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Reference functions and solutions to bargaining problems with and without claims

Anke Gerber

Institute for Empirical Research in Economics, University of Zurich, Blümlisalpstr. 10, 8006 Zurich, Switzerland (e-mail: agerber@iew.unizh.ch)

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Abstract. Following an idea due to Thomson (*Journal of Economic Theory*, 1981, 25: 431–441) we examine the role of reference functions in the axiomatic approach to the solution of bargaining problems with and without claims. A reference function is a means of summarizing essential features of a bargaining problem. Axioms like *Independence of Irrelevant Alternatives* and *Monotonicity* are then reformulated with respect to this reference function. Under some weak conditions on the reference function we obtain characterizations of different parametrized classes of solutions. We present several examples of reference functions and thereby recover many well-known solutions to bargaining problems with and without claims.

1 Introduction

The purpose of this paper is to provide a unifying approach to the solution of bargaining problems with and without claims. To this end we study the role of *reference functions* in the definition and axiomatization of bargaining solutions. Reference points like the status quo and the ideal point have ever played an important role in characterizations of bargaining solutions but the first formal introduction of the concept of a reference function in the context of two-person bargaining problems is due to Thomson [12]. A reference point represents an origin from which relative utility gains or losses can be

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measured and it is meant to summarize features of the bargaining problem that are regarded as essential, either by the players or by some impartial arbitrator.

In this paper we examine reference functions mainly in the context of n-person bargaining problems with claims. Solutions to this class of problems also utilize reference points like the status quo and the claims point. And, as in the case of traditional bargaining problems, we might wish to use more general reference points that, for example, take into account some properties of the feasible set. Thus, like Thomson [12], we consider very general reference functions and show how solutions to bargaining problems with claims can be classified apart from differences in the reference function. By formulating axioms like *Independence of Irrelevant Alternatives* and *Monotonicity* with respect to the given reference function g we obtain characterizations of three different classes of solutions, all parametrized by g: Nash-, egalitarian- and proportional-type solutions. In this way we can point out the common features between several solutions that have been analyzed independently of each other so far.

Although our focus is on bargaining problems with claims, as a corollary we obtain characterization results for solutions to bargaining problems without claims. We present several examples of reference functions that fulfill the conditions needed for the characterization results, but do not argue in favor of a particular function which would clearly go beyond the scope of this paper. For particular choices of the reference function we do not only recover well-known bargaining solutions including the *Nash*, *egalitarian*, *Kalai-Smorodinsky*, *equal-loss*, *proportional* and *claim-egalitarian solution*, but we also find interesting new solution concepts.

The paper is organized as follows. Section 2 provides the basic definitions. In Sects. 3, 4, and 5 we present the characterization results for egalitarian-, Nash- and proportional-type solutions, respectively. Section 6 concludes the paper with some final remarks.

2 Notation and definitions

In the following, $\mathbb{N}$ will denote the set of positive integers and $\mathbb{R}$ will denote the set of real numbers. By $\mathbb{R}^n, n \in \mathbb{N}$, we denote the n-dimensional euclidean space. Vector inequalities in $\mathbb{R}^n$ are denoted by $\geq, >, \gg$.[1] By $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ we denote the set of nonnegative and strictly positive vectors in $\mathbb{R}^n$, respectively, i.e. $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x \geq 0\}$ and $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n | x \gg 0\}$. Weak set inclusion is denoted by $\subset$. Convergence of a sequence of subsets of $\mathbb{R}^n$ is defined in terms of the Hausdorff topology. By $x \cdot y$ we denote the scalar product of x and y in

[1] For $x, y \in \mathbb{R}^n$ we write $x \geq y$ if $x_i \geq y_i$ for all i, $x > y$ if $x \geq y$ and $x \neq y$, and $x \gg y$ if $x_i > y_i$ for all i.

$\mathbb{R}^n$. A set $A \subset \mathbb{R}^n$ is called *comprehensive* if $x \in A$ and $x > y$ imply that $y \in A$. For $A \subset \mathbb{R}^n$ let

$$\text{WPO}\ (A) = \{x \in A | y \in \mathbb{R}^n,\ y \gg x \Rightarrow y \notin A\}$$

be the set of *weakly Pareto optimal* points in A and let

$$\text{PO}\ (A) = \{x \in A | y \in \mathbb{R}^n,\ y > x \Rightarrow y \notin A\}$$

denote the set of *Pareto optimal* points in A.

The set $N = \{1, \ldots, n\}, n \in \mathbb{N}$, will denote the *player set*. If $\pi : N \to N$ is a permutation, then π defines the mapping $\pi : \mathbb{R}^n \to \mathbb{R}^n$, which we denote by the same symbol, via $(\pi(x))_i = x_{\pi^{-1}(i)}$, $i \in N, x \in \mathbb{R}^n$. For a set $A \subset \mathbb{R}^n$ let $\pi(A) = \{y \,|\, \exists x \in A \text{ with } y = \pi(x)\}$. We call $A \subset \mathbb{R}^n$ *symmetric* if $\pi(A) = A$ for all permutations π. A mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ is called a *positive affine transformation* if there exist $a \in \mathbb{R}^n_{++}$ and $b \in \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$ and all $i, (L(x))_i = a_i x_i + b_i$. Let $\mathscr{L}$ be the class of all positive affine transformations on $\mathbb{R}^n$. For a set $A \subset \mathbb{R}^n$ and $L \in \mathscr{L}$ let $L(A) = \{y \,|\, \exists x \in A \text{ with } y = L(x)\}$.

Bargaining problems were first studied by Nash [9] and are defined as follows.

Definition 2.1. *An* ***n-person bargaining problem*** *is a tuple* (S, d), *where*

1. $S \subset \mathbb{R}^n$ *is convex, closed and comprehensive,*
2. $d \in S$,
3. $\{x \in S \,|\, x \geq d\}$ *is bounded.*

A bargaining problem is characterized by a set S of feasible utility allocations, measured in von Neumann-Morgenstern scales, and a point d, called "threatpoint" or "disagreement point" or "status quo," which is the outcome of the game if the players do not agree on a utility allocation in the feasible set. Thus, the status quo d can be unilaterally enforced by any player. Let $\mathbf{\Sigma}$ be the class of all n-person bargaining problems. A *solution* on a class of bargaining problems $\mathbf{D} \subset \mathbf{\Sigma}$ is a mapping $f : \mathbf{D} \to \mathbb{R}^n$ such that $f(S, d) \in S$ for all $(S, d) \in \mathbf{D}$.

