COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA 30 (1) (2010) 105–123
DOI: 10.1007/s00493-010-2436-z

ONLINE VERTEX-COLORING GAMES IN RANDOM GRAPHS*

MARTIN MARCINISZYN, RETO SPÖHEL†

Received September 7, 2007

Consider the following one-player game. The vertices of a random graph on n vertices are revealed to the player one by one. In each step, also all edges connecting the newly revealed vertex to preceding vertices are revealed. The player has a fixed number of colors at her disposal, and has to assign one of these to each vertex immediately. However, she is not allowed to create any monochromatic copy of some fixed graph F in the process.

For $n \to \infty$, we study how the limiting probability that the player can color all n vertices in this online fashion depends on the edge density of the underlying random graph. For a large family of graphs F, including cliques and cycles of arbitrary size, and any fixed number of colors, we establish explicit threshold functions for this edge density. In particular, we show that the order of magnitude of these threshold functions depends on the number of colors, which is in contrast to the corresponding offline coloring problem.

1. Introduction

Consider the following one-player game on a random graph $G_{n,p}$. (Throughout, $G_{n,p}$ denotes the standard binomial random graph on n vertices in which each edge is included independently with probability $p=p(n)$.) The vertices of $G_{n,p}$ are revealed one by one to a player called Painter. In each step, also all edges connecting the newly revealed vertex to preceding vertices are revealed. Painter has a fixed number r of colors at her disposal, and has to assign one of these to each vertex immediately. Her goal is to do so without

Mathematics Subject Classification (2000): 05C80, 05C15

* A preliminary version of this paper appeared in the proceedings of the 18th annual Symposium on Discrete Algorithms (SODA'07) [6].

† The author was supported by Swiss National Science Foundation, grant 200021-108158.

creating a monochromatic copy of some fixed graph F in the process. She loses as soon as the subgraph induced by the vertices of any color contains a copy of F, and she wins if she can avoid this until all n vertices are colored. We ask for which densities p of the underlying random graph Painter can win the game asymptotically almost surely (a.a.s., with probability tending to 1 as n tends to infinity).

We answer this question for a large class of graphs F, including cliques and cycles of arbitrary size, and any fixed number of colors r: we give an explicit threshold function $p_0 = p_0(F,r,n)$ such that, using the right strategy, Painter a.a.s. succeeds in coloring the entire random graph $G_{n,p}$ for any function $p = o(p_0)$ (also denoted by $p \ll p_0$ in the following), but a.a.s. fails to do so with *any* strategy if $p = \omega(p_0)$ (also denoted by $p \gg p_0$).

1.1. Our results

In order to state our main theorem, we need to introduce some notation. For any graph F, let

$$m_1(F) := \max_{H \subseteq F} \frac{e_H}{v_H - 1} \tag{1}$$

denote the maximum so-called 1-density of a subgraph H of F. The thresholds we establish in this paper are determined by the density measures $\overline{m}_1^r(F)$, which are inductively defined as follows:

$$\overline{m}_1^r(F) := \begin{cases} \max\limits_{H \subseteq F} \dfrac{e_H}{v_H} & \text{if } r = 1, \\ \max\limits_{H \subseteq F} \dfrac{e_H + \overline{m}_1^{r-1}(F)}{v_H} & \text{if } r \geq 2. \end{cases} \tag{2}$$

Theorem 1 (Main Result). *Let F be a nonempty graph that has an induced subgraph $F^\circ \subset F$ on $v_F - 1$ vertices satisfying*

$$m_1(F^\circ) \leq \overline{m}_1^2(F). \tag{3}$$

Then for all $r \geq 1$, the threshold for the online F-avoidance vertex-coloring game with r colors is

$$p_0(F,r,n) = n^{-1/\overline{m}_1^r(F)}.$$

The side condition (3) is only used in the upper bound proof – we prove a lower bound of $n^{-1/\overline{m}_1^r(F)}$ in full generality.

If F is 1-balanced, i.e., if we have $m_1(F) = e_F/(v_F - 1)$, Theorem 1 can be simplified as follows (cf. Lemma 9 below).

Corollary 2. *Let F be a 1-balanced nonempty graph that has an induced subgraph $F^\circ \subset F$ on $v_F - 1$ vertices satisfying*

$$m_1(F^\circ) \le m_1(F)\left(1 - v_F^{-2}\right).$$

Then for all $r \ge 1$, the threshold for the online F-avoidance vertex-coloring game with r colors is

$$p_0(F, r, n) = n^{-\frac{1}{m_1(F)\left(1 - v_F^{-r}\right)}}.$$

In particular, this implies the following thresholds for the game with cliques and cycles respectively.

Corollary 3 (Clique-avoidance games). *For all $\ell \ge 2$ and $r \ge 1$, the threshold for the online K_ℓ-avoidance vertex-coloring game with r colors is*

$$p_0(\ell, r, n) = n^{-\frac{2}{\ell(1-\ell^{-r})}}.$$

Corollary 4 (Cycle-avoidance games). *For all $\ell \ge 3$ and $r \ge 1$, the threshold for the online C_ℓ-avoidance vertex-coloring game with r colors is*

$$p_0(\ell, r, n) = n^{-\frac{\ell-1}{\ell(1-\ell^{-r})}}.$$

The maximization over r potentially different subgraphs in (2) gives rise to the curious phenomenon that a disconnected graph F may have a higher threshold than each of his components. Consider for example $r = 2$ and F the disjoint union of a triangle K_3 and a cycle of length 6 with one extra edge connecting two opposite vertices, denoted by C_6^+. Corollary 2 yields that the individual components have thresholds $p_0(K_3, 2, n) = n^{-3/4} = n^{-0.75}$ and $p_0(C_6^+, 2, n) = n^{-36/49} \approx n^{-0.735}$ respectively, whereas our lower bound proof yields that $p_0(F, 2, n)$ is at least $n^{-1/\overline{m}_1^2(F)} = n^{-18/25} = n^{-0.72}$.

1.2. From subgraph appearance to Ramsey properties

It is instructive to compare these threshold functions to the lower and upper bounds that follow from known offline results. Clearly, Painter is only in danger if the underlying random graph contains a copy of F. The following well-known theorem due to Bollobás states a threshold for this event.

