - V(b;b)]\gamma dt\} + (x-b)\gamma dt, \quad x > b.$$

Either decomposition leads to Eqs. 4 and 5.

5 An auxiliary function

In the spirit of Gerber et al. [8], we introduce a function $h(x), -\infty<x<\infty$. It is unique only up to a constant factor and defined by the following property: Let $-\infty<x<y<\infty$. Given the initial surplus x, the expectation of a discounted contingent payment of 1 at the time when the surplus reaches the level y, provided that bankruptcy has not occurred in the meantime, is $h(x)/h(y)$. In the framework of Lévy processes the function $h(x)$, apart from a constant factor, is the scale function. The function $h(x)$ is a solution of the differential equations

$$\frac{\sigma^2}{2}h''(x) + \mu h'(x) - [\delta + \omega(x)]h(x) = 0, \quad x<0, \tag{6}$$

$$\frac{\sigma^2}{2}h''(x) + \mu h'(x) - \delta h(x) = 0, \quad x \geq 0, \tag{7}$$

with the requirement that $h(x)$ and $h'(x)$ are continuous, and that $h(-\infty) = 0$. From (7) it follows that

$$h(x) = Ae^{rx} + Be^{sx}, \quad x \geq 0, \tag{8}$$

where $r > 0$ and $s < 0$ are the solutions of the characteristic equation

$$\frac{\sigma^2}{2}\xi^2 + \mu\xi - \delta = 0. \tag{9}$$

Because of the continuity conditions at $x = 0$, we can express A and B by $h(0)$ and $h'(0)$. We find that

$$A = \frac{h'(0) - sh(0)}{r-s}, \quad B = \frac{rh(0) - h'(0)}{r-s}. \tag{10}$$

Remark If there is a finite x_0 as in Remark (ii) of Sect. 2, then the condition $h(-\infty) = 0$ is changed to $h(x_0) = 0$ and the variable x in Eq. 6 is restricted to $x_0 < x < 0$.

6 The determination of *V*(*x*; *b*)

By interpretation, we see that

$$V(x;b) = \frac{h(x)}{h(b)} V(b;b), \quad x \le b. \tag{11}$$

Hence, we have the factorization formula

$$V(x;b) = C(b)h(x), \quad x \le b, \tag{12}$$

with $C(b) = V(b;b)/h(b)$.

A particular solution of the inhomogeneous differential equation (3) is the linear function

$$\eta(x) = \frac{\gamma}{\delta+\gamma}[x - b + V(b;b)] + \frac{\mu\gamma}{(\delta+\gamma)^2}, \quad x > b. \tag{13}$$

Hence,

$$V(x;b) = De^{s_\gamma(x-b)} + \eta(x), \quad x > b, \tag{14}$$

where $s_\gamma < 0$ is the negative solution of the characteristic equation

$$\frac{\sigma^2}{2}\xi^2 + \mu\xi - (\delta+\gamma) = 0. \tag{15}$$

Setting $x = b$ in (14), we obtain

$$\begin{aligned} D &= V(b;b) - \eta(b) \\ &= \frac{\delta}{\delta+\gamma} V(b;b) - \frac{\mu\gamma}{(\delta+\gamma)^2}. \end{aligned} \tag{16}$$

Then, from (12), (14) and (16), and the continuity of $V(x;b)$ at $x = b$, we find that

$$\begin{aligned} V(x;b) = {} & \left(\frac{\delta}{\delta+\gamma} C(b)h(b) - \frac{\mu\gamma}{(\delta+\gamma)^2}\right) e^{s_\gamma(x-b)} \\ & + \frac{\gamma}{\delta+\gamma}[x - b + C(b)h(b)] + \frac{\mu\gamma}{(\delta+\gamma)^2}, \ x > b. \end{aligned} \tag{17}$$

Finally, the continuity of $V'(x;b)$ at $x = b$ leads to the condition that

$$C(b)h'(b) = s_\gamma \left(\frac{\delta}{\delta+\gamma} C(b)h(b) - \frac{\mu\gamma}{(\delta+\gamma)^2}\right) + \frac{\gamma}{\delta+\gamma}, \tag{18}$$

which yields

$$C(b) = \frac{\frac{\gamma}{\delta+\gamma} - s_\gamma \frac{\mu\gamma}{(\delta+\gamma)^2}}{h'(b) - s_\gamma \frac{\delta}{\delta+\gamma} h(b)}. \tag{19}$$

After substitution in (12) and (17), we have closed-form expressions for $V(x;b)$.

