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The class of algebraically closed p-semilattices is finitely axiomatizable

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Abstract. We prove our title, and thereby establish the base for a positive solution of Albert and Burris' problem on the finite axiomatizability of the model companion of the class of all pseudocomplemented semilattices.

1. Introduction

The purpose of this paper is to prove its title. It is based on [11] and essentially combines, in a new setting, results of Regula Rupp's PhD Thesis [10] with results of Joel Adler's PhD Thesis [1].

The motivation for this work comes from the problem posed by Albert and Burris in the final paragraph of [3]: "Does the class of pseudo-complemented semilattices have a finitely axiomatizable model companion?"

Together with Adler's 2012 preprint [2], the present paper will provide a positive answer to Albert and Burris' question. In fact, we show here that the class of all algebraically closed pseudocomplemented semilattices—for short: a.c. p-semilattices—is finitely axiomatizable in the first-order language of p-semilattices, by providing four axioms—one of which is distributivity—characterizing this class.

Recall that the model companion mentioned above consists precisely of all existentially complete—for short, e.c.—p-semilattices and is thus a subclass of the class of all a.c. p-semilattices. Adler's preprint [2] provides finitely many additional axioms singling out the e.c. members within all a.c. p-semilattices, and thus will settle the problem.

The paper is organized as follows. Section 2 collects the basic algebraic notions concerning p-semilattices, while Section 3 provides a short summary of the relevant model-theoretic concepts, adapted to our setting.

In Section 4, we consider distributive meet-semilattices. The main result of the section is that in a distributive p-semilattice P, an arbitrary—not necessarily distributive—finite p-subsemilattice $F \trianglelefteq P$ can be extended to a finite distributive p-semilattice F_0 such that $F \trianglelefteq F_0 \trianglelefteq P$.

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In Section 5, we specify an axiom (A1) guaranteeing that F_0, as obtained in Section 4, can be extended to a finite distributive p-semilattice F_1 with $F_0 \trianglelefteq F_1 \trianglelefteq P$ such that the dense elements of F_1 form a boolean meet-semilattice under the induced order.

In Section 6, another axiom, (A2), is introduced, and it is shown that F_1, as obtained in Section 5, can be extended to a finite p-semilattice F_2 with $F_1 \trianglelefteq F_2 \trianglelefteq P$ such that F_2 is isomorphic to a direct product of subdirectly irreducible p-semilattices, provided P satisfies (A2).

In Section 7, it is shown that in a p-semilattice P satisfying an additional axiom (A3), any finite p-subsemilattice F_2, as obtained in Section 6, can be extended to a p-subsemilattice $F_3 \trianglelefteq P$ isomorphic to a direct product with finitely many factors, each of them being either the two-element boolean p-semilattice or the unique countable atom-free boolean algebra with a new top element added.

Section 8 establishes the necessity of the above axioms for a p-semilattice to be algebraically closed. Finally, Section 9 formulates our main theorem.

2. Pseudocomplemented semilattices

A pseudocomplemented semilattice (for short: p-semilattice) $(P;\ \wedge,^*, 0, 1)$ is a meet-semilattice $(P; \wedge)$ with least element 0 and top element 1, equipped with an unary operation $a \mapsto a^*$ such that for all $x \in P$, $x \wedge a = 0$ iff $x \leq a^*$. It is a nontrivial fact that the class **PCS** of all p-semilattices can be (finitely) axiomatized by identities in the first-order language $\mathcal{L}_{PCS} = \{\wedge,^*, 0, 1\}$, making **PCS** a variety; see [5]. We freely write P for the p-semilattice $(P; \wedge,^*, 0, 1)$ (and similarly for algebraic structures in general whenever the operations and relations under consideration are clear from the context). An element $d \in P$ satisfying $d^* = 0$ is called *dense*. $\mathrm{D}(P)$ denotes the set of all dense elements of P; moreover, $(\mathrm{D}(P); \wedge, 1)$ is a subsemilattice—in fact, a filter—of $(P; \wedge, 1)$. Further, $d \in \mathrm{D}(P)$ is called *maximally dense* iff $d \neq 1$ and $d \leq d' \leq 1$ implies $d' = d$ or $d' = 1$. An element $s \in P$ is called *skeletal* iff $s^{**} = s$. The set of all skeletal elements of P is denoted by $\mathrm{Sk}(P)$; it is a subalgebra of the p-semilattice P. Within $\mathrm{Sk}(P)$ the supremum of two elements exists w.r.t. to the order inherited from P; in fact, $\sup_{\mathrm{Sk}}\{a, b\} = (a^* \wedge b^*)^*$ for $a, b \in \mathrm{Sk}(P)$. Setting $a \sqcup b = (a^* \wedge b^*)^*$, $(\mathrm{Sk}(P); \wedge, \sqcup,^*, 0, 1)$ is a boolean algebra. The set of all atoms of P is denoted by $\mathrm{At}(P)$.

For any p-semilattice P, a p-semilattice $\hat{P}$ is obtained from P by adding a new top element. In most cases, the top element of P will be renamed to e and 1 will stand for the new top element. We write **2** for the two-element boolean algebra and **A** for the unique countable atom-free boolean algebra.

The class of p-semilattices P that are generated (as p-semilattices) by their skeletal and dense elements—that is, $P = \langle \mathrm{Sk}(P) \cup \mathrm{D}(P) \rangle_{\mathbf{PCS}}$—play an important rôle in our context. They are called *representable*; equivalently, P is representable iff every $x \in P$ admits a (not necessarily unique) representation of the the form $x = x^{**} \wedge d$ for some $d \in \mathrm{D}(P)$. Obviously, in this case, $P = \{\, b \wedge d : b \in \mathrm{Sk}(P), d \in \mathrm{D}(P) \,\}$.

Although there is only one binary operation in a p-semilattice P, a notion of distributivity can be introduced: Call P *distributive* if for all $a, b, c \in P$ with $c \geq a \wedge b$, there exist $x, y \in P$ satisfying $x \geq a$, $y \geq b$, and $x \wedge y = c$. Distributivity in p-semilattices—in particular, its relationship with representability—will be considered in detail in Section 4.

For a p-semilattice P and a skeletal element $a \in P$, the binary relation $x \; \theta_a \; y :\Longleftrightarrow a \wedge x = a \wedge y$ is a PCS-congruence. The factor algebra P/θ_a is isomorphic to $(\{\, a \wedge x : x \in P \,\}; \cdot, ', 0, a)$, where $(a \wedge x) \cdot (a \wedge y)$ is defined as $a \wedge (x \wedge y)$ and $(a \wedge x)'$ as $a \wedge x^*$. Furthermore, the map $f_a \colon P \to P/\theta_a$ defined by $f_a(x) = a \wedge x$ is a surjective homomorphism. The following special case will frequently occur: Consider a direct product $P = \prod_{i \in I} P_i$ of p-semilattices, and a subset $J \subseteq I$. Then $\prod_{i \in J} P_i \cong P/\theta_a$, where $a \in P$ is given by $(a)_i = 1$ iff $i \in J$, and by $(a)_i = 0$ iff $i \in I \setminus J$.

In a general meet-semilattice $(S; \wedge)$, $\downarrow_S x$ (or simply $\downarrow x$ if S is clear from the context) stands for $\{\, y \in S : y \leq x \,\}$, the *down-set* generated by x in S, where x is any element of S. We write $\mathcal{O}(S)$ for the (distributive) lattice of all down-sets of S ordered by set inclusion. Note that if S is *finite*, then $\downarrow x$ is actually a lattice under its induced order for any x, and we thus will call $x \in S$ *join-irreducible* iff x is such in $\downarrow x$, for S finite. We write $\mathcal{J}(F)$ for the set of all (non-zero) join-irreducibles of a finite meet-semilattice F. Whenever there is no danger of confusion, $\mathcal{J}(F)$ also stands for the *poset* of all join-irreducibles under the order inherited from F.

Finally, a meet-semilattice $(S; \wedge)$ is called *boolean* iff it is the $\wedge$-reduct of a boolean algebra. We use $Q \trianglelefteq P$ (respectively $P \trianglerighteq Q$) freely to indicate that Q is a subalgebra of P in whatever signature P and Q are considered at the moment. More background on (p-)semilattices may be found in [5] and [7], or in [4].

3. Model theory

For a given p-semilattice P, let $\mathcal{L}^P_{PCS}$ be the language obtained from $\mathcal{L}_{PCS}$ by adding bijectively a new constant symbol for each $a \in P$ to $\mathcal{L}_{PCS}$. P is called *algebraically closed*—abbreviated by a.c.—(in **PCS**) if P satisfies every positive existential $\mathcal{L}^P_{PCS}$-sentence that holds in some extension $P \trianglelefteq P'$ with $P' \in \mathbf{PCS}$. In plainer terms, P is a.c. iff every finite system of PCS-equations with coefficients from P that is solvable in some extension $P \trianglelefteq P' \in \mathbf{PCS}$ already has a solution in P. The stronger notion of being *existentially complete*—not considered in this paper but crucial in the problem posed by Albert and Burris in [3]—just differs from a.c. by allowing also (finitely many) negated equations; the *model companion* of **PCS** mentioned in the introduction is then just the class of all existentially complete algebras in **PCS**. For more background on the model theory relevant here, the reader is referred to [6], especially Chapter 7. We use ω to denote the set of all natural numbers.

In [11] the following characterization of algebraically closed p-semilattices is established.

Theorem 3.1. *A p-semilattice P is algebraically closed iff for any finite subalgebra $F \trianglelefteq P$, there exists a p-semilattice F' isomorphic to $\mathbf{2}^r \times (\hat{\mathbf{A}})^s$ for some $r, s \in \omega$ such that $F \trianglelefteq F' \trianglelefteq P$.*

Note that the trivial one-element p-semilattice is a.c., since it only can be embedded into itself. Write $\mathbf{A}(\mathbf{PCS})$ for the class of all a.c. members of $\mathbf{PCS}$. The main result of this paper is a finite list of $\mathcal{L}_{PCS}$-sentences that hold in $P \in \mathbf{PCS}$ iff P is a.c.; what actually will be shown is that these sentences hold in P iff P has the extension property specified in Theorem 3.1 above.

The remainder of this section collects some results from [1], providing evidence that a finite axiomatization of $\mathbf{A}(\mathbf{PCS})$ should exist. So far, the only members of $\mathbf{A}(\mathbf{PCS})$ identified immediately by Theorem 3.1 are the direct products $\mathbf{2}^r \times (\hat{\mathbf{A}})^s$ for some $r, s \in \omega$. There are others:

Let Q be the subalgebra of $(\hat{\mathbf{A}})^\omega$ jointly generated by $\mathrm{Sk}((\hat{\mathbf{A}})^\omega)$ and $D_e := \{\, d \in \mathrm{D}((\hat{\mathbf{A}})^\omega) : (d)_i = e$ for at most finitely many $i \in \omega \,\}$.

It is easy to see that $Q = \{\, a \wedge d : a \in \mathrm{Sk}((\hat{\mathbf{A}})^\omega)$ and $d \in \mathrm{D}_e \,\}$, since the latter set evidently is closed under $\wedge$ and $(a \wedge d)^{**} = a^{**} \wedge d^{**} = a^{**} \wedge 1 = a^{**} \in \mathrm{Sk}((\hat{\mathbf{A}})^\omega)$.

Note that Q is not isomorphic to any direct product with factors $\mathbf{2}$ or $\hat{B}$ (B any boolean algebra), since such a product has either a finite or uncountable number of dense elements while $\mathrm{D}(Q) = D_e$ is countable.

