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Comment on "Combined Forced and Free Convective Flow in a Vertical Porous Channel: The Effects of Viscous Dissipation and Pressure Work" by A. Barletta and D. A. Nield, Transport in Porous Media, DOI 10.1007/s11242-008-9320-y, 2009

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Abstract In a recent article by Barletta and Nield (Transport in Porous Media, DOI 10.1007/s11242-008-9320-y, 2009), the title problem for the fully developed parallel flow regime was considered assuming isoflux/isothermal wall conditions. For the limiting cases of the forced and the free convection, analytical solutions were reported; for the general case, numerical solutions were reported. The aim of the present note is (i) to give an analytical solution for the full problem in terms of the Weierstrass elliptic P-function, (ii) to illustrate this general approach by two easily manageable examples, and (iii) to rise a couple of questions of basic physical interest concerning the interplay between the viscous dissipation and the pressure work. In this context, the concept of "eigenflow" introduced by Barletta and Nield is discussed in some detail.

Keywords Mixed convection · Laminar flow · Porous media · Viscous dissipation · Pressure work · Analytical solution

The basic boundary value problem of the approach of Barletta and Nield (2009) (hereinafter referred to as BN2009) for the dimensionless temperature field $T = T(y)$ is specified by the following differential equation and boundary conditions

$$T'' - Br\,(\varepsilon + R)\,(1 + RT)\,(\gamma + T) + Br\,(1 + RT)^2 = 0 \tag{1}$$

$$T'(0) = -1 \text{ (isoflux left wall)}, \quad T(1) = 0 \text{ (isothermal right wall)}, \tag{2}$$

where the prime denotes differentiation with respect to the transverse coordinate y. In terms of $T = T(y)$, the flow velocity u is obtained as $u = 1 + RT$. All the quantities involved in the above equations are dimensionless and everywhere the notations of BN2009 are used. The parameters γ and ε are related to the temperature scales, and R is the buoyancy parameter. The second and the third term on the left-hand side of Eq. 1 represents the contribution of the pressure work and the viscous dissipation, respectively. Both effects scale with the Brinkman

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number Br so that both are canceled simultaneously as $Br \to 0$. This is the conduction regime with solution $T = 1 - y$.

(i) As a first step toward the exact analytical solution, we transcribe Eq. 1 in the form

$$T'' = A_1 + A_2 T + A_3 T^2, \tag{3}$$

$$A_1 = \left[\gamma (\varepsilon + R) - 1\right] Br, \quad A_2 = \left[\gamma R (\varepsilon + R) + \varepsilon - R\right] Br, \quad A_3 = \varepsilon R Br. \tag{4}$$

The second, and at the same time, the key step is to notice that Eq. 3 admits the first integral

$$T'^2 = 2A_0 + 2A_1 T + A_2 T^2 + (2A_3/3)\, T^3, \tag{5}$$

where A_0 is a constant of integration. Setting $y = 1$ in Eq. 5 and bearing in mind the Eq. 2, we obtain the value of A_0 in terms of the heat flux $q_1 = -T'(1)$ through the right isothermal wall as $A_0 = q_1^2/2 \geq 0$. In the limiting case of the forced convection plug flow $u = 1$ corresponding to $R = 0$, we have $A_3 = 0$. In this case Eq. 1 is linear, and its exact solution has already been given in BN2009. Thus, we assume here that $A_3 \neq 0$ and change from T to a new dependent variable P in Eq. 5 by the substitution

$$T(y) = -A_2/(2A_3) + (6/A_3)\, P(y). \tag{6}$$

After some algebra, we obtain

$$P'^2 = 4P^3 - g_2 P - g_3, \tag{7}$$

$$g_2 = \left(A_2^2 - 4A_1 A_3\right)/12, \quad g_3 = \left(6A_1 A_2 A_3 - A_2^3 - 12 A_0 A_3^2\right)/6^3. \tag{8}$$

