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ORIGINAL PAPER

The growth function of Coxeter garlands in $\mathbb{H}^4$

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Abstract The growth function $W(t)$ of a Coxeter group W relative to a Coxeter generating set is always a rational function. We prove by an explicit construction that there are infinitely many cocompact Coxeter groups W in hyperbolic 4-space with the following property. All the roots of the denominator of $W(t)$ are on the unit circle except exactly two pairs of real roots.

Keywords Coxeter group · Growth Function · Polynomials and location of zeros · Hyperbolic polytope

Mathematical Subject Classification (2010) Primary 20F55 · 26C10 · 52B11

1 Introduction

Steinberg (1968) proved that the growth function $W(t)$ of a Coxeter group W relative to a Coxeter generating set is always a rational function $W(t) = R(t)/Q(t)$ with (relatively prime) polynomials $R(t)$, $Q(t)$ in $\mathbb{Z}[t]$. Serre (1971) and Charney and Davis (1991) observed that the growth function of every cocompact hyperbolic Coxeter group is reciprocal for even-dimensional hyperbolic spaces and antireciprocal for odd-dimensional hyperbolic spaces. Parry (1993) generalized results of Cannon and Wagreich (1992) and proved the following.

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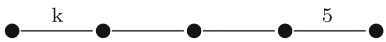
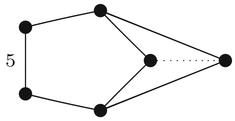
If W is a cocompact hyperbolic Coxeter group in the hyperbolic space $\mathbb{H}^2$ or $\mathbb{H}^3$ (that is, W is generated by reflections in the facets of a compact hyperbolic polygon or polyhedron) then the polynomial Q is a product of distinct irreducible cyclotomic polynomials and exactly one Salem polynomial P. The polynomial P is palindromic and all its roots are on the unit circle except exactly one pair of real roots $(\lambda, 1/\lambda)$.

Of course, this leads to the question whether and how these facts can be generalized to cocompact hyperbolic Coxeter groups in $\mathbb{H}^4$.

For the cocompact 5-generator groups it is easy to see that the denominator $Q(t)$ of the growth function $W(t)$ can be written as a product of cyclotomic polynomials and exactly one irreducible palindromic polynomial P which has two pairs of real roots outside the unit circle (P is not a Salem polynomial). The groups



for $k = 3, 4$ and 5 show that all properly complex roots are not necessarily on the unit circle. Investigating the cocompact hyperbolic 6-generator groups in $\mathbb{H}^4$ classified by Kaplinskaja (1974) and Esselmann (1996) we find that the palindromic polynomial P has exactly two pairs of real roots outside the unit circle for all groups except one. The polynomial P of the group



has four pairs of real roots outside the unit circle.

The results mentioned about the cocompact Coxeter groups in $\mathbb{H}^4$ are experimental observations only. For a general and systematic investigation of the growth functions of these groups we refer to Perren (2009).

In this paper we prove by an explicit construction that there are infinitely many cocompact Coxeter groups in $\mathbb{H}^4$ such that the denominator $Q(t)$ of $W(t)$ itself has the following property. All the roots of the polynomial $Q(t)$ are on the unit circle except exactly two pairs of real roots. This construction is motivated by the work of Makarov (1968). Using the process of truncating and pasting together he constructed infinite series of bounded Coxeter polytopes, called garlands, in $\mathbb{H}^4$ (and $\mathbb{H}^5$).

The paper is organized as follows. In Sect. 2 we prove a preliminary result about the roots of a palindromic polynomial. In Sect. 3 we recall important facts about Coxeter systems and its growth functions. The construction of garlands is described in Sect. 4. Finally, in Sect. 5, we state and prove the main theorems (Theorems 1 and 2).

2 Palindromic polynomials

A polynomial $f(t)$ in $\mathbb{Z}[t]$ with degree n is called *palindromic* if $f(t) = t^n f(t^{-1})$. It is called *anti-palindromic* if $f(t) = -t^n f(t^{-1})$. In other words, palindromic polynomials are integer polynomials that read the same whether read backwards or forwards.

