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Thurston equivalence of topological polynomials

by

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Contents

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1. Introduction

Consider the "rabbit" polynomial $f_R(z) \approx z^2 + (-0.1226 + 0.7449i)$, whose critical point 0 is on a periodic orbit of length 3. Up to affine transformations, there are exactly two other polynomials with the same action on post-critical points (with the same *ramification graph*), called the "corabbit" $f_C \approx z^2 + (-0.1226 - 0.7449i)$ and the "airplane" $f_R \approx z^2 - 1.7549$. Furthermore, by a result of W. Thurston (see below), every branched covering with the same ramification graph is equivalent to precisely one of f_R, f_C or f_A.

Consider now a Dehn twist T of $\mathbf{C}$ around the two non-critical values of the f_R-orbit of 0. The map $T^m f_R$ is again a branched covering, and it has the same ramification graph as f_R; therefore it is equivalent (i.e. conjugate up to homotopies) to one of f_R, f_C or f_A. Which one?

This question was asked by J. Hubbard; see [9]. The answer, as we shall show (Theorem 4.7), is the following. Write m in base 4, as

$$m = \sum_{i=0}^{\infty} m_i 4^i,$$

with $m_i \in \{0, 1, 2, 3\}$ and almost all $m_i = 0$ if m is non-negative, and almost all $m_i = 3$ if m is negative. If one of the m_i's is 1 or 2, then $T^m f_R$ is equivalent to f_A. Otherwise, it is equivalent to f_R if m is non-negative, and to f_C if m is negative.

For example, $T^{-1} f_R$ and $T^{-4} f_R$ are equivalent to f_C, since

$$-1 = \dots 333 \quad \text{and} \quad -4 = \dots 3330.$$

Consider now the polynomial $f_i = z^2 + i$, whose finite critical point has orbit

$$0 \longmapsto i \longmapsto i - 1 \longmapsto -i \longmapsto i - 1.$$

A branched covering with that ramification graph is either equivalent to f_i, or to f_{-i}, or is not equivalent to any rational map (it is *obstructed*). A. Douady and J. Hubbard ask ([4, p. 293]) to determine, as a function of a Dehn twist D, when $f_i \cdot D$ is obstructed, and if not, whether it is equivalent to f_i or to f_{-i}. The answer (see Theorem 6.1) depends on the image of D in a finite group of order 100.

Thurston's theorem does not tell us when two obstructed maps are equivalent. We may however also answer that question: Corollary 6.11 shows that there are infinitely many inequivalent obstructed maps with the same ramification graph as f_i, and gives an algorithm to determine equivalence among obstructed maps.

The first construction of infinitely many non-equivalent Thurston maps with the same ramification graph was presented in [3, Proposition 2.12].

1.1. Thurston's theorem

Consider more generally a branched covering f of $\mathbf{S}^2$, with set of critical points C_f, and let P_f be its *post-critical set*:

$$P_f = \bigcup_{n\geqslant 1} f^{\circ n}(C_f).$$

Let us suppose that P_f is finite, in which case f is called *post-critically finite*, or a *Thurston map*. Two Thurston maps f and g are *equivalent* if there exist orientation-preserving homeomorphisms $\phi_0, \phi_1\colon (\mathbf{S}^2, P_f) \to (\mathbf{S}^2, P_g)$ such that ϕ_0 and ϕ_1 are isotopic relative to P_f, and $\phi_0 f = g\phi_1$. Recall also that a branched covering is a *topological polynomial* if $f^{-1}(\infty) = \{\infty\}$.

A *multicurve* is a system $\Gamma = \{\gamma_1, ..., \gamma_n\}$ of simple, closed, disjoint, non-homotopic, non-peripheral curves on $\mathbf{S}^2 \setminus P_f$. Here a curve is called *peripheral* if one of the two parts into which it divides the sphere contains less than two post-critical points. The multicurve Γ is *stable* if for all $\gamma \in \Gamma$, all non-peripheral elements of $f^{-1}(\gamma)$ are homotopic to elements of Γ. There is then an induced map

$$\begin{aligned} f_\Gamma\colon \mathbf{R}^\Gamma &\longrightarrow \mathbf{R}^\Gamma, \\ \gamma_i &\longmapsto \sum_{\delta \in f^{-1}(\gamma_i)} \frac{[\delta]}{\deg f|_\delta}, \end{aligned}$$

where $[\delta]$ is, if it exists, the element of Γ homotopic to δ, and is 0 otherwise.

THEOREM 1.1. (Thurston's criterion; see [4]) *A Thurston map f with hyperbolic orbifold*(1) *is equivalent to a rational function if and only if the spectral radius of f_Γ is less than 1 for all stable multicurves Γ.*

In that case, the rational function equivalent to f is unique up to conjugation by a Möbius transformation.

A stable multicurve Γ is an *obstruction* if the spectral radius of f_Γ is greater than or equal to 1. Therefore, Thurston's theorem says that a Thurston map is equivalent to a rational map if and only if there are no obstructions.

If the Thurston map f is a topological polynomial, then the structure of obstructions is better understood. Namely, every Thurston obstruction of a topological polynomial contains a *Levy cycle* (see [2]), i.e. a multicurve $\{\gamma_0, \gamma_1, ..., \gamma_{n-1}\}$ such that the only non-peripheral component $\widetilde{\gamma}_{i-1}$ of $f^{-1}(\gamma_i)$ is homotopic to γ_{i-1} and the maps $f\colon \widetilde{\gamma}_{i-1} \to \gamma_i$ are of degree 1 for every i (here the indices i are considered modulo n).

(1) For a definition of a hyperbolic orbifold see [4].

Thurston's theorem does not, in principle, provide an algorithmic answer to the question when a Thurston map f is equivalent to a rational function; nor does it construct the rational function. Many attempts were made to that end, notably [9] and [6]. The present paper may be seen as another step in this direction.

1.2. Sketch of the method

Given a post-critically finite branched covering f, we associate with it a finitely generated group acting faithfully on a rooted tree. Its action is given recursively by self-similar tree isometries. This group, the *iterated monodromy group* of f, is an invariant for the branched covering up to equivalence.(2) In favourable ("contracting") cases, the "nucleus" of its action is a finite-state automaton characterising the group. This gives an effective method to solve J. Hubbard's question for any given m.

Let P be the post-critical set of f. The (pure) mapping class group $\mathcal{G}_\mathcal{C}$ of $\mathcal{C}=\mathbf{C}\setminus P$ acts on the branched coverings with post-critical set P by pre- and post-composition. The action of $\mathcal{G}_\mathcal{C}$ can be pushed through the recursion to give a self-map $\overline{\psi}$ (almost a homomorphism) of $\mathcal{G}_\mathcal{C}$ such that $f\cdot g$ and $f\cdot\overline{\psi}(g)$ are combinatorially equivalent. The map $\overline{\psi}$ is contracting in the case of the rabbit polynomial: there is a finite set $L\subset\mathcal{G}_\mathcal{C}$ such that for any $g\in\mathcal{G}_\mathcal{C}$ we have $\overline{\psi}^n(g)\in L$ for some n.

It remains to compute the iterated monodromy group of $f\cdot g$ for all $g\in L$ to obtain a general answer to Douady's and Hubbard's questions.

1.3. Outline of the paper

We explain in §2 the construction of iterated monodromy groups, the fundamental notion of "contracting" actions, and nuclei. We apply that construction to the study of Thurston equivalence and study the post- and pre-composition actions of the mapping class group on the covering in §3.

We consider in more detail in §4 the rabbit polynomial, for which we obtain the recursions as described above, and we specialise in §4.3 to Dehn twists around the rabbit's ears, obtaining an answer to Hubbard's question.

We re-express the solution in §5 in more classical terms of iterations of the pull-back map on Teichmüller and moduli spaces. This gives another approach to the problem, which is however less algorithmical.

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(2) It is a complete invariant for combinatorial equivalence, if equipped with additional algebraic data; see [8, Theorem 6.5.2] and Proposition 3.1.

We finally consider in §6 and §7 the other two classes of degree-two topological polynomials with three finite post-critical points.

§6 deals with the polynomials whose ramification graph is the same as that of z^2+i (period 2, preperiod 1). In particular, we classify the obstructed examples up to combinatorial equivalence. The last section considers the case of period 1 and preperiod 2.

1.4. A remark on notation

We compose in most cases transformations as if they acted on the right: in a product $f\cdot g$ the transformation f acts before g. In particular, if g_1 and g_2 are elements of a group, then $g_1^{g_2}=g_2^{-1}g_1g_2$. However, we sometimes have to also consider left actions. Therefore, if we write $f\circ g$, then we mean left composition, in which g acts before f.

2. Iterated monodromy groups

We give here an overview of techniques and results in self-similar actions and iterated monodromy groups. More details and proofs can be found in [8].

2.1. Partial self-coverings and monodromy action

A *covering* is a continuous surjective map $f:\mathcal{C}_1\to\mathcal{C}$ of topological spaces, with the property that for every $z\in\mathcal{C}$ there exists an open neighborhood $\mathcal{U}$ of z such that $f^{-1}(\mathcal{U})$ is a disjoint union of open sets each of which is mapped homeomorphically onto $\mathcal{U}$ by f. We say that the covering f is *d-fold* if there exists $d\in\mathbf{N}$ such that $|f^{-1}(z)|=d$ for all $z\in\mathcal{C}$.

A *partial self-covering* is a covering map $f:\mathcal{C}_1\to\mathcal{C}$ such that $\mathcal{C}_1$ is an open subspace of a path connected and locally path connected space $\mathcal{C}$.

Let $f:\mathcal{C}_1\to\mathcal{C}$ be a d-fold partial self-covering. Then the nth iteration $f^n:\mathcal{C}_n\to\mathcal{C}$ of f is a d^n-fold partial self-covering with domain $\mathcal{C}_n$, usually smaller than $\mathcal{C}_1$.

Choose a basepoint $t\in\mathcal{C}$. Then the fundamental group $\pi_1(\mathcal{C},t)$ acts naturally by monodromy on each of the sets $f^{-n}(t)$. The action that it induces on the disjoint union $\bigsqcup_{n\geqslant 0} f^{-n}(t)$ is called the *iterated monodromy action*.

The set $\bigsqcup_{n\geqslant 0} f^{-n}(t)$ has a natural structure of a rooted d-regular tree. The root of this tree is the point $t\in f^{-0}(t)=\{t\}$, and each vertex $z\in f^{-n}(t)$, for $n\geqslant 1$, is connected to the vertex $f(z)\in f^{-(n-1)}(t)$.

The *iterated monodromy group* of the partial self-covering f, denoted by $\mathrm{IMG}(f)$, is the quotient of the fundamental group $\pi_1(\mathcal{C},t)$ by the kernel of the iterated monodromy action. This kernel, by definition, consists of the loops $\gamma\in\pi_1(\mathcal{C},t)$ such that for every n all f^n-preimages of γ are also loops.

In other words, the iterated monodromy group of f is the group of all automorphisms of the tree $\bigsqcup_{n\geqslant 0} f^{-n}(t)$ which are induced by elements of $\pi_1(\mathcal{C},t)$.

The iterated monodromy group can be effectively computed using the following recursive formula. Choose an alphabet X of d letters. Then every rooted d-regular tree is isomorphic to the tree X^* of finite words over the alphabet X. In this tree the empty word is the root and every word $v\in X^*$ is connected by edges to all the words of the form vx, where $x\in X$ is a letter.

We will use throughout the paper the following notation. If γ is a path in $\mathcal{C}$ and z is an f-preimage of the startpoint of γ, then

$$f^{-1}(\gamma)[z] \tag{1}$$

denotes the unique f-preimage of γ starting at z.

Choose a bijection $\Lambda\colon X\to f^{-1}(t)$ and a path ℓ_x in $\mathcal{C}$ starting at t and ending in $\Lambda(x)$ for every $x\in X$. Extend the bijection $\Lambda\colon X\to f^{-1}(t)$ to an isomorphism of the rooted trees $\Lambda\colon X^*\to\bigsqcup_{n\geqslant 0} f^{-n}(t)$ inductively by the condition that

$$\Lambda(xv) \text{ be the end of the path } f^{-|v|}(\ell_x)[\Lambda(v)].$$

It is not hard to prove that Λ is a well-defined isomorphism of the rooted trees (see [8, Proposition 5.2.1]). We then have the following recursive description of the iterated monodromy action (see [8, Proposition 5.2.2]).

Proposition 2.1. *Define an action of $\pi_1(\mathcal{C},t)$ on X^* by conjugating the iterated monodromy action by Λ. Then for all $\gamma\in\pi_1(\mathcal{C},t)$, $v\in X^*$ and $x\in X$ we have*

$$(xv)^\gamma = y(v^{\ell_x\gamma_x\ell_y^{-1}}),$$

where $\Lambda(y)$ is the end of the path $\gamma_x=f^{-1}(\gamma)[\Lambda(x)]$.

We multiply the paths in the natural order: in a product $\gamma_1\gamma_2$ the path γ_1 is followed before the path γ_2. Therefore $\pi_1(\mathcal{C},t)$ and $\mathrm{IMG}(f)$ act from the right on the tree of preimages and on X^*. (Note that this is different from the convention in [8].)

2.2. Wreath recursions

The recursive formula in Proposition 2.1 can be interpreted either as a *wreath recursion*, or as the description of an automaton.

A *wreath recursion* is a homomorphism Φ from a group G to the wreath product $G\wr\mathfrak{S}(X)$, where $\mathfrak{S}(X)$ is the symmetric group on X. Recall that the *wreath product* $G\wr\mathfrak{S}(X)$ is, by definition, the semidirect product $G^X\rtimes\mathfrak{S}(X)$, where $\mathfrak{S}(X)$ acts on the direct power G^X by permutation of the coordinates. If $X=\{1,...,d\}$, we write the elements of the wreath product in the form $\langle\!\langle g_1,g_2,...,g_d\rangle\!\rangle\pi$, where $\langle\!\langle g_1,g_2,...,g_d\rangle\!\rangle$ is an element of the direct power G^X and π is an element of the symmetric group $\mathfrak{S}(X)$. The elements of the wreath product are multiplied according to the rule

$$\langle\!\langle g_1,g_2,...\,g_d\rangle\!\rangle\varrho\langle\!\langle h_1,h_2,...,h_d\rangle\!\rangle\tau=\langle\!\langle g_1h_{1^\varrho},g_2h_{2^\varrho},...,g_dh_{d^\varrho}\rangle\!\rangle\varrho\tau.$$

We denote by $g|_x$ the xth coordinate of $\Phi(g)$, for $g\in G$ and $x\in X$. Inductively, we put

$$g|_{vx}=(g|_v)|_x$$

for all $v\in X^*$ and $x\in X$.

We let G act on X by post-composing Φ with the natural homomorphism

$$G\wr\mathfrak{S}(X)\longrightarrow\mathfrak{S}(X).$$

We extend this action to the *associated action of G on X^**, defined by the recursion

$$(xv)^g=x^g(v^{g|_x}).$$

The associated action and restrictions satisfy the following relations:

$$(gh)|_v=g|_vh|_{v^g}\quad\text{and}\quad g|_{vw}=g|_v|_w \tag{2}$$

for all $g,h\in G$ and $v,w\in X^*$.

If $X=\{1,2,...,d\}$, then Proposition 2.1 can be expressed in terms of wreath recursions in the following way.

PROPOSITION 2.2. *The action of $\pi_1(\mathcal{C},t)$ on X^* is the action associated with the wreath recursion $\Phi\colon\pi_1(\mathcal{C},t)\to\pi_1(\mathcal{C},t)\wr\mathfrak{S}(X)$ given by*

$$\Phi(\gamma)=\langle\!\langle \ell_1\gamma_1\ell_{k_1}^{-1},\ell_2\gamma_2\ell_{k_2}^{-1},...,\ell_d\gamma_d\ell_{k_d}^{-1}\rangle\!\rangle\varrho,$$

where $\gamma_i=f^{-1}(\gamma)[\Lambda(i)]$, $\Lambda(k_i)$ is the endpoint of γ_i, and ϱ is the permutation $i\mapsto k_i$.

2.3. Virtual endomorphisms

A wreath recursion $\Phi\colon G\to G\wr\mathfrak{S}(X)$ can be constructed using the *associated virtual endomorphism.* A virtual endomorphism $\phi\colon G\dashrightarrow G$ is a homomorphism $\operatorname{Dom}\phi\to G$ from a subgroup of finite index into G. The subgroup $\mathrm{Dom}\,\phi$ is called the *domain* of the virtual endomorphism ϕ.

If $\Phi\colon G\to G\wr\mathfrak{S}(X)$ is a wreath recursion and $x_0\in X$ is a letter, then the domain of the associated virtual endomorphism $\phi=\phi_{x_0}$ is the stabilizer of the letter x_0 (with respect to the action of G on X); and the virtual endomorphism is defined as the restriction

$$\phi(g)=g|_{x_0}.$$

It follows from the relations (2) that $\phi\colon\mathrm{Dom}\,\phi\to G$ is a homomorphism.

Suppose that the action of G on $X=\{1,2,...,d\}$ is transitive, choose $x_0=1$, and choose some $r_i\in G$, for all $i\in X$, such that $1^{r_i}=i$. Write $h_i=r_i|_1$. Then the wreath recursion can be reconstructed by the formula

$$\Phi(g)=\langle\!\langle h_1^{-1}\phi(r_1gr_{k_1}^{-1})h_{k_1},h_2^{-1}\phi(r_2gr_{k_2}^{-1})h_{k_2},...,h_d^{-1}\phi(r_dgr_{k_d}^{-1})h_{k_d}\rangle\!\rangle\varrho,$$

where $h_i=r_i|_1$, ϱ is the permutation $i\mapsto k_i$, and the indices k_i are uniquely defined by the condition $r_igr_{k_i}^{-1}\in\mathrm{Dom}\,\phi$.

If we change the elements h_i, or if we change $\{r_i\}_{i\in X}$ to another left coset transversal, then we change Φ to its post-composition by an inner automorphism of $G\wr\mathfrak{S}(X)$, and therefore conjugate the associated actions of G on X^* by an automorphism of X^*. More precisely (see [8, §2.3 and §2.5]), there will exist an automorphism Δ of the d-regular rooted tree, conjugating G into the new action; and Δ will satisfy a recursion of the form

$$\Phi(\Delta)=\langle\!\langle g_1\Delta,...,g_d\Delta\rangle\!\rangle\pi,$$

where $\langle\!\langle g_1,...,g_d\rangle\!\rangle\pi$ is the element defining the inner automorphism of $G\wr\mathfrak{S}(X)$.

2.4. Contraction

A wreath recursion $\Phi\colon G\to G\wr\mathfrak{S}(X)$ is *contracting* if there is a finite set $\mathcal{N}\subset G$ such that for every $g\in G$ there exists $n_0\in\mathbf{N}$ with $g|_v\in\mathcal{N}$ for all words $v\in X^*$ of length greater than n_0. This property ensures that many calculations regarding arbitrary elements of G can be reduced, via the wreath recursion, to considerations on a finite set. For example, the "word problem" (determining if a given product of N generators is trivial) can be solved in polynomial time in contracting groups.

If G is generated by a finite symmetric set $S=S^{-1}$, then a subset $\mathcal{N}\subset G$ satisfies the above condition if and only if $1\in\mathcal{N}$ and there exists $n_0\in\mathbf{N}$ such that

$$(gs)|_v\in\mathcal{N}$$

for all $g \in \mathcal{N}$, $s \in S$ and words $v \in X^*$ of length greater than n_0.

The smallest set $\mathcal{N}$ satisfying the condition of these definitions is called the *nucleus* of the contracting action.

If $\phi: G \dashrightarrow G$ is a virtual endomorphism of a finitely generated group, then its *spectral radius* is equal to

$$r(\phi) = \limsup_{n\to\infty} \sqrt[n]{\limsup_{\substack{g\in \mathrm{Dom}\,\phi^n \\ l(g)\to\infty}} \frac{l(\phi^n(g))}{l(g)}},$$

where $l(g)$ denotes the word length of g with respect to some fixed generating set of G.

PROPOSITION 2.3. ([8, Proposition 2.11.11]) *Let $\Phi: G \to G \wr \mathfrak{S}(X)$ be a wreath recursion and let ϕ be an associated virtual endomorphism.*

If Φ is contracting, then $r(\phi) < 1$. If the action of G on X^* is transitive on every level X^n (in particular, if it is transitive on X^1 and ϕ is onto) and $r(\phi) < 1$, then the wreath recursion Φ is contracting.

It is proved in [8, Theorem 5.5.3] that if a partial self-covering $f: \mathcal{C}_1 \to \mathcal{C}$ is expanding, then the associated wreath recursion Φ_f on $\pi_1(\mathcal{C})$, as defined in Proposition 2.2, is contracting.