Equation (7) coincides exactly with the differential equation of Weierstrass' elliptic function $P(y) = P(y; g_2, g_3)$ (for the properties of $P(y; g_2, g_3)$ see e.g., Abramowitz and Stegun (1972), Chap. 18). The general solution of Eq. 7 can be written in the form $P(y) = P(y + y_0; g_2, g_3)$, where y_0 is the second constant of integration of the problem. Accordingly, the general solution of the problem (1) and (2) for the temperature field $T(y)$ is

$$T(y) = -A_2/(2A_3) + (6/A_3)\, P(y + y_0; g_2, g_3). \tag{9}$$

The two constants of integration y_0 and A_0 can be determined with the aid of the boundary conditions (2), which now lead to the equations

$$P'(y_0; g_2, g_3) = -A_3/6, \; P(1 + y_0; g_2, g_3) = A_2/12. \tag{10}$$

(ii) When its discriminant $\Delta = g_2^3 - 27 g_3^2$ is non-vanishing, the P-function can be expressed in terms of the Jacobian elliptic functions sn, cn and dn, and when $\Delta = 0$, in terms of elementary functions. A comprehensive discussion of the solution (9) on this general background is beyond the aim and scope of this article. However, in order to illustrate how the approach works, we will consider here two special realizations of the case $\Delta = 0$ shortly. To this end, we first write the solution of equation $\Delta = 0$ in the parametric form $g_2 = 4a^4/3$, $g_3 = 8a^6/27$ with a as parameter. The simplest realization of $\Delta = 0$ is obtained for $a = 0$, when $g_2 = g_3 = 0$ and the P-function reduces to

$$P(y + y_0; 0, 0) = (y + y_0)^{-2} \quad (a = 0). \tag{11}$$

When, however, $a \neq 0$, the P-function with vanishing discriminant, $\Delta = 0$, has the form

$$P\left(y + y_0; 4a^4/3, 8a^6/27\right) = -\left(a^2/3\right) + a^2 \left[\sin(ay + ay_0)\right]^{-2} \quad (a \neq 0). \tag{12}$$

There are several easily manageable cases of the temperature solution (9) with the P-function given by Eq. 12. In this article, however, we consider only the case related to a special solution, which was named in BN2009 as "*eigenflow*" solution. This issue will be discussed under the point (iii) below. In the case of the P-function (11), we obtain for the temperature solution (9) and the heat flux $q(y) = -dT/dy$, the expressions

$$T(y) = \frac{y_0^3}{2(1+y_0)^2}\left[\left(\frac{1+y_0}{y+y_0}\right)^2 - 1\right], \quad q(y) = \left(\frac{y_0}{y+y_0}\right)^3, \tag{13}$$

where y_0 is the real root of the cubic equation $Ry_0^3 - 2(1+y_0)^2 = 0$,

$$y_0 = \frac{2}{3R}\left[1 + \frac{4(1+3R) + \left(8+36R+27R^2+\sqrt{27(8+27R)R^3}\right)^{2/3}}{\left(8+36R+27R^2+\sqrt{27(8+27R)R^3}\right)^{1/3}}\right] > 0. \tag{14}$$

We see that in this special case, a given value of R fixes also the value of y_0 uniquely. The same holds for γ, as well as for the product εBr, which are determined by the relationships $\gamma = 1/R$ and $\varepsilon Br = 12/\left(Ry_0^3\right)$, respectively. In spite of these restrictions (which originate from the simplifying assumption $g_2 = g_3 = 0$), the above equations yield quite reasonable values. Indeed, taking, e.g., $R = 0.025$ and $Br = 0.05$, we obtain $y_0 = 81.96$, $\gamma = 40$, and $\varepsilon = 0.0174$. In this case, $T(y)$ decreases monotonically from $T(0) = 0.982$ to $T(1) = 0$ and $q(y)$ from $q(0) = 1$ to $q(1) = 0.964$. The corresponding mixed convection temperature profile (13) is close to the conduction profile $T = 1 - y$.

(iii) We now turn to the third issue mentioned in the Abstract and ask the four questions listed below. Our answers will be supported by exact analytical calculations.

(1) Is it possible that the heat generated by viscous dissipation is fully consumed by the pressure work, so that the incoming heat flux $q_0 = 1$ passes the porous channel unchanged, i.e., $q_1 = q_0 = 1$?
(2) Could it happen that the outgoing heat flux q_1 becomes larger than the incoming one, e.g., twice as much as this, i.e., $q_1 = 2q_0 = 2$?
(3) Is it possible that the heat flux q_1 through the right wall becomes equal but opposite to the heat flux q_0 prescribed at the left wall, i.e., $q_1 = -q_0 = -1$?
(4) Is there a possibility that the isothermal right wall becomes adiabatic, i.e., $q_1 = 0$?