We will use the following notations if f is palindromic:

$$f(t) = \begin{cases} \sum_{j=0}^{\frac{n}{2}-1} a_{n-j}(t^{n-j} + t^j) + a_{\frac{n}{2}} t^{\frac{n}{2}} = [a_n, a_{n-1}, \ldots, a_{\frac{n}{2}}]_+^e & \text{if } n \text{ even} \\ \sum_{j=0}^{\frac{n-1}{2}} a_{n-j}(t^{n-j} + t^j) = [a_n, a_{n-1}, \ldots, a_{\frac{n+1}{2}}]_+^o & \text{if } n \text{ odd} \end{cases}$$

If f is anti-palindromic we write:

$$f(t) = \begin{cases} \sum_{j=0}^{\frac{n}{2}-1} a_{n-j}(t^{n-j} - t^j) + a_{\frac{n}{2}} t^{\frac{n}{2}} = [a_n, a_{n-1}, \ldots, a_{\frac{n}{2}}]_-^e & \text{if } n \text{ even} \\ \sum_{j=0}^{\frac{n-1}{2}} a_{n-j}(t^{n-j} - t^j) = [a_n, a_{n-1}, \ldots, a_{\frac{n+1}{2}}]_-^o & \text{if } n \text{ odd} \end{cases}$$

If f is a nonlinear palindromic polynomial of odd degree, then $f(-1) = -f(-1)$ and so -1 is a root of f. It follows that a nonlinear irreducible palindromic polynomial has even degree. The roots of a monic irreducible palindromic polynomial of even degree appear in pairs $\{z, 1/z\}$. Furthermore, if z is a root, so is $\bar{z}$ and if $|z| = 1$ we have $\bar{z} = 1/z$.

If f is a nonlinear anti-palindromic polynomial of any degree, then $f(1) = -f(1)$ and so 1 is a root of f. It follows that f is reducible over $\mathbb{Z}$.

We are particularly interested in polynomials over $\mathbb{Z}$ with many roots on the unit circle. In general, it is impossible to decide by numerical computations, if a polynomial has roots exactly on the unit circle. In our situation, we can simplify this problem by using rational transformations. The following theorem is an adaption of the process developed by Kempner (1935) to palindromic polynomials.

Proposition 1 *Let f in $\mathbb{Z}[t]$ be a palindromic polynomial of (even) degree $n \geq 2$ with $f(\pm 1) \neq 0$ and let*

$$g(t) = (t-i)^n f\left(\frac{t+i}{t-i}\right) = (t+i)^n f\left(\frac{t-i}{t+i}\right).$$

Then g is a polynomial in $\mathbb{Z}[t]$ of degree n and an even function. Furthermore, the roots of f and g are related as follows.

(i) *f has $2k$ roots on the unit circle if and only if g has k positive real roots.*
(ii) *f has $2l$ real roots if and only if g has l positive imaginary roots.*

Proof The two equations for g follow from the fact that $f(t) = t^n f(t^{-1})$ for palindromic polynomials. Further, it is easy to check that $\overline{g}(t) = g(t)$ for all $t \in \mathbb{R}$ and $g(-t) = g(t)$ for all $t \in \mathbb{C}$. Hence, g is an even polynomial over $\mathbb{Z}$.

(i) The map $w(t) = \frac{t-i}{t+i}$ maps the real axis in the t-plane bijectively on the unit circle (positively oriented) in the w-plane. More precisely, $w((-\infty, 0)) = \{e^{i\eta} : \eta \in (0, \pi)\}$, $w(0) = -1$ and $w((0, \infty)) = \{e^{i\eta} : \eta \in (\pi, 2\pi)\}$. Hence each pair $\{z, \bar{z} = 1/z\}$ of roots of f on the unit circle is mapped by w^{-1} to a pair of real roots (one positive and one negative).

(ii) The map $w(t) = \frac{t-i}{t+i}$ maps the positive imaginary axis in the t-plane on the interval $(-1, 1)$ in the w-plane. More precisely, $w((0, i)) = (-1, 0)$, $w(i) = 0$ and $w((i, \infty i)) = (0, 1)$. Hence each pair $\{\epsilon, 1/\epsilon\}$ of real roots of f, where exactly one is in $(-1, 1)$, is mapped by w^{-1} to a pair of imaginary roots (one positive and one negative). □

As g is an even function we can restate the conclusion of Proposition 1 as follows.