Theorem 5 **([1]).** *Let F be a nonempty graph, and let $\mathcal{P}=$ 'G contains a copy of F'. Then*

$$\lim_{n\to\infty} \mathbb{P}[G_{n,p} \in \mathcal{P}] = \begin{cases} 1 & \text{if } p \gg n^{-1/m(F)} \\ 0 & \text{if } p \ll n^{-1/m(F)}, \end{cases}$$

where

$$m(F) := \max_{H\subseteq F} \frac{e_H}{v_H}.$$

Thus, for $p \ll n^{-1/m(F)}$ there is a.a.s. no copy of F in $G_{n,p}$, and Painter cannot possibly lose the game. In fact, since by definition we have $\overline{m}_1^1(F) = m(F)$, Theorem 1 yields the statement of Theorem 5 for $r=1$. It is easy to see that for $r\geq 2$, $\overline{m}_1^r(F)$ is strictly larger than $m(F)$ and defines a higher threshold.

On the other hand, knowing the entire graph in advance would ease Painter's situation. Thus, the following result of Łuczak, Ruciński, and Voigt concerning offline Ramsey properties of random graphs yields an upper bound on the threshold of the online game.

Theorem 6 **([5]).** *Let $r\geq 2$ and F be a nonempty graph that in the case $r=2$ is not a matching. Moreover, let $\mathcal{P}=$ 'every r-vertex-coloring of G contains a monochromatic copy of F'. Then there exist positive constants $c=c(F,r)$ and $C=C(F,r)$ such that*

$$\lim_{n\to\infty} \mathbb{P}[G_{n,p} \in \mathcal{P}] = \begin{cases} 1 & \text{if } p > Cn^{-1/m_1(F)} \\ 0 & \text{if } p < cn^{-1/m_1(F)}, \end{cases}$$

where $m_1(F)$ is defined as in (1).

It follows that Painter a.a.s. has no chance of winning the game if $p \gg n^{-1/m_1(F)}$ since no proper coloring exists at all. Note that the order of magnitude of this offline threshold does not depend on the number of colors r. Comparing our results to that, we see that for $r\to\infty$, the exponent in Corollary 2 tends to the exponent for the offline case, and it can be shown (cf. Lemma 7) that this convergence holds in general, i.e., that every graph satisfies

$$\lim_{r\to\infty} \overline{m}_1^r(F) = m_1(F).$$

Thus, the online threshold, which does depend on the number of colors r, approaches the offline threshold as r grows.

Our proofs show that it is impossible to strengthen the online thresholds to 'semi-sharp' thresholds as in Theorem 6. This suggests that online colorability is essentially determined by local properties, and in some sense closer related to Theorem 5 than to Theorem 6 (cf. [2] for a discussion of sharp thresholds and global vs. local graph properties).

1.3. Edge-colorings

Online Ramsey games in random graphs were first considered for edge-colorings. Investigating algorithmic Ramsey properties of triangles, Friedgut *et al.* [3] introduced and solved the online triangle-avoidance edge-coloring game with two colors. In [7,8] this result was extended to a theorem similar to Theorem 1, covering the game with two colors. It was conjectured that the theory generalizes to the game with more colors analogously to the vertex-coloring case – in fact, the corresponding lower bound was shown to hold in full generality.

1.4. Organization of this paper

We explain our notation and prove some auxiliary results in Section 2. In Sections 3 and 4 we prove that $n^{-1/\overline{m}_1^r(F)}$ is a lower and an upper bound on the threshold respectively.

2. Preliminaries and notation

The expression $f \asymp g$ means that the functions f and g differ at most by a multiplicative constant. All graphs are simple and undirected. We denote a clique on ℓ vertices by K_ℓ and a cycle on ℓ vertices by C_ℓ. The number of vertices of a graph G is denoted by v_G or $v(G)$, and similarly the number of edges by e_G or $e(G)$.

2.1. Density measures

The standard density measure for graphs is $d(G) := e_G/v_G$, which is exactly half of the average degree. Besides $d(G)$, we also use the so-called 1-density $d_1(G) := e_G/(v_G - 1)$. For the sake of completeness, we also define $d(G) = d_1(G) := 0$ if G is empty. For a given density function d_i, we let $m_i(G) := \max_{H \subseteq G} d_i(H)$. We say that G is *balanced with respect to* d_i if $m_i(G) = d_i(G)$. We simply write *balanced* for balancedness w.r.t. d, and 1-*balanced* for balancedness w.r.t. d_1.

For nonempty graphs F and G and any integer $r \geq 2$, we define

$$\overline{d}_1^r(F,G) := \frac{e_G + \overline{m}_1^{r-1}(F)}{v_G},$$

where $\overline{m}_1^{r-1}(F)$ is defined as in (2). We set $\overline{d}_1^r(F,G) := 0$ if F or G is empty. Note that

$$\overline{m}_1^r(F) = \max_{H \subseteq F} \overline{d}_1^r(F,H)$$

for all $r \geq 2$, and recall that $\overline{m}_1^1(F) = m(F)$ by definition. We say that F is balanced w.r.t. $\overline{d}_1^r$ if $\overline{m}_1^r(F) = \overline{d}_1^r(F,F)$. To simplify notation, we often write $\overline{d}_1$ for $\overline{d}_1^2$.

The maximum density measures m and m_1 are well-known and motivated by Theorems 5 and 6 respectively. It is also well-known that every nonempty graph satisfies $m(F) < m_1(F)$, and that every 1-balanced graph is balanced. In contrast, the maximum densities $\overline{m}_1^r(F)$ seem to have not been studied before. The next lemma shows that they interpolate between m and m_1 in some sense.

Lemma 7. *Let F be a nonempty graph.*

(i) We have

$$m(F) = \overline{m}_1^1(F) < \overline{m}_1^2(F) < \cdots < \overline{m}_1^r(F) < \cdots < m_1(F).$$

(ii) We have

$$\lim_{r \to \infty} \overline{m}_1^r(F) = m_1(F).$$

(iii) If F is 1-balanced, it is balanced with respect to $\overline{d}_1^r$ for all $r \geq 1$.

(iv) For all $r \geq 2$, if F is balanced with respect to $\overline{d}_1^r$, it is balanced with respect to $\overline{d}_1^{r-1}$.

For the proof we use the following observation, which we state separately for further reference.