In Case 1 the channel would act as an *ideal heat conductor*, in Case 2 as a *heat multiplier*, (heating device) in Case 3 as a *heat absorber* (cooling device), and in Case 4 as an *ideal heat insulator*. Bearing in mind that the active work is done by the driving pressure gradient (and in a downward flow additionally by the gravity), there is no violation of the First Principle threats. For the sake of greatest transparency, we first give the answer for the questions 1–4 in the limiting case of the forced convection plug flow, $R = 0$, $u = 1$, since in this case the temperature solution is available in a simple analytical form (see Eq. 18) of BN2009). Based on this solution, the answer for all the questions 1–4 is decidedly YES regarding that the parameters γ, ε, and Br satisfy certain conditions. These conditions are collected in Table 1 together with the corresponding values of the wall temperature $T(0) \equiv T_0$. Concerning these rigorous mathematical results, however, several comments are in order.

A hasty inspection of Fig. 2 of BN2009 would suggest that all the forced convection temperature profiles decrease monotonically from some $T_0 > 0$ to $T(1) = 0$, and that all of them lie *above* the conduction profile $T(y) = 1 - y$. This, however, is not always so. In our Case 3, none of these two features holds, whereas in Case 4, only the former property is true, but the corresponding temperature profile lies markedly below of $T(y) = 1 - y$ as shown in Fig. 1. Indeed, in Case 3, Eq. 18 of BN2009 goes over in

Table 1 In the limiting case of the forced convection plug flow, $R = 0$, $u = 1$, the answer for all the four questions 1–4 is YES, when the indicated conditions are satisfied

Case	$q_1 \equiv q(1)$	$T_0 \equiv T(0)$	Conditions
1	1	$\frac{2}{\sigma}\tanh\frac{\sigma}{2}$	$\gamma = \frac{1}{\varepsilon} - \frac{1}{\sigma}\tanh\frac{\sigma}{2}$
2	2	$\frac{3}{\sigma}\tanh\frac{\sigma}{2}$	$\gamma = \frac{1}{\varepsilon} - \frac{1}{\sigma}\tanh\frac{\sigma}{2} - \frac{1}{\sigma\tanh\sigma}$
3	-1	0	$\gamma = \frac{1}{\varepsilon} + \frac{1}{\sigma\tanh\frac{\sigma}{2}}$
4	0	$\frac{1}{\sigma}\tanh\frac{\sigma}{2}$	$\gamma = \frac{1}{\varepsilon} + \frac{1}{\sigma\sinh\sigma}$

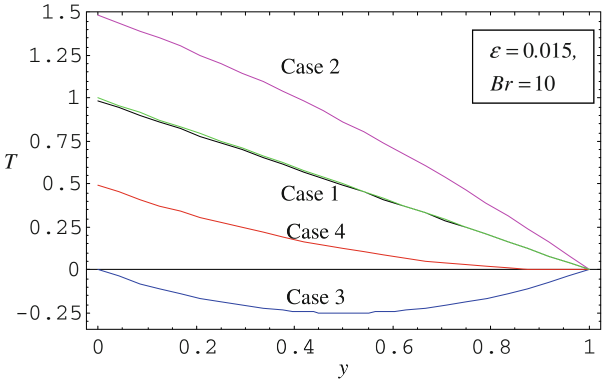

Fig. 1 Shown are the forced convection temperatures in the four cases of Table 1 for $\varepsilon = 0.015$ and $Br = 10$. The corresponding γ-values are $\gamma_1 = 66.17$, $\gamma_2 = 59.18$, $\gamma_3 = 80.17$, and $\gamma_4 = 73.17$, respectively. The temperature $T(y) = 1 - y$ of the conduction regime separates the temperature profiles corresponding to $q(1) > 1$ and $q(1) \leq 1$ from each other. In Case 1, with $q(1) = 1$, the temperature profile (*black curve*) lies very close to $T(y) = 1 - y$ (*green line*)

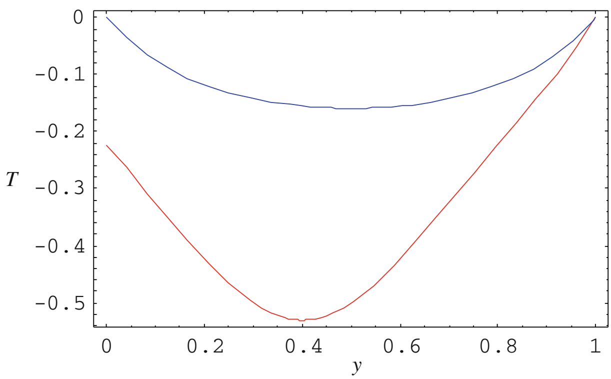

Fig. 2 Shown is the symmetric temperature profile (19) of the mixed convection eigenflow corresponding to the eigenvalue $a_0 = \pi/2$ (*blue curve*), as well as its nonsymmetric dual counterpart (*red curve*)