Let $F \trianglelefteq Q$ be finite. There exists a least $n_F \in \omega$ such that $(x)_i \neq e$ for all $x \in F$ and $i > n_F$. Define an element $a \in Q$ by $(a)_i = 1$ for $i \leq n_F$ and $(a)_i = 0$ for $i > n_F$. Let $Q_a = Q/\theta_a$ (see Section 2) and $F_a = F/\theta_a \cap (F \times F)$; define Q_{a^*} and F_{a^*} analogously. Now $Q \cong Q_a \times Q_{a^*}$ canonically, $F_a \trianglelefteq Q_a$, $F_{a^*} \trianglelefteq Q_{a^*}$, and thus $F \cong F' \trianglelefteq F_a \times F_{a^*}$ for some copy F' of F. It is clear that $Q_a \cong (\hat{\mathbf{A}})^n$; moreover, F_{a^*} is a finite boolean subalgebra of $\mathrm{Sk}(Q_{a^*})$, and thus $F_{a^*} \cong \mathbf{2}^k$ for some $k \in \omega$. Hence, $F \cong F' \trianglelefteq F_a \times F_{a^*} \trianglelefteq (\hat{\mathbf{A}})^n \times F_{a^*} \trianglelefteq Q_a \times Q_{a^*}$. Under the canonical isomorphism $Q \cong Q_a \times Q_{a^*}$, the algebra $(\hat{\mathbf{A}})^n \times F_{a^*}$ corresponds to a subalgebra of Q of the form required by Theorem 3.1.

That the class of all *existentially complete* p-semilattices—*alias* the model companion of $\mathbf{PCS}$—can be axiomatized by $\mathcal{L}_{PCS}$-sentences follows from general model-theoretic properties of $\mathbf{PCS}$, viz., the fact that $\mathbf{PCS}$ is a finitely generated universal Horn class with both the amalgamation and joint embedding properties; see [3] for details. No such general argument seems to apply to the (wider) class $\mathbf{A}(\mathbf{PCS})$. In fact, the mere axiomatizability of $\mathbf{A}(\mathbf{PCS})$ was first established in [1].

Now, an axiomatizable class of $\mathcal{L}_{PCS}$-structures is *finitely axiomatizable* iff both the class itself as well as its complementary class are closed under elementary equivalence and ultraproducts. So partial evidence for the finite axiomatizability of $\mathbf{A}(\mathbf{PCS})$ is provided by [1, Theorem 4.1], which states that an ultraproduct of finite p-semilattices that are not a.c. cannot be a.c. either.

4. Distributivity

There is a natural notion of distributivity for meet-semilattices, see Subsection 4.2 below. Generally, a subsemilattice of a distributive meet-semilattice need not be distributive. However, a finite subsemilattice $F \trianglelefteq S$ of any distributive meet-semilattice S can always be expanded to a *finite* distributive meet-semilattice F' such that $F \trianglelefteq F' \trianglelefteq S$—a very crucial fact in our context, as we shall see. This fact is well known; to the best of our knowledge, it appeared first in print as Fact 4 in [9]. What we actually need is a p-semilattice version of this result, which does not follow immediately from [9]. Therefore, we present an exposition based on so-called *minimal boolean extensions*; moreover, the specific properties of such extensions will be crucial in Section 5 when they are used to construct successive distributive extensions by destroying comparabilities between join-irreducibles.

4.1. Minimal boolean extensions. Every semilattice $(S; \wedge)$ embeds—as a meet-semilattice—into a boolean algebra: Indeed, the map $x \longmapsto \downarrow x$ embeds S into the power set algebra $\mathcal{P}(S)$. If S is *finite*, so is $\mathcal{P}(S)$, and there exists, therefore, a uniquely determined—up to isomorphism—smallest boolean algebra containing S as a meet-subsemilattice, denoted by B_S in the sequel.

So let $(F; \wedge)$ be an arbitrary but fixed *finite* meet-semilattice, and put $\mathrm{At}(B_F) = \{q_1, \ldots, q_n\}$, thus $B_F \cong \mathcal{P}\{q_1, \ldots, q_n\}$. We identify B_F with its canonical copy $\mathcal{P}\{q_1, \ldots, q_n\}$ in the sequel, and fix an embedding $e_F \colon F \to B_F$.

Given $q_i \in \mathrm{At}(B_F)$, define $y_i \in F$ by $y_i = \bigwedge \{ x \in F : q_i \in e_F(x) \}$. The doubleton $J_i := \{\emptyset, \{q_i\}\}$ is a nontrivial ideal in B_F, so F will no longer embed into B_F/J_i. With $p \colon B_F \to B_F/J_i$ the canonical (boolean) epimorphism, we thus find $u \neq v \in F$ such that $(p \circ e_F)(u) = (p \circ e_F)(v)$. It follows that, say, (i) $q_i \in e_F(u)$ but (ii) $q_i \notin e_F(v)$. We infer (i) $u \geq y_i$ and (ii) $v \ngeq y_i$, so $y_i = u \wedge y_i > u \wedge y_i \wedge v = y_i \wedge v$. But

$$\begin{aligned}(p \circ e_F)(y_i) &= (p \circ e_F)(y_i) \cap (p \circ e_F)(u) \\ &= (p \circ e_F)(y_i) \cap (p \circ e_F)(v) = (p \circ e_F)(y_i \wedge v),\end{aligned}$$

and we conclude that the sets $e_F(y_i)$ and $e_F(y_i \wedge v)$ differ exactly in the point q_i. In other words, y_i has $y_i \wedge v$ as its unique lower neighbor in F, that is, $y_i \in \mathcal{J}(F)$.

Conversely, consider $y \in F$ join-irreducible with lower neighbor y'. Suppose we find atoms $q_i \neq q_j$ in $e_F(y) \setminus e_F(y')$. Put $J = \{\emptyset, \{q_j\}\}$, J is a nontrivial ideal in B_F. With $p \colon B_F \to B_F/J$ the canonical epimorphism, it follows that $p \circ e_F$ is a monomorphism, contradicting the minimality of B_F. This shows that $e_F(y) \setminus e_F(y')$ must be a singleton.

Summing up, we have established a bijective correspondence between $\mathcal{J}(F)$ and $\mathrm{At}(B_F)$, and we will identify the two sets in the sequel. This means that B_F is taken to be the powerset algebra $\mathcal{P}(\mathcal{J}(F))$. Next, define $\mathsf{O}_F\colon F \to \mathcal{P}(\mathcal{J}(F))$ by $\mathsf{O}_F(x) = {\downarrow}x \cap \mathcal{J}(F)$. Since $x = \sup_{\downarrow x} \mathsf{O}_F(x)$, we see that O_F is injective. Also, since $y \le x_1 \wedge x_2$ iff $y \le x_1$ and $y \le x_2$ for $x_1, x_2 \in F$, and as $y \in \mathcal{J}(F)$, we have $\mathsf{O}_F(x_1 \wedge x_2) = \mathsf{O}_F(x_1) \cap \mathsf{O}_F(x_2)$ for all $x_1, x_2 \in F$. So, O_F actually is an embedding of F into $\mathcal{P}(\mathcal{J}(F))$.

Definition 4.1. The pair $(\mathcal{P}(\mathcal{J}(F)), \mathsf{O}_F) = (B_F, \mathsf{O}_F)$ is the (canonical) minimal boolean extension of a finite meet-semilattice F.

For easier reference, we also write $\hat{x}$ instead of ${\downarrow}x \cap \mathcal{J}(F) = \mathsf{O}_F(x)$ for elements x of finite meet-semilattices F (there is no danger of confusion with the notation $\hat{P}$ introduced for p-semilattices in Section 2).

4.2. Distributivity in meet-semilattices. The canonical notion of distributivity for meet-semilattices is captured by:

Definition 4.2. A meet-semilattice S is distributive iff for all a, b, c in S, the following holds: Whenever $c \ge a \wedge b$, there exist $x, y \in S$ such that $x \ge a$, $y \ge b$, and $x \wedge y = c$.

It is clear that this property can be expressed by a sentence (DIST) in (the $\wedge$-reduct of) $\mathcal{L}_{PCS}$.

The above definition of distributivity in meet-semilattices is closely related to distributivity in lattices:

Remark 4.3. For any lattice $(L; \wedge, \vee)$, its meet-semilattice reduct $(L; \wedge)$ satisfies (DIST) iff L is distributive as a lattice. Alternatively, $(S; \wedge)$ is distributive as a meet-semilattice iff the poset of all nonempty filters of S, ordered by set inclusion, is a distributive lattice.

Note that distributivity in meet-semilattices is not necessarily inherited by subsemilattices: Let $\mathbf{2}$ be the 2-element chain $0 < 1$. Then $\mathbf{2} \times \mathbf{2} \setminus \{(1,1)\}$ is a nondistributive meet-subsemilattice of the distributive lattice $\mathbf{2} \times \mathbf{2}$.

Lemma 4.4. *A distributive p-semilattice is representable.*

Proof. Obviously, $x \ge 0 = x^{**} \wedge x^*$. Using distributivity we find $a, b \in P$ such that $a \ge x^{**}$, $b \ge x^*$ and $a \wedge b = x$. Meeting both sides of the last equation with x^{**} we obtain $a \wedge x^{**} \wedge b = x \wedge x^{**}$, that is, $x^{**} \wedge b = x$. But $b \in \mathrm{D}(P)$, since $b \ge x, x^*$ and thus $b^* \le x^*, x^{**}$, that is, $b \le 0 = x^* \wedge x^{**}$. □

The converse of Lemma 4.4 does not hold as easy examples show. However, the distributivity of a representable p-semilattice depends only on its dense elements, as we will show presently.

For the purpose of this paper, call an element x of an arbitrary p-semilattice P *distributive* iff for any $a, b \in P$, $x \ge a \wedge b$ implies the existence of $x_a, x_b \in P$ such that $x_a \ge a$, $x_b \ge b$ and $x_a \wedge x_b = x$. It is routine to check that the meet of two distributive elements is distributive in any p-semilattice.

Lemma 4.5. *A representable p-semilattice P is distributive iff every $d \in \mathrm{D}(P)$ is distributive.*

Proof. Note first that skeletal elements are distributive in any p-semilattice: Indeed, consider $a, b \in P$ and $c \in \mathrm{Sk}(P)$ such that $c \geq a \wedge b$. This implies $c = c^{**} \geq a^{**} \wedge b^{**}$. By boolean distributivity, we obtain $(c \sqcup a^{**}) \wedge (c \sqcup b^{**}) = c \sqcup (a^{**} \wedge b^{**}) = c$, with $c \vee a^{**} \geq a^{**} \geq a$ and $c \vee b^{**} \geq b^{**} \geq b$. Since P is representable, we have $x = x^{**} \wedge d_x$ with suitable $d_x \in \mathrm{D}(P)$ for any $x \in P$. So x as the meet of two distributive elements is distributive provided every $d \in \mathrm{D}(P)$ is such. $\square$

Given a distributive meet-semilattice S and a subsemilattice $F \trianglelefteq S$, it is trivial to find a distributive semilattice F' such that $F \trianglelefteq F' \trianglelefteq S$: Just take $F' = S$. It turns out to be less trivial to find, for F *finite*, a *finite* distributive F' extending F within S. Proposition 4.9 asserts that this is always possible. Moreover, in Proposition 4.11, we will show that the same is true within the class of all pseudocomplemented meet-semilattices.

Lemma 4.6. *Let S be a distributive meet-semilattice, and let $a, a_1, \dots, a_n, b, c$ be elements of S.*

(i) *If $a \wedge b \leq c \leq b$, there exists $x \in S$ such that $x \geq a$ and $x \wedge b = c$.*
(ii) *If $a_1 \wedge c = \dots = a_n \wedge c$, there exists $x \in S$ such that $x \geq a_i$ $(1 \leq i \leq n)$ and $x \wedge c = a_1 \wedge c$.*

Proof. (i): Let $a \wedge b \leq c \leq b$. Using distributivity, we find $x, y \in S$ with $x \geq a, y \geq b$, and $x \wedge y = c$. Since $b \geq c$, we obtain $c = x \wedge y = x \wedge y \wedge b = x \wedge b$.

