Transformation Groups, Vol. 16, No. 4, 2011, pp. 1137–1142 ©Birkhäuser Boston (2011)

ROOTS OF THE AFFINE CREMONA GROUP

ALVARO LIENDO

Mathematisches Institut
Universität Basel
Rheinsprung 21
CH-4051 Basel, Switzerland

alvaro.liendo@unibas.ch

Abstract. Let $\mathbf{k}^{[n]} = \mathbf{k}[x_1, \ldots, x_n]$ be the polynomial algebra in n variables and let $\mathbb{A}^n = \operatorname{Spec} \mathbf{k}^{[n]}$. In this note we show that the root vectors of $\operatorname{Aut}^*(\mathbb{A}^n)$, the subgroup of volume preserving automorphisms in the affine Cremona group $\operatorname{Aut}(\mathbb{A}^n)$, with respect to the diagonal torus are exactly the locally nilpotent derivations $\mathbf{x}^{\boldsymbol{\alpha}}(\partial/\partial x_i)$, where $\mathbf{x}^{\boldsymbol{\alpha}}$ is any monomial not depending on x_i. This answers a question posed by Popov.

Introduction

Letting $\mathbf{k}$ be an algebraically closed field of characteristic zero, we let $\mathbf{k}^{[n]} = \mathbf{k}[x_1, \ldots, x_n]$ be the polynomial algebra in n variables, and $\mathbb{A}^n = \operatorname{Spec} \mathbf{k}^{[n]}$ be the affine space. The affine Cremona group $\operatorname{Aut}(\mathbb{A}^n)$ is the group of automorphisms of $\mathbb{A}^n$, or equivalently, the group of $\mathbf{k}$-automorphisms of $\mathbf{k}^{[n]}$. We define $\operatorname{Aut}^*(\mathbb{A}^n)$ as the subgroup of volume preserving automorphisms, i.e.,

$$\operatorname{Aut}^*(\mathbb{A}^n) = \left\{ \gamma \in \operatorname{Aut}(\mathbb{A}^n) \;\middle|\; \det\left(\frac{\partial}{\partial x_i}\gamma(x_j)\right)_{i,j} = 1 \right\}.$$

The groups $\operatorname{Aut}(\mathbb{A}^n)$ and $\operatorname{Aut}^*(\mathbb{A}^n)$ are infinite dimensional algebraic groups [Sha66, Kam79].

It follows from [BB66, BB67] that the maximal dimension of an algebraic torus contained in $\operatorname{Aut}^*(\mathbb{A}^n)$ is $n-1$. Moreover, every algebraic torus of dimension $n-1$ contained in $\operatorname{Aut}^*(\mathbb{A}^n)$ is conjugated to the diagonal torus

$$\mathbf{T} = \{\gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_n) \in \operatorname{Aut}^*(\mathbb{A}^n) \mid \gamma_1 \cdots \gamma_n = 1\}\ .$$

A $\mathbf{k}$-derivation ∂ on an algebra A is called *locally nilpotent* (LND for short) if for every $a \in A$ there exists $k \in \mathbb{Z}_{\geq 0}$ such that $\partial^k(a) = 0$. If $\partial : \mathbf{k}^{[n]} \to \mathbf{k}^{[n]}$ is an LND on the polynomial algebra, then $\exp(t\partial) \in \operatorname{Aut}^*(\mathbb{A}^n)$, for all $t \in \mathbf{k}$ [Fre06]. Hence, ∂ belongs to the Lie algebra $\operatorname{Lie}(\operatorname{Aut}^*(\mathbb{A}^n))$.

In analogy with the notion of root from the theory of algebraic groups [Spr98], Popov introduced the following definitions; see [Pop$_1$05], [Pop$_2$05]. A nonzero LND

DOI: 10.1007/s00031-011-9140-y

Received October 22, 2010. Accepted January 25, 2011. Published online May 12, 2011.

