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On non-rank facets of the stable set polytope of claw-free graphs and circulant graphs

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Abstract. We deal with non-rank facets of the stable set polytope of claw-free graphs. We extend results of Giles and Trotter [7] by (i) showing that for any nonnegative integer a there exists a circulant graph whose stable set polytope has a facet-inducing inequality with $(a,a+1)$-valued coefficients (rank facets have only coefficients 0, 1), and (ii) providing new facets of the stable set polytope with up to five different non-zero coefficients for claw-free graphs. We prove that coefficients have to be consecutive in *any* facet with *exactly* two different non-zero coefficients (assuming they are relatively prime). Last but not least, we present a complete description of the stable set polytope for graphs with stability number 2, already observed by Cook [3] and Shepherd [18].

Key words: Stable sets, Claw-free graphs, Quasi-line graphs, Circulant graphs, Clique family inequalities

1 Introduction

Claw-free graphs are a superclass of line graphs and many crucial properties of the matching problem extend to the stable set problem in claw-free graphs. Nevertheless, while there exists a polynomially bounded algorithm for finding a maximum weighted stable set in a claw-free graph [10,11], the problem of characterizing the stable set polytope (ssp) of a claw-free graph G, denoted by $STAB(G)$, is a long-standing open problem (see [7,12]).

In fact, the nice polyhedral properties of the matching polytope do *not* extend to the ssp of a claw-free graph. While a minimal system of inequalities defining the matching polytope contains only *rank* inequalities, Giles and Trotter [7] in 1980 proved that a minimal defining system for $STAB(G)$, G claw-free, must contain:

(i) inequalities with $\{a, a+1\}$ coefficients, for any integer $a \geq 0$;
(ii) inequalities with three (consecutive) different non-zero coefficients.

This result is quite negative in a sense, and, by now, we are still far from solving the problem of providing a "... decent linear description of $STAB(G)$" [8] for claw-free graphs.

A possible line of attack on the problem is that of dealing with classes of claw-free graphs where the ssp has an easier structure: possibly, this is the case with *quasi-line* graphs. A conjecture on a linear description of the ssp of a quasi-line graph, called *Ben Rebea's Conjecture*, has been introduced in [12]. Ben Rebea's Conjecture claims that all the non-trivial facets of the ssp of a quasi-line graph belong to one class of inequalities, called the *clique family inequalities*. Clique family inequalities have (at most) two non-zero different coefficients, which have to be consecutive (if they are relatively prime): in fact, they contain the inequalities (i) above; while the inequalities (ii) do not arise for quasi-line graphs.

Proving Ben Rebea's Conjecture seems to be hard. In fact, quasi-line graphs are not *rank-perfect* (i.e. the ssp of quasi-line graphs can have non-rank facets). To this date no linear description of the ssp is known for graphs which are not rank-perfect (an exception is given by near-bipartite graphs [17], but their non-rank facets are trivial liftings of rank facets). Roughly speaking, we do not know much about non-rank facets, while, for instance, a complete characterization of the rank facets of the ssp of claw-free graphs has been given by Galluccio and Sassano [6].

It is therefore reasonable to check the conjecture on some classes of quasi-line graphs. This has already been done in [12] for the classes of quasi-line graphs for which a linear description of the ssp is known: line graphs [5] and circulants with clique number equal to three [4]. Not surprisingly, both classes are rank-perfect. What about the other circulants?

In this paper we deal with non-rank facets of the ssp of claw-free graphs. First we consider the class of circulants and show that the statement (i) above holds even if we restrict to their ssp, so circulants are far from being rank-perfect. We then move to general claw-free graphs. We prove that indeed coefficients have to be consecutive in *any* facet with *exactly* two different non-zero coefficients (assuming they are relatively prime). We provide new facets with more than two (up to five!) different non-zero coefficients, none of which appears in quasi-line graphs. Interestingly, all these facets always have consecutive coefficients. This suggests a simple research question: When the non-zero coefficients of a facet of $STAB(G)$ are relatively prime, must they be consecutive? Finally, we present a complete description of the ssp for graphs with stability number 2. This description was first observed by Cook [3] and Shepherd [18].