(ii): Suppose $a_1 \wedge c = \dots = a_n \wedge c$ and consider $a_1 \wedge c = a_2 \wedge c$. Using distributivity on $a_1 \geq a_2 \wedge c$, find $u_2, u_c \in S$ such that $u_2 \geq a_2$, $u_c \geq c$, and $u_2 \wedge u_c = a_1$ (thus, $u_2 \geq a_1, a_2$). Analogously, $a_2 \geq a_1 \wedge c$ gives the existence of $v_1, v_c \in S$ satisfying $v_1 \geq a_1$, $v_c \geq c$, and $v_1 \wedge v_c = a_2$ (thus, $v_1 \geq a_1, a_2$). Put $x_{12} = v_1 \wedge u_2$; then $x_{12} \geq a_1, a_2$. Moreover,

$$\begin{aligned} x_{12} \wedge c &= x_{12} \wedge u_c \wedge v_c \wedge c = v_1 \wedge u_2 \wedge u_c \wedge v_c \wedge c \\ &= a_1 \wedge a_2 \wedge c = a_1 \wedge c = a_c \wedge c. \end{aligned}$$

Repeat this process suitably often, first proceeding with $x_{12} \wedge c = a_3 \wedge c = \dots = a_n \wedge c$. $\square$

Corollary 4.7. *A distributive meet-semilattice S is upwards directed, that is, any two elements have a common upper bound in S. If S is also finite, then it is a distributive lattice under its natural order.*

Proof. Putting $c = b$ in Lemma 4.6(i), obtain x as a common upper bound for a and b. If S is finite, it will thus contain a greatest element, and thus the supremum of any two elements. $\square$

Going back to the minimal boolean extension B_F of a finite meet-semilattice F, note that $\mathsf{O}_F(x)$ is actually a *down-set* in $\mathcal{J}(F)$. Hence, O_F embeds

F into the sublattice $\mathcal{O}(\mathcal{J}(F))$ of $\mathcal{P}(\mathcal{J}(F))$. For easier reference, we also write L_F for the distributive lattice $\mathcal{O}(\mathcal{J}(F))$; L_F is generated, as a lattice, by $\{\, \mathsf{O}_F(y) : y \in \mathcal{J}(F)\,\}$ and is (up to isomorphism) the uniquely determined minimal distributive lattice embedding F. By Corollary 4.7, L_F can also be characterized as the (unique up to isomorphism) minimal distributive meet-semilattice embedding F. Note also that $B_{L_F} \cong B_F$ canonically.

Corollary 4.8. *For any finite meet-semilattice F, O_F provides an embedding of F into L_F. Moreover, F is distributive iff O_F is an isomorphism between F and L_F.*

4.3. Distributive extensions. The basic result, for our purposes, is:

Proposition 4.9. *Assume S is a distributive meet-semilattice and $F \trianglelefteq S$ a finite subsemilattice of S. Then there exists a finite distributive semilattice F_0 such that $F \trianglelefteq F_0 \trianglelefteq S$. In fact, we find such F_0 satisfying $F_0 \cong L_F$.*

Proof. Obviously, F has a least element 0_F, since it is finite. Moreover, we can assume without loss of generality that F has a greatest element 1_F: If not, there is an upper bound s for F within S by Corollary 4.7 as S is distributive. Clearly, $F_s := F \cup \{s\}$ is a subsemilattice of S extending F, and we can proceed by replacing F by F_s.

Suppose F is *not* distributive. Then the embedding $\mathsf{O}_F \colon F \to L_F$ cannot be surjective by Corollary 4.8, so there exists a down-set $H \subseteq \mathcal{J}(F)$ such that $H \notin im\mathsf{O}_F$. Pick H_0 *minimal* with this property. This means that $H_0 \neq \hat{w}$ for any $w \in F$, but $H = \widehat{w_H}$ for a unique w_H whenever H is a down-set in $\mathcal{J}(F)$ strictly contained in H_0. Let $\{j_1, \ldots, j_r\}$ be the complete list of all maximal elements in H_0. It follows that $r \geq 2$ for otherwise $H_0 = \widehat{j_1}$. Put $u = \sup_F H_0 = \sup_F\{j_1, \ldots, j_r\}$, which exists, since F has a greatest element. Note that $u \notin \mathcal{J}(F)$: $u \neq j_k$ for all $1 \leq k \leq r$, since $r \geq 2$, so if $u \in \mathcal{J}(F)$ with lower cover u^-, then $u^- \geq j_k$ for all $1 \leq k \leq r$, contradicting $u = \sup_F H_0$.

Let $U = \{\, x \in F : \hat{x} \supseteq H_0 \,\}$, $L = \{\, x \in F : \hat{x} \subseteq H_0 \,\}$, and $I = F \setminus (U \cup L)$. Note that U is a nonempty up-set in F ($1_F \in U$), L is nonempty down-set in F ($0_F \in L$), and $L \cap U = \emptyset$ by the choice of H_0. Also, $I \neq \emptyset$, since otherwise $H_0 = \hat{u}$. Pick $x \in I$. It follows that $\hat{x} \cap H_0 \subset H_0$, and thus $\hat{x} \cap H_0 = \widehat{w_x}$ for some $w_x \in L$. So $w_x \neq x$ and $\widehat{w_x} \subseteq \hat{x}$, which implies $w_x < x$, since $\hat{}$ is an embedding of F into L_F by Corollary 4.8. Further, consider j_k with $1 \leq k \leq r$. Since $\widehat{j_k} \subseteq H_0$, we have $\hat{x} \cap \widehat{j_k} \subseteq \hat{x} \cap H_0 = \widehat{w_x}$. Invoking the embedding property of $\hat{}$ again, we conclude that $x \wedge j_k \leq w_x$.

Using distributivity of S and Lemma 4.6(i), we find, for $1 \leq k \leq r$, an element $a_k \in S$ satisfying $a_k \geq j_k$ and $a_k \wedge x = w_x < x$. By Lemma 4.6(ii), we then find $b_x \in S$ satisfying $b_x \geq a_k$ ($\geq j_k$) for $1 \leq k \leq r$ and $b_x \wedge x = w_x$; moreover, $b_x \ngeq x$ for otherwise $w_x = b_x \wedge x = x$.

Define $b \in S$ by $b = \bigwedge_{x \in I} b_x \ \wedge \ u$. We claim that for any $y \in F$, either $b \wedge y = b$ or $b \wedge y \in F$: If $y \in U$, then $y \geq u \geq b$, and thus $b \wedge y = b$. If $y \in L$, then also $b \wedge y \in L$, since L is a down-set in F. If $y \in I$, then

$b \wedge y \leq b_y \wedge y = w_y \in L$, and thus $b \wedge y \in L$ again. It follows that $F_b := F \cup \{b\}$ is a subsemilattice of S containing F.

Consider $j \in \mathcal{J}(F)$ such that $j \leq b$: If $j \in U$, then $j \geq u$, in fact, $j > u$, since $u \notin \mathcal{J}(F)$. Hence, $j \not\leq b \leq u$. If $j \in L$, then $j \leq u$ and $j \leq j_k$ for some $1 \leq k \leq r$, which implies $j \leq b_x$ for all $x \in I$, thus $j \leq b$. If $j \in I$, then $j \not\leq b_j$ as shown above, thus $j \not\leq b$. Summing up, the down-set induced by b in $\mathcal{J}(F)$ is $L \cap \mathcal{J}(F) = H_0$. Moreover, $\mathsf{O}_F \cup \{(b, H_0)\}$ obviously is the canonical isomorphism O_{F_b} between F_b and the subsemilattice generated within $L_F = \mathcal{O}(\mathcal{J}(F))$ by $\operatorname{im} \mathsf{O}_F \cup \{H_0\}$. Repeat the procedure with F_b and iterate; the process breaks off with an isomorphism between some subsemilattice F_0 of S and L_F, making F_0 distributive. □

Since our concern is with p-semilattices, the natural question is whether any finite p-subsemilattice F of a distributive p-semilattice P can be extended, within P, to a finite *distributive* p-subsemilattice F' of P.

The starting observation is that $L_F = \mathcal{O}(\mathcal{J}(F))$, as a finite distributive lattice, is pseudocomplemented for any meet-semilattice F. Indeed, for any down-set $H \subseteq \mathcal{J}(F)$ the set $H^+ := \{ j \in \mathcal{J}(F) : \hat{\jmath} \cap H = \emptyset \}$ is its pseudocomplement.

Lemma 4.10. *Let F be a finite p-semilattice. Then the canonical embedding $\mathsf{O}_F \colon F \to L_F$ preserves pseudocomplements.*

Proof. We have to show that $\widehat{x^*} = \hat{x}^+$ for all $x \in F$. Now, $j \leq x^*$ iff $j \wedge x = 0$ for all $j \in \mathcal{J}(F)$, and thus

$$\begin{aligned} \hat{x}^+ &= \{ j \in \mathcal{J}(F) : \hat{\jmath} \cap \hat{x} = \emptyset \} = \{ j \in \mathcal{J}(F) : j \wedge x = 0 \} \\ &= \{ j \in \mathcal{J}(F) : j \leq x^* \} = \widehat{x^*}. \end{aligned}$$

□

This is enough to prove the following result.

Corollary 4.11. *Assume P is a distributive p-semilattice and F a finite p-subsemilattice of P. Then there exists a finite distributive p-semilattice F_0 such that $F \trianglelefteq F_0 \trianglelefteq P$. In fact, we find such F_0 satisfying $F_0 \cong L_F$.*

Proof. Consider F_b as in the proof of Proposition4.9 (note that 1_F exists and equals 1_P). All we need to show is that F_b is closed under pseudocomplements and that $\mathsf{O}_F \cup \{(b, H_0)\}$ preserves pseudocomplements. We claim that $b^* = u^*$, and thus $b^* \in F \subseteq F_b$. Indeed, since $b \leq u$, we have (1): $b^* \geq u^*$. Further, $j_k \leq b$ $(1 \leq k \leq r)$, thus $j_k^* \geq b^*$, and so (2): $j_1^* \wedge \cdots \wedge j_r^* \geq b^*$. Also, $j_1^* \wedge \cdots \wedge j_r^* \leq j_k^*$ $(1 \leq k \leq r)$, hence $(j_1^* \wedge \cdots \wedge j_r^*)^* \geq j_k^{**} \geq j_k$ for all k, which implies $(j_1^* \wedge \cdots \wedge j_r^*)^* \geq u$, and finally (3): $(j_1^* \wedge \cdots \wedge j_r^*)^{**} \leq u^*$. But certainly, (4): $j_1^* \wedge \cdots \wedge j_r^* \leq (j_1^* \wedge \cdots \wedge j_r^*)^{**}$. Putting all together, we obtain

$$j_1^* \wedge \cdots \wedge j_r^* \leq (j_1^* \wedge \cdots \wedge j_r^*)^{**} \leq u^* \leq b^* \leq j_1^* \wedge \cdots \wedge j_r^*,$$

using (4), (3), (1), and (2), respectively; this proves our claim. Finally, that $\mathsf{O}_F \cup \{(b, H_0)\}$ preserves pseudocomplements is immediate. □

5. Making the dense filter boolean

Suppose F_0 is a finite distributive p-subsemilattice of a distributive p-semilattice P. The purpose of this section is to show that we can find, provided P satisfies a certain condition (A1), a finite distributive p-subsemilattice $F_1 \trianglelefteq P$ such that $F_0 \trianglelefteq F_1 \trianglelefteq P$ and the dense filter $\mathrm{D}(F_1)$ is boolean.

We start by characterizing finite distributive p-semilattices F with boolean dense filters $\mathrm{D}(F)$ in terms of their associated posets $\mathcal{J}(F)$. Write $\mathcal{J}(F)_{\min}$ for the set of all *minimal* elements of $\mathcal{J}(F)$.

