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Spectral Series of the Schrödinger Operator with Delta-Potential on a Three-Dimensional Spherically Symmetric Manifold

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Abstract. The spectral series of the Schrödinger operator with a delta-potential on a three-dimensional compact spherically symmetric manifold in the semiclassical limit as $h \to 0$ are described.

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1. INTRODUCTION

Many physical and mathematical works treat Schrödinger operators with delta-potentials (point potentials, zero-range potentials). The model of point potentials can be used to describe short-range impurities, admixtures, defects, and similar phenomena in diverse systems. One of the first works in which the zero-range potentials were used to study the band spectrum of periodic systems was the paper [32], where a model of nonrelativistic electron moving in a rigid crystalline lattice was considered. Since then, the model has become very popular, especially in atomic and nuclear physics (see, e.g., [22, 7, 26–29, 31, 33, 9]).

A rigorous mathematical justification of the method of delta-potentials was given in [3], where it was suggested to use Krein's formula to describe the resolvents of operators with point perturbations. For an extensive bibliography of the works devoted to applications of the method of point potential, see the monographs [20, 21]. In [23–25, 12], using the theory of extensions, spectral properties of operators with delta-potentials and of operators close to them on singular spaces have been studied.

In the present paper, we describe the spectral series of the Schrödinger operator with delta-potential of the form $H = -\frac{h^2}{2}\Delta + \alpha\delta_{x_0}(x), \alpha \in \mathbb{R}$, in the semiclassical limit as $h \to 0$ on a three-dimensional compact surface admitting a spherical symmetry. For a large class of equations with smooth coefficients, the semiclassical theory was developed by Maslov (see, e.g., [13, 14]); in particular, this theory implies the following result. Let N be a Riemannian manifold and $V\colon N \to \mathbb{R}$ a smooth function (the potential). If the Hamiltonian system in T^*N defined by the Hamiltonian $(1/2)|p|^2 + V$ is completely integrable, then the corresponding Liouville tori Λ define semiclassical spectral series of the operator $H = -\frac{h^2}{2}\Delta + V(x)$ (here $x \in N$ and (x,p) stand for the standard coordinates on T^*N). Namely, the asymptotic behavior as $h \to 0$ of the eigenvalues of H is calculated from the Bohr–Sommerfeld–Maslov conditions

$$\frac{1}{2\pi h}\int_\gamma (p, dx) + (1/4)\mu(\gamma) = m \in \mathbb{Z}, \tag{1}$$

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where γ is an arbitrary cycle on Λ, μ stands for the Maslov index, and $m = O(1/h)$. The formal asymptotic behavior of the eigenfunctions (quasimodes) is of the form $\psi = K_\Lambda(1)$, where K_Λ stands for the Maslov canonical operator on the torus Λ satisfying quantization conditions.

In general, the construction of the canonical operator cannot be applied to operators with delta-potentials; at present, the geometry of the corresponding classical problem remains only slightly investigated. Below we describe the invariant Lagrangian manifolds corresponding to the spectral series of the above operator with delta-potential and obtain quantization conditions determining the asymptotic behavior of the eigenvalues. In general, these conditions are nonstandard; both for large and for small values of the coefficient α, they pass to equations of the form (1) with diverse values of the Maslov index μ; possibly, this indicates the presence of a more complicated geometric objects associated with the semiclassical theory of operators with singular coefficients. The paper continues the works [18] and [16]; in those papers, a similar problem was studied for the standard sphere (in this case, the spectrum can be calculated exactly) and for the two-dimensional surface of revolution.

2. SETTING OF THE PROBLEM

2.1. Spectral Problem

Consider the spectral problem

$$\Big(-\frac{h^2}{2}\Delta + \alpha\delta_{x_0}(x)\Big)\Psi = E\Psi, \quad x \in N, \quad \alpha \in \mathbb{R}, \tag{2}$$

where $\delta_{x_0}(x)$ stands for the Dirac delta function concentrated at the point x_0, in the semiclassical limit as $h \to 0$ on a three-dimensional manifold in $\mathbb{R}^4$,

$$N = (f(z)\cos\theta\cos\varphi, f(z)\cos\theta\sin\varphi, f(z)\sin\theta, z),$$

where $z \in [z_0, z_1]$, $0 \leqslant \varphi \leqslant 2\pi$, and $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$. We impose the following conditions on the function $f(z)$.

