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## Private Computation: $k$ -Connected versus 1-Connected Networks\*

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**Abstract.** We study the role of connectivity of communication networks in private computations under information theoretical settings in the honest-but-curious model. We show that some functions can be 1-privately computed even if the underlying network is 1-connected but not 2-connected. Then we give a complete characterisation of non-degenerate functions that can be 1-privately computed on non-2-connected networks.

Furthermore, we present a technique for simulating 1-private protocols that work on arbitrary (complete) networks on  $k$ -connected networks. For this simulation, at most  $(1 - k/(n - 1)) \cdot L$  additional random bits are needed, where  $L$  is the number of bits exchanged in the original protocol and  $n$  is the number of players.

Finally, we give matching lower and upper bounds for the number of random bits needed to compute the parity function on  $k$ -connected networks 1-privately, namely  $\lceil (n - 2)/(k - 1) \rceil - 1$  random bits for networks consisting of  $n$  players.

**Key words.** Private computation, Secure multi-party computation, Secure function evaluation, Connectivity, Parity, Randomness.

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## 1. Introduction

Consider a set of players, each knowing an individual secret. The goal is to compute a function depending on these secrets such that after the computation none of the players knows anything about the secrets of the other players that cannot be derived from the function value and its own secret. An example for such a computation is the “secret ballot problem”: The members of a committee wish to decide whether the majority votes for yes or no. However, after the vote nobody should know anything about the opinions of the other committee members, not even about the exact number of votes for yes and no, except for whether the majority has voted for yes or no. To come to a decision, any two members can talk to each other in private. If however the members are distributed in a network, then only those members that are connected by a link can talk to each other. In this work we investigate the influence of the underlying network on the ability to perform private computations.

Let  $f$  be an  $n$ -ary Boolean function and let  $x_1, \dots, x_n$  be bits distributed among  $n$  players. A protocol for computing  $f(x_1, \dots, x_n)$  is called  $t$ -private if after executing the protocol all players know  $f(x_1, \dots, x_n)$ , but no group of at most  $t$  players learns anything about the bits of the other players except for what they can deduce from the function value and their own bits.

Depending on the computational power of the players we distinguish between cryptographically secure privacy and information theoretically secure privacy. In the first case we assume that no player is able to gain any information about the input bits of the other players within polynomial time [28], [29]. In the second case we do not restrict the computational power of the players. This notion of privacy (sometimes called unconditional privacy) has been introduced by Ben-Or et al. [3] and Chaum et al. [8]. Private computation has been examined with two different types of players. Malicious players (also called Byzantine players) may arbitrarily deviate from the protocol in order to jam the correctness or the privacy constraint [19], [28], [29]. Honest-but-curious players follow the protocol precisely but are allowed to “gossip” afterwards [19], [23].

We are concerned with 1-privacy in the information theoretically secure setting with honest-but-curious players.

### 1.1. Previous Results

Private computation has been the subject of a considerable amount of research. In the information theoretically secure model, all  $n$ -ary Boolean functions can  $t$ -privately be computed if  $t < n/2$  in the case of honest-but-curious players and if  $t < n/3$  in the case of malicious players [3], [8]. In the cryptographically secure model, this holds for  $t \leq n$  in the case of honest-but-curious players and  $t < n/2$  in the case of malicious players [19], [28] (assuming that trapdoor functions exist). Canetti and Ostrovsky [7] proved that in the cryptographically secure setting, it can be tolerated that all parties deviate from the protocol under the restriction that most parties do not risk being detected by other parties. (Malicious players do not care about being detected.)

Traditionally, one investigates the number of rounds and random bits as complexity measures for private protocols. According to Canetti et al. [6], the quantification of the

amount of randomness needed in cryptographically secure privacy is not meaningful, since it can be reduced using pseudorandom generators [4], [21].

The following papers deal with information theoretical private computation with honest-but-curious players: Chor and Kushilevitz [12] have studied the number of rounds necessary for privately computing the sum modulo an integer. This function has also been investigated by Blundo et al. [5] and Chor et al. [10]. The number of random bits needed for privately computing the parity function has been examined by Kushilevitz and Mansour [25] and Kushilevitz and Rosén [27]. Gál and Rosén [16] have shown that the parity function cannot be computed by any private protocol in  $o(\log n / \log d)$  rounds using  $d$  random bits. They have also given an almost tight randomness-round tradeoff for private computations of arbitrary Boolean functions depending on their sensitivity. Bounds on the maximum number of rounds needed for privately computing a function have also been given by Bar-Ilan and Beaver [2] and by Kushilevitz [24]. Gál and Rosén [17] have proved an upper and lower bound for the number of random bits needed for  $t$ -privately computing parity.

The number of random bits necessary for privately computing a Boolean function is closely related to its circuit size. Kushilevitz et al. [26] have shown that every function can be computed with linear circuit size if and only if it can be privately computed with a constant number of random bits.

Chor and Kushilevitz [11] have characterised the class of Boolean functions that can be  $t$ -privately computed for some  $t \geq n/2$ . Any such function can already be  $n$ -privately computed.

Chor et al. [9], [10] have extended the field of private computation to functions defined over finite domains. Kilian et al. [22], [23] have introduced a notion of reduction and completeness in private computation.

All the papers mentioned above do not restrict the communication capabilities of the players. In other words, they use complete graphs as underlying communication networks. However, most realistic parallel architectures have a restricted connectivity and nodes of bounded degree. Franklin and Yung [15] have been the first who studied the role of connectivity in private computations. They have presented a protocol for  $k$ -connected bus networks that simulates communication steps of a private protocol that was originally written for a complete graph. To simulate a single communication step, their protocol uses  $O(n)$  additional random bits. Franklin and Wright [14] have examined which functions are still privately computable, if the players are malicious and the network connectivity is low.

## 1.2. Our Results

As mentioned above, we are concerned with 1-privacy in the information theoretically secure setting with honest-but-curious players. In the following we use the term “private” for “1-private”.