Lemma 5.1. *The dense filter of a finite distributive p-semilattice F is boolean iff $\mathcal{J}(F) \setminus \mathcal{J}(F)_{\min}$ is an antichain.*

Proof. Using $F \cong L_F \cong \mathcal{O}(\mathcal{J}(F))$, it is immediate that $\mathcal{J}(F)_{\min}$ represents the unique minimal dense element of F, and that $\mathrm{D}(F)$ is isomorphic to the collection of all down-sets $H \subseteq \mathcal{J}(F)$ containing $\mathcal{J}(F)_{\min}$ (under set inclusion), which in turn is isomorphic to the collection of all down-sets in $\mathcal{J}(F) \setminus \mathcal{J}(F)_{\min}$ (again under set inclusion). But for any finite poset Q, one has that $\mathcal{O}(Q)$ is boolean iff Q is an antichain. □

Consider any poset Q with order relation $\leq$ and $a, b \in Q$ such that b covers a w.r.t. to $\leq$. It is easy to see that $\leq' := \leq \setminus \{(a, b)\}$ is also an order relation on Q: Dropping (a, b) from $\leq$ does neither affect reflexivity nor antisymmetry, and since b covers a w.r.t. $\leq$, (a, b) cannot be forced back into $\leq'$ by applying transitivity to $\leq'$. We will use the short-hand notation Q'_{ab} for the resulting poset $(Q; \leq')$. We want to describe $\mathcal{O}(Q'_{ab})$:

Lemma 5.2. *Consider Q, a, b as above and let M the uniquely determined maximal down-set in $\mathcal{O}(Q)$ not containing a. Then*

$$\mathcal{O}(Q'_{ab}) = \mathcal{O}(Q) \cup \{\, U \cup \{b\} : U \in \mathcal{O}(Q) \text{ and } {\downarrow} b \cap M \subseteq U \subseteq M \,\}.$$

Proof. Since $\mathcal{O}(Q'_{ab}) \supseteq \mathcal{O}(Q)$ is clear, a down-set $V \in \mathcal{O}(Q'_{ab}) \setminus \mathcal{O}(Q)$ must contain b but not a. If $x \in V \setminus \{b\}$, then $x \in M$, for otherwise, $x \geq a$, putting a in V. Moreover, $b >' x$ implies $b > x$, and thus $x \in V$, so $V \setminus \{b\} \supseteq {\downarrow} b \cap M$. □

The property (A1) of p-semilattices is defined as follows:

$$\begin{aligned}(\forall d_0, d_1, d_2 \in \mathrm{D}(P), t \in P)(\exists x \in P)&\\ \big((d_0 < d_1 < d_2 \;\&\; t \wedge d_0 < t \wedge d_1 < t \wedge d_2) \Longrightarrow&\\ (d_0 < x < d_2 \;\&\; x \wedge d_1 = d_0 \;\&\; t \wedge d_0 < t \wedge x < t \wedge d_2)\big).&\end{aligned} \tag{A1}$$

Our present aim is to prove

Lemma 5.3. *Let P be a distributive p-semilattice satisfying* (A1) *and $F \trianglelefteq P$ a finite distributive p-subsemilattice of P. Let there be $j_1, j_2 \in \mathcal{J}(F) \setminus \mathcal{J}(F)_{\min}$ such that j_2 covers j_1. Then there exists a finite distributive p-semilattice F' satisfying $F \trianglelefteq F' \trianglelefteq P$ such that $\mathcal{J}(F')$ is order-isomorphic to $\mathcal{J}(F)'_{j_1 j_2}$.*

Proof. Let H_0 be the unique maximal down-set in $\mathcal{J}(F)$ *not* containing j_1. Certainly, $H_0 \supseteq \mathcal{J}(F)_{\min}$, so H_0 corresponds to an element $d_0 \in \mathrm{D}(F)$ under the canonical isomorphism $\mathsf{O}_F\colon F \to L_F$, that is, $\widehat{d_0} = H_0$. It follows that $w \not\geq j_1$ iff $w \leq d_0$, for any $w \in F$. Note that if $j \in \mathcal{J}(F)$ and $j < j_2$, then either $j = j_1$ or $j \leq d_0$ (since j_2 covers j_1 in $\mathcal{J}(F)$). Hence, there are $d_1, d_2 \in \mathrm{D}(F)$ such that $\widehat{d_1} = H_0 \cup \{j_1\}$ and $\widehat{d_2} = H_0 \cup \{j_1, j_2\}$.

We conclude that (i) $d_0 < d_1 < d_2$ and (ii) $j_2 \wedge d_0 < j_2 \wedge d_1 < j_2 \wedge d_2 = j_2$. Using (A1) with $t = j_2$, we thus find $x \in P$ such that $d_0 < x < d_2$, $d_1 \wedge x = d_0$, and $j_2 \wedge d_0 < j_2 \wedge x < j_2 \wedge d_2$. Note that $x \in \mathrm{D}(P)$, since $x > d_0 \in \mathrm{D}(F) \subseteq \mathrm{D}(P)$.

Let $F' = \langle F \cup \{x\} \rangle_P$ be the p-subsemilattice of P generated by F and x. Now, $F \cup \{\, w \wedge x : w \in F \,\} \subseteq P$ is obviously closed under meets; moreover, $(w \wedge x)^* = (w \wedge x)^{***} = (w^{**} \wedge x^{**})^* = (w^{**} \wedge 1)^* = w^{***} = w^* \in F$ as $x \in \mathrm{D}(P)$. We conclude that $F' = F \cup \{\, w \wedge x : w \in F \,\}$.

We analyze the structure of $F' \setminus F$: Suppose $w \in F$ but $w \wedge x \notin F$. Since $x \leq d_2$, we have $w \wedge x = w \wedge d_2 \wedge x$, hence—replacing w by $w \wedge d_2$—we can assume without loss of generality that $w \leq d_2$. So let $w \leq d_2$ and assume, towards a contradiction, that $w \leq d_1$. Then $w \wedge x \leq d_1 \wedge x = d_0$, and thus $w \wedge x = w \wedge x \wedge d_0 = w \wedge d_0 \in F$, contradicting $w \wedge x \notin F$. So we can assume without loss of generality that $w \not\leq d_1$. But, working in L_{F_0}, $\hat{w} \subseteq \widehat{d_2}$ and $\hat{w} \nsubseteq \widehat{d_1}$ are equivalent to $j_2 \in \hat{w}$, which translates into $j_2 \leq w$. Summing up, $F' \setminus F \subseteq \{\, w \wedge x : j_2 \leq w \leq d_2 \,\}$.

Conversely, let $w \in F$, $j_2 \leq w \leq d_2$, and suppose $w \wedge x \in F$. Then also $j_2 \wedge w \wedge x = j_2 \wedge x \in F$ (as $j_2 \leq w$). Now, $j_2 \wedge d_0 < j_2 \wedge x < j_2 \wedge d_2 = j_2$ by (A1). Looking at L_F, it is immediate that the unique down-set in $\mathcal{J}(F)$ situated strictly between $\widehat{j_2} \cap \widehat{d_0}$ and $\widehat{j_2}$ is given by $(\widehat{j_2} \cap \widehat{d_0}) \cup \widehat{j_1}$, and thus contains j_1. Translated back to F, this means that $j_1 \leq j_2 \wedge x$, and thus $j_1 \leq x$. But also $j_1 \leq d_1$ by construction of d_1, so $j_1 \leq x \wedge d_1 = d_0$, contradicting the choice of d_0. We conclude that $w \wedge x \notin F$ whenever $w \in F$ and $j_2 \leq w \leq d_2$. Summing up, we have established $F' \setminus F = \{\, w \wedge x : j_2 \leq w \leq d_2 \,\}$.

Finally, assume $w_1, w_2 \in F$, $j_2 \leq w_1, w_2 \leq d_2$, and $w_1 \wedge x = w_2 \wedge x$. This implies $w_1 \wedge d_0 = w_1 \wedge x \wedge d_1 = w_2 \wedge x \wedge d_1 = w_2 \wedge d_0$. Working in L_F, we have $\widehat{d_2} \setminus \widehat{d_0} = \{j_1, j_2\}$, and we conclude, observing that $j_1 \leq j_2 \leq w_1, w_2$, that $\widehat{w_1} = (\widehat{w_1} \cap \widehat{d_0}) \cup \{j_1, j_2\} = (\widehat{w_2} \cap \widehat{d_0}) \cup \{j_1, j_2\} = \widehat{w_2}$, that is, $w_1 = w_2$.

Define a map h from the interval $[j_2, d_2] \subseteq F$ to $F' \setminus F$ by $h(w) = w \wedge x$. So h is onto and injective by the above, and clearly order-preserving. Assume $w \wedge x \leq w' \wedge x$. Then $w \wedge w' \wedge x = w \wedge x$, and thus $w \wedge w' = w$ or $w \leq w'$. So h^{-1} also preserves order and the final result is that h is an order-isomorphism between $[j_2, d_2] \subseteq F$ and $F' \setminus F$.

Using h, we define a map $\phi\colon F' \to B_F = \mathcal{P}(\mathcal{J}(F))$ by

$$\phi(z) = \begin{cases} \hat{z} & \text{for } z \in F, \\ \hat{w} \setminus \{j_1\} & \text{for } z = h(w) \in F' \setminus F. \end{cases}$$

We want to determine $\phi[F'] \subseteq B_F$. Since $\phi(z) = \mathsf{O}_F(z)$ for $z \in F$, we certainly have $L_F \subseteq \phi[F']$. So consider $z = h(w) \in F' \setminus F$. Then $w \in [j_2, d_2] \subseteq F$, and thus $\hat{w} = \hat{u} \cup \{j_1, j_2\}$ for a uniquely determined $u \in [j_2 \wedge d_0, d_0] \subseteq F$, namely $u = w \wedge d_0$. Consequently, $\phi(z) = \hat{u} \cup \{j_2\} \in B_F \setminus L_F$. Conversely, any set $\hat{u} \cup \{j_2\} \in B_F$ with $u \in [j_2 \wedge d_0, d_0]$ has a unique preimage under ϕ, given by $z = w \wedge x$ where $\hat{w} = \hat{u} \cup \{j_1, j_2\}$. Since h is an order isomorphism, ϕ restricted to $F' \setminus F$ thus provides an order isomorphism between $F' \setminus F$ and $\{\, \hat{u} \cup \{j_2\} : u \in [j_2 \wedge d_0, d_0] \,\}$ (the latter ordered by set inclusion).

Putting this all together, we see—where $\cong$ is an order isomorphism—

$$F' \cong L_F \cup \{\, U \cup \{j_2\} : U \in \mathcal{O}(\mathcal{J}(F)) \text{ and } {\downarrow} j_2 \cap H_0 \subseteq U \subseteq H_0 \,\}.$$

By Lemma 5.2, the latter is just $\mathcal{O}((\mathcal{J}(F)'_{j_1 j_2})$, a (distributive) down-set lattice. An order isomorphism between lattices is always a lattice isomorphism, so $F' \cong \mathcal{O}(\mathcal{J}(F)'_{j_1 j_2})$ as lattices and $\mathcal{J}(F') \cong \mathcal{J}(F)'_{j_1 j_2}$ as posets. □

Corollary 5.4. *Assume F_0 is a finite distributive p-subsemilattice of a distributive p-semilattice P satisfying* (A1). *Then there exists a finite distributive p-semilattice F_1 such that $F_0 \trianglelefteq F_1 \trianglelefteq P$ and the dense filter* $\mathrm{D}(F_1)$ *is boolean.*

Proof. Let $G_0 = F_0$ and for $i \geq 0$, obtain G_{i+1} from G_i by applying Lemma 5.3 w.r.t. a covering pair of nonmimimal dense elements in G_i. The process stops when no such pair can be found; the final G_{i_0} has a boolean dense filter by Lemma 5.1. Put $F_1 = G_{i_0}$. □

6. Adding "central" elements to the skeleton

In this section, we assume that P is an arbitrary distributive p-semilattice, and $F \trianglelefteq P$ a finite distributive non-boolean p-subsemilattice whose dense filter $\mathrm{D}(F)$ is boolean, that is, $\mathrm{D}(F) \cong \mathbf{2}^n$ for some $n \geq 1$. It follows that $\mathrm{D}(F)$ contains n different maximally dense elements. Let $\mathrm{D}_{\max}(F) = \{d_1, \dots, d_n\}$.