Proposition 8. *For a, c, $C \in \mathbb{R}$ and $b > d > 0$, we have*

$$\frac{a}{b} \geq C \ \wedge \ \frac{c}{d} \leq C \quad \Longrightarrow \quad \frac{a-c}{b-d} \geq C$$

and

$$\frac{a-c}{b-d} \geq \frac{a}{b} \quad \Longleftrightarrow \quad \frac{c}{d} \leq \frac{a}{b}.$$

Proof of Lemma 7. We prove (i) by induction on r, showing that $\overline{m}_1^r(F) < \overline{m}_1^{r+1}(F)$ and $\overline{m}_1^r(F) < m_1(F)$. In the base case $r=1$, these statements follow from the inequalities $d(H) < \overline{d}_1^2(F,H)$ and $d(H) < d_1(H)$ for all nonempty subgraphs $H \subseteq F$. For $r \geq 2$, the first inequality follows from

$$\overline{m}_1^r(F) = \max_{H \subseteq F} \frac{e_H + \overline{m}_1^{r-1}(F)}{v_H} \overset{\text{Ind.}}{<} \max_{H \subseteq F} \frac{e_H + \overline{m}_1^r(F)}{v_H} = \overline{m}_1^{r+1}(F).$$

The second inequality follows with $e_H/(v_H - 1) \leq m_1(F)$ from

$$\overline{m}_1^r(F) = \max_{H \subseteq F} \frac{e_H + \overline{m}_1^{r-1}(F)}{v_H} \overset{\text{Ind.}}{<} \max_{H \subseteq F} \frac{m_1(F)(v_H - 1) + m_1(F)}{v_H} = m_1(F).$$

This concludes the proof of (i).

Clearly, (i) implies that $\overline{m}_1^\infty(F) := \lim_{r\to\infty} \overline{m}_1^r(F)$ exists and is at most $m_1(F)$. On the other hand, letting $r \to \infty$ in (2) yields that $\frac{e_H + \overline{m}_1^\infty(F)}{v_H} \leq \overline{m}_1^\infty(F)$ for all $H \subseteq F$, which by elementary calculations is equivalent to $\overline{m}_1^\infty(F) \geq m_1(F)$. This proves (ii).

An easy calculation shows that $\overline{d}_1^r(F,F) \leq d_1(F)$ is equivalent to $\overline{m}_1^{r-1}(F) \leq d_1(F)$. Hence, F satisfies $\overline{d}_1^r(F,F) \leq d_1(F)$ by (i). If F is 1-balanced, we obtain for every subgraph $H \subsetneq F$ with $2 \leq v_H < v_F$ that

$$\begin{aligned}\frac{(e_F + \overline{m}_1^{r-1}(F)) - (e_H + \overline{m}_1^{r-1}(F))}{v_F - v_H} &= \frac{e_F - e_H}{(v_F - 1) - (v_H - 1)} \\ &\overset{\text{Prop.8}}{\geq} d_1(F) \geq \overline{d}_1^r(F,F),\end{aligned}$$

which again by Proposition 8 implies that $\overline{d}_1^r(F,H) \leq \overline{d}_1^r(F,F)$, concluding the proof of (iii).

Claim (iv) follows analogously from

$$\begin{aligned}\frac{(e_F + \overline{m}_1^{r-1}(F)) - (e_H + \overline{m}_1^{r-1}(F))}{v_F - v_H} &= \frac{(e_F + \overline{m}_1^{r}(F)) - (e_H + \overline{m}_1^{r}(F))}{v_F - v_H} \\ &\overset{\text{Prop.8}}{\geq} \overline{d}_1^r(F,F) \geq \overline{d}_1^{r-1}(F,F)\end{aligned}$$

for every subgraph $H \subsetneq F$ with $2 \leq v_H < v_F$. ∎

The inductive definition of $\overline{m}_1^r(F)$ can be written in the following explicit form, which yields the threshold formula in Corollary 2.

Lemma 9. *For all nonempty graphs F and $r \geq 1$, we have*

$$\overline{m}_1^r(F) = \max_{H_1,\ldots,H_r \subseteq F} \frac{\sum_{i=1}^r e(H_i) \prod_{j=1}^{i-1} v(H_j)}{\prod_{i=1}^r v(H_i)}, \tag{4}$$

and the graphs $H_1, \dots, H_r$ maximizing (4) can be calculated recursively via

$$H_r = H_r(F) := \begin{cases} \operatorname{argmax}_{H \subseteq F} \dfrac{e_H}{v_H} & \text{if } r = 1, \\ \operatorname{argmax}_{H \subseteq F} \dfrac{e_H + \overline{m}_1^{r-1}(F)}{v_H} & \text{if } r \geq 2. \end{cases} \tag{5}$$

If F is 1-balanced, we have

$$\overline{m}_1^r(F) = m_1(F)\left(1 - v_F^{-r}\right).$$

Proof. To prove the first statement, we apply induction on r. For $r=1$, the claim follows directly from (2). For $r \geq 2$, we have

$$\begin{aligned} \overline{m}_1^r(F) &= \max_{H_r \subseteq F} \frac{e(H_r) + \overline{m}_1^{r-1}(F)}{v(H_r)} \\ &\overset{\text{Ind.}}{=} \max_{H_1,\dots,H_r \subseteq F} \frac{e(H_r) + \frac{\sum_{i=1}^{r-1} e(H_i) \prod_{j=1}^{i-1} v(H_j)}{\prod_{i=1}^{r-1} v(H_i)}}{v(H_r)} \\ &= \max_{H_1,\dots,H_r \subseteq F} \frac{\sum_{i=1}^{r} e(H_i) \prod_{j=1}^{i-1} v(H_j)}{\prod_{i=1}^{r} v(H_i)}. \end{aligned}$$

If F is 1-balanced, by Lemma 7 it is also balanced w.r.t. $\overline{d}_1^r$, $r \geq 1$, and we have $H_1 = \dots = H_r = F$. Plugging this into (4), we thus obtain

$$\overline{m}_1^r(F) = \frac{e_F}{v_F} \sum_{j=0}^{r-1} v_F^{-j} = \frac{e_F}{v_F}\left(v_F^{-(r-1)} \frac{v_F^r - 1}{v_F - 1}\right) = \frac{e_F}{v_F - 1}\left(1 - v_F^{-r}\right). \qquad \blacksquare$$

2.2. Janson's inequality

Janson's inequality is a very useful tool in probabilistic combinatorics. In many cases, it yields an exponential bound on lower tails where the second moment method only gives a bound of $o(1)$. Here we formulate a version tailored to random graphs.