In this paper we investigate the number of random bits needed to compute functions by private protocols on  $k$ -connected networks. We present a simulation of private protocols designed for arbitrary networks on arbitrary  $k$ -connected networks (for  $k \geq 2$ ) in Section 3. For this simulation, only  $(1 - k/(n - 1)) \cdot \min\{L, (k - 2)/(k - 1) \cdot (n^2 -$

$n) + L/(k - 1)$  additional random bits are needed, where  $L$  is the total number of bits sent in the original protocol.

In Section 4 we study the parity function to a greater extent. For every  $k$ -connected graph with  $k \geq 2$ , we design a private protocol for computing the parity function that uses only  $\lceil (n - 2)/(k - 1) \rceil - 1$  random bits. This considerably reduces the number of random bits compared with the general simulation technique of Section 3 for the specific case of the parity function. This result is tight: There are  $k$ -connected graphs on which every private protocol needs that many random bits to compute the parity function.

All of the above results hold for  $k \geq 2$ . In Section 5 we investigate graphs that are not 2-connected. Our first insight is the following: The parity function over  $n > 2$  bits cannot be computed by a private protocol on any network that is not 2-connected. This can be generalised to a large class of non-degenerate functions. An  $n$ -ary Boolean function is called *non-degenerate* if it depends on all of its  $n$  input bits. It turns out that there are functions that can be privately computed, even if the underlying network is not 2-connected. An example is the following non-degenerate function  $f : \{0, 1\}^{2n+1} \rightarrow \{0, 1\}$  (for  $n \geq 2$ ):

$$f(x, y, z) = \left( x \wedge \bigwedge_{i=1}^n y_i \right) \vee \left( \bar{x} \wedge \bigwedge_{i=1}^n z_i \right).$$

Here,  $x$  is a single bit and both  $y$  and  $z$  are bit strings of length  $n$ . We construct a communication network  $G$  for  $f$  as follows: Let  $G_y$  and  $G_z$  be complete networks with  $n$  players each. Then connect another player  $P_x$  with all players in both  $G_y$  and  $G_z$ . The network obtained is not 2-connected. Using a slight modification of the protocol presented by Kushilevitz et al. [26], one can privately compute the subfunctions

$$\begin{aligned} f_y(x, y) &= x \wedge \bigwedge_{i=1}^n y_i \quad \text{and} \\ f_z(x, z) &= \bar{x} \wedge \bigwedge_{i=1}^n z_i \end{aligned}$$

on the networks  $G_y$  with  $P_x$  and  $G_z$  with  $P_x$ , respectively. Overall, the protocol is private as will be shown in Section 5.

We fully characterise the class of non-degenerate functions that can be privately computed on non-2-connected networks. It turns out that the above example is fairly representative: Each such function has this *if-then-else* structure. The corresponding non-2-connected network consists of two 2-connected components of appropriate sizes.

## 2. Preliminaries

For  $n \in \mathbb{N}$  let  $[n] = \{1, \dots, n\}$ . A graph  $G$  is called  $k$ -connected if, after deleting an arbitrary subset of at most  $k - 1$  nodes, the resulting node-induced graph remains connected. Equivalently, for any two nodes  $u$  and  $v$  of  $G$ , there are at least  $k$  pairwise node-disjoint paths between  $u$  and  $v$ . A *block* of  $G$  is a maximum node-induced subgraph of  $G$  that is 2-connected.

We consider the computation of Boolean functions  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  on a network of  $n$  players. In the beginning each player knows a single bit of the input  $x$ . The players can send messages to other players using secure links where the link topology is given by an undirected graph  $G = (V, E)$ . When the computation stops, all players know the value  $f(x)$ . The goal is to compute  $f(x)$  such that no player learns anything about the other input bits in an information theoretical sense, i.e. with unbounded computational power, except for the information he can deduce from his own bit and the result. Such a protocol is called private. (Recall that private in this paper means 1-private and that we are considering honest players.)

**Definition 2.1.** Let  $C_i$  be a random variable of the communication string seen by player  $P_i$  and let  $R_i$  be his random string. A protocol  $\mathcal{A}$  for computing a function  $f$  is *private with respect to player  $P_i$*  if for every pair of input vectors  $x$  and  $y$  with  $f(x) = f(y)$  and  $x_i = y_i$ , the following conditions hold:

1. for all  $r$ ,  $\Pr(R_i = r \mid x) = \Pr(R_i = r \mid y)$ , and
2. for all  $r$  with  $\Pr(R_i = r \mid x) > 0$  and for all  $c$ ,

$$\Pr(C_i = c \mid R_i = r, x) = \Pr(C_i = c \mid R_i = r, y).$$

(The probabilities are taken over the random strings of all other players.) A protocol  $\mathcal{A}$  is *private* if it is private with respect to all players.

If the number of random bits each player uses is independent of the input, we can omit the first condition since in this case all these probabilities are equal. However, if the number of random bits a player uses depends on the input of the other players and on their random bits, this player might be able to learn something from his random string if  $\Pr(R_i = r \mid x) \neq \Pr(R_i = r \mid y)$ .

In all the protocols presented here it is known in advance how many random bits a player uses. The lower bound for the number of random bits needed for computing parity however holds also for the more general case when this is not known in advance.

We call a protocol *oblivious* if the communication takes place in rounds, each message consists of a single bit, and the number of bits (which is then either zero or one) that  $P_i$  sends to  $P_j$  in round  $t$  depends only on  $i$ ,  $j$ , and  $t$ , but not on the input and the random strings. For an oblivious protocol  $\mathcal{A}$  let  $L(P_i, P_j, \mathcal{A})$  be the total number of bits sent from  $P_i$  to  $P_j$  in  $\mathcal{A}$  and

$$L(\mathcal{A}) = \sum_{i \in [n]} \sum_{j \in [n] \setminus \{i\}} L(P_i, P_j, \mathcal{A}).$$

We distribute the input bits among the nodes of the graph. For convenience, we call the node that gets bit  $x_i$  player  $P_i$ . The players  $P_i$  and  $P_j$  can communicate directly with each other if and only if they are connected by an edge in the graph.