Our purpose is to show that there exists a finite distributive p-subsemilattice $F' \trianglelefteq P$ such that $F \trianglelefteq F' \trianglelefteq P$ and $F' \cong \mathbf{2}^r \times \prod_{i=1}^n \hat{B}_i$ for some $r \in \omega$ and B_i a boolean algebra for $1 \leq i \leq n$, provided P satisfies the following property (A2) of p-semilattices:

$$\begin{gathered}(\forall a \in \mathrm{Sk}(P), d, d' \in \mathrm{D}(P), p, p', x \in P)(\exists z \in \mathrm{Sk}(P)) \\ \big((d \| d' \;\&\; p \leq d' \;\&\; p' \leq d \;\&\; p' \not\leq d' \;\&\; a \leq d \;\&\; a^* \wedge p \not\leq d \;\&\; x^* \leq d') \qquad \text{(A2)} \\ \Longrightarrow (a \leq z \leq d \;\&\; z^* \wedge p \not\leq d \;\&\; z \wedge p' \not\leq d' \;\&\; (z \wedge x)^* \leq d')\big).\end{gathered}$$

Roughly speaking, boolean elements as provided by (A2) will be used to manufacture a finite extension F' of F containing a decomposition $\{u_1, \dots, u_t\}$ of $1_{F'} = 1_F$ into finitely many pairwise disjoint boolean elements such that F'/θ_{u_i} is isomorphic to either $\mathbf{2}$ or $\hat{B}_i$ for $1 \leq i \leq t$.

We start by constructing, for every $d_i \in \mathrm{D}_{\max}(F)$, an element $k_i \in \mathrm{Sk}(P)$ satisfying certain properties, using (A2). This is accomplished in several steps.

Observe first that the set $H_i := \{ x \in F : x \not\leq d_i \}$ is closed under meets for every $d_i \in \mathrm{D}_{\max}(F)$. Indeed, assume towards a contradiction that $x, x' \in H_i$ but $x \wedge x' \leq d_i$. By distributivity, we find $y, y' \in F$ such that $y \geq x$, $y' \geq x'$, and $y \wedge y' = d_i$. So, $y, y' \in \mathrm{D}(F)$, and by maximality of d_i, we have $y = d_i$ or $y' = d_i$. Without loss of generality, assume $y = d_i$. But this violates $x \not\leq d_i$, proving our claim.

For $1 \leq i \leq n$, define $m_i = \bigwedge H_i$; it follows that m_i is the smallest element of F not below d_i. Observe further that $m_i \leq d_j$ for any $j \neq i$, $1 \leq j \leq n$: we have $d_j \not\leq d_i$ by the maximality of d_j, hence $m_i \leq d_j$ by the minimality of m_i.

These properties of the d_i and m_i together with (A2) prove the following lemma.

Lemma 6.1. *Assume* (A2) *and suppose that* $k \in \mathrm{Sk}(P)$ *and* $d_i \in \mathrm{D}_{\max}(F)$ *satisfy* $k \leq d_i$ *and* $k^* \wedge m_i \not\leq d_i$. *Then for any* $d_j \in \mathrm{D}_{\max}(F)$ *with* $j \neq i$, *there exists* $z \in \mathrm{Sk}(P)$ (*depending on* j) *such that* $k \leq z \leq d_i$, $z^* \wedge m_i \not\leq d_i$, $z \wedge m_j \not\leq d_j$, *and satisfying that for all* $x \in F$, $x^* \leq d_j \Rightarrow (z \wedge x)^* \leq d_j$.

Proof. Let $X_j = \{ x \in F : x^* \leq d_j \} = \{x_{j1}, \ldots, x_{jn(j)}\}$. Use (A2) with $a := k$, $d := d_i$, $d' := d_j$, $p := m_i$, $p' := m_j$, and $x := x_{j1}$. (It is routine to check that the assumptions in (A2) are all satisfied). So, by (A2), there exists $z_1 \in \mathrm{Sk}(P)$ such that $k \leq z_1 \leq d_i$, $z_1^* \wedge m_i \not\leq d_i$, $z_1 \wedge m_j \not\leq d_j$, and $(z_1 \wedge x_{j1})^* \leq d_j$.

Now apply (A2) with d, d', p, p' as above but with $a := z_1$ and $x := x_{j2}$ (again, all the assumptions in (A2) are satisfied). So we find $z_2 \in \mathrm{Sk}(P)$ such that $z_1 \leq z_2 \leq d_i$, $z_2^* \wedge m_i \not\leq d_i$, $z_2 \wedge m_j \not\leq d_j$, and $(z_2 \wedge x_{j2})^* \leq d_j$. Note that $z_1 \leq z_2$ implies $z_1 \wedge x_{j1} \leq z_2 \wedge x_{j1}$, and thus $(z_2 \wedge x_{j1})^* \leq (z_1 \wedge x_{j1})^* \leq d_j$; consequently, z_2 also satisfies $(z_2 \wedge x_{j1})^* \leq d_j$.

Continue until X_j is exhausted. The final $z_{n(j)}$ has all of the properties required by the lemma, so put $z = z_{n(j)}$. □

Lemma 6.2. *Assume* (A2). *Then for every* $d_i \in D_{\max}$, *there exists* $k_i \in \mathrm{Sk}(P)$ *such that* $k_i \leq d_i$, $k_i^* \wedge m_i \not\leq d_i$, $k_i \wedge m_j \not\leq d_j$ (*for all* $j \neq i$), *and satisfying that for all* $x \in F$, $x^* \leq d_j \Rightarrow (k_i \wedge x)^* \leq d_j$ (*for all* $j \neq i$).

Proof. Assume, without loss of generality, that $i = 1$, and put $h_1 = 0$. This means that $h_1 \in \mathrm{Sk}(P)$, $h_1 \leq d_1$ and $h_1^* \wedge m_1 \not\leq d_1$. Put $j = 2$ and apply Lemma 6.1 in order to obtain an element z satisfying $0 = h_1 \leq z \leq d_1$, $z^* \wedge m_1 \not\leq d_1$, $z \wedge m_2 \not\leq d_2$ and $\forall x \in F(x^* \leq d_2 \Rightarrow (z \wedge x)^* \leq d_2)$. We put $h_2 := z$.

Put $j = 3$ and repeat to obtain $h_3 \in \mathrm{Sk}(P)$ satisfying $h_2 \leq h_3 \leq d_1$, $h_3^* \wedge m_1 \not\leq d_1$, $h_3 \wedge m_3 \not\leq d_3$, and for all $x \in F$, $x^* \leq d_3 \Rightarrow (h_3 \wedge x)^* \leq d_3$.

Now h_3 works also for d_2: indeed, since $h_2 \leq h_3$, we have $h_2 \wedge m_2 \leq h_3 \wedge m_2$, and so $h_3 \wedge m_2 \not\leq d_2$, since $h_2 \wedge m_2 \not\leq d_2$. Similarly, for any $x^* \leq d_2$, we have $h_2 \wedge x \leq h_3 \wedge x$, and so $(h_3 \wedge x)^* \leq (h_2 \wedge x)^* \leq d_2$.

Continuing, we finally obtain $h_n \in \mathrm{Sk}(P)$ satisfying $h_n \leq d_1$, $h_n^* \wedge m_1 \not\leq d_1$, $h_n \wedge m_j \not\leq d_j$ (for all $j \neq 1$), and for all $x \in F$, $x^* \leq d_j \Rightarrow (h_n \wedge x)^* \leq d_j$) (for all $j \neq 1$). Put $k_1 = h_n$. □

Note that $k_i^* \wedge m_i \not\leq d_i$ is equivalent to $k_i^* \wedge z \not\leq d_i$ *for all* $z \in H_i$. One direction is clear, since $m_i \in F$ and $m_i \not\leq d_i$. For the other, assume $z \in F$ and $z \not\leq d_i$. Then $z \geq m_i$ by definition of m_i, and thus $k_i^* \wedge z \geq k_i^* \wedge m_i$. So if $k_i^* \wedge m_i \not\leq d_i$, then $k_i^* \wedge z \not\leq d_i$ *a fortiori*. The same argument shows that $k_i \wedge m_j \not\leq d_j$ $(j \neq i)$ is equivalent to $k_i \wedge z \not\leq d_j$ for all $z \in H_j$.

This gives the final description of the elements $k_i \in \mathrm{Sk}(P)$ we are after.

Lemma 6.3. *For each element $d_i \in \mathrm{D}_{max(F)}$, there exists $k_i \in \mathrm{Sk}(P)$ such that*

(i) $k_i \leq d_i$,
(ii) *for* $z \in F$, $z \not\leq d_i \;\Rightarrow\; k_i^* \wedge z \not\leq d_i$,
(iii) *for* $j \neq i$ *and* $z \in F$, $z \not\leq d_j \;\Rightarrow\; k_i \wedge z \not\leq d_j$,
(iv) *for* $j \neq i$ *and* $x \in F$, $x^* \leq d_j \;\Rightarrow\; (k_i \wedge x)^* \leq d_j$.

For easier reference, we list some consequences of the preceding lemma.

Corollary 6.4. *The elements k_i described in Lemma 6.3 have the following additional properties:*

(ii-bis) *for* $z \in F$, $k_i^* \wedge z \leq d_i \;\Rightarrow\; z \leq d_i$,
(iii-bis) *for* $j \neq i$ *and* $z \in F$, $k_i \wedge z \leq d_j \;\Rightarrow\; z \leq d_j$,
(v) *for* $y \in \mathrm{Sk}(F)$, $y \leq d_i \;\Rightarrow\; k_i \sqcup y \leq d_i$,
(vi) *for* $y \in \mathrm{Sk}(F)$, $y \leq d_j \;\Rightarrow\; k_i^* \sqcup y \leq d_j$.

Proof. (ii-bis) and (iii-bis) are just the contrapositions of (ii) and (iii), respectively, in the preceding lemma.

(v): Assume $y \in \mathrm{Sk}(F), y \leq d_i$, and put $z = k_i \sqcup y$. Then $k_i^* \wedge z = k_i^* \wedge (k_i \sqcup y) = k_i^* \wedge y \leq y \leq d_i$. Using (ii-bis), we obtain $z \leq d_i$ as desired.

(vi): Assume $y \in \mathrm{Sk}(F), y \leq d_j$, and put $z = k_i^* \sqcup y$. Then $k_i \wedge z = k_i \wedge (k_i^* \sqcup y) = k_i \wedge y \leq y \leq d_j$, which implies $z \leq d_j$, using (iii-bis). □

Next, consider $F[k_i]$, the p-semilattice generated in P by $F \cup \{k_i\}$. Write $\mathrm{Sk}(F)[k_i]$ for the (boolean) subalgebra of $\mathrm{Sk}(P)$ generated by $\mathrm{Sk}(F) \cup \{k_i\}$; it is easy to see that $\mathrm{Sk}(F[k_i]) = \mathrm{Sk}(F)[k_i]$. Moreover, $\mathrm{D}(F[k_i]) = \mathrm{D}(F)$. Since F is distributive, and thus representable by Lemma 4.4, it follows that also $F[k_i]$ is representable.

Lemma 6.5. *$F[k_i]$ is distributive.*

Proof. Using Lemma 4.5, it suffices to show that each $d \in \mathrm{D}(F[k_i]) = \mathrm{D}(F)$ is distributive. But $\mathrm{D}(F)$ is boolean and finite, so every $d \in \mathrm{D}(F)$ is the meet of all $d_j \in \mathrm{D}_{\max}(F)$ covering d. Since the meet of distributive elements is always distributive, it remains to check that every $d_j \in \mathrm{D}_{\max}(F)$ is distributive in $F[k_i]$.