(1) $f(z_0) = f(z_1) = 0$, $f(z) > 0$ for $z \in (z_0, z_1)$;

(2) $f(z) = \sqrt{(z_1 - z)(z - z_0)}\,\omega(z)$, where $\omega(z)$ is a polynomial.

Under these assumptions, the surface N is an analytic manifold diffeomorphic to the three-dimensional sphere; the points x_0 and x_1 corresponding to the values $z = z_0, z_1$ of the parameter Z are the poles of this surface (and the delta function is concentrated at one of the poles).

Remark 1. Condition (2) can be weakened. Seemingly, it is sufficient to assume that f is analytic in a neighborhood of the closed interval $[z_0, z_1]$, except for the points z_0 and z_1 at which f has a root singularity.

Below we present a formal definition of the operator with δ-potential on the surface N.

2.2. Formal Definition of the Operator H

The operator

$$H = -\frac{h^2}{2}\Delta + \alpha\delta_{x_0}(x), \quad x \in N, \quad \alpha \in \mathbb{R}, \tag{3}$$

in the space $L_2(N)$ is defined by the construction of self-adjoint extensions (see [3]). Namely, H is constructed in such a way that the following conditions hold.

(1) The operator H is self-adjoint.

(2) On the functions vanishing at the point x_0, H coincides with the operator $H_0 = -\frac{h^2}{2}\Delta$, where Δ stands for the Laplace–Beltrami operator.

To be more precise, consider a self-adjoint operator H_0 with the domain $D(H_0) = W_2^2(N)$, where $W_2^2(N)$ stands for the Sobolev space of second order. Restricting the operator H_0 to functions $\psi(x)$ such that $\psi(x_0) = 0$, we obtain a symmetric operator $H_0|_{\psi(x_0)=0}$.

Definition 1. By the operator $H = -\frac{h^2}{2}\Delta + \alpha\delta_{x_0}(x)$ we mean the self-adjoint extension of the operator $H_0|_{\psi(x_0)=0}$.

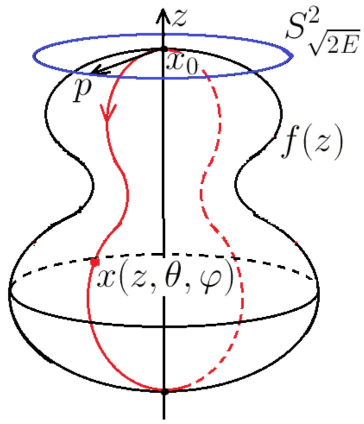

Fig. 1. The Lagrangian manifold corresponding to problem (2).

Remark 2. All extensions of this kind are parametrized by a single real parameter α, which can naturally be interpreted as the coefficient at the δ-potential in (3) (in particular, for $\alpha = 0$, we obtain $H = H_0$).

Every extension of this kind is defined by a boundary condition at the point x_0; to be more precise, the domain of the operator H consists of the functions of the form $\psi = \psi_0 + c_1 G(x, x_0; i) + c_2 G(x, x_0, -i)$, where $\psi_0 \in W_2^2(N)$, $\psi_0(x_0) = 0$, and $G(x, y, \lambda)$ stands for Green's function of the operator Δ, i.e., the integral kernel of the resolvent, $(\Delta - \lambda)^{-1} f = \int_M G(x, y; z) f(y)\Omega$ (Ω stands for the volume form on N).

The functions of the above form have a singularity at the point x_0; namely, the following expansion holds:

$$\psi(x) = -\frac{a}{4\pi} d(x, x_0)^{-1} + b + o(1), \tag{4}$$

where $a, b \in \mathbb{C}$ and $d(x, x_0)$ stands for the geodesic distance between x and x_0 on N. The domain of the extension H corresponding to the parameter α consists of the functions satisfying the boundary condition

$$a = \frac{2\alpha}{h^2} b. \tag{5}$$

3. FORMULATION OF THE RESULT

3.1. Description of the Lagrangian Manifold

The semiclassical asymptotic behavior of the eigenvalues of the operator H is evaluated from the quantization condition on the Lagrangian manifold which we shall now describe. Let $(x, p) \in T^*N$, where T^*N stands for the cotangent bundle of N, $x \in N$, and p a vector cotangent to N (the momentum). Consider the Hamiltonian system (the geodesic flow)