### 3. Private Computation on $k$ -Connected Networks

Most known private protocols are written for specific networks. A simulation of such a private protocol on a different network can be done in such a way that each player of the

new network simulates a player of the original network step-by-step. Hence, we have to find a way to realise the communication steps between all players that are not directly connected. Franklin and Yung [15] have presented a strategy to simulate a transmission of one single bit on a hypergraph by using  $O(n)$  additional random bits. Thus, the whole simulation presented by them requires  $O(m + nL(\mathcal{A}))$  random bits where  $m$  is the number of random bits used by the original protocol. If we consider 2-connected graphs we can simulate each communication step between two players  $P_i$  and  $P_j$  by one additional random bit  $r$  as follows: Assume  $P_i$  has to send bit  $b$  to  $P_j$ . Then  $P_i$  chooses two disjoint paths to  $P_j$  and sends  $r$  to  $P_j$  along the first path and  $r \oplus b$ , the parity of  $r$  and  $b$ , along the second path. In this way,  $m + L(\mathcal{A})$  random bits are sufficient. To reduce the number of random bits even further, we consider the following optimisation problem.

**Definition 3.1** (Max-Neighbour-Embedding). Let  $G = (V, E)$  be a complete graph with edge weights  $\sigma : E \rightarrow \mathbb{N}$  and let  $G' = (V', E')$  be a graph with  $|V| = |V'|$ . Let  $\pi : V \rightarrow V'$  be a bijective mapping. Then the performance of  $\pi$  is defined as

$$\rho(\pi) = \sum_{\{\pi(u), \pi(v)\} \in E'} \sigma(\{u, v\}).$$

The aim is to find a bijection  $\pi : V \rightarrow V'$  that maximises  $\rho(\pi)$  over all bijections.

By reduction from 3-Dimensional-Matching [18, SP1], it can be shown that the decision problem corresponding to Max-Neighbour-Embedding is  $\mathcal{NP}$ -hard, even if  $\sigma$  is  $\{0, 1\}$ -valued, the graph consisting of the weight one edges of  $G$  has maximum degree four, and  $G'$  has maximum degree four. In the following lemma we estimate the performance for the case that  $G'$  is  $k$ -connected.

**Lemma 3.2.** Let  $G = (V, E)$  be a graph with  $n$  nodes and edge weights  $\sigma$ . Let  $G' = (V', E')$  be a  $k$ -connected graph with  $n$  nodes. Then we have

$$\max_{\pi : V \rightarrow V', \pi \text{ is bijective}} \rho(\pi) \geq \frac{k}{n-1} \cdot \sum_{e \in E} \sigma(e).$$

**Proof.** The graph  $G'$  is  $k$ -connected. Thus, every node in  $V'$  has degree at least  $k$ . Let  $\Pi$  be a random bijection from  $V$  to  $V'$ . Since every node in  $V'$  has degree at least  $k$ , the probability that two arbitrary nodes  $u$  and  $v$  are neighbours under  $\Pi$ , i.e.  $\{\Pi(u), \Pi(v)\} \in E'$ , is at least  $k/(n-1)$ . Thus, the edge  $e = \{u, v\} \in E$  yields weight  $\sigma(e)$  with probability at least  $k/(n-1)$  and its expected weight is at least  $k/(n-1) \cdot \sigma(e)$ . Hence, the expected performance  $\rho(\Pi)$  fulfils

$$E(\rho(\Pi)) \geq \sum_{e \in E} \frac{k}{n-1} \cdot \sigma(e) = \frac{k}{n-1} \cdot \sum_{e \in E} \sigma(e).$$

Thus, there exists a bijection with performance at least  $k/(n-1) \cdot \sum_{e \in E} \sigma(e)$ .  $\square$

A bijection that fulfils the requirements of the above lemma can be computed in polynomial time using the method of conditional expectation (see, e.g. [1]).

In the simulation described below, the graph  $G$  is the network for which a given protocol was designed. The edge weights are the number of bits exchanged over each edge (with weight zero if there is no edge in the original network). The graph  $G'$  is the  $k$ -connected network on which we want to simulate the protocol.

The main idea is that for all nodes  $P_i$  and  $P_j$  in a  $k$ -connected graph, we have  $k$  node-disjoint paths connecting these two nodes. Thus, we can simulate  $k - 1$  bits sent from  $P_i$  to  $P_j$  as follows: First,  $P_i$  sends a random bit to  $P_j$  on one path. Then he uses this random bit to encode  $k - 1$  bits sent along the other  $k - 1$  paths.

**Theorem 3.3.** *Every oblivious private protocol  $\mathcal{A}$  using  $m$  random bits can be simulated with  $m + (1 - k/(n - 1)) \cdot \min\{L(\mathcal{A}), (k - 2)/(k - 1) \cdot (n^2 - n) + L(\mathcal{A})/(k - 1)\}$  random bits on every  $k$ -connected graph.*

**Proof.** Let  $G = (V, E)$  be the network used in protocol  $\mathcal{A}$  and let  $G' = (V', E')$  be the  $k$ -connected network. To simulate  $\mathcal{A}$  we first choose a bijection between the players in  $G$  and the players in  $G'$ . For every edge  $\{P_i, P_j\} \in E$  let  $\sigma(\{P_i, P_j\}) = L(P_i, P_j, \mathcal{A}) + L(P_j, P_i, \mathcal{A})$ . In Lemma 3.2 we have seen that there exists a bijection  $\pi : V \rightarrow V'$  with performance  $\rho(\pi) \geq (k/(n - 1))L(\mathcal{A})$ . Using this bijection, at least  $k/(n - 1) \cdot L(\mathcal{A})$  bits of the total communication in  $\mathcal{A}$  are sent between players that are also neighbours in  $G'$ . Thus, this part of the communication can be simulated directly without additional random bits.

For the remaining  $(1 - k/(n - 1)) \cdot L(\mathcal{A})$  bits we proceed as follows: Let  $P_i$  and  $P_j$  be two players that are not directly connected in  $G'$ . Then  $P_i$  partitions the bits he will send to  $P_j$  into blocks  $B_1, \dots, B_{\lceil L(P_i, P_j, \mathcal{A})/(k-1) \rceil}$  of size at most  $k - 1$ . Furthermore,  $P_i$  chooses  $k$  node-disjoint paths from  $P_i$  to  $P_j$ .  $P_i$  uses a separate random bit  $r_\ell$  for each block  $B_\ell$ . He sends  $r_\ell$  along the first path and  $b \oplus r_\ell$  for each  $b \in B_\ell$  along the remaining paths, each bit on a separate path.

