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SINGULAR PERTURBATIONS OF SOME NONLINEAR PROBLEMS

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Dedicated to Professor V. V. Zhikov on the occasion of his 70th birthday

We deal with singular perturbations of nonlinear problems depending on a small parameter $\varepsilon > 0$. First we consider the abstract theory of singular perturbations of variational inequalities involving some nonlinear operators, defined in Banach spaces, and describe the asymptotic behavior of these solutions as $\varepsilon \to 0$. Then these abstract results are applied to some boundary value problems. Bibliography: 15 titles.

1 Introduction

The goal of this paper is to study the asymptotic behavior of singular perturbations problems as a parameter ε goes towards 0. Our results are very general, but we have more particularly in mind anisotropic cases where ε only acts on some variables of a domain $\Omega \subset \mathbb{R}^n$ (n is an integer) where we consider the partial differential equations. To be more precise we can take, as a model, the diffusion problem defined in the unit square $\Omega = (0,1) \times (0,1)$

$$\begin{cases} -\varepsilon^2 \partial^2_{x_1} u_\varepsilon - \partial^2_{x_2} u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\varepsilon > 0$ and f represents the source term. We assume that the diffusion in the x_1-direction is negligible with respect to the other direction as $\varepsilon \to 0$. Formally, the natural limit of u_ε is a function u_0 defined on the sections $\{x_1\} \times (0,1)$ for a.e. $x_1 \in (0,1)$ as a solution to the problem

$$\begin{cases} -\partial_{x_2}^2 u_0(x_1,\cdot) = f(x_1,\cdot) \quad \text{in} \quad (0,1), \\ u_0(x_1,\cdot) = 0 \quad \text{on} \quad \{0,1\}. \end{cases} \tag{1.2}$$

Note that the variable x_1 plays a role of a parameter. It is clear that if f (not identically equal to 0) is independent of x_1, i.e., $f = f(x_2)$, then $u_0 \notin H_0^1(\Omega)$. This prevents the convergence $u_\varepsilon \to u_0$ to occur in $H^1(\Omega)$. From this remark we may discuss many issues concerning this convergence.

In this paper, we begin by dealing with abstract singular perturbations problems of variational inequalities. Our approach has the advantage to include in a short theory a wide class of problems spread in the literature. We give then some applications of it.

In the literature, linear elliptic, parabolic, and hyperbolic problems defined on arbitrary domains are analyzed in different contexts and the convergence $u_\varepsilon \to u_0$ is obtained in different norms. A boundary layer may occur at the lateral boundary of cylindrical domains ($\{0,1\} \times (0,1)$ for the above example). The convergence in Sobolev spaces may be shown in regions far from this lateral boundary. We may see this clearly when our perturbed problem satisfies some cylindrical symmetries. This means that $f = f(x_2)$ in the above example. In this case, u_ε converges towards u_0 at an exponential rate. For more details we refer the reader to [1]–[10].

An abstract approach to this theory was also given in [11, 12] where the following operator equation is considered:

$$\varepsilon A u_\varepsilon + B u_\varepsilon = f \tag{1.3}$$

with A and B linear operators defined on Hilbert spaces. This approach covers diagonal structure problems as the problem (1.1). The authors also showed, as in the case of partial differential equations, that u_ε converges towards u_0 solution to the equation

$$B u_0 = f$$

as $\varepsilon \to 0$. There are also some previous works on singular perturbations of variational inequalities, i.e., when (1.3) is replaced by

$$(\varepsilon A u_\varepsilon, v - u_\varepsilon) + (B u_\varepsilon, v - u_\varepsilon) \geqslant (f, v - u_\varepsilon) \quad \forall\, v \in K \tag{1.4}$$

where K is some nonempty closed convex set (cf. [13]–[15]). In [15], this abstract approach is established to investigate the isotropic singular perturbations problems.

In order to cover a larger class of problems by an abstract theory, we deal with the variational inequality (1.4) when A and B are nonlinear operators defined on different Banach spaces V and W respectively, which, in particular, applies to the anisotropic singular perturbations problems. This is what we will see in the next section. In the last section, the first example is devoted to show that these results also cover the isotropic case. Then some examples of anisotropic singular perturbations problems are introduced in order to illustrate some points of the theory as, for instance, the lack of compactness.

2 Abstract Singular Perturbations Problems

Let V and W be two reflexive separable Banach spaces equipped with the norms $|\cdot|_V$ and $|\cdot|_W$ respectively. We suppose that the space $V \cap W$ is dense in V and W, and is equipped with the norm

$$|\cdot|_{V\cap W} = |\cdot|_V + |\cdot|_W .$$

Of course, $V \cap W$ is a Banach space equipped with the previous norm. For any space X we denote by $\langle\cdot,\cdot\rangle_X$ the duality pairing between X' and X, where X' is the dual of X. It is clear that

$$V\cap W \subset V, W \quad \text{and} \quad V', W' \subset (V\cap W)' .$$

Moreover, one can check that $(V\cap W)' = V' + W'$. We consider two nonlinear operators A and B such that

$$A : V \to V', \quad B : W \to W'.$$

We suppose that A, B are monotone, that is to say that

$$\langle Au - Av, u - v\rangle_V \geqslant 0 \quad \forall\, u, v \in V, \tag{2.1}$$

$$\langle Bu - Bv, u - v\rangle_W \geqslant 0 \quad \forall\, u, v \in W. \tag{2.2}$$

We denote by $K \neq \varnothing$ a closed convex set of $V \cap W$ and for A, B we make the following coerciveness assumption. We suppose that for some $v_0 \in K$

$$\frac{\langle Au - Av_0, u - v_0\rangle_V}{|u - v_0|_V} \to +\infty \quad \text{as } |u - v_0|_V \to +\infty, \quad u \in K, \tag{2.3}$$

$$\frac{\langle Bu - Bv_0, u - v_0\rangle_W}{|u - v_0|_W} \to +\infty \quad \text{as } |u - v_0|_W \to +\infty, \quad u \in K. \tag{2.4}$$

Remark 2.1. If K is bounded in V (respectively, in W) we will not need the assumption (2.3) (respectively, (2.4)). Note also that for some $v_0 \in K$ they are equivalent with

$$\frac{\langle Au, u - v_0\rangle_V}{|u - v_0|_V} \to +\infty \quad \text{as } |u - v_0|_V \to +\infty, \quad u \in K, \tag{2.5}$$

$$\frac{\langle Bu, u - v_0\rangle_W}{|u - v_0|_W} \to +\infty \quad \text{as } |u - v_0|_W \to +\infty, \quad u \in K. \tag{2.6}$$

In addition, we assume that

$$A \text{ sends bounded sets of } V \text{ in bounded sets of } V', \tag{2.7}$$

$$B \text{ sends bounded sets of } W \text{ in bounded sets of } W', \tag{2.8}$$

$$A \text{ and } B \text{ are hemicontinuous on } V \text{ and } W \text{ respectively.} \tag{2.9}$$

This last assumption means that – for instance, for A –

$$t \mapsto \langle A(u + tv), w\rangle_V \quad \text{is continuous on } \mathbb{R} \text{ for all } u, v, w \in V.$$

Under the assumptions above, we have the following assertion.