Theorem 10 ([4]). *Consider a family (potentially a multi-set) $\mathcal{F} = \{H_i \mid i \in I\}$ of graphs on the vertex set $\{v_1, \dots, v_n\}$. For each $H_i \in \mathcal{F}$, let X_i denote the indicator random variable for the event $H_i \subseteq G_{n,p}$, and for each* *pair $H_i, H_j \in \mathcal{F}$, $i \neq j$, write $H_i \sim H_j$ if H_i and H_j are not edge-disjoint. Let*

$$X = \sum_{H_i \in \mathcal{F}} X_i \,,$$
$$\mu = \mathbb{E}[X] = \sum_{H_i \in \mathcal{F}} p^{e(H_i)},$$
$$\Delta = \sum_{\substack{H_i, H_j \in \mathcal{F} \\ H_i \sim H_j}} \mathbb{E}[X_i X_j] = \sum_{\substack{H_i, H_j \in \mathcal{F} \\ H_i \sim H_j}} p^{e(H_i)+e(H_j)-e(H_i \cap H_j)}.$$

Then for all $0 \leq \delta \leq 1$ we have

$$\mathbb{P}[X \leq (1-\delta)\mu] \leq e^{-\frac{\delta^2 \mu^2}{2(\mu+\Delta)}}.$$

In particular, Janson's inequality yields the following counting version of Theorem 6. The proof is essentially the same as the one for the 1-statement of Theorem 6 given in [5].

Theorem 11. *Let $r \geq 2$ and F be a nonempty graph. Then there exist positive constants $C = C(F,r)$ and $a = a(F,r)$ such that for*

$$p = p(n) \geq C n^{-1/m_1(F)},$$

the random graph $G_{n,p}$ a.a.s. satisfies the property that in every r-vertex-coloring there are at least $a n^{v_F} p^{e_F}$ monochromatic copies of F.

3. Lower bound

In this section we prove a general lower bound of $n^{-1/\overline{m}_1^r(F)}$ for the game with an arbitrary graph F and an arbitrary number of colors r. We provide an explicit strategy which a.a.s. succeeds in coloring $G_{n,p}$ online if $p \ll n^{-1/\overline{m}_1^r(F)}$.

Consider the game with two colors, say red and blue, and assume that Painter uses one color, say red, in every move if this does not create an entirely red copy of F. Clearly, if Painter loses she does so with a blue copy of F, which was forced by a surrounding red structure. More precisely, after her losing move the board contains a blue copy of F, each vertex of which completes a red subgraph to a copy of F. We say that the graph formed by these copies is a dangerous graph for Painter. Figure 1 shows two such dangerous graphs for the case $F = K_4$.

This simple greedy strategy yields the claimed lower bound if F is balanced w.r.t. $\overline{d}_1^2$. For general graphs, it may be smarter to play the greedy

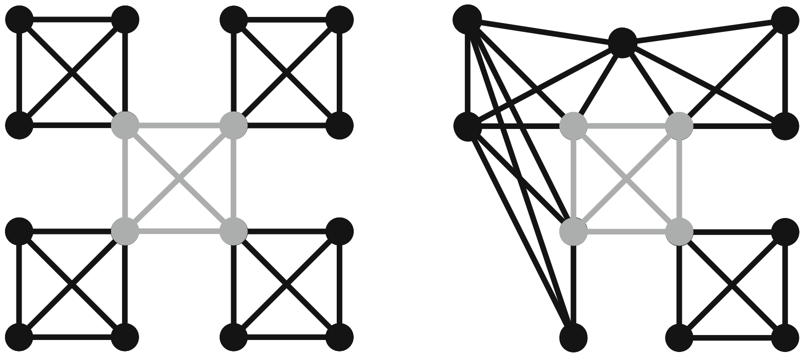

Figure 1. Two graphs from the class $\mathcal{F}(K_4, K_4)$, which are dangerous to a greedy Painter in the K_4-avoidance game with two colors. The graph on the left hand side is the unique graph K_4^*. In both graphs, the inner vertices and edges are shaded.

strategy with respect to an appropriately chosen subgraph H of F. For an example, consider the graph F consisting of a triangle with one edge attached to it. Here it turns out that greedily avoiding triangles and forgetting about the additional edge guarantees longer survival than greedily avoiding F itself. If Painter follows this greedy strategy for a fixed $H \subseteq F$, the game ends with a blue copy of F, each vertex of which completes a red subgraph to a copy of H.

For arbitrary nonempty graphs H_1 and H_2, let $\mathcal{F}(H_1, H_2)$ denote the class of all graphs that have an 'inner' (blue) copy of H_1, each vertex of which also completes an 'outer' (red) copy of H_2. Here the colors should only provide the intuitive connection to the greedy strategy, the members of the family $\mathcal{F}(H_1, H_2)$ are not associated with a coloring. We say that the inner copy of H_1 is formed by inner vertices and edges, and refer to the surrounding elements as outer vertices and edges respectively. Formally, we define the family of graphs $\mathcal{F}(H_1, H_2)$ as follows.

Definition 12. For all graphs $H_1 = (V, E)$ and H_2, let

$$\begin{aligned}\mathcal{F}(H_1, H_2) := \big\{ F' = (V \,\dot\cup\, U, E \,\dot\cup\, D) \colon\ & F' \text{ is a } \textit{minimal} \text{ graph} \\ & \text{such that for all } v \in V \text{ there are sets } U(v) \subseteq U \text{ and } D(v) \subseteq D \text{ with} \\ & \big(\{v\} \,\dot\cup\, U(v), D(v)\big) \cong H_2 \big\}.\end{aligned}$$

The inner vertices V and edges E form the inner of H_1. Every vertex $v \in V$ together with $U(v)$ and $D(v)$ forms a copy of H_2. Hence, $|U(v)| = v(H_2) - 1$ and $|D(v)| = e(H_2)$. We take F' as a minimal element with respect to subgraph inclusion, i.e., F' does not have a subgraph which satisfies the same properties. This ensures in particular that $\mathcal{F}(H_1, H_2)$ is finite.