2.5. Automata

It is convenient to describe wreath recursions and nuclei of contracting wreath recursions in terms of *automata*.

A subset $A \subset G$ is *state-closed* if for every $g \in A$ and $x \in X$ we have $g|_x \in A$. It is easy to see that the nucleus of a contracting wreath recursion is a state-closed set.

If A is a state-closed set, then we interpret it as an automaton, which, when it is in a state $g \in A$ and it reads a letter $x \in X$ on the input tape, prints the letter x^g on the output tape and goes to the state $g|_x$. Then the automaton A with initial state g transforms any word $v \in X^*$ to the word v^g and thus describes the associated action of G on X^*.

We draw state-closed sets as graphs (*Moore diagrams*) with vertex set A. The vertex $g \in A$ is marked by its image in $\mathfrak{S}(X)$, i.e. by the permutation it induces on X, and for every $g \in A$ and $x \in X$ we draw an arrow from g to $g|_x$ labeled by x. Then the graph completely describes the restriction of the wreath recursion to A.

We will always have in our paper $X = \{0, 1\}$; then the symmetric group $\mathfrak{S}(X)$ consists of two elements: 1 and $\sigma = (0, 1)$. If an element of G^X or $\mathfrak{S}(X)$ is trivial, then we usually do not write it, so that $\langle\langle g_0, g_1\rangle\rangle = \langle\langle g_0, g_1\rangle\rangle 1$ and $\sigma = \langle\langle 1, 1\rangle\rangle\sigma$. The elements of $G \wr \mathfrak{S}(X)$ are either written $g = \langle\langle g_0, g_1\rangle\rangle$ (they are then called *inactive*), or $g = \langle\langle g_0, g_1\rangle\rangle\sigma$ (and are then called *active*). All computations in $G\wr\mathfrak{S}(X)$ are based then on two rules:

$$\langle\!\langle g_0,g_1\rangle\!\rangle\cdot\langle\!\langle h_0,h_1\rangle\!\rangle=\langle\!\langle g_0h_0,g_1h_1\rangle\!\rangle\quad\text{and}\quad\sigma\cdot\langle\!\langle g_0,g_1\rangle\!\rangle=\langle\!\langle g_1,g_0\rangle\!\rangle\sigma.$$

In drawing Moore diagrams, we indicate active states by a grey dot and inactive states by a white dot.

3. Post-critically finite topological polynomials

3.1. Homotopy in terms of wreath recursion

Let $P_0\subset\mathbf{C}$ be a finite set of complex numbers and suppose that a map $f_P\colon P_0\to P_0$ and a point $c_0\in P_0$ are given such that $P_0=\{f_P^{\circ n}(c_0):n\geqslant 0\}$. Let $\mathcal{F}$ be the set of all degree-two orientation-preserving branched coverings $f\colon\widehat{\mathbf{C}}\to\widehat{\mathbf{C}}$, with critical points c_0 and ∞, whose restriction to P_0 coincides with f_P and $f^{-1}(\infty)=\{\infty\}$.

Let us denote by P the set of *post-critical points* of f, i.e. $P=\{f^{\circ n}(c_0):n\geqslant 1\}\cup\{\infty\}$. Note that $P=P_0\cup\{\infty\}$ if c_0 is periodic (i.e. if $f_P^{\circ n}(c_0)=c_0$ for some positive n) and $P=(P_0\setminus\{c_0\})\cup\{\infty\}$ otherwise. The *ramification graph* of f is the directed graph with vertex set P and an arrow from p to $f(p)$ for each $p\in P$.

If $f\in\mathcal{F}$, then $\mathrm{IMG}(f)$ is defined as the iterated monodromy group of $f\colon\mathcal{C}_1\to\mathcal{C}$, where $\mathcal{C}=\widehat{\mathbf{C}}\setminus P$ and $\mathcal{C}_1=f^{-1}(\mathcal{C})$.

Let us denote by $\mathfrak{F}$ the set of homotopy classes (within $\mathcal{F}$) of branched coverings $f\in\mathcal{F}$; in other words, $\mathfrak{F}$ is the set of path-connected components of the space $\mathcal{F}$.

Choose a basepoint $t\in\mathcal{C}$, imagined close to infinity. Note that $\pi_1(\mathcal{C},t)$ is a free group of rank $|P|-1$; we can take its generators to be loops going around the finite post-critical points in the positive direction.

Let a be a small simple closed loop in $\widehat{\mathbf{C}}$ going around ∞ in the negative direction and based at t (on $\mathbf{C}$ it is a big loop going in the positive direction around all the finite post-critical points). The loop a divides the sphere $\widehat{\mathbf{C}}$ into two parts. One contains ∞ and the other contains $P\setminus\{\infty\}$.

We may assume, after changing the map f to a homotopic one, that $f(a)=a$. Then f maps a onto itself by a degree-two covering. We call a the *circle at infinity*. We may also assume that $f(t)=t$ (again after changing f to a homotopic map). Let us denote t by $+\infty$ and the f-preimage of t different from t by $-\infty$. The loop a has two f-preimages. One is the subcurve of a starting at $+\infty$ and ending at $-\infty$. The other is the curve starting at $-\infty$ and ending at $+\infty$.

Let the connecting path ℓ_0 be the trivial path starting and ending in the basepoint $+\infty$. Let ℓ_1 be the f-preimage of a starting at $+\infty$ and ending in $-\infty$ (the *upper semi-circle at infinity*). Let $\Phi_f\colon\pi_1(\mathcal{C},t)\to\mathfrak{S}(X)\wr\pi_1(\mathcal{C},t)$ be the wreath recursion defined by f

and the given choice of connecting paths, and let $\Lambda_f\colon \pi_1(\mathcal{C}, t) \to \mathrm{Aut}(X^*)$ be the associated iterated monodromy action on the tree X^* (see Proposition 2.2 and the definition of the associated action before it).

Note that $\Lambda_f(a)$ is always equal to the standard adding machine $\tau = \langle\!\langle 1, \tau \rangle\!\rangle \sigma$.

PROPOSITION 3.1. *Two branched coverings $f_0, f_1 \in \mathcal{F}$ are homotopic if and only if there exists $n \in \mathbf{Z}$ such that for every $\gamma \in \pi_1(\mathcal{C}, +\infty)$ we have*

$$\Phi_{f_0}(\gamma) = \Phi_{f_1}(a^n \cdot \gamma \cdot a^{-n}).$$

Proof. Suppose that the branched coverings f_0 and f_1 are homotopic. Let f_x be a homotopy between them, where x varies from 0 to 1. The basepoint $t = +\infty$ has two preimages under f_x. The first preimage $t_0(x)$ draws, as x ranges over $[0, 1]$, a path g_0 starting at $+\infty$. The other preimage $t_1(x)$ draws a path g_1 starting at $-\infty$.

Either $t_0(1) = +\infty$ and $t_1(1) = -\infty$, and then g_0 and g_1 are loops, or $t_0(1) = -\infty$ and $t_1(1) = +\infty$, and then g_0 goes from $+\infty$ to $-\infty$ and g_1 goes from $-\infty$ to $+\infty$. We have chosen $t = +\infty$ to be close to the point $\infty \in P$ and the homotopies must fix the point ∞, so $t_0(x)$ and $t_1(x)$ remain close to ∞. Consequently, if g_0 and g_1 are loops, then $g_0 = a^n$ and $g_1 = \ell_1^{-1} a^m \ell_1$ for some $n, m \in \mathbf{Z}$, and if g_0 and g_1 are not loops, then they are of the form $a^n \ell_1$ and $\bar{\ell}_1 a^m$, where $\bar{\ell}_1$ is the lower semi-circle at infinity (we have $a = \ell_1 \bar{\ell}_1$). Given two paths b and c, we write $b = c$ here and below to indicate that they are homotopic.

One of the f_x-preimages of the loop a is a path $\ell(x, \cdot\,)\colon [0, 1] \to \mathcal{C}$ starting in $t_0(x)$ and ending in $t_1(x)$. We have $\ell(0, \cdot\,) = \ell_1$. The path $\ell(1, \cdot\,)$ is equal to ℓ_1 if g_0 and g_1 are loops, and to the lower semicircle $\bar{\ell}_1$ otherwise. Note also that $\ell(\,\cdot\,, 0) = g_0$ and $\ell(\,\cdot\,, 1) = g_1$.

The path $\ell(x, y)$ depends continuously on x, and therefore defines a continuous map $\ell(x, y)\colon [0, 1] \times [0, 1] \to \mathcal{C}$. It follows that if g_0 and g_1 are loops, then $g_0 = \ell_1 \cdot g_1 \cdot \ell_1^{-1}$, and if not, then $g_0 = \ell_1 \cdot g_1 \cdot (\bar{\ell}_1)^{-1}$.

Consequently, if $g_0 = a^n$ then $g_1 = \ell_1^{-1} a^n \ell_1$, and if $g_0 = a^n \ell_1$ then $g_1 = \ell_1^{-1} a^n \ell_1 \bar{\ell}_1 = \ell_1^{-1} a^{n+1}$.

Take an arbitrary loop $\gamma \in \pi_1(\mathcal{C}, t)$, and let $\gamma_0 = f_0^{-1}(\gamma)[+\infty]$ and $\gamma_1 = f_0^{-1}(\gamma)[-\infty]$ denote their preimages starting at $+\infty$ and $-\infty$, respectively. The homotopy from f_0 to f_1 deforms the paths γ_i continuously, giving paths $\gamma_{0,x} = f_x^{-1}(\gamma)\,[t_0(x)]$ and $\gamma_{1,x} = f_x^{-1}(\gamma)\,[t_1(x)]$.

Suppose first that γ_0, γ_1, $g_0 = a^n$ and $g_1 = \ell_1^{-1} a^n \ell_1$ are loops. Then we get a homotopy transforming γ_0 via $t_0([0, x]) \gamma_{0,x} t_0([0, x])^{-1}$ to the loop

$$g_0 \gamma_0' g_0^{-1} = a^n \gamma_0' a^{-n},$$

where $\gamma_0' = f_1^{-1}(\gamma)[+\infty]$.

Similarly, the loop $\ell_1\gamma_1\ell_1^{-1}$ is homotopic to

$$\ell_1 g_1\gamma_1' g_1^{-1}\ell_1^{-1}=a^n\ell_1\gamma_1'\ell_1^{-1}a^{-n},$$

where $\gamma_1'=f_1^{-1}(\gamma)[-\infty]$.

Since we assumed that γ_i are loops, $\Phi_{f_1}(\gamma)=\langle\!\langle\gamma_0',\ell_1\gamma_1'\ell_1^{-1}\rangle\!\rangle$. We then compute

$$\begin{aligned}\Phi_{f_0}(\gamma)&=\langle\!\langle\gamma_0,\ell_1\gamma_1\ell_1^{-1}\rangle\!\rangle=\langle\!\langle a^n\gamma_0'a^{-n},a^n\ell_1\gamma_1'\ell_1^{-1}a^{-n}\rangle\!\rangle\\&=\langle\!\langle a^n,a^n\rangle\!\rangle\cdot\Phi_{f_1}(\gamma)\cdot\langle\!\langle a^{-n},a^{-n}\rangle\!\rangle=\Phi_{f_1}(a^{2n}\cdot\gamma\cdot a^{-2n}).\end{aligned}$$

If on the other hand $\Phi_{f_1}(\gamma)=\langle\!\langle\widetilde{\gamma}_0',\widetilde{\gamma}_1'\rangle\!\rangle\sigma$, then

$$\Phi_{f_1}(\gamma\cdot a)=\langle\!\langle\widetilde{\gamma}_0',\widetilde{\gamma}_1'\rangle\!\rangle\sigma\langle\!\langle 1,a\rangle\!\rangle\sigma=\langle\!\langle\widetilde{\gamma}_0'a,\widetilde{\gamma}_1'\rangle\!\rangle.$$

We then obtain $\Phi_{f_0}(\gamma a)=\Phi_{f_1}(a^{2n}\cdot\gamma a\cdot a^{-2n})$, so again

$$\Phi_{f_0}(\gamma)=\Phi_{f_1}(a^{2n}\cdot\gamma\cdot a^{-2n}).$$

Consider now the case $g_0=a^n\ell_1$ and $g_1=\ell_1^{-1}a^{n+1}$. Take a loop $\gamma\in\pi_1(\mathcal{C},+\infty)$ such that its f_1-preimages are loops γ_0 and γ_1, so that we have $\Phi_{f_0}(\gamma)=\langle\!\langle\gamma_0,\ell_1\gamma_1\ell_1^{-1}\rangle\!\rangle$. Let γ_0' and γ_1' be the f_1-preimages of γ, so that $\Phi_{f_1}(\gamma)=\langle\!\langle\gamma_0',\ell_1\gamma_1'\ell_1^{-1}\rangle\!\rangle$. Then, as before, γ_0 is homotopic to $g_0\gamma_1'g_0^{-1}=a^n\ell_1\gamma_1'\ell_1^{-1}a^{-n}$, and $\ell_1\gamma_1\ell_1^{-1}$ is homotopic to $\ell_1g_1\gamma_0'g_1^{-1}\ell_1^{-1}=\ell_1\ell_1^{-1}a^{n+1}\gamma_0'a^{-n-1}\ell_1\ell_1^{-1}=a^{n+1}\gamma_0'a^{-n-1}$. We then have

$$\begin{aligned}\Phi_{f_0}(\gamma)&=\langle\!\langle\gamma_0,\ell_1\gamma_1\ell_1^{-1}\rangle\!\rangle=\langle\!\langle a^n\ell_1\gamma_1'\ell_1^{-1}a^{-n},a^{n+1}\gamma_0'a^{-n-1}\rangle\!\rangle\\&=\langle\!\langle a^n,a^{(n+1)}\rangle\!\rangle\sigma\cdot\langle\!\langle\gamma_0',\ell_1\gamma_1'\ell_1^{-1}\rangle\!\rangle\cdot\sigma\langle\!\langle a^{-n},a^{-(n+1)}\rangle\!\rangle=\Phi_{f_1}(a^{2n+1}\cdot\gamma\cdot a^{-2n-1}).\end{aligned}$$

If $\Phi_{f_1}(\gamma)$ is of the form $\langle\!\langle\widetilde{\gamma}_0',\widetilde{\gamma}_1'\rangle\!\rangle\sigma$, we reduce to the previous case by multiplying by a, as before. We also obtain $\Phi_{f_0}(\gamma)=\Phi_{f_1}(a^{2n+1}\cdot\gamma\cdot a^{-2n-1})$.

Suppose now, in order to prove the proposition in the other direction, that there exists $n\in\mathbf{Z}$ such that

$$\Phi_{f_0}(\gamma)=\Phi_{f_1}(a^n\cdot\gamma\cdot a^{-n})\tag{3}$$

holds for all $\gamma\in\pi_1(\mathcal{C},+\infty)$.

For $i\in\{0,1\}$, denote by $G_i\leqslant\pi_1(\mathcal{C},+\infty)$ the set of loops whose preimages under f_i are loops. The set G_i is an index-two subgroup, isomorphic both to $\pi_1(f_i^{-1}(\mathcal{C}),+\infty)$ and to $\pi_1(f_i^{-1}(\mathcal{C}),-\infty)$, where the isomorphisms $G_i\to\pi_1(f_i^{-1}(\mathcal{C}),\pm\infty)$ are the maps $\gamma\mapsto f_i^{-1}(\gamma)[\pm\infty]$. Note that if $\gamma\in G_i$, then

$$\Phi_{f_i}(\gamma)=(f_i^{-1}(\gamma)[+\infty],\ell_1\cdot f_i(\gamma)[-\infty]\cdot\ell_1^{-1}).$$

Condition (3) implies that $G_0{=}G_1$ and that for all $\gamma\in G_0{=}G_1$,

$$f_0^{-1}(\gamma)[+\infty] \quad \text{is homotopic to} \quad f_1^{-1}(a^n\gamma a^{-n})[+\infty] \quad \text{in } \mathcal{C} \tag{4}$$

and

$$\ell_1\cdot f_0^{-1}(\gamma)[-\infty]\cdot\ell_1^{-1} \quad \text{is homotopic to} \quad \ell_1\cdot f_1^{-1}(a^n\gamma a^{-n})[-\infty]\cdot\ell_1^{-1}. \tag{5}$$

The equality $G_0{=}G_1$ implies that there are homeomorphisms

$$h_+, h_-\colon f_0^{-1}(\mathcal{C})\longrightarrow f_1^{-1}(\mathcal{C})$$

such that $f_0{=}f_1\circ h_\pm$, where h_+ fixes the preimages $+\infty$ and $-\infty$ of the basepoint, and h_- permutes them.

The homeomorphisms $h_\pm$ then fix the points c_0, c_1, c_2 and can be extended in a unique way to homeomorphisms $\tilde{h}_\pm\colon\mathcal{C}\to\mathcal{C}$.

Suppose first that n is even. Then $f_0^{-1}(a^n)[+\infty]{=}a^{n/2}$ and we have from (4),

$$\tilde{h}_+(f_0^{-1}(\gamma)[+\infty]) = f_1^{-1}(\gamma)[+\infty] = f_0^{-1}(a^{-n}\gamma a^n)[+\infty] = a^{-n/2}\cdot f_0^{-1}(\gamma)[+\infty]\cdot a^{n/2},$$

where all equalities are homotopies in $\mathcal{C}$. We then have

$$\tilde{h}_+(\gamma) = a^{-n/2}\gamma a^{n/2}$$

for all $\gamma\in\pi_1(\mathcal{C},+\infty)$.

Suppose now that n is odd. Then $f_0^{-1}(a^{-n})[-\infty]{=}\ell_1^{-1}a^{(-n+1)/2}$. We have

$$\tilde{h}_-(f_0^{-1}(\gamma)[+\infty]) = f_1^{-1}(\gamma)[-\infty]$$

for every $\gamma\in G_0$. Let us identify the group $\pi_1(\mathcal{C},+\infty)$ with $\pi_1(\mathcal{C},-\infty)$, by identifying the loop $f_1^{-1}(\gamma)[-\infty]\in\pi_1(\mathcal{C},-\infty)$ with the loop

$$\ell_1\cdot f_1^{-1}(\gamma)[-\infty]\cdot\ell_1^{-1} = \ell_1\cdot f_0^{-1}(a^{-n}\gamma a^n)[-\infty]\cdot\ell_1^{-1} = a^{-(n-1)/2}\cdot f_0^{-1}(\gamma)[+\infty]\cdot a^{(n-1)/2}.$$

We see that for every $\gamma\in\pi_1(\mathcal{C},+\infty)$ the loop $\tilde{h}_-(\gamma)$ is identified with $a^{-(n-1)/2}\gamma a^{(n-1)/2}$.

Therefore in all cases we can find a homeomorphism $h\colon\mathcal{C}\to\mathcal{C}$ satisfying $f_0{=}f_1\circ h$ and such that the induced homomorphism h_* on the fundamental group $\pi_1(\mathcal{C},+\infty)$ is inner. But if a homeomorphism of a surface induces an inner automorphism of the fundamental group, then this homeomorphism is isotopic to identity. Consequently, f_0 and f_1 are homotopic. $\square$

3.2. The set $\mathfrak{F}$ as a bimodule over the mapping class group

Denote by $\mathcal{G}_\mathcal{C}$ the (pure) mapping class group of $\mathcal{C}=\widehat{\mathbf{C}}\setminus P$, i.e. the set of isotopy classes relative to P of homeomorphisms $h:\widehat{\mathbf{C}}\to\widehat{\mathbf{C}}$ fixing P pointwise. The group $\mathcal{G}_\mathcal{C}$ is the quotient of the pure braid group on $|P|-1$ strings by its centre. For instance, if $|P|=4$, then $\mathcal{G}_\mathcal{C}$ is a free group on two generators.

The set $\mathfrak{F}$ has a natural structure of a permutational $\mathcal{G}_\mathcal{C}$-bimodule, i.e. a set $\mathfrak{F}$ equipped with commuting left- and right-actions of the group $\mathcal{G}_\mathcal{C}$. If $f\in\mathfrak{F}$ and $h\in\mathcal{G}_\mathcal{C}$, then we just set fh and hf to be the corresponding compositions. (Here in the composition fh the map f acts before h.) It is easy to see that the left and the right actions of $\mathcal{G}_\mathcal{C}$ on $\mathfrak{F}$ are well defined and commute.

There exists a natural homomorphism $\mathcal{G}_\mathcal{C}\to\mathrm{Out}(\pi_1(\mathcal{C}))$ mapping every element $h\in\mathcal{G}_\mathcal{C}$ to the automorphism h_* of the fundamental group of $\mathcal{C}$ (which is defined uniquely up to an inner automorphism of $\pi_1(\mathcal{C})$). It is well known that this homomorphism is injective (see [11, Theorem 5.13.1]).

