Ark. Mat., **45** (2007), 71–82
DOI: 10.1007/s11512-006-0029-2

On the behavior of strictly plurisubharmonic functions near real hypersurfaces

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Abstract. We describe the behavior of certain strictly plurisubharmonic functions near some real hypersurfaces in $\mathbf{C}^n$, $n\geq 3$. Given a hypersurface we study continuous plurisubharmonic functions which are zero on the hypersurface and have Monge–Ampère mass greater than one in a one-sided neighborhood of the hypersurface. If we can find complex curves which have sufficiently high contact order with the hypersurface then the plurisubharmonic functions we study cannot be globally Lipschitz in the one-sided neighborhood.

1. Introduction

The potential theory for the complex Monge–Ampère operator is not as well understood as potential theory for the Laplace operator. In the theory of one complex variable the subharmonic functions are important. In several complex variables the subharmonic functions are not the correct class to study since this class is not invariant under holomorphic coordinate changes. One should study the class of subharmonic functions which are invariant under holomorphic coordinate changes, that is the class of plurisubharmonic functions which we shall denote $\mathcal{PSH}$. The subharmonic functions can be characterized as the functions satisfying $\Delta u\geq 0$ in the distribution sense. In the theory for plurisubharmonic functions the complex Monge–Ampère operator plays the role the Laplace operator does in the theory of subharmonic functions. However, the complex Monge–Ampère operator is nonlinear and this makes the definition of the operator for nonsmooth plurisubharmonic functions delicate. For C^2-functions the definition is

$$\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right).$$

In the language of currents a plurisubharmonic function is a function u which has the property that

$$dd^c u = 2i \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \, dz_j \wedge d\bar{z}_k$$

is a positive current of bidegree $(1,1)$. Here $d=\partial+\bar{\partial}$ and $d^c=i(\partial-\bar{\partial})$. Notice that when u is a C^2-function we have no problems defining $(dd^c u)^n$, the nth exterior power of $dd^c u$, and get

$$(dd^c u)^n = 4^n n! \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) d\lambda,$$

where $d\lambda$ denotes Lebesgue measure. Here we have made the natural identification between the volume form and the Lebesgue measure. So for a plurisubharmonic function u of class C^2 we see that $(dd^c u)^n$ is a positive Borel measure. Therefore it is natural to try to extend the definition of $(dd^c u)^n$ for nonsmooth u as a positive Borel measure. In [2] Bedford and Taylor obtained such a definition for locally bounded plurisubharmonic functions. It is known that $(dd^c u)^n$ cannot be defined as a positive Borel measure for all plurisubharmonic functions, see Kiselman's paper [14]. Recently in [9] Cegrell has given a definition of $(dd^c u)^n$ with domain of definition as large as possible.

Assume that μ is a positive Borel measure on a domain Ω and φ is some function on the boundary of Ω. Central to pluripotential theory is the study of the Dirichlet problem

$$\begin{cases} (dd^c u)^n = \mu & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

In this paper we shall always have $\mu = f\, d\lambda$, where f is at least a continuous function of z. We shall be considering the question of how regularity of f implies regularity of u.

First consider $(dd^c u)^n = 0$. One realizes that this equation can have very irregular solutions since any plurisubharmonic function which depend on $n-1$ variables only solves the equation. However, if one demands that the boundary data be continuous then it can be proved in certain domains, as it was done by Walsh in [16], that the solution is continuous. Put

$$PB_\varphi(z) = \sup\Big\{ v(z) \,;\, v \in \mathcal{PSH}(\Omega) \text{ and } \limsup_{z\to z_0} v(z) \le \varphi(z_0) \text{ for all } z_0 \in \partial\Omega \Big\}.$$

It had been observed by Bremermann in [7] that if the problem

$$\begin{cases} (dd^c u)^n = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

is solvable the solution is *the Perron–Bremermann envelope*

$$(PB_\varphi)^*(z) = \limsup_{\zeta\to z} PB_\varphi(\zeta).$$

The result Walsh obtained is the following.

Theorem 1.1. *Suppose that Ω is a bounded domain in $\mathbf{C}^n$ and $\varphi\in C(\partial\Omega)$. Assume that*

$$\liminf_{z\to z_0} PB_\varphi(z) = \limsup_{z\to z_0} PB_\varphi(z) = \varphi(z_0)$$

for all $z_0\in\partial\Omega$. Then $PB_\varphi\in C(\overline{\Omega})$.