Consider now a bargaining situation in which the players have claims that are not compatible with each other. Assume that the claims are credible or verifiable and that all players agree that they should be taken into account by any (fair) solution to the problem at issue. The classic example for such a situation is a bankruptcy problem. While in the latter utility is transferable, general bargaining problems with claims, including those with nontransferable utility, were introduced by Chun and Thomson [3].

Definition 2.2. *An* ***n-person bargaining problem with claims*** *is a triple* (S, d, c), *where*

1. $(S, d) \in \mathbf{\Sigma}$,
2. $c \in \mathbb{R}^n \setminus S$, $c > d$.

Let $\mathbf{\Sigma^c}$ be the class of all n-person bargaining problems with claims. A *solution* on a class of bargaining problems with claims $\mathbf{D^c} \subset \mathbf{\Sigma^c}$ is a mapping $F : \mathbf{D^c} \to \mathbb{R}^n$ such that $F(S,d,c) \in S$ for all $(S,d,c) \in \mathbf{D^c}$. In the following sections we will study solutions for bargaining problems with and without claims.

3 Egalitarian-type solutions

Consider a *reference function* given by a mapping $g : \mathbf{\Sigma^c} \to \mathbb{R}^n$. As explained in the introduction a reference function is a means of summarizing "essential" features of a bargaining problem and a reference point will serve as an origin from which relative utility gains or losses are measured. Observe that we do not necessarily require the reference point to be a feasible utility allocation.

Let $e = (1,\ldots,1) \in \mathbb{R}^n$ and define the diagonal Δ in $\mathbb{R}^n$ by $\Delta = \{x | x = \lambda e \text{ forsome } \lambda \in \mathbb{R}\}$. We impose the following assumptions on the reference function $g : \mathbf{\Sigma^c} \to \mathbb{R}^n$.

(E1) **Covariance with respect to translations.** Let $(S,d,c) \in \mathbf{\Sigma^c}$ and let $L \in \mathscr{L}$ be a translation, i.e. there exists $b \in \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$ and for all i, $(L(x))_i = x_i + b_i$. Then $g(L(S,d,c)) = L(g(S,d,c))$.[2]

(E2) **Invariance with respect to the restriction to a symmetric subset.** Let $(S,d,c) \in \mathbf{\Sigma^c}$ be such that $g(S,d,c) \in \Delta$. Then there exists $(S',d',c') \in \mathbf{\Sigma^c}$ such that $S' \subset S$, S' is symmetric, $S' \cap \Delta = S \cap \Delta$, and $g(S',d',c') = g(S,d,c)$.

(E3) **Invariance with respect to approximation.** Let $(S,d,c) \in \mathbf{\Sigma^c}$ with $g(S,d,c) \in \Delta$. Then there exists a sequence $((S^n,d^n,c^n))_n \subset \mathbf{\Sigma^c}$ such that $S^n \to S$, $S \subset S^n$, WPO $(S^n) \cap \Delta =$ PO $(S^n) \cap \Delta$, and $g(S^n,d^n,c^n) = g(S,d,c)$ for all n.

E1 is a weakening of the standard *covariance with respect to positive affine transformations* axiom. In a context, where utility is measured in von Neumann-Morgenstern utility scales and where there is interpersonal comparison of utility, we consider two bargaining problems as equivalent if one is obtained from the other by a shift in each player's origin of utility measurement. Thus, we would like the reference function to comply with such shifts, i.e., be covariant under translations. **E2** and **E3** are minimal technical requirements we have to impose on the reference function in order to obtain our characterization result below. It is understood that a reference function not satisfying these axioms can nevertheless be reasonable. Observe that we are not concerned with justifying or characterizing reference functions, which we take as exogenously given, but rather we aim at characterizing bargaining solutions that fulfill some properties with respect to a given reference function.[3]

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[2]Any $L \in \mathscr{L}$ induces the mapping $L : \mathbf{\Sigma^c} \to \mathbf{\Sigma^c}$, denoted by the same symbol, via $L(S,d,c) = (L(S),L(d),L(c))$ for all $(S,d,c) \in \mathbf{\Sigma^c}$.

[3]For further discussion we refer the reader to our concluding remarks in Sect. 6.

Hence the axioms we impose upon the reference function are qualitatively different from the axioms we ask to be satisfied by a bargaining solution. All we have to make sure is that our axioms are satisfied by the common reference functions that have emerged in the literature, so that our characterization is not void. We will see later that this is indeed the case for the axioms we impose.

Consider the following axioms for a solution $F : \mathbf{D^c} \to \mathbb{R}^n$ on a class $\mathbf{D^c} \subset \mathbf{\Sigma^c}$.

(WPO) Weak Pareto optimality. $F(S,d,c) \in \mathrm{WPO}\ (S)$ for all $(S,d,c) \in \mathbf{D^c}$.

(SY$_\mathbf{g}$) Symmetry with respect to g. If $(S,d,c) \in \mathbf{D^c}$ is such that S is symmetric and $g(S,d,c) \in \Delta$, then $F(S,d,c) \in \Delta$.

(TRANS) Covariance with respect to translations. For all $(S,d,c) \in \mathbf{D^c}$, if $L \in \mathscr{L}$ is a translation and if $L(S,d,c) \in \mathbf{D^c}$, then $F(L(S,d,c)) = L(F(S,d,c))$.

(RMON$_\mathbf{g}$) Restricted monotonicity with respect to g. Let (S,d,c), $(S',d',c') \in \mathbf{D^c}$ with $g(S',d',c') = g(S,d,c)$ and $S \subset S'$. Then $F(S,d,c) \leq F(S',d',c')$.

WPO is a standard axiom in bargaining theory and is certainly a minimal requirement for any reasonable solution. **SY$_\mathbf{g}$** is the symmetry axiom (cf. Nash [9]) with the reference point substituted for the disagreement point. For an interpretation of **TRANS**, see our remarks about **E1**. Finally, **RMON$_\mathbf{g}$** is Kalai's [7] monotonicity axiom with the reference point substituted for the disagreement point. Consider the following solution on the class $\mathbf{\Sigma^c}$.

Definition 3.1. *Let $g : \mathbf{\Sigma^c} \to \mathbb{R}^n$ be a reference function. The egalitarian solution with respect to g is the function $\mathscr{E}^g : \mathbf{\Sigma^c} \to \mathbb{R}^n$, given by*

$$\mathscr{E}^g(S,d,c) = g(S,d,c) + \bar{\lambda}\, e,$$

where $\bar{\lambda} = \max\{\lambda \in \mathbb{R} | g(S,d,c) + \lambda\, e \in S\}$, $(S,d,c) \in \mathbf{\Sigma^c}$.