Using this fact and Proposition 3.1 one can describe the structure of the bimodule $\mathfrak{F}$. Take an arbitrary $f\in\mathcal{F}$ and a homeomorphism $h:\widehat{\mathbf{C}}\to\widehat{\mathbf{C}}$ acting trivially on P (we will also denote by $f\in\mathfrak{F}$ and $h\in\mathcal{G}_\mathcal{C}$ the corresponding homotopy classes). Recall that a denotes a small circle in the neighbourhood of ∞.

PROPOSITION 3.2. *For $f\in\mathfrak{F}$ and $g,h\in\mathcal{G}_\mathcal{C}$, the following conditions are equivalent*:

(i) $fg=hf$;

(ii) *there exists $n\in\mathbf{Z}$ such that*

$$\Phi_f(\gamma^{g^{-1}})=\Phi_f(\gamma^{a^n})^{h^{-1}}$$

for all $\gamma\in\pi_1(\mathcal{C},+\infty)$;

(iii) *there exists $n\in\mathbf{Z}$ such that*

$$\Phi_f(\gamma)^h=\Phi_f(\gamma^{ga^n})$$

for all $\gamma\in\pi_1(\mathcal{C},+\infty)$;

(iv) *there exists $n\in\mathbf{Z}$ such that*

$$\phi_f(\gamma)^h=\phi_f(\gamma^{ga^n})$$

for all $\gamma\in\mathrm{Dom}\,\phi_f$, where ϕ_f is the virtual endomorphism associated with the first coordinate of the wreath recursion Φ_f.

Proof. It is obvious that (ii) is equivalent to (iii). The equivalence of (iii) and (iv) follows directly from the definition of the virtual endomorphism associated with a wreath recursion. The equivalence of (i) and (ii) follows directly from Propositions 3.1 and 2.2, and the specific definition of Φ_f that we gave in §3.1. □

Definition 3.1. Two branched coverings $f_1, f_2 \in \mathcal{F}$ are *combinatorially equivalent* (also called *Thurston equivalent*) if there exists a homeomorphism h of $\widehat{\mathbf{C}}$ fixing P pointwise, such that $f_1 \cdot h$ and $h \cdot f_2$ are homotopic.

In other words, two elements $f_1, f_2 \in \mathcal{F}$ are equivalent if there exists an element h of the mapping class group of $\widehat{\mathbf{C}} \setminus P$ such that $f_1 h = h f_2$ in $\mathfrak{F}$ (recall that $\mathfrak{F}$ is the set of *homotopy classes* of branched coverings).

COROLLARY 3.3. *If two branched coverings $f_0, f_1 \in \mathcal{F}$ are combinatorially equivalent, then*

$$\Lambda_{f_0}(\pi_1(\mathcal{C}, t)) = \Lambda_{f_1}(\pi_1(\mathcal{C}, t)),$$

as subsets of $\mathrm{Aut}(X^*)$.

(Recall that $\Lambda_f : \pi_1(\mathcal{C}, t) \to \mathrm{Aut}(X^*)$ is the associated iterated monodromy action.)

Proof. Let first $\Phi: G \to G \wr \mathfrak{S}(X)$ be a wreath recursion, and let h be an automorphism of G. Consider the wreath recursion $\Phi^h: G \to G \wr \mathfrak{S}(X)$ given by $\Phi^h(g) = \Phi(g^{h^{-1}})^h$, where h acts on $G \wr \mathfrak{S}(X)$ by the diagonal action on G^X. Let Λ_Φ denote the action of G on X^* defined by the recursion Φ: if $g \in G$, then $\Lambda_\Phi(g)$ maps $x_1 \dots x_n \in X^*$ to $y_1 \dots y_n$, where $\Phi(g) = \langle\langle g_1, \dots, g_d \rangle\rangle \pi$ and $x_1^\pi = y_1$, and $\Lambda_\Phi(g_{x_1})$ maps $x_2 \dots x_n$ to $y_2 \dots y_n$.

We easily check by induction that $\Lambda_\Phi(g) = \Lambda_{\Phi^h}(g^h)$: the respective actions on the first level coincide, and the xth coordinate $g^h|_x$ of $\Phi^h(g^h)$ is equal to $(g|_x)^h$, where $g|_x$ is the xth coordinate of $\Phi(g)$.

Let now h be an element of the mapping class group of $\widehat{\mathbf{C}} \setminus P_{f_0}$ such that $f_0 \cdot h$ and $h \cdot f_1$ are homotopic. Then, by Proposition 3.1,

$$\Phi_{f_0}(\gamma^{h^{-1}}) = \Phi_{f_1}(\gamma^{a^n})^{h^{-1}}$$

for some n. Consequently, $\Phi_{f_0}^h(\gamma) = \Phi_{f_0}(\gamma^{h^{-1}})^h = \Phi_{f_1}(\gamma^{a^n})$. By the first two paragraphs, we have $\Lambda_{\Phi_{f_0}^h}(\gamma) = \Lambda_{f_0}(\gamma^{h^{-1}})$.

The wreath recursion $\gamma \mapsto \Psi(\gamma) := \Phi_{f_1}(\gamma^{a^n})$ is the conjugate by $\Phi_{f_1}(a^n)$ of Φ_{f_1}, so $\Lambda_\Psi(\gamma)$ is the conjugate of $\Lambda_{f_1}(\gamma)$ by the automorphism Δ of X^* given by the recursion

$$\Delta = \tau^n \cdot \langle\langle \Delta, \Delta \rangle\rangle,$$

where $\tau = \langle\langle 1, \tau \rangle\rangle \sigma = \Lambda_{f_i}(a)$ is the adding machine. Now the element $\tau^n \Delta$ satisfies the recursion

$$\tau^n \Delta = \tau^{2n} \langle\langle \Delta, \Delta \rangle\rangle = \langle\langle \tau^n \Delta, \tau^n \Delta \rangle\rangle,$$

so $\tau^n \Delta = 1$, and therefore Δ equals τ^{-n}.

Consequently, $\Lambda_{f_0}(\gamma^{h^{-1}}) = \Lambda_{f_1}(\gamma^{a^{-n}})$, hence $\Lambda_{f_0}(\gamma) = \Lambda_{f_1}(\gamma^{ha^{-n}})$, which implies that $\Lambda_{f_0}(\pi_1(\mathcal{C}, t)) = \Lambda_{f_1}(\pi_1(\mathcal{C}, t))$. □

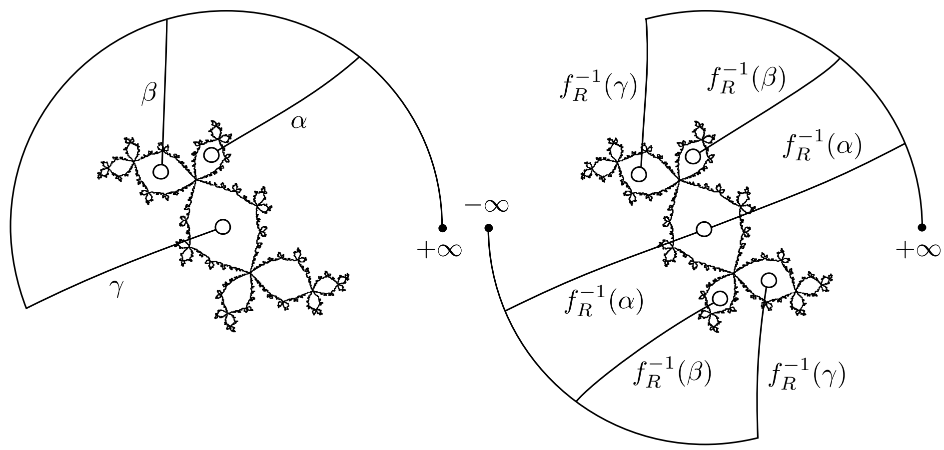

Figure 1. Computing $\mathrm{IMG}(f_R)$.

4. Twisted rabbits

We consider now some concrete examples of bimodules $\mathfrak{F}$. We will consider the cases when f is a quadratic polynomial whose set of finite post-critical points has size 3.

Let us consider first the "Douady rabbit" [7, Figure 35]. It is the polynomial $f_R \approx z^2+(-0.1226+0.7449i)$. The two other polynomials inducing the same permutation of their post-critical set are the "corabbit" $f_C \approx z^2+(-0.1226-0.7449i)$ and the "airplane" $f_A \approx z^2-1.7549$.

We choose as usual $+\infty$ as the basepoint. Let α, β and γ be the loops going around c, c^2+c and 0, respectively, in the positive direction and connected to the basepoint as shown in the left part of Figure 1. Let $P=\{\infty, 0, c, c^2+c\}$ be the post-critical set of f_R.

The rabbit's wreath recursion Φ_{f_R} is defined by

$$\Phi_{f_R}(\alpha)=\langle\langle \alpha^{-1}\beta^{-1}, \gamma\beta\alpha\rangle\rangle\sigma, \quad \Phi_{f_R}(\beta)=\langle\langle \alpha, 1\rangle\rangle \quad \text{and} \quad \Phi_{f_R}(\gamma)=\langle\langle \beta, 1\rangle\rangle.$$

The preimages of the paths α, β and γ are shown in the right part of Figure 1. Note that $\tau=\gamma\beta\alpha=\langle\langle 1, \gamma\beta\alpha\rangle\rangle\sigma$ is the standard adding machine.

4.1. The mapping class group action

The mapping class group $\mathcal{G}_{\mathcal{C}}$ is generated by the left-handed (counterclockwise) Dehn twist T about the curve encircling the points c and c^2+c, and by the left-handed twist S about the curve encircling the points 0 and c^2+c (see Figure 2).

The twists T and S are defined by their action on the fundamental group of the

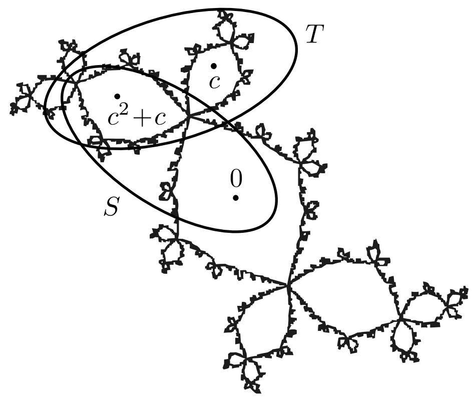

Figure 2. The generators of $\mathcal{G}_{\mathcal{C}}$.

punctured plane by the rules

$$\begin{array}{lll} \alpha^T=\alpha^{\beta\alpha}, & \beta^T=\beta^{\beta\alpha}=\beta^{\alpha}, & \gamma^T=\gamma, \\ \alpha^S=\alpha, & \beta^S=\beta^{\gamma\beta}, & \gamma^S=\gamma^{\gamma\beta}=\gamma^{\beta}. \end{array}$$

Their inverses act by the rules

$$\begin{array}{lll} \alpha^{T^{-1}}=\alpha^{\alpha^{-1}\beta^{-1}}=\alpha^{\beta^{-1}}, & \beta^{T^{-1}}=\beta^{\alpha^{-1}\beta^{-1}}, & \gamma^{T^{-1}}=\gamma, \\ \alpha^{S^{-1}}=\alpha, & \beta^{S^{-1}}=\beta^{\beta^{-1}\gamma^{-1}}=\beta^{\gamma^{-1}}, & \gamma^{S^{-1}}=\gamma^{\beta^{-1}\gamma^{-1}}. \end{array}$$

Proposition 4.1. *Let ψ be the virtual endomorphism of the group $\mathcal{G}_{\mathcal{C}}$ defined on the subgroup $H=\langle T^2, S, S^T\rangle$ of index 2 by*

$$\psi(T^2)=S^{-1}T^{-1}, \quad \psi(S)=T \quad \textit{and} \quad \psi(S^T)=1.$$

Consider the map

$$\bar{\psi}\colon g\longmapsto \begin{cases} \psi(g), & \textit{if } g \textit{ belongs to the domain of } \psi, \\ T\psi(gT^{-1}), & \textit{otherwise.} \end{cases}$$

Then for every $g\in\mathcal{G}_{\mathcal{C}}$ the branched coverings $f_R\cdot g$ and $f_R\cdot\bar{\psi}(g)$ are combinatorially equivalent.

The subgroup H is generated by those Dehn twists about curves that encircle the critical value c an even number of times. They are therefore those mapping classes that can be lifted through f_R. The map ψ is precisely that lift.

Proof. We first claim that for all $g\in H$ the maps $f_R g$ and $\psi(g) f_R$ are homotopic. It suffices to check this on the generators $\{T^2, S, S^T\}$ of H, and this is done below.

Note then that $f_R\cdot g=\psi(g)\cdot f_R=\psi(g)\cdot(f_R\cdot\psi(g))\cdot\psi(g)^{-1}$, i.e. that $f_R\cdot g$ and $f_R\cdot\psi(g)$ are combinatorially equivalent.

If g does not belong to the domain of ψ, then gT^{-1} does, and

$$f_R\cdot g=f_R\cdot gT^{-1}T=\psi(gT^{-1})\cdot f_R\cdot T=\psi(gT^{-1})\cdot(f_R\cdot T\psi(gT^{-1}))\cdot\psi(gT^{-1})^{-1},$$

i.e. $f_R\cdot g$ and $f_R\cdot\overline{\psi}(g)$ are combinatorially equivalent.

Let now $\phi=\phi_{f_R}$ be the virtual endomorphism associated with the first coordinate of the wreath recursion Φ_{f_R}. We have

$$\begin{array}{ccc} \alpha=\phi(\beta), & \beta=\phi(\gamma), & \gamma=\phi(\alpha^{2\beta^{-1}\gamma^{-1}}), \\ \gamma^{\beta\alpha}=\phi(\alpha^2), & \phi(\beta^\alpha)=1, & \phi(\gamma^\alpha)=1, \end{array}$$

from which we compute

$$\begin{aligned} \phi(\beta^{T^2})&=\phi(\beta^{\alpha\beta\alpha})=\phi(\beta^{\alpha^2\cdot\beta^\alpha})=\alpha^{\beta^{-1}\gamma\beta\alpha}, \\ \phi(\gamma^{T^2})&=\phi(\gamma)=\beta, \\ \phi((\alpha^{2\beta^{-1}\gamma^{-1}})^{T^2})&=\phi(\alpha^{2\beta^{-1}\beta\alpha\beta\alpha\gamma^{-1}})=\phi(\alpha^{2\beta^\alpha\cdot\gamma^{-1}})=\gamma^{\beta\alpha\beta^{-1}}. \end{aligned}$$

We see that for every $\delta\in\pi_1(\mathcal{C},+\infty)$ we have $\phi(\delta^{T^2})=\phi(\delta)^h$, where the automorphism h is given on the generators by

$$\alpha^h=\alpha^{\beta^{-1}\gamma\beta\alpha},\quad \beta^h=\beta\quad\text{and}\quad\gamma^h=\gamma^{\beta\alpha\beta^{-1}}. \tag{6}$$

The automorphism h is equal to the product $S^{-1}T^{-1}a$, where a is conjugation by $\gamma\beta\alpha$:

$$\begin{array}{llll} \alpha \xmapsto{S^{-1}} \alpha & \xmapsto{T^{-1}} \alpha^{\beta^{-1}} & \xmapsto{a} \alpha^{\beta^{-1}\gamma\beta\alpha}, \\ \beta \xmapsto{S^{-1}} \beta^{\gamma^{-1}} & \xmapsto{T^{-1}} \beta^{\alpha^{-1}\beta^{-1}\gamma^{-1}} & \xmapsto{a} \beta, \\ \gamma \xmapsto{S^{-1}} \gamma^{\beta^{-1}\gamma^{-1}} & \xmapsto{T^{-1}} \gamma^{\beta^{-\alpha^{-1}\beta^{-1}}\gamma^{-1}} & \xmapsto{a} \gamma^{\beta\alpha\beta^{-1}}. \end{array}$$

Consequently, by Proposition 3.2 (iv) with $n=1$,

$$f_R\cdot T^2=S^{-1}T^{-1}\cdot f_R.$$

We have next

$$\phi(\beta^S)=\phi(\beta^{\gamma\beta})=\alpha^{\beta\alpha},\quad \phi(\gamma^S)=\phi(\gamma^\beta)=\beta^\alpha\quad\text{and}\quad\phi((\alpha^{2\beta^{-1}\gamma^{-1}})^S)=\phi(\alpha^{2\beta^{-1}\gamma^{-1}})=\gamma,$$

so

$$f_R\cdot S=T\cdot f_R.$$

Let us finally compute the action of S^T on the generators $\{\alpha,\beta,\gamma\}$ of the fundamental group:

$$\begin{array}{llll}\alpha \xmapsto{T^{-1}} \alpha^{\beta^{-1}} & \xmapsto{S} \alpha^{\beta^{-\gamma\beta}} & \xmapsto{T} \alpha^{\beta\alpha\beta^{-\alpha\gamma\beta^{\alpha}}} = \alpha^{\gamma^{-1}\alpha^{-1}\beta^{-1}\alpha\gamma\alpha^{-1}\beta\alpha}, \\ \beta \xmapsto{T^{-1}} \beta^{\alpha^{-1}\beta^{-1}} & \xmapsto{S} \beta^{\gamma\beta\alpha^{-1}\beta^{-1}\gamma^{-1}\beta^{-1}\gamma\beta} & \xmapsto{T} \beta^{\alpha\gamma\alpha^{-1}\gamma^{-1}\alpha^{-1}\beta^{-1}\alpha\gamma\alpha^{-1}\beta\alpha}, \\ \gamma \xmapsto{T^{-1}} \gamma & \xmapsto{S} \gamma^{\beta} & \xmapsto{T} \gamma^{\alpha^{-1}\beta\alpha}.\end{array}$$

Let us apply the virtual endomorphism ϕ to the conjugators:

$$\begin{aligned}\phi(\gamma^{-1}\cdot\alpha^{-1}\beta^{-1}\alpha\cdot\gamma\cdot\alpha^{-1}\beta\alpha) &= \beta^{-1}\beta = 1, \\ \phi(\alpha\gamma\alpha^{-1}\cdot\gamma^{-1}\cdot\alpha^{-1}\beta^{-1}\alpha\cdot\gamma\cdot\alpha^{-1}\beta\alpha) &= \beta^{-1}\beta = 1, \\ \phi(\alpha^{-1}\beta\alpha) &= 1.\end{aligned}$$

Consequently,

$$\phi((\alpha^2)^{T^{-1}ST}) = \phi(\alpha^2), \quad \phi(\beta^{T^{-1}ST}) = \phi(\beta) \quad \text{and} \quad \phi(\gamma^{T^{-1}ST}) = \phi(\gamma),$$

which implies that

$$f_R\cdot S^T = f_R. \qquad \square$$

Proposition 4.2. *The map $\bar\psi\colon\mathcal{G}_{\mathcal{C}}\to\mathcal{G}_{\mathcal{C}}$ is contracting: for every $g\in\mathcal{G}_{\mathcal{C}}$ there exists $n\in\mathbf{N}$ such that $\bar\psi^n(g)\in\{1,T,T^{-1}\}$.*

Proof. Consider the wreath recursion for $\mathcal{G}_{\mathcal{C}}$ given by

$$\Phi(T) = \langle\langle 1, S^{-1}T^{-1}\rangle\rangle\sigma \quad \text{and} \quad \Phi(S) = \langle\langle T, 1\rangle\rangle. \tag{7}$$

It is straightforward to check that $\bar\psi$ may be recovered from the recursion as follows: if $\Phi(g)=\langle\langle g_0,g_1\rangle\rangle$, then $\bar\psi(g)=g_0$; if $\Phi(g)=\langle\langle g_0,g_1\rangle\rangle\sigma$, then $\bar\psi(g)=Tg_0$. Since $T|_0=1$, we obtain inductively for all $n\in\mathbf{N}$,

$$\bar\psi^n(g) = g|_v \quad \text{or} \quad \bar\psi^n(g) = Tg|_v \quad \text{for some } v\in X^n.$$

We first claim that the recursion (7) is contracting. For that purpose, it suffices to compute the nucleus of (7). A simple calculation shows that it is

$$\mathcal{N} = \{1, S, T, TS, S^{-1}, T^{-1}, S^{-1}T^{-1}\}.$$

It follows that for all $g\in\mathcal{G}_{\mathcal{C}}$ we have $\bar\psi^n\in\mathcal{N}\cup T\mathcal{N}$ for all sufficiently large n, and therefore that $\bar\psi^n(g)$ lands on a $\bar\psi$-cycle. Direct computations then show that the only $\bar\psi$-periodic elements in $\mathcal{N}\cup T\mathcal{N}$ are the fixed points 1 and T, and the cycle

$$T^{-1} \longmapsto T^2S \longmapsto S^{-1} \longmapsto T^{-1}. \qquad \square$$

4.2. Contraction along the subgroup $\langle T\rangle$

Every integer m has a unique 4-*adic* expansion

$$m=\sum_{k=0}^{\infty} m_k 4^k,$$

with $m_k\in\{0,1,2,3\}$, and almost all $m_k=0$ if m is non-negative, and almost all $m_k=3$ if m is negative.(3)

Proposition 4.3. *If the* 4*-adic expansion of the number m has digits* 1 *or* 2*, then the branched covering $f_R{\cdot}T^m$ is equivalent to $f_R{\cdot}T$. Otherwise it is equivalent to f_R for non-negative m and to $f_R{\cdot}T^{-1}$ for negative m.*

Proof. Let us iterate the map $\bar\psi$ on the cyclic subgroup $\langle T\rangle$. We have

$$\bar\psi^3(T^{4k})=\bar\psi^2(S^{-1}T^{-1})^{2k}=\bar\psi^2((S^{-1}\cdot S^{-T}\cdot T^{-2})^k)=\bar\psi(T^{-1}\cdot 1\cdot TS)^k=\bar\psi(S^k)=T^k,$$

so $f_R{\cdot}T^{4k}$ is equivalent to $f_R{\cdot}T^k$. Similarly

$$\begin{aligned}\bar\psi^3(T^{4k+1})&=\bar\psi^2(T\psi(T^{4k}))\\&=\bar\psi^2(T(S^{-1}T^{-1})^{2k})\\&=\bar\psi(T\psi(T(S^{-1}T^{-1})^{2k}T^{-1}))\\&=\bar\psi(T\psi((TS^{-1}T^{-1}\cdot S^{-1}\cdot T^{-2})^k))\\&=\bar\psi(T(1\cdot T^{-1}\cdot TS)^k)\\&=\bar\psi(TS^k)\\&=T\psi(TS^kT^{-1})\\&=T,\end{aligned}$$

so all branched coverings $f_R{\cdot}T^{4k+1}$ are equivalent to $f_R{\cdot}T$. Next,

$$\begin{aligned}\bar\psi^3(T^{4k+2})&=\bar\psi^2(S^{-1}T^{-1})^{2k+1}\\&=\bar\psi(T\psi((S^{-1}T^{-1})^{2k+1}T^{-1}))\\&=\bar\psi(T\psi((S^{-1}\cdot S^{-T}\cdot T^{-2})^k\cdot S^{-1}\cdot T^{-2}))\\&=\bar\psi(T(T^{-1}\cdot 1\cdot TS)^k\cdot T^{-1}\cdot TS)\\&=\bar\psi(TS^{k+1})\\&=T\psi(TS^{k+1}T^{-1})\\&=T,\end{aligned}$$

(3) For example, $m=-1$ corresponds to $m_k=3$ for all k.