High order regularity is harder for the equation $(dd^c u)^n=0$. We give the example $u(z_1,z_2)=\max\{|z_1|^2-\frac{1}{2},|z_2|^2-\frac{1}{2},0\}^2$. This function is plurisubharmonic, satisfies $(dd^c u)^2=0$, is smooth on the boundary of the unit ball but is not smooth in the unit ball. For more examples of lack of high order regularity see Bedford's and Fornæss' paper [1]. The first result on high order regularity was obtained in 1985 by Caffarelli, Kohn, Nirenberg and Spruck in [8]. Note that positivity of f is crucial in view of the example above. Actually, we state only a special case of the theorem that Caffarelli, Kohn, Nirenberg and Spruck proved.

Theorem 1.2. *Suppose that Ω is a bounded, strongly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary. Let $f\in C^\infty(\overline{\Omega}\times\mathbf{R})$ be a strictly positive function which is increasing in the second variable. Suppose that $\varphi\in C^\infty(\partial\Omega)$. Then the problem*

$$\begin{cases} \det\left(\dfrac{\partial^2 u}{\partial z_j\partial\bar{z}_k}\right)=f(z,u) & \text{in } \Omega,\\ u=\varphi & \text{on } \partial\Omega,\\ u\in\mathcal{PSH}(\Omega)\cap C^2(\Omega)\cap C(\overline{\Omega}), \end{cases}$$

has a unique solution. Moreover $u\in C^\infty(\overline{\Omega})$.

Remark 1.3. When we say that a function $g\colon\mathbf{R}\to\mathbf{R}$ is *increasing* we mean that $x\le x'$ implies that $g(x)\le g(x')$. If $x<x'$ implies that $g(x)<g(x')$ we say that g is *strictly increasing*. Finally *smooth* will always mean C^∞-smooth.

A domain Ω in $\mathbf{C}^n$ is called *hyperconvex* if for every $z_0\in\partial\Omega$ there exists $v\in\mathcal{PSH}(\Omega)$ such that $v<0$ and $\lim_{z\to z_0} v(z)=0$. Kerzman and Rosay showed in [13] that for bounded domains it is equivalent to say that there exists a smooth bounded strictly plurisubharmonic exhaustion function ρ in Ω, that is a strictly negative plurisubharmonic ρ satisfying $\lim_{z\to z_0\in\partial\Omega}\rho(z)=0$. This was improved upon by

Błocki in [4] so that we can choose a smooth bounded strictly plurisubharmonic exhaustion function ρ satisfying

$$\det\left(\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}\right)\geq 1.$$

If we do not demand that the solutions be smooth we can get the following result, which was proved by Błocki in [3].

Theorem 1.4. *Let Ω be a bounded, hyperconvex domain in $\mathbf{C}^n$. Assume that f is nonnegative, continuous and bounded in Ω. Suppose that φ is continuous on $\partial\Omega$ and that it can be continuously extended to a plurisubharmonic function on Ω. Then there exists a unique solution to the following problem*

$$\begin{cases} (dd^cu)^n=f(z) & \textit{in } \Omega,\\ u=\varphi & \textit{on } \partial\Omega,\\ u\in\mathcal{PSH}(\Omega)\cap C(\overline{\Omega}). \end{cases}$$

Błocki has also given a sufficient condition for a smooth solution in convex domains in [6]. This result has also been announced in [5].

Theorem 1.5. *Let Ω be a bounded, convex domain in $\mathbf{C}^n$. Assume that f is a strictly positive, smooth function in Ω such that*

$$\sup\left\{\left|\frac{\partial f^{1/n}}{\partial x_l}(z)\right|; z\in\Omega\right\}<\infty.$$

Then there exists a unique solution to the following problem

$$\begin{cases} \det\left(\dfrac{\partial^2 u}{\partial z_j\partial\bar{z}_k}\right)=f(z) & \textit{in } \Omega,\\ \lim\limits_{z\to z_0} u(z)=0 & \textit{for all } z_0\in\partial\Omega,\\ u\in\mathcal{PSH}(\Omega)\cap C^\infty(\Omega). \end{cases}$$

Note that a convex domain is hyperconvex since a convex domain has a bounded exhaustion function which is convex and convex functions are plurisubharmonic. We also see that a hyperconvex domain is pseudoconvex since $\tilde{\rho}(z)=-\log(-\rho(z))$ is plurisubharmonic and $\lim_{z\to z_0}\tilde{\rho}(z)=\infty$, where ρ is a bounded plurisubharmonic exhaustion function for the hyperconvex domain.