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p}, \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x}, \tag{6}$$

where $\mathcal{H} = |p|^2/2$, and consider the trajectories of the system, $x = X(\omega, t)$, $p = P(\omega, t)$, $\omega \in S^2_{\sqrt{2E}}$, $t \in \mathbb{R}$, given by the initial conditions

$$x(0) = x_0, \quad p(0) = \omega, \qquad \omega \in S^2_{\sqrt{2E}}, \quad |\omega| = \sqrt{2E}. \tag{7}$$

Here $S^2_{\sqrt{2E}}$ stands for the two-dimensional sphere of radius $\sqrt{2E}$ in the cotangent space at the point x_0. Thus, trajectories of the Hamiltonian system are issued from the point x_0 (at which the delta-potential is concentrated) along the surface N with the momentum "running" around the sphere $S^2_{\sqrt{2E}}$ (i.e., $p \in \Lambda_0$, where $\Lambda_0 = \{x = x_0, |p| = \sqrt{2E}\}$). These trajectories are contained in the manifold $\Lambda = \bigcup_t g_t \Lambda_0$ (g_t stands for the Hamiltonian phase flow) diffeomorphic to $S^2 \times S^1$ (see Fig. 1). It describes the classical motions corresponding to the quantum problem. The projections of the trajectories to N are geodesics.

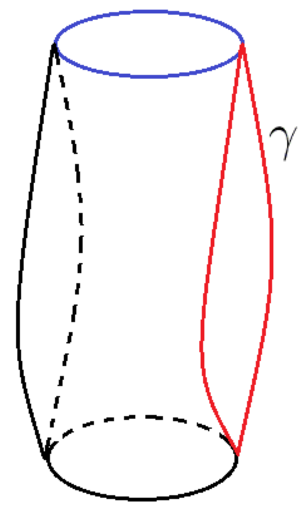

Fig. 2. Classical trajectories on the surface N corresponding to the quantum problem (2).

There is only one basis cycle γ on the manifold Λ, and it is precisely the integrals over γ that contribute to the spectrum. For γ, we can take the closed trajectory of the Hamiltonian system (6) with the initial conditions (7) (see Fig. 2).

The projection of Λ to the x-space is arranged as follows: for every point x of N, except for x_0 and x_1, there are two points of Λ of the form (x, p) and $(x, -p)$ that are projected to x. A two-dimensional sphere is projected to each of the points x_0 and x_1.

3.2. Formulation of the Result: the Quantization Condition

The quantization condition on the manifold Λ with respect to the cycle γ is just the desired equation for the spectrum of the problem. More precisely, the following assertion holds.

Theorem 1. *Let $Ch^{-\epsilon} < \frac{\alpha}{h^3} < Ch^{\epsilon}$ for some sufficiently small $\epsilon > 0$. Let there be a number $E = O(1)$ satisfying the quantization condition $\tan((1/(2h)) \oint_\gamma (p, dx) + O(h)) = 2h^3/(\sqrt{2E}\alpha)$, where γ stands for the cycle indicated above (the closed trajectory) on the Lagrangian manifold Λ. Then there is an eigenvalue E_0 of the operator H such that $|E - E_0| = o(h)$ as $h \to 0$.*

Remark 3. The formula $\tan((1/h) \int_{z_0}^{z_1} \sqrt{2E(f'^2 + 1)}dz + O(h)) = 2h^3/(\sqrt{2E}\,\alpha)$ is the explicit analytic form of the quantization condition.

Remark 4. The asymptotic behavior of the eigenfunction, outside an arbitrarily small neighborhood of x_0 independent of h, is of the form $K_{\tilde{\Lambda}}(1)$, where K stands for the Maslov canonical operator and $\tilde{\Lambda}$ is a noncompact Lagrangian manifold obtained from Λ by deleting the two-dimensional sphere projected to the point x_0 (this manifold is obviously homeomorphic to the cylinder $S^2 \times \mathbb{R}$).