∂ on $\mathbf{k}^{[n]}$ is called a *root vector* of $\operatorname{Aut}^*(\mathbb{A}^n)$ with respect to the diagonal torus $\mathbf{T}$ if there exists a character χ of $\mathbf{T}$ such that

$$\gamma \circ \partial \circ \gamma^{-1} = \chi(\gamma) \cdot \partial, \quad \text{for all} \quad \gamma \in \mathbf{T}\,.$$

The character χ is called the *root* of $\operatorname{Aut}^*(\mathbb{A}^n)$ with respect to $\mathbf{T}$ corresponding to ∂.

Letting $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we let $\mathbf{x}^{\boldsymbol{\alpha}}$ be the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. In this note we apply the results in [Lie10] to prove the following theorem. This answers a question posed by Popov [$\text{Pop}_1$05], [$\text{Pop}_2$05].

Theorem 1. *The root vectors of* $\operatorname{Aut}^*(\mathbb{A}^n)$ *with respect to the diagonal torus* $\mathbf{T}$ *are exactly the LNDs*

$$\partial = \lambda \cdot \mathbf{x}^{\boldsymbol{\alpha}} \cdot \frac{\partial}{\partial x_i}\,,$$

where $\lambda \in \mathbf{k}^*$, $i \in \{1, \ldots, n\}$, *and* $\boldsymbol{\alpha} \in \mathbb{Z}^n_{\geq 0}$ *with* $\alpha_i = 0$. *The corresponding root is the character*

$$\chi : \mathbf{T} \to \mathbf{k}^*, \quad \gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_n) \mapsto \gamma_i^{-1} \prod_{j=1}^{n} \gamma_j^{\alpha_j}\,.$$

Acknowledgments. The author is grateful to Ivan Arzhantsev who pointed out this problem to us, and to Vladimir Popov, Polina Kotenkova and both referees for useful remarks.

Proof of Theorem 1

It is well known that the group $\chi(\mathbf{T})$ of characters of $\mathbf{T}$ forms a lattice whose dual lattice is the group $\lambda(\mathbf{T})$ of one-parameter subgroups of $\mathbf{T}$. It is customary to consider these lattices in additive notation. In this case we denote $\chi(\mathbf{T})$ by M and $\lambda(\mathbf{T})$ by N. To avoid confusion between the addition in M and that in the algebra of regular functions on $\mathbf{T}$, the character of $\mathbf{T}$ corresponding to an element $m \in M$ is denoted by χ^m. The composition of the canonical isomorphism $\mathbb{Z}^n \to \chi((\mathbf{k}^*)^n)$ with the restriction map $\chi((\mathbf{k}^*)^n) \to \chi(\mathbf{T})$, $f \mapsto f|_{\mathbf{T}}$, induces the isomorphism of lattices $\mathbb{Z}^n/\mathbb{1} \cdot \mathbb{Z} \xrightarrow{\sim} M$, where $\mathbb{1} = (1, \ldots, 1) \in \mathbb{Z}^n$. We identify them by means of this isomorphism: $M = \mathbb{Z}^n/\mathbb{1} \cdot \mathbb{Z}$. Correspondingly, we put $N = \ker(p \mapsto p(\mathbb{1})) \subseteq (\mathbb{Z}^n)^*$.

The natural $\mathbf{T}$-action on $\mathbb{A}^n$ gives rise to an M-grading on $\mathbf{k}^{[n]}$ given by

$$\mathbf{k}^{[n]} = \bigoplus_{m \in M} B_m, \quad \text{where} \quad B_m = \left\{ f \in \mathbf{k}^{[n]} \mid \gamma(f) = \chi^m(\gamma) f,\, \forall \gamma \in \mathbf{T} \right\}.$$

An LND ∂ on $\mathbf{k}^{[n]}$ is called homogeneous if it sends homogeneous elements into homogeneous elements. Let ∂ be a homogeneous LND on $\mathbf{k}^{[n]}$, and let $f \in \mathbf{k}^{[n]} \setminus \ker \partial$ be homogeneous. We define the degree of ∂ as $\deg \partial = \deg(\partial(f)) - \deg(f) \in M$. This definition does not depend on the choice of f; see [Lie10, Sect. 1.2].