$\sum_{i \in [n], j \in [n] \setminus \{i\}} \lceil L(P_i, P_j, \mathcal{A})/(k - 1) \rceil \leq (k - 2)/(k - 1) \cdot (n^2 - n) + L(\mathcal{A})/(k - 1)$  holds, since we round at most  $n^2 - n$  fractions with denominator  $k - 1$ . (This is a worst-case estimate. Given a concrete protocol, additional knowledge about the distribution of the bits on the links may be used to get a better bound.) However, we never need more than  $(1 - k/(n - 1)) \cdot L(\mathcal{A})$  bits altogether. Both observations together imply the bound proposed.  $\square$

#### 4. Computing Parity on $k$ -Connected Networks

It is well known that the parity function of  $n$  bits can be privately computed on a Hamiltonian cycle by using only one random bit. On the other hand, using our simulation presented in Section 3 we get an upper bound of  $n - 1$  random bits for arbitrary 2-connected networks. The aim of this section is to close this gap. We present a private protocol for parity that uses  $\lceil (n - 2)/(k - 1) \rceil - 1$  random bits and show that there are  $k$ -connected networks on which parity cannot be computed with less than  $\lceil (n - 2)/(k - 1) \rceil - 1$  random bits.

Note that in the proof of the following lemma, we make no assumptions about how many random bits any player uses or that the number of random bits is known in advance.

Thus, the lower bound holds also for the more general case where the number of random bits each player uses can depend on the input and the other player's random tapes.

**Lemma 4.1.** *There exist  $k$ -connected networks with  $n \geq 2k$  players on which the parity function cannot be computed by a private protocol with less than  $\lceil (n-2)/(k-1) \rceil - 1$  random bits.*

**Proof.** We consider the bipartite graph  $K_{k,n-k}$ , which is  $k$ -connected, and show that every private protocol that computes the parity function on this network needs at least  $\lceil (n-2)/(k-1) \rceil - 1$  random bits. Let  $\{P_1, P_2, \dots, P_k\}$  and  $\{P_{k+1}, P_{k+2}, \dots, P_n\}$  be the two sets of nodes of  $K_{k,n-k}$ . For every  $i = 1, \dots, k$  and  $j = k+1, \dots, n$  we have an edge  $\{P_i, P_j\}$  in  $K_{k,n-k}$ . Now assume to the contrary that there exists a private protocol  $\mathcal{A}$  on  $K_{k,n-k}$  using less than  $\lceil (n-2)/(k-1) \rceil - 1$  random bits.

Let  $r = \langle r_1, \dots, r_n \rangle$  be the contents  $r_1, \dots, r_n$  of all random tapes. For a string  $x \in \{0, 1\}^n$  and  $i \in [n]$ , let  $\mathcal{C}_i(x, r)$  be a full description of the communication received by  $P_i$  during the execution of  $\mathcal{A}$  with random bits  $r$  on input  $x$ . Moreover, let

$$\begin{aligned} \mathcal{C}(x) &= \{\langle c_1, c_2, \dots, c_k \rangle \mid \exists r \forall i \in [k] : c_i = \mathcal{C}_i(x, r)\} \quad \text{and} \\ \mathcal{C}_i(x) &= \{c \mid \exists r : c = \mathcal{C}_i(x, r)\}. \end{aligned}$$

We consider computations of  $\mathcal{A}$  on inputs

$$X = \left\{ x \mid x_1 = x_2 = \dots = x_k = 0 \text{ and } \bigoplus_{i=1}^n x_i = 0 \right\}.$$

For every  $x \in X$  and every communication  $c_1$  we define

$$\mathcal{C}(c_1, x) = \{\langle c_2, \dots, c_k \rangle \mid \langle c_1, c_2, \dots, c_k \rangle \in \mathcal{C}(x)\}.$$

**Claim 4.2.**  $\exists c_1 \forall x \in X : \mathcal{C}(c_1, x) \neq \emptyset$ .

**Proof.** Let  $x \in X$ . Because  $x$  is a valid input for the protocol  $\mathcal{A}$ , there exists at least one tuple  $\langle c_1, \dots, c_k \rangle$  in  $\mathcal{C}(x)$ . Hence, there exists at least one  $c_1$  with  $\mathcal{C}(c_1, x) \neq \emptyset$ . If for some  $y \in X$  the set  $\mathcal{C}(c_1, y)$  is empty, then this violates the privacy constraint.  $\square$

We also need the following claim, which follows from work by Kushilevitz and Rosén [27]. For the sake of completeness we give a proof though.

**Claim 4.3.** *Let  $d$  be the maximum number of random bits used. Then for all  $i \in [k]$ , we have  $|\bigcup_{x \in X} \mathcal{C}_i(x)| \leq 2^d$ .*

**Proof.** We start by considering any fixed  $x \in X$  and show that  $|\mathcal{C}_i(x)| \leq 2^d$ . We view the execution of  $\mathcal{A}$  such that in each round first  $P_1$ , then  $P_2, \dots$ , and finally  $P_n$  performs his computation. This can be done, since these computations do not depend on each other. Viewing the computation this way, only one random bit is read at any time. The



claim follows from the following observations: Any random bit has two outcomes, the player who reads the next random bit is determined by the previous random bits and  $x$ , and the players read at most  $d$  random bits.