3.3. Jump of the Maslov Index

Consider the limit cases of the quantization condition described in the theorem. Let $\alpha h^{-3} \to 0$; then the quantization conditions acquire the standard form $(2\pi h)^{-1} \oint_\gamma (p, dx) = k + 1/2$ (note that the Maslov index of the cycle γ is equal to two). Suppose now that $\alpha h^{-3} \to \infty$; then we have $(2\pi h)^{-1} \oint_\gamma (p, dx) = k$. This equation also has the form of the Bohr–Sommerfeld–Maslov condition; however, in this case, the "Maslov index" of the cycle γ is equal to zero. Thus, when passing through the critical value $\alpha = O(h^3)$, we face a jump of the integral-valued invariant, which coincides with the Maslov index in the case of a smooth potential; the presence of the delta function leads to the change of this invariant by 2. This possibly indicates the existence of some topological construction (still unclear to us) which generalizes the Maslov canonical operator to the case of singular coefficients.

Remark 5. The theorem remains valid even if the above condition on α fails to hold. However, outside the interval indicated above, the quantization condition always has the form of the Bohr–Sommerfeld–Maslov rule; moreover, in the course of proving the theorem, it can happen in general that several corrections to the asymptotic behavior are to be constructed (the corresponding procedure is similar to that described below), and the number of corrections depends on the order of the ratio α/h^3.

The remaining part of the paper is devoted to the proof of Theorem 1.

4. PROOF OF THEOREM 1

4.1. Separation of Variables

We construct an asymptotic solution of the problem

$$H\psi(x) = E\psi(x) + o(h) \tag{8}$$

in the form

$$\psi(x) = e_1\psi_1(x) + e_2\psi_2(x), \tag{9}$$

where e_1, e_2 is a partition of unity, $e_1 + e_2 = 1$, where the support of the function $\psi_1(x)$ is placed near x_0 and the support of $\psi_2(x)$ is outside x_0. At the points of intersection of the supports, the functions must coincide mod $o(h)$.

The metric on the surface of revolution N is of the form

$$g_{ij} = \begin{pmatrix} f'^2(z)+1 & 0 & 0 \\ 0 & f^2(z) & 0 \\ 0 & 0 & \cos^2\theta f^2(z) \end{pmatrix}, \qquad \sqrt{g} = f^2(z)\sqrt{f'^2(z)+1}\,\cos\theta.$$

The Laplace operator in the coordinates is represented as follows:

$$\Delta = \frac{1}{f^2(z)\sqrt{f'^2(z)+1}}\frac{\partial}{\partial z}\frac{f^2(z)}{\sqrt{f'^2(z)+1}}\frac{\partial}{\partial z} + \frac{1}{f^2(z)}\Delta_0,$$

where Δ_0 stands for the Laplace operator on the two-dimensional sphere,

$$\Delta_0 = \frac{1}{\cos\theta}\left(\frac{\partial}{\partial\theta}\cos\theta\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\right).$$

We seek a solution of (9) in the form $\psi = u(z)Y$, where $-\Delta_0 Y = m(m+1)Y$. After the substitution, equation (8) becomes

$$u''(z) + \left(\frac{2f'(z)}{f(z)} - \frac{f'(z)f''(z)}{f'^2(z)+1}\right)u'(z) + (f'^2(z)+1)\left(\frac{2E}{h^2} - \frac{m(m+1)}{f^2(z)}\right)u(z) = 0. \tag{10}$$

4.2. Structure of Solutions in a Neighborhood of Singular Points

Equation (10) has two regular singular points, z_0 and z_1. Let us find out the local structure of solutions in a neighborhood of these points; to be definite, let $z \to z_0$. Substitute the expansion $f(z) = \sqrt{z-z_0}\sum_{j=0}^{\infty} a_j(z-z_0)^j$ into equation (10) and consider the terms of the order of $(z-z_0)^{-1}$ only. We obtain

$$u''(z) - \frac{2}{z_0-z}u'(z) + \frac{a_0^2}{4(z_0-z)}\left(\frac{2E}{h^2} - \frac{m(m+1)}{a_0^2(z_0-z)}\right)u(z) = 0. \tag{11}$$

To evaluate the characteristic exponents (see, e.g., [17]), we substitute $u(z) = (z - z_0)^\rho r(z)$ into equation (11), where $r(z) = \sum_{k=0}^{\infty} r_k (z - z_0)^k$. Equating the coefficient at $(z - z_0)^{\rho-2}$ to zero, we obtain $\rho(\rho - 1) + 2\rho - (m(m + 1))/4 = 0$, i.e., $\rho = (-1 \pm \sqrt{1 + m(m + 1)})/2$, and, using [17], we see that equation (10) has two linearly independent solutions of the following form: $R_1(z) = (z - z_0)^{\rho_1}\varphi_1(z)$ and $R_2(z) = (z - z_0)^{\rho_2}\varphi_2(z)$, where $\varphi_1(z)$ and $\varphi_2(z)$ are holomorphic in a neighborhood of z_0.