Lemma 2. *An LND on* $\mathbf{k}^{[n]}$ *is a root vector of* $\mathrm{Aut}^*(\mathbb{A}^n)$ *with respect to the diagonal torus* $\mathbf{T}$ *if and only if* ∂ *is homogeneous with respect to the* M*-grading on* $\mathbf{k}^{[n]}$ *given by* $\mathbf{T}$*. Furthermore, the corresponding root is the character* $\chi^{\deg \partial}$*.*

Proof. Let ∂ be a root vector of $\mathrm{Aut}^*(\mathbb{A}^n)$ with root χ^e, so that

$$\partial = \chi^{-e}(\gamma) \cdot \gamma \circ \partial \circ \gamma^{-1}, \quad \forall \gamma \in \mathbf{T}.$$

We consider a homogeneous element $f \in B_{m'}$ and we let $\partial(f) = \sum_{m \in M} g_m$, where $g_m \in B_m$, so that

$$\sum_{m \in M} g_m = \partial(f) = \chi^{-e}(\gamma) \cdot \gamma \circ \partial \circ \gamma^{-1}(f) = \chi^{-e-m'}(\gamma) \sum_{m \in M} \chi^m(\gamma) \cdot g_m, \quad \forall \gamma \in \mathbf{T}.$$

This equality holds if and only if $g_m = 0$ for all but one $m \in M$, i.e., if ∂ is homogeneous. In this case, $\partial(f) = g_m = \chi^{-e-m'+m}(\gamma) \cdot \partial(f)$, and so $e = m - m' = \deg(\partial(f)) - \deg(f) = \deg \partial$. □

In [AH06], a combinatorial description of a normal affine M-graded domain A is given in terms of polyhedral divisors, and in [Lie10] a description of the homogeneous LNDs on A is given in terms of these combinatorial data in the case where $\mathrm{tr.deg}\, A = \mathrm{rank}\, M + 1$. In the following we apply these results to compute the homogeneous LNDs on the M-graded algebra $\mathbf{k}^{[n]}$. First, we give a short presentation of the combinatorial description in [AH06] in the case where $\mathrm{tr.deg}\, A = \mathrm{rank}\, M + 1$. For a more detailed treatment see [Lie10, Sect. 1.1].

The combinatorial description in [AH06] deals with the following data: A pointed polyhedral cone $\sigma \subseteq N_{\mathbb{Q}} := N \otimes \mathbb{Q}$ dual to the weight cone $\sigma^\vee \subseteq M_{\mathbb{Q}} := M \otimes \mathbb{Q}$ of the M-grading; a smooth curve C; and a divisor $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ on C whose coefficients Δ_z are polyhedra in $N_{\mathbb{Q}}$ having tail cone σ. For every $m \in \sigma^\vee$ the evaluation of $\mathfrak{D}$ at m is the $\mathbb{Q}$-divisor given by

$$\mathfrak{D}(m) = \sum_{z \in C} \min\{p(m) \mid p \in \Delta_z\} \cdot z\,.$$

Furthermore, in the case where C is projective we ask for the following two conditions:

(i) For every $m \in \sigma^\vee$, $\deg \mathfrak{D}(m) \geq 0$; and
(ii) If $\deg \mathfrak{D}(m) = 0$, then m is in the boundary of $\sigma^\vee$ and a multiple of $\mathfrak{D}(m)$ is principal.

We define the M-graded algebra

$$A[\mathfrak{D}] = \bigoplus_{m \in \sigma^\vee \cap M} A_m \chi^m, \quad \text{where} \quad A_m = H^0(C, \mathcal{O}_C(\mathfrak{D}(m)))\,, \tag{1}$$

and χ^m is the corresponding character of the torus $\mathrm{Spec}\, \mathbf{k}[M]$ seen as a rational function on $\mathrm{Spec}\, A$ via the embedding $\mathrm{Frac}\, \mathbf{k}[M] \hookrightarrow \mathrm{Frac}\, A[\mathfrak{D}] = \mathrm{Frac}\, \mathbf{k}(C)[M]$.