We have $F[k_i] = \{\, b \wedge d : b \in \mathrm{Sk}(F[k_i]), d \in \mathrm{D}(F[k_i]) \,\}$, since $F[k_i]$ is representable. Using conjunctive normal form for boolean terms and $\mathrm{D}(F[k_i]) = \mathrm{D}(F)$, this boils down to

$$F[k_i] = \{\, (a \sqcup k_i) \wedge (b \sqcup k_i^*) \wedge d : a, b \in \mathrm{Sk}(F), d \in \mathrm{D}(F) \,\}.$$

So assume $d_j \geq v \wedge w$ with $d_j \in \mathrm{D}_{\max}(F)$ and $v, w \in F[k_i]$. We want to find $v', w' \in F[k_i]$ such that $v' \geq v$, $w' \geq w$, and $v' \wedge w' = d_j$. Explicitly, v, respectively, w is given as $v = (v_1 \sqcup k_i) \wedge (v_2 \sqcup k_i^*) \wedge d_v$, respectively, $w = (w_1 \sqcup k_i) \wedge (w_2 \sqcup k_i^*) \wedge d_w$ with $v_1, v_2, w_1, w_2 \in \mathrm{Sk}(F)$ and $d_v, d_w \in \mathrm{D}(F)$.

Case 1: $j \neq i$. We have

$$\begin{aligned} d_j &\geq ((v_1 \sqcup k_i) \wedge (v_2 \sqcup k_i^*) \wedge d_v) \wedge ((w_1 \sqcup k_i) \wedge (w_2 \sqcup k_i^*) \wedge d_w) \\ &= ((v_1 \wedge w_1) \sqcup k_i) \wedge ((v_2 \wedge w_2) \sqcup k_i^*) \wedge d_v \wedge d_w \geq k_i \wedge (v_2 \wedge w_2) \wedge d_v \wedge d_w. \end{aligned}$$

Putting $z = (v_2 \wedge w_2) \wedge d_v \wedge d_w$, we have $d_j \geq k_i \wedge z$. Observe that $z \in F$; so Corollary 6.4(iii-bis) applies and gives $z \leq d_j$, that is, $d_j \geq (v_2 \wedge w_2) \wedge (d_v \wedge d_w)$. Using distributivity of F, we find $d_1 \geq v_2 \wedge w_2$, $d_2 \geq d_v \wedge d_w$ such that $d_1 \wedge d_2 = d_j$. This makes d_1, d_2 dense and thus $d_1 = d_j$ or $d_2 = d_j$, since $d_j \in \mathrm{D}_{\max}(F)$.

Suppose $d_j = d_2$. Then $d_j \geq d_v \wedge d_w$, and by distributivity of F again, we find $d'_v \geq d_v$, $d'_w \geq d_w$ such that $d'_v \wedge d'_w = d_j$. By maximality of d_j, we must have $d'_v = d_j$ or $d'_w = d_j$. In the first case, we obtain $d_j = d'_v \geq d_v \geq v$. Putting $v' = d_j$, $w' = 1$, we realize $v' \geq v$, $w' \geq w$, and $v' \wedge w' = d_j$, as desired. If $d'_w = d_j$, the analogous argument shows that $v' = 1$ and $w' = d_j$ work as well.

It remains to consider the case $d_j = d_1$. This time, we have $d_j \geq v_2 \wedge w_2$, and distributivity of F provides $d'_v \geq v_2$, $d'_w \geq w_2$ such that $d'_v \wedge d'_w = d_j$. Again, $d_j = d'_v$, and thus $d_j \geq v_2$, or $d_j = d'_w$, and thus $d_j \geq w_2$.

Use Lemma 6.3(iv) with $x = 1$ to obtain $d_j \geq k_i^*$. Applying Corollary 6.4(vi), we deduce that $d_j \geq v_2 \sqcup k_i^*$ or $d_j \geq w_2 \sqcup k_i^*$. In the first case, $d_j \geq (v_2 \sqcup k_i^*) \wedge (v_1 \sqcup k_i) \wedge d_v = v$; in the second, $d_j \geq (w_2 \sqcup k_i^*) \wedge (w_1 \sqcup k_i) \wedge d_w = v$. This shows that $v' = d_j$ and $w' = 1$, respectively $v' = 1$ and $w' = d_j$, have the desired properties.

Case 2: $j = i$. The arguments have the same structure as in the case $j \neq i$, so we give only an outline. Start from

$$\begin{aligned} d_i &\geq ((v_1 \sqcup k_i) \wedge (v_2 \sqcup k_i^*) \wedge d_v) \wedge ((w_1 \sqcup k_i) \wedge (w_2 \sqcup k_i^*) \wedge d_w) \\ &= ((v_1 \wedge w_1) \sqcup k_i) \wedge ((v_2 \wedge w_2) \sqcup k_i^*) \wedge d_v \wedge d_w \geq (v_1 \wedge w_1) \wedge k_i^* \wedge d_v \wedge d_w. \end{aligned}$$

Put $z = (v_1 \wedge w_1) \wedge d_v \wedge d_w$ to obtain $d_i \geq k_i^* \wedge z$. Since $z \in F$, Corollary 6.4(ii-bis) applies and gives $z \leq d_i$, that is, $d_i \geq (v_1 \wedge w_1) \wedge (d_v \wedge d_w)$. By distributivity of F, find $d_1 \geq v_1 \wedge w_1$, $d_2 \geq d_v \wedge d_w$ such that $d_1 \wedge d_2 = d_i$, thus $d_1 = d_i$ or $d_2 = d_i$.

If $d_i = d_1$, obtain $v' = d_i$ and $w' = 1$, respectively $v' = 1$ and $w' = d_i$, as in Case 1. If $d_i = d_2$, the same arguments work, using Lemma 6.3(i) and Corollary 6.4(v). □

We are now ready to construct $F' \trianglelefteq P$ such that we have $F \trianglelefteq F' \trianglelefteq P$ and $F' \cong \mathbf{2}^r \times \prod_{i=1}^n \hat{B}_i$. Observe that $F[k_i]$ is a finite distributive p-subsemilattice of P containing F and having the same dense filter as F. So we can iterate the construction of $F[k_i]$ with some other $d_{i'} \in \mathrm{D}_{\max}(F)$, finding an element

$k_{i'} \in \mathrm{Sk}(P)$ that has all the required properties with respect to $F[k_i]$ and thus, a fortiori, with respect to F.

Explicitly, let $G_0 = F$ and for $1 \le i \le n$, put $G_i = G_{i-1}[k_i]$. Then define $F' = G_n$. F' is distributive, $\mathrm{D}(F') = \mathrm{D}(F)$, and in particular, F' has the following properties: for every element $d_i \in \mathrm{D}_{\max}(F')$, there is $k_i \in \mathrm{Sk}(F')$ with

(i′) $k_i \le d_i$ by 6.3(i),
(ii′) $k_i^* \not\le d_i$ by 6.3(ii), setting $z = 1$,
(iii′) $k_i \not\le d_j$ by 6.3(iii) for $j \ne i$, setting $z = 1$,
(iv′) $k_i^* \le d_j$ by 6.3(iv) for $j \ne i$, setting $x = 1$.

Proposition 6.6. *For some $r \in \omega$, $F' \cong \mathbf{2}^r \times \prod_{i=1}^n \hat{B}_i$, with B_i a boolean algebra for $1 \le i \le n$.*

Proof. Let $C = \{\, a \in \mathrm{Sk}(F') : a \le d \in \mathrm{D}(F') \text{ implies } d = 1 \,\}$. $C \ne \emptyset$, since $1 \in C$. Moreover, C is closed under meets. Let $a, b \in C$ and $a \wedge b \le d \in \mathrm{D}(F')$. By distributivity, we find d_a, d_b such that $d_a \ge a$, $d_b \ge b$, and $d_a \wedge d_b = d$. Hence, $d_a, d_b \in \mathrm{D}(F')$,and so $d_a = d_b = 1$, thus $d = 1$. Since F' is finite, $c_0 := \bigwedge C$ exists and is the smallest element of C. Note that $c_0 \ne 0$ (otherwise F' would be boolean).

The elements of F' have a canonical form. For $b \in \mathrm{Sk}(F')$, let $\Delta_b = \{\, d_l \in \mathrm{D}_{\max} : b \not\le d_l \,\}$. Since F' is representable and $\mathrm{D}(F')$ is boolean, it is clear that for every $u \in F'$, there exists a representation $u = b \wedge \bigwedge Q$ with $b \in \mathrm{Sk}(F')$ and some subset $Q \subseteq \Delta_b$ (note that $\bigwedge Q = 1$ iff $Q = \emptyset$). Assume $u = b \wedge \bigwedge Q = b' \wedge \bigwedge Q'$ are two different representations of this type. Applying **, we obtain $u^{**} = b = b'$, so $Q \ne Q'$, and we find, without loss of generality, an element $d \in Q \setminus Q'$. Note that $Q' \ne \emptyset$ for otherwise $\bigwedge Q' = 1$, and so $b = b \wedge 1 = b \wedge \bigwedge Q$, implying $b \le \bigwedge Q \le d$. Writing $Q' = \{d'_1, \dots, d'_t\}$, we obtain $d \ge b \wedge d'_1 \wedge \cdots \wedge d'_t$. By distributivity, there are $v, w \in F'$ such that $v \ge b$, $w \ge d'_1 \wedge \cdots \wedge d'_t$, and $v \wedge w = d$. By maximality of d, it follows that $v = d$ or $w = d$. But $v = d$ is not possible, since $d \not\ge b$, so we must have $w = d$, that is, $d \ge d'_1 \wedge \cdots \wedge d'_t$. This implies the existence of y, z such that $y \ge d'_1$, $z \ge d'_2 \wedge \cdots \wedge d'_t$, and $y \wedge z = d$. Repeat the procedure, using distributivity successively, to obtain finally that $d = d'_s$ for some $d'_s \in Q'$, which contradicts our choice of d. It follows that $Q = Q'$. So there is a unique subset $Q_u \subseteq \Delta_{u^{**}}$ such that $u = u^{**} \wedge \bigwedge Q_u$, and obviously $Q_u = \{\, d_l \in \mathrm{D}_{\max} : u \le d_l \text{ and } u^{**} \not\le d_l \,\}$. Consequently, the correspondence $u \longleftrightarrow (u^{**}, Q_u)$ is bijective.

For $1 \le i \le n$, define $a_i = k_i \sqcup c_0^* \in \mathrm{Sk}(F')$. We claim that $a_i \sqcup a_j = 1$ for $i \ne j$: Let $a_i \sqcup a_j \le d \in D(F')$. If $d \ne 1$, there exists $d_k \in \mathrm{D}_{\max}(F')$ such that $d \le d_k$, implying $k_i \le a_i \le d_k$ and $k_j \le a_j \le d_k$. By (iii′) above, we have $i = k$ and $j = k$, contradicting $i \ne j$. Thus, $d = 1$ and, consequently, $a_i \sqcup a_j \ge c_0$ by the definition of c_0. But $a_i, a_j \ge c_0^*$ by the definition of the a_i, so $a_i \sqcup a_j \ge c_0 \sqcup c_0^* = 1$, as claimed.

By definition, $a_i^* = (k_i \sqcup c_0^*)^* = k_i^* \wedge c_0$. Note that $a_i^* \neq 0$: Otherwise, $k_i^* \wedge c_0 = 0$, which implies $c_0 \leq k_i^{**} = k_i \leq d_i$ (by (i′) above), contradicting the definition of c_0. We have $a_i^* \wedge a_j^* = (a_i^* \wedge a_j^*)^{**} = (a_i \sqcup a_j)^* = 1^* = 0$ for $i \neq j$. Moreover, $a_i^* \wedge c_0^* = k_i^* \wedge c_0 \wedge c_0^* = 0$ for $1 \leq i \leq n$.

On the other hand, $a_1^* \sqcup \cdots \sqcup a_n^* = (k_1^* \sqcup \cdots \sqcup k_n^*) \wedge c_0 \leq c_0$. If we have that $(k_1^* \sqcup \cdots \sqcup k_n^*) \wedge c_0 < c_0$, there exists $1 \neq d \in \mathrm{D}(F')$—and with that, $d_l \in \mathrm{D}_{\max}(F')$—such that $(k_1^* \sqcup \cdots \sqcup k_n^*) \wedge c_0 \leq d \leq d_l$. Using distributivity, we find $d_1 \geq k_1^* \sqcup \cdots \sqcup k_n^*$ and $d_2 \geq c_0$ such that $d_1 \wedge d_2 = d_l$. Hence, $d_1, d_2 \in \mathrm{D}(F')$, and thus $d_2 = 1$, which gives $d_1 = d_l$. But $d_l \geq k_1^* \sqcup \cdots \sqcup k_n^*$ implies $d_l \geq k_l^*$, contradicting (ii′) above. Consequently, $a_1^* \sqcup \cdots \sqcup a_n^* = (k_1^* \sqcup \cdots \sqcup k_n^*) \wedge c_0 = c_0$. Summing up, we see that $\{c_0^*, a_1^*, \ldots, a_n^*\}$ provides a boolean partition of 1.