We must find a solution of (10) which is analytic in a neighborhood of the point z_1 and has a singularity of the form $(z - z_0)^{-1}$ in a neighborhood of the point z_0. It is clear that such a solution can exist only if $m = 0$; in what follows, we restrict ourselves to this case. Then we have $R_1(z) = \varphi_1(z)$ and $R_2(z) = (z - z_0)^{-1}\varphi_2(z)$. The structure of solutions in a neighborhood of the point z_1 is similar.

Remark 6. If $m \neq 0$, then the equation (4.3) admits a solution vanishing at z_0. A solution of this kind defined an eigenfunction of the Laplace–Beltrami operator (without the delta-potential), and the asymptotic behavior can be constructed in this case in accordance with the standard lines of reasoning [14].

4.3. The Standard Equation in a Neighborhood of z_0

By the formula for the arc length on a surface defined parametrically, we have $d(z, z_0) = \int_z^{z_0} \sqrt{f'^2(z) + 1}\, dz$, $z \in (z_1, z_0)$, where $d(z, z_0) \sim (z_0 - z)\sqrt{f'^2(z) + 1}$ near z_0.

Let us study the fundamental system of solutions of equation (11) with regard to the condition $m = 0$,

$$P''(z) - \frac{2}{z_0 - z}P'(z) + \frac{a_0^2}{4(z_0 - z)}\frac{2E}{h^2}P(z) = 0. \tag{12}$$

Make the change $t = (a_0\sqrt{2E}/h)\sqrt{z_0 - z}$; then equation (12) becomes

$$P_{tt}(t) + \frac{3}{t}P_t(t) + P(t) = 0. \tag{13}$$

This is a Bessel equation (see, e.g., [2]), and its solution near the point z_0 is a linear combination of the Bessel function and the first-order Neumann function multiplied by a power of the argument, namely,

$$P(t) = At^{-1}N_1(t) + Bt^{-1}C_1(t). \tag{14}$$

When constructing asymptotic formulas for the solution of equation (10) in a neighborhood of the point z_0, we shall use these very functions.

4.4. Asymptotics of a Solution in a Neighborhood of the Point z_0*. A Langer Variable*

Consider the general form of equation (10),

$$u''(z) + f'(z)\left(\frac{2f'(z)}{f(z)} - \frac{f'(z)f''(z)}{f'^2(z) + 1}\right)u'(z) + (f'^2(z) + 1)\frac{2E}{h^2}u(z) = 0. \tag{15}$$

This is an equation with two regular singular points z_0 and z_1. For convenience, write

$$p(z) = (z - z_0)\left(\frac{2f'(z)}{f(z)} - \frac{f'(z)f''(z)}{f'^2(z) + 1}\right), \tag{16}$$

$$q(z) = (z - z_0)(f'^2(z) + 1), \tag{17}$$

and then equation (15) becomes

$$(z - z_0)u''(z) + p(z)u'(z) + q(z)\frac{2E}{h^2}u(z) = 0, \tag{18}$$

where $p(z)$ and $q(z)$ are analytic functions as $z \to z_0$.

Using the Langer–Wasow approach, we seek an asymptotic solution of equation (18) in the form

$$u(z) = F(\tau) + \cdots, \tag{19}$$

where $\tau = S(z)/h^2$ is a Langer variable and $S(z)$ is an unknown function analytic in a neighborhood of the point z_0. Choose functions F and S in such a way that u have a singularity of a desired form at the point z_0. Substituting (19) into (18), we obtain

$$\tau F_{\tau\tau} + \frac{p(z)S(z)}{S'(z)(z-z_0)} F_\tau + \frac{2ES(z)q(z)}{S'^2(z)(z-z_0)} F + \frac{S''S}{(S')^2} F_\tau + \cdots = 0. \tag{20}$$