$$T(y) = -\frac{2\sinh(\sigma y/2)\sinh[\sigma(1-y)/2]}{\sigma\sinh(\sigma/2)}, \quad \sigma \equiv \sqrt{\varepsilon Br}, \tag{15}$$

which clearly shows that $T(y)$ is negative everywhere across the flow, while at the walls $T(0) = T(1) = 0$ holds. Thus, the temperature (15) is the forced convection counterpart of the mixed convection "eigenflow" temperatures shown in Figures 6 and 7 of BN2009 and associated (for $\gamma = 50$ and $\varepsilon = 0.01$) with the "eigenvalues" $Br = 0.45212$ (for $R = 0.1$) and $Br = 0.048125$ (for $R = 1$). However, according to Figure 10 of BN2009 (and the corresponding explanation at the end of Sect. 6), a mixed convection eigenflow solution can only exist for $R > 0.01$, but not in the forced convection limit $R = 0$. Hence, according to this numerical finding, the present forced convection eigenflow solution (15) should not exist at all. The explanation of this contradiction can easily be found in the "Condition" corresponding to the Case 3 of Table 1. Indeed, this relationship shows that (for $R = 0$) an eigenvalue of Br only can exist when $\gamma > 1/\varepsilon$ (in case of the corresponding curve of Fig. 1, where $1/\varepsilon = 1/0.015 = 66.67$ and $\gamma = 80.1663$, the eigenvalue is exactly $Br = 10$). When, however, $(1/\varepsilon) \to \gamma$, the eigenvalue Br goes to infinity, and for $\gamma < 1/\varepsilon$, it disappears. Therefore, the lower limit $R > 0.01$ for the existence of the mixed convection eigenflows reported by Barletta and Nield (2009) is in fact a consequence of their choice of the parameter values $\gamma = 50$ and $\varepsilon = 0.01$ with the property $\gamma < 1/\varepsilon$, regardless of the intrinsic R-dependence of the governing equations. The mixed convection eigenflows must exist down to the forced convection limit $R = 0$, when $\gamma > 1/\varepsilon$ is chosen. Then, the corresponding eigenvalue Br is the root of the transcendental equation

$$(1/\varepsilon) + \left[(\varepsilon Br)^{1/2} \tanh (\varepsilon Br/4)^{1/2}\right]^{-1} = \gamma \tag{16}$$

The fact that already the eigenflow temperature (15) is everywhere smaller than the wall temperatures $T(0) = T(1) = 0$ in the forced convection regime ($R = 0$), is a quite surprising result. Indeed, in Case 3, both heat fluxes $q(0) = 1$ and $q(1) = -1$ are directed from the respective walls toward the fluid. In spite of this, the fluid temperature becomes everywhere lower than the boundary temperatures. This can only happen when the upward moving fluid experiences an extremely vigorous expansion, so that the expansion work does consume (i) the whole incoming heat supplied by both the walls, (ii) the whole heat generated by viscous dissipation and in addition, (iii) it also withdraws a part of the internal energy of the fluid, diminishing in this way its temperature. Whether this *expansion cooling with heat supply* is actually compatible with the *incompressibility assumption* adopted in BN2009 is still an open question. It is also worth emphasizing here that the symmetry of the "eigenflow" temperature profiles with respect to the mid-plane of the channel is a necessary consequence of the corresponding boundary values $T(0) = T(1) = 0$ and $q(0) = -q(1) = 1$ which render the "left" and "right" walls physically undistinguishable. In the case of the forced convection profile (15) this feature becomes clearly manifest. Indeed, it is seen that the transformation $y \to 1 - y$ which interchanges the left and right boundaries, leaves the expression (15) invariant.

As already mentioned above, the temperature profile of Case 4, although monotonically decreasing, lies not above but below the conduction profile. This later feature holds not only for Cases 3 and 4, but also for all $Br \neq 0$ whenever the heat flux q_1 is smaller than or equal to 1 (i.e. in all cases of Table 1, except for Case 2, as it can be seen in Fig. 1).

