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Comment on "Flow of a Weakly Conducting Fluid in a Channel Filled with a Porous Medium" by A. Pantokratoras and T. Fang, *Transport in Porous Media*, DOI 10.1007/s11242-009-9470-6

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Abstract In a recent article by Pantokratoras and Fang (*Transport in Porous Media*, doi:10.1007/s11242-009-9470-6, 2009), the title problem for the fully developed flow regime was considered, and exact solutions for two types of boundary conditions were reported. However, for a certain parameter combination, some terms of these exact solutions become singular. As a consequence, the solutions cannot be evaluated by a direct substitution of the respective parameter values. The aim of the present note is to (i) show how the singularities can be removed, (ii) give the corresponding nonsingular solutions in an explicit form, and (iii) discuss the physical meaning of the singular cases.

Keywords Mixed convection · Fully developed flow · Porous media · Lorentz force · Riga plate · Analytical solution

The two boundary value problems of the approach of Pantokratoras and Fang (2009) (hereinafter referred to as PF2009) for the dimensionless velocity filed $U = U(Y)$ are specified by the differential equation

$$\frac{d^2U}{dY^2} - DaU = -A - Qe^{-BY} \tag{1}$$

and the boundary conditions

$$U(0) = 0, \quad U(1) = 1 \text{ (Couette flow conditions)} \tag{2}$$

and

$$U(0) = 0, \quad U(1) = 0 \text{ (Poiseuille flow conditions)}, \tag{3}$$

respectively. All the quantities involved in the above equations are dimensionless, and everywhere the notation of PF2009 is used. The parameter A is a nondimensionalized form of the constant pressure gradient, B is the ratio between the channel width h and the characteristic

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length a/π of the *Riga plate* which is the bottom boundary of the channel, Q stands for the Chandrasekhar number, and Da is the Darcy number,

$$A = -\frac{h^2}{\mu_{\text{eff}} u_r}\frac{\mathrm{d}p}{\mathrm{d}x}, \quad B = \frac{h}{(a/\pi)}, \quad Q = \frac{\pi j_0 M_0 h^2}{8\mu_{\text{eff}} u_r}, \quad Da = \frac{\mu}{\mu_{\text{eff}}}\frac{h^2}{K} \tag{4}$$

The general solution of Eq. 1 has been written in PF2009 in the form

$$U(Y) = \frac{A}{Da} + \frac{Q}{Da - B^2}e^{-BY} + C_1 e^{\sqrt{Da}Y} + C_2 e^{-\sqrt{Da}Y} \tag{5}$$

For the Couette flow boundary conditions (2), the coefficients C_1 and C_2 are given in Eqs. 10 and 11 of PF2009 as

$$C_1 = -\frac{Q}{Da - B^2} - \frac{A}{Da} - \frac{1 + \frac{Q\left(e^{\sqrt{Da}} - e^{-B}\right)}{Da - B^2} + \frac{A\left(e^{\sqrt{Da}} - 1\right)}{Da}}{e^{-\sqrt{Da}} - e^{\sqrt{Da}}} \tag{6}$$

and

$$C_2 = \frac{1 + \frac{Q\left(e^{\sqrt{Da}} - e^{-B}\right)}{Da - B^2} + \frac{A\left(e^{\sqrt{Da}} - 1\right)}{Da}}{e^{-\sqrt{Da}} - e^{\sqrt{Da}}}, \tag{7}$$

respectively. For the Poiseuille flow boundary conditions (3), the expressions of C_1 and C_2 are slightly different from the above ones and are given in Eqs. 12 and 13 of PF2009.

(i) A simple inspection of Eqs. 5–7 shows that for the parameter values which satisfy the equality

$$B = \sqrt{Da} \tag{8}$$

in the mentioned equations, singular terms occur. The same holds for the coefficients C_1 and C_2 corresponding to the Poiseuille flow boundary conditions (3), given in Eqs. 12 and 13 of PF2009. Accordingly, the solution (5) cannot be evaluated by a direct substitution of the parameter values B and Da when Eq. 8 holds. More precisely, a numerical evaluation of Eq. 5 is possible in any arbitrarily small neighborhood of the equality (8), but not exactly for $B = \sqrt{Da}$. A close analysis of this problem is the main issue of the present note.

The type of singularities described above is frequently encountered in engineering problems. The forced oscillations of an undamped spring-mass system offer the simplest example in this respect (see below). The most important feature of this phenomenon, however, is that the singularities are only apparent singularities and can be removed from the mathematical description. There are two different ways to do this. One of them is to take the limiting values of the corresponding expressions, as the critical parameter values are approached. In this case, this procedure is not straightforward since the coefficient (7), as well as the second term on the right-hand side of Eq. 5 do not admit limiting values as $B \to \sqrt{Da}$. Thus, the limiting operation $B \to \sqrt{Da}$ must be carried out in Eqs. 5–7 simultaneously, i.e., after the expressions (6) and (7) of the coefficients have been substituted in (5). This is a tedious and formal exercise. The second way to remove the apparent singularities is to solve the boundary value problems (1)–(3) from the very beginning for the critical case (8). The preferred approach of this article is the latter one since, in contrast to the formal limiting procedure, it reveals the physical origin the apparent singularities. Now, we turn to a detailed analysis of this issue.