It follows from [AH06] that $A[\mathfrak{D}]$ is a normal affine domain and that every normal affine M-graded domain A with $\mathrm{tr.deg}\, A = \mathrm{rank}\, M + 1$ is equivariantly isomorphic to $A[\mathfrak{D}]$ for some polyhedral divisor on a smooth curve; see also [Lie10, Theorem 1.4].

We turn back now to our particular case where we deal with the polynomial algebra $\mathbf{k}^{[n]}$ graded by $M = \mathbb{Z}^n/\mathbb{1}\cdot\mathbb{Z}$. Letting $\{\mu_1,\ldots,\mu_n\}$ be the canonical basis of $\mathbb{Z}^n$ the M-grading on $\mathbf{k}^{[n]}$ is given by $\deg x_i = \mu_i$, for all $i\in\{1,\ldots,n\}$. Now let $\{\nu_1,\ldots,\nu_n\}$ be the dual basis of $(\mathbb{Z}^n)^*$, and let $\Delta \subseteq N_{\mathbb{Q}} = \ker(p\mapsto p(\mathbb{1}))\subseteq(\mathbb{Q}^n)^*$ be the convex hull of the set $\{\nu_1-\nu_n,\ldots,\nu_{n-1}-\nu_n,\bar{0}\}$.

Lemma 3. *The M-graded algebra $\mathbf{k}^{[n]}$ is equivariantly isomorphic to $A[\mathfrak{D}]$, where $\mathfrak{D}$ is the polyhedral divisor $\mathfrak{D}=\Delta\cdot[0]$ on $\mathbb{A}^1$.*

Proof. By [AH06], the M-graded algebra $\mathbf{k}^{[n]}$ is isomorphic to $A[\mathfrak{D}]$ for some polyhedral divisor $\mathfrak{D}$ on a smooth curve C. Since the weight cone $\sigma^\vee$ of $\mathbf{k}^{[n]}$ is $M_{\mathbb{Q}}$, the coefficients of $\mathfrak{D}$ are just bounded polyhedra in $N_{\mathbb{Q}}$.

Since $\mathbb{A}^n$ is a toric variety and the torus $\mathbf{T}$ is a subtorus of the big torus, we can apply the method in [AH06, Sect. 11]. In particular, C is a toric curve. Thus $C=\mathbb{A}^1$ or $C=\mathbb{P}^1$. Furthermore, the graded piece $B_{\bar{0}}\supsetneq\mathbf{k}$ and so C is not projective by (1). Hence $C=\mathbb{A}^1$.

The only divisor in $\mathbb{A}^1$ invariant by the big torus is [0], so $\mathfrak{D}=\Delta\cdot[0]$ for some bounded polyhedron Δ in $N_{\mathbb{Q}}$. Finally, applying the second equation in [AH06, Sect. 11], a routine computation shows that Δ can be chosen as the the convex hull of $\{\nu_1-\nu_n,\ldots,\nu_{n-1}-\nu_n,\bar{0}\}$. □

Remark 4.

(i) The polyhedron $\Delta\subseteq N_{\mathbb{Q}}$ is the standard $(n-1)$-simplex in the basis $\{\nu_1-\nu_n,\ldots,\nu_{n-1}-\nu_n\}$.

(ii) Letting $\mathbb{A}^1=\operatorname{Spec}\mathbf{k}[t]$, it is possible to show by a direct computation that the isomorphism $\mathbf{k}^{[n]}\simeq A[\mathfrak{D}]$ is given by $x_i=\chi^{\mu_i}$, for all $i\in\{1,\ldots,n-1\}$, and $x_n=t\chi^{\mu_n}$. This provides a proof of Lemma 3 that avoids the reference to [AH06].