Case 2 is also quite surprising. Indeed, while in Case 1, the effect of viscous dissipation and the pressure work compensate each other exactly, in Case 2, their simultaneous effect results in a net heat production, leading in turn to an outgoing heat flux q_1 which is twice as much as the incoming one q_0. Moreover, with increasing values of Br, the outgoing heat flux q_1 may become arbitrary large. This can be seen in the case of the plug flow with uniform core temperature, flanked by two narrow temperature boundary layers arising for $B >> 1$. Indeed, for $Br >> 1$, Eq. 18 of BN2009 reduces to

$$T(y) = \left(\frac{1}{\varepsilon} - \gamma\right)\left(1 - e^{-\sigma(1-y)}\right) + \frac{e^{-\sigma y}}{\sigma} \quad \text{for} \quad y << 1, \tag{17}$$
$$\text{and} \quad \left(\frac{1}{\varepsilon} - \gamma\right)\left(1 - e^{-\sigma(1-y)}\right) \text{ else.}$$

Thus, the outgoing heat flux $q_1 = (1 - \varepsilon\gamma)\sqrt{(Br/\varepsilon)}$ can become arbitrarily large as $Br \to \infty$.

A deeper insight into the peculiar nature of the mixed convection eigenflows can be gained with the aid of the exact solution (9) related to the case (12) of the P-function. Indeed, choosing for a and y_0 the values

$$a \equiv a_n = (2n+1)\,\pi/2, \quad y_0 \equiv y_{0,n} = \pi/(4a_n), \quad n = 0, 1, 2, \ldots \tag{18}$$

the A's become $A_1 = 4a_n$, $A_2 = 20a_n^2$, $A_3 = 24a_n^3$, and the corresponding solutions are

$$T(y) \equiv T_n(y) = -\frac{1}{2a_n}\,\frac{\sin(2a_n y)}{1 + \sin(2a_n y)}. \tag{19}$$

It is easy to see that all these solutions yield the wall values $T(0) = 0$, $q(1) = -1$, and thus, they are actually mixed convection eigensolutions. Therefore, from the mathematical point of view, the name eigenflow coined by Barletta and Nield (2009) is in fact a reasonable, properly chosen name and not "just a conventional word." However, except for the case $n = 0$, the eigensolutions (19) always become singular so that, from physical point of view, the cases with $n = 1, 2, 3, \ldots$ must be excluded. In this way, the eigenvalue spectrum (18) reduces to the single eigenvalue $a = a_0 = \pi/2$. Due to this circumstance, the proper meaning of "eigenvalues" and "eigensolutions" goes down in a numerical study. The mixed convection eigensolution (19) corresponding to the eigenvalue $a_0 = \pi/2$ shares all the qualitative features of the forced convection eigensolution (15), namely, it is symmetric with respect to the mid-plane of the channel (invariant under $y \to 1 - y$) and negative in the whole variation range of y. A special feature of the eigensolution (19) is its apparent "universality," in the sense that it does not contain the physical parameters γ, ε, R, and $Br$of the problem explicitly. Only the corresponding eigenvalue $a_0 = \pi/2$ is related to these parameters via Eqs. 5 and $A_1 = 4a_n$, $A_2 = 20a_n^2$, $A_3 = 24a_n^3$. As the last point of this article, this connection will be discussed here shortly.

The solution of the system of three equations (5) for the four parameters γ, ε, R, and Br is not unique. Three of the parameters can be expressed in terms of the fourth them. Taking Br as the "fourth" parameter, we obtain from the eigenvalue $a = a_0 = \pi/2$, the following two sets of solutions for R, ε, and γ

$$\left\{R_1 = \frac{3\pi}{2}, \varepsilon_1 = \frac{2\pi^2}{Br}, \gamma_1 = \frac{2\,(2\pi + Br)}{\pi\,(4\pi + 3Br)}\right\}, \quad \left\{R_2 = \pi, \varepsilon_2 = \frac{3\pi^2}{Br}, \gamma_2 = \frac{2\pi + Br}{\pi\,(3\pi + Br)}\right\}. \tag{20}$$

We are thus faced with an astonishing result. The two different sets of values (20) of $\{R, \varepsilon, \gamma\}$ are associated with *one and the same eigensolution* $T_0(y)$ given by Eq. 19, and this for all values of Br. Moreover, this eigensolution possesses also a dual counterpart which is obtained for the "initial value" (wall temperature) $T_0 = -0.223078$ and, similarly to (19), it is also the same for both sets of values (20) (see Fig. 2).

The above considerations show that the problems (1) and (2) possess a very rich mathematical and physical content which is waiting for further research effort. Especially, the property of doubly periodicity of the P-function could reveal subtle features of the flow behavior.

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