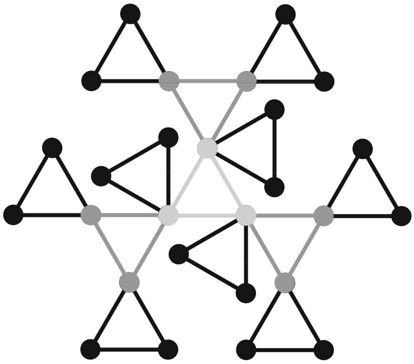

Figure 2. The unique graph from the class $\mathcal{F}(K_3, K_3, K_3)$ in which no outer copies overlap.

In the game with three colors, say, with colors yellow (3), red (2) and blue (1), a greedy Painter first avoids some subgraph H_3 in yellow, H_2 in red, and eventually H_1 in blue. We call this strategy the greedy $\langle H_3, H_2, H_1\rangle$-avoidance strategy. By the same argument as before, if Painter loses, the board contains a red-blue copy of a member from the family $\mathcal{F}(H_1, H_2)$, every vertex of which completes an entirely yellow copy of H_3. We denote the class of all such graphs by $\mathcal{F}(H_1, H_2, H_3)$. Figure 2 depicts a graph from the class $\mathcal{F}(K_3, K_3, K_3)$.

This motivates the following inductive definition for general r.

Definition 13. For any graph H_1, let $\mathcal{F}(H_1) := \{H_1\}$. For $r \geq 2$ and arbitrary graphs $H_1, \ldots, H_r$, let

$$\mathcal{F}(H_1, \ldots, H_r) := \left\{F^r \in \mathcal{F}(F^{r-1}, H_r) \colon F^{r-1} \in \mathcal{F}(H_1, \ldots, H_{r-1})\right\}.$$

By the same argument as before, if Painter loses the game with r colors following the greedy $\langle H_r, \ldots, H_1\rangle$-avoidance strategy, the board contains a copy of a graph from $\mathcal{F}(H_1, \ldots, H_r)$. With this observation at hand, the lower bound in Theorem 1 is an immediate consequence of the next, purely deterministic lemma.

Lemma 14. *Let F be a nonempty graph, and let $r \geq 1$. If the subgraphs $H_1, \ldots, H_r \subseteq F$ are chosen according to (5), all graphs $F^r \in \mathcal{F}(H_1, \ldots, H_r)$ satisfy*

$$m(F^r) \geq \overline{m}_1^r(F).$$

Proof of Theorem 1 (Lower Bound). Since the class $\mathcal{F}(H_1, \ldots, H_r)$ is finite, Lemma 14 implies by Theorem 5 that $G_{n,p}$ a.a.s. contains no graph from this class if $p \ll n^{-1/\overline{m}_1^r(F)}$. Therefore, Painter will a.a.s. not lose the game if she follows the greedy $\langle H_r, \dots, H_1\rangle$-avoidance strategy. ∎

Before proving Lemma 14, we give some intuition for it. Consider again the case $r = 2$. Among the graphs $F' \in \mathcal{F}(H_1, H_2)$, there are some distinguished ones F^* which have vertex-disjoint outer copies. Such a graph is depicted in Figure 1 on the left hand side. Clearly, these graphs have exactly $v(H_1)(v(H_2)-1)+v(H_1) = v(H_1)v(H_2)$ vertices and $v(H_1)e(H_2)+e(H_1)$ edges. If H_1 and H_2 are chosen according to (5), this yields

$$d(F^*) = \frac{v(H_1)e(H_2) + e(H_1)}{v(H_1)v(H_2)} = \frac{e(H_2) + d(H_1)}{v(H_2)} = \overline{m}_1^2(F).$$

More generally, the density of a 'nice' member of the class $\mathcal{F}(H_1, \dots, H_r)$ with vertex-disjoint outer copies is given by the fraction on the right hand side of (4) (cf. Figure 2 for an example). Thus the statement of Lemma 14 is essentially that members of the family $\mathcal{F}(H_1, \dots, H_r)$ that contain overlapping substructures are at least as dense as the 'nice' members which have vertex-disjoint outer copies.

We shall prove Lemma 14 by induction on r using Lemma 15, which essentially proves the case $r = 2$. Recall that we abbreviate $\overline{d}_1^2$ by $\overline{d}_1$.

Lemma 15. *Let S and H be graphs satisfying $m(H) \le \overline{d}_1(S, H)$. Then all $S' \in \mathcal{F}(S, H)$ satisfy*

$$m(S') \ge \overline{d}_1(S, H).$$

With Lemma 15 at hand, Lemma 14 follows easily.

Proof of Lemma 14. We proceed by induction on r. For $r = 1$, we have $\mathcal{F}(F) = \{F\}$ and $m(F) = \overline{m}_1^1(F)$ by definitions.

Now suppose we have $r \ge 2$, and let F^r be any graph from $\mathcal{F}(H_1, H_2, \dots, H_r)$. Then by definition there is a graph $F^{r-1} \in \mathcal{F}(H_1, H_2, \dots, H_{r-1})$ such that $F^r \in \mathcal{F}(F^{r-1}, H_r)$, and we have

$$\begin{aligned}\overline{d}_1(F^{r-1}, H_r) = \frac{e(H_r) + m(F^{r-1})}{v(H_r)} &\overset{\text{Ind.}}{\ge} \frac{e(H_r) + \overline{m}_1^{r-1}(F)}{v(H_r)} \\ &= \overline{d}_1^r(F, H_r) = \overline{m}_1^r(F)\end{aligned} \tag{6}$$

by the induction hypothesis. Due to

$$m(H_r) \le m(F) \overset{\text{L. 7}}{\le} \overline{m}_1^r(F) \overset{(6)}{\le} \overline{d}_1(F^{r-1}, H_r),$$

we can apply Lemma 15 with $S \leftarrow F^{r-1}$ and $H \leftarrow H_r$ to obtain that

$$m(F^r) \overset{\text{L. 15}}{\geq} \overline{d}_1(F^{r-1}, H_r) \overset{(6)}{\geq} \overline{m}_1^r(F). \qquad \blacksquare$$