Theorem 2.1. *For $f \in (V\cap W)'$ and $\varepsilon > 0$ there exists a solution u_ε to the problem*

$$\begin{cases} \varepsilon\,\langle Au_\varepsilon, v - u_\varepsilon\rangle_V + \langle Bu_\varepsilon, v - u_\varepsilon\rangle_W \geqslant \langle f, v - u_\varepsilon\rangle_{V\cap W} \quad \forall\, v \in K, \\ u_\varepsilon \in K. \end{cases} \tag{2.10}$$

Moreover, if A or B is strictly monotone (i.e., if one of the inequalities (2.1), (2.2) *is strict for $u \neq v$), the solution is unique.*

Proof. We consider the operator A_ε defined by

$$A_\varepsilon : V \cap W \to (V \cap W)' = V' + W', \qquad v \mapsto \varepsilon A v + B v.$$

This operator is monotone, hemicontinuous, and coercive on K. For this last point, by the coerciveness assumptions on A and B, for every $M > 0$ there exist $\delta_1(M)$, $\delta_2(M) \geqslant 1$ such that

$$|u - v_0|_V \geqslant \delta_1(M) \Rightarrow \frac{\langle \varepsilon A u, u - v_0 \rangle_V}{|u - v_0|_V} \geqslant M, \tag{2.11}$$

$$|u - v_0|_W \geqslant \delta_2(M) \Rightarrow \frac{\langle B u, u - v_0 \rangle_W}{|u - v_0|_W} \geqslant M. \tag{2.12}$$

Since A and B are bounded, there exist constants C_A and C_B such that

$$|u - v_0|_V \leqslant \delta_1(M) \Rightarrow |\langle \varepsilon A u, u - v_0 \rangle_V| \leqslant C_A(M),$$
$$|u - v_0|_W \leqslant \delta_2(M) \Rightarrow |\langle B u, u - v_0 \rangle_W| \leqslant C_B(M).$$

Choose

$$|u - v_0|_V + |u - v_0|_W \geqslant 2\delta_1(M) + 2\delta_2(M) + \delta_1(2M + 2C_B(M)) + \delta_2(2M + 2C_A(M)).$$

Of course, one has either $|u - v_0|_V \geqslant \delta_1(M)$ or $|u - v_0|_W \geqslant \delta_2(M)$. Suppose, for instance, that $|u - v_0|_V \geqslant \delta_1(M)$, the other case being the same. If, moreover, $|u - v_0|_W \geqslant \delta_2(M)$, from (2.11) and (2.12) one has

$$\begin{aligned} \frac{\langle \varepsilon A u, u - v_0 \rangle_V + \langle B u, u - v_0 \rangle_W}{|u - v_0|_V + |u - v_0|_W} &= \frac{|u - v_0|_V}{|u - v_0|_V + |u - v_0|_W} \cdot \frac{\langle \varepsilon A u, u - v_0 \rangle_V}{|u - v_0|_V} \\ &\quad + \frac{|u - v_0|_W}{|u - v_0|_V + |u - v_0|_W} \cdot \frac{\langle B u, u - v_0 \rangle_W}{|u - v_0|_W} \geqslant M. \end{aligned}$$

If $|u - v_0|_W \leqslant \delta_2(M)$, then $|u - v_0|_V \geqslant \delta_2(M)$, $\delta_1(2M + 2C_B(M))$, so that

$$\begin{aligned} \frac{\langle \varepsilon A u, u - v_0 \rangle_V + \langle B u, u - v_0 \rangle_W}{|u - v_0|_V + |u - v_0|_W} &\geqslant \frac{|u - v_0|_V}{|u - v_0|_V + |u - v_0|_W} \{2M + 2C_B(M)\} - C_B(M) \\ &\geqslant \frac{1}{2} \{2M + 2C_B(M)\} - C_B(M) \geqslant M. \end{aligned}$$

This shows the coerciveness of A_ε. The existence of u_ε follows from the classical theory of variational inequalities. □

Remark 2.2. Let $K = V \cap W$. Taking $v = u_\varepsilon \pm w, w \in K$, one sees that u_ε is a solution to the problem

$$\begin{cases} \varepsilon A u_\varepsilon + B u_\varepsilon = f, \\ u_\varepsilon \in V \cap W. \end{cases} \tag{2.13}$$

We are now interested in studying the behavior of u_ε as $\varepsilon \to 0$. Note that this is not possible in general. Indeed, taking, for instance, V a Hilbert space, $A =$ the identity, $B = 0, f \in V' = V$, we can see that the solution to (2.13) is given by $u_\varepsilon = f/\varepsilon$ and $(u_\varepsilon)_\varepsilon$ has no limit. In what follows, we will assume that

$$f \in W'. \tag{2.14}$$

The essential convergences are given as follows.

Theorem 2.2. *Suppose that $f \in W'$ and u_ε is a solution to* (2.10)*. Then, as $\varepsilon \to 0$,*

(i) *u_ε is bounded in W independently of ε,*

(ii) *$\varepsilon u_\varepsilon \to 0$ in V,*

(iii) *$\varepsilon A u_\varepsilon \to 0$ in V',*

(iv) $\langle \varepsilon A u_\varepsilon, u_\varepsilon \rangle_V \to 0$.

Proof. (i) Choose $v_0 \in K$ such that (2.5) and (2.6) hold. Suppose that $|u_\varepsilon - v_0|_W$ is unbounded. Then for some sequence $\varepsilon_k \to 0$

$$|u_{\varepsilon_k} - v_0|_W \to +\infty.$$

Taking $v = v_0$ in (2.10), we derive

$$\varepsilon_k \langle A u_{\varepsilon_k}, u_{\varepsilon_k} - v_0 \rangle_V + \langle B u_{\varepsilon_k}, u_{\varepsilon_k} - v_0 \rangle_W \leqslant \langle f, u_{\varepsilon_k} - v_0 \rangle_W \leqslant |f|_{W'} \, |u_{\varepsilon_k} - v_0|_W \, .$$

It follows that

$$\frac{\varepsilon_k \langle A u_{\varepsilon_k}, u_{\varepsilon_k} - v_0 \rangle_V}{|u_{\varepsilon_k} - v_0|_W} + \frac{\langle B u_{\varepsilon_k}, u_{\varepsilon_k} - v_0 \rangle_W}{|u_{\varepsilon_k} - v_0|_W} \leqslant |f|_{W'} \, . \tag{2.15}$$

If $|u_{\varepsilon_k} - v_0|_V$ is bounded, then

$$\frac{\varepsilon_k \langle A u_{\varepsilon_k}, u_{\varepsilon_k} - v_0 \rangle_V}{|u_{\varepsilon_k} - v_0|_W} \to 0$$

else, by the coerciveness of A, this term is nonnegative for some k large enough. In both cases, due to the coerciveness of B, the left-hand side of (2.15) is unbounded, which is impossible. This proves assertion (i).