1.1 Notations and definitions

Our graphs will be always finite, undirected and without loops. We denote by $N(v)$ the *neighbourhood* of a vertex $v \in V$, i.e., $N(v)$ is the set of vertices that are adjacent to v. A *claw* is a graph with vertex set $\{u, v, w, z\}$ and edge set $\{uv, uw, uz\}$. A graph is said to be *claw-free* if it does not contain a claw as an induced subgraph. A graph is *quasi-line* if the neighbourhood of any vertex can be partitioned into two cliques.

Definition 1. *An* (n,p)-circulant, $p \leq \lfloor \frac{n}{2} \rfloor$, *is the graph with vertex set* $\{v_1, v_2, \ldots, v_n\}$ *and edge set* $\{v_i v_{i+j} : 1 \leq i \leq n, 1 \leq j \leq p-1\}$ (*sums are taken modulo* n). *The clique number of an* (n,p)*-circulant is* p.

The *complete join* of two graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ is the graph $G = (V, E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{u_1 u_2 : u_1 \in V_1, u_2 \in V_2\}$, we write $G = G_1 \times G_2$.

Let $STAB(G)$ be the convex hull of the incidence vectors of all stable sets of a graph $G = (V, E)$. Among all valid inequalities for $STAB(G)$, special attention is given to the class of *rank* inequalities. The *rank inequality* associated with a subset T of V is an inequality of the form $\sum_{j \in T} x_j \leq \alpha(T)$, where we denote by $\alpha(T)$ the cardinality of a maximum stable set in T. A graph is *rank-perfect* [20] if its ssp has only rank facets.

Among valid non-rank inequalities for $STAB(G)$, we give special attention to the class of *clique family inequalities.*

Definition 2. *Let* $\mathcal{F}$ *be a family of* n (*inclusion-wise maximal*) *cliques of* G, $n \geq 3$, *and let* $p \leq n$ *be an integer. Define*

$$I(\mathcal{F}, p) := \{v \in V : |\{F \in \mathcal{F} : v \in F\}| \geq p\},$$
$$O(\mathcal{F}, p) := \{v \in V : |\{F \in \mathcal{F} : v \in F\}| = p-1\},$$
$$r := n - p\left\lfloor \frac{n}{p} \right\rfloor = n \bmod p.$$

The clique family inequality *associated with* $\mathcal{F}$, n *and* p *is defined as*

$$(p-r) \cdot \sum_{v \in I(\mathcal{F},p)} x_v + (p-r-1) \cdot \sum_{v \in O(\mathcal{F},p)} x_v \leq (p-r) \cdot \left\lfloor \frac{n}{p} \right\rfloor.$$

Clique family inequalities are valid for $STAB(G)$ [12].

2 Circulants

Circulants are a subclass of quasi-line – and therefore claw-free – graphs. In 1999 Dahl [4] proved that the circulants with clique number three are rank-perfect. We show in the following that, in general, circulants are far from being rank-perfect, since a description of their ssp must include facets with coefficients a and $a+1$, for any integer $a \geq 0$. A similar result was proved for the larger class of quasi-line graphs in [7].

Let a be a positive integer and define $p = 2(a+2)$ and $n = (p-1)^2$. Let $C^a = (V^a, E^a)$ be an (n,p)-circulant where $V^a = \{0, \ldots, n-1\}$. We denote $R^a = \{i \in V^a : k(p-1) \leq i \leq k(p-1) + a$ for some $k \in \{0, \ldots, p-2\}\}$.

The graph C^1 is shown in Fig. 1. The vertices of R^1 are colored in black. For readability, only the connections from a vertex to its furthest neighbours are drawn.

Theorem 3. *For each* $a \geq 1$, *the inequality*

$$a \sum_{j \notin R^a} x_j + (a+1) \sum_{j \in R^a} x_j \leq (a+1)(2a+1) \tag{1}$$

is a facet of $STAB(C^a)$.