Proof of Lemma 15. First, we argue that it suffices to prove Lemma 15 for the case when S is balanced. Suppose it is not, and let $G \subset S$ be a (balanced) subgraph satisfying $d(G) = m(S)$. Then we have $\overline{d}_1(S,H) = \overline{d}_1(G,H)$. Hence, if S and H satisfy the assumption of the lemma, so do G and H. Moreover, every graph $S' \in \mathcal{F}(S,H)$ contains a subgraph $G' \in \mathcal{F}(G,H)$. Therefore, if Lemma 15 holds for balanced graphs, it follows that

$$m(S') \geq m(G') \overset{\text{L. 15}}{\geq} \overline{d}_1(G,H) = \overline{d}_1(S,H).$$

For the rest of the proof, we assume that S is balanced and thus

$$\overline{d}_1(S,H) = \frac{e_H + d(S)}{v_H} = \frac{v_S e_H + e_S}{v_S v_H}. \tag{7}$$

In order to prove $m(S') \geq \overline{d}_1(S,H)$ for all $S' \in \mathcal{F}(S,H)$, we show the slightly stronger assertion $d(S') \geq \overline{d}_1(S,H)$.

Recall that in every graph $S' = (V \dot\cup U, E \dot\cup D)$, the inner copy (V,E) is isomorphic to S. By definition, for each inner vertex $v \in V$, we can identify sets of outer vertices $U(v) \subseteq U$ and outer edges $D(v) \subseteq D$ such that $\widehat{H}(v) := (\{v\} \dot\cup U(v), D(v))$ is isomorphic to H. While these sets are not necessarily unique, for the rest of the proof we fix one choice of appropriate sets $U(v)$ and $D(v)$. The minimality condition in Definition 12 ensures that every vertex of U is contained in one of the sets $U(v)$, $v \in V$, and that every edge of D is contained in one of the sets $D(v)$, $v \in V$. Let $\ell := v_S$, and fix an arbitrary order $v_1, \ldots, v_\ell$ on the inner vertices of S'. For $2 \leq i \leq \ell$, let

$$J_i := \widehat{H}(v_i) \cap \left(\bigcup_{j=1}^{i-1} \widehat{H}(v_j) \right)$$

denote the intersection of the ith outer copy of H with the preceding $i-1$ copies. For $0 \leq k \leq \ell$, let

$$S_k := (V,E) \cup \bigcup_{i=1}^{k} \widehat{H}(v_i).$$

Note that $S_0 \cong S$ and $S_\ell = S'$.

With induction on k and using that

$$V\big(S_{k-1} \cap \widehat{H}(v_k)\big) = \{v_k\} \dot\cup V(J_k),$$

we obtain that, for $0 \le k \le \ell$,

$$\begin{aligned} v(S_k) &= v(S_{k-1}) + v\big(\widehat{H}(v_k)\big) - v\big(S_{k-1} \cap \widehat{H}(v_k)\big) \\ &\overset{\text{Ind.}}{=} \left(v_S + (k-1)(v_H - 1) - \sum_{i=2}^{k-1} v(J_i)\right) + v_H - (1 + v(J_k)) \\ &= v_S + k(v_H - 1) - \sum_{i=2}^{k} v(J_i). \end{aligned}$$

Similarly, we obtain with

$$E\big(S_{k-1} \cap \widehat{H}(v_k)\big) = E(J_k)$$

that, for $0 \le k \le \ell$,

$$e(S_k) = e_S + ke_H - \sum_{i=2}^{k} e(J_i).$$

For $k = \ell (= v_S)$ this yields

$$v(S') = v(S_\ell) = v_S v_H - \sum_{i=2}^{\ell} v(J_i) \tag{8}$$

and

$$e(S') = e(S_\ell) = e_S + v_S e_H - \sum_{i=2}^{\ell} e(J_i). \tag{9}$$

Since we chose H maximizing (2), and since each J_i is a subgraph of H, we have for $2 \le i \le \ell$ that

$$\frac{e(J_i)}{v(J_i)} \le \frac{e(J_i) + d(S)}{v(J_i)} \le \frac{e_H + d(S)}{v_H} \overset{(7)}{=} \overline{d}_1(S, H). \tag{10}$$

Using (10) and applying Proposition 8 repeatedly, we obtain from (8) and (9) that

$$\frac{e(S')}{v(S')} = \frac{e_S + v_S e_H - \sum_{i=2}^{\ell} e(J_i)}{v_S v_H - \sum_{i=2}^{\ell} v(J_i)} \ge \frac{e_S + v_S e_H}{v_S v_H} \overset{(7)}{=} \overline{d}_1(S, H). \qquad \blacksquare$$

4. Upper bound

In this section we prove an upper bound of $n^{-1/\overline{m}_1^r(F)}$ for the game with a graph F satisfying the precondition of Theorem 1, and an arbitrary number of colors r. We show that Painter a.a.s. fails to color $G_{n,p}$ online regardless of her strategy if $p \gg n^{-1/\overline{m}_1^r(F)}$, provided that there exists an induced subgraph $F^\circ \subset F$ on $v_F - 1$ vertices satisfying

$$m_1(F^\circ) \le \overline{m}_1^2(F)$$

(cf. (3)). We will need this assumption in order to apply Theorem 11 to F°.

Assume that $p \gg n^{-1/\overline{m}_1^r(F)}$ is given. We combine a two-round approach with induction on r to prove that every strategy a.a.s. fails to color all vertices. We partition the vertex set of $G = G_{n,p}$ into two sets $V_1 := \{v_1, \dots, v_{\lfloor n/2 \rfloor}\}$ and $V_2 := \{v_{\lfloor n/2 \rfloor + 1}, \dots, v_n\}$ and relax the game to a *semi-online* game, revealing the subgraph $G[V_1]$ to Painter all at once. She may color the vertices of V_1 *offline*. In the second round, the vertices of V_2 are revealed one by one, and Painter has to color them online as before.

Suppose that Painter's coloring of the vertex set V_1 is fixed, and consider the set of edges $E_G(V_1, V_2)$ generated between V_1 and V_2, but hidden from Painter's view. For each color $s \in \{1, \dots, r\}$, this edge set defines a vertex set $\mathrm{Base}(s) \subseteq V_2$ consisting of all vertices in V_2 that complete a copy of F in color s. Obviously, Painter may not assign color s to any vertex in $\mathrm{Base}(s)$ in the second round. The next claim asserts that there exists some color class $s_0 \in \{1, \dots, r\}$ such that $\mathrm{Base}(s_0)$ is large enough to apply the induction hypothesis.