Since for all $(S,d,c) \in \mathbf{\Sigma^c}$ the set S is closed and comprehensive and the set $\{x \in S | x \geq d\}$ is bounded, $\mathscr{E}^g$ is well defined on $\mathbf{\Sigma^c}$. If $g(S,d,c) \in S$ ($g(S,d,c) \notin S$), then $\mathscr{E}^g$ equalizes the gains (losses) from the reference point (see Fig. 1).

Theorem 3.1. *If $g : \mathbf{\Sigma^c} \to \mathbb{R}^n$ satisfies* **E1**, **E2**, **E3**, *then $\mathscr{E}^g$ is the unique solution on $\mathbf{\Sigma^c}$ which satisfies* **WPO**, **SY$_\mathbf{g}$**, **TRANS** *and* **RMON$_\mathbf{g}$**.

Proof. Using **E1**–**E3** we can apply a method of proof similar to Kalai's [7] characterization of the proportional solution on $\mathbf{\Sigma}$ and Bossert's [1] characterization of the claim-egalitarian solution on $\mathbf{\Sigma^c}$. It is straightforward to see that $\mathscr{E}^g$ satisfies the axioms. Let $F : \mathbf{\Sigma^c} \to \mathbb{R}^n$ be a solution which satisfies **WPO**, **SY$_\mathbf{g}$**, **TRANS**, **RMON$_\mathbf{g}$**, and let $(S,d,c) \in \mathbf{\Sigma^c}$. By **TRANS** and **E1** we can assume that $g(S,d,c) = 0$. Then $\mathscr{E}^g(S,d,c) = x^* \in \Delta$. By **E2** there exists $(S',d',c') \in \mathbf{\Sigma^c}, S' \subset S$, S' symmetric, such that $S \cap \Delta = S' \cap \Delta$ and $g(S',d',c') = g(S,d,c)$. Therefore, $x^* \in S'$ and by **RMON$_\mathbf{g}$** $F(S',d',c') \leq$

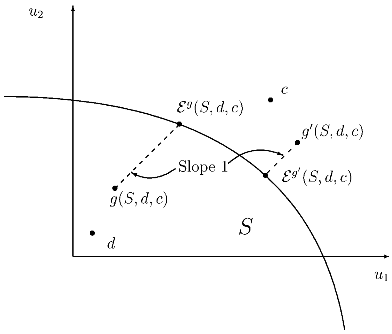

Fig. 1. Egalitarian solution with respect to g and g'. $\mathscr{E}^g$ equalizes the gains and $\mathscr{E}^{g'}$ equalizes the losses from the reference point

$F(S,d,c)$. By $\mathbf{SY_g}$ and **WPO** we have $F(S',d',c') = x^*$. If $x^* \in$ PO (S) this implies $F(S,d,c) = x^*$.

If $x^* \in$ WPO $(S) \setminus$ PO(S) by **E3** there exists a sequence $((S^n,d^n,c^n))_n \subset \mathbf{\Sigma^c}$ such that $S^n \to S, S \subset S^n$, WPO $(S^n) \cap \Delta =$ PO $(S^n) \cap \Delta$, and $g(S^n,d^n,c^n) = g(S,d,c)$ for all n. Therefore, $\mathscr{E}^g(S^n,d^n,c^n) \in$ PO $(S^n) \cap \Delta$ and by the above we conclude that $F(S^n,d^n,c^n) = \mathscr{E}^g(S^n,d^n,c^n)$ for all n. Thus, by $\mathbf{RMON_g}$ it is true that $x^* = F(S',d',c') \le F(S,d,c) \le F(S^n,d^n,c^n) = \mathscr{E}^g(S^n,d^n,c^n)$ for all n. Since $g(S^n,d^n,c^n) = g(S,d,c)$ for all n and $S^n \to S$ we have that $\mathscr{E}^g(S^n,d^n,c^n) \to x^*$. This implies $F(S,d,c) = x^* = \mathscr{E}^g(S,d,c)$. ■

A straightforward corollary of Theorem 3.1 is a characterization of egalitarian-type solutions on the class $\mathbf{\Sigma}$. To see this, for any function $h : \mathbf{\Sigma} \to \mathbb{R}^n$ we denote by $\bar{h}$ its trivial extension to $\mathbf{\Sigma^c}$, namely, $\bar{h} : \mathbf{\Sigma^c} \to \mathbb{R}^n$, defined by $\bar{h}(S,d,c) = h(S,d), (S,d,c) \in \mathbf{\Sigma^c}$. Let $g : \mathbf{\Sigma} \to \mathbb{R}^n$ be a reference function on the class $\mathbf{\Sigma}$ and let $f : \mathbf{D} \to \mathbb{R}^n$ be a solution on the class $\mathbf{D} \subset \mathbf{\Sigma}$. By **WPO***, $\mathbf{SY_g^*}$, **TRANS***, $\mathbf{RMON_g^*}$, we denote the counterparts of **WPO**, $\mathbf{SY_g}$, **TRANS**, $\mathbf{RMON_g}$, on the class **D**. For example,

($\mathbf{SY_g^*}$) If $(S,d) \in \mathbf{D}$ is such that S is symmetric and $g(S,d) \in \Delta$, then

$$f(S,d) \in \Delta.$$

For $g : \mathbf{\Sigma} \to \mathbb{R}^n$ define the solution $\varepsilon^g : \mathbf{\Sigma} \to \mathbb{R}^n$ by

$$\varepsilon^g(S,d) = g(S,d) + \bar{\lambda} e,$$

where $\bar{\lambda} = \max\{\lambda \in \mathbb{R} | g(S,d) + \lambda e \in S\}, (S,d) \in \mathbf{\Sigma}$. The following result then immediately follows from Theorem 3.1, if one notes that, whenever $f : \mathbf{\Sigma} \to \mathbb{R}^n$ fulfills **WPO***, $\mathbf{SY_g^*}$, **TRANS*** and $\mathbf{RMON_g^*}$, then $\bar{f} : \mathbf{\Sigma^c} \to \mathbb{R}^n$ satisfies **WPO**, $\mathbf{SY_g}$, **TRANS** and $\mathbf{RMON_g}$.

Corollary 3.1. *If $g : \mathbf{\Sigma} \to \mathbb{R}^n$ is such that $\bar{g}$ fulfills* **E1**, **E2** *and* **E3**, *then ε^g is the unique solution on $\mathbf{\Sigma}$ that satisfies* **WPO***, $\mathbf{SY_g^*}$, **TRANS*** *and* $\mathbf{RMON_g^*}$.