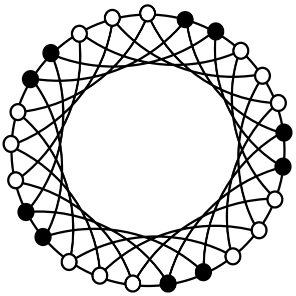

Fig. 1. Facet of C^1: $\sum_{v\in\circ} x_v + 2\sum_{v\in\bullet} x_v \le 6$

Proof. Let a be a positive integer. We first show that (1) is a valid inequality for $STAB(C^a)$.

Consider the family $\mathcal{F} := \{K_j : j \in R^a\}$ of maximal cliques of C^a starting in vertices from R^a, i.e., K_j is the graph induced by the vertices $\{j, \ldots, j+p-1\}$ (indices are taken modulo n). Then $|\mathcal{F}| = |R^a| = (a+1)(p-1) = (a+1)(2a+3)$. Using the notation introduced in Subsection 1, we have:

$$I(\mathcal{F}, a+2) = \{v \in V^a : |\{F \in \mathcal{F} : v \in F\}| \ge a+2\} = R^a,$$
$$O(\mathcal{F}, a+2) = \{v \in V^a : |\{F \in \mathcal{F} : v \in F\}| = a+1\} = V^a \setminus R^a.$$

Let r be the remainder of $|\mathcal{F}|$ divided by $a+2$. Then the following inequality is valid for $STAB(C^a)$ [12]:

$$(a+2-r)\cdot \sum_{v\in I(\mathcal{F},a+2)} x_v + (a+1-r)\cdot \sum_{v\in O(\mathcal{F},a+2)} x_v \le (a+2-r)\cdot \left\lfloor \frac{|\mathcal{F}|}{a+2} \right\rfloor$$

We easily derive (1) from this inequality by observing that

$$|\mathcal{F}| = (2a+1)(a+2)+1 \tag{2}$$

and thus, $r = 1$ and $\left\lfloor \frac{|\mathcal{F}|}{a+2} \right\rfloor = 2a+1$.

It remains to find n affinely independent stable sets that are tight for (1). We will use the same technique as [7]. We define the sets of vertices

$$S_{ij} = \{i(p-1)+kp : -j \le k \le -j+2a+1, k \in \mathbb{Z}\}$$

for each $i = 0, \ldots, p-2$ and $j = a+1, \ldots, 0$.

The following table illustrates the S_{ij} for the graph C^1.

Table 1. Example for the graph C^1

Vertex	13	19	0	6	12	18	24	5	11	17	23	...
Type of vertex	○	○	●	●	○	○	○	●	●	○	○	...
S_{02}	+	+	+	+								
S_{01}		+	+	+	+							
S_{00}			+	+	+	+						
S_{12}						+	+	+	+			
S_{11}							+	+	+	+		
S_{10}								+	+	+	+	

One can check that each S_{ij} is a stable set of C^a and it contains $a+1$ vertices of R^a and $a+1$ vertices of $V^a \setminus R^a$. Thus it follows that S_{ij} is tight for (1). Furthermore, the incidence matrix of the sets S_{ij} in $V^a \setminus R^a$ has a circulant structure, i.e., the sets S_{ij} in $V^a \setminus R^a$ correspond to the maximal cliques of an (n', p')-circulant with $n' = |V_a \setminus R_a| = (p-1)(a+2) = (p+1)(a+1)+1$ and $p' = a+1$, i.e., n' and p' are relatively prime. Hence, we have $(p-1)(a+2)$ linearly independent stable sets on $V^a \setminus R^a$.

Moreover, the subgraph induced by the vertices of R^a is a $\{|R^a|, a+2\}$-circulant. Since $|R^a| = |\mathcal{F}|$ and $a+2$ are relatively prime (2), the inequality:

$$\sum_{j \in R^a} x_j \leq \left\lfloor \frac{|R^a|}{a+2} \right\rfloor = (2a+1)$$

is a facet of STAB(R^a) [19]. The $|R^a| = (a+1)(p-1)$ corresponding stable sets satisfy (1) at equality and they are linearly independent on R^a. We denote by $\mathcal{S}'$ this collection of stable sets.