Since $S = h^2\tau$, it follows that the last summand is small (for the bound, see below); deleting it and writing

$$n(z) = \frac{S(z)}{S'(z)} \left(\frac{2f'(z)}{f(z)} - \frac{f'(z)f''(z)}{f'^2(z)+1} \right), \tag{21}$$

$$k(z) = \frac{2ES(z)}{S'^2(z)} (f'^2(z)+1), \tag{22}$$

we represent equation (20) in the form

$$\tau F_{\tau\tau} + n(z)F_\tau + k(z)F = 0. \tag{23}$$

An arbitrary solution of equation (20) is a linear combination of the following functions:

$$F_1 = \tau^{\frac{1-n(z)}{2}} N_{n(z)-1}(2\sqrt{k(z)\tau}), \tag{24}$$

$$F_2 = \tau^{\frac{1-n(z)}{2}} C_{n(z)-1}(2\sqrt{k(z)\tau}), \tag{25}$$

where $N_\nu(t)$ stands for the Neumann function and $C_\nu(t)$ for the Bessel function of order ν (see, e.g.,[2]).

For the function F to depend on τ only, the quantities $k(z)$ and $n(z)$ must be constant. One may assume that $k(z) = 1$; this equation leads to the Hamilton–Jacobi equation of the form $2ES(z)S'^{-2}(z)(f'^2(z)+1) = 1$. Whence $S(z) = \big((1/2)\int_{z_0}^z \sqrt{2E(f'^2(z)+1)dz}\big)^2$. Note that this function is holomorphic in a neighborhood of the point z_0 and that $S(z_0) = 0$ and $S'(z_0) \neq 0$. In general, the function $n(z)$ is not constant; however, it can readily be seen that changing $n(z)$ by $n(z_0)$ in equation (23) leads to adding a small summand. Indeed,

$$(n(z) - n(z_0))F_\tau = \frac{n(z)-n(z_0)}{S(z)} S(z)F_\tau = h^2 \frac{n(z)-n(z_0)}{S(z)} \tau F_\tau$$

(a rigorous estimate is given below).

The direct calculation shows that $n(z_0) = 2$; thus, we represent the function F in the form

$$F(\tau) = A_1\tau^{-\frac{1}{2}} N_1(2\sqrt{\tau}) + A_2\tau^{-\frac{1}{2}} C_1(2\sqrt{\tau}), \tag{26}$$

where A_1 and A_2 are constants. The relationship between these constants can be found from the boundary condition at the point z_0. Note that

$$\sqrt{\tau} = \sqrt{\frac{S(z)}{h^2}} = \frac{1}{2h} \int_{z_0}^z \sqrt{2E(f'^2(z)+1)}dz. \tag{27}$$

Let us substitute (27) into (26); we obtain an asymptotic solution in a neighborhood of the point z_0 in the form

$$u_1(z) = A_1\tau^{-\frac{1}{2}} N_1\Big(\frac{\sqrt{2E}}{h} \int_{z_0}^z \sqrt{f'^2(z)+1}dz\Big) + A_2\tau^{-\frac{1}{2}} C_1\Big(\frac{\sqrt{2E}}{h} \int_{z_0}^z \sqrt{f'^2(z)+1}dz\Big).$$

Let $A_2 = 1$; considering the asymptotic behavior of the Bessel and Neumann functions as $\tau \to 0$ (see, e.g., [1]), we obtain $u_1|_{z\to z_0} = -A_1(h/(\pi\sqrt{2E}))(1/d(z_0,z)) - 1 + o(1)$. Let us use the expansion (4) and the boundary condition (5),

$$A_1 = -\sqrt{2E}\alpha/(2h^3). \tag{28}$$

Substitute (28) into (26) and write out the asymptotic solution of (15) that belongs to the domain of the operator H, because it satisfies the boundary condition (5),

$$u_1(z) = -\frac{\sqrt{2E}\alpha}{2h^3}\tau^{-\frac{1}{2}} N_1(2\sqrt{\tau}) + \tau^{-\frac{1}{2}} C_1(2\sqrt{\tau}), \tag{29}$$

where $2\sqrt{\tau} = \frac{\sqrt{2E}}{h}\int_{z_0}^{z}\sqrt{(f'^2(z)+1)}dz$.

Since we consider the case $m = 0$, the corresponding function $\psi_1(x)$ coincides with u_1 (i.e., does not depend on the angles θ and φ).