Remarks

(i) The classical model of continuous dividends can be retrieved as the limiting case $\gamma \to \infty$. In the limit, (19) reduces to $C(b) = 1/h'(b)$, and hence (12) to

$$V(x;b) = \frac{h(x)}{h'(b)}, \quad x \le b. \tag{20}$$

(ii) If bankruptcy is defined in the traditional sense, $V(0;\, b) = 0$ of course. The results for $V(x;\, b)$ remain valid, if we set $h(x) = e^{rx} - e^{sx}$, or $B = -A$.

7 The optimal dividend barrier

The optimal dividend barrier b^* is defined as the value of b which maximizes $V(x;\, b)$ in (12), that is, which maximizes $C(b)$. We shall assume that $b^* > 0$. Then the condition for b^* is that the derivative of the denominator in (19) vanishes,

$$h''(b^*) - s_\gamma \frac{\delta}{\delta + \gamma} h'(b^*) = 0. \tag{21}$$

Using (8), we obtain

$$b^* = \frac{1}{r - s} \ln \frac{B\left(-s^2 + s_\gamma \frac{\delta}{\delta+\gamma} s\right)}{A\left(r^2 - s_\gamma \frac{\delta}{\delta+\gamma} r\right)}, \tag{22}$$

with A and B given by (10).

Remarks

(i) Let r_γ denote the positive solution of the quadratic equation (15). Then

$$\frac{\delta}{\delta + \gamma} = \frac{rs}{r_\gamma s_\gamma}. \tag{23}$$

After substitution in (22), we obtain an alternative expression for the optimal dividend barrier:

$$b^* = \frac{1}{r - s} \ln \frac{-Bs^2(r_\gamma - r)}{Ar^2(r_\gamma - s)}. \tag{24}$$

It is instructive to write this expression as a sum:

$$b^* = \frac{1}{r - s} \ln \frac{s^2}{r^2} + \frac{1}{r - s} \ln \frac{-B}{A} + \frac{1}{r - s} \ln \frac{r_\gamma - r}{r_\gamma - s}. \tag{25}$$

This has the following interpretation. The first term on the right-hand side is the optimal dividend barrier in the classical model, where bankruptcy is defined in the traditional way and dividends are continuous; see, for example, formula (6.2) of Gerber and Shiu [9]. The second and third term are negative adjustment terms. They show nicely the separate effects of the bankruptcy rate function and the discreteness of the dividends on lowering the optimal dividend barrier. For example, formula (25) shows that b^* is an increasing function of r_γ and with that of γ.

(ii) In Albrecher et al. [1] it is shown that for constant bankruptcy rate $\omega(x) \equiv \gamma$ the optimal strategy for maximizing the expectation of discounted dividend payments until bankruptcy is indeed a barrier strategy. We note that the search for the optimal barrier is also meaningful in cases where the optimal strategy is not a barrier strategy.

8 Results for $V(x; b)$ at $x = b = b^*$

The first result is that

$$V'(b^*; b^*) = 1. \tag{26}$$

In the case of continuous dividends, $V'(b;\ b) = 1$ for any $b > 0$; this follows immediately from (20). However, if γ is finite, (26) is not obvious and has to be verified. From (12) and (19), we have

$$\begin{aligned} V'(b^*; b^*) &= C(b^*)h'(b^*) \\ &= \frac{\frac{\gamma}{\delta+\gamma} - s_\gamma \frac{\mu\gamma}{(\delta+\gamma)^2}}{h'(b^*) - s_\gamma \frac{\delta}{\delta+\gamma} h(b^*)} h'(b^*). \end{aligned} \tag{27}$$

Using (7) and the optimality condition (21), we find that

$$\begin{aligned} \delta h(b^*) &= \frac{\sigma^2}{2} h''(b^*) + \mu h'(b^*) \\ &= \left(\frac{\sigma^2}{2} s_\gamma \frac{\delta}{\delta + \gamma} + \mu \right) h'(b^*). \end{aligned} \tag{28}$$

Hence,

$$V'(b^*; b^*) = \frac{\frac{\gamma}{\delta+\gamma} - s_\gamma \frac{\mu\gamma}{(\delta+\gamma)^2}}{1 - \frac{\sigma^2}{2} s_\gamma^2 \frac{\delta}{(\delta+\gamma)^2} - s_\gamma \frac{\mu}{\delta+\gamma}}. \tag{29}$$

As a solution of the quadratic equation (15), s_γ satisfies

$$\frac{\sigma^2}{2} s_\gamma^2 = -\mu s_\gamma + (\delta + \gamma). \tag{30}$$

Upon substitution in (29) and simplification we obtain indeed (26).