Claim 16. *After the first round was played, there a.a.s. exists a color $s_0 \in \{1, \dots, r\}$ such that we have*

$$|\,\mathrm{Base}(s_0)| = \Omega\big(n^{\overline{m}_1^{r-1}(F)/\overline{m}_1^r(F)}\big).$$

With Claim 16 at hand, the upper bound in Theorem 1 is easy to prove.

Proof of Theorem 1 (Upper Bound). We proceed by induction on r. The base case $r = 1$ follows directly from Theorem 5. For $r \ge 2$, Claim 16 implies that in the second round, Painter must a.a.s. color a binomial random graph $G_{\widetilde{n},\widetilde{p}}$ on $\widetilde{n} = \Omega(n^{\overline{m}_1^{r-1}(F)/\overline{m}_1^r(F)})$ vertices with edge probability

$$\widetilde{p} = p \gg n^{-1/\overline{m}_1^r(F)} = \Omega\big(\widetilde{n}^{-1/\overline{m}_1^{r-1}(F)}\big)$$

with just $r - 1$ colors left in an online fashion. Applying the induction hypothesis yields that she a.a.s. fails to do so no matter which strategy she employs. ∎

It remains to prove Claim 16.

Proof of Claim 16. We will prove a lower bound on $\mathbb{P}[v\in\mathrm{Base}(s_0)]$ by an application of Theorem 10 to the random edges between V_1 and V_2 generated in the second round. For this calculation to work out we need certain properties to hold for the random graph on V_1 generated in the first round. In the following we specify these properties and prove that they hold a.a.s.

In order to simplify notation, let $\beta := \overline{m}_1^{r-1}(F)/\overline{m}_1^r(F)$. Note that we have $\beta < 1$ due to Lemma 7. For any graph $J \subseteq F$ with $v_J \geq 1$ and $p \gg n^{-1/\overline{m}_1^r(F)}$, we have

$$\begin{aligned} n^{v_J} p^{e_J} &\gg n^{v_J - e_J/\overline{m}_1^r(F)} \geq n^{v_J - \frac{e_J v_J}{e_J + \overline{m}_1^{r-1}(F)}} \\ &= n^{\frac{v_J e_J + v_J \overline{m}_1^{r-1}(F) - e_J v_J}{e_J + \overline{m}_1^{r-1}(F)}} = n^{\frac{v_J}{e_J + \overline{m}_1^{r-1}(F)} \overline{m}_1^{r-1}(F)} \\ &\geq n^{\overline{m}_1^{r-1}(F)/\overline{m}_1^r(F)} = n^{\beta}. \end{aligned} \tag{11}$$

Consider a fixed induced subgraph $F^\circ \subset F$ on $v_F - 1$ vertices satisfying $m_1(F^\circ) \leq \overline{m}_1^2(F) \leq \overline{m}_1^r(F)$ (cf. (3) and Lemma 7). Let $s_0 \in \{1, \ldots, r\}$ denote the color for which the number of monochromatic copies of F° in $G[V_1]$ is largest, and let M denote the number of copies in this color. In the following, we label these copies by F_i°, $i=1, \ldots, M$. By Theorem 11, we a.a.s. have

$$M = \Omega\big(n^{v(F^\circ)} p^{e(F^\circ)}\big). \tag{12}$$

For any vertex $v \in V_2$ and $i = 1, \ldots, M$, let $T_{v,i} \subseteq V_1 \times \{v\}$ be a set of potential edges that connect F_i° and v such that they form a copy of F. If there are several such sets, pick one arbitrarily. Thus $|T_{v,i}| = \deg_F(u)$ for all v and i, where u denotes the vertex that was removed from F to obtain F°.

For $v \in V_2$ and any pair of indices i, j, $1 \leq i, j \leq M$, let

$$J_{v,ij} := (F_i^\circ \cup T_{v,i}) \cap (F_j^\circ \cup T_{v,j})$$

denote the graph in which the two potential copies of F intersect. Furthermore, for $J \subseteq F$ let $M_{v,J}$ denote the number of pairs i, j, $i \neq j$, for which $T_{v,i} \cap T_{v,j} \neq \emptyset$ and $J_{v,ij} \cong J$. Note that $M_{v,J}$ is bounded by a constant times the number of subgraphs in $G[V_1]$ formed by two copies of F° intersecting in a copy of $J^\circ := J \cap F^\circ$. (Here the constant accounts for the fact that the same subgraph of $G[V_1]$ might correspond to different overlapping pairs of copies of F°.) The expected number of such subgraphs in $G[V_1]$ is of order

$$\begin{aligned} n^{2v(F^\circ) - v(J^\circ)} p^{2e(F^\circ) - e(J^\circ)} &= \big(n^{v(F^\circ)} p^{e(F^\circ)}\big)^2 n^{-v_J + 1} p^{-e_J + \deg_J(u)} \\ &\overset{(11)}{\ll} \big(n^{v_F - 1} p^{e(F^\circ)}\big)^2 n^{1-\beta} p^{\deg_J(u)}. \end{aligned}$$

Thus it follows with Markov's inequality that

$$M_{v,J} \ll \left(n^{v_F-1}p^{e(F^\circ)}\right)^2 n^{1-\beta} p^{\deg_J(u)} \tag{13}$$

a.a.s. Since the number of subgraphs $J^\circ \subseteq F^\circ$ is a constant only depending on F, a.a.s. this bound holds simultaneously for all subgraphs $J \subseteq F$ and all $v \in V_2$ after the first round.