In the following we present some examples of reference functions satisfying **E1**–**E3**. The first 2 examples give rise to well-known bargaining solutions while the last 3 examples lead to new solutions. Our characterization results above show that all solutions obey the same principles, the most prominent one being a monotonicity axiom, while they differ in the associated reference function.

For later usage, for any $(S, d) \in \mathbf{\Sigma}$ we define the set of *individually rational points* by $\mathrm{IR}\ (S, d) = \{x \in S | x \geq d\}$ and the *utopia point* $u(S, d)$ by $u_i(S, d) = \max\{x_i | x \in \ \mathrm{IR}\ (S, d)\}$ for all i.

Example 3.1. For $0 \leq \alpha \leq 1$ let $g^\alpha : \mathbf{\Sigma^c} \to \mathbb{R}^n$ be given by

$$g^\alpha(S, d, c) = \alpha d + (1 - \alpha)c \text{ for all } (S, d, c) \in \mathbf{\Sigma^c}.$$

For $\alpha = 0$ we find that $\mathscr{E}^{g^0}$ is the *claim-egalitarian solution* proposed by Bossert [1]. If we define the reference function $g : \mathbf{\Sigma} \to \mathbb{R}^n$ by $g(S, d) = d$ for all $(S, d) \in \mathbf{\Sigma}$, then $\bar{g} = g^1$ and ε^g is the *egalitarian solution* proposed by Kalai [7].

Example 3.2. Let $u : \mathbf{\Sigma} \to \mathbb{R}^n$ be the reference function that assigns to each bargaining problem (S, d) the utopia point $u(S, d)$. Then ε^u is the *equal-loss solution* proposed by Chun [2].

Example 3.3. Let $m : \mathbf{\Sigma} \to \mathbb{R}^n$ be given by

$$m_i(S, d) = \min\{x_i | x \in \ \mathrm{PO}\ (S) \cap \ \mathrm{IR}\ (S, d)\}, \quad i = 1, \ldots, n,$$

for $(S, d) \in \mathbf{\Sigma}$. For two-person games the *point of minimal expectations* $m(S, d)$ was introduced by Roth [11]. One can argue that for the resolution of the conflict the only relevant agreements are in that part of the Pareto frontier which dominates the disagreement point. Therefore, $m_i(S, d)$ is the minimum player i can expect to achieve in the game $(S, d) \in \mathbf{\Sigma}$. Observe that if $(S, d) \in \mathbf{\Sigma}$ is such that $\mathrm{WPO}\ (S) \cap \ \mathrm{IR}(S, d) \subset \ \mathrm{PO}(S)$, then $m(S, d) = d$. It is straightforward to see that $\overline{m}$ satisfies **E1**, while **E2** and **E3** are satisfied only if $n = 2$.[4]

Example 3.4. Let $M : \mathbf{\Sigma^c} \to \mathbb{R}^n$ be given by

$$M_i(S, d, c) = \min\{x_i | x \in \ \mathrm{PO}\ (S),\ d \leq x \leq c\}, \quad i = 1, \ldots, n,$$

for $(S, d, c) \in \mathbf{\Sigma^c}$. The reference point $M(S, d, c)$ is a natural extension of m (see Example 3.3) to the class of bargaining problems with claims. If any agreement is expected to be individually rational, bounded by the claims point and such that there are no further gains from cooperation, then $M_i(S, d, c)$ is the minimum payoff player i a priori expects to achieve. Observe, however, that choosing $M(S, d, c)$ as a reference point is not sufficient for the

[4] If $n = 2$ the reference point m uniquely defines two points on the relative boundary of the set of Pareto optimal and individually rational points, which is not the case in general. Therefore, the case $n = 2$ is qualitatively different from the case $n > 2$. We already pointed out that originally m was only defined for two-person games (Roth [11]).

final agreement to be individually rational and bounded by the claims point. We discuss the issue of boundedness in Remark 3.1. For the same reason as in Example 3.3 conditions **E2** and **E3** can only be verified for $n = 2$.

Example 3.5. Let $t : \mathbf{\Sigma^c} \to \mathbb{R}^n$ be given

$$t_i(S, d, c) = \max\{d_i, \max\{x_i | (x_i, c_{-i}) \in S\}\}, \quad i = 1, \dots, n,^5$$

for $(S, d, c) \in \mathbf{\Sigma^c}$. The *adjusted threatpoint* $t(S, d, c)$ is a minimally equitable agreement for a bargaining problem with claims, given that no player can expect someone else to settle with less than what is necessary to satisfy the claims of the other players, and given that no rational player will accept any payoff below the disagreement utility level. In the context of bankruptcy problems Curiel et al. [4] call $t_i(S, d, c)$ the "minimal right of claimant i." The adjusted threatpoint was also used by Herrero [6] to define the *adjusted proportional rule* as we will see in Sect. 5.

Remark 3.1. A major shortcoming of the egalitarian-type solutions is the fact that, in general, they do not satisfy boundedness, i.e. $d \leq \mathscr{E}^g(S, d, c) \leq c$ for all $(S, d, c) \in \mathbf{\Sigma^c}$. However, as it was already shown by Bossert [1] for the special case of the claim-egalitarian solution, $\mathscr{E}^g$ can be modified in a straightforward way such as to satisfy boundedness.

To this end let $g : \mathbf{\Sigma^c} \to \mathbb{R}^n$ be a reference function that satisfies $d \leq g(S, d, c) \leq c$ for all $(S, d, c) \in \mathbb{R}^n$. If $g(S, d, c) \notin S$, define the *extended egalitarian solution with respect to* g in an iterative way as follows. To simplify the notation, let $g = g(S, d, c)$. Then let $\lambda_0 = \max\{\lambda \in \mathbb{R} | g + \lambda e \in S \text{ and } g + \lambda e \geq d\}$. If $g + \lambda_0 e \in$ WPO (S), define $\overline{\mathscr{E}}^g(S, d, c) = g + \lambda_0 e$. Otherwise, let $M_1 = \{i | g_i + \lambda_0 > d_i\}$ and let $\lambda_1 = \max\{\lambda \in \mathbb{R} | g + \lambda_0 e + \lambda e_{M_1} \in S \text{ and } g + \lambda_0 e + \lambda e_{M_1} \geq d\}$.[6] If $g + \lambda_0 e + \lambda_1 e_{M_1} \in$ WPO (S), define $\overline{\mathscr{E}}^g(S, d, c) = g + \lambda_0 e + \lambda_1 e_{M_1}$. Otherwise, repeat the procedure until after $T \leq n - 1$ steps $g + \lambda_0 e + \lambda_1 e_{M_1} + \cdots + \lambda_T e_{M_T} \in$ WPO (S), in which case we define

$$\overline{\mathscr{E}}^g(S, d, c) = g + \lambda_0 e + \lambda_1 e_{M_1} + \ldots + \lambda_T e_{M_T}.$$

Then, by definition $d \leq \overline{\mathscr{E}}^g(S, d, c) \leq c$. Analogously, we can define $\overline{\mathscr{E}}^g(S, d, c)$ for $g(S, d, c) \in S$ such that $\overline{\mathscr{E}}^g(S, d, c)$ is bounded by the disagreement and claims point.