(ii) The two terms on the right-hand side of Eq. 1 correspond to the driving pressure gradient and the Lorentz force, respectively. Substituting in Eq. 1

$$U\left(Y\right)=\frac{A}{Da}+V\left(Y\right) \tag{9}$$

the pressure gradient term A can be removed, and Eq. 1 reduces to

$$\frac{\mathrm{d}^2 V}{\mathrm{d}Y^2}-DaV=-Qe^{-BY} \tag{10}$$

This step, i.e. the change from Eqs. 1–10, is essential in understanding the singularity occurring in the general solution (5) when Eq. 8 holds. Indeed, when $B=\sqrt{Da}$, Eq. 10 becomes

$$\frac{\mathrm{d}^2 V}{\mathrm{d}Y^2}-DaV=-Qe^{-\sqrt{Da}Y} \tag{11}$$

This equation has the remarkable property that its right-hand side is a solution of its left-hand side. For this reason, the particular solution of Eq. 11 is not simply proportional to the exponential e^{-BY} (with $B=\sqrt{Da}$) as in the case $B\neq\sqrt{Da}$ tacitly assumed in PF2009, but includes also an algebraic dependence on the transverse coordinate Y. More precisely, the particular solution in this case is

$$V_{\mathrm{part}}=\frac{Q}{2\sqrt{Da}}Ye^{-\sqrt{Da}Y} \tag{12}$$

Therefore, when the relationship (8) holds, the general solution of Eq. 1 has the form

$$U(Y)=\frac{A}{Da}+\frac{Q}{2\sqrt{Da}}Ye^{-\sqrt{Da}Y}+C_3e^{\sqrt{Da}Y}+C_4e^{-\sqrt{Da}Y} \tag{13}$$

The integration constants C_3 and C_4 can be determined from the boundary conditions (2) and (3), respectively. In addition, as already mentioned in PF2009, in the case of Poiseuille flow boundary conditions, by an adequate choice of the velocity scale, the constant A may be set equal to 1 without any loss of generality. Thus, after some algebric calculations, we obtain

$$C_3=\frac{1}{2\sinh\sqrt{Da}}\left[1+\frac{A}{Da}\left(e^{-\sqrt{Da}}-1\right)-\frac{Q}{2\sqrt{Da}}e^{-\sqrt{Da}}\right], \tag{14}$$

$$C_4=\frac{1}{2\sinh\sqrt{Da}}\left[-1+\frac{A}{Da}\left(1-e^{\sqrt{Da}}\right)+\frac{Q}{2\sqrt{Da}}e^{-\sqrt{Da}}\right] \tag{15}$$

for the Couette flow boundary conditions (2), and

$$C_3=\frac{1}{2\sinh\sqrt{Da}}\left[\frac{1}{Da}\left(e^{-\sqrt{Da}}-1\right)-\frac{Q}{2\sqrt{Da}}e^{-\sqrt{Da}}\right], \tag{16}$$

$$C_4=\frac{1}{2\sinh\sqrt{Da}}\left[\frac{1}{Da}\left(1-e^{\sqrt{Da}}\right)+\frac{Q}{2\sqrt{Da}}e^{-\sqrt{Da}}\right] \tag{17}$$

for the Poiseuille flow boundary conditions (3).

From mathematical point of view, the situation occurring in Eqs. 10 and 11 is similar to the case of *forced oscillations* of a linear oscillator described by the differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}+\omega_0^2x=a\cos\left(\omega t\right) \tag{18}$$

Indeed, when the eigenfrequency ω_0 of the oscillator differs from the frequency ω of the forcing, the general solution of Eq. 18 is

$$x(t) = \frac{a}{\omega_0^2 - \omega^2} \cos(\omega t) + C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t} \; (\omega \neq \omega_0) \tag{19}$$

When $\omega \to \omega_0$, the solution (19) becomes singular, similar to the behavior occurring in Eq. 5 when $B \to \sqrt{Da}$. In this case, the forcing term $a\cos(\omega_0 t)$ on the right-hand side of Eq. 18 is a solution of the left-hand side. This is the *resonance regime* of the forced linear oscillator. Accordingly, the general solution of Eq. 18 which describes the resonance regime of the oscillator will include also an algebraic time dependence and has the form

$$x(t) = \frac{a}{2\omega_0} t \sin(\omega_0 t) + C_3 e^{i\omega_0 t} + C_4 e^{-i\omega_0 t} \; (\omega = \omega_0) \tag{20}$$

Equation 20 is the (formal) counterpart of our Eq. 13.

(iii) Let us now consider the physical meaning of Eq. 8. In Eq. 4 of B, the quantity a/π represents a characteristic length of the Riga plate (see Fig. 1 of PF2009) and at the same time the *logarithmic decrement* δ_{Lorentz} of the Lorentz force in the transverse direction of the channel,

$$\frac{a}{\pi} \equiv \delta_{\text{Lorentz}} \tag{21}$$

Thus, Eq. 8 becomes

$$\delta_{\text{Lorentz}} = \sqrt{K \frac{\mu_{\text{eff}}}{\mu}} \tag{22}$$

Therefore, the critical "*quasi-resonance regime*" (8) of our channel flow occurs when the logarithmic decrement of the Lorentz force equals an intrinsic *characteristic length* of the problem, which depends on the permeability K of the porous medium and the viscosity ratio μ_{eff}/μ according to Eq. 22. It is worth emphasizing here that the condition (22) is independent of both the channel width h and the driving pressure gradient.

Reference

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