Corollary 1 *If $u = t^2$ then*

(i) *$f(t)$ has $2k$ roots on the unit circle if and only if $g(u)$ has k positive real roots.*
(ii) *$f(t)$ has $2l$ real roots if and only if $g(u)$ has l negative real roots.*

3 Coxeter systems and growth functions

A *Coxeter system* is a triple $(W, S, \mathcal{R})$ consisting of a group W and a set $S \subset W$ of generators, subject only to relations

$$\mathcal{R} = \{(ss')^{m(s,s')} = id | s, s' \in S\}$$

where $m(s, s) = 1$, $m(s, s') = m(s', s)$ for $s \neq s'$ in S. If there is no relation between a pair s, s', we write $m(s, s') = m(s', s) = \infty$. The group W is called a *Coxeter group*. For all subsets $I \subset S$ we denote by W_I the (Coxeter) subgroup of W generated by the set I.

The *nerve* $N = N(W)$ of $(W, S, \mathcal{R})$ is the partially ordered set (poset) of those subsets $I \subset S$ such that W_I is finite. The partial ordering is by inclusion. For all $I \subset S$ we define $L_I = \{\tau \in N : I < \tau\}$. The set $L_\emptyset$ is called the *proper nerve* of $(W, S, \mathcal{R})$. It is a simplicial complex. More precisely, it is isomorphic to the poset of simplices of a simplicial complex with vertex set S. Moreover, for $I \neq \emptyset$, the simplicial complex L_I can be identified with the link of I in $L_\emptyset$ (see Charney and Davis 1991).

In the following, we will consider Coxeter systems allowing a special partition of generators and relations. A so-called *one-generator orthogonal Coxeter system* is a Coxteter system $(W, S, \mathcal{R})$ such that the generators $s_1, \ldots, s_m, s_{m+1}, \ldots, s_k, s$ in S can be arranged in such a way that

$$\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4$$

where the set $\mathcal{R}_0$ contains all relations between the elements $s_1, \ldots, s_m$, the set $\mathcal{R}_1$ contains all relations between the elements $s_{m+1}, \ldots, s_k$ and the set $\mathcal{R}_2$ contains all relations between one element in $\{s_1, \dots, s_m\}$ and one element in $\{s_{m+1}, \dots, s_k\}$. The sets $\mathcal{R}_3$ and $\mathcal{R}_4$ are given by

$$\mathcal{R}_3 = \{(s_j s)^2 = (s s_j)^2 = id \mid j = 1, \dots, m\}$$
$$\mathcal{R}_4 = \{(s_j s)^\infty = (s s_j)^\infty \mid j = m+1, \dots, k\}.$$

The generator s is called an *orthogonal generator* of the system $(W, S, \mathcal{R})$. Further, the Coxeter subsystems $(A, \{s_1, \dots, s_m\}, \mathcal{R}_0)$ and $(A_\bullet, \{s_1, \dots, s_m, s\}, \mathcal{R}_0 \cup \mathcal{R}_3)$, where $A = A(s)$ and $A_\bullet = A_\bullet(s)$ are the subgroups of W generated by $s_1, \dots, s_m$ and $s_1, \dots, s_m, s$ respectively, are called *principal subsystems* of $(W, S, \mathcal{R})$. In a geometric realisation of W, the hyperplane H corresponding to the generator s is orthogonal to all hyperplanes of W intersecting H.

Let $(W, S, \mathcal{R})$ be a Coxeter system and X a subgroup of W. For $n \geq 0$ we denote by a_n the number of elements in X of length n. The power series

$$X(t) = \sum_{n \geq 0} a_n t^n$$

is called the *Poincaré series* of X. For $X = W$ we get the Poincaré series $W(t)$ of the group W.

The Poincaré series $W(t)$ is an explicitly computable rational function (more precisely, is generated by a rational function) in t. This generating function is called the *growth function* of the group W and also denoted by $W(t)$.

The growth function $W(t)$ can be defined recursively by the growth functions of special subgroups of W. Let $(W, S, \mathcal{R})$ be a Coxeter system with nerve N. Then we have

$$\frac{1}{W(t)} = \sum_{I \in N} (-1)^{|I|} \frac{1}{W_I(t^{-1})} \tag{1}$$

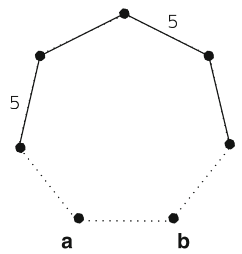
(see Steinberg 1968, p. 14).