The second result is that

$$V(b^*; b^*) = \frac{\mu}{\delta} - \frac{\mu}{\delta + \gamma} + \frac{1}{s_\gamma}. \tag{31}$$

This generalizes the classical result

$$V(b^*; b^*) = \frac{\mu}{\delta} \tag{32}$$

in the case of continuous dividends ($\gamma \to \infty$), which has been found by Gerber [7] and is (7.1) in Gerber and Shiu [9]. It is remarkable that $V(b^*; b^*)$, unlike b^*, does not depend on the bankruptcy rate function $\omega(x)$.

For a proof of (31), we use the formula

$$V(b^*; b^*) = \frac{h(b^*)}{h'(b^*)}. \tag{33}$$

To verify it, we differentiate (11), set $x = b = b^*$ in the resulting equation, and use (26). Next, we combine (7) and (21) to see that

$$\left(\frac{\sigma^2}{2} s_\gamma \frac{\delta}{\delta + \gamma} + \mu\right) h'(b^*) - \delta h(b^*) = 0. \tag{34}$$

From this and (33) it follows that

$$V(b^*; b^*) = \frac{\mu}{\delta} + \frac{\sigma^2}{2} s_\gamma \frac{1}{\delta + \gamma}.$$

Finally, to obtain (31), we substitute for $\frac{\sigma^2}{2} s_\gamma$ according to (30).

Remarks

(i) From (26) and (11), it follows that

$$V(x; b^*) = \frac{h(x)}{h'(b^*)}, \quad x \le b^*. \tag{35}$$

This formula should be compared with (20), which is for arbitrary b, but valid only in the limit $\gamma \to \infty$. The function h does not depend on the value of γ, hence formula (35) is valid for any γ. The dependence on γ comes in through b^*, which is a function of γ.

(ii) The optimal dividend barrier b^* is at the same time the optimal financial capital in the following sense. Let $P(x; b)$ denote the expected discounted profit if the barrier strategy with parameter b is applied, that is,

$$P(x; b) = V(x; b) - x.$$

Equation 26 shows that $P'(b^*; b^*) = 0$, and from (2), (26) and (31) we see that

$$V''(b^*; b^*) = -\frac{2\delta}{\sigma^2}\left(\frac{\mu}{\delta + \gamma} - \frac{1}{s_\gamma}\right).$$

Thus $P''(b^*; b^*) = V''(b^*; b^*)$ is negative. It follows that $P(x; b^*)$ is maximal for the financial capital $x = b^*$. Because this result is in line with intuition, it is also an indirect explanation of $V'(b^*; b^*) = 1$.

(iii) Formula (31) can be derived without (26) as a starting point. From (12) and (19) we find that

$$V(b^*;b^*) = C(b^*)h(b^*) = \frac{\frac{\gamma}{\delta+\gamma} - s_\gamma \frac{\mu\gamma}{(\delta+\gamma)^2}}{\frac{h'(b^*)}{h(b^*)} - s_\gamma \frac{\delta}{\delta+\gamma}}.$$

Now we substitute according to (34) and use basic algebra to obtain (31).

(iv) From the continuity of $V(x;\,b)$ and $V'(x;\,b)$ and Eqs. 2 and 3 it follows that $V''(x;\,b)$ is continuous at $x = b$. Now we differentiate (2) and (3) to see that the discontinuity of $V'''(x;\,b)$ at $x = b$ is

$$V'''(b+;b) - V'''(b-;b) = \frac{2\gamma}{\sigma^2}[V'(b;b) - 1].$$

Because of (26), this discontinuity vanishes if $b = b^*$. Hence $V'''(x;\,b^*)$ is continuous at $x = b^*$. Such a condition is called a smooth-pasting condition in literature on optimal stopping and a high contact condition in finance literature.