(ii) Since u_ε is bounded in W and, consequently, $B u_\varepsilon$ is bounded in W', from (2.10) written for $v = v_0$ we derive that

$$\varepsilon \langle A u_\varepsilon, u_\varepsilon - v_0 \rangle_V \leqslant C \tag{2.16}$$

for some constant C independent of ε. If $(u_\varepsilon - v_0)$ is bounded in V, it is clear that $\varepsilon u_\varepsilon = \varepsilon (u_\varepsilon - v_0) + \varepsilon v_0 \to 0$. Also, from (2.5) and (2.16) we have – up to a subsequence –

$$\varepsilon \, |u_\varepsilon - v_0|_V \leqslant C \frac{|u_\varepsilon - v_0|_V}{\langle A u_\varepsilon, u_\varepsilon - v_0 \rangle_V} \to 0,$$

and the result follows as in the previous case.

(iii) and (iv) We first show that $\varepsilon A u_\varepsilon \rightharpoonup 0$ in V'. Let $v \in V$. By the monotonicity of A,

$$\varepsilon \langle A u_\varepsilon - A v, u_\varepsilon - v \rangle_V \geqslant 0. \tag{2.17}$$

Hence

$$\varepsilon \langle A u_\varepsilon, v \rangle_V \leqslant \varepsilon \langle A u_\varepsilon, u_\varepsilon \rangle_V + \langle A v, \varepsilon (v - u_\varepsilon) \rangle_V \, . \tag{2.18}$$

For $v_0 \in K$ from (2.16) we derive

$$\varepsilon \langle A u_\varepsilon, u_\varepsilon \rangle_V \leqslant \langle \varepsilon A u_\varepsilon, v_0 \rangle_V + C.$$

Using (2.18), we get

$$\varepsilon \langle A u_\varepsilon, v - v_0 \rangle_V \leqslant C + \langle A v, \varepsilon (v - u_\varepsilon) \rangle_V \, , \tag{2.19}$$

where C is a constant independent of ε. Choosing $v \in v_0 + \mathscr{B}_1$, where $\mathscr{B}_1$ is the unit ball of V, we arrive to

$$\varepsilon \langle Au_\varepsilon, v_1 \rangle_V \leqslant C' \quad \forall\, v_1 \in \mathscr{B}_1,$$

where C' is independent of ε. Thus, $\varepsilon A_\varepsilon$ is bounded in V' and – for some subsequence –

$$\varepsilon A u_\varepsilon \rightharpoonup \psi \text{ in } V'.$$

Passing to the limit in (2.19) we derive

$$\langle \psi, v - v_0 \rangle_V \leqslant C \quad \forall v \in V.$$

Thus, $\psi = 0$. By the uniqueness of the possible limits, we have shown that

$$\varepsilon A u_\varepsilon \rightharpoonup 0 \text{ in } V'.$$

For any $v \in K$, by (2.10) and the monotonicity of B, we have

$$\begin{aligned} \varepsilon \langle Au_\varepsilon, u_\varepsilon \rangle_V &\leqslant \langle \varepsilon Au_\varepsilon, v \rangle_V + \langle f, u_\varepsilon - v \rangle_W + \langle Bu_\varepsilon, v - u_\varepsilon \rangle_W \\ &\leqslant \langle \varepsilon Au_\varepsilon, v \rangle_V + \langle f, u_\varepsilon - v \rangle_W + \langle Bv, v - u_\varepsilon \rangle_W . \end{aligned} \tag{2.20}$$

Let $(\varepsilon_k)_k$ be a sequence such that

$$\varepsilon_k \langle Au_{\varepsilon_k}, u_{\varepsilon_k} \rangle_V \to \limsup_{\varepsilon \to 0} \varepsilon \langle Au_\varepsilon, u_\varepsilon \rangle_V .$$

Since u_{ε_k} is bounded in W, one can suppose – extracting, if necessary, another subsequence – that

$$u_{\varepsilon_k} \rightharpoonup \widetilde{u} \quad \text{in } W.$$

Passing to the limit in (2.20) written for ε_k, we get

$$\limsup_{\varepsilon \to 0} \varepsilon \langle Au_\varepsilon, u_\varepsilon \rangle_V \leqslant \langle f, \widetilde{u} - v \rangle_W + \langle Bv, v - \widetilde{u} \rangle_W \quad \forall v \in K. \tag{2.21}$$

It is clear that $\widetilde{u}$ belongs to $\overline{K}^W$, the weak closure of K in W which coincides with its strong closure since K is convex. Thus, there exists a sequence $v_n \in K$ such that

$$v_n \to \widetilde{u} \quad \text{in } W.$$

Taking $v = v_n$ in (2.21) and passing to the limit, we derive

$$\limsup_{\varepsilon \to 0} \varepsilon \langle Au_\varepsilon, u_\varepsilon \rangle_V \leqslant 0.$$

Passing to the limit in (2.18), we also have

$$\liminf_{\varepsilon \to 0} \varepsilon \langle Au_\varepsilon, u_\varepsilon \rangle_V \geqslant 0,$$

which proves (iv).

To complete the proof, going back to (2.18), for every $v_1 \in \mathscr{B}_1$ one has

$$\varepsilon \langle Au_\varepsilon, v_1 \rangle_V \leqslant \varepsilon \langle Au_\varepsilon, u_\varepsilon \rangle_V + |Av_1|_{V'} (\varepsilon + |\varepsilon u_\varepsilon|_V) \leqslant \varepsilon \langle Au_\varepsilon, u_\varepsilon \rangle_V + C\,(\varepsilon + |\varepsilon u_\varepsilon|_V) \to 0$$

where C is independent of v_1. This completes the proof of the theorem. □

Remark 2.3. In the case where $K = V \cap W$, from Equation (2.13) one derives that

$$Bu_\varepsilon - f \to 0 \quad \text{in } V'. \tag{2.22}$$

In addition, we have the following assertion.

Theorem 2.3. *Suppose that for some sequence $\varepsilon_k \to 0$*

$$u_{\varepsilon_k} \rightharpoonup \widetilde{u} \quad \text{in } W. \tag{2.23}$$

Then $\widetilde{u}$ is a solution to the variational inequality

$$\begin{cases} \langle B\widetilde{u}, v - \widetilde{u}\rangle_W \geqslant \langle f, v - \widetilde{u}\rangle_W \quad \forall\, v \in \overline{K}^W, \\ \widetilde{u} \in \overline{K}^W. \end{cases} \tag{2.24}$$

Moreover,

$$Bu_{\varepsilon_k} \rightharpoonup B\widetilde{u} \quad \text{in } W', \quad \langle Bu_{\varepsilon_k}, u_{\varepsilon_k}\rangle_W \to \langle B\widetilde{u}, \widetilde{u}\rangle_W\,. \tag{2.25}$$