In [Lie10] the homogeneous LNDs on a normal affine M-graded domain are classified into 2 types: fiber type and horizontal type. In the case where the weight cone is $M_{\mathbb{Q}}$, there are no LNDs of fiber type. Thus, $\mathbf{k}^{[n]}$ admits only homogeneous LNDs of horizontal type. The homogeneous LNDs of horizontal type are described in [Lie10, Theorem 3.28]. In the following, we specialize this result to the particular case of $A[\mathfrak{D}]\simeq\mathbf{k}^{[n]}$.

Let $v_i=\nu_i-\nu_n$, $i\in\{1,\ldots,n-1\}$ and $v_n=\bar{0}$, so that $\{v_1,\ldots,v_n\}$ is the set of vertices of Δ. For every $\lambda\in\mathbf{k}^*$, $i\in\{1,\ldots,n\}$, and $e\in M$ we let $\partial_{\lambda,i,e}:\operatorname{Frac}A[\mathfrak{D}]\to\operatorname{Frac}A[\mathfrak{D}]$ be the derivation given by

$$\partial_{\lambda,i,e}(t^r\cdot\chi^m)=\lambda(r+v_i(m))\cdot t^{r-v_i(e)-1}\cdot\chi^{m+e},\quad\forall(m,r)\in M\times\mathbb{Z}\,.$$

Lemma 5 ([Lie10, Theorem 3.28])**.** *If ∂ is a nonzero homogeneous LND of $A[\mathfrak{D}]\simeq\mathbf{k}^{[n]}$, then $\partial=\partial_{\lambda,i,e}|_{A[\mathfrak{D}]}$ for some $\lambda\in\mathbf{k}^*$, some $i\in\{1,\ldots,n\}$, and some $e\in M$ satisfying $v_j(e)\geq v_i(e)+1$, $\forall j\neq i$. Furthermore, e is the degree* $\deg\partial$.

Proof of Theorem 1. By Lemma 2 the root vectors of $\mathbf{k}^{[n]}$ correspond to the homogeneous LNDs in the M-graded algebra $\mathbf{k}^{[n]}$. But the homogeneous LNDs on $A[\mathfrak{D}]\simeq\mathbf{k}^{[n]}$ are given in Lemma 5, so we need only to translate the homogeneous

LND $\partial = \partial_{\lambda,i,e}|_{A[\mathfrak{D}]}$ in Lemma 5 in terms of the explicit isomorphism given in Remark 4(ii).

Let $e = (e_1, \ldots, e_n) \in M$ and $i \in \{1, \ldots, n-1\}$, so that $v_i = \nu_i - \nu_n$. Since $\mathbb{1}$ is in the class of zero in M, we may and will assume $e_i = -1$. Then, the condition $v_j(e) \geq v_i(e) + 1$ yields $e_j \geq 0$, $\forall j \neq i$. Furthermore, $\partial(x_k) = \partial(\chi^{\mu_k}) = 0$, for all $k \neq i$, $k \in \{1, \ldots, n-1\}$, $\partial(x_n) = \partial(t\chi^{\mu_n}) = 0$, and

$$\partial(x_i) = \partial(\chi^{\mu_i}) = \lambda t^{e_n} \chi^{e+\mu_i} = \lambda \prod_{j \neq i} x_j^{e_j} = \lambda \mathbf{x}^{\boldsymbol{\alpha}},$$

where $\alpha_i = 0$, and $\alpha_j = e_j \geq 0$, for all $j \neq i$. Hence, $\partial = \lambda \cdot \mathbf{x}^{\boldsymbol{\alpha}} \cdot (\partial/\partial x_i)$, for some $\lambda \in \mathbf{k}^*$, some $i \in \{1, \ldots, n-1\}$, and some $\boldsymbol{\alpha} \in \mathbb{Z}^n_{\geq 0}$ such that $\alpha_i = 0$.

Now let $e = (e_1, \ldots, e_n) \in M$ and $i = n$, so