As R^a and $V^a \setminus R^a$ are orthogonal subspaces and the elements of $\mathcal{S}'$ have zero coefficients in $V^a \setminus R^a$, we have $(a+1)(p-1)+(p-1)(a+2) = n$ linearly independent stable sets that are tight for (1). $\square$

The class of facets we provide in this paper is only part of all non-rank facets for circulant graphs. Indeed, we are still far from being able to characterize the stable set polytope for circulants. Pulleyblank and Shepherd [15] gave a compact projective formulation for the polytope of distance claw-free graphs (which include circulants). Nevertheless, it is still unclear how to derive from this formulation a description (in the original space) of the ssp of circulants.

The example depicted in Fig. 2 illustrates a facet which does not belong to the class of inequalities (1). However, this facet belongs to the class of clique family inequalities.

3 Non-rank facets in claw-free graphs

We believe that Theorem 3 prompts the important question of providing a linear description of the ssp of circulants. Since circulants are quasi-line graphs, a possible answer is given by Ben Rebea's Conjecture.

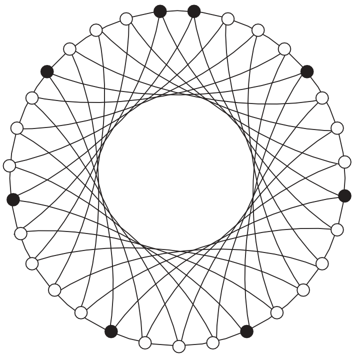

Fig. 2. Facet of $\{31, 10\}$-circulant: $\sum_{v \in \circ} x_v + 2 \sum_{v \in \bullet} x_v \leq 4$

Conjecture 4 (*Ben Rebea's Conjecture* [12]) *Let $G = (V, E)$ be a quasi-line graph. Then all facets of $STAB(G)$ are induced by clique family inequalities.*

Observe that in any clique family inequality there are at most two non-zero coefficients, which must be consecutive if they are relatively prime. Hence, possible counterexamples to the conjecture would be given by facets of the ssp of a quasi-line graph with:

(i) either three or more different non-zero coefficients;
(ii) two different non-zero coefficients which are relatively prime but non-consecutive.

But actually, as we show in the following, the latter configuration is forbidden even in the larger class of claw-free graphs.

3.1 Facets with two non-zero coefficients

First, we note the following facts, whose simple proofs are omitted. Let the *support* $I(a)$ of an inequality $\sum_{i\in V} a_i x_i \le b$ be the set $\{v \in V : a_v > 0\}$.

Fact 5 *Let $\sum_{i\in V} a_i x_i \le b$ be a facet of $STAB(G)$. Then the inequality $\sum_{i\in I(a)} a_i x_i \le b$ is a facet of $STAB(G[I(a)])$.*

Fact 6 *Let $\sum_{i\in V} a_i x_i \le b$ be a facet of $STAB(G)$ with $a_i, b \in \mathbb{Z}$. Then*

$$\gcd(\{a_i : i \in I(a)\}, b) = \gcd(\{a_i : i \in I(a)\}).$$

Theorem 7. *Let $G = (V, E)$ be a claw-free graph and*

$$p \sum_{i\in P} x_i + q \sum_{i\in Q} x_i \le \gamma$$

a facet-inducing inequality of $STAB(G)$ with $p, q \in \mathbb{Z}$, $\gcd(p, q, \gamma) = 1$, $p < q$, and $P \cap Q = \phi$. Then $p + 1 = q$, i.e., *p and q are consecutive.*