We claim now that $H(e_1\psi_1 + e_2\psi_2) = E(e_1\psi_1 + e_2\psi_2) + o(h)$ on the interval on which $\psi_1 \neq 0$ and $\psi_2 = 0$. The function $\psi_1 = u_1$ satisfies the equation $H\psi_1 = E\psi_1$ up to two summands, each of which is of the form $h^2 w(z)\tau F_\tau$, where the function $w(z)$ is holomorphic in a neighborhood of z_0 and $\tau = S(z)/h^2$. Let us estimate the norm of this function in L_2,

$$\|h^2 w(z)\tau F_\tau\|_{L_2}^2 \leqslant Ch^4\int_{z_0}^{z_0+\delta}(\tau F_\tau)^2 w^2 dz \leqslant h^{3-2\epsilon}C_1\delta,$$

because $|\tau F_\tau| \leqslant \frac{Ch^{-\epsilon}}{\sqrt{h}}$. Hence, $\|h^2 w\tau F_\tau\| \leqslant h^{3/2-\epsilon}\sqrt{C_1\delta} = o(h)$, as was to be proved.

Let us state the main result of this subsection.

Lemma 1. *The function ψ_1 (see (29)) satisfies the relation $H\psi_1 = E\psi_1 + o(h)$ in some neighborhood of x_0 independent of h.*

4.5. The Solution Outside a Neighborhood of z_0

An equation of the form (4.8) has a solution outside a neighborhood of the singular point z_0, which is analytic in a neighborhood of the point z_1, and the asymptotic formula for this solution can be constructed by using the complex WKB method up to $O(h^2)$ (see [17]). When using this method, the condition $H\psi_2 = E\psi_2 + o(h)$ holds by construction. The solution can be represented, outside an arbitrary small neighborhood of Z_1 independent of h, in the form ([17])

$$\psi_2(z) = \nu_1\omega_1(z) + \nu_2\omega_2(z), \tag{30}$$

where ν_1 and ν_2 are constants and ω_1 and ω_2 are fundamental asymptotic solutions of equation (15) as $h \to 0$. Namely, $\omega_{1,2} = \omega_{1,2}^o(z)(1 + O(h))$ and

$$\omega_{1,2}^o(z) = (f'^2(z)+1)^{\frac14}\left(\frac{f(z)}{f(z_1)}\Big(\frac{f'^2(z)+1}{f'^2(z_1)+1}\Big)^{-\frac12}\right)^{-\frac12}\exp\Big(\pm i\frac{\sqrt{2E}}{h}\int_{z_1}^{z}\sqrt{f'^2(z)+1}\,dz\Big).$$

Let us find constants ν_1 and ν_2 that are the components of an eigenvector of the monodromy matrix of (4.8) which corresponds to the point z_1. The asymptotic behavior of this matrix is also evaluated in [17]; it is of the form $\left(\begin{smallmatrix} 2 & D \\ -(D)^{-1} & 0\end{smallmatrix}\right)$, where $D = -\exp\big(\pm(\sqrt{2E}/h)\int_\beta\sqrt{f'^2(t)+1}\,dt\big)$. Here β stands for a contour enclosing the point z_1. This gives $\left(\begin{smallmatrix}\nu_1\\ \nu_2\end{smallmatrix}\right) = \left(\begin{smallmatrix}-CD\\ C\end{smallmatrix}\right)$, where C is a constant. Further, we substitute the expressions for the functions and constants into (30) and obtain an asymptotic solution outside a neighborhood of the point z_0,

$$\psi_2(z) = \chi(z)\cos\Big(\frac{\sqrt{2E}}{h}\int_\beta\sqrt{f'^2(t)+1}\,dt + \frac{\sqrt{2E}}{h}\int_{z_0}^{z}\sqrt{f'^2(t)+1}\,dt\Big), \tag{31}$$

$$\text{where } \chi(z) = 2C(f'^2(z)+1)^{\frac14}\Big(\frac{f(z)}{f(z_1)}\Big(\frac{f'^2(z)+1}{f'^2(z_0)+1}\Big)^{-\frac12}\Big)^{-\frac12}.$$

Let us state the main result of this subsection.

Lemma 2. *There is a function $\psi_2(x)$ defined outside an arbitrary neighborhood of x_0 independent of h and satisfying the relation $\|H\psi_2 - E\psi_2\|_{L_2} \leqslant A_1Ch^2$ outside such a neighborhood, where A_1 stands for a constant. Outside an arbitrary neighborhood of x_1 independent of h, this function is of the form* (31).