If $(W, S, \mathcal{R})$ is a finite Coxeter system (the group W is finite) with degrees $d_1, \dots, d_n$ (see Humphreys 1990, p.59), the growth function of W is a polynomial with the following nice factorization

$$W(t) = \prod_{j=1}^{n} \frac{t^{d_j} - 1}{t - 1} = \prod_{j=1}^{n} (t^{d_j - 1} + t^{d_j - 2} + \cdots + 1). \tag{2}$$

(see Humphreys 1990, p. 73). In the following we will abbreviate $t^{d_j-1} + t^{d_j-2} + \cdots + 1$ by $[d_j]$ for $1 \leq j \leq n$.

Example 1 Applying formulae (1) and (2) we get for the Coxeter group with graph



the growth function

$$W(t) = \frac{[2]^2[6][10]}{[1, -3, 1, 0, 0, 1, 0, 1, 0]_+^e}.$$

4 Construction of garlands

Let $(W, S, \mathcal{R})$ and $(W', S', \mathcal{R}')$ be two one-generator orthogonal Coxeter systems with

$$S = \{s_1, \ldots, s_m, s_{m+1}, \ldots, s_k, s\}$$
$$S' = \{s'_1, \ldots, s'_m, s'_{m+1}, \ldots, s'_l, s'\}$$

and, after renumbering the generators, $\mathcal{R}_0(s_1, \ldots, s_m) = \mathcal{R}'_0(s'_1, \ldots, s'_m)$ or simply $\mathcal{R}_0 = \mathcal{R}'_0$. Then of course $\mathcal{R}_3 = \mathcal{R}'_3$. The two principal subgroups A and A' of W and W' respectively are canonically isomorphic and we can glue together the two Coxeter systems $(W, S, \mathcal{R})$ and $(W', S', \mathcal{R}')$ along A to construct the following new Coxeter system, called a *garland*,

$$(W, S, \mathcal{R}) *_A (W', S', \mathcal{R}') = (W *_A W', S *_A S', \mathcal{R} *_A \mathcal{R}')$$

with

$$S *_A S' = \{s_1, \ldots, s_m, s_{m+1}, \ldots, s_k, s'_{m+1}, \ldots, s'_l\}$$
$$\mathcal{R} *_A \mathcal{R}' = \mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}'_1 \cup \mathcal{R}_2 \cup \mathcal{R}'_2 \cup \mathcal{R}_5$$

where

$$\mathcal{R}_5 = \{(s_i s'_j)^\infty = (s'_j s_i)^\infty = id | i = m+1, \ldots, k;\ j = m+1, \ldots, l\}.$$

Proposition 2 *We have*

$$\frac{1}{(W *_A W')(t)} = \frac{1}{W(t)} + \frac{1}{W'(t)} - \frac{2}{A_\bullet(t)} + \frac{1}{A(t)}.$$

Proof We will use the recursion formula (1) for each of the five groups $W *_A W'$, W, W', A and $A_\bullet$. The relations determining the nerve of each of these groups are listed in the following table.

nerve	relations					
$N(W)$	$\mathcal{R}_0$	$\mathcal{R}_1$		$\mathcal{R}_2$		$\mathcal{R}_3$
$N(W')$	$\mathcal{R}_0' = \mathcal{R}_0$		$\mathcal{R}_1'$		$\mathcal{R}_2'$	$\mathcal{R}_3' = \mathcal{R}_3$
$N(A_\bullet)$	$\mathcal{R}_0$					$\mathcal{R}_3$
$N(A)$	$\mathcal{R}_0$					
$N(W *_A W')$	$\mathcal{R}_0$	$\mathcal{R}_1$	$\mathcal{R}_1'$	$\mathcal{R}_2$	$\mathcal{R}_2'$	

The relations $\mathcal{R}_4$ for W, $\mathcal{R}_4'$ for W' and $\mathcal{R}_5$ for $W *_A W'$ do not contribute to the nerves $N(W)$, $N(W')$ and $N(W *_A W')$ respectively. As formula (1) only depends on the nerves of the five groups, the theorem follows. □

Of course, $A_\bullet$ can be written as the direct product of the group A and the Coxeter group with one generator. It follows that $A_\bullet(t) = (1 + t)A(t)$.