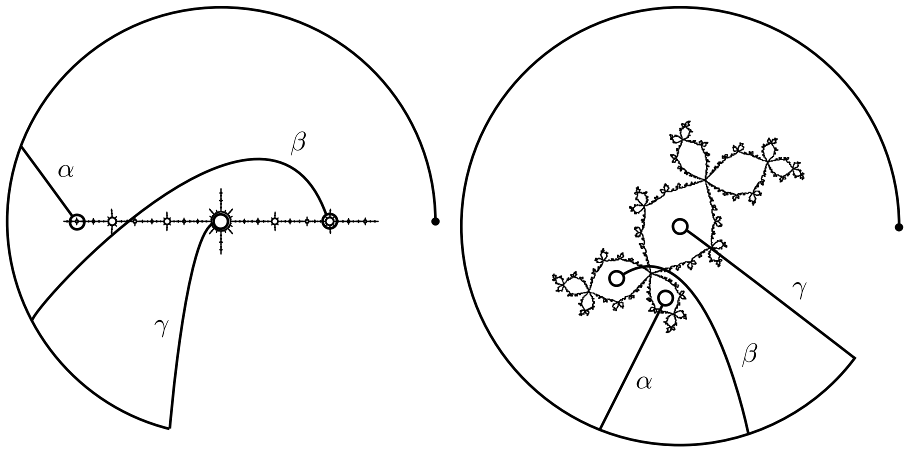

Figure 3. The standard generators of IMG(f_A) and IMG(f_C).

so all branched coverings $f_R{\cdot}T^{4k+2}$ are equivalent to $f_R{\cdot}T$. Finally

$$\begin{aligned}\bar\psi^3(T^{4k+3}) &= \bar\psi^2(T\psi(T^{4k+2})) = \bar\psi^2(T(S^{-1}T^{-1})^{2k+1})\\ &= \bar\psi^2(TS^{-1}T^{-1}\cdot(S^{-1}\cdot S^{-T}\cdot T^{-2})^k) = \bar\psi(1\cdot(T^{-1}\cdot 1\cdot TS)^k) = \bar\psi(S^k) = T^k,\end{aligned}$$

so $f_R{\cdot}T^{4k+3}$ is equivalent to $f_R{\cdot}T^k$. The statement now easily follows. □

4.3. Solving the problem for all $m\in\mathbf{Z}$

The wreath recursion for the airplane polynomial f_A is given by

$$\Phi_{f_A}(\alpha) = \langle\!\langle \alpha^{-1}, \gamma\alpha\rangle\!\rangle\sigma, \quad \Phi_{f_A}(\beta) = \langle\!\langle \alpha, 1\rangle\!\rangle \quad \text{and} \quad \Phi_{f_A}(\gamma) = \langle\!\langle 1, \beta^{\gamma^{-1}}\rangle\!\rangle. \tag{8}$$

Here α, β and γ are loops going in the positive direction around c, c^2+c and $(c^2+c)^2+c$, respectively, as shown in the left part of Figure 3.

The wreath recursion for the corabbit f_C is given by

$$\Phi_{f_C}(\alpha) = \langle\!\langle \alpha^{-1}\beta^{-1}, \gamma\beta\alpha\rangle\!\rangle\sigma, \quad \Phi_{f_C}(\beta) = \langle\!\langle \alpha^{\beta\alpha}, 1\rangle\!\rangle \quad \text{and} \quad \Phi_{f_C}(\gamma) = \langle\!\langle \beta^{\alpha}, 1\rangle\!\rangle. \tag{9}$$

Here α, β and γ are loops going in the positive direction around c, c^2+c and $(c^2+c)^2+c$, respectively, as shown in the right part of Figure 3. Note that for $f=f_A, f_C$, just as for $f=f_R$, we have $\Phi_f(\gamma\beta\alpha)=\langle\!\langle 1, \gamma\beta\alpha\rangle\!\rangle\sigma$.

Let us identify the planes of f_A and f_C with the plane of f_R by identifying their respective loops α, β and γ (the definition of α, β and γ in the plane of f_R is given in the left part of Figure 1).

Let T denote, as before, the left Dehn twist about the curve around the points c and c^2+c in the plane of f_R. Then, from the definition of T, we get the following wreath recursion for the standard iterated monodromy action for $T^m \cdot f_R$:

$$\begin{aligned}\Phi_{T^m\cdot f_R}(\alpha) &= \Phi_{f_R}(\alpha)^{T^{-m}} = \langle\!\langle \alpha^{-1}\beta^{-1}, \gamma\beta\alpha\rangle\!\rangle\sigma,\\ \Phi_{T^m\cdot f_R}(\beta) &= \Phi_{f_R}(\beta)^{T^{-m}} = \langle\!\langle \alpha^{(\alpha^{-1}\beta^{-1})^m}, 1\rangle\!\rangle,\\ \Phi_{T^m\cdot f_R}(\gamma) &= \Phi_{f_R}(\gamma)^{T^{-m}} = \langle\!\langle \beta^{(\alpha^{-1}\beta^{-1})^m}, 1\rangle\!\rangle,\end{aligned} \tag{10}$$

where T^{-m} acts on $\pi_1(\mathcal{C}, +\infty) \wr \mathfrak{S}(X)$ by the diagonal action:

$$(\langle\!\langle x, y\rangle\!\rangle\sigma^k)^{T^{-m}} = \langle\!\langle x^{T^{-m}}, y^{T^{-m}}\rangle\!\rangle\sigma^k.$$

Corollary 3.3 makes it possible to solve Hubbard's question algorithmically for every given m in the following way.

Thurston's Theorem 1.1 implies that $T^m \cdot f_R$ is combinatorially equivalent to exactly one polynomial in the set $\{f_R, f_A, f_C\}$. There are no obstructions, since the only obstructions for polynomials are Levy cycles, which cannot exist in the case of a periodic critical point. Corollary 3.3 then tells us that $\Lambda_{T^m\cdot f_R}(\pi_1(\mathcal{C}))$ coincides with the iterated monodromy group of the associated polynomial. One can prove that these groups are different (as sets), and therefore, if we prove that $\Lambda_{T^m\cdot f_R}(\pi_1(\mathcal{C}))$ coincides with a given group $\mathrm{IMG}(f_*)$ for $*\in\{R, A, C\}$, then we can conclude that $T^m \cdot f_R$ is equivalent to the respective f_*.

We therefore prove that the $\mathrm{IMG}(f_*)$ are all distinct. This is done by computing their nuclei, and checking that they are distinct as finite automata; this is done in Figure 4.

Proposition 4.4. *The group* $\mathrm{IMG}(T\cdot f_R) = \Lambda_{T\cdot f_R}(\pi_1(\mathcal{C}))$ *coincides with* $\mathrm{IMG}(f_A)$. *Indeed the homeomorphism* $h = TS^{-1}a$ *conjugates* $T\cdot f_R$ *with* f_A, *if the planes of* f_R *and* f_A *are identified as above.*

Proof. Let α, β and γ be the generators of $\mathrm{IMG}(f_A)$. They are defined now as the automorphisms of X^* satisfying the recursion (compare with (8))

$$\alpha = \langle\!\langle \alpha^{-1}, \gamma\alpha\rangle\!\rangle\sigma, \quad \beta = \langle\!\langle \alpha, 1\rangle\!\rangle \quad \text{and} \quad \gamma = \langle\!\langle 1, \beta^{\gamma^{-1}}\rangle\!\rangle.$$

Let α_1, β_1 and γ_1 be the generators of $\mathrm{IMG}(T\cdot f_R)$. They are given by the recursion (10):

$$\alpha_1 = \langle\!\langle \alpha_1^{-1}\beta_1^{-1}, \gamma_1\beta_1\alpha_1\rangle\!\rangle\sigma, \quad \beta_1 = \langle\!\langle \alpha_1^{\beta_1^{-1}}, 1\rangle\!\rangle \quad \text{and} \quad \gamma_1 = \langle\!\langle \beta_1^{\alpha_1^{-1}\beta_1^{-1}}, 1\rangle\!\rangle.$$

We claim that

$$\alpha_1 = \alpha^h = \alpha^{\beta^{\gamma^{-1}}\alpha\gamma\beta\alpha}, \quad \beta_1 = \beta^h = \beta^{\gamma^{-1}\alpha\gamma\beta\alpha} \quad \text{and} \quad \gamma_1 = \gamma^h = \gamma^\alpha.$$

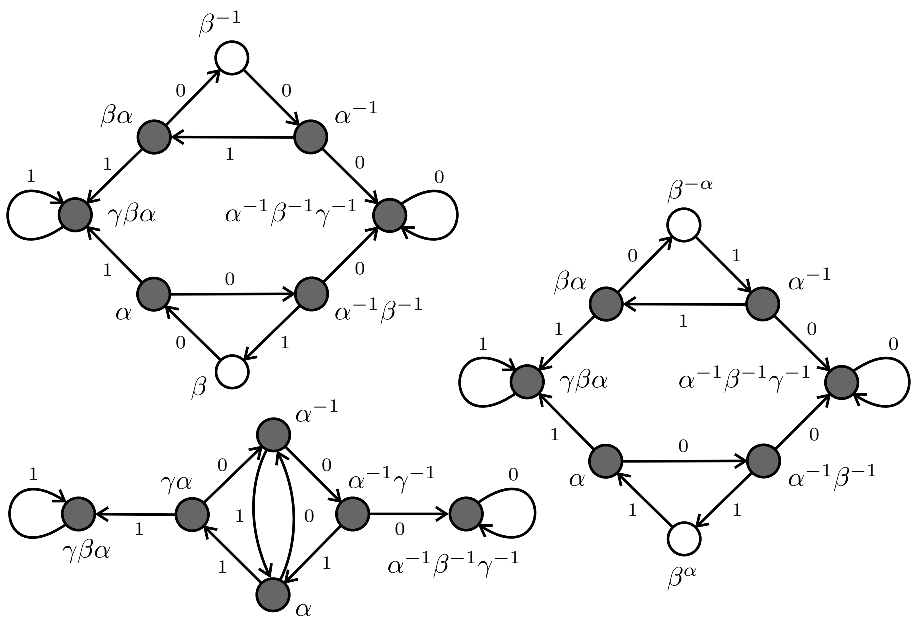

Figure 4. Nuclei of the rabbit (top), the corabbit (right) and the airplane (bottom).

For that purpose, it suffices to show that the right-hand sides of these equalities satisfy the same recursions as α_1, β_1 and γ_1. Note that $\gamma^h\beta^h\gamma^h=(\gamma\beta\alpha)^h=\gamma\beta\alpha$. We have

$$\beta^{\gamma^{-1}}\alpha\gamma\beta\alpha=\langle\!\langle\alpha,1\rangle\!\rangle\langle\!\langle\alpha^{-1},\gamma\alpha\rangle\!\rangle\sigma\langle\!\langle 1,\gamma\beta\alpha\rangle\!\rangle\sigma=\langle\!\langle\gamma\beta\alpha,\gamma\alpha\rangle\!\rangle,$$

hence

$$\begin{aligned}\alpha^{\beta^{\gamma^{-1}}\alpha\gamma\beta\alpha}&=(\langle\!\langle\alpha^{-1},\gamma\alpha\rangle\!\rangle\sigma)^{\langle\!\langle\gamma\beta\alpha,\gamma\alpha\rangle\!\rangle}\\&=\langle\!\langle\alpha^{-1}\beta^{-1}\gamma^{-1}\alpha^{-1}\gamma\alpha,\alpha^{-1}\gamma^{-1}\gamma\alpha\gamma\beta\alpha\rangle\!\rangle\sigma\\&=\langle\!\langle\alpha^{-1}\beta^{-1}\gamma^{-1}\alpha^{-1}\gamma\alpha,\gamma\beta\alpha\rangle\!\rangle\sigma\\&=\langle\!\langle(\gamma\beta\alpha)^{-h}\gamma^h,(\gamma\beta\alpha)^h\rangle\!\rangle\sigma\\&=\langle\!\langle\alpha^{-h}\beta^{-h},\gamma^h\beta^h\alpha^h\rangle\!\rangle\sigma,\end{aligned}$$

$$\begin{aligned}\beta^{\gamma^{-1}\alpha\gamma\beta\alpha}&=\beta^{\gamma^{-1}\beta^{\gamma^{-1}}\alpha\gamma\beta\alpha}=\langle\!\langle\alpha,1\rangle\!\rangle^{\langle\!\langle\gamma\beta\alpha,\gamma\alpha\rangle\!\rangle}=\langle\!\langle\alpha^{\gamma\beta\alpha},1\rangle\!\rangle\\&=\langle\!\langle(\alpha^{\gamma\beta\gamma^{-1}\alpha\gamma\beta\alpha})^{\beta^{-\gamma^{-1}\alpha\gamma\beta\alpha}},1\rangle\!\rangle=\langle\!\langle(\alpha^h)^{\beta^{-h}},1\rangle\!\rangle,\end{aligned}$$

$$\begin{aligned}\gamma^\alpha&=\langle\!\langle 1,\beta^{\gamma^{-1}}\rangle\!\rangle^{\langle\!\langle\alpha^{-1},\gamma\alpha\rangle\!\rangle\sigma}=\langle\!\langle\beta^\alpha,1\rangle\!\rangle\\&=\langle\!\langle(\beta^{\alpha^{-\gamma\beta}\beta^{-1}})^{\gamma^{-1}\alpha\gamma\beta\alpha},1\rangle\!\rangle=\langle\!\langle(\beta^h)^{\alpha^{-h}\beta^{-h}},1\rangle\!\rangle.\end{aligned}$$

This shows the required relations between α, β, γ and α_1, β_1, γ_1, so $\mathrm{IMG}(f_A)=\mathrm{IMG}(T\cdot f_R)$. Proposition 3.1 now implies that $T\cdot f_R$ is homotopic to $h^{-1}\cdot f_A\cdot h$. $\square$

PROPOSITION 4.5. *The group*

$$\mathrm{IMG}(T^{-1}\cdot f_R)=\Lambda_{T^{-1}\cdot f_R}(\pi_1(\mathcal{C}))$$

coincides with $\mathrm{IMG}(f_C)$*. Moreover,* $T^{-1}\cdot f_R$ *and* f_C *are homotopic if the planes of* f_R *and* f_C *are identified as above.*

Proof. Let α, β and γ be the generators of the iterated monodromy group of the corabbit. They are defined by the recursion (compare with (9))

$$\alpha=\langle\langle\alpha^{-1}\beta^{-1},\gamma\beta\alpha\rangle\rangle\sigma,\quad \beta=\langle\langle\alpha^{\beta\alpha},1\rangle\rangle\quad\text{and}\quad\gamma=\langle\langle\beta^{\alpha},1\rangle\rangle.$$

Let α_{-1},β_{-1} and γ_{-1} be the generators of $\mathrm{IMG}(T^{-1}\cdot f_R)$. They are given by the recursion (10):

$$\alpha_{-1}=\langle\langle\alpha_{-1}^{-1}\beta_{-1}^{-1},\gamma_{-1}\beta_{-1}\alpha_{-1}\rangle\rangle\sigma,\quad \beta_{-1}=\langle\langle\alpha_{-1}^{\beta_{-1}\alpha_{-1}},1\rangle\rangle\quad\text{and}\quad\gamma_{-1}=\langle\langle\beta_{-1}^{\alpha_{-1}},1\rangle\rangle.$$

Since these two recursions are the same, we have $\alpha_{-1}=\alpha$, $\beta_{-1}=\beta$ and $\gamma_{-1}=\gamma$. Proposition 3.1 now implies that f_C and $T^{-1}\cdot f_R$ are homotopic. $\square$

COROLLARY 4.6. *The branched coverings* $T\cdot f_R$ *and* $f_R\cdot T$ *are equivalent to* f_A *and the branched coverings* $T^{-1}\cdot f_R$ *and* $f_R\cdot T^{-1}$ *are equivalent to* f_C*.*

The last corollary together with Proposition 4.3 prove the following solution of the "twisted rabbit question".

THEOREM 4.7. *If the* 4*-adic expansion of the number* m *has digits* 1 *or* 2*, then the branched covering* $f_R\cdot T^m$ *is equivalent to the airplane* f_A*. Otherwise it is equivalent to the rabbit* f_R *for non-negative* m *and to the corabbit* f_C *for negative* m*.*

The general case of any element of the mapping class group $\mathcal{G}_\mathcal{C}$ is treated using Proposition 4.2 in the following theorem.

THEOREM 4.8. *Let* $g\in\mathcal{G}_\mathcal{C}$ *be an arbitrary homeomorphism fixing the post-critical set pointwise. Let* $\bar\psi\colon\mathcal{G}_\mathcal{C}\to\mathcal{G}_\mathcal{C}$ *be the map defined in Proposition* 4.1*. The orbit of* g *under the iteration of* $\bar\psi$ *will land either on* 1*, on* T*, or on* T^{-1}*. In the first case* $g\cdot f_R$ *is equivalent to the rabbit, in the second case it is equivalent to the airplane and in the last to the corabbit.*

This theorem gives an algorithm solving the general "twisted rabbit question". Note that due to the fact that $\bar\psi$ is contracting on $\mathcal{G}_\mathcal{C}$, this algorithm has linear complexity with respect to the word-length of elements of $\mathcal{G}_\mathcal{C}$.

Note also that in Theorem 4.7 the typical answer is airplane, and exponentially few values of m yield "rabbits" or "corabbits". This seems to happen quite often on cyclic subgroups of $\mathcal{G}_{\mathcal{C}}$. On the other hand, on the cyclic subgroup $\langle ST^2\rangle$ all twists are "rabbits"; and on the subgroup $\langle ST\rangle$ there are roughly as many "rabbits" as "airplanes": $(ST)^m\cdot f_R$ is equivalent to

$$\begin{cases} f_R, & \text{if } m<-2,\ m=2n-3 \text{ and } n\text{'s 4-adic expansion has only 0's and 3's},\\ f_C, & \text{if } m>-2,\ m=2n-1 \text{ and } n\text{'s 4-adic expansion has only 0's and 3's},\\ f_R, & \text{if } n\equiv 0 \pmod 2,\\ f_A, & \text{if } n\equiv 1 \pmod 2 \text{ in the cases not covered above.}\end{cases}$$

This is because $\bar{\psi}((ST)^{2m})=S^{-mT^{-1}}$ and $\bar{\psi}((ST)^{2m+1})=T^2S^{-m}$; calculations are similar to those of Proposition 4.3.