In [12] the author studied the regularity of plurisubharmonic solutions to the problem

$$\begin{cases} \det\left(\dfrac{\partial^2 u}{\partial z_j\partial\bar{z}_k}\right)=f(z,u) & \text{in } \Omega,\\ \lim\limits_{z\to z_0} u(z)=0 & \text{for all } z_0\in\partial\Omega, \end{cases}$$

where $f \in C^\infty(\overline{\Omega} \times \mathbf{R})$ is a strictly positive function which is increasing in the second variable. This problem was studied in a certain type of hyperconvex domains. The proof of the main theorem in that paper [12, Theorem 5.1] rests on an a priori estimate of the C^2-norms of solutions by Schulz [15]. Błocki has discovered an error in the argument leading to this estimate and therefore the proof of Theorem 5.1 in [12] is not complete. If we examine the argument in [12] we find that the following can be proven.

Theorem 1.6. *Let Ω be a bounded hyperconvex domain in $\mathbf{C}^n$ and $f \in C^\infty(\overline{\Omega} \times \mathbf{R})$ be a strictly positive function which is increasing in the second variable. The problem*

$$\begin{cases} \det\left(\dfrac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) = f(z,u) & \text{in } \Omega, \\ \lim\limits_{z \to z_0} u(z) = 0 & \text{for all } z_0 \in \partial\Omega, \end{cases}$$

has a unique strictly plurisubharmonic solution u which is globally Lipschitz if we can find a smooth plurisubharmonic function that is globally Lipschitz and ρ satisfies $\lim_{z \to z_0} \rho(z) = 0$ for all $z_0 \in \partial\Omega$ and

$$\det\left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\right) \geq 1.$$

The purpose of this paper is to better understand when we can find such a plurisubharmonic function ρ. We have the following two comparison principles which will be useful in what follows. A proof of the first can be found in [2] and a proof of the second in [12].

Lemma 1.7. *Suppose that Ω is a bounded domain in $\mathbf{C}^n$ and $v, w \in C(\overline{\Omega}) \cap \mathcal{PSH}(\Omega)$. Assume that $(dd^c v)^n \geq (dd^c w)^n$. Then*

$$\min\{w(z) - v(z)\,;\, z \in \overline{\Omega}\} = \min\{w(z) - v(z)\,;\, z \in \partial\Omega\}.$$

Lemma 1.8. *Let Ω be a bounded domain in $\mathbf{C}^n$. Assume that $f \in C(\Omega \times \mathbf{R})$ is a nonnegative function which is increasing in the second variable. Suppose that $v, w \in C(\overline{\Omega}) \cap \mathcal{PSH}(\Omega)$. Then*

$$(dd^c w)^n \leq f(z,w), \quad f(z,v) \leq (dd^c v)^n$$

and $v \leq w$ on $\partial\Omega$ implies that $v \leq w$ in Ω.

2. Local behavior of strictly plurisubharmonic functions near real hypersurfaces

In this section our analysis will be local and therefore we shall formulate our results for smooth real hypersurfaces in $\mathbf{C}^n$ rather than for domains in $\mathbf{C}^n$ with smooth boundary. For a smooth real hypersurface M we can find a real-valued function $\rho \in C^\infty(U)$, $U \subseteq \mathbf{C}^n$, such that $M = \{z \in U; \rho(z) = 0\}$ and $d\rho \neq 0$ on M. We say that ρ is a *defining function for* M. Given such a ρ we define $M^- = \{z \in U; \rho(z) < 0\}$.