Corollary 2 *We have*

$$\frac{1}{(W *_A W')(t)} = \frac{1}{W(t)} + \frac{1}{W'(t)} + \frac{t-1}{t+1}\frac{1}{A(t)}.$$

The name garland is motivated by hyperbolic Coxeter groups. If the two Coxeter systems $(W, S, \mathcal{R})$ and $(W', S', \mathcal{R}')$ from above are realizable as geometric Coxeter groups in the hyperbolic space, then the corresponding Coxeter polytopes can be glued together along an isomorphic face and the resulting subset of the hyperbolic space is itself a hyperbolic Coxeter polytope (see Vinberg 1993, p. 213).

Example 2 We consider the two one-generator orthogonal Coxeter systems $(W, S, \mathcal{R})$ and $(W', S', \mathcal{R}')$ given by the following two Coxeter graphs $\sum$ and $\sum'$. The Coxeter groups W and W' are realizable as geometric Coxeter groups in the space $\mathbb{H}^4$ (Fig. 1).

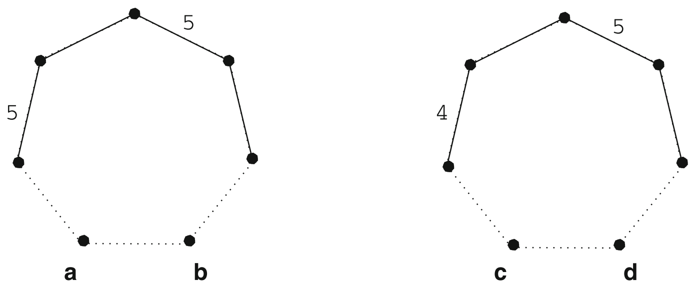


Fig. 1 The graphs $\sum$ and $\sum'$

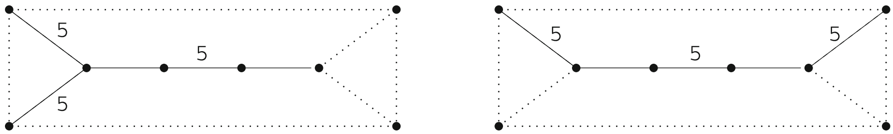


Fig. 2 $W *_{A(a)} W$ and $W *_{A(a)-} W$

As orthogonal generators s and s' we can choose a or b in $(W, S, \mathcal{R})$ and c or d in $(W', S', \mathcal{R}')$.

There are several possibilities to glue together these two groups. For example, there are the two possibilities $W *_{A(a)} W$ and $W *_{A(a)-} W$ to glue together $(W, S, \mathcal{R})$ and $(W, S, \mathcal{R})$ along the subgroup $A(a)$ (the graph of $A(a)$ has an inner symmetry). The resulting Coxeter groups are constructed in Fig. 2.

For $W *_{A(a)} W$ we get by Corollary 2 can be written as

$$\begin{aligned}\frac{1}{(W *_{A(a)} W)(t)} &= 2\frac{1}{W(t)} + \frac{t-1}{t+1}\frac{1}{A(t)} \\ &= 2\frac{[1,-3,1,0,0,1,0,1,0]^e_+}{[2]^2[6][10]} + \frac{t-1}{t+1}\frac{-[1,-2,1,0,-1,2]^o_-}{[2][6][1,0,0]^o_+} \\ &= \frac{[1,-4,1,0,1,1,0,2,0]^e_+}{[2]^2[5][6][1,0,0]^o_+}\end{aligned}$$

5 The growth function of some garlands

The Coxeter group $G = W *_{A(a)} W$ from Example 2 has a one-generator orthogonal Coxeter system; more precisely, there are two distinct orthogonal generators. We can glue together in an obvious way an arbitrary number $n \geq 1$ of these groups to construct a garland.