4.6. Gluing Local Asymptotic Solutions Together. Quantization Condition

The asymptotic behavior of the Bessel and Neumann functions as $z \to \infty$ is well known (see, e.g., [1]), namely, $C_\alpha(x) \to \sqrt{2/(\pi x)}\cos(x-\alpha\pi/2-\pi/4)$ and $N_\alpha(x) \to \sqrt{2/(\pi x)}\sin(x-\alpha\pi/2-\pi/4)$.

Let us substitute these formulas into the solution (29) obtained in a neighborhood of the point z_0, which gives

$$\psi_1(z)|_{\tau\to\infty} = -\frac{\sqrt{2E}\alpha}{2h^3}\Big[\tau^{-1/2}\sqrt{\frac{2}{2\pi\sqrt{\tau}}}\sin(2\sqrt{\tau}-3\pi/4)+\tau^{-1/2}\sqrt{\frac{2}{2\pi\sqrt{\tau}}}\cos(2\sqrt{\tau}-3\pi/4)\Big]+O(\tau^{-3/4}).$$

After manipulations, we obtain

$$\psi_1(z)|_{\tau\to\infty} = \sqrt{A^2+1}\left[\frac{\tau^{-\frac{3}{4}}}{\sqrt{\pi}}\cos\left(\frac{\sqrt{2E}}{h}\int_{z_0}^{z}\sqrt{f'^2(z)+1}\,dz-\frac{3\pi}{4}-\operatorname{arcctg}A\right)+O(\tau^{-\frac{3}{4}})\right], \quad (32)$$

where $A = -\sqrt{2E}\alpha/(2h^3)$.

Equating the arguments of the functions (31) and (32) outside neighborhoods of the points z_0 and z_1, we obtain the equation $\tan((1/(2h))\oint_\gamma(p,dx)+O(h)) = 2h^3/(\sqrt{2E}\alpha)$ for the spectrum (in the form described in Theorem 1), where γ is the aforementioned cycle on the Lagrangian manifold Λ and $(p,dx) = \sqrt{2E(f'^2(z)+1)}\,dz$. If the equation holds, then the functions ψ_1 and ψ_2 coincide mod $O(h^{3/2-\epsilon})$ outside arbitrarily small neighborhoods of the points z_0 and z_1.

Suppose that $(H-E)\psi(x) = r$. Let us estimate the L_2-norm of the remainder term r on the entire interval $[z_0,z_1]$. Substitute $\psi(x) = e_1\psi_1(x)+(1-e_1)\psi_2$ into equation (8). We have $\|r\|^2 = \int_{z_1}^{z_0}|r|^2dz \leqslant I_1^2+I_2^2+I_3^2$, where $I_1 = \int_{z_0}^{z_0+\delta}|r|^2dz$, $I_2 = \int_{z_0+\delta}^{z_1-\delta}|r|^2dz$, and $I_3 = \int_{z_1-\delta}^{z_1}|r|^2dz$.

As was proved above, $I_1 \leqslant h^{3/2-\epsilon}A_2$ and $I_3 \leqslant h^{2-\epsilon}A_1$, where A_1 and A_2 are constants. On the interval $(z_0+\delta, z_1-\delta)$, we have $\psi_1(z)\neq 0$, $\psi_2(z)\neq 0$, and $\psi_1 = \psi_2+O(h^{3/2-\epsilon})$, and thus, at least $I_2 \leqslant h^{3/2-\epsilon}A_3$, where A_3 is a constant. Therefore, $\|r\| \leqslant \sqrt{h^{3-2\epsilon}A_2^2+h^{3-2\epsilon}A_3^2+h^{4-2\epsilon}4A_1^2} \leqslant h^{3/2-\epsilon}\sqrt{A_2^2+A_3^2+hA_1^2} = o(h)$. Thus, we have constructed a solution such that

$$(H-E)\psi(x) = o(h). \quad (33)$$

To complete the proof of Theorem 1, we use the following lemma.

Lemma 3 (see, e.g., [14]). *Let H be a self-adjoint operator and let $(H-E)\psi = f$, where $\psi \in D(H)$, $\|\psi\| = 1$. Then there is a point E_0 belonging to the spectrum of the operator H for which $|E-E_0| \leqslant \|f\|$.*

This lemma, together with (33), proves the theorem.

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