These phenomena can ultimately be traced to the following reason: the subgroup $\operatorname{Dom}\psi^3$ of $\mathcal{G}_{\mathcal{C}}$ has index 8:

$$\operatorname{Dom}\psi^3=\langle S^4,T^4,ST^2,S^{-2}T^2S,S^{-1}T^2S^2,S^T,S^{T^{-1}},S^{TS},S^{T^{-1}S}\rangle.$$

The map ψ^3 is defined on these generators by $\psi^3(S^4)=S$, $\psi^3(T^4)=T$ and all other generators are mapped to 1.

5. Dynamics on the moduli space

In the previous sections, we have computed a virtual endomorphism ψ of the mapping class group $\mathcal{G}_{\mathcal{C}}$ associated with the bimodule $\mathfrak{F}$ of branched coverings with the ramification graph of the rabbit polynomial. This virtual endomorphism is given by

$$\psi(T^2)=S^{-1}T^{-1},\quad \psi(S)=T \quad\text{and}\quad \psi(S^T)=1.$$

This virtual endomorphism is associated to a self-similar action of $\mathcal{G}_{\mathcal{C}}$ on the binary tree. Let us show that this action coincides with the iterated monodromy action associated with the rational function $F(w)=1-1/w^2$.

The critical points of $1-1/w^2$ are 0 and ∞. Their orbit is

$$0\longmapsto\infty\longmapsto 1\longmapsto 0.$$

Let us take the fixed point $t\approx 0.8774+0.7449i$ as our basepoint. The fundamental group of $\widehat{\mathbf{C}}\setminus\{0,1,\infty\}$ is generated by the loops going in the *negative* direction around 0 and 1. Let us denote the first by X and the second by Y (see the solid lines on the left- and right-hand sides of Figure 5, respectively). Let us choose the connecting paths ℓ_0 and ℓ_1, so that ℓ_0 is the trivial path at t, and ℓ_1 connects t to $-t$ passing above the puncture 0, as shown in the right-hand side of Figure 5.

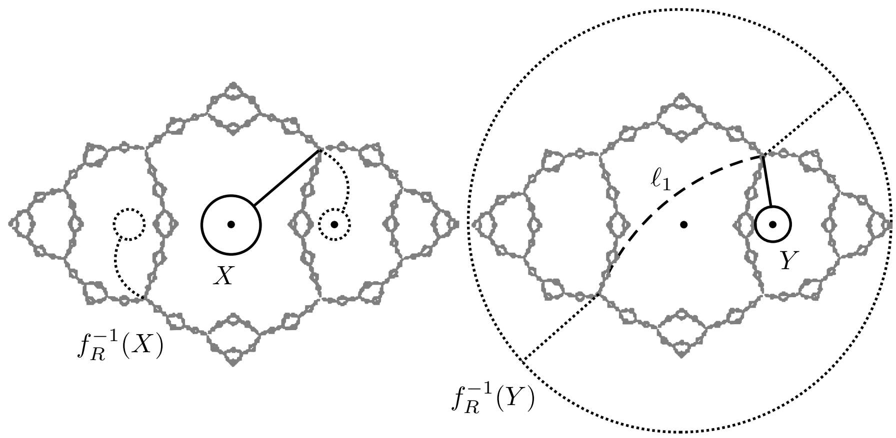

Figure 5.

We then get the following recursion for the iterated monodromy group of F:

$$X=\langle\langle Y,1\rangle\rangle \quad\text{and}\quad Y=\langle\langle 1,X^{-1}Y^{-1}\rangle\rangle\sigma.$$

Next $Y^2=\langle\langle X^{-1}Y^{-1},X^{-1}Y^{-1}\rangle\rangle$ and $Y^{-1}XY=\langle\langle 1,Y\rangle\rangle$. We see that the virtual endomorphism associated with the wreath recursion is given by

$$X\longmapsto Y,\quad Y^2\longmapsto X^{-1}Y^{-1}\quad\text{and}\quad Y^{-1}XY\mapsto 1.$$

It is therefore precisely the virtual endomorphism ψ associated to $\mathfrak{F}$, if we identify X with S and Y with T.

5.1. Moduli and Teichmüller spaces

This coincidence has a nice explanation in terms of Teichmüller theory. Let $P\subset\mathbf{S}^2$ be a finite subset of the sphere. Then the *Teichmüller space* $\mathcal{T}_P$ modelled on $(\mathbf{S}^2,P)$ is the space of homeomorphisms $\tau\colon\mathbf{S}^2\to\widehat{\mathbf{C}}$, where τ_1 and τ_2 are identified if there exists a biholomorphic isomorphism $\Theta\colon\widehat{\mathbf{C}}\to\widehat{\mathbf{C}}$ (i.e. an element of the Möbius group) such that $\Theta\circ\tau_1=\tau_2$ on P and $\Theta\circ\tau_1$ is isotopic to τ_2 relative to P.

The *moduli space* $\mathcal{M}_P$ of $(\mathbf{S}^2,P)$ is the space of all injective maps $P\hookrightarrow\widehat{\mathbf{C}}$ modulo post-compositions with elements of the Möbius group. The Teichmüller space $\mathcal{T}_P$ is the universal cover of the moduli space $\mathcal{M}_P$, where the covering map is the restriction map of $\tau\in\mathcal{T}_P$ to P.

Let now $f\colon \mathbf{S}^2 \to \mathbf{S}^2$ be a branched covering with post-critical set P. Then for every $\tau \in \mathcal{T}_P$ there exists a unique element $\tau' \in \mathcal{T}_P$ such that we have a commutative diagram

$$\begin{tikzcd} \mathbf{S}^2 \arrow[r, "f"] \arrow[d, "\tau'"] & \mathbf{S}^2 \arrow[d, "\tau"] \\ \widehat{\mathbf{C}} \arrow[r, "f_\tau"] & \widehat{\mathbf{C}} \end{tikzcd} \tag{11}$$

and $f_\tau = (\tau')^{-1} \cdot f \cdot \tau\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is a rational function. Let us write $\tau' = \sigma_f(\tau)$. The map $\sigma_f\colon \mathcal{T}_P \to \mathcal{T}_P$ is analytic and weakly (i.e. non-uniformly) contracting (see [10] or [4]).

Let us return to the case when P is the post-critical set of the rabbit polynomial. The moduli space $\mathcal{M}_P$ is the set of maps $\tau|_P\colon \{0, c, c^2+c, \infty\} \hookrightarrow \widehat{\mathbf{C}}$ modulo post-compositions with Möbius transformations. We may assume, applying an appropriate element of the Möbius group, that 0 is mapped by τ to 0, c to 1 and ∞ to ∞. Then the points of the moduli space are uniquely determined by the value of $\tau|_P(c^2+c) = w$. We have $w \notin \{0, 1, \infty\}$, since the map $\tau|_P$ is injective.

Therefore, the moduli space is isomorphic to the punctured sphere $\widehat{\mathbf{C}} \setminus \{0, 1, \infty\}$, where the point $w \in \widehat{\mathbf{C}} \setminus \{0, 1, \infty\}$ corresponds to the element $\tau|_P$ such that $\tau|_P(\infty) = \infty$, $\tau|_P(0) = 0$, $\tau|_P(c) = 1$ and $\tau|_P(c^2+c) = w$.

Let $f \in \mathcal{F}$ be arbitrary. Recall that $\mathcal{F}$ is the set of degree-two branched coverings of $\widehat{\mathbf{C}}$ with critical points 0 and ∞, whose ramification graph coincides with that of the rabbit polynomial.

Let τ be arbitrary. Suppose that the projection of $\tau' = \sigma_f(\tau)$ on the moduli space is given by the point $w_0 \in \widehat{\mathbf{C}} \setminus \{0, 1, \infty\}$, and the projection of τ is given by $w_1 \in \widehat{\mathbf{C}} \setminus \{0, 1, \infty\}$. Then the rational function f_τ in the diagram (11) is a degree-two map having critical points at 0 and ∞, and satisfying

$$f_\tau(\infty) = \infty, \quad f_\tau(0) = 1, \quad f_\tau(1) = w_1 \quad \text{and} \quad f_\tau(w_0) = 0,$$

since $f_\tau|_P = (\tau'|_P)^{-1} \cdot f|_P \cdot \tau|_P$.

We conclude that f_τ is a quadratic polynomial. It is of the form az^2+1, since 0 is critical and $f_\tau(0) = 1$. We get therefore

$$a+1 = w_1 \quad \text{and} \quad aw_0^2+1 = 0,$$

hence $a = -1/w_0^2$, so that

$$w_1 = 1 - \frac{1}{w_0^2}.$$

We have thus obtained the following description of the action of the pull-back map σ_f on the moduli space.

PROPOSITION 5.1. *The correspondence* $\sigma_f(\tau)\mapsto\tau$ *on the Teichmüller space is projected on the moduli space* $\mathcal{M}_P=\widehat{\mathbf{C}}\setminus\{0,1,\infty\}$ *to the rational function*

$$F\colon w\longmapsto 1-\frac{1}{w^2}.$$

Suppose now that $h\in\mathcal{G}_\mathcal{C}$ is an arbitrary element of the mapping class group of $\mathcal{C}=\widehat{\mathbf{C}}\setminus P$. It defines an automorphism of the Teichmüller space by pre-composition: $\tau\mapsto h\cdot\tau$. The mapping class group $\mathcal{G}_\mathcal{C}$ is the fundamental group of the moduli space $\mathcal{M}_P$ and the action of h on $\mathcal{T}_P$ coincides with the corresponding deck transformation. Note that if we identify elements of $\mathcal{T}_P$ with paths in $\mathcal{M}_P$, then the action of $\mathcal{G}_\mathcal{C}$ by deck transformations is given by pre-composition of paths; therefore both actions of $\mathcal{G}_\mathcal{C}$ are left actions.

Let $f=f_R$ denote the rabbit polynomial. The corresponding point of the moduli space is given by the identity map $\{0,c,c^2+c,\infty\}\to\widehat{\mathbf{C}}$. After normalization, we see that the corresponding point of $\mathcal{M}_P=\mathbf{C}\setminus\{0,1\}$ is the fixed point $(c^2+c)/c=t\approx0.8774+0.7449i$ of the rational function $1-1/w^2$.

Let $\tau_0\in\mathcal{T}_P$ be the point of the Teichmüller space given by the identity map $\widehat{\mathbf{C}}\to\widehat{\mathbf{C}}$. It is projected onto the point t of the moduli space $\mathcal{M}_P$. Every point $\tau\in\mathcal{T}_P$ can be identified with the homotopy class of a path ℓ_τ in $\mathcal{M}_P$ starting at t, ending at the projection of τ and equal to the image of a path in $\mathcal{T}_P$ starting at τ_0 and ending at τ. The homotopy class ℓ_τ is uniquely defined and we have $\ell_{h\cdot\tau}=\gamma_h\cdot\ell_\tau$, where $\gamma_h\in\pi_1(\mathcal{M}_P,t)$ is the loop corresponding to $h\in\mathcal{G}_\mathcal{C}$.

PROPOSITION 5.2. *For all* $\tau\in\mathcal{T}_P$, $h\in\mathcal{G}_\mathcal{C}$ *and* $f\in\mathcal{F}$ *the following equalities hold*:

$$\sigma_{h\cdot f}(\tau)=h\cdot\sigma_f(\tau)\quad\textit{and}\quad\sigma_{f\cdot h}(\tau)=\sigma_f(h\cdot\tau).$$

If $\tau\in\mathcal{F}$ *corresponds to a path* ℓ_τ *in* $\mathcal{M}_P$, *then* $\sigma_{f_R}(\tau)$ *is represented by the path* $F^{-1}(\ell_\tau)\,[t]$, *where* $F(w)=1-1/w^2$.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathbf{S}^2 \arrow[r, "h"] \arrow[d, "h\cdot\sigma_f(\tau)"] & \mathbf{S}^2 \arrow[r, "f"] \arrow[d, "\sigma_f(\tau)"] & \mathbf{S}^2 \arrow[r, "h"] \arrow[d, "\tau"] & \mathbf{S}^2 \arrow[d, "h^{-1}\cdot\tau"] \\
\widehat{\mathbf{C}} \arrow[r, "\mathrm{id}"] & \widehat{\mathbf{C}} \arrow[r, "f_\tau"] & \widehat{\mathbf{C}} \arrow[r, "\mathrm{id}"] & \widehat{\mathbf{C}}
\end{tikzcd} \tag{12}$$

It implies that $\sigma_{h\cdot f}(\tau)=h\cdot\sigma_f(\tau)$ and $\sigma_{f\cdot h}(h^{-1}\cdot\tau)=\sigma_f(\tau)$ for all $h\in\mathcal{G}_\mathcal{C}$ and $\tau\in\mathcal{T}_P$. The last equality implies that $\sigma_{f\cdot h}(\tau)=\sigma_f(h\cdot\tau)$ for all $\tau\in\mathcal{T}_P$.

The second statement is a direct corollary of Proposition 5.1. □

Consider now an arbitrary element $h\in\mathcal{G}_{\mathcal{C}}$ and the composition $h\cdot f_R$. It is known (see [4]) that the orbit of τ_0 under iteration of $\sigma_{h\cdot f_R}$ will converge to a point τ such that $\sigma_{h\cdot f_R}(\tau)=\tau$, and the polynomial f_τ in the diagram (11) for $f=h\cdot f_R$ is the polynomial which is Thurston equivalent to $h\cdot f_R$.

Let γ_h be the loop in $\mathcal{M}_P$ corresponding to $h\in\mathcal{G}_{\mathcal{C}}$. By Proposition 5.2, we have $\sigma_{h\cdot f_R}(\tau_0)=h\cdot\sigma_{f_R}(\tau_0)=h\cdot\tau_0$, hence the path representing $\sigma_{h\cdot f_R}(\tau)$ is γ_h.

If ℓ_n is the path representing $\sigma_{h\cdot f_R}^{\circ n}(\tau_0)$, then the path representing the point

$$\sigma_{h\cdot f_R}^{\circ(n+1)}(\tau_0)=h\cdot\sigma_{f_R}(\sigma_{h\cdot f_R}^{\circ n}(\tau_0))$$

is $\gamma_h\cdot F^{-1}(\ell_n)[t]$, by Proposition 5.2. Consequently, the path representing the limit point τ is

$$\ell_\tau=\gamma_h^{(0)}\gamma_h^{(1)}\gamma_h^{(2)}\gamma_h^{(3)}\ldots,$$

where $\gamma_h^{(0)}=\gamma_h$ and $\gamma_h^{(n)}$ is the preimage of γ_h under $F^{\circ n}$ which starts at the end of $\gamma_h^{(n-1)}$.

The endpoint of the path ℓ_τ is one the three fixed points of $1-1/w^2$. It is easy to see that the fixed point $t\approx0.8774+0.7449i$ corresponds to the rabbit, the point $\bar{t}\approx 0.8774-0.7449i$ corresponds to the corabbit and the point ≈-0.7549 corresponds to the airplane.

See for instance Figure 6, where the paths ℓ_τ, for $h=Y=T$ and $h=Y^{-1}=T^{-1}$, are indicated.

One can see that in the first case the path ℓ_τ converges to the fixed point corresponding to the airplane and in the second case it converges to $\bar{t}$, which corresponds to the corabbit. This can be shown after a detailed analysis of the dynamics of the Fatou components of the rational function F.

Proposition 4.1 can also be interpreted in these terms: namely, we have $\psi=\phi_F$. If h belongs to the domain of the virtual endomorphism ψ, then the corresponding loop $\gamma_h\in\pi_1(\mathcal{M}_P,t)$ belongs to the domain of the virtual endomorphism ϕ_F associated with F. We have then that the path converging to the fixed point of $\sigma_{h\cdot f_R}$ is of the form

$$\gamma_h\cdot F^{-1}(\gamma_h)[t]\cdot F^{-2}(\gamma_h)[t_1]\cdot F^{-3}(\gamma_h)[t_2]\ldots,$$

where t_n is the endpoint of $F^{-n}(\gamma_h)[t_{n-1}]$. This path is equal to

$$\gamma_h\gamma_g\cdot F^{-1}(\gamma_g)[t]\cdot F^{-2}(\gamma_g)[t_1']\cdot F^{-3}(\gamma_g)[t_2']\ldots,$$

where $g=\phi_F(h)=F^{-1}(h)[t]$ and t_n' is the endpoint of $F^{-n}(\gamma_g)[t_{n-1}']$. This proves that $h\cdot f_R$ and $\psi(h)\cdot f_R$ are combinatorially equivalent. Similar arguments work also in the case when h does not belong to the domain of ψ (see [8, Theorem 6.6.3]).

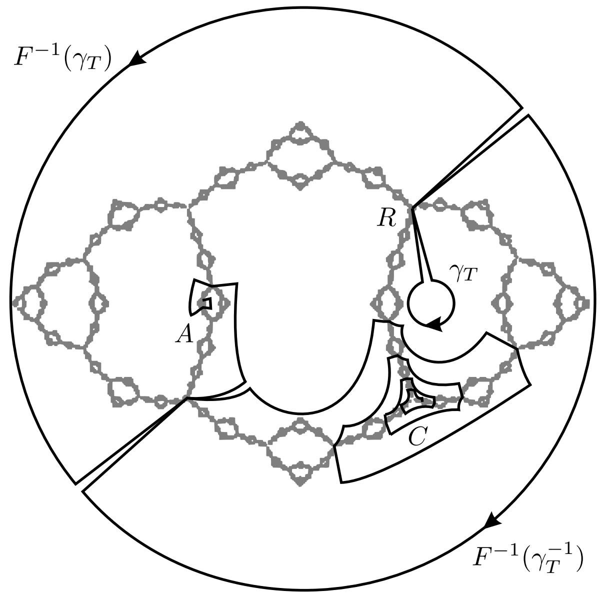

Figure 6. The path ℓ_τ for $h{=}T$ and T^{-1}.

6. Preperiod 1, period 2

There are three families of quadratic topological polynomials with three post-critical points: the first contains the rabbit and the airplane, and its ramification graph is a cycle of length 3. The next family has ramification graph with preperiod 1 and period 2; it contains the polynomial $f_i(z){=}z^2{+}i$ and $f_{-i}{=}z^2{-}i$, as well as obstructed topological polynomials. The last family has ramification graph with preperiod 2 and a fixed post-critical point; it contains the polynomials $\approx z^2-1.5434$ and $\approx z^2-0.2282\pm1.1151i$, and is dealt with in §7.

6.1. The iterated monodromy group

Let us consider first the polynomial $f_i(z){=}z^2{+}i$. The dynamics of f_i on its post-critical set is

$$i\longmapsto -1+i\longmapsto -i\longmapsto -1+i.$$

Let us compute the iterated monodromy group of f_i. We again take $+\infty$ as the basepoint. Let α be the loop going around i in the positive direction and connected to the basepoint by the external ray $R_{1/6}$ and the arc $\left[0,\frac{1}{6}\right]$ of the circle at infinity. The loops β and γ go around the points $-1+i$ and $-i$, and are connected to the circle at infinity by the rays $R_{1/3}$ and $R_{2/3}$, and to the point $+\infty$ by the arcs $\left[0,\frac{1}{3}\right]$ and $\left[0,\frac{2}{3}\right]$,

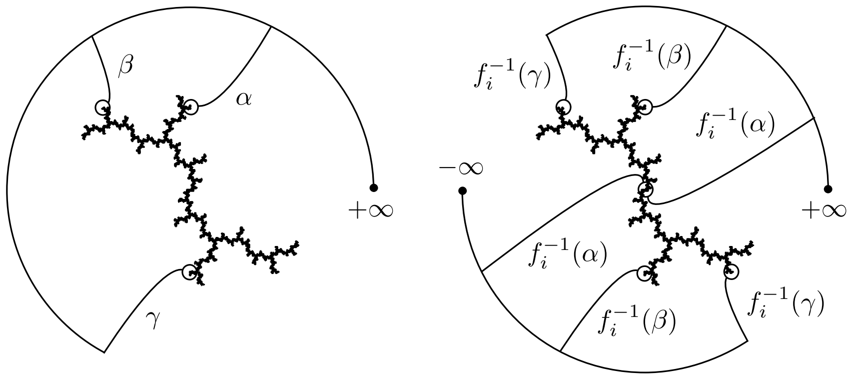

Figure 7. Computation of IMG(z^2+i).

respectively. The connecting paths ℓ_0 and ℓ_1 are, as usual, the trivial path and the upper semicircle. See the loops α, β and γ and their preimages in Figure 7. Computation of the wreath recursion gives

$$\Phi_{f_i}(\alpha) = \langle\langle \alpha^{-1}\beta^{-1}, \beta\alpha \rangle\rangle\sigma, \quad \Phi_{f_i}(\beta) = \langle\langle \alpha, \gamma \rangle\rangle \quad \text{and} \quad \Phi_{f_i}(\gamma) = \langle\langle \beta, 1 \rangle\rangle.$$

We see that the corresponding elements α, β and γ of IMG(f_i) are of order 2.