We omit the characterization of the extended egalitarian-type solutions which requires additional assumptions on the reference function and is straightforward but more involved than the characterization of the egalitarian-type solutions provided in this section (cf., Theorem 5 in Bossert [1]). ♦

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[5] For $x \in \mathbb{R}^n, a \in \mathbb{R}$, and $i \in \{1, \dots, n\}$, the vector (a, x_{-i}) is defined to be the vector $y \in \mathbb{R}^n$ such that $y_i = a$ and $y_j = x_j$ for all $j \neq i$. Also, by definition $\max(\emptyset) = -\infty$.

[6] For $T \subset N$ let $e_T \in \mathbb{R}^n$ be given by $(e_T)_i = 1$ if $i \in T$, and $(e_T)_i = 0$ else.

4 Nash-type solutions

The result and method of this section are an adaptation of Thomson [12] to our context of bargaining problems with claims and n players. Analogous to [12] we impose the following axioms on the reference function $g : \mathbf{\Sigma^c} \to \mathbb{R}^n$.

(N1) Covariance with respect to positive affine transformations. Let $(S,d,c) \in \mathbf{\Sigma^c}$ and let $L \in \mathscr{L}$. Then $g(L(S,d,c)) = L(g(S,d,c))$.

(N2) Invariance with respect to symmetrization of almost symmetric problems. Let $(S,d,c) \in \mathbf{\Sigma^c}$ with $g(S,d,c) \in \Delta$ and $e \cdot x^* \geq e \cdot y$ for all $y \in S$ where $x^* \in$ WPO $(S) \cap \Delta$. Then there exists $(S',d',c') \in \mathbf{\Sigma^c}$ with S' symmetric, $S \subset S'$, $e \cdot x^* \geq e \cdot y$ for all $y \in S'$, and $g(S',d',c') = g(S,d,c)$.

N1 is a natural assumption for a reference function given that we want relative utility gains over (or losses from) the reference point to be covariant under equivalent utility representations. By comparison, **N2** is a purely technical assumption (see our remarks about **E2** and **E3** in the last section). It requires that a bargaining problem with claims which already exhibits some symmetric structure (symmetric reference point and symmetric supporting hyperplane on the diagonal) can be replaced by a symmetric one without changing the essential features of the problem.

Let $F : \mathbf{D^c} \to \mathbb{R}^n$ be a solution on the class $\mathbf{D^c} \subset \mathbf{\Sigma^c}$. Together with **WPO**, and $\mathbf{SY_g}$ the following conditions are the original Nash axioms (Nash [9]) with the reference point substituted for the disagreement point (see also Thomson [12]). Hence they require no further discussion.

(COV) Covariance with respect to positive affine transformations. For all $(S,d,c) \in \mathbf{D^c}$ and $L \in \mathscr{L}$, if $L(S,d,c) \in \mathbf{D^c}$, then $F(L(S,d,c)) = L(F(S,d,c))$.

($\mathbf{IA_g}$) Independence of alternatives other than $\mathbf{g(S,d,c)}$. If (S,d,c), $(S',d',c') \in \mathbf{D^c}$ with $S \subset S', g(S,d,c) = g(S',d',c')$ and $F(S',d',c') \in S$, then $F(S,d,c) = F(S',d',c')$.

For any reference function $g : \mathbf{\Sigma^c} \to \mathbb{R}^n$ let

$$\mathbf{\Sigma^c_g} = \{(S,d,c) \in \mathbf{\Sigma^c} | \exists x \in S, x \gg g(S,d,c)\}.$$

Definition 4.1. *Let $g : \mathbf{\Sigma^c} \to \mathbb{R}^n$ be a reference function. The Nash solution with respect to g is the function $\mathscr{N}^g : \mathbf{\Sigma^c_g} \to \mathbb{R}^n$, given by*

$$\mathscr{N}^g(S,d,c) = \operatorname{argmax} \left\{ \prod_{i=1}^n (x_i - g_i(S,d,c)) \,\middle|\, x \in S, x \geq g(S,d,c) \right\},$$

$(S,d,c) \in \mathbf{\Sigma^c_g}$.

$\mathscr{N}^g$ (see Fig. 2) is well defined since $\{x \in S | x \geq g(S,d,c)\}$ is bounded for all $(S,d,c) \in \mathbf{\Sigma^c_g}$. The following characterization of $\mathscr{N}^g$ is an immediate adaptation of Thomson's result [12] to the domain of bargaining problems with claims. Hence we omit the proof which is analogous to Nash's own proof [9].

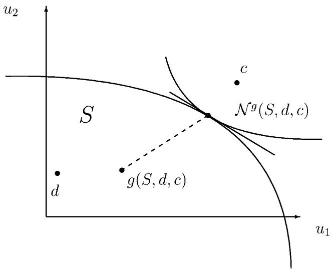

Fig. 2. Nash solution with respect to g

Theorem 4.1. *If* $g : \mathbf{\Sigma^c} \to \mathbb{R}^n$ *satisfies* **N1** *and* **N2**, *then* $\mathcal{N}^g$ *is the unique solution on* $\mathbf{\Sigma_g^c}$ *which satisfies* **WPO**, $\mathbf{SY_g}$, **COV** *and* $\mathbf{IA_g}$.

Again, a characterization of Nash-type solutions on the class $\mathbf{\Sigma}$ is obtained as a straightforward corollary of Theorem 4.1. For $g : \mathbf{\Sigma} \to \mathbb{R}^n$ let

$$\mathbf{\Sigma_g} = \{(S,d) \in \mathbf{\Sigma} \mid \exists x \in S,\, x \gg g(S,d)\},$$

and define $\nu^g : \mathbf{\Sigma_g} \to \mathbb{R}^n$ by

$$\nu^g(S,d) = \operatorname{argmax}\left\{ \prod_{i=1}^{n} (x_i - g_i(S,d)) \,\middle|\, x \in S,\, x \geq g(S,d) \right\},$$

$(S,d) \in \mathbf{\Sigma_g}$. As before, the counterparts of **WPO**, $\mathbf{SY_g}$, **COV**, $\mathbf{IA_g}$, on a class $\mathbf{D} \subset \mathbf{\Sigma}$ are marked with an asterisk *.