Proof. Up to a subsequence – still labelled by ε_k – one can assume that

$$Bu_{\varepsilon_k} \rightharpoonup \chi \quad \text{in } W'.$$

Passing to the limit in (2.10) written for ε_k, we obtain (cf. Theorem 2.2)

$$\lim_{\varepsilon_k \to 0} \sup \langle Bu_{\varepsilon_k}, u_{\varepsilon_k}\rangle_W \leqslant \langle \chi, v\rangle_W + \langle f, \widetilde{u} - v\rangle_W \quad \forall\, v \in K. \tag{2.26}$$

Considering a sequence $v = v_n \to \widetilde{u}$ as above, we obtain

$$\lim_{\varepsilon_k \to 0} \sup \langle Bu_{\varepsilon_k}, u_{\varepsilon_k}\rangle_W \leqslant \langle \chi, \widetilde{u}\rangle_W\,.$$

From the monotonicity of B we have

$$\langle Bu_{\varepsilon_k}, u_{\varepsilon_k}\rangle_W \geqslant \langle Bu_{\varepsilon_k}, v\rangle_W + \langle Bv, u_{\varepsilon_k} - v\rangle_W \quad \forall\, v \in W.$$

Then

$$\lim_{\varepsilon_k \to 0} \inf \langle Bu_{\varepsilon_k}, u_{\varepsilon_k}\rangle_W \geqslant \langle \chi, v\rangle_W + \langle Bv, \widetilde{u} - v\rangle_W \quad \forall\, v \in W. \tag{2.27}$$

It follows – taking $v = \widetilde{u}$ – that

$$\lim_{\varepsilon_k \to 0} \langle Bu_{\varepsilon_k}, u_{\varepsilon_k}\rangle_W = \langle \chi, \widetilde{u}\rangle_W\,.$$

From (2.27) we derive

$$\langle \chi - Bv, \widetilde{u} - v\rangle_W \geqslant 0 \quad \forall v \in W.$$

Replacing v by $\widetilde{u} + tw$ and letting $t \to 0$, we obtain

$$\langle \chi - B\widetilde{u}, w\rangle_W \geqslant 0 \quad \forall w \in W,$$

i.e., $\chi = B\widetilde{u}$. It follows that the whole sequence Bu_{ε_k} converges toward $B\widetilde{u}$. Moreover, (2.26) becomes

$$\langle B\widetilde{u}, v - \widetilde{u}\rangle_W \geqslant \langle f, v - \widetilde{u}\rangle_W \quad \forall v \in K.$$

Since $\overline{K}^W$ is closed (weakly closed), $\widetilde{u} \in \overline{K}^W$ and the above inequality holds also for every $v \in \overline{K}^W$. This completes the proof of the theorem. □

Remark 2.4. (i) We have proved that the only possible limits for the subsequences of $(u_\varepsilon)_\varepsilon$ are solutions to the variational inequality (2.24). In particular, if the solution is unique, then

$$u_\varepsilon \rightharpoonup \widetilde{u} \quad \text{in } W,$$
$$Bu_\varepsilon \rightharpoonup B\widetilde{u} \quad \text{in } W'.$$

This is the case where B is strictly monotone.

(ii) In the case where $K = V \cap W$, we have $\overline{K}^W = W$ and $\widetilde{u}$ is a solution to the equation

$$B\widetilde{u} = f.$$

Corollary 2.1. (i) *Suppose that A is strongly coercive in the sense that*

$$\langle Av, v\rangle_V \geqslant \lambda \, |v|_V^\alpha \quad \forall v \in V, \tag{2.28}$$

for some constants $\lambda > 0$ and $\alpha > 1$. Then

$$\varepsilon^{1/\alpha} u_\varepsilon \to 0 \quad \textit{in } V. \tag{2.29}$$

(ii) *If B is strongly monotone in the sense that for some $\delta > 0$ and $\beta > 1$*

$$\langle Bu - Bv, u - v\rangle_W \geqslant \delta \, |u - v|_W^\beta \quad \forall v, u \in W, \tag{2.30}$$

then the solution $\widetilde{u}$ to (2.24) is unique and

$$u_\varepsilon \to \widetilde{u} \quad \textit{in } W.$$

Proof. (i) follows directly from the inequality

$$\varepsilon \, \langle Au_\varepsilon, u_\varepsilon\rangle_V \geqslant \lambda\varepsilon \, |u_\varepsilon|_V^\alpha$$

and Theorem 2.2, (iv).

For (ii), by (2.30), $u_\varepsilon \in K$, and (2.25), one has

$$\begin{aligned} \delta \, |\widetilde{u} - u_\varepsilon|_W^\beta &\leqslant \langle B\widetilde{u} - Bu_\varepsilon, \widetilde{u} - u_\varepsilon\rangle_W \leqslant \langle f, \widetilde{u} - u_\varepsilon\rangle_W - \langle Bu_\varepsilon, \widetilde{u} - u_\varepsilon\rangle_W \\ &= \langle f, \widetilde{u} - u_\varepsilon\rangle_W + \langle Bu_\varepsilon, u_\varepsilon\rangle_W - \langle Bu_\varepsilon, \widetilde{u}\rangle_W \to 0. \end{aligned}$$

□

Remark 2.5. If only the basic coerciveness (2.3) of A is assumed, then the convergence result (ii) is sharp since, if α approaches 1 in (2.29), the exponent of ε tends to 1.

In the following assertion, some monotonicity property of $(u_\varepsilon)_\varepsilon$ is shown.

Corollary 2.2. *Let $\varepsilon > \varepsilon' > 0$. Then*

$$\langle Au_\varepsilon, u_\varepsilon\rangle_V \leqslant \langle Au_\varepsilon, u_{\varepsilon'}\rangle_V . \tag{2.31}$$

Proof. Indeed, setting $v = u_\varepsilon$ (respectively, $v = u_{\varepsilon'}$) in (2.10), written for ε (respectively, ε'), we get

$$\varepsilon \, \langle Au_\varepsilon, u_\varepsilon - u_{\varepsilon'}\rangle_V - \varepsilon' \, \langle Au_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'}\rangle_V + \langle Bu_\varepsilon - Bu_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'}\rangle_W \leqslant 0.$$

Using the monotonicity of A and B, it comes

$$\varepsilon \, \langle Au_\varepsilon, u_\varepsilon - u_{\varepsilon'}\rangle_V \leqslant \varepsilon' \, \langle Au_\varepsilon, u_\varepsilon - u_{\varepsilon'}\rangle_V - \varepsilon' \, \langle Au_\varepsilon - Au_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'}\rangle_V \leqslant \varepsilon' \, \langle Au_\varepsilon, u_\varepsilon - u_{\varepsilon'}\rangle_V .$$

Then (2.31) follows because $\varepsilon > \varepsilon'$. □

Remark 2.6. The above characterization is more clear if A is linear. For instance, if V is a Hilbert space and $A = I_d$, then (2.31) yields

$$|u_\varepsilon|_V \leqslant |u_{\varepsilon'}|_V \quad \text{for } \varepsilon' < \varepsilon.$$

Next we pay attention to more regular problems, i.e., when some solutions to (2.24) are in V.