Finally, we have  $C_i(x) = C_i(y)$  for all  $x, y \in X$ , since  $P_i$  must not be able to distinguish  $x$  and  $y$ .  $\square$

Since the number of random bits used by the protocol is less than  $(n - k - 1)/(k - 1)$ , we have  $|\bigcup_{x \in X} C_i(x)| < 2^{(n-k-1)/(k-1)}$ . Hence, we have

$$\left| \bigcup_{x \in X} C(c_1, x) \right| \leq \prod_{j=2}^k \left| \bigcup_{x \in X} C_j(x) \right| < 2^{n-k-1}.$$

Since  $|X| = 2^{n-k-1}$  and by Claim 4.2, we get

$$\exists c_1, c_2, \dots, c_k \exists x, y \in X: x \neq y \text{ and } \langle c_2, \dots, c_k \rangle \in C(c_1, x) \cap C(c_1, y).$$

This means that there are two different strings  $x, y \in X$  such that on either string the players  $P_1, \dots, P_k$  receive  $c_1, \dots, c_k$ , respectively. Let  $i$ , with  $k + 1 \leq i \leq n$ , be a position where  $x_i \neq y_i$ . Let  $r = \langle r_1, \dots, r_n \rangle$  and  $r' = \langle r'_1, \dots, r'_n \rangle$  be the contents of the random tapes such that  $c_i = C_i(x, r) = C_i(y, r')$  for all  $1 \leq i \leq k$ .

During a computation of protocol  $\mathcal{A}$  on input  $x_1 \cdots x_{i-1} y_i x_{i+1} \cdots x_n$  with random strings  $\langle r_1, \dots, r_{i-1}, r'_i, r_{i+1}, \dots, r_n \rangle$  the players  $P_1, P_2, \dots, P_k$  again receive the communication strings  $c_1, c_2, \dots, c_k$ . This is because the graph is bipartite and  $P_i$  can only communicate with  $P_1, \dots, P_k$ . Hence, for this input they compute the same result as for  $x$ , a contradiction.  $\square$

Now we show that this bound is best possible. To obtain a private protocol that computes the parity function with  $\lceil (n - 2)/(k - 1) \rceil - 1$  random bits, we use the following result by Egawa et al. [13].

**Lemma 4.4** [13]. *Let  $G$  be a  $k$ -connected graph,  $k \geq 2$ , with minimum degree  $d$  and at least  $2d$  vertices. Let  $V'$  be an arbitrary set of  $k$  vertices of  $G$ . Then  $G$  has a cycle of length at least  $2d$  that contains every vertex of  $V'$ .*

**Lemma 4.5.** *Let  $G = (V, E)$  be a  $k$ -connected graph with  $|V| \geq 2k$  and  $k \geq 2$ . Then for every subset  $V' \subseteq V$  with  $|V'| = k$ , there exists a simple cycle of length at least  $2k$  containing all nodes in  $V'$ .*

**Proof.** Since  $G$  is  $k$ -connected, every node has degree at least  $k$ . Thus,  $G$  contains a simple cycle of length at least  $2k$  running through all nodes in  $V'$  by Lemma 4.4.  $\square$

**Lemma 4.6.** *Let  $G = (V, E)$  be a  $k$ -connected graph,  $k \geq 2$ , with  $|V| \geq 2k$ . Then for every subset  $V' \subseteq V$  with  $|V'| = k + 1$ , there exists a simple path containing all nodes in  $V'$ .*

**Proof.** By Lemma 4.5,  $G$  contains a cycle  $C$  running through  $k$  of the nodes in  $V'$ . If the last node  $v$  of  $V'$  is also on  $C$ , we simply delete one edge of  $C$  and are done. Otherwise, since  $G$  is connected there is a path from  $v$  to a node  $u$  of  $C$  such that each internal node of this path is not in  $C$ . By deleting one edge of  $C$  incident with  $u$ , we obtain the desired path.  $\square$

**Lemma 4.7.** *Let  $G = (V, E)$  be a  $k$ -connected graph,  $k \geq 2$ , with  $|V| \geq 2k + 1$ . Then  $G$  has a simple path with at least  $2k + 1$  nodes.*

**Proof.** By Lemma 4.5,  $G$  has a cycle  $C$  of length at least  $2k$ . If this length is strictly greater than  $2k$ , we delete one of its edges and are done. Otherwise, there is a node  $v$  not in  $C$ . Since  $G$  is connected there is a path from  $v$  to a node  $u$  of  $C$  such that each internal node of this path is not in  $C$ . By deleting one edge of  $C$  incident with  $u$ , we obtain the desired path.  $\square$

Now we present a protocol for computing parity on arbitrary  $k$ -connected networks  $G$ . We first assume that  $G$  has at least  $2k + 1$  nodes. Basically, our protocol works as follows. Each player is either red or black. Initially, all players are red. A player is red as long as he holds some (input or random) bit that has not contributed to parity yet. Otherwise, he is black. Using Lemmas 4.5–4.7, we find paths or cycles containing a certain number of red players, who then contribute their bits. For each such path or cycle, we need one random bit.

1. Mark all nodes in  $G$  red. Set  $z_i := x_i$  for each player  $P_i$ .
2. Choose a path in  $G$  of length  $2k + 1$ . According to Lemma 4.7 such a path exists. The first player  $P_i$  in the path generates a random bit  $r$ . Then  $P_i$  computes  $r \oplus z_i$ , sends the result to the next player in the path, and sets  $z_i := r$ .  
Each internal player  $P_j$  on the path receives a bit  $b$  from his predecessor in the path, computes  $b \oplus z_j$ , sends this bit to his successor, and changes his colour to black.  
The last player  $P_\ell$  on the path receives a bit  $b$  from his predecessor and computes  $z_\ell := z_\ell \oplus b$ .  
After this step,  $2k - 1$  players have changed their colour.
3. We repeat the following step  $\lceil (n - 3k + 1)/(k - 1) \rceil$  times.  
Choose  $k + 1$  red nodes and a path in  $G$  containing all these nodes. According to Lemma 4.6 such a path exists. We can assume that the start and the end node of the path are among the  $k + 1$  given players, hence both are red. Then the first player  $P_i$  on this path generates a random bit  $r$ , computes  $r \oplus z_i$ , sends the result to the next player in the path, and sets  $z_i := r$ .  
Each internal player of the path  $P_j$  receives a bit  $b$  from his predecessor in the path. If  $P_j$  is a black player, he sends  $b$  to his successor. If  $P_j$  is red, he computes  $b \oplus z_j$ , sends this bit to his successor, and changes his colour to black.  
The last player  $P_\ell$  on the path receives a bit  $b$  from its predecessor and computes  $z_\ell := z_\ell \oplus b$ .