(v) Originally, the expression on the right-hand side of (32) is interpreted as the present value of a perpetuity-certain at rate μ. By observing that $1/\delta$ is also the expectation of an exponentially distributed random variable with parameter δ, we can rewrite (32) in a form that is appealing within the framework of the second alternative model of Sect. 4:

$$V(b^*;b^*) = \mu E(T_\delta),$$

where T_δ is an exponentially distributed termination time of the process, i.e., under the optimal barrier strategy and with the initial surplus at the optimal barrier, the expected present value of dividend payments equals the undiscounted sum of a continuous payment stream of rate μ until the expected termination time of the process.

9 Constant and piecewise constant bankruptcy rate functions

We first look at the case where $\omega(x) \equiv \omega$ (constant). Then (6) is a differential equation with constant coefficients, and we may set $h(x) = e^{r_\omega x},\ x<0$, where r_ω is the positive solution of the characteristic equation

$$\frac{\sigma^2}{2}\xi^2 + \mu\xi - (\delta + \omega) = 0.$$

According to (10), we have

$$A = \frac{r_\omega - s}{r - s}, \quad B = -\frac{r_\omega - r}{r - s}.$$

By (25), the optimal dividend barrier is

$$b^* = \frac{1}{r-s}\ln\frac{s^2}{r^2} + \frac{1}{r-s}\ln\frac{r_\omega - r}{r_\omega - s} + \frac{1}{r-s}\ln\frac{r_\gamma - r}{r_\gamma - s}. \tag{36}$$

The symmetry between the roles of r_ω and r_γ is remarkable and somewhat unexpected. Note that for $\omega = \gamma$, formula (36) is the diffusion limit of formula (24) in Albrecher et al. [2], where a compound Poisson process with exponential jumps was considered.

Now suppose that $\omega(x)$ is piecewise constant,

$$\omega(x) = \omega_k, \quad x_{k-1} < x < x_k,$$

$k = 1, \ldots, n$, where $x_0 = -\infty, x_n = 0$ and $x_1 < x_2 < \cdots < x_{n-1} < 0$. Typically,

$$\omega_1 > \omega_2 > \cdots > \omega_n > 0. \tag{37}$$

It follows that

$$h(x) = A_k e^{r_k x} + B_k e^{s_k x}, \quad x_{k-1} < x < x_k,$$

where $r_k > 0$ and $s_k < 0$ are the solutions of the equation

$$\frac{\sigma^2}{2}\xi^2 + \mu\xi - (\delta + \omega_k) = 0.$$

Note that under (37) we have $r_{k+1} < r_k$ and $s_{k+1} > s_k$ for $k = 1, \ldots, n-1$. We can determine the coefficients recursively, starting with $A_1 = 1$, $B_1 = 0$. From the continuity of $h(x)$ and $h'(x)$ at $x = x_k$, it follows that

$$A_{k+1}e^{r_{k+1}x_k} + B_{k+1}e^{s_{k+1}x_k} = A_k e^{r_k x_k} + B_k e^{s_k x_k},$$
$$A_{k+1}r_{k+1}e^{r_{k+1}x_k} + B_{k+1}s_{k+1}e^{s_{k+1}x_k} = A_k r_k e^{r_k x_k} + B_k s_k e^{s_k x_k}.$$

Thus,

$$A_{k+1}(r_{k+1} - s_{k+1})e^{r_{k+1}x_k} = A_k(r_k - s_{k+1})e^{r_k x_k} + B_k(s_k - s_{k+1})e^{s_k x_k} \tag{38}$$

and

$$B_{k+1}(s_{k+1} - r_{k+1})e^{s_{k+1}x_k} = A_k(r_k - r_{k+1})e^{r_k x_k} + B_k(s_k - r_{k+1})e^{s_k x_k}. \tag{39}$$

In view of (25), the ultimate goal is to calculate the ratio $\rho = -B/A$, where $A = A_n$, $B = B_n$. Hence it is useful to establish a direct recursion for $\rho_k = -B_k/A_k$. From (38) and (39) we see that

$$\rho_{k+1} = e^{(r_{k+1}-s_{k+1})x_k}\frac{(r_k - r_{k+1})e^{r_k x_k} + \rho_k(r_{k+1} - s_k)e^{s_k x_k}}{(r_k - s_{k+1})e^{r_k x_k} + \rho_k(s_{k+1} - s_k)e^{s_k x_k}}, \quad k = 1, \ldots, n-1,$$

with starting value $\rho_1 = 0$.