Let $\mathfrak{F}$ be the set of homotopy classes of branched coverings which have the same ramification graph as f_i. Namely, $\mathfrak{F}$ is the set of homotopy classes of degree-two topological polynomials f such that the finite post-critical set of f has three different points c_1, c_2 and c_3 which are mapped in the following way:

$$c_1 \longmapsto c_2 \longmapsto c_3 \longmapsto c_2$$

(where we assume that the same points c_1, c_2 and c_3 are chosen for all elements of $\mathfrak{F}$).

Among quadratic polynomials $f(z)=z^2+c$, only z^2+i and z^2-i have this ramification graph, since the set of the roots of the equation $f^3(c)=f(c)$ is $\{0, -1, i, -i, -2\}$, but z^2, z^2-1 and z^2-2 have different post-critical dynamics.

Note, however, that there exist obstructed topological polynomials in $\mathfrak{F}$. A way to construct one is shown in Figure 8. It describes the post-critical points c_1, c_2 and c_3 and curves connecting them to infinity in the right part of the figure. The left-hand side shows the preimages of the points and curves. The map folds the horizontal line in two and maps the critical point $f_*^{-1}(c_1)$ to c_1. It is a homeomorphism from each of the upper and lower half-planes to the complement of the line connecting c_1 to infinity.

Consider the simple closed curve Γ around the points c_2 and c_3. It has two f_*-preimages. One is peripheral, and the other is homotopic to Γ. The map f_* is of degree 1 on the non-peripheral preimage of Γ, so the curve Γ is an obstruction.

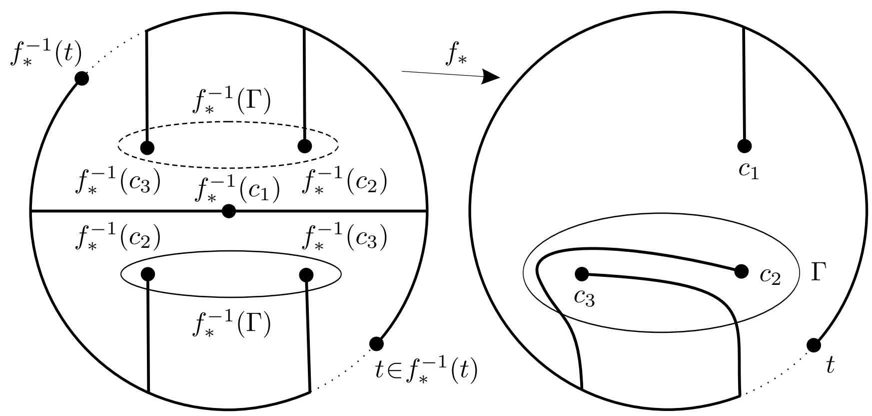

Figure 8. An obstructed topological polynomial.

Let us compute the iterated monodromy group of the map f_*. Let t be the fixed point of f_* on the circle at infinity, as shown in Figure 8, and let the connecting paths from t to its preimages be the trivial path ℓ_0 at t and the semicircle ℓ_1 going in the positive direction from t to its other preimage. Let α, β and γ be generators of $\pi_1(\widehat{\mathbf{C}}\setminus P_{f_*}, t)$ following the circle at infinity and the arcs towards the points c_1, c_2 and c_3, respectively, encircling the point in the positive direction, and returning back to t, as shown in the right part of Figure 8.

It is easy to see from Figure 8 that the wreath recursion is

$$\Phi_{f_*}(\alpha) = \langle\!\langle \alpha^{-1}, \alpha \rangle\!\rangle \sigma, \quad \Phi_{f_*}(\beta) = \langle\!\langle \alpha, \gamma \rangle\!\rangle \quad \text{and} \quad \Phi_{f_*}(\gamma) = \langle\!\langle 1, \gamma\beta\gamma^{-1} \rangle\!\rangle. \tag{13}$$

Direct computation shows that the corresponding elements α, β and γ of the iterated monodromy group are of order 2 and that β and γ commute. Conjugating $\mathrm{IMG}(f_*)$ by $\Delta = \langle\!\langle \alpha\Delta, \Delta \rangle\!\rangle$, we see that the generators of $\mathrm{IMG}(f_*)$ can be defined by the recursion

$$\bar\alpha = \sigma, \quad \bar\beta = \langle\!\langle \bar\alpha, \bar\gamma \rangle\!\rangle \quad \text{and} \quad \bar\gamma = \langle\!\langle 1, \bar\beta \rangle\!\rangle.$$

This is one of the Grigorchuk groups G_ω from [5], namely that given by $\omega = (01)^\infty$. One of the ways to see that it is not the iterated monodromy group of a rational function (and thus that the map f_* is not combinatorially equivalent to a polynomial) is to show that its limit orbispace (see [8, §4.6]) has an isotropy group isomorphic to the Klein group $C_2 \times C_2$, namely $\langle \bar\beta, \bar\gamma \rangle$.

Note also that the composition of f_* with a power of the Dehn twist about the curve Γ is also obstructed (with the same obstruction Γ).

6.2. The moduli space

Let us compute the virtual endomorphism ψ on the mapping class group $\mathcal{G}_{\mathcal{C}}$ of the punctured plane $\mathbf{C}\setminus P=\mathbf{C}\setminus\{i,i-1,-i\}$, via the moduli space approach.

Let $\mathfrak{F}$ be, as before, the set of homotopy classes of branched coverings having the same ramification graph as z^2+i. The moduli space of $\widehat{\mathbf{C}}\setminus P$ is again a plane with two punctures. We normalize the mappings $P\hookrightarrow\widehat{\mathbf{C}}$ so that ∞ is mapped to ∞, 0 is the critical value of $f\in\mathfrak{F}$ and $f(0)=1$. Let us compute the action of the inverse of σ_f on the moduli space in this case. We have

$$f_\tau(0)=1, \quad f_\tau(1)=w_1 \quad \text{and} \quad f_\tau(w_0)=1.$$

We also have that 0 is the critical value of the quadratic polynomial f_τ. Hence f_τ is of the form $(az-1)^2$, where $1/a$ is its critical point. The last equality implies that $aw_0-1=1$ (as $aw_0-1=-1$ would give $w_0=0$), hence $a=2/w_0$, so

$$w_1=f_\tau(1)=\left(\frac{2-w_0}{w_0}\right)^2.$$

Therefore, the correspondence $\sigma_f(\tau)\mapsto\tau$ is projected to the rational function

$$F(w)=\left(\frac{2-w}{w}\right)^2$$

on the moduli space (compare with Proposition 5.1).

The fixed points of the rational function $F(w)$ are $w=1$ and $w=\pm 2i$. The point 1 belongs to the post-critical set and is the puncture of the moduli space. Thus, it does not correspond to any quadratic polynomial (and, as we will see, corresponds to obstructed topological polynomials). The point $2i$ corresponds to the polynomial

$$\frac{4}{(2i)^2}\left(z-\frac{2i}{2}\right)^2=-(z-i)^2,$$

which is conjugate to z^2+i. The point $-2i$ corresponds to $-(z+i)^2$, which is conjugate to z^2-i.

The critical points of the rational function $F(w)$ are $w=2$ and $w=0$. We have the following dynamics on the post-critical orbit:

$$2\longmapsto 0\longmapsto\infty\longmapsto 1\longmapsto 1.$$

Therefore, the post-critical set of $F(w)$ is $\{0,1,\infty\}$. The Julia set of F is the whole sphere, since there are no superattracting cycles. Actually, the Thurston orbifold $\mathcal{O}_F$ of

F is $(2,4,4)$, i.e. the orbifold of the action on $\mathbf{C}$ of the group of affine transformations $\{z\mapsto i^k z+a+bi : k,a,b\in\mathbf{Z}\}$. Moreover, F is induced on the orbifold $\mathcal{O}_F$ by the expanding map $z\mapsto(1+i)z$ of $\mathbf{C}$.

To show this explicitly, consider the Weierstrass function

$$\wp(z)=\frac{1}{z^2}+\sum_{\omega\in\mathbf{Z}[i]\setminus\{0\}}\left[\frac{1}{(z+\omega)^2}-\frac{1}{\omega^2}\right]$$

associated to the lattice $\mathbf{Z}[i]$.

It follows from its definition that $\wp$ is an even function and that, by the choice of the lattice $\mathbf{Z}[i]$, we have $\wp(iz)=-\wp(z)$. Consequently, $i\wp'(iz)=-\wp'(z)$ and thus

$$\wp'(iz)=i\wp'(z).$$

It is known that

$$(\wp'(z))^2=4\wp^3(z)-g_2\wp(z)-g_3,\ (^4)$$

with $g_2=60s_4$ and $g_3=140s_6$, where $s_m=\sum_{\omega\in\mathbf{Z}[i]\setminus\{0\}}\omega^{-m}$ are the Eisenstein series. It is clear that $s_6=0$ because $\mathbf{Z}[i]$ has a 4-fold symmetry, so $g_3=0$.

Another classical fact is the addition formula

$$\wp(z_1+z_2)=-\wp(z_1)-\wp(z_2)+\frac{1}{4}\left(\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right)^2.$$

Let us then compute $\wp((i+1)z)$:

$$\wp(iz+z)=-\wp(iz)-\wp(z)+\frac{1}{4}\left(\frac{\wp'(iz)-\wp'(z)}{\wp(iz)-\wp(z)}\right)^2=\frac{1}{4}\left(\frac{(i-1)\wp'(z)}{-2\wp(z)}\right)^2=-\frac{i}{8}\frac{4\wp^2(z)-g_2}{\wp(z)}.$$

The function $\wp\colon\mathbf{C}\to\widehat{\mathbf{C}}$ realizes the universal covering of the orbifold $(2,2,2,2)$. The group of deck transformations of this covering is the group of all affine transformations of the form $z\mapsto\pm z+c$, with $c\in\mathbf{Z}[i]$. In other words, this group is the group of holomorphic automorphisms $\alpha\colon\mathbf{C}\to\mathbf{C}$ such that $\wp(\alpha(z))=\wp(z)$.

It follows that the function $\wp^2\colon\mathbf{C}\to\widehat{\mathbf{C}}$ realizes the universal covering of the orbifold $(2,4,4)$ for which the group G of deck transformations is the group of affine transformations of the form $z\mapsto i^k z+c$, with $c\in\mathbf{Z}[i]$.

We have

$$\wp^2((i+1)z)=-\frac{1}{64}\left(\frac{4\wp^2(z)-g_2}{\wp(z)}\right)^2=-\frac{(\wp^2(z)-g_2/4)^2}{4\wp^2(z)}.$$

(^{4}) Here $\wp^3(z)=(\wp(z))^3$.

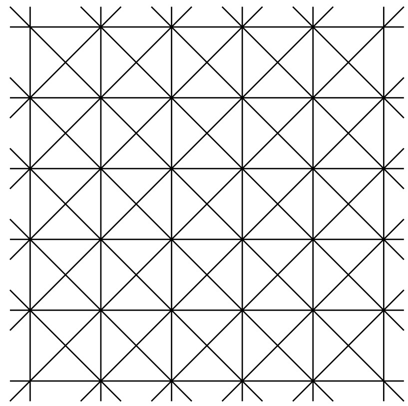

Figure 9. The fundamental group of the orbifold $(2, 4, 4)$.

It follows that the map $z \mapsto (i+1)z$ on the universal cover $\mathbf{C}$ induces on the orbifold $G\backslash\mathbf{C}$ the rational map $t \mapsto -(t-g_2/4)^2/4t$. Performing the change of variables $t=(g_2/4)/(1-w)$, we see that this rational map is conjugate to $w \mapsto ((2-w)/w)^2$.

We conclude that the iterated monodromy group of the rational function

$$w \longmapsto \left(\frac{2-w}{w}\right)^2$$

is isomorphic to the group of affine transformations $z \mapsto i^k z + c$, $c \in \mathbf{Z}[i]$ (see [8, §6.3.2.2] and [1, §5]).

This group is the group of the orientation-preserving automorphisms of the tiling of the plane by triangles shown in Figure 9. The triangles are fundamental domains of the group action. We assume that the vertices of the grid coincide with the Gaussian integers $\mathbf{Z}[i]$.

Let us cut the Riemann sphere $\widehat{\mathbf{C}}$ along the ray $[0, +\infty]$ consisting of the non-negative reals and infinity. This ray will then contain the post-critical set of $F(z)=((2-z)/z)^2$. It is also easy to see that the preimage of this cut in the universal cover $\mathbf{C}$ of the orbifold $\mathcal{O}_F$ is precisely the union of the lines of the tiling in Figure 9. In particular, the preimage of $\widehat{\mathbf{C}}\backslash[0, \infty]$ is the disjoint union of the open triangles of the tiling.

6.3. The iterated monodromy group on the moduli space

Let us compute the recursion defining $\mathrm{IMG}(F)$. We take $t=2i$ as basepoint. The iterated monodromy group is generated by the loops a and b going in the positive direction around 0 and 1, respectively, as shown in Figure 10. The loops a and b correspond to

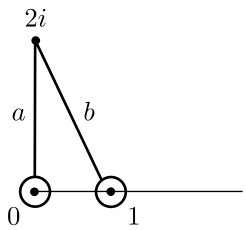

Figure 10. The generators of IMG(F).

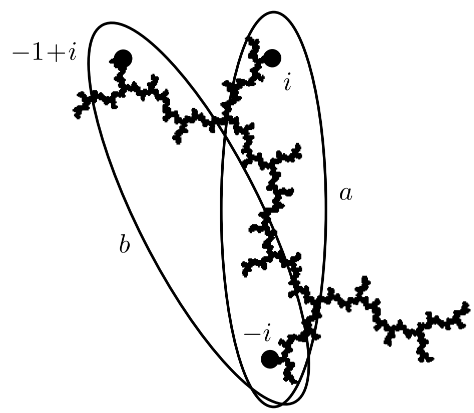

Figure 11. The generators a and b as Dehn twists.

the right Dehn twists about the curves going around i, $-i$ and $-1+i$, $-i$, respectively (see Figure 11).

The preimages of $2i$ under F are $2i$ and $\frac{1}{5}(4-2i)$. Let ℓ_0 be the trivial path at $2i$ and let ℓ_1 be the path connecting $2i$ to $\frac{1}{5}(4-2i)$ passing between 1 and 2, as shown in Figure 12.

The preimages of the loops a and b are shown in Figure 12. We see that the wreath recursion is

$$\Phi(a) = \sigma \quad \text{and} \quad \Phi(b) = \langle\!\langle b^{-1}a^{-1}, b \rangle\!\rangle. \tag{14}$$

The virtual endomorphism associated with the first coordinate of the wreath recursion Φ on the mapping class group is given by

$$\psi(a^2) = 1, \quad \psi(b) = b^{-1}a^{-1} \quad \text{and} \quad \psi(b^a) = b, \tag{15}$$

since

$$\Phi(a^2) = 1, \quad \Phi(b) = \langle\!\langle b^{-1}a^{-1}, b \rangle\!\rangle \quad \text{and} \quad \Phi(b^a) = \langle\!\langle b, b^{-1}a^{-1} \rangle\!\rangle.$$

This virtual endomorphism has the property that $f_i \cdot g = \psi(g) \cdot f_i$ for every g in the index-two subgroup $\operatorname{Dom}\psi = \langle a^2, b, b^a \rangle$ of the mapping class group $\mathcal{G}_{\mathcal{C}}$. The proof is the same as in the case of the rabbit polynomial.

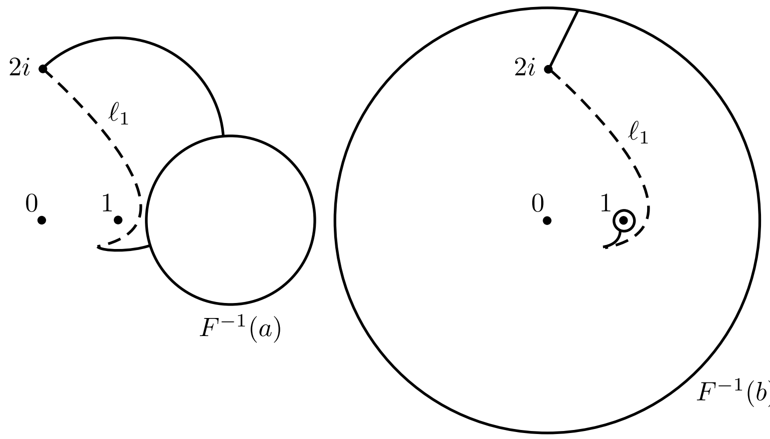

Figure 12. Computation of $\mathrm{IMG}(F)$.

Let us see now what happens on the universal cover of the orbifold $\mathcal{O}_F$. For every path γ in the moduli space $\mathcal{M}_P$ and for every preimage $\zeta\in\mathbf{C}$ of the startpoint of γ in the universal cover of the orbifold $\mathcal{O}_F$, there exists a unique preimage of γ in $\mathbf{C}$ which starts at ζ.

Let $t\in\mathbf{C}$ be the preimage of $2i$ under the covering map $\mathbf{C}\to\mathcal{O}_F$ which belongs to the triangle Δ with vertices 0, 1 and $\frac{1}{2}(1+i)$. Note that 0 and $1\in\mathbf{C}$ are preimages of $1\in\mathcal{O}_F$, that $\frac{1}{2}(1+i)$ is a preimage of $\infty\in\mathcal{O}_F$, and that $\frac{1}{2}$ is a preimage of $0\in\mathcal{O}_F$.

Then the preimages of the loops a and b starting at t go in the positive direction around the points $\frac{1}{2}$ and 1, if we assume that the universal covering map $\mathbf{C}\to\mathcal{O}_F$ is orientation-preserving (see Figure 13). We use here the fact that the segment $\left[\frac{1}{2},1\right]$ is a preimage of the interval $[0,1]\subset\mathcal{O}_F$. It follows that the elements a and b of the fundamental group of $\mathcal{M}_P$ act on the plane $\mathbf{C}$ via the affine maps

$$A(z)=-z+1 \quad\text{and}\quad B(z)=iz+1-i,$$

respectively. Recall that this has to be a left action.

The element $b^{-1}a^{-1}$ of the fundamental group $\pi_1(\mathcal{M}_P,2i)$ is a small loop around ∞ connected to the basepoint by a curve disjoint from the positive ray. It follows that the preimage of the loop $b^{-1}a^{-1}$ in the triangle Δ is the top curve shown in Figure 13.

We have

$$B^{-1}A^{-1}(z)=iz+1 \quad\text{and}\quad B^{A}(z)=iz.$$

The preimage of the loop b^a in the triangle Δ goes around the vertex 0.

Let $\mathcal{T}_P$ be the Teichmüller space modeled on $(\widehat{\mathbf{C}},P)$ for $P=\{i,-1+i,-i,\infty\}$. We

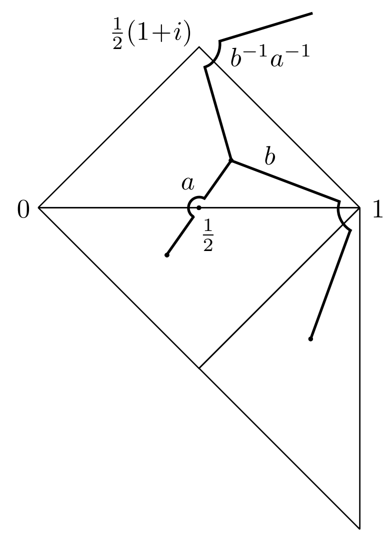

Figure 13. Loops a and b in the universal cover $\mathbf{C}$.

have the following commutative diagram

$$\begin{array}{ccc} \mathcal{T}_P & \longrightarrow & \mathbf{C} \\ \downarrow & & \downarrow \\ \mathcal{M}_P & \hookrightarrow & \mathcal{O}_F, \end{array}$$

where $\mathcal{T}_P \to \mathcal{M}_P$ is the natural projection, $\mathcal{M}_P \hookrightarrow \mathcal{O}_F = \widehat{\mathbf{C}}$ is the identical embedding and $\mathbf{C} \to \mathcal{O}_F$ is the universal covering map. The map $\mathcal{T}_P \to \mathbf{C}$ exists, since $\mathcal{T}_P$ is the universal cover of $\mathcal{M}_P$. It is defined uniquely up to an element of the group $G = \{i^k z + c : c \in \mathbf{Z}[i]\}$ acting on $\mathbf{C}$.