After this step, at least  $k - 1$  players have changed their colour. Hence, after  $\lceil (n - 3k + 1)/(k - 1) \rceil$  iterations of this step we have at least

$$\left\lceil \frac{n - 3k + 1}{k - 1} \right\rceil \cdot (k - 1) + 2k - 1 \geq n - k$$

black players. Thus, at most  $k$  are red.

4. Choose a cycle in  $G$  containing all red nodes. According to Lemma 4.5 such a cycle exists. Let  $P_{i_0}$  be a red player. Then  $P_{i_0}$  generates a random bit  $r$ , computes  $r \oplus z_{i_0}$ , and sends the result to the next player in the cycle.

Each other player  $P_j$  on the cycle receives a bit  $b$  from its predecessor. If  $P_j$  is black, he sends  $b$  to its successor. If  $P_j$  is red, he computes  $b \oplus z_j$ , sends this bit to his successor, and changes his colour to black.

If  $P_{i_0}$  receives a bit  $b$ , he computes  $b \oplus r$ . The result of this step is the result of the parity function.

Let us count the number of random bits used in the protocol above. In the second and in the last step we use one random bit. In the third step we need  $\lceil (n - 3k + 1)/(k - 1) \rceil$  random bits. Hence, the total number of random bits is

$$\left\lceil \frac{n - 3k + 1}{k - 1} \right\rceil + 2 = \left\lceil \frac{n - 2}{k - 1} \right\rceil - 1.$$

It remains to show that the protocol is private and computes the parity function. Correctness follows from the fact that each input bit  $x_i$  is stored by exactly one red player and each random bit is stored by either none or two players that are red after each step. By storing a bit  $b$  we mean that a player  $P_i$  knows a value  $z_i$  that depends on  $b$ . Since  $P_{i_0}$  is the last red player, he knows the result of the parity function.

Every bit received by some player in the second and third steps is masked by a separate random bit. Hence, none of these players can learn anything from these bits. The same holds for all players except for player  $P_{i_0}$  in the last step. So we have to analyse the bits sent and received by  $P_{i_0}$  more carefully. In the last step  $z_{i_0}$  is either  $x_{i_0}$ , a random bit, or the parity of a subset of input bits masked by a random bit. In neither case does  $P_{i_0}$  learn anything about the other input bits from the bit he receives and the value of  $z_{i_0}$  except for what can be derived from the result of the function and  $x_{i_0}$ .

**Theorem 4.8.** *Let  $G$  be an arbitrary  $k$ -connected network,  $k \geq 2$ , with  $n$  nodes such that  $n \geq 2k$ . Then the parity of  $n$  bits can be computed by a private protocol on  $G$  using at most  $\lceil (n - 2)/(k - 1) \rceil - 1$  random bits. If  $n < 2k$ , then the parity can be computed with one random bit.*

*For all  $k \geq 2$  and  $n > k$ , there exists a  $k$ -connected network on  $n$  nodes for which this bound is best possible.*

**Proof.** The case  $n \geq 2k + 1$  has already been demonstrated. If  $n \leq 2k$ , then every node in  $G$  has degree at least  $n/2$ . Thus,  $G$  contains a Hamiltonian cycle due to Dirac's theorem, see, e.g. [20], and parity can be computed using one random bit.

The lower bound follows from Lemma 4.1 for  $n \geq 2k$ . For  $n < 2k$ , only one random bit is needed, which is optimal.  $\square$

## 5. Private Computation on Non-2-Connected Networks

In this section we characterise the class of non-degenerate Boolean functions that can be privately computed on networks that are 1-connected but not 2-connected.

A function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  is non-degenerate if for all  $1 \leq i \leq n$ , there are  $x, y \in \{0, 1\}^n$  that differ only at the  $i$ th position and  $f(x) \neq f(y)$ .

Let  $f$  be a non-degenerate  $n$ -ary Boolean function. We say that a variable  $x_i$  *dominates*  $f$  if there is a partition  $Y, Z$  of the variables  $\{x_1, \dots, x_n\} \setminus \{x_i\}$  with  $Y, Z \neq \emptyset$ , such that  $f(x_1, \dots, x_n)$  depends only on variables in  $Y$  if  $x_i = 0$  and only on variables in  $Z$  if  $x_i = 1$ . (This partition is unique, since  $f$  is non-degenerate.) We call  $\text{bal}_f(x_i) = \min\{|Y|, |Z|\}$  the *balance* of  $x_i$  in  $f$ . If  $f$  is dominated by a variable  $x$ , then we can reorder the variables of  $f$  and find  $g_0$  and  $g_1$  such that

$$f(x, y, z) = \begin{cases} g_0(y) & \text{if } x = 0 \text{ and} \\ g_1(z) & \text{if } x = 1. \end{cases}$$

For  $k \leq n/2$ , we denote by  $B_{n,k}$  the class of all networks with  $n$  nodes and with exactly two blocks such that one block consists of  $k + 1$  and the other block consists of  $n - k$  nodes. (The bridge node belongs to both components.)

**Lemma 5.1.** *Let  $f$  be a non-degenerate Boolean function. Then  $f$  cannot be both  $x$ - and  $y$ -dominated.*

**Proof.** Let  $f$  depend on  $x, y$ , and  $z_1, \dots, z_n$ . Assume that  $f$  is dominated by  $x$  and  $y$ . Then for  $x = 0$ ,  $f$  depends on variables  $X_0 \subseteq \{y, z_1, \dots, z_n\}$ , and for  $x = 1$ ,  $f$  depends on variables  $X_1 = \{y, z_1, \dots, z_n\} \setminus X_0$ , since  $f$  is non-degenerate. Similarly, we have two disjoint sets  $Y_0$  and  $Y_1$  of variables with  $Y_0 \cup Y_1 = \{x, z_1, \dots, z_n\}$ .