10 Other bankruptcy rate functions

For more general bankruptcy rate functions $\omega(x)$, one has two possibilities. On the one hand, one can approximate $\omega(x)$ by a piecewise constant function and then follow the procedure of the previous section to obtain an approximation for $\rho_n = -B/A$. In fact one can obtain upper and lower bounds by using piecewise constant upper and lower bounds for $\omega(x)$.

Alternatively, for certain specific forms of $\omega(x)$, one may be able to solve the differential equation (6) explicitly, which then gives the exact expression of

$$-\frac{B}{A}=\frac{rh(0)-h'(0)}{sh(0)-h'(0)}=\frac{rh(0)/h'(0)-1}{sh(0)/h'(0)-1}.$$

This is illustrated by two examples:

(i) If $\omega(x)=-x$, the differential equation (6) with boundary condition $h(-\infty)=0$ has the solution (apart from an arbitrary multiplicative constant)

$$h(x)=\exp\left(-\frac{\mu}{\sigma^2}x\right)\mathrm{Ai}\left(\frac{-2x+2\delta+\mu^2/\sigma^2}{4^{1/3}\sigma^{2/3}}\right),$$

where

$$\mathrm{Ai}(z)=\frac{1}{\pi}\int_0^\infty\cos\left(\frac{t^3}{3}+zt\right)dt$$

is the Airy function (see e.g. [11, p. 7]). This results in

$$-\frac{B}{A}=\frac{\left((\mu+r\sigma^2)\mathrm{Ai}(z_0)+2^{1/3}\sigma^{4/3}\mathrm{Ai}'(z_0)\right)\left((\mu+s\sigma^2)\mathrm{Ai}(z_0)+2^{1/3}\sigma^{4/3}\mathrm{Ai}'(z_0)\right)}{\left(\mu\mathrm{Ai}(z_0)+2^{1/3}\sigma^{4/3}\mathrm{Ai}'(z_0)\right)^2}$$

with $z_0=(2\delta+\mu^2/\sigma^2)/(4^{1/3}\sigma^{2/3})$.

(ii) If

$$\omega(x)=\begin{cases}\frac{1}{1+x}, & -1<x<0,\\ \infty, & x\leq-1,\end{cases}$$

survival is only possible for a surplus exceeding $x_0=-1$. The differential equation (6) for $-1<x<0$ with boundary condition $h(-1)=0$ has the solution (apart from an arbitrary multiplicative constant)

$$h(x)=e^{-(x\mu+(1+x)z_1)/\sigma^2}(1+x)M\left(1-\frac{1}{z_1},2,2(1+x)z_1/\sigma^2\right)$$

with $z_1=\sqrt{\mu^2+2\delta\sigma^2}$ (see e.g. [11, p. 21]). Here

$$M(a,b,z)=\sum_{k=0}^{\infty}\frac{a(a+1)(a+k-1)}{b(b+1)(b+k-1)}\frac{z^k}{k!}$$

denotes the Kummer confluent hypergeometric function. Using $M'(a,b,z)=\frac{a}{b}M(a+1,b+1,z)$, the factor $-B/A$ is then given by

$$-\frac{B}{A}=\frac{(g_1-g_2)(g_3-g_4)}{(g_5-g_6)^2}$$

with

$$
\begin{aligned}
g_1 &= (z_1 - 1)\, M\left(2 - \frac{1}{z_1}, 3, \frac{2z_1}{\sigma^2}\right),\\
g_2 &= (\mu + (r-1)\sigma^2 + z_1)\, M\left(1 - \frac{1}{z_1}, 2, \frac{2z_1}{\sigma^2}\right),\\
g_3 &= (\mu + (s-1)\sigma^2 + z_1)\, M\left(1 - \frac{1}{z_1}, 2, \frac{2z_1}{\sigma^2}\right),\\
g_4 &= (z_1 - 1)\, M\left(2 - \frac{1}{z_1}, 3, \frac{2z_1}{\sigma^2}\right),\\
g_5 &= (\sigma^2 - \mu + z_1)\, M\left(1 - \frac{1}{z_1}, 2, \frac{2z_1}{\sigma^2}\right),\\
g_6 &= (z_1 + 1)\, M\left(1 - \frac{1}{z_1}, 3, \frac{2z_1}{\sigma^2}\right).
\end{aligned}
$$

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