Proof. Without loss of generality we may assume that $V = P \cup Q$ (Fact 5) and that $\gcd(p, q) = 1$ (Fact 6). Since $F := \{x \in STAB(G) \mid p\sum_{i\in P} x_i + q\sum_{i\in Q} x_i = \gamma\}$ is a non-rank facet, there exist tight stable sets of different cardinality, i.e., there are tight stable sets S and T that are adjacent on F such that $|S| < |T|$. Then the symmetric difference $S \triangle T$ is connected, see [2]. Since G is claw-free, $S \triangle T$ is the node-disjoint union of alternating cycles and paths [1]. Hence, $S \triangle T$ is an alternating path since $|S| \ne |T|$. Consequently, $|T| - |S| = 1$. Define $s := |S \cap P|$, $s' := |S \cap Q|$ and $t := |T \cap P|$, $t' := |T \cap Q|$. Then $t = s + k + 1$ and $t' = s' - k$ for some $k \in \mathbb{Z}$. It follows

$$\begin{aligned} \gamma &= p \cdot t + q \cdot t' = p \cdot (s + k + 1) + q \cdot (s' - k) \\ &= p \cdot s + q \cdot s' + k(p - q) + p \\ &= \gamma + k(p - q) + p \end{aligned}$$

and thus, $k(q - p) = p$. Since $\gcd(q - p, p) = 1$ we obtain $q - p = 1$ and $k = p$, i.e., $p + 1 = q$. □

The previous theorem advises that, in order to disprove Ben Rebea's Conjecture, a reasonable strategy would be to look for facets of the ssp of a quasi-line graph with more than two different non-zero coefficients. Actually, we point out that all the facets that are known for the ssp of a quasi-line graph have at most two coefficients, *including zero coefficients*!

3.2 New non-rank facets for claw-free graphs

If we move from quasi-line to the larger class of claw-free graphs, facets with more than two different non-zero coefficients are known. Giles and Trotter gave an example with three different coefficients in [7]. The complement of this graph G is shown in Fig. 3. They proved that the inequality

$$\sum_{v\in\square} x_v + 2\sum_{v\in\circ} x_v + 3\sum_{v\in\bullet} x_v \leq 4$$

is a facet of $STAB(G)$.

Further examples are:

- Let G' denote the complement of the graph in Fig. 4. G' is claw-free and it is straightforward to check that the inequality

$$\sum_{v\in\square} x_v + 2\sum_{v\in\diamond} x_v + 3\sum_{v\in\circ} x_v + 4\sum_{v\in\bullet} x_v \leq 5$$

  is a facet of $STAB(G')$.

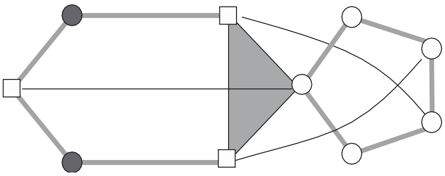

Fig. 3. Example of Giles and Trotter

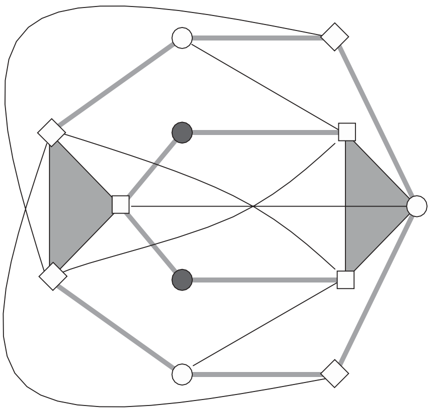

Fig. 4. "Fish in a net"

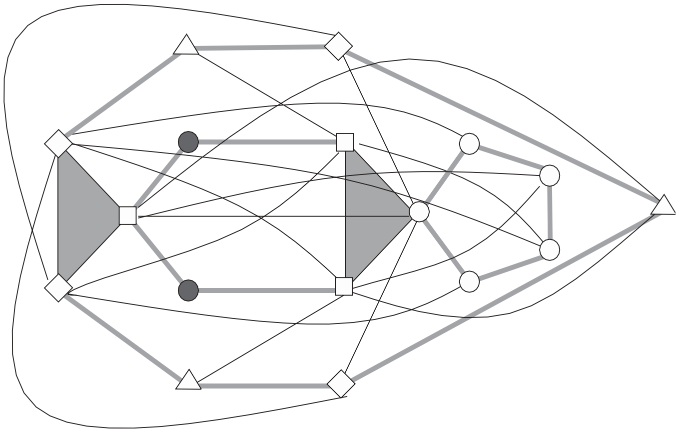

Fig. 5. "Fish in a net with bubble"