We shall investigate the behavior of the normal derivative of any smooth strictly plurisubharmonic function u which is zero on a given smooth real hypersurface and satisfies

$$\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) \geq 1.$$

In general such a plurisubharmonic function does not exist for a given smooth real hypersurface. Namely if the Levi form

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k$$

has at least one strictly negative eigenvalue on the complex tangent space of M,

$$T_p^{\mathbf{C}}(M) = \left\{\xi \in \mathbf{C}^n; \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p)\xi_j = 0\right\},$$

such a plurisubharmonic function u would violate

$$\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) \geq 1.$$

On the other hand if the Levi form is positive semidefinite such a function exists. This is because such a smooth real hypersurface has a defining function which is plurisubharmonic in M^-, see Diederich's and Fornæss' paper [11]. Intersect M^- with a small ball. This domain is hyperconvex. In a hyperconvex domain such a plurisubharmonic function exists, which was proved by Błocki in [4]. In [12] the author introduced a notion that he called the non-precipitousness condition [12, Definition 1.7]. We introduce a local version of this condition.

Definition 2.1. Let M be a smooth real hypersurface, $p \in M$ and ρ be a defining function for M. Assume that the Levi form is positive semidefinite on $T_q^{\mathbf{C}}(M)$ for all $q \in M$. We say that $p \in M$ satisfies *the local non-precipitousness condition*, or for short *the local NP-condition*, if we can find an open neighborhood U of p and a smooth plurisubharmonic function u defined on M^- that is globally Lipschitz and satisfies $\lim_{z\to z_0\in M} u(z)=0$ and

$$(dd^c u)^n \geq 1.$$

We shall now investigate the behavior of first derivatives of defining functions of ellipsoids. Let $a=(a_1, ..., a_n)\in\mathbf{R}^n, a_j>0$, and put

$$\Omega_a = \left\{ z\in\mathbf{C}^n\,;\sum_{j=1}^n \frac{|z_j|^2}{a_j^2} < 1 \right\}.$$

A defining function for Ω_a is $\rho_a=\left(\sum_{j=1}^n |z_j|^2/a_j^2\right)-1$. We see that

$$\det\left(\frac{\partial^2\rho_a}{\partial z_j\partial\bar{z}_k}\right)=\prod_{j=1}^n a_j^{-2}$$

and

$$\frac{\partial\rho_a}{\partial z_j}=\frac{1}{a_j}\left(\frac{\bar{z}_j}{a_j}\right),\quad \frac{\partial\rho_a}{\partial\bar{z}_j}=\frac{1}{a_j}\left(\frac{z_j}{a_j}\right).$$

Let $\tilde{\rho}_a=\left(\prod_{j=1}^n a_j^{2/n}\right)\rho_a$. We get

$$\det\left(\frac{\partial^2\tilde{\rho}_a}{\partial z_j\partial\bar{z}_k}\right)=1$$

and

$$\frac{\partial\tilde{\rho}_a}{\partial z_j}=\frac{\prod_{l=1}^n a_l^{2/n}}{a_j}\left(\frac{\bar{z}_j}{a_j}\right),\quad \frac{\partial\tilde{\rho}_a}{\partial\bar{z}_j}=\frac{\prod_{l=1}^n a_l^{2/n}}{a_j}\left(\frac{z_j}{a_j}\right).$$

In particular we see that

$$\frac{\partial\tilde{\rho}_a}{\partial z_1}(a_1,0,...,0)=\frac{\partial\tilde{\rho}_a}{\partial\bar{z}_1}(a_1,0,...,0)=a_1^{2/n-1}\prod_{j=2}^n a_j^{2/n}.$$

Thus we see that the normal derivative at this boundary point depends on the lengths of the semi-axes. Now we investigate the boundary behavior of a smooth plurisubharmonic function satisfying

$$\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) \geq 1$$

in the polydisk D^n, $n \geq 3$, and $\lim_{z \to z_0} u(z) = 0$ for all $z_0 \in \partial D^n$. Let us study the normal derivative of u at the boundary point $(1, 0, ..., 0)$. Fix $a_2, ..., a_n$ so that $\Omega_a \subseteq D^n$ for all small a_1. We see that $\widetilde{\Omega}_a = \Omega_a + (1 - a_1, 0, ..., 0) \subseteq D^n$ and $(1, 0, ..., 0) \in \partial\widetilde{\Omega}_a \cap \partial D^n$. By the comparison principle, Lemma 1.7, we get that

$$\tilde{\rho}_a(z_1 + a_1 - 1, z_2, ..., z_n) \geq u(z) \quad \text{in } \widetilde{\Omega}_a.$$