Corollary 2.3. *If the variational inequality* (2.24) *has a solution* $\widehat{u} \in K$ *satisfying*

$$\liminf \langle Au, u - \widehat{u}\rangle_V > 0 \quad \text{as} \quad |u|_V \to +\infty, \quad u \in K, \tag{2.32}$$

then u_ε *is bounded in* V *and there exists a sequence* u_{ε_k} *such that*

$$u_{\varepsilon_k} \rightharpoonup \widetilde{u} \quad \text{in } V \text{ and } W, \tag{2.33}$$

where $\widetilde{u} \in K$ *is a solution to* (2.24), *i.e., all the accumulation points of* $(u_\varepsilon)_\varepsilon$ *belong to* K *and are solutions to* (2.24).

In addition, if B *satisfies* (2.30), *then*

$$|u_\varepsilon - \widetilde{u}|_W = o(\varepsilon^{1/\beta}). \tag{2.34}$$

Proof. Taking $v = \widehat{u}$ in (2.10), we derive

$$\varepsilon \langle Au_\varepsilon, u_\varepsilon - \widehat{u}\rangle_V \leqslant \langle f, u_\varepsilon - \widehat{u}\rangle_W - \langle Bu_\varepsilon, u_\varepsilon - \widehat{u}\rangle_W \leqslant -\langle Bu_\varepsilon - B\widehat{u}, u_\varepsilon - \widehat{u}\rangle_W \leqslant 0. \tag{2.35}$$

Thus, $\langle Au_\varepsilon, u_\varepsilon - \widehat{u}\rangle_V \leqslant 0$ for all $\varepsilon > 0$ and

$$\limsup_{\varepsilon\to 0} \langle Au_\varepsilon, u_\varepsilon - \widehat{u}\rangle_V \leqslant 0.$$

By (2.32), u_ε must be bounded in V, and one can find a sequence ε_k such that

$$u_{\varepsilon_k} \rightharpoonup \widetilde{u} \ \text{ in } W,\ V, \text{ and } V \cap W.$$

In fact, since u_{ε_k} is bounded in V, W, and $W \cap V$ one can assume that – up to a subsequence –

$$u_{\varepsilon_k} \rightharpoonup u \ \text{ in } V, \quad u_{\varepsilon_k} \rightharpoonup u' \ \text{ in } W, \quad u_{\varepsilon_k} \rightharpoonup u'' \ \text{ in } V \cap W.$$

If $h \in V' \subset V' + W'$, then

$$\langle h, u_{\varepsilon_k}\rangle_{V\cap W} \to \langle h, u\rangle_{V\cap W}, \quad \langle h, u_{\varepsilon_k}\rangle_{V\cap W} \to \langle h, u''\rangle_{V\cap W}.$$

Hence

$$\langle h, u\rangle_V = \langle h, u''\rangle_V \quad \forall\, h \in V'.$$

Similarly, one can show that

$$\langle h, u'\rangle_W = \langle h, u''\rangle_W \quad \forall\, h \in W'.$$

It follows that

$$u = u' = u'' = \widetilde{u}, \tag{2.36}$$

and $\widetilde{u}$ is necessarily a solution to (2.24).

For the last part of the corollary, since $\widehat{u} = \widetilde{u}$, by the uniqueness of the solution to (2.24), from (2.35), one has

$$\begin{aligned} \delta\, |\widetilde{u} - u_\varepsilon|_W^\beta &\leqslant \langle B\widetilde{u} - Bu_\varepsilon, \widetilde{u} - u_\varepsilon\rangle_W \leqslant -\varepsilon \langle Au_\varepsilon, u_\varepsilon - \widetilde{u}\rangle_V \\ &= -\varepsilon \langle Au_\varepsilon - A\widetilde{u}, u_\varepsilon - \widetilde{u}\rangle_V + \varepsilon \langle A\widetilde{u}, u_\varepsilon - \widetilde{u}\rangle_V \leqslant \varepsilon \langle A\widetilde{u}, u_\varepsilon - \widetilde{u}\rangle_V = o(\varepsilon), \end{aligned}$$

and the result follows. □

Remark 2.7. If we assume that $f = 0$, $B(0) = 0, 0 \in K$, and B satisfies a hypothesis as (2.28), then $u_\varepsilon \to 0$ in W. Indeed, taking $v = 0$ in (2.10)

$$\varepsilon \langle Au_\varepsilon, u_\varepsilon \rangle_V + \langle Bu_\varepsilon, u_\varepsilon \rangle_W \leqslant 0,$$

and by the monotonicity of A we have

$$\lambda |u_\varepsilon|_W^\beta \leqslant \varepsilon \langle Au_\varepsilon - A(0), u_\varepsilon \rangle_V + \langle Bu_\varepsilon, u_\varepsilon \rangle_W \leqslant -\varepsilon \langle A(0), u_\varepsilon \rangle_V .$$

The convergence follows by Theorem 2.2.

3 Some Applications

It is interesting to note that, using a priori estimates in the previous section, there is no need to have some compactness assumptions to pass to the limit in the nonlinear terms. In order to illustrate this, we consider here three nonlinear elliptic boundary value problems as examples of the abstract theory above. We will apply the theory to some anisotropic singular perturbations problems in the last two examples. To also see the power of our abstract analysis in general, we consider a very classical case of nonlinear obstacle problems.

3.1 Nonlinear obstacle problems

We denote by $a(\xi) = (a_i(\xi))$ a continuous vector field in $\mathbb{R}^n$. We suppose that a is such that for some $\lambda, \Lambda > 0$ and $c \in \mathbb{R}$

$$\begin{aligned} &a(\xi) \cdot \xi \geqslant \lambda |\xi|^2 + c, \\ &|a(\xi)| \leqslant \Lambda |\xi| \quad \forall\, \xi \in \mathbb{R}^n \end{aligned} \tag{3.1}$$

and, in addition,

$$(a(\xi) - a(\zeta)) . (\xi - \zeta) \geqslant 0 \quad \forall\, \xi, \zeta \in \mathbb{R}^n. \tag{3.2}$$

Then, for $f \in L^2(\Omega)$ there exists a unique solution u_ε to

$$\begin{cases} u_\varepsilon \in K_0 = \left\{ v \in H_0^1(\Omega) \,|\, v(x) \geqslant 0, \quad \text{a.e. } \; x \in \Omega \right\}, \\ \varepsilon \displaystyle\int_\Omega a(\nabla u_\varepsilon) \cdot \nabla (v - u_\varepsilon)\, dx + \int_\Omega u_\varepsilon (v - u_\varepsilon)\, dx \geqslant \int_\Omega f (v - u_\varepsilon)\, dx \quad \forall\, v \in K_0, \end{cases} \tag{3.3}$$

where Ω is a bounded open subset in $\mathbb{R}^n$. Then setting

$$V = H_0^1(\Omega), \quad W = L^2(\Omega), \quad Au = -\operatorname{div}(a(\nabla u)), \quad B = I_d,$$

our results apply and we get

$$u_\varepsilon \to f^+ \quad \text{in } L^2(\Omega),$$

where f^+ (respectively, f^-) denotes the positive (respectively, negative) part of f. Indeed, thanks to Theorems 2.2, 2.3 and Corollary 2.1 we see that $u_\varepsilon \to \widetilde{u}$ in $L^2(\Omega)$ where $\widetilde{u}$ is the unique solution to the problem

$$\begin{cases} \widetilde{u} \in \overline{K}_0 = \left\{ v \in L^2(\Omega) \,|\, v(x) \geqslant 0, \quad \text{a.e. } \; x \in \Omega \right\}, \\ \displaystyle\int_\Omega \widetilde{u} (v - \widetilde{u})\, dx \geqslant \int_\Omega f (v - \widetilde{u})\, dx \quad \forall\, v \in \overline{K}_0. \end{cases} \tag{3.4}$$