We next determine the structure of the factor algebras $F'/\theta_{a_i^*}$ for $1 \leq i \leq n$. Now, $F'/\theta_{a_i^*} \cong \{\, u \wedge a_i^* : u \in F' \,\}$ (the latter with the operations given in Section 2). So we have to compute the meets $u^{**} \wedge \bigwedge Q_u \wedge a_i^*$ for $u \in F'$. Now, $a_i^* = k_i^* \wedge c_0$, and $k_i^* \leq d_l$ for $l \neq i$ by (iv′) above, so $u \wedge a_i^* = u^{**} \wedge a_i^*$ if $d_i \notin Q_u$, and $u \wedge a_i^* = u^{**} \wedge a_i^* \wedge d_i$ if $d_i \in Q_u$. We distinguish the cases $u^{**} \geq a_i^*$, respectively, $u^{**} \ngeq a_i^*$.

First, assume that $u^{**} \geq a_i^*$. Then $u \wedge a_i^* = a_i^*$ if $d_i \notin Q_u$, and $u \wedge a_i^* = a_i^* \wedge d_i$ if $d_i \in Q_u$. We claim that $a_i^* \wedge d_i < a_i^*$. If not, $a_i^* = k_i^* \wedge c_0 \leq d_i$. But then, by distributivity, there are v, w such that $k_i^* \leq v$, $c_0 \leq w$, and $v \wedge w = d_i$, implying $v = d_i$ or $w = d_i$. Now $w = d_i$ yields $c_0 \leq d_i$, which is not possible, and $v = d_i$ means $k_i^* \leq d_i$, violating (ii′). Thus, $a_i^* \wedge d_i < a_i^*$ as claimed. Since $(a_i^* \wedge d_i)^{**} = a_i^*$, we see that $a_i^* \wedge d_i \notin \mathrm{Sk}(F')$.

Next, suppose $u^{**} \ngeq a_i^*$. Then $u^{**} \wedge a_i^* < a_i^*$. We have $(u^{**} \wedge a_i^*) \sqcup k_i \ngeq c_0$. (Since meeting $(u^{**} \wedge a_i^*) \sqcup k_i \geq c_0$ on both sides with k_i^* gives $u^{**} \wedge a_i^* \wedge k_i^* \geq k_i^* \wedge c_0 = a_i^*$, violating $u^{**} \wedge a_i^* < a_i^*$.) So there exists d_l such that $d_l \geq (u^{**} \wedge a_i^*) \sqcup k_i$, whence $d_l \geq k_i$, and thus $l = i$ by (iii′). We conclude that $d_i \geq (u^{**} \wedge a_i^*) \sqcup k_i \geq u^{**} \wedge a_i^*$. So $u^{**} \wedge a_i^* \leq a_i^* \wedge d_i$ (in fact, $u^{**} \wedge a_i^* < a_i^* \wedge d_i$, since $a_i^* \wedge d_i$ is non-boolean), and we obtain $u \wedge a_i^* = u^{**} \wedge a_i^*$ whether $d_i \in Q_u$ or not.

In other words, $\{\, u \wedge a_i^* : u \in F' \,\}$ consists of a_i^*, $a_i^* \wedge d_i$, and all $b' \in \mathrm{Sk}(F')$ with $b' < a_i^*$, and the latter all satisfy $b' \leq a_i^* \wedge d_i$. Moreover, $a_i^* \wedge d_i$ is the only non-boolean element occurring. It follows that $F'/\theta_{a_i^*} \cong \hat{B}_i$ with B_i a finite boolean algebra.

It remains to compute $F'/\theta_{c_0^*}$. Consider $d_l \in \mathrm{D}_{\max}(F')$. Clearly, $d_l \geq 0 = c_0 \wedge c_0^*$, so there are u, v such that $u \geq c_0$, $v \geq c_0^*$, and $u \wedge v = d_l$. This implies $u = 1$ (since $u \in \mathrm{D}(F')$ and $c_0 \in C$), thus $v = d_l$. This shows that $d_l \geq c_0^*$ for all $d_l \in \mathrm{D}_{\max}(F')$.

Again, $F'/\theta_{c_0^*} \cong \{\, u \wedge c_0^* : u \in F' \,\}$ (the latter with the operations given in Section 2). So we have to compute the meets $u^{**} \wedge \bigwedge Q_u \wedge c_0^*$ for $u \in F'$. But since $d_l \geq c_0^*$ for any $d_l \in \mathrm{D}_{\max}(F')$, so $u^{**} \wedge \bigwedge Q_u \wedge c_0^* = u^{**} \wedge c_0^*$, and so we have $u \wedge c_0^* = u^{**} \wedge c_0^*$ for all $u \in F'$, whence

$$\{\, u \wedge c_0^* : u \in F' \,\} = \{\, b \in \mathrm{Sk}(F') : b \leq c_0^* \,\}.$$

It follows that $F'/\theta_{c_0^*}$ is a finite boolean algebra. Note that $F'/\theta_{c_0^*}$ is the trivial one-element algebra if $c_0 = 1$.

Let the canonical homomorphism $h\colon F' \to F'/\theta_{c_0^*} \times \prod_{i=1}^n F'/\theta_{a_i^*}$ be given by $h(u) := (u \wedge c_0^*, u \wedge a_1^*, \dots, u \wedge a_n^*)$; it is injective and surjective.

For injectivity, consider $v, w \in F'$, $v \neq w$. Suppose first that $v^{**} \neq w^{**}$. Since $\{c_0^*, a_1^*, \dots, a_n^*\}$ is a partition, so $v^{**} \wedge c_0^* \neq w^{**} \wedge c_0^*$ or $v^{**} \wedge a_l^* \neq w^{**} \wedge a_l^*$ for some $1 \leq l \leq n$. In the first case, we are done, since $u \wedge c_0^* = u^{**} \wedge c_0^*$ for all $u \in F'$, and thus $v \wedge c_0^* \neq w \wedge c_0^*$. In the second, remember that $u \wedge a_i^*$ equals a_i^* or $a_i^* \wedge d_i$ if $u^{**} \geq a_i^*$, and $u^{**} \wedge a_i^*$ if $u^{**} \ngeq a_i^*$ (and then $u^{**} \wedge a_i^* < a_i^* \wedge d_i$). Since $v^{**} \wedge a_l^* \neq w^{**} \wedge a_l^*$, we cannot have $v^{**}, w^{**} \geq a_l^*$, so suppose, without loss of generality, that $v^{**} \ngeq a_l^*$, which implies $v \wedge a_l^* = v^{**} \wedge a_l^*$. If also $w^{**} \ngeq a_l^*$, then $w \wedge a_l^* = w^{**} \wedge a_l^* \neq v^{**} \wedge a_l^* = v \wedge a_l^*$, and we are done. If $w^{**} \geq a_l^*$, then $v \wedge a_l^* = v^{**} \wedge a_l^* < a_l^* \wedge d_l \leq w \wedge a_l^*$, settling also this case.

Now suppose $v^{**} = w^{**}$. This implies $Q_v \neq Q_w$, so assume, without loss of generality, that there is $d_l \in Q_w \backslash Q_v$. It follows that $v \leq d_l$ but $w \nleq d_l$.We infer that $v \wedge a_l^* \leq a_l^* \wedge d_l$. Suppose, towards a contradiction, that $v \wedge a_l^* = w \wedge a_l^*$. Then $w \wedge a_l^* \leq d_l$ and, by distributivity, we find x, y such that $x \geq v$,$y \geq a_l^*$ and $x \wedge y = d_l$. As usual, we must have $x = d_l$ or $y = d_l$, which forces the contradiction $v \leq d_l$, respectively, $a_l^* \leq d_l$. Thus,$v \wedge a_l^* \neq w \wedge a_l^*$, as desired.

For surjectivity, consider, without loss of generality,

$$w = (b_0, b_1, \dots, b_k, b_{k+1} \wedge d_{k+1}, \dots, b_n \wedge d_n)$$

in $F'/\theta_{c_0^*} \times \prod_{i=1}^n F'/\theta_{a_i^*}$ with $b_0, \dots, b_n \in \mathrm{Sk}(F')$, $b_0 \leq c_0$, and $b_j \leq a_j^*$ for $1 \leq j \leq n$, and $d_l \in \mathrm{D}_{\max}(F')$ for $k+1 \leq l \leq n$. It follows that $w^{**} = (b_0, \dots, b_n)$. Put $x = (b_0 \sqcup \cdots \sqcup b_n) \wedge d_{k+1} \wedge \cdots \wedge d_n$. Then $h(x) = w$. □

Corollary 6.7. *Assume P is an arbitrary distributive p-semilattice satisfying* (A2)*, and $F_1 \trianglelefteq P$ a finite distributive p-semilattice such that* $\mathrm{D}(F_1) \cong \mathbf{2}^n$ *for some $n \geq 1$. Then there exists a finite distributive p-semilattice F_2 such that $F_1 \trianglelefteq F_2 \trianglelefteq P$ and $F_2 \cong \mathbf{2}^r \times \prod_{i=1}^n \hat{B}_i$, for some $r \in \omega$ and B_i a boolean algebra for $1 \leq i \leq n$.*

Proof. Use $F_2 := F'$ as provided by Proposition 6.6. □

7. Extending factors $\hat{F}_n$ to $\hat{\mathbf{A}}$

Assume that P is an arbitrary distributive p-semilattice, $F \trianglelefteq P$ a finite distributive p-semilattice of the form $F \cong \mathbf{2}^r \times \prod_{i=1}^s \hat{B}_i$ for some $r \in \omega$, and B_i a finite boolean algebra for $1 \leq i \leq s$. We will show that F can be extended to a p-semilattice F' such that $F \trianglelefteq F' \trianglelefteq P$ and $F' \cong \mathbf{2}^r \times (\hat{\mathbf{A}})^s$ (with $\mathbf{A}$ the countable atom-free boolean algebra), provided P satisfies the following property (A3):

$$(\forall b_1 \in \mathrm{Sk}(P),\, d \in \mathrm{D}(P))(\exists b_2 \in \mathrm{Sk}(P)) \big(b_1 < d < 1 \Longrightarrow b_1 < b_2 < d \;\&\; b_1 \sqcup b_2^* < d\big). \tag{A3}$$

The key ingredient needed to prove the above statement is contained in the following lemma.

Lemma 7.1. *Let $F \cong \hat{F}_k \times F'$, with $k \geq 1$ and F' any finite distributive p-semilattice. If P is any distributive p-semilattice satisfying* (A3) *and $F \trianglelefteq P$, there exists $F^+ \cong \hat{F}_{k+1} \times F'$ such that $F \trianglelefteq F^+ \trianglelefteq P$. Such F^+ can be obtained by "splitting" any atom of $\hat{F}_k$.*

Proof. Assume $F \cong \hat{F}_k \times F' \trianglelefteq P$ and $k \geq 2$. Let $c = (1,0)$ and $c^* = (0,1)$ be the central elements of F associated with the direct product decomposition of F specified above. Pick an atom of F such that $a \leq c$. It follows that $a \in \mathrm{Sk}(F)$; moreover, a^* is a coatom of $\mathrm{Sk}(F)$ and $a^* \geq c^*$. Further, let $e \in \mathrm{D}(F)$ be the unique dense element satisfying $e \neq 1$ and $e \geq c^*$. Now use (A3) to find $u^* \in \mathrm{Sk}(P)$ such that $a^* < u^* < e$ and $a^* \sqcup u < e$.