Therefore

$$\lim_{t \to 1-} \frac{u(1, 0, ..., 0) - u(t, 0, ..., 0)}{1 - t} \geq 2a_1^{2/n-1} \prod_{j=2}^{n} a_j^{2/n}.$$

This estimate holds for all small a_1 and if we let a_1 tend to zero we see that

$$\lim_{t \to 1-} \frac{u(1, 0, ..., 0) - u(t, 0, ..., 0)}{1 - t} = \infty$$

and we conclude that D^n does not satisfy the NP-condition for $n \geq 3$. Notice that this argument only works if $n \geq 3$. However, using a different argument the author has proved that D^2 does not satisfy the NP-condition, see [12]. We now use the interplay between a_1 and $a_2, ..., a_n$ to describe the behavior of strictly plurisubharmonic functions near some real hypersurfaces which do not necessarily contain complex lines. We shall use the order of contact between a hypersurface M and complex curves in the ambient space $\mathbf{C}^n$. A *complex curve* is a holomorphic mapping γ from an open neighborhood of $0 \in \mathbf{C}$ to $\mathbf{C}^n$ such that $\partial\gamma \neq 0$. The *order of contact* between M and γ at $p \in M$ is l if $d_M(q) \leq C d(p, q)^l$ near p and l is the largest such number.

Theorem 2.2. *Let M be a hypersurface in $\mathbf{C}^n$ which is pseudoconvex at $p \in M$. Assume that there are complex curves $\gamma_2, ..., \gamma_n$ whose order of contact with M is $2l_2, ..., 2l_n$, respectively, and that $\gamma_2', ..., \gamma_n'$ are linearly independent. Suppose that*

$$\sum_{j=2}^{n} \frac{1}{l_j} < n - 2.$$

Then $p \in M$ does not satisfy the local NP-condition, see Definition 2.1.

Proof. After a rotation and a translation we can assume that p is the origin and that $\nabla\rho(p) = (1, 0, ..., 0)$, where all γ_j', $2 \leq j \leq n$, are tangent to $M = \{z \in \mathbf{C}^n; \rho(z) = 0\}$.

We want to change coordinates holomorphically so that in new coordinates we can approximate M by

$$\partial\widetilde{\Omega}_a=(-a_1,0,...,0)+\left\{z\in\mathbf{C}^n\,;\sum_{j=1}^{n}\frac{|z_j|^2}{a_j^2}=1\right\}$$

near p. Define the invertible holomorphic mapping

$$F(\zeta_1,\zeta_2,...,\zeta_n)=(\zeta_1,0,...,0)+\gamma_2(\zeta_2)+...+\gamma_n(\zeta_n).$$

In the coordinates $(\zeta_1,...,\zeta_n)$ the curves $\gamma_2,...,\gamma_n$ are the coordinate axes. Working in these coordinates we show that $p\in M$ does not satisfy the local NP-condition. We now want to put ellipsoids in M^- so that the intersection of the boundary of the ellipsoid and M^- is the origin. We can choose $a_2,...,a_n$ so that $\widetilde{\Omega}_a=\Omega_a+(-a_1,0,...,0)\subseteq M^-$ and $\partial\widetilde{\Omega}_a\cap\partial M^-=\{0\}$. Let $\tilde{\rho}_a(\zeta)$ be the plurisubharmonic function that satisfies

$$\det\left(\frac{\partial\tilde{\rho}_a}{\partial\zeta_j\partial\bar{\zeta}_k}\right)=1$$

in $\widetilde{\Omega}_a$ and $\tilde{\rho}_a=0$ on $\partial\widetilde{\Omega}_a$. Using that the order of contact of γ_j with M is $2l_j$ one sees that we can choose $a_j=K_ja_1^{1/2l_j}$ for $j=2,...,n$ and some constants K_j, $0<K_j<1$. For these $a_1,...,a_n$ we get that

$$\frac{\partial\tilde{\rho}_a}{\partial\zeta_1}(0)=a_1^{2/n-1+\sum_{j=2}^{n}(1/nl_j)}\prod_{j=2}^{n}K_j^{2/n}$$

and

$$\frac{\partial\tilde{\rho}_a}{\partial\bar{\zeta}_1}(0)=a_1^{2/n-1+\sum_{j=2}^{n}(1/nl_j)}\prod_{j=2}^{n}K_j^{2/n}.$$

We see that a negative smooth plurisubharmonic function which satisfies

$$\det\left(\frac{\partial^2u}{\partial\zeta_j\partial\bar{\zeta}_k}\right)\geq 1$$

and $\lim_{\zeta\to\zeta_0}u(\zeta)=0$ for all $\zeta_0\in M$ also satisfies $u(\zeta)\leq\tilde{\rho}_a(\zeta_1+a_1,\zeta_2,...,\zeta_n)$ in $\widetilde{\Omega}_a$. We get

$$\lim_{t\to 0-}\frac{u(0)-u(t,0,...,0)}{-t}\geq 2a_1^{2/n-1+\sum_{j=2}^{n}(1/nl_j)}\prod_{j=2}^{n}K_j^{2/n}.$$

If $2/n-1+\sum_{j=2}^{n}(1/nl_j)<0$ and we let a_1 tend to zero we see that

$$\lim_{t\to 0-}\frac{u(0)-u(t,0,...,0)}{-t}=\infty.$$

Hence $0\in M$ does not satisfy the local NP-condition if

$$\frac{2}{n}-1+\sum_{j=2}^{n}\frac{1}{nl_j}<0.$$

This condition can be rewritten as

$$\sum_{j=2}^{n}\frac{1}{l_j}<n-2. \quad \square$$

Remark 2.3. The condition

$$\sum_{j=2}^{n}\frac{1}{l_j}<n-2$$

can also be understood in the following way. If at least two of the γ_j's have order of contact with M greater than 2 then it is satisfied.

Remark 2.4. It is not clear what happens when the inequality

$$\sum_{j=2}^{n}\frac{1}{l_j}<n-2$$

is not satisfied. It cannot be a necessary and sufficient condition for the local NP-condition to fail. This is because the bidisk D^2 does not satisfy the NP-condition, see [12], and in this case the inequality is not met.

3. Discussion

In Remark 2.4 we noted that Theorem 2.2 only gives a sufficient condition for the NP-condition to fail. In fact Theorem 2.2 gives us no information for domains in $\mathbf{C}^2$. In $\mathbf{C}^n$, $n\geq 3$, the theorem gives us plenty of examples of domains which do not satisfy the NP-condition. The problem is to find the curves $\gamma_2, ..., \gamma_n$. A naive guess to decide which order of contact a complex curve can have with M might be to look at the Levi form of M or maybe the Taylor expansion of a defining function ρ. However, the problem of determining the optimal order of contact, denoted $\Delta^1_{\mathrm{reg}}(M,p)$, turns out to be harder than that. The example $M=\{z\in\mathbf{C}^2; \operatorname{Re} z_1-|z_1|^2+\operatorname{Re} z_2^2+|z_2|^4=0\}$ shows this. A guess might be that $\Delta^1_{\mathrm{reg}}(M,0)=4$. However, M contains the complex curve $(\zeta^2, i\zeta)$ and hence $\Delta^1_{\mathrm{reg}}(M,0)=\infty$. A good reference for methods for deciding best possible order of contact is D'Angelo's book [10]. Now it should be noted that in order to apply

Theorem 2.2 the curves $\gamma_2, ..., \gamma_n$ need not have optimal order of contact. Looking at the Taylor expansion of a defining function ρ it is not hard to derive a lower bound for $\Delta^1_{\text{reg}}(M,p)$. For example if $M \subseteq \mathbf{C}^3$ has a defining function ρ whose Taylor expansion is

$$z_1 + \bar{z}_1 + |z_2|^4 + |z_3|^6 + o(|z|^6)$$

then the curves $\gamma_2(\zeta) = (\zeta^4, \zeta, 0)$ and $\gamma_3(\zeta) = (\zeta^6, 0, \zeta)$ have order of contact 4 and 6, respectively, with M. We see this by studying the Taylor expansion of $K_j(\zeta) = \rho(\gamma_j(\zeta))$, $j=2,3$, around zero. We now use Theorem 2.2 to see that $0 \in M$ does not satisfy the local NP-condition.

Acknowledgements. The author would like to thank Christer Kiselman and Tobias Ekholm for their help during the writing of this paper. I would also like to thank Frank Kutzschebauch for making it possible for me to visit the University of Bern.

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Received December 19, 2005
published online February 8, 2007