But it is clear that

$$\int_{\Omega} f^{+}\left(v-f^{+}\right)dx=\int_{\Omega}\left(f+f^{-}\right)\left(v-f^{+}\right)dx$$

$$=\int_{\Omega} f\left(v-f^{+}\right)dx+\int_{\Omega} f^{-}v dx\geqslant\int_{\Omega} f\left(v-f^{+}\right)dx\quad\forall\, v\in\overline{K}_{0}$$

and $\widetilde{u}=f^{+}$. As a consequence of Theorems 2.2, 2.3 and Corollary 2.1 we can state the following.

Corollary 3.1. *As $\varepsilon\to 0$, we have*

$$u_{\varepsilon}\to f^{+}\quad in\ L^{2}(\Omega),\quad \varepsilon u_{\varepsilon}\to 0\quad in\ H_{0}^{1}(\Omega),$$

$$-\varepsilon\partial_{x_{i}}\left(a\left(\nabla u_{\varepsilon}\right)\right)\to 0\quad in\ H^{-1}(\Omega),\quad i=1,\cdots,n,$$

$$\varepsilon\int_{\Omega} a\left(\nabla u_{\varepsilon}\right)\cdot\nabla u_{\varepsilon}dx\to 0.$$

Remark 3.1. Note that, as in (3.1), we may add a constant $c\in\mathbb{R}$ in (2.28) since it will be neglected once it is multiplied by ε, i.e.,

$$\langle Av,v\rangle_{V}\geqslant\lambda\left|v\right|_{V}^{\alpha}+c\quad\forall v\in V.$$

Of course, here the strong convergence of $\sqrt{\varepsilon}\nabla u_{\varepsilon}$ comes from the last convergence in the above corollary, i.e., $\sqrt{\varepsilon}\nabla u_{\varepsilon}\to 0$ in $L^{2}(\Omega)$.

3.2 Semilinear elliptic problems

Let Ω be a bounded open subset of $\mathbb{R}^{n}$ with sufficiently smooth boundary. We split the components of a point $x\in\mathbb{R}^{n}$ into the q first components and the $n-q$ last ones, i.e.,

$$X_{1}=(x_{1},\ldots,x_{q})\quad\text{and}\quad X_{2}=(x_{q+1},\ldots,x_{n}),$$

where q is a positive integer such that $q<n$. We denote by $\Pi_{X_{1}}$ (respectively, $\Pi_{X_{2}}$) the orthogonal projection from $\mathbb{R}^{n}$ onto the space $X_{2}=0$ (respectively, $X_{1}=0$). For any $X_{1}\in\Pi_{1}:=\Pi_{X_{1}}(\Omega)$ and $X_{2}\in\Pi_{2}:=\Pi_{X_{2}}(\Omega)$ we denote by $\Omega_{X_{1}}$ (respectively, $\Omega_{X_{2}}$) the section of Ω above X_{1} (respectively, X_{2}) i.e.,

$$\Omega_{X_{1}}=\{\,X_{2}\mid(X_{1},X_{2})\in\Omega\,\}\quad\text{and}\quad\Omega_{X_{2}}=\{\,X_{1}\mid(X_{1},X_{2})\in\Omega\,\}.$$

With this notation we set

$$\nabla u=(\partial_{x_{1}}u,\ldots,\partial_{x_{n}}u)^{T}=\begin{pmatrix}(\partial_{x_{1}}u,\ldots,\partial_{x_{q}}u)^{T}\\(\partial_{x_{q+1}}u,\ldots,\partial_{x_{n}}u)^{T}\end{pmatrix}=\begin{pmatrix}\nabla_{X_{1}}u\\\nabla_{X_{2}}u\end{pmatrix}.$$

We consider the following semilinear elliptic problem:

$$\begin{cases}-\varepsilon\Delta_{X_{1}}u_{\varepsilon}-\Delta_{X_{2}}u_{\varepsilon}+g\left(x,u_{\varepsilon}\right)=f\quad\text{in }\Omega,\\u_{\varepsilon}\in H_{0}^{1}(\Omega)\cap L^{p}(\Omega),\end{cases}\tag{3.5}$$

where

$$\Delta_{X_1} = \sum_{i=1}^{i=q} \frac{\partial^2}{\partial^2 x_i}, \quad \Delta_{X_2} = \sum_{i=q+1}^{i=n} \frac{\partial^2}{\partial^2 x_i}, \quad p > 1, \quad f \in L^2(\Omega) + L^{p'}(\Omega),$$

where p' is the conjugate of p. In order to apply the abstract approach, we assume that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathédeory function and nondecreasing in the second variable, i.e.,

$$x \mapsto g(x,t) \text{ is measurable on } \Omega \text{ for all } t \in \mathbb{R},$$
$$t \mapsto g(x,t) \text{ is continuous and nondecreasing on } \mathbb{R} \text{ for a.e. } x \in \Omega$$

and there exist $c, c' \geqslant 0$ such that

$$|g(x,t)| \leqslant c\,|t|^{p-1} + c' \quad \forall t \in \mathbb{R} \quad \text{a.e. } x \in \Omega, \tag{3.6}$$

$$g(x,t)\,t \geqslant |t|^p \quad \forall t \in \mathbb{R} \quad \text{a.e. } x \in \Omega. \tag{3.7}$$