Without loss of generality we assume that  $x \in Y_0$  and  $y \in X_0$ . Now consider  $x = 1$ . Since  $y \notin X_1$ ,  $f$  does not depend on  $y$  when setting  $x = 1$ . Hence,  $f$  depends only on  $X_1 = Y_0 \cap Y_1 = \emptyset$  for  $x = 1$ , because we assumed that  $f$  is  $y$ -dominated. Thus,  $x$  does not dominate  $f$ .  $\square$

**Lemma 5.2.** *Let  $f$  be a non-degenerate  $n$ -ary Boolean function,  $n \geq 3$ . Let  $G$  be a network in  $B_{n,k}$ . Assume that either*

1.  $x$  does not dominate  $f$  or
2.  $x$  dominates  $f$  but  $\text{bal}_f(x) \neq k$  or
3.  $x$  dominates  $f$  with partition  $Y$  and  $Z$ ,  $\text{bal}_f(x) = k$  but both blocks hold input bits from both  $Y$  and  $Z$ .

*Then  $f$  cannot be privately computed on  $G$  when the bridge player  $P_x$  holds  $x$ .*

**Proof.** Let  $G_y$  and  $G_z$  be the two blocks of  $G$ . The vectors of input bits for  $G_y$  without  $P_x$  and  $G_z$  without  $P_x$  are  $y$  and  $z$ , respectively. In all three cases,  $f(0, y, z)$  or  $f(1, y, z)$

depends on both  $y$  and  $z$ . Without loss of generality assume that  $f(0, y, z)$  depends on both  $y$  and  $z$ . Then there exist  $y'$  and  $z'$  such that  $f(0, y', z)$  depends on  $z$  and  $f(0, y, z')$  depends on  $y$ . Thus, there exist  $y''$  and  $z''$  such that

$$f(0, y', z'') = f(0, y'', z') \neq f(0, y', z').$$

Now consider any protocol for computing  $f$  on the given network. We fix some arbitrary content of  $P_x$ 's random tape and  $x = 0$ .

In the following,  $m_y^t$  denotes a message received by  $P_x$  from  $G_y$  in round  $t$ . Analogously,  $m_z^t$  denotes a message received by  $P_x$  from  $G_z$  in round  $t$ . We assume that in any round first  $P_x$  receives  $m_y^t$ , then  $P_x$  receives  $m_z^t$ , and finally  $P_x$  sends messages to  $G_y$  and  $G_z$ . (Formally, this means splitting up one round into three.) Then  $m_y^t$  does not depend on  $m_z^t$  and  $m_z^t$  does not depend on  $m_y^{t+1}$ . Let  $c_y^t = (m_y^1, \dots, m_y^t)$  and  $c_z^t = (m_z^1, \dots, m_z^t)$ . We call a certain  $c_y^t$  *undecided* if  $P_x$  can observe  $c_y^t$  both on input  $y'$  and  $y''$  for  $G_y$ . Otherwise, we call  $c_y^t$  *decided*. For  $c_z^t$ , the terms decided and undecided are analogously defined. The intuition behind these terms is as follows: If  $c_y^t$  is decided, then  $P_x$  has learned that either  $y'$  or  $y''$  is not  $G_y$ 's input. On the other hand, if  $c_y^t$  is undecided, then we can change  $G_y$ 's input from  $y'$  to  $y''$  or vice versa and modify its random bits such that  $P_x$  does not perceive any differences. Clearly,  $c_y^0$  and  $c_z^0$  are undecided.

We start our protocol on  $y', z'$  as input for  $G_y, G_z$ . Now we prove two things: First, if both  $c_y^t$  and  $c_z^t$  are undecided for all  $t$ , then we can fool the protocol such that it computes a wrong function value. Second, if eventually  $c_y^t$  or  $c_z^t$  is decided, then the protocol is not private with respect to  $P_x$ .

Assume that  $c_y^t$  and  $c_z^t$  are undecided for all  $t$ . Our protocol eventually outputs  $f(0, y', z')$  and  $c_y^t$  and  $c_z^t$  are still undecided. Then we can replace  $y'$  by  $y''$  and adjust  $G_y$ 's random bits such that  $P_x$  does not notice a difference. Thus, our protocol has computed  $f(0, y', z') \neq f(0, y'', z')$ , but  $f(0, y'', z')$  would have been the right value.

So consider the first  $t$  on which  $c_y^t$  or  $c_z^t$  is decided. Due to symmetry, we restrict ourselves to considering the first case.  $P_x$  has learned that  $y''$  is no longer possible as input for  $G_y$ . Since  $P_x$  receives  $m_z^t$  after  $m_y^t$ , the current  $c_z^{t-1}$  is still undecided. Thus, we can replace  $z'$  with  $z''$ . When the protocol terminates,  $P_x$  knows the function value  $f(0, y', z'') = f(0, y'', z')$ . In addition, he knows that  $(y'', z')$  has not been the input. Thus, the protocol is not private.  $\square$

**Lemma 5.3.** *Let  $f$  be a non-degenerate  $n$ -ary Boolean function that is dominated by  $x$  with  $\text{bal}_f(x) = 1$ . Then  $f$  cannot be privately computed on any non-2-connected network.*

**Proof.** Due to Lemma 5.2, the only possibility for computing  $f$  is a network from  $B_{n,1}$  with bridge node  $P_x$ . Let  $P_y$  be the other player of the block of size 2. Without loss of generality we assume that for  $x = 0$ ,  $f(x, y, z) = g(z)$ , and for  $x = 1$ ,  $f(x, y, z)$  is either  $y$  or  $\bar{y}$ . We assume that  $f(1, y, z) = y$ .

We show how to compute the conjunction of two variables (namely  $x$  and  $y$ ) privately. Consider  $f$  on some input  $z_0$  with  $g(z_0) = 0$ . Then  $f(x, y, z_0) = x \wedge y$ . If  $P_x$  and  $P_y$  could compute  $f$  privately, then a single player would be able to simulate the behaviour of the large block on input  $z_0$  and  $P_x$  on  $x$  while another player would be able to simulate

$P_y$  on  $y$ . This would yield a protocol for privately computing  $x \wedge y$ , which is impossible for two players [24].  $\square$

**Theorem 5.4.** *Let  $f$  be a non-degenerate  $n$ -ary Boolean function,  $n \geq 3$ , and let  $G$  be a connected network of  $n$  nodes. Then  $f$  cannot be privately computed on  $G$ , if one of the following conditions holds:*

1.  $G \in B_{n,k}$ , but there is no variable  $x$  that dominates  $f$  with  $\text{bal}_f(x) = k$ .
2.  $G$  consists of more than two blocks.
3.  $f$  is  $x$ -dominated with  $\text{bal}_f(x) = 1$ .