We have $u < a$, since $u^* > a^*$, and $u \neq 0$ (for otherwise $u^* = 1 \not\leq e$), hence $0 < u < a$. Consider $a \wedge u^*$: $a \wedge u^* = (a^* \sqcup u)^* \neq 0$ (for otherwise $(a^* \sqcup u)^{**} = a^* \sqcup u = 1 \not\leq e$), and $a \wedge u^* \neq u^*$ (for otherwise $u^* \leq a$, whence $a^* \leq u^{**} = u < a$, and thus $a = 1$). Summing up, we have $0 < a \wedge u^* < u^*$ and obviously $u \wedge (a \wedge u^*) = 0$ and $u \sqcup (a \wedge u^*) = a$. So, u and $a \wedge u^*$ provide a proper splitting of the atom a.

Let $F[u]$ be the p-semilattice generated $F \cup \{u\}$ within P. It is clear that $F[u]$ is representable, being generated by $\mathrm{Sk}(F)[u]$ (the p-semilattice generated by $\mathrm{Sk}(F) \cup \{u\}$ within $\mathrm{Sk}(P)$) together with $\mathrm{D}(F)$, and that $\mathrm{Sk}(F[u]) = \mathrm{Sk}(F)[u]$. So we start by describing $\mathrm{Sk}(F)[u]$.

Note that every $x \in \mathrm{Sk}(F)$ has a unique representation $x = x_1 \sqcup x_2$ with $x_1 \leq c$ and $x_2 \leq c^*$: take $x_1 = x \wedge c$ and $x_2 = x \wedge c^*$. The same holds for u with $u_1 = u$ and $u_2 = 0$, and u^* with $(u^*)_1 = u^* \wedge c$ and $(u^*)_2 = u^* \wedge c^* = c^*$. Define $S \subseteq \mathrm{Sk}(F)[u]$ by $s \in S$ iff $s = s_1 \sqcup s_2$, where s_1 is x_1 or $x_1 \sqcup u$ or $x_1 \wedge u^*$ for some $x_1 \in F$ with $x_1 \leq c$, and where $s_2 = x_2$ for some $x_2 \in F$ with $x_2 \leq c^*$. It is routine to see that S is closed under $\wedge$, $\sqcup$, and * by checking cases (this boils down to checking that $S_1 = \{\, s_1 : s \in S \,\}$ is closed under $\wedge$, $\sqcup$, and $'$ where $s_1' = s_1^* \wedge c$). Moreover, S contains u, so $S = \mathrm{Sk}(F)[u]$.

For any member of $\mathrm{D}(F') = \{\, \delta_t : t \in T \,\}$, define $d_t \in \mathrm{D}(F)$ to be the dense element associated with $(1, \delta_t)$ in the direct product decomposition of F. It follows that every $d \in \mathrm{D}(F)$ can be written as $d_1 \wedge d_2$ with $d_1 \in \{e, 1\}$ and $d_2 = d_t$ for some $t \in T$. Finally, since $F[u]$ is representable, any $w \in F[u]$ can be written as $w = s \wedge d_1 \wedge d_2$ with $s \in S$ and d_1, d_2 as specified just above.

We determine $F[u]/\theta_c \cong \{\, w \wedge c : w \in F[u] \,\}$ (the latter under the operations specified in Section 2). Now, $w \wedge c = (s_1 \sqcup s_2) \wedge d_1 \wedge d_2 \wedge c$, which reduces to $s_1 \wedge d_1$, since $s_1 \leq c \leq d_2$ and $c \wedge s_2 = 0$. Let Q be the set of all atoms of $\mathrm{Sk}(F)$ lying below c; then $a \in Q$ and Q has k elements. If $s_1 < c$, then s_1 is the boolean join of a proper subset of $(Q \setminus \{a\}) \cup \{u, a \wedge u^*\}$ that contains $k+1$ elements, and we conclude that $\mathrm{Sk}(F[u]/\theta_c) \cong \mathbf{2}^{k+1}$. By construction, all such $s_1 < c$ are below $e \wedge c$, which is thus the only non-boolean element in $F[u]/\theta_c$. It follows that $F[u]/\theta_c \cong \hat{F}_{k+1}$.

Working analogously from $w \wedge c^* = (s_1 \sqcup s_2) \wedge d_1 \wedge d_2 \wedge c^*$, which simplifies to $s_2 \wedge d_2$ due to $s_2 \leq c^* \leq d_1$ and $c^* \wedge s_1 = 0$, we obtain directly that $F[u]/\theta_{c^*} \cong F'$.

Finally, consider the map $h\colon F[u] \to F[u]/\theta_c \times F[u]/\theta_{c^*}$, which we define by $h((s_1 \sqcup s_2) \wedge d_1 \wedge d_2)) = (s_1 \wedge d_1, s_2 \wedge d)$. Since the component maps of h are the canonical projections of $F[u]$ onto $F[u]/\theta_c$, respectively, $F[u]/\theta_{c^*}$, h is a homomorphism that is bijective by construction.

It remains to check the case $k = 1$. Use (A3) to obtain $u^* \in \mathrm{Sk}(P)$ such that $c^* < u^* < e$ and $c^* \sqcup u < e$. Proceed analogously (but more simply) as above to show that $F[u]/\theta_c \cong \hat{F}_2$ and that $F[u] \cong \hat{F}_2 \times F'$. □

Corollary 7.2. *Let P be an arbitrary distributive p-semilattice satisfying* (A3)*, and $F_2 \trianglelefteq P$ a finite distributive nonboolean p-semilattice of the form $F_2 \cong \mathbf{2}^r \times \prod_{i=1}^{n} \hat{B}_i$, with $r \in \omega$ and B_i a finite boolean algebra for $1 \leq i \leq n$. Then there exists a p-semilattice F_3 such that $F_2 \trianglelefteq F_3 \trianglelefteq P$ and $F_3 \cong \mathbf{2}^r \times (\hat{\mathbf{A}})^n$ (with $\mathbf{A}$ the countable atom-free boolean algebra).*

Proof. Put $G_0 = F_2$ and, for $m \in \omega$, obtain G_{m+1} from G_m by splitting every (boolean) atom in each of the s factors of G_m of type $\hat{B}_i$, using Lemma 7.1 repeatedly. Then let $F_3 = \bigcup_{m \in \omega} G_m$. □

8. Necessity

In this section, we show that the axioms (DIST), (A1), (A2), and (A3) are also necessary for a p-semilattice P to be algebraically closed. This is done by extending any finite set $\{a_1, \ldots, a_n\}$ of elements of an a.c. p-semilattice P satisfying the assumptions of such an axiom, to a p-subsemilattice $S \trianglelefteq P$ isomorphic to a direct product $\mathbf{2}^r \times \widehat{A}^s$, $r, s \in \omega$. This is possible, since the class **PCS** is locally finite, which makes Theorem 3.1 applicable. It will be shown then that within S, any element whose existence is postulated by the axiom under consideration actually can be found.

We carry this out in detail for (A2) only, as the procedure is rather straightforward for (DIST), (A1), and (A3).

So assume that $a \in \mathrm{Sk}(P)$, $d, d' \in \mathrm{D}(P)$, and $p, p', x \in P$ satisfy all of the following: $d \| d'$, $p \leq d'$, $p' \leq d$, $p' \not\leq d'$, $a \leq d$, $a^* \wedge p \not\leq d$, and $x^* \leq d'$. Since P is a.c., there is a p-sub-semilattice $S \trianglelefteq P$ of P isomorphic to a direct product $\mathbf{2}^r \times \widehat{A}^s$, for $r, s \in \omega$, containing $\{a, d, d', p, p', x\}$.

We will now define an element $z \in \mathrm{Sk}(S)$ satisfying $a \leq z \leq d$, $z^* \wedge p \not\leq d$, $z \wedge p' \not\leq d'$, and $(z \wedge x)^* \leq d'$, by specifying its components $(z_1, \ldots, z_{r+s})$. We distinguish four cases according to the values of d_i and d'_i for $1 \leq i \leq r + s$.

(1) $d_i = d'_i = 1$: Put $z_i = a_i$.
(2) $d_i = 1$, $d'_i = e$: Put $z_i = 1$.
(3) $d_i = e$, $d'_i = 1$: Put $z_i = a_i$.
(4) $d_i = d'_i = e$

(a) $a_i \wedge x_i \neq 0$: Put $z_i = a_i$.
(b) $a_i \wedge x_i = 0$: This implies $a_i \leq x_i^* < d_i' = e$ (remember $x^* \leq d'$). Since $\pi_i[S] = \hat{\mathbf{A}}$, there is $b_i \in \pi_i[S]$ with $x_i^* < b_i < e$ (where π_i stands for the projection of S onto its i-th factor). Put $z_i = b_i$.

We claim that z has all of the required properties:

- $a \leq z \leq d$ obviously is satisfied.
- $z^* \wedge p \not\leq d$: Since $a^* \wedge p \not\leq d$ and $p \leq d'$, there is $i \in \{r+1, \ldots, r+s\}$ such that $a_i^* \wedge p_i \not\leq d$, thus $a_i^* = p_i = 1$, $d_i = e$, and $d_i' = 1$. This is Case 3 of the definition of z, hence $z_i = a_i$, and $(z^* \wedge p)_i = z_i^* \wedge p_i = a_i^* \wedge p_i = 1 \not\leq e = d_i$, thus $z^* \wedge p \not\leq d$.
- $z \wedge p' \not\leq d'$: Since $p' \not\leq d'$, there is $i \in \{r+1, \ldots, r+s\}$ such that $p_i' = 1$ and $d_i' = e$. Because $p' \leq d$, we have $d_i = 1$. This is Case 2 of the definition of z, thus $z_i = 1$. We obtain $(z \wedge p')_i = z_i \wedge p_i' = 1 \not\leq e = d_i'$, thus $z \wedge p' \not\leq d'$.
- $(z \wedge x)^* \leq d'$: We can assume $d_i' = e$, since if $d_i' = 1$, we trivially have $(z_x \wedge x)_i^* \leq d_i'$. So it remains to consider Cases 2, 4a, and 4b of the definition of z.

  In Case 2, we have $z_i = 1$, and so $(z \wedge x)_i^* = x_i^*$. But $x^* \leq d'$ by assumption, thus $(z \wedge x)_i^* \leq d_i'$.

  In Case 4a, $a < d$ implies $0 < a_i \wedge x_i \leq a_i < d_i = e$, and so also $0 < (a_i \wedge x_i)^* < e$, and thus $(z \wedge x)_i^* < e = d_i'$.

  In Case 4b, we have $z_i \wedge x_i \leq z_i = b_i < e$. Moreover, $z_i \wedge x_i \neq 0$ (for otherwise $z_i = b_i \leq x_i^* < b_i$). It follows that $(z_i \wedge x_i)^* < e = d_i'$.

9. Main theorem

The results obtained so far are enough to prove our main theorem:

Theorem 9.1. *A p-semilattice P is algebraically closed if and only if it is distributive and satisfies axioms* (A1)*,* (A2)*, and* (A3).

Proof. Necessity is established in the preceding section. For sufficiency, let F be a finite p-subsemilattice of a distributive p-semilattice P satisfying (A1)–(A3). By Corollary 4.11, F can be extended within P to a finite distributive p-subsemilattice F_0. By Corollary 5.4, F_0 can be extended within P to a finite distributive p-subsemilattice F_1 whose dense filter is boolean. If F_1 is boolean, we are done, since then $F_1 \cong \mathbf{2}^r$ and $s = 0$. Otherwise, using Corollary 6.7, F_1 can be extended within P to a p-subsemilattice F_2 such that $F_2 \cong \mathbf{2}^r \times \prod_{i=1}^n \hat{B}_i$, for some $r \in \omega$ and B_i a boolean algebra for $1 \leq i \leq n$. Finally, by Corollary 7.2, F_2 can be extended within P to a subsemilattice F_3 such that $F_3 \cong \mathbf{2}^r \times (\hat{\mathbf{A}})^n$ (with $\mathbf{A}$ the countable atom-free boolean algebra). □

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