- Let G'' denote the complement of the graph in Fig. 5. Using the same reasoning, one can check that the inequality

$$2\sum_{v\in\square} x_v + 3\sum_{v\in\diamond} x_v + 4\sum_{v\in\circ} x_v + 5\sum_{v\in\triangle} x_v + 6\sum_{v\in\bullet} x_v \leq 8$$

  is a facet of $STAB(G'')$.

Remark 8. *In the three different figures, the tight stable sets are represented by dark cliques: dark edges for stable sets of size* 2 *and dark triangles for stable sets of size* 3.

We believe that the previous examples show how difficult the problem of providing a linear description of the ssp of a claw-free graph is. Nevertheless, a seemingly simple research question arises from the observation of all the known facets of this polytope.

Question 9. *Let $G = (V, E)$ be a claw-free graph. When the non-zero coefficients of a facet of $STAB(G)$ are relatively prime, must they be consecutive ?*

As for quasi-line graphs, the question can be even refined to include zero-coefficients. In fact, we do not know any facet for the ssp of a claw-free graph with non-consecutive coefficients, including zero coefficients!

Question 10. *Let $G = (V, E)$ be a claw-free graph. When the coefficients of a facet of $STAB(G)$ are relatively prime, must they be consecutive ?*

3.3 The case with stability number two

As we show in the following, if $\alpha(G) = 2$, i.e., when the maximum cardinality of a stable set in G is 2, then the answers to Questions 9 and 10 are in the affirmative. First, we need some definitions.

Definition 11. *A graph G is a* sequential lifting *of a graph H if there exists a sequence of vertices $\{v_1, \dots, v_p\}$ and subgraphs $H = G_0 \subset G_1 \subset \dots G_p = G$ such that, for each $i = 1, \dots, p$:*

- $G_{i+1} = G_i \cup \{v_{i+1}\}$;
- $\alpha(G_i) - \alpha(G_i \setminus N(v_{i+1})) = 1$.

It follows from [2] and [13] that, if $x(H) \leq \alpha(H)$ is a facet of $STAB(H)$ and G is a sequential lifting of H then $x(G) \leq \alpha(H)$ is a facet of $STAB(G)$. Moreover, rank facets of $STAB(H_1)$ and $STAB(H_2)$ lead to facets of the stable set polytope of the complete join $H_1 \times H_2$: If $x(H_i) \leq \alpha(H_i)$ is a facet of $STAB(H_i)$, $i = 1, 2$, then $\alpha(H_2)x(H_1) + \alpha(H_1)x(H_2) \leq \alpha(H_1)\alpha(H_2)$ is a facet of $STAB(H_1 \times H_2)$.

Definition 12. *A graph G is a* combination of odd antiholes *if G can be built up from odd antiholes by iterating sequential lifting and complete join operations, i.e.,*

- G is a lifting of $H_1 \times H_2 \times \dots \times H_q$;
- for each $j = 1, \dots, q$, H_j is either an odd antihole or a combination of odd antiholes.

Examples of combinations of odd antiholes are

- $G_{10} := C_5 \times C_5$;
- $G_{11} := G_{10}+$ one vertex adjacent to all vertices of G_{10} but one
- $G_{16} := G_{11} \times C_5$
- $G_{17} := G_{16}+$ one vertex adjacent to all vertices of G_{16} but one

Hence, if $G = (V, E)$ is a combination of odd antiholes, then $x(V) \leq 2$ is a facet of $STAB(G)$.