**Proof.** Items 1 and 3 follow immediately from Lemmas 5.2 and 5.3, respectively.

Now assume that  $G$  consists of more than two blocks. There are two possibilities: either all blocks share one bridge node or we have at least two bridge nodes. In both cases our aim is to apply Lemma 5.2. This is not directly possible since Lemma 5.2 only speaks about networks with two blocks. Note however that if one cannot privately compute a function on a given network  $H$ , then one cannot privately compute it on any subnetwork of  $H$ . Hence, if we cannot privately compute a function on a network  $H$  with two blocks, we cannot privately compute it on any network that is obtained by splitting up each of the two blocks into several new blocks.

First, we treat the case that there is only one bridge node  $P_x$  holding variable  $x$ . Since  $f$  is non-degenerate, for either  $x = 0$  or  $x = 1$  the function value depends on input bits of at least two blocks  $B_1$  and  $B_2$ . Let  $G'$  be the network with two blocks such that one block is  $B_1$  and the other block is the complete graph on the remaining nodes with the bridge node  $P_x$ . If  $f$  could be privately computed on  $G$ , then  $f$  could also be privately computed on  $G'$ , but this contradicts Lemma 5.2.

Second, assume that there are two bridge nodes. If  $f$  could be privately computed on  $G$ , then there must be two variables  $x$  and  $x'$  that dominate  $f$  due to Lemma 5.2. (Here, we again unite blocks to end up with two blocks as above.) This contradicts Lemma 5.1.  $\square$

Many well-known Boolean functions like and, or, majority, and parity are not dominated and thus cannot be privately computed on non-2-connected networks.

**Theorem 5.5.** *Let  $f$  be a non-degenerate  $n$ -ary Boolean function,  $n \geq 5$ , that is dominated by  $x$  with  $\text{bal}_f(x) = k > 1$ . Then  $f$  can be privately computed on  $B_{n,k}$ .*

**Proof.** The protocol works the same as the one presented in Section 1.2. Let  $f$  be of the form

$$f(x, y, z) = \begin{cases} g_0(y) & \text{if } x = 0 \\ g_1(z) & \text{if } x = 1 \end{cases} \quad \text{and}$$

for some  $g_0$  and  $g_1$ . Then  $f(x, y, z) = (\bar{x} \wedge g_0(y)) \vee (x \wedge g_1(z))$ . Assume that  $y$  contains  $k$  variables and  $z$  contains  $n - k - 1$  variables. Let the bridge player  $P_x$ , which is part of both components, hold  $x$ . We share the  $k$  variables of  $y$  among the  $k$  remaining nodes

of the first component and the  $n - k - 1$  variables of  $z$  among the remaining nodes of the second component. Then we privately compute  $(\bar{x} \wedge g_0(y))$  within the first block and  $(x \wedge g_1(z))$  within the second block. This can be done since both blocks consist of at least three nodes. (Every Boolean function can be privately computed on a complete network of at least three players [3] and henceforth on any 2-connected network with at least three players.) Finally,  $P_x$  knows the result.

It remains to prove the protocol is private. It is clearly private with respect to all players except for  $P_x$ , since no player needs to learn anything about  $(\bar{x} \wedge g_0(y))$  or  $(x \wedge g_1(z))$ . Let  $x = 0$  ( $x = 1$  follows analogously due to symmetry). Then  $(x \wedge g_1(z)) = 0$  and thus  $P_x$  does not learn anything about  $z$ . Furthermore,  $P_x$  only learns  $(\bar{x} \wedge g_0(y)) = g_0(y)$  about  $y$ , which is just  $f(x, y, z)$ .  $\square$

Note that when the conditions of Theorem 5.5 are not fulfilled, we can always apply Theorem 5.4. Furthermore, there is no function on three or four variables that can be privately computed on a non-2-connected network: either the function is not dominated or the balance is one.

## 6. Conclusions and Open Problems

We have investigated the relation between the connectivity of networks and the possibility of computing functions by private protocols on these networks. Special emphasis has been put on the amount of randomness needed.

We have presented a general simulation technique that allows us to transfer every oblivious private protocol on an arbitrary network into an oblivious private protocol on a given  $k$ -connected network of the same size, where  $k \geq 2$ . The new protocol needs  $(1 - k/(n - 1)) \cdot \min\{L, (k - 2)/(k - 1) \cdot (n^2 - n) + L/(k - 1)\}$  additional random bits, where  $L$  is the total number of bits sent in the original protocol. A future goal is either to reduce the number of additional random bits further or to prove general lower bounds.

The parity function can be computed on a cycle using only one random bit and only one message per link. (Strictly speaking, an additional message per link is necessary to broadcast the result in the end. However, we do not need to use any random bits to encode this broadcast, hence we can assume that  $n$  bits are sent altogether.) Thus,  $1 + n - kn/(n - 1) \leq n - k + 1$  random bits are sufficient to compute the parity function on an arbitrary  $k$ -connected graph by a private protocol using our simulation. We have strengthened this bound by showing that on every  $k$ -connected graph, parity can be computed by an oblivious private protocol using at most  $\lceil (n - 2)/(k - 1) \rceil - 1$  random bits. Furthermore, there exist  $k$ -connected networks for which this bound is tight.

While every Boolean function can be computed on a 2-connected network by a private protocol, this is no longer true for 1-connected networks. Starting from this observation, we have completely characterised the functions that can be computed by a private protocol on non-2-connected networks.

Our simulation results focus on the extra amount of randomness needed. It would also be interesting to bound the number of rounds of the simulation in terms of the number of rounds of the original protocol and, say, the diameter of the new network.

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