For $v \in V_2$, let

$$\mathcal{F}_v := \{T_{v,1}, \dots, T_{v,M}\}.$$

Note that $\mathcal{F}_v$ might be a multiset, since the same set of edges may complement distinct monochromatic copies of F° in $G[V_1]$ to a copy of F. For $i = 1, \dots, M$, let $X_{v,i}$ denote the indicator random variable for the event $T_{v,i} \subseteq G_{n,p}$, and set

$$X_v := \sum_{i=1}^{M} X_{v,i}\,.$$

Clearly, the vertex v is contained in $\mathrm{Base}(s_0)$ if $X_v \geq 1$. We apply Theorem 10 to the family $\mathcal{F}_v$ in order to obtain a lower bound on the probability of this event. Conditioning on the outcome of the first round as specified, i.e., on (12) and (13), we obtain

$$\mu = \mathbb{E}[X_v] = M \cdot p^{\deg_F(u)} \overset{(12)}{=} \Omega\left(n^{v_F-1}p^{e_F}\right) \overset{(11)}{=} \omega\left(n^{\beta-1}\right) \tag{14}$$

and

$$\begin{aligned}
\Delta &= \sum_{\substack{i,j=1\\ T_{v,i}\sim T_{v,j}}}^{M} \mathbb{E}[X_{v,i}X_{v,j}] \\
&= \sum_{\substack{J\subseteq F\\ u\in V(J)}} \sum_{\substack{i,j=1\\ T_{v,i}\sim T_{v,j}, J_{v,ij}\cong J}}^{M} \mathbb{E}[X_{v,i}X_{v,j}] \\
&= \sum_{\substack{J\subseteq F\\ u\in V(J)}} M_{v,J} \cdot p^{2\deg_F(u)-\deg_J(u)} \\
&\overset{(13)}{=} o\left((n^{v_F-1}p^{e_F})^2 n^{1-\beta}\right) \overset{(14)}{=} o\left(\mu^2 n^{1-\beta}\right).
\end{aligned}$$

Therefore, Theorem 10 yields that

$$\begin{aligned}
\mathbb{P}[v \notin \mathrm{Base}(s_0)] \leq \mathbb{P}[X_v = 0] &\leq \exp\left\{-\frac{\mu^2}{2(\mu+\Delta)}\right\} \\
&= \exp\left\{-\omega\left(n^{\beta-1}\right)\right\} = 1 - \omega\left(n^{\beta-1}\right).
\end{aligned}$$

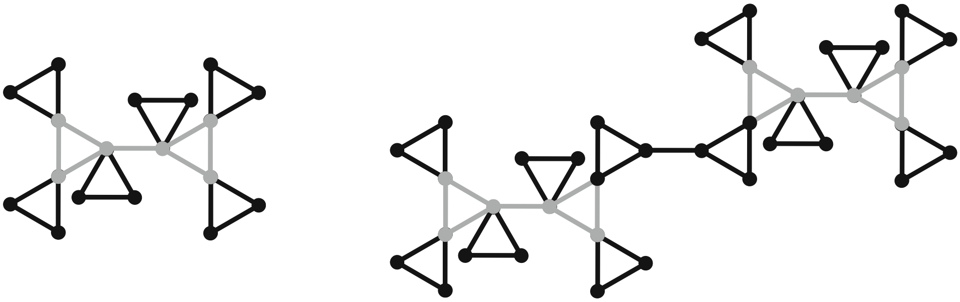

Figure 3. Illustrations for the bowtie example, cf. text.

Since for a fixed outcome of the first round, the events $\{u \in \mathrm{Base}(s_0)\}$ and $\{v \in \mathrm{Base}(s_0)\}$ are independent for $u \neq v \in V_2$, standard Chernoff bounds imply that a.a.s. we have

$$|\,\mathrm{Base}(s_0)| = \Omega\Bigg(\sum_{v \in V_2} \mathbb{P}\left[v \in \mathrm{Base}(s_0)\right]\Bigg) = \Omega\big(n^{\beta}\big). \qquad \blacksquare$$

5. Outlook

It is tempting to dismiss the precondition (3) as an artifact of our upper bound proof technique and conjecture that $n^{-1/\overline{m}_1^r(F)}$ is in fact the threshold of the game for all graphs F and all $r \geq 1$. We conclude this work by presenting an example that shows that this is not the case.

Consider the case $r = 2$ and the 'bowtie graph' B formed by two triangles which are connected by an edge. Our lower bound proof shows that greedily avoiding triangles (say, greedily using red, and using blue to avoid red triangles) allows Painter to win the game if $p \ll n^{-1/\overline{m}_1^2(B)} = n^{-18/25} \approx n^{-0.72}$. This value corresponds to the number of edges and vertices of a 'nice' graph from $\mathcal{F}(F, K_3)$, depicted on the left hand side of Figure 3. With some case checking, one can show that switching back to red when the previous strategy would complete a blue copy of B (thus completing a red triangle instead) improves the lower bound to $n^{-36/51} = n^{-12/17} \approx n^{-0.706}$. This value is given by the number of edges and vertices of the graph depicted on the right hand side of Figure 3.

References

[1] B. Bollobás: Threshold functions for small subgraphs, *Math. Proc. Cambridge Philos. Soc.* **90(2)** (1981), 197–206.

[2] E. Friedgut: Hunting for sharp thresholds, *Random Structures & Algorithms* **26(1–2)** (2005), 37–51.

[3] E. Friedgut, Y. Kohayakawa, V. Rödl, A. Ruciński and P. Tetali: Ramsey games against a one-armed bandit, *Combinatorics, Probability and Computing* **12(5–6)** (2003), 515–545.

[4] S. Janson: Poisson approximation for large deviations, *Random Structures & Algorithms* **1(2)** (1990), 221–229.

[5] T. Łuczak, A. Ruciński and B. Voigt: Ramsey properties of random graphs, *J. Combin. Theory Ser. B* **56(1)** (1992), 55–68.

[6] M. Marciniszyn and R. Spöhel: Online vertex colorings of random graphs without monochromatic subgraphs, in: *Proceedings of the 18th annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2007)*, pages 486–493, 2007.

[7] M. Marciniszyn, R. Spöhel and A. Steger: Online Ramsey games in random graphs, *Combinatorics, Probability and Computing* **18(1–2)** (2009), 271–300.

[8] M. Marciniszyn, R. Spöhel and A. Steger: Upper bounds for online Ramsey games in random graphs, *Combinatorics, Probability and Computing* **18(1–2)** (2009), 259–270.

Martin Marciniszyn, Reto Spöhel

Institute of Theoretical Computer Science
ETH Zürich
Universitätsstrasse 6
8092 Zürich
Switzerland

mmarcini@inf.ethz.ch, rspoehel@inf.ethz.ch