It is clear that, if $u \in L^p(\Omega)$, then $g(\cdot, u(\cdot)) \in L^{p'}(\Omega)$. So g defines an operator (still labelled by g) from $L^p(\Omega)$ into $L^{p'}(\Omega)$ by

$$u \mapsto g(\cdot, u(\cdot)), \tag{3.8}$$

which is bounded, monotone and hemicontinuous. Then we choose the suitable Banach spaces

$$V = \left\{ u \in L^2(\Omega) \,\middle|\, \begin{array}{l} \nabla_{X_1} u \in \left[L^2(\Omega)\right]^q, \\ u(\cdot, X_2) \in H_0^1(\Omega_{X_2}), \text{ a.e. } X_2 \in \Pi_2 \end{array} \right\}, \tag{3.9}$$

equipped with the norm

$$|v|_V := |\nabla_{X_1} v|_{L^2(\Omega)}$$

and

$$W = \left\{ u \in L^2(\Omega) \cap L^p(\Omega) \,\middle|\, \begin{array}{l} \nabla_{X_2} u \in \left[L^2(\Omega)\right]^{n-q}, \\ u(X_1, \cdot) \in H_0^1(\Omega_{X_1}) \quad \text{a.e. } X_1 \in \Pi_1 \end{array} \right\}, \tag{3.10}$$

equipped with the norm

$$|v|_W := |\nabla_{X_2} v|_{L^2(\Omega)} + |v|_{L^p(\Omega)}.$$

We can easily check that V and W are separable reflexive Banach spaces. Next we set

$$A = -\Delta_{X_1} \quad \text{and} \quad B = -\Delta_{X_2} + g(x, \cdot).$$

Then the operator $A : V \to V'$ is linear, bounded, and coercive. Since the operator $B : W \to W'$ is a sum of a linear operator, satisfying the same properties as A, and the operator defined in (3.8), it is bounded, monotone, and coercive. In this example, the limit problem is defined for a.e. $X_1 \in \Pi_1$ as

$$\begin{cases} -\Delta_{X_2} \widetilde{u}(X_1, \cdot) + g((X_1, \cdot), \widetilde{u}(X_1, \cdot)) = f(X_1, \cdot) \quad \text{in } \Omega_{X_1}, \\ \widetilde{u}(X_1, \cdot) = 0 \quad \text{on } \partial\Omega_{X_1}. \end{cases} \tag{3.11}$$

Then it remains to precise the connection between the boundary conditions, which is the subject of the following proposition.

Proposition 3.1. *Let V and W be the spaces defined in* (3.9) *and* (3.10) *respectively. Then, if the boundary of Ω is smooth, we have*

$$V \cap W = H_0^1(\Omega) \cap L^p(\Omega).$$

Proof. The first inclusion $H_0^1(\Omega)\cap L^p(\Omega)\subset V\cap W$ is easy. For $u\in H_0^1(\Omega)\cap L^p(\Omega)$ there exists a sequence $(u_n)_n\subset\mathscr{D}(\Omega)$ such that $u_n\to u$ in $H_0^1(\Omega)\cap L^p(\Omega)$. In particular, we have

$$|\nabla(u_n-u)|_{L^2(\Omega)}\to 0.$$

By the Lebesgue theorem, we get – up to a subsequence – for a.e. $X_1\in\Pi_1$ and $X_2\in\Pi_2$

$$|\nabla(u_n(X_1,\cdot)-u(X_1,\cdot))|_{L^2(\Omega_{X_1})}\to 0,$$
$$|\nabla(u_n(\cdot,X_2)-u(\cdot,X_2))|_{L^2(\Omega_{X_2})}\to 0.$$

This means that $u\in V$ and $u\in W$.

For the converse inclusion, we take $u\in V\cap W$ and consider the elliptic problem

$$\begin{cases}-\varepsilon\Delta v_\varepsilon+v_\varepsilon=u & \text{in } \Omega,\\ v_\varepsilon=0 & \text{on } \partial\Omega.\end{cases}\tag{3.12}$$

Since Ω is sufficiently regular and of course $V\cap W\subset H^1(\Omega)\cap L^p(\Omega)$, we have $v_\varepsilon\in H^2(\Omega)$. According to Corollary 2.1, we derive

$$v_\varepsilon\to u\quad\text{in } L^2(\Omega).\tag{3.13}$$

Then, applying the Laplace operator to the first equation in (3.12) and taking $-v_\varepsilon$ as a test function, we obtain

$$\varepsilon\left\langle\Delta^2 v_\varepsilon,v_\varepsilon\right\rangle_{H_0^1(\Omega)}-\int_\Omega\Delta v_\varepsilon v_\varepsilon dx=-\left\langle\Delta u,v_\varepsilon\right\rangle_{H_0^1(\Omega)}.$$

It is clear that $\Delta u\in H^{-1}(\Omega)$, $\Delta^2 v_\varepsilon\in H^{-1}(\Omega)$ since

$$-\Delta v_\varepsilon=\frac{u-v_\varepsilon}{\varepsilon}\in H^1(\Omega).\tag{3.14}$$

It follows that

$$-\varepsilon\int_\Omega\nabla(\Delta v_\varepsilon)\cdot\nabla v_\varepsilon dx+|\nabla v_\varepsilon|^2_{L^2(\Omega)}=\int_\Omega\nabla u\cdot\nabla v_\varepsilon dx.$$

Hence

$$\begin{aligned}&-\varepsilon\int_{\Pi_1}\int_{\Omega_{X_2}}\nabla_{X_1}(\Delta v_\varepsilon)\cdot\nabla_{X_1}v_\varepsilon dX_1dX_2-\varepsilon\int_{\Pi_2}\int_{\Omega_{X_1}}\nabla_{X_2}(\Delta v_\varepsilon)\cdot\nabla_{X_2}v_\varepsilon dX_2dX_1+|\nabla v_\varepsilon|^2_{L^2(\Omega)}\\ &=\int_\Omega\nabla u\cdot\nabla v_\varepsilon dx\leqslant\frac{1}{2}|\nabla u|^2_{L^2(\Omega)}+\frac{1}{2}|\nabla v_\varepsilon|^2_{L^2(\Omega)}.\end{aligned}\tag{3.15}$$

Since $v_\varepsilon\in H_0^1(\Omega)$ and $u\in V\cap W$ in (3.14), for a.e. $X_1\in\Pi_1$ and a.e. $X_2\in\Pi_2$ (cf. [4])

$$\Delta v_\varepsilon(X_1,\cdot)\in H_0^1(\Omega_{X_2})\quad\text{and}\quad\Delta v_\varepsilon(\cdot,X_2)\in H_0^1(\Omega_{X_1}).$$

Thus, we can write (3.15) as

$$2\varepsilon\int_{\Pi_1}\int_{\Omega_{X_2}}\Delta v_\varepsilon\Delta_{X_1}v_\varepsilon dX_1dX_2+2\varepsilon\int_{\Pi_2}\int_{\Omega_{X_1}}\Delta v_\varepsilon\Delta_{X_2}v_\varepsilon dX_2dX_1+|\nabla v_\varepsilon|^2_{L^2(\Omega)}\leqslant|\nabla u|^2_{L^2(\Omega)}.$$

Hence

$$2\varepsilon\left|\Delta v_{\varepsilon}\right|_{L^{2}(\Omega)}^{2}+\left|\nabla v_{\varepsilon}\right|_{L^{2}(\Omega)}^{2}\leqslant\left|\nabla u\right|_{L^{2}(\Omega)}^{2}. \tag{3.16}$$

It follows that v_ε is bounded in $H_0^1(\Omega)$. Then – up to a subsequence – its weak limit is in $H_0^1(\Omega)$, and due to (3.13) this limit is u. Thus, $u \in H_0^1(\Omega)$, which completes the proof. □