Corollary 4.1. *If* $g : \mathbf{\Sigma} \to \mathbb{R}^n$ *is such that* $\bar{g}$ *fulfills* **N1** *and* **N2**, *then* ν^g *is the unique solution on* $\mathbf{\Sigma_g}$ *which satisfies* $\mathbf{WPO}^*$, $\mathbf{SY_g^*}$, $\mathbf{COV}^*$ *and* $\mathbf{IA_g^*}$.

Again we present some examples of reference functions satisfying **N1** and **N2**, some of which lead to well-known bargaining solutions.

Example 4.1. Consider the reference functions $m : \mathbf{\Sigma} \to \mathbb{R}^n$ and $M : \mathbf{\Sigma^c} \to \mathbb{R}^n$, defined in Examples 3.3 and 3.4, which assign to each bargaining problem the point of minimal expectations in the respective context. The solution ν^m was first studied by Roth [11]. In contrast to **E2** and **E3**, **N2** is satisfied by $\bar{m}$ and M for all n.

Example 4.2. For $0 \leq \alpha \leq 1$ let $g^\alpha : \mathbf{\Sigma^c} \to \mathbb{R}^n$ be the reference function defined in Example 3.1. As a special case we get $g^1(S,d,c) = d$ for all $(S,d,c) \in \mathbf{\Sigma^c}$. If we define the reference function $g : \mathbf{\Sigma} \to \mathbb{R}^n$ by $g(S,d) = d$ for all $(S,d) \in \mathbf{\Sigma}$, then $\bar{g} = g^1$ and ν^g is the *Nash solution* (Nash [9]).

Example 4.3. The adjusted threatpoint $t : \mathbf{\Sigma^c} \to \mathbb{R}^n$ defined in Example 3.5 satisfies **N1** and **N2** and gives rise to the bargaining solution $\mathcal{N}^t$ that has not been studied so far.

Remark 4.1. The solution $\mathcal{N}^g$ shares a shortcoming with the egalitarian-type solutions discussed before, namely that it may assign a utility allocation which is not bounded by the claims point. Hence, we propose a modification of the Nash solution with respect to g which is bounded by the claims point. For any reference function $g : \mathbf{\Sigma^c} \to \mathbb{R}^n$ and $(S, d, c) \in \mathbf{\Sigma^c}$ let $\mathrm{BD}_g(S, d, c) = \{x \in S | g(S, d, c) \leq x \leq c\}$ and $\mathbf{\Sigma^c_g} = \{(S, d, c) \in \mathbf{\Sigma^c} \mid \exists x \in \mathrm{BD}_g(S, d, c), x \gg g(S, d, c)\}$. Then, the *bounded Nash solution with respect to g* is the function $\overline{\mathcal{N}}^g : \mathbf{\Sigma^c_g} \to \mathbb{R}^n$, given by

$$\overline{\mathcal{N}}^g(S, d, c) = \operatorname{argmax} \left\{ \prod_{i=1}^{n} (x_i - g_i(S, d, c)) \middle| x \in \mathrm{BD}_g(S, d, c) \right\},$$

$(S, d, c) \in \mathbf{\Sigma^c_g}$. A characterization of the bounded Nash solution is easily obtained under a slight modification of the axioms **N2**, $\mathbf{SY_g}$ and $\mathbf{IA_g}$.[7] If, in addition, the reference function satisfies individual rationality, i.e. $g(S, d, c) \geq d$ for all $(S, d, c) \in \mathbf{\Sigma^c_g}$, then $\overline{\mathcal{N}}^g$ is individually rational as well. ♦

5 Proportional-type solutions

In order to characterize proportional-type solutions we consider a different kind of reference function: Let $g : \mathbf{\Sigma^c} \to \mathbb{R}^n \times \mathbb{R}^n_+$, where $g(S, d, c) = (g^r(S, d, c), g^p(S, d, c))$ and $\|g^p(S, d, c)\| = 1$ for all $(S, d, c) \in \mathbf{\Sigma^c}$.[8] Thus, the reference function g assigns to any bargaining problem with claims not only a reference point $g^r(S, d, c)$ but also a vector of weights $g^p(S, d, c)$. These weights can be interpreted as relative "bargaining strengths" of the players which are deduced from the bargaining game (S, d, c).[9] We will see later that several well-known bargaining solutions are defined with respect to this type of reference function.

We impose the following assumptions on $g : \mathbf{\Sigma^c} \to \mathbb{R}^n \times \mathbb{R}^n_+$.

(P1) Covariance with respect to positive affine transformations. Let $(S, d, c) \in \mathbf{\Sigma^c}$ and let $L \in \mathscr{L}$ be given by $L_i(x) = a_i x_i + b_i$ for all i and all $x \in \mathbb{R}^n$, where $a \in \mathbb{R}^n_{++}$ and $b \in \mathbb{R}^n$. Then

---

[7] **N2** is modified as follows: *Let* $(S, d, c) \in \mathbf{\Sigma^c}$ *with* $g(S, d, c) \in \Delta$ *and let* $x^* \in \mathrm{BD}_g(S, d, c) \cap \Delta$ *satisfy* $e \cdot x^* \geq e \cdot y$ *for all* $y \in \mathrm{BD}_g(S, d, c)$. *Then there exists* $(S', d', c') \in \mathbf{\Sigma^c}$ *with* S' *symmetric, such that* $\mathrm{BD}_g(S, d, c) \subset \mathrm{BD}_g(S', d', c')$, $e \cdot x^* \geq e \cdot y$ *for all* $y \in \mathrm{BD}_g(S', d', c')$, *and* $g(S', d', c') = g(S, d, c)$. In $\mathbf{SY_g}$ add the requirement that $x^* \leq c$ for $x^* \in$ WPO $(S) \cap \Delta$, and in $\mathbf{IA_g}$ change the requirements that $S \subset S'$ and $F(S', d', c') \in S$ to $\mathrm{BD}_g(S, d, c) \subset \mathrm{BD}_g(S', d', c')$ and $F(S', d', c') \in \mathrm{BD}_g(S, d, c)$, respectively.