Let $\tau_0 \in \mathcal{T}_P$ be the original complex structure on $\widehat{\mathbf{C}}$. It is projected to $2i \in \mathcal{M}_P$ and we may assume that its image ζ in $\mathbf{C}$ belongs to the triangle Δ. This fixes uniquely the map $\mathcal{T}_P \to \mathbf{C}$.

We know that the correspondence $\tau \mapsto \sigma_{f_i}(\tau)$ on $\mathcal{T}_P$ is semi-conjugated via the map $\mathcal{T}_P \to \mathbf{C}$ to an affine map

$$\Sigma : z \longmapsto i^k \frac{z}{1+i} + \xi$$

for some $k \in \mathbf{Z}$ and $\xi \in \mathbf{C}$ (since F is induced by multiplication by $(1+i)$ on $\mathbf{C}$). The point ζ is fixed by Σ, since τ_0 is fixed under σ_{f_i}.

The preimage of the positive ray under the action of F is the whole real line. The preimage of the negative ray in the universal cover $\mathbf{C}$ divides the triangle Δ into two

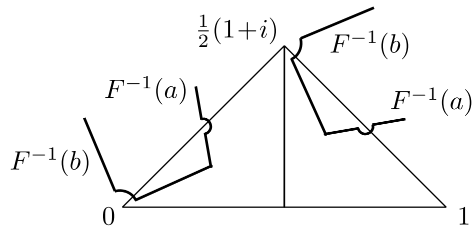

Figure 14.

similar triangles; see Figure 14. This figure also displays the preimages of the curves $F^{-1}(a)$ and $F^{-1}(b)$.

We know from the wreath recursions (14) that the F-preimage of b which starts at $2i$ coincides with $b^{-1}a^{-1}$, which is a loop around infinity. This shows that the point ζ belongs to the right triangle in Figure 14.

It follows that $\Sigma(z)=\frac{1}{2}(i-1)z+1$ and, therefore,

$$\zeta=\frac{1+i}{2+i}=\frac{3+i}{5}.$$

6.4. The answer

We are now in a position to determine the combinatorial equivalence class of the Thurston maps considered in the non-obstructed case. The obstructed maps will be considered later.

Let π be the homomorphism from the mapping class group $\mathcal{G}_\mathcal{C}$ onto the affine group $G=\{z\mapsto i^k z+c : k\in\mathbf{Z} \text{ and } c\in\mathbf{Z}[i]\}$ defined on the generators by

$$\pi(a)=A\colon z\longmapsto -z+1 \quad \text{and} \quad \pi(b)=B\colon z\longmapsto iz+1-i.$$

We have shown above that π is the canonical epimorphism of $\mathcal{G}_\mathcal{C}$ onto the iterated monodromy group $\mathrm{IMG}(((2-z)/z)^2)$. Note that multiplication in the affine group G corresponds to left action, i.e. $\pi(g_1g_2)=\pi(g_1)\circ\pi(g_2)$.

THEOREM 6.1. *Let $h\in\mathcal{G}_\mathcal{C}$ be an element of the mapping class group. Write $\pi(h)$ as the affine transformation $\pi(h)(z)=i^kz+c$. Then*

- *if $k=0$ and $(c-i-1)/(2+i)\notin\mathbf{Z}[i]$, then $f_i\cdot h$ is equivalent to f_i;*
- *if $k=1$ and $(c-i-1)/(1+2i)\notin\mathbf{Z}[i]$, then $f\cdot h$ is equivalent to f_{-i};*
- *in all other cases, f is obstructed.*

Let Q be the group $(\mathbf{Z}[i]/5\mathbf{Z}[i])\rtimes\mathbf{Z}/4$ of order 100, where the action of $\mathbf{Z}/4$ is by multiplication by i. Then the answer (in $\{f_i, f_{-i}, \text{obstructed}\}$) depends only on the image of h in Q under the homomorphism $\mathcal{G}_\mathcal{C}\to Q$ mapping a to $(-1,2)$ and b to $(1-i,1)$.

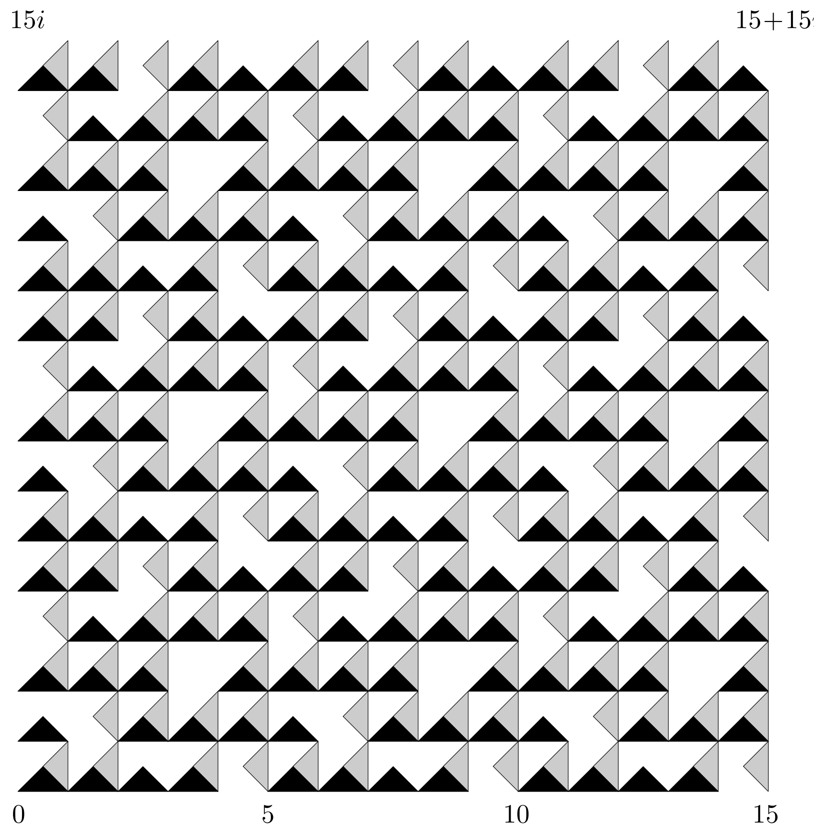

Figure 15.

Therefore, unlike in the rabbit/airplane case, the answer is periodic. See Figure 15 for dependence of the answer on the element $\pi(h)$ of the group G. The lower left black triangle is the origin (it corresponds to the trivial element of G). For every other triangle Δ of the picture there is a unique element h mapping the original triangle to Δ. If Δ is black, then $f_i \cdot h$ is equivalent to z^2+i, if Δ is grey, then $f_i \cdot h$ is equivalent to z^2-i. White triangles correspond to obstructed maps. The picture is periodic with period 5 in both directions.

Proof of Theorem 6.1. By Proposition 5.2, the map $\sigma_{f_i \cdot h}$ is projected to the affine transformation $\Sigma \circ \pi(h)$ of $\mathbf{C}$. The fixed point of $\sigma_{f_i \cdot h}$, if it exists, is mapped to the fixed point of $\Sigma \circ H(z) = \frac{1}{2}(i-1)(i^k z + c) + 1$ in $\mathbf{C}$, which is

$$\xi = \frac{\frac{1}{2}(i-1)c+1}{1-\frac{1}{2}(i-1)i^k} = -\frac{c-i-1}{i+1+i^k}.$$

Let us consider the possible values for k in turn.

If $k=0$ then $\xi = -(c-i-1)/(i+2)$, which is in the same G-orbit as $(c+1)/(i+2)$. The possible residues modulo $i+2$ are 0, $\pm i$ and ± 1, which, modulo multiplication by powers of i, are either 0 or 1. Consequently, if $(c+1)/(i+2)\in\mathbf{Z}[i]$, then either ξ belongs to the G-orbit of 0, or it belongs to the G-orbit of $1/(i+2)$, which coincides with the G-orbit of $\zeta=\frac{1}{5}(3+i)$. If ξ belongs to the orbit of 0, then the map $f_i\cdot h$ is obstructed. If it belongs to the orbit of ζ, then $f_i\cdot h$ is equivalent to $f_i(z)=z^2+i$.

If $k=1$ then $\xi=-(c-i-1)/(2i+1)$, which is in the same G-orbit as $(c+i)/(2i+1)$. Here again, if $(c+i)/(2i+1)\in\mathbf{Z}[i]$, then $f_i\cdot h$ is obstructed, otherwise it is equivalent to z^2-i.

If $k=2$ then $\xi=-(c-i-1)/i$, which is integral, and we get an obstructed map.

If $k=3$ then $\xi=-c+i+1$, and we also get an obstructed map. $\square$

6.5. The obstructed cases

No description of combinatorial classes of obstructed Thurston maps follows directly from Thurston's Theorem 1.1; therefore we have to treat the obstructed cases separately.

Let $\mathcal{F}$ be the set of quadratic topological polynomials $f:\widehat{\mathbf{C}}\to\widehat{\mathbf{C}}$ with finite critical value i, and such that $f(i)=-1+i$, $f(-1+i)=-i$ and $f(-i)=-1+i$. Let $\mathfrak{F}$ be the set of homotopy classes of elements of $\mathcal{F}$ relative to $P=\{\infty,i,-1+i,-i\}$. Let $\mathcal{G}_\mathcal{C}$ be the mapping class group of $\widehat{\mathbf{C}}\setminus P$. Recall that $\mathfrak{F}$ is a $\mathcal{G}_\mathcal{C}$-bimodule by pre- and post-composition.

Proposition 6.2. *For every $f\in\mathfrak{F}$ there exist elements $h,g\in\mathcal{G}_\mathcal{C}$ such that $h\cdot f\cdot g$ is equal to the polynomial $f_i(z)=z^2+i$.*

In the terminology of [8], this means that the $\mathcal{G}_\mathcal{C}$-bimodule $\mathfrak{F}$ is *irreducible*.

Proof. Let $\mathcal{T}_P$ and $\mathcal{M}_P$ denote the Teichmüller and the moduli space of $\widehat{\mathbf{C}}\setminus P$, respectively. We identify, as above, $\mathcal{M}_P$ with $\widehat{\mathbf{C}}\setminus\{0,1,\infty\}$, so that the correspondence $\sigma_f(\tau)\mapsto\tau$ on $\mathcal{T}_P$ is projected to the rational function $F(w)=((2-w)/w)^2$ on $\mathcal{M}_P$.

Let $\zeta=2i$ be the fixed point of F corresponding to the polynomial z^2+i. Let τ_i be an arbitrary preimage of ζ in $\mathcal{T}_P$. Then the image of $\sigma_f(\tau_i)$ in $\mathcal{M}_P$ is equal to one of the F-preimages of ζ, i.e. either to $2i$ or to $\frac{1}{5}(4-2i)$.

The path $\sigma_f(a)$ is a lift to $\mathcal{T}_P$ of an F-preimage of the loop a in $\mathcal{M}_P$. We know, by (14), that both F-preimages of a are paths starting at one of the points $2i$, $\frac{1}{5}(4-2i)$ and ending at the other.

Consequently, if the image of $\sigma_f(\tau_i)$ in $\mathcal{M}_P$ is equal to $\frac{1}{5}(4-2i)$, then the image of $\sigma_{f\cdot a}(\tau_i)=\sigma_f(a\cdot\tau_i)$ is equal to $2i$ (see Proposition 5.2).

Hence, either $\sigma_f(\tau_i)$ or $\sigma_{f\cdot a}(\tau_i)$ belongs to the $\pi_1(\mathcal{M}_P)$-orbit of τ_i, i.e. there exists a homeomorphism $h\in\mathcal{G}_\mathcal{C}$ such that $\sigma_{h\cdot f}(\tau_i)=h\cdot\sigma_f(\tau_i)=\tau_i$ or $\sigma_{h\cdot f\cdot a}(\tau_i)=h\cdot\sigma_{f\cdot a}(\tau_i)=\tau_i$. But this implies that either $h\cdot f$ or $h\cdot f\cdot a$ is combinatorially equivalent to f_i. $\square$

Corollary 6.3. *For every branched covering $f\in\mathfrak{F}$ there exists $g\in\mathcal{G}_\mathcal{C}$ such that $f=f_i\cdot g$.*

Proof. This follows from Proposition 6.2 and the fact that the associated virtual endomorphism ψ is onto; see (15). □

It will be more convenient for us to proceed with the branched covering $f_i\cdot a$, which is obstructed by Theorem 6.1. We now show that it is equivalent to the branched covering f_* shown in Figure 8.

For that purpose, note that the twist a acts on the generators α, β and γ of the fundamental group of the punctured plane in the same way as the element h in (6). Applying that twist to the wreath recursion Φ_{f_i}, we see that the wreath recursion $\Phi_{f_i a}$ is given by (13), so that the maps $f_i a$ and f_* are equivalent, by Proposition 3.1.

Let ϕ denote the virtual endomorphism of $\mathcal{G}_\mathcal{C}$ such that $f_*\cdot g=\phi(g)\cdot f_*$. It follows from the wreath recursion Φ defined by (14) that ϕ is the virtual endomorphism associated with the second coordinate of the wreath recursion Φ, and thus has domain $H=\langle a^2, b, b^a\rangle$ and is given by

$$\phi(a^2)=1,\quad \phi(b)=b \quad\text{and}\quad \phi(b^a)=b^{-1}a^{-1}. \tag{16}$$

Let us introduce the following subgroups $\mathcal{E}_n\triangleleft\mathcal{G}_\mathcal{C}$ (see [8, §2.13.1]). We set $\mathcal{E}_0=\{1\}$ and

$$\mathcal{E}_n=\{g\in H<\mathcal{G}_\mathcal{C} : \phi(g) \text{ and } \phi(g^a) \text{ belong to } \mathcal{E}_{n-1}\}.$$

In other words, $\mathcal{E}_n$ is the kernel of the wreath recursion $\Phi^n\colon\mathcal{G}_\mathcal{C}\to\mathcal{G}_\mathcal{C}\wr\mathfrak{S}(X^n)$ describing the action and the restrictions of $\mathcal{G}_\mathcal{C}$ on words of length n.

It follows that $\mathcal{E}_n$ are normal subgroups and that $\mathcal{E}_n\geqslant\mathcal{E}_{n-1}$. Let

$$\mathcal{E}_\infty=\bigcup_{n\geqslant 0}\mathcal{E}_n.$$

Lemma 6.4. *If $g\in\mathcal{E}_\infty$ then for all $f\in\mathfrak{F}$ the branched coverings $f\cdot g$ and f are combinatorially equivalent.*

Proof. We have $f=f_*\cdot h$ for some $h\in\mathcal{G}_\mathcal{C}$, by Corollary 6.3. Let us prove the statement by induction on n such that $g\in\mathcal{E}_n$. The statement is trivial for $n=0$. We have

$$f_*\cdot hg=f_*\cdot hgh^{-1}\cdot h=\phi(hgh^{-1})\cdot f_*\cdot h=\phi(hgh^{-1})\cdot(f_*\cdot h\phi(hgh^{-1}))\cdot\phi(hgh^{-1})^{-1}.$$

Since $hgh^{-1}\in\mathcal{E}_n$, we have $\phi(hgh^{-1})\in\mathcal{E}_{n-1}$. By the inductive hypothesis, $f_*\cdot h\phi(hgh^{-1})$ is combinatorially equivalent to $f_*\cdot h$. Thus, it follows that $f_*\cdot hg$ is combinatorially equivalent to $f_*\cdot h$. □

COROLLARY 6.5. *The combinatorial equivalence class of a branched covering $f_*\cdot g$ depends only on the image of g in the quotient group $\widetilde{G}=\mathcal{G}_\mathcal{C}/\mathcal{E}_\infty$.*

It is easy to see that the wreath recursion Φ induces a well-defined wreath recursion (which we will also denote by Φ) on $\widetilde{G}=\mathcal{G}_\mathcal{C}/\mathcal{E}_\infty$, given by the same formula on the generators. We also denote by ϕ the virtual endomorphism induced on $\widetilde{G}$ by the virtual endomorphism ϕ of G.

We have shown earlier that the iterated monodromy group of $F(w)=((2-w)/w)^2$ is the quotient of $\mathcal{G}_\mathcal{C}$ given by the presentation

$$\mathrm{IMG}(F)=G=\mathbf{Z}[i]\rtimes\mathbf{Z}/4=\langle A,B\mid A^2,(AB)^4,B^4\rangle. \tag{17}$$

The virtual endomorphism ϕ induces a virtual endomorphism of G (still denoted ϕ), which is contracting on G.

In $\mathcal{G}_\mathcal{C}$ we have the equalities

$$\phi(a^2)=1,\quad \phi((ab)^4)=b^{-1}a^{-2}b \quad\text{and}\quad \phi(a^{-1}(ab)^4a)=a^{-2}, \tag{18}$$

which imply that $a^2\in\mathcal{E}_1$ and $(ab)^4\in\mathcal{E}_2$.

The following lemma computes the nucleus of the wreath recursion on $\widetilde{G}$.

LEMMA 6.6. *For every $g\in\widetilde{G}$ there exists n such that, for every word $v\in X^n$ of length greater than n, the restriction $g|_v$ (computed with respect to the wreath recursion Φ) belongs to the set*

$$\mathcal{N}=\{a,ab,ab^2,ab^3,a^b,b^a,(ab)^2,b^{-2}ab,abab^2\}^{\pm1}\cup\{b^k\}_{k\in\mathbf{Z}}.$$

Proof. It is sufficient to show that $\mathcal{N}$ is state-closed, and that there exists m such that modulo the relations $a^2=(ab)^4=1$ obtained in (18) we have $A|_v\subseteq\mathcal{N}$ for all words v of length m, where

$$A=\mathcal{N}\cdot\{1,a,b,b^{-1}.\}$$

Figure 16 shows the Moore diagram of the set $\mathcal{N}$.

We see that $\mathcal{N}$ is state-closed. Therefore it is sufficient to check our condition for $A=\mathcal{N}\cdot\{a,b,b^{-1}\}\setminus\mathcal{N}$. Moreover, multiplication by a from either side does not change the restrictions, therefore, we may take A equal to

$$\begin{aligned}\mathcal{N}\cdot\{b,b^{-1}\}\setminus\mathcal{N}=\{&ab^4,abab^{-1},b^{-2}ab^2,abab^3,b^{-1}ab^{-1},b^{-2}ab^{-1},b^{-3}ab,b^{-3}ab^{-1},\\ &ab^{-1}ab,ab^{-1}ab^{-1},b^{-1}ab^3,b^{-2}ab^{-1}ab,b^{-2}ab^{-1}ab^{-1}=b^{-1}aba\}.\end{aligned}$$

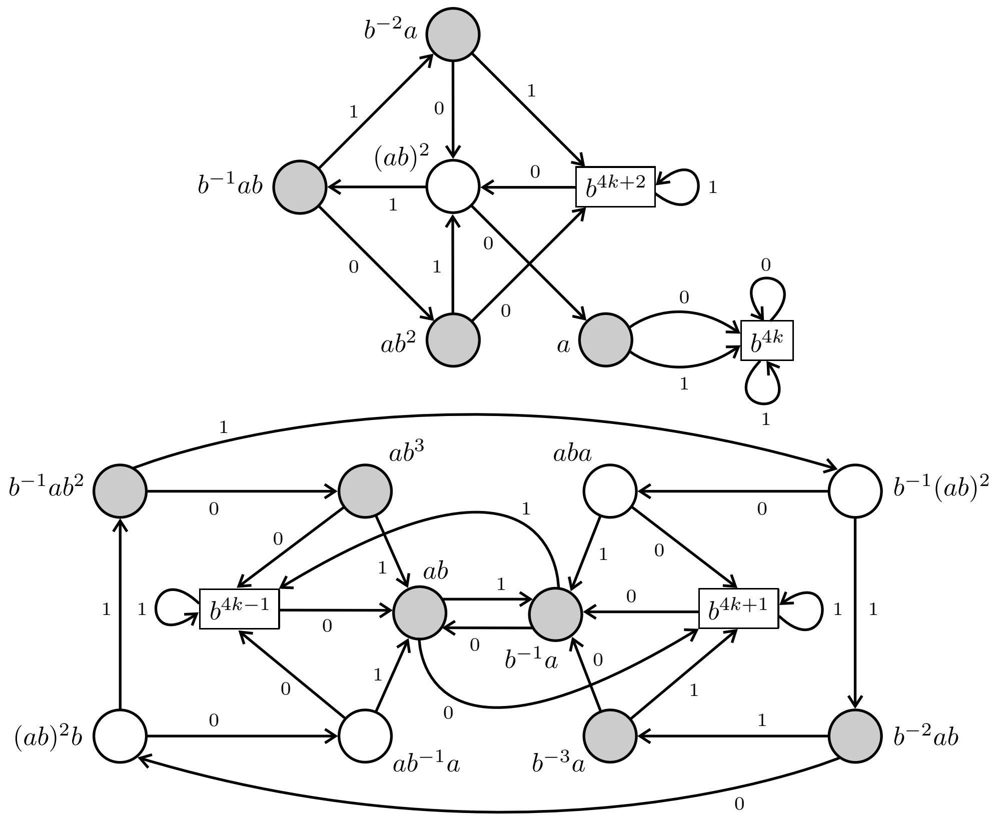

Figure 16. Nucleus of $\mathcal{G}_{\mathcal{C}}$.