On the other hand, Galluccio and Sassano [6] showed that, if $x(V) \leq 2$ is a facet of $STAB(G)$, then G is a combination of odd antiholes. Actually, it is quite easy to show that, if $x(Q) \leq 2$ is a facet of $STAB(G)$ and $\alpha(G) = 2$, then $Q = V(G)$. Hence, we can state the following corollary of their result.

Corollary 13. *Let $G = (V, E)$ be a graph wit*h $\alpha(G) = 2$. *If $x(Q) \leq 2$ is a facet of $STAB(G)$, then $Q = V$ and G is a combination of odd antiholes.*

It follows that all the *rank* facets of $STAB(G)$ are either of the form $x(P) \leq 1$ for some maximal clique P or of the form $x(V) \leq 2$ and G is a combination of odd antiholes.

The following lemma gives a characterization for the *non-rank* facets.

Lemma 14. *Let $G = (V, E)$ be a graph with $\alpha(G) = 2$. Then all the non-rank facets of $STAB(G)$ are of the form*

$$2\sum_{i \in P} x_i + \sum_{i \in Q} x_i \leq 2$$

where P and Q are non-empty disjoint subsets of V such that: P is a clique; Q is a combination of odd antiholes; P is totally joined to Q.

Proof. Let $\sum_{i \in V} a_i x_i \leq b$ be a non-rank facet of $STAB(G)$. Hence, there are stable sets of different cardinality which are tight for this inequality. In particular, due to $\alpha(G) = 2$, there will be a tight stable set (of size one) $\{w\}$ for some $w \in V$. Then w is totally joined to $I(a) \setminus \{w\}$ and $a_w = b$. Define $Q := \{i \in I(a) : a_i < b\} \neq \emptyset$ and $P := I(a) \setminus Q$; it follows that P is a clique totally joined to Q. Then $\sum_{i \in Q} a_i x_i \leq b$ is a facet of the ssp of $G' := G[Q]$ and all its tight stable sets have size 2, i.e., $\sum_{i \in Q} a_i x_i \leq b$ is a nonnegative multiple of $\sum_{i \in Q} x_i \leq 2$. Hence, from Corollary 13, Q is a combination of odd antiholes. Finally, $\sum_{i \in V} a_i x_i \leq b$ is a nonnegative multiple of $2\sum_{i \in P} x_i + \sum_{i \in Q} x_i \leq 2$. □

Thus we have the following theorem, which was first observed by Cook [3] and Shepherd [18].

Theorem 15. *The facets of the stable set polytope of a graph* $G = (V, E)$ with $\alpha(G) = 2$ *are given by the following inequalities*:

(*i*) $x_v \geq 0$ *for each* $v \in V$,
(*ii*) $2 \sum_{v \in P} x_v + \sum_{v \in Q} x_v \leq 2$

for each pair (P, Q) *such that* P *is a clique*, Q *is a combination of odd antiholes*, P *is totally joined to* Q, *and* P *and* Q *are maximal with respect to these properties*.

Trivially, if $\alpha(G) = 2$, then the stable sets of G are $O(|V(G)|^2)$. Hence, a maximum stable set can be found in polynomial time by enumeration.

Analogously, the separation problem can be solved via the general approach of antiblocking polyhedra (see for instance [16]) by listing all stable sets. Nevertheless, separating by *facets* seems to be hard. For instance, finding a clique inequality which is violated by a point x, asks for the solution of a maximum clique problem on $G[I(x)]$, and this problem is known to be hard [14].

Corollary 16. Let G be a quasi-line graph with $\alpha(G) = 2$. Then G is rank-perfect.

Proof. Suppose G is not rank-perfect. It follows from Lemma 14 that there exist non-empty disjoint subsets P and Q of V such that: P is a clique; Q is a combination of odd antiholes; P is totally joined to Q. Consequently, P has to be totally joined to an odd antihole, in contradiction to G being quasi-line. □

Observe that the previous result does not extend to quasi-line graphs with stability number equal to three, as is shown by the graph in Fig. 2.

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