[8] $\| \cdot \|$ denotes the euclidean norm in $\mathbb{R}^n$.

[9] The type of reference function we consider here is in fact equivalent to a function which assigns to each bargaining problem with claims *two* reference points (a good example would be the status quo and the claims point). However, for technical reasons it is more convenient to work with the type of reference function defined before.

$$g^r(L(S,d,c)) = L(g^r(S,d,c)) \text{ and } g^p(L(S,d,c)) = \lambda L^a(g^p(S,d,c)),$$

where $L^a \in \mathscr{L}$ is given by $L_i^a(x) = a_i x_i$ for all i and $x \in \mathbb{R}^n$, and $\lambda = \|L^a(g^p(S,d,c))\|^{-1}$.

(P2) **Invariance with respect to the restriction to a symmetric subset.** Let $(S,d,c) \in \mathbf{\Sigma^c}$ be such that $g(S,d,c) \in \Delta \times \Delta$. Then there exists $(S',d',c') \in \mathbf{\Sigma^c}$ such that $S' \subset S$, S' is symmetric, $S' \cap \Delta = S \cap \Delta$, and $g(S',d',c') = g(S,d,c)$.

(P3) **Invariance with respect to approximation.** Let $(S,d,c) \in \mathbf{\Sigma^c}$ be such that $g(S,d,c) \in \Delta \times \Delta$. Then there exists a sequence $((S^n,d^n,c^n))_n \subset \mathbf{\Sigma^c}$ such that $S^n \to S, S \subset S^n$, WPO $(S^n) \cap \Delta =$ PO $(S^n) \cap \Delta$, and $g(S^n,d^n,c^n) = g(S,d,c)$ for all n.

P2 and **P3** are the exact counterparts of **E2** and **E3** for the type of reference function we consider in this section. Similarly, **P1** is the analogue of **N1**. Observe that the covariance condition we impose upon $g^p(\cdot)$ is very natural: the origin of utility measurement should not influence a player's relative bargaining strength, while if we scale up or down a player's utility by a given factor, then his relative bargaining strength should be scaled up or down accordingly.

Let $F : \mathbf{D^c} \to \mathbb{R}^n$ be a solution on the class $\mathbf{D^c} \subset \mathbf{\Sigma^c}$. By $\mathbf{SY2_g}$ and $\mathbf{RMON2_g}$ we denote the analogue of $\mathbf{SY_g}$ and $\mathbf{RMON_g}$ for the type of reference function we are considering here (in $\mathbf{SY_g}$ simply replace the condition $g(S,d,c) \in \Delta$ by $g(S,d,c) \in \Delta \times \Delta$). For any reference function $g : \mathbf{\Sigma^c} \to \mathbb{R}^n \times \mathbb{R}^n_+$ let

$$\overline{\mathbf{\Sigma}}^{\mathbf{c}}_{\mathbf{g}} = \{(S,d,c) \in \mathbf{\Sigma^c} | g^p(S,d,c) \in \mathbb{R}^n_{++}\}.$$

Definition 5.1. *Let* $g : \mathbf{\Sigma^c} \to \mathbb{R}^n \times \mathbb{R}^n_+$ *be a reference function. The proportional solution with respect to g is the function* $\mathscr{P}^g : \overline{\mathbf{\Sigma}}^{\mathbf{c}}_{\mathbf{g}} \to \mathbb{R}^n$, given by

$$\mathscr{P}^g(S,d,c) = g^r(S,d,c) + \bar{\lambda} g^p(S,d,c),$$

where $\bar{\lambda} = \max\{\lambda \in \mathbb{R} | g^r(S,d,c) + \lambda g^p(S,d,c) \in S\}, \quad (S,d,c) \in \overline{\mathbf{\Sigma}}^{\mathbf{c}}_{\mathbf{g}}$.

Observe that the term "proportional" does not mean that relative utility gains are exogenously given. Rather, the vector of weights arises endogenously from the bargaining problem. In particular, the egalitarian-type solutions analyzed in Sect. 3 do not belong to the class of proportional-type solutions defined here. Given the assumptions on the class $\overline{\mathbf{\Sigma}}^{\mathbf{c}}_{\mathbf{g}}$ it is straightforward to see that $\mathscr{P}^g$ (see Fig. 3) is well defined.

We omit the proof of the following theorem which is similar to the proof of Theorem 3.1 if one observes that $(S,d,c) \in \overline{\mathbf{\Sigma}}^{\mathbf{c}}_{\mathbf{g}}$ for any $(S,d,c) \in \mathbf{\Sigma^c}$ with $g^p(S,d,c) \in \Delta$.

Theorem 5.1. *Let* $g : \mathbf{\Sigma^c} \to \mathbb{R}^n \times \mathbb{R}^n_+$ *with* $g(S,d,c) = (g^r(S,d,c), g^p(S,d,c))$ and $\|g^p(S,d,c)\| = 1$ *for all* $(S,d,c) \in \mathbf{\Sigma^c}$, *satisfy* **P1**, **P2**, **P3**. *Then* $\mathscr{P}^g$ *is the unique solution on* $\overline{\mathbf{\Sigma}}^{\mathbf{c}}_{\mathbf{g}}$ *which satisfies* **WPO**, $\mathbf{SY2_g}$, **COV** *and* $\mathbf{RMON2_g}$.

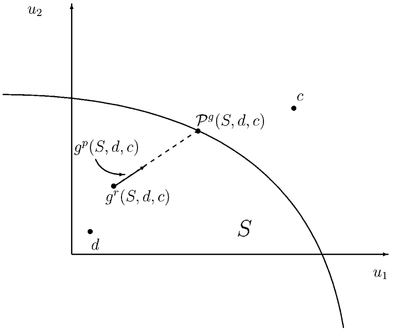

Fig. 3. Proportional solution with respect to g

As a corollary we again obtain a characterization of proportional-type solutions on the class $\boldsymbol{\Sigma}$: Let $g : \boldsymbol{\Sigma} \to \mathbb{R}^n \times \mathbb{R}^n_+$ be such that $g(S,d) = (g^r(S,d), g^p(S,d))$ with $\|g^p(S,d)\| = 1$ for all $(S,d) \in \boldsymbol{\Sigma}$. As before, we denote by $\mathbf{WPO}^*$, $\mathbf{SY2}^*_\mathbf{g}$, $\mathbf{COV}^*$ and $\mathbf{RMON2}^*_\mathbf{g}$ the counterparts of **WPO**, $\mathbf{SY2_g}$, **COV** and $\mathbf{RMON2_g}$ for a reference function $g : \boldsymbol{\Sigma} \to \mathbb{R}^n \times \mathbb{R}^n_+$, and a solution $f : \mathbf{D} \to \mathbb{R}^n$ defined on a subclass $\mathbf{D} \subset \boldsymbol{\Sigma}$. Let $\overline{\boldsymbol{\Sigma}}_\mathbf{g} = \{(S,d) \in \boldsymbol{\Sigma} | g^p(S,d) \in \mathbb{R}^n_{++}\}$. For $g : \boldsymbol{\Sigma} \to \mathbb{R}^n \times \mathbb{R}^n_+$ we define the solution $\rho^g : \overline{\boldsymbol{\Sigma}}_\mathbf{g} \to \mathbb{R}^n$ by