Theorem 1 *Let G^n be the garland glued from n groups $G = W *_{A(a)} W$ and $G^n(t)$ be the corresponding growth function. Then we have*

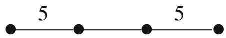
$$G^n(t) = \frac{[2]^2[5][6][1,0,0]^o_+}{D_n(t)}$$

with

$$\begin{aligned}D_n(t) &= t^{16} - 2(n+1)t^{15} + t^{14} + (n-1)t^{13} + t^{12} + nt^{11} + (n-1)t^{10} + 2t^9 \\ &\quad + 2(n-1)t^8 + 2t^7 + (n-1)t^6 + nt^5 + t^4 + (n-1)t^3 + t^2 - 2(n+1)t + 1 \\ &= [1, -2(n+1), 1, (n-1), 1, n, (n-1), 2, 2(n-1)]^e_+\end{aligned}$$

Proof For $n = 1$ the formula is true for $G^n(t) = G(t)$ by direct calculations (see Example 2). Now we assume that the formula is true for some n. Then by Corollary 2 we get

$$\frac{1}{G^{n+1}(t)} = \frac{1}{G^n(t)} + \frac{1}{G(t)} + \frac{t-1}{t+1}\frac{1}{A(t)}$$

where A is the Coxeter group with scheme  and

$$A(t) = -\frac{[2]\,[10]\,[1,0]^o_+}{[1,-2,1,0,-1,1,-1]^o_-}.$$

With the growth functions $G(t)$ and $G^n(t)$ (induction hypothesis) the result follows immediately by induction. □

Theorem 2 *The polynomials*

$$D_n(t) = [1, -2(n+1), 1, (n-1), 1, n, (n-1), 2, 2(n-1)]^e_+$$

have for all $n \geq 1$ the following properties.

(i) *$D_n(t)$ has exactly six pairs of roots on the unit circle.*
(ii) *$D_n(t)$ has exactly two pairs of real roots $(\tau_n, \frac{1}{\tau_n})$ and $(\gamma_n, \frac{1}{\gamma_n})$ with*

$$0 < \tau_n < \gamma_n < 1 < \frac{1}{\gamma_n} < \frac{1}{\tau_n}.$$

Furthermore, the sequence $\{\tau_n\}$ is strictly decreasing to 0 as $n \to \infty$.

Proof First of all, the roots of the palindromic polynomial D_n appear in pairs $(z, 1/z)$. For all $n \geq 1$ let

$$K_n(t^2) = (t-i)^{16} D_n\left(\frac{t+i}{t-i}\right).$$

For $u = t^2$ we get

$$\begin{aligned}K_n(u) &= nu^8 + (9+68n)u^7 + (135-892n)u^6 + (-2083+2108n)u^5\\ &\quad + (6115-10n)u^4 + (-5829-2116n)u^3 + (2500+900n)u^2\\ &\quad + (-225-60n)u + n + 1.\end{aligned}$$

By direct calculations we can localize the eight real roots of $K_n(u)$ in the following intervals. For example, we get

$$K_n(2.6) = K_n(\frac{26}{10}) = -\frac{42'061'824}{390'625}n - \frac{671'615'872}{78'125} < 0$$

for all $n \geq 1$. Indeed, the intervals are independent of the parameter n.

u	-86	-3.1	0	0.03	0.15	0.43	1	2.6	8.5
$sign(K_n(u))$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$

Hence $K_n(u)$ has exactly two negative and six positive real roots. It follows from Corollary 1 that $D_n(t)$ has exactly four real roots $\tau_n \le \gamma_n \le \frac{1}{\gamma_n} \le \frac{1}{\tau_n}$ and twelve properly complex roots on the unit circle.

For an estimation of the real roots we first observe that $D_n(0) = 1 > 0$, $D_n(\frac{1}{2}) = \frac{11977}{65536} - \frac{13415}{16384}n < 0$ and $D_n(1) = 4n > 0$. Hence we have $0 < \tau_n < \frac{1}{2} < \gamma_n < 1 < \frac{1}{\gamma_n} < \frac{1}{\tau_n}$. By a direct calculation we finally see that $D_n(\frac{1}{2n+2}) > 0$ and $D_n(\frac{1}{2n+1}) < 0$, that is

$$\frac{1}{2n+2} < \tau_n < \frac{1}{2n+1}$$

for all $n \ge 1$. □

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