As is known, we need a pointwise convergence to pass to the limit in the nonlinear term $g(\cdot, u_\varepsilon)$. But the estimates that one has, i.e.,

$$\left|\nabla_{X_2}u_\varepsilon\right|_{L^2(\Omega)}, \left|u_\varepsilon\right|_{L^p(\Omega)} \quad \text{are bounded,}$$

are not sufficient to get the pointwise limit of $(u_\varepsilon)_\varepsilon$ since the embedding $W \subset L^2(\Omega)$ is not compact. So, in this case, the monotonicity hypothesis is necessary, and as an obvious consequence of Theorems 2.2, 2.3 and Corollary 2.1 we have

Corollary 3.2. *As $\varepsilon \to 0$, we have*

$$u_\varepsilon \to \widetilde{u}, \quad \nabla_{X_2}u_\varepsilon \to \nabla_{X_2}\widetilde{u}, \quad \sqrt{\varepsilon}\nabla_{X_1}u_\varepsilon \to 0 \quad \text{in } L^2(\Omega)$$

where $\widetilde{u}$ and u_ε are the solutions to (3.11) *and* (3.5) *respectively. Moreover, if g is strongly monotone, then*

$$u_\varepsilon \to \widetilde{u} \quad \text{in } L^p(\Omega).$$

Remark 3.2. Even if B is not strongly monotone, the first two convergences hold strongly. This is due to the following monotone type inequality:

$$\langle \Delta_{X_2}v - \Delta_{X_2}u, v-u\rangle_W + \int_\Omega (g(x,v)-g(x,u))(v-u)\,dx \geqslant \left|\nabla_{X_2}(v-u)\right|_{L^2(\Omega)}^2 \quad \forall\, u,v \in W.$$

3.3 $p-$Laplacian type problem

The second application of the abstract theory, in the anisotropic case, is the following quasilinear elliptic equation:

$$\begin{cases} -\varepsilon\Delta_{p_1,X_1}u_\varepsilon - \Delta_{p_2,X_2}u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.17}$$

where $p_1, p_2 > 1$ are real constants and Δ_{p_1,X_1}, Δ_{p_2,X_2} are the p_i-Laplace operators in X_1 and X_2 respectively, i.e.,

$$\Delta_{p_1,X_1}\cdot = \nabla_{X_1}\cdot\left(\left|\nabla_{X_1}\cdot\quad\right|^{p_1-2}\nabla_{X_1}\cdot\right),$$
$$\Delta_{p_2,X_2}\cdot = \nabla_{X_2}\cdot\left(\left|\nabla_{X_2}\cdot\quad\right|^{p_2-2}\nabla_{X_2}\cdot\right).$$

We assume that $f \in L^{p_2'}(\Omega)$ (p_2' is the conjugate of p_2). In this case, we set

$$V = \left\{ u \in L^{p_1}(\Omega) \,\middle|\, \begin{array}{l} \nabla_{X_1}u \in [L^{p_1}(\Omega)]^q, \\ u(\cdot, X_2) \in W_0^{1,p_1}(\Omega_{X_2}), \quad \text{a.e. } X_2 \in \Pi_2 \end{array} \right\},$$

equipped with the norm

$$|v|_V = |\nabla_{X_1} v|_{L^{p_1}(\Omega)}$$

and

$$W = \left\{ u \in L^{p_2}(\Omega) \;\middle|\; \begin{array}{l} \nabla_{X_2} u \in [L^{p_2}(\Omega)]^{n-q}, \\ u(X_1,\cdot) \in W_0^{1,p_2}(\Omega_{X_1}) \quad \text{a.e. } X_1 \in \Pi_1 \end{array} \right\},$$

equipped with the norm

$$|v|_W = |\nabla_{X_2} v|_{L^{p_2}(\Omega)}.$$

We can easily show that V and W are separable reflexive Banach spaces. Then we define the operators $A : V \to V'$ and $B : W \to W'$ as

$$A = -\Delta_{p_1,X_1} \quad \text{and} \quad B = -\Delta_{p_2,X_2}.$$

It is easy to see that A and B are coercive, bounded, and hemicontinuous. The monotonicity of A and B is shown by the following lemma (cf. [2, 13]).

Lemma 3.1. *For all $p > 1$ and $\xi, \eta \in \mathbb{R}^n$*

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right) \cdot (\xi - \eta) \geqslant c_p \{|\xi| + |\eta|\}^{p-2} |\xi - \eta|^2$$

with a constant $c_p > 0$. If $p \geqslant 2$, then

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right) \cdot (\xi - \eta) \geqslant c_p |\xi - \eta|^p,$$

where $|\cdot|$ is the usual Euclidean norm in $\mathbb{R}^n$ and "$\cdot$" is the scalar product.

Thus, the operator A (respectively, B) is strictly monotone for all $p_1 > 1$ (respectively, $p_2 > 1$) and strongly monotone if $p_1 \geqslant 2$ (respectively, $p_2 \geqslant 2$). The limit problem is defined for a.e. $X_1 \in \Pi_1$ as

$$\begin{cases} -\Delta_{p_2,X_2} \widetilde{u}(X_1,\cdot) = f(X_1,\cdot) & \text{in } \Omega_{X_1}, \\ \widetilde{u}(X_1,\cdot) = 0 & \text{on } \partial\Omega_{X_1}. \end{cases} \tag{3.18}$$

Finally, as in the previous subsection, we can show that $(V \cap W) \subset W_0^{1,\min(p_1,p_2)}(\Omega)$. More precisely, we have

$$V \cap W = \left\{ u \in L^{\max(p_1,p_2)}(\Omega) \;\middle|\; \begin{array}{l} \nabla_{X_1} u \in [L^{p_1}(\Omega)]^q, \quad \nabla_{X_2} u \in [L^{p_2}(\Omega)]^{n-q}, \\ u|_{\partial\Omega} = 0 \end{array} \right\},$$

which gives a sense to the boundary conditions. Then, by Theorems 2.2, 2.3 and Corollary 2.1, the following assertion holds.

Corollary 3.3. *For all $p_1, p_2 > 1$,*

$$\begin{aligned} &u_\varepsilon \rightharpoonup \widetilde{u} \quad \text{in } W, \\ &\varepsilon \nabla_{X_1} u_\varepsilon \to 0 \quad \text{in } L^{p_1}(\Omega), \\ &\varepsilon \Delta_{p_1,X_1} u_\varepsilon \to 0 \quad \text{in } V', \\ &\Delta_{p_2,X_2} u_\varepsilon \rightharpoonup f \quad \text{in } W', \end{aligned} \tag{3.19}$$

where u_ε and $\widetilde{u}$ are the solutions to (3.17) and (3.18) respectively. Moreover, if $p_1 \geqslant 2$, then

$$\varepsilon^{1/p_1} \nabla_{X_1} u_\varepsilon \to 0 \quad \text{in } L^{p_1}(\Omega), \tag{3.20}$$

and, if $p_2 \geqslant 2$, then

$$u_\varepsilon \to \widetilde{u}, \nabla_{X_2} u_\varepsilon \to \nabla_{X_2} \widetilde{u} \quad \text{in } L^{p_2}(\Omega). \tag{3.21}$$

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