$$\rho^g(S,d) = g^r(S,d) + \bar{\lambda} g^p(S,d),$$

where $\bar{\lambda} = \max\{\lambda \in \mathbb{R} | g^r(S,d) + \lambda g^p(S,d) \in S\}$, $(S,d) \in \boldsymbol{\Sigma}$.

Corollary 5.1. *Let* $g : \boldsymbol{\Sigma} \to \mathbb{R}^n \times \mathbb{R}^n_+$ *with* $g(S,d) = (g^r(S,d), g^p(S,d))$ *and* $\|g^p(S,d)\| = 1$ *for all* $(S,d) \in \overline{\boldsymbol{\Sigma}}_\mathbf{g}$, *be such that* $\bar{g}$ *fulfills* **P1**, **P2** *and* **P3**. *Then* ρ^g *is the unique solution on* $\overline{\boldsymbol{\Sigma}}_\mathbf{g}$ *that satisfies* $\mathbf{WPO}^*$, $\mathbf{SY2}^*_\mathbf{g}$, $\mathbf{COV}^*$ *and* $\mathbf{RMON2}^*_\mathbf{g}$.

In the following we present some examples of reference functions which satisfy **P1**–**P3** and which give rise to well-known bargaining solutions. Again our results show that all these solutions obey the same principles and only differ in the associated reference function.

Example 5.1. Let $g : \boldsymbol{\Sigma}^\mathbf{c} \to \mathbb{R}^n \times \mathbb{R}^n_+$ be given by

$$g^r(S,d,c) = d, \; g^p(S,d,c) = (c-d)\|c-d\|^{-1}, \quad (S,d,c) \in \boldsymbol{\Sigma}^\mathbf{c}.$$

Observe that g is well defined since $c > d$ for all $(S,d,c) \in \boldsymbol{\Sigma}^\mathbf{c}$. For this choice of g the solution $\mathscr{P}^g$ is the *proportional solution* proposed by Chun and Thomson [3].

Example 5.2. Let $g : \boldsymbol{\Sigma} \to \mathbb{R}^n \times \mathbb{R}^n_+$ be given by

$$g^r(S,d) = d \text{ and } g^p(S,d) = \begin{cases} \lambda(u(S,d) - d), & \text{if } u(S,d) > d \\ \sqrt{\frac{1}{n}} e, & \text{else} \end{cases},$$

where $\lambda = \|u(S,d) - d\|^{-1}, (S,d) \in \boldsymbol{\Sigma}$. For this choice of g the solution ρ^g is the *Kalai-Smorodinsky solution* which was proposed by Raiffa [10] and was later axiomatically characterized by Kalai and Smorodinsky [8].

Example 5.3. Let $g : \boldsymbol{\Sigma}^{\mathbf{c}} \to \mathbb{R}^n \times \mathbb{R}^n_+$ be given by

$$g^r(S,d,c) = t(S,d,c), \quad g^p(S,d,c) = \lambda(c - t(S,d,c)),$$

where $\lambda = \|c - t(S,d,c)\|^{-1}$, $(S,d,c) \in \boldsymbol{\Sigma}^{\mathbf{c}}$. (For a definition of $t(S,d,c)$ see Example 3.5.) Since $t(S,d,c) \in S$ and $c \notin S$ the function g is well defined. Then, the solution $\mathscr{P}^g$ is the *adjusted proportional solution* proposed by Herrero [6].

Example 5.4. Let $g : \boldsymbol{\Sigma}^{\mathbf{c}} \to \mathbb{R}^n \times \mathbb{R}^n_+$ be given by

$$g^r(S,d,c) = t(S,d,c),$$
$$g^p(S,d,c) = \begin{cases} \lambda(u(S,t(S,d,c)) - t(S,d,c)), \text{ if } u(S,t(S,d,c)) > t(S,d,c) \\ \sqrt{\frac{1}{n}}e, \text{ else} \end{cases},$$

where $\lambda = \|u(S,t(S,d,c)) - t(S,d,c)\|^{-1}, (S,d,c) \in \boldsymbol{\Sigma}^{\mathbf{c}}$. For this choice of g the solution $\mathscr{P}^g$ is the *extended Kalai–Smorodinsky solution* $\mathscr{K}^t$ proposed by Gerber [5].

Remark 5.1. Again, for a general reference function g satisfying **P1–P3** the proportional solution with respect to g may violate individual rationality or boundedness by the claims point. However, similar to the case of egalitarian-type solutions (see Remark 3.1) it is straightforward to define and characterize extended proportional-type solutions which satisfy boundedness.[10] ♦

6 Conclusion

We have proposed a unifying approach to the solution of bargaining problems with and without claims by using the concept of a reference function. This function was taken to be exogenously given, reflecting that in many real world situations there exist "focal points" as perceived by an arbitrator, by a group of players or by society as a whole. Although a reference function sometimes may be naturally given, from a theoretical point of view, we might be interested in the choice of the reference function. A reference function g can be characterized either directly or indirectly, i.e., either by imposing a set of axioms on g itself or on the solution which is defined with respect to g. It seems safe to conjecture that it is not possible to uniquely characterize a given reference function.[11] Rather, we can hope to characterize a certain range in

[10] In Remark 3.1 simply replace e by $g^p(S,d,c)$ for $(S,d,c) \in \overline{\boldsymbol{\Sigma}}^{\mathbf{c}}_{\mathbf{g}}$.

[11] At least not without using an axiom that has the flavor of prescribing which reference function to choose.

which the reference points will lie. The main reason for this indeterminacy is that a reference function, in general, does not assign a weakly Pareto optimal outcome to a bargaining problem. Often the reference point is not even a feasible utility allocation. A different question is whether for a given bargaining solution there is a "natural" reference function that can be associated with the solution. These questions are left for future research.

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