Taking into account again that multiplication by a does not change the restrictions, and replacing some of the elements by their inverses (as $\mathcal{N}=\mathcal{N}^{-1}$), we reduce our checking to the set

$$A=\{bab^{-1}, b^{-2}ab^2, bab^3, b^{-1}ab^3, b^{-2}ab^{-1}ab\}.$$

But we have

$$\begin{aligned} bab^{-1} &= \langle\langle b^{-1}ab^{-1}, bab\rangle\rangle\sigma = \langle\langle (ab)^2a, a(ab)^2\rangle\rangle\sigma,\\ b^{-2}ab^2 &= \langle\langle abab^3, b^{-3}ab^{-1}a\rangle\rangle\sigma = \langle\langle a\cdot bab^3, (a\cdot bab^3)^{-1}\rangle\rangle\sigma,\\ bab^3 &= \langle\langle b^{-1}ab^3, bab\rangle\rangle\sigma = \langle\langle b^{-1}ab^3, a(ab)^2\rangle\rangle\sigma,\\ b^{-1}ab^3 &= \langle\langle a\cdot b^4, b^{-1}ab\rangle\rangle\sigma,\\ b^{-2}ab^{-1}ab &= \langle\langle (ab)^2\cdot a, b^{-2}\rangle\rangle, \end{aligned}$$

hence we may take $m=4$, since the longest chain staying outside $\mathcal{N}$ is

$$b^{-2}ab^2 \longmapsto abab^3 \longmapsto b^{-1}ab^3 \longmapsto ab^4 \longmapsto b^4. \qquad \square$$

LEMMA 6.7. *Let* $P=\bigcap_{n\geqslant 1}\operatorname{Dom}\phi^n$ *be stabilizer of the path* $111\ldots$ *in the group* $\widetilde{G}$. *For every* $g\in P$ *there exists* n *such that* $\phi^n(g)\in\langle b\rangle$.

Proof. Let $g\in P$ stabilize the path $111\ldots$. There exists n such that $\phi^n(g)\in\mathcal{N}$, where $\mathcal{N}$ is given by Lemma 6.6. The sequence $(\phi^m(g))_{m\geqslant n}$ must pass through inactive (white) states of the Moore diagram of $\mathcal{N}$ (Figure 16), and follow arrows labeled by 1. The only such infinite paths in the Moore diagram are the loops attached to the powers of b. Consequently, $\mathcal{N}\cap P=\langle b\rangle$. □

PROPOSITION 6.8. *Every branched covering* $f\in\mathfrak{F}$ *is combinatorially equivalent either to* z^2+i, *or to* z^2-i, *or to* $f_*\cdot b^k$ *for some* $k\in\mathbf{Z}$, *where* b *is, as before, the Dehn twist about the curve* Γ *encircling the points* $-1+i$ *and* $-i$. *The branched coverings* $f_*\cdot b^k$ *are obstructed, with obstruction* Γ.

Proof. Consider the map $\bar{\phi}:\widetilde{G}\to\widetilde{G}$ defined by

$$\bar{\phi}: g\mapsto\begin{cases}\phi(g), & \text{if } w \text{ belongs to the domain of } \phi,\\ a\phi(ga), & \text{otherwise.}\end{cases}\tag{19}$$

Following the line of reasoning used for the rabbit polynomial (see Proposition 4.1), and using Corollary 6.5, we check the identity $f_*\cdot g=\psi(g)\cdot f_*$. It then follows that the branched coverings $f_*\cdot g$ and $f_*\cdot\bar{\phi}(g)$ are combinatorially equivalent for all $g\in\widetilde{G}$.

Lemma 6.6 and the argument of Proposition 4.2 show that it is sufficient to study the dynamics of the map induced by $\bar{\phi}$ on the set $\mathcal{N}\cup a\mathcal{N}\subset\widetilde{G}$.

Direct computations show that the map $\bar{\phi}$ on $\mathcal{N}\cup a\mathcal{N}$ has fixed points a and b^k, for $k\in\mathbf{Z}$, and cycles $ab\leftrightarrow b^{-a}$ and $a^b\mapsto b^{-2a}\mapsto abab\mapsto a^b$.

This shows that for every $g\in\widetilde{G}$ there exists n such that $\bar{\phi}^n(g)\in\{a,ab,a^b\}\cup\{b^k\}_{k\in\mathbf{Z}}$. Consequently, by Corollary 6.5, every element $f\in\mathfrak{F}$ is combinatorially equivalent either to $f_*\cdot a=f_i\cdot a^2$, or to $f_*\cdot ab=f_i\cdot a^2b$, or to $f_*\cdot b^{-1}ab=f_i\cdot ab^{-1}ab$, or to $f_*\cdot b^k$ for some $k\in\mathbf{Z}$.

The branched coverings $f_i\cdot a^2$, $f_i\cdot a^2b$ and $f_i\cdot ab^{-1}ab$ are combinatorially equivalent to f_i, f_{-i} and f_i, respectively, by Theorem 6.1. The branched coverings $f_*\cdot b^k=f_i\cdot ab^k$ are obstructed, also by Theorem 6.1. □

The following proposition completely describes the group $\widetilde{G}=\mathcal{G}_{\mathcal{C}}/\mathcal{E}_\infty$.

PROPOSITION 6.9. $\widetilde{G}$ *is a (non-split) extension of* G *by the* G*-module* $K=\mathbf{Z}[\langle B\rangle\backslash G]$. *It may be given by the presentation*

$$\widetilde{G}=\langle a,b\,|\,a^2,(ab)^4,[b^4,b^{4w}]\ \textit{for all}\ w\in\widetilde{G}\rangle.$$

We will only need the fact that the order of b in $\widetilde{G}$ is infinite, so we give only a sketch of the proof.

Proof. Let N denote the normal closure in $\mathcal{G}_{\mathcal{C}}$ of $\langle a^2,(ab)^4,[b^4,b^{4w}]$ for all $w\in\mathcal{G}_{\mathcal{C}}\rangle$. We wish to prove that $N=\mathcal{E}_\infty$.

By comparing $\mathcal{G}_{\mathcal{C}}/N$ with the presentation (17), we see that the map $a\mapsto A$, $b\mapsto B$ defines a natural epimorphism $\mathcal{G}_{\mathcal{C}}/N\to G$, whose kernel $K=\langle b^{4w}:w\in\mathcal{G}_{\mathcal{C}}\rangle N$ is abelian, isomorphic as a G-module to $\mathbf{Z}[\langle B\rangle\backslash G]$.

A straightforward argument shows that the wreath recursion Φ on $\mathcal{G}_{\mathcal{C}}$ induces a wreath recursion (still denoted Φ) on the group $\mathcal{G}_{\mathcal{C}}/N$. Furthermore, the virtual endomorphism $\phi\colon\langle B,B^A\rangle\to G$ is a bijection, and induces a bijection $\Phi\colon K\to K\times K$, given by

$$\Phi(b^{4w})=\begin{cases}(1,b^{4w_1}), & \text{if } \Phi(w)=(w_0,w_1),\\ (b^{4w_1},1), & \text{if } \Phi(w)=(w_0,w_1)\sigma.\end{cases}\tag{20}$$

We first prove that $N\subseteq\mathcal{E}_\infty$. We already checked in (18) that the first two generators of N lie in $\mathcal{E}_\infty$. We may view equation (20) as a relation in $\widetilde{G}$; Lemma 6.6 then implies that for every $w\in\widetilde{G}$ there exists $n_0(w)\in\mathbf{N}$ such that if $n\geqslant n_0(w)$ then all coordinates of $\Phi^n(b^{4w})\in\widetilde{G}^{2^n}$ are trivial, except for one, which is equal to b^4. Therefore, the elements b^{4w_1} and b^{4w_2} commute for all $w_1,w_2\in\widetilde{G}$.

We next prove $\mathcal{E}_\infty\subseteq N$. Assume for contradiction that $\mathcal{E}_n\not\supseteq N$ for some minimal n; and choose $r\in\mathcal{E}_n\setminus N$. Then clearly $r\in K$, because the action of r on X^* is trivial, so the image of r in G is trivial. We have $\Phi(r)=(r_0,r_1)$, with $r_0,r_1\in\mathcal{E}_{n-1}$. Since Φ is a bijection $K\to K\times K$, one of r_0 and r_1 is non-trivial in $\mathcal{G}_{\mathcal{C}}/N$, and we have contradicted the minimality of n. □

PROPOSITION 6.10. *None of the maps $f_*\cdot b^k$, for $k\in\mathbf{Z}$, are equivalent to each other.*

Proof. We may work inside the group $\widetilde{G}$, by Corollary 6.5.

Suppose that $f_*\cdot b^r$ and $f_*\cdot b^s$ are equivalent. Then there exists $h\in\widetilde{G}$ such that $h\cdot f_*\cdot b^r=f_*\cdot b^s h$ in $\mathfrak{F}$. We get $hb^r\cdot f_*=f_*\cdot b^s h$, which implies that $hb^r=\phi(b^s h)$, or

$$b^{-s}hb^r=\phi(h),\tag{21}$$

by definition of ϕ and since the left $\mathcal{G}_{\mathcal{C}}$-action on the bimodule $\mathfrak{F}$ is free (which follows, for instance, from the fact that the $\mathcal{G}_{\mathcal{C}}$-bimodule $\mathfrak{F}$ is isomorphic to the bimodule associated with the self-covering $F(z)=((2-z)/z)^2$ of the moduli space).

By induction, we get from (21) that for every n,

$$\phi^n(h)=b^{-sn}hb^{rn}.$$

By Lemma 6.7, there exists n such that $\phi^n(h)\in\langle b\rangle$. Consequently, $h=b^m$ for some $m\in\mathbf{Z}$, hence $b^{-s}hb^r=h$ and $b^{r-s}=1$. But this implies that $r=s$, since the order of b is infinite. □

COROLLARY 6.11. *Let f be a degree-two topological polynomial with preperiod of length* 1 *and period of length* 2. *Then f is combinatorially equivalent to precisely one of f_i, f_{-i}, or $f_*\cdot b^n$ for some $n\in\mathbf{Z}$.*

Theorem 6.1 *describes the equivalence classes of the polynomials f_i and f_{-i}. If $f_*\cdot g=f_i\cdot ag$ is an obstructed topological polynomial, then it is equivalent to $f_*\cdot b^n$ if and only if there exists $m\in\mathbf{N}$ such that $\bar\phi^m(g)=b^n$ in $\widetilde G$, where $\bar\phi\colon\widetilde G\to\widetilde G$ is given by* (16) *and* (19).

7. Preperiod 2, period 1

7.1. Iterated monodromy groups

We next consider the degree-two topological polynomials with ramification graph

$$c_1\longmapsto c_2\longmapsto c_3\longmapsto c_3,$$

where c_1 is the critical value. The calculations are close to those for the rabbit and airplane polynomials, so will be given in a more condensed manner.

If $f(z)=z^2+c$ is a polynomial whose ramification graph has preperiod 2 and period 1, then the parameter c must be a root of the polynomial x^3+2x^2+2x+2, i.e. one of approximately

$$-0.2282+1.1151i,\quad -0.2282-1.1151i\quad\text{or}\quad -1.5437.$$

The corresponding points of the Mandelbrot set are the landing points of the external rays at angles $\frac14$, $\frac34$ and $\frac{5}{12}$, respectively. The last angle is particular because, under doubling, we have

$$\tfrac{5}{12}\longmapsto\tfrac56\longmapsto\tfrac23\longmapsto\tfrac13\longmapsto\tfrac23;$$

but the dynamical rays with angles $\frac13$ and $\frac23$ land at the same point of the Julia set. Let us denote the corresponding polynomials by $f_{1/4}$, $f_{3/4}$ and $f_{5/12}$.

The wreath recursions are given by

$$\begin{aligned}
\Phi_{f_{1/4}}(\alpha)&=\langle\!\langle\alpha^{-1}\beta^{-1},\beta\alpha\rangle\!\rangle\sigma, &\Phi_{f_{1/4}}(\beta)&=\langle\!\langle\alpha,1\rangle\!\rangle, &\Phi_{f_{1/4}}(\gamma)&=\langle\!\langle\gamma,\beta\rangle\!\rangle;\\
\Phi_{f_{3/4}}(\alpha)&=\langle\!\langle\beta^{-1}\alpha^{-1},\alpha\beta\rangle\!\rangle\sigma, &\Phi_{f_{3/4}}(\beta)&=\langle\!\langle1,\alpha\rangle\!\rangle, &\Phi_{f_{3/4}}(\gamma)&=\langle\!\langle\gamma,\beta\rangle\!\rangle;\\
\Phi_{f_{5/12}}(\alpha)&=\langle\!\langle\alpha^{-1}\gamma^{-1},\gamma\alpha\rangle\!\rangle\sigma, &\Phi_{f_{5/12}}(\beta)&=\langle\!\langle\alpha,1\rangle\!\rangle, &\Phi_{f_{5/12}}(\gamma)&=\langle\!\langle\gamma^{\alpha},\beta\rangle\!\rangle.
\end{aligned}$$

Here α of f_t is a small loop going positively around the critical value and connected to the circle at infinity by the external ray at angle t, β is connected by the ray at angle $2t$, and γ at angle 0 (for $f_{1/4}$ and $f_{3/4}$) or $\frac23$ (for $f_{5/12}$). See for example the loops α, β and γ for the polynomial $f_{1/4}$ in Figure 17.

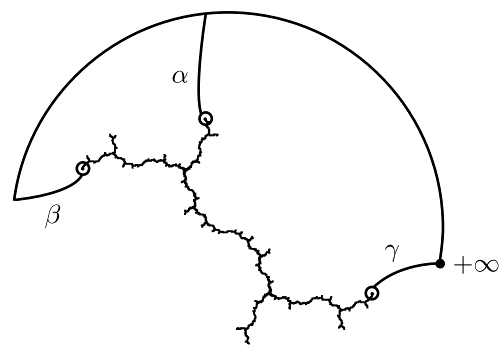

Figure 17.

Their respective nuclei are

$$\mathcal{N}_{1/4}=\{1,\alpha,\beta,\gamma,\gamma^{\alpha\beta},\alpha\beta,\beta\alpha,(\beta\alpha\gamma)^{\pm1}\},$$
$$\mathcal{N}_{3/4}=\{1,\alpha,\beta,\gamma,\gamma^{\beta\alpha},\alpha\beta,\beta\alpha,(\alpha\beta\gamma)^{\pm1}\},$$
$$\mathcal{N}_{5/12}=\{1,\alpha,\beta,\gamma,\gamma^{\alpha},\beta^{\gamma\alpha},\alpha\gamma,\gamma\alpha,(\beta\gamma\alpha)^{\pm1}\}.$$

Direct verification shows that the nuclei are different as automata. It is not hard to prove that there is no possible obstruction, so every degree-two branched covering with the given post-critical dynamics is equivalent to precisely one of $f_{1/4}$, $f_{3/4}$ or $f_{5/12}$.

7.2. The iterated monodromy group on the moduli space

As before, we force the post-critical sets to be of the form $\{0,1,w\}$, so that

$$w\in\widehat{\mathbf{C}}\setminus\{0,1,\infty\}$$

represents a point in the moduli space. A polynomial with critical value 0 and orbit $0\mapsto1\mapsto w$ is of the form $f(z)=(az-1)^2$; then we have

$$(a-1)^2=w_1 \quad\text{and}\quad (aw_0-1)^2=w_1,$$

so $aw_0-1=-a+1$, hence $a=2/(w_0+1)$ and

$$F(w_0)=w_1=\left(\frac{w_0-1}{w_0+1}\right)^2.$$

The fixed points of F are approximately

$$-0.6478+1.7214i,\quad -0.6478-1.7214i \quad\text{and}\quad 0.2956.$$

The fixed point w corresponds to the polynomial $(2z/(w+1)-1)^2$, which is conjugate to $z^2-2/(1+w)$. Direct computation shows that the above fixed points correspond to the polynomials $f_{1/4}, f_{3/4}$ and $f_{5/12}$, respectively.

Let us consider the first point $t\approx-0.6478+1.7214i$, and let a and b be small loops going around the points 0 and 1 in the positive direction and connected to t by straight segments. The F-preimages of t are t itself and $1/t\approx-0.1915-0.5089i$. Take the connecting path ℓ_0 to be the trivial path at t, and let ℓ_1 be a path connecting t to $1/t$, intersecting the real line only once, with the intersection point strictly between 0 and 1.

Then the iterated monodromy group of F is given by the recursions

$$\Phi(a)=\langle\!\langle 1,b\rangle\!\rangle\sigma \quad \text{and} \quad \Phi(b)=\langle\!\langle b^{-1}a^{-1},a\rangle\!\rangle.$$

This wreath recursion is contracting on the free group $\langle a,b\rangle$, and has nucleus

$$\mathcal{N}=\{1,a,b,ab,a^{-1}b\}^{\pm 1}.$$

Let ψ denote the virtual endomorphism of $\mathcal{G}_{\mathcal{C}}$ corresponding to projection on the first coordinate, i.e.

$$\psi(a^2)=b, \quad \psi(b)=b^{-1}a^{-1} \quad \text{and} \quad \psi(aba^{-1})=a.$$

Define $\bar{\psi}\colon\mathcal{G}_{\mathcal{C}}\to\mathcal{G}_{\mathcal{C}}$ by

$$\bar{\psi}\colon g\mapsto\begin{cases}\psi(g), & \text{if } g \text{ belongs to the domain of } \psi,\\ a\psi(ga^{-1}), & \text{otherwise.}\end{cases} \tag{22}$$

LEMMA 7.1. *For every $g\in\mathcal{G}_{\mathcal{C}}$, there is $n\in\mathbf{N}$ such that $\bar{\psi}^n(g)\in\{1,a,a^{-1}b\}$.*

Proof. It suffices to compute the orbits of $\bar{\psi}$ on $\mathcal{N}\cup a\mathcal{N}$. There are fixed points 1 and a, and a cycle $ab^{-1}a\leftrightarrow a^{-1}b$. All the other elements of $\mathcal{N}\cup a\mathcal{N}$ are attracted to these cycles. □

The generators a and b of the mapping class group $\mathcal{G}_{\mathcal{C}}=\pi_1(\widehat{\mathbf{C}}\setminus\{0,1,\infty\})$ correspond to the right Dehn twists shown in Figure 18, since they correspond to the fixed post-critical point c_3 moving in the positive direction around the critical value c_1 and the other post-critical point c_2, respectively.

Computation of the iterated monodromy groups of the maps $f_{1/4}\cdot a$ and $f_{1/4}\cdot a^{-1}b$ and their nuclei show that the first branched covering is equivalent to $f_{5/12}$ and the second to $f_{3/4}$.

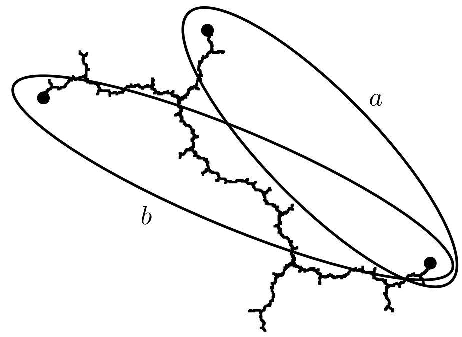

Figure 18. Twisting $f_{1/4}$.

THEOREM 7.2. *Let* $g\in\langle a,b\rangle$ *be an arbitrary element of the mapping class group. Then* $f_{1/4}\cdot g$ *is equivalent to* $f_{1/4}$, $f_{3/4}$ *or* $f_{5/12}$ *if and only if the orbit of* g *under iteration of* $\bar{\psi}$ *(given by* (22)*) eventually reaches the fixed point* 1, a, *or the cycle* $ab^{-1}a\leftrightarrow a^{-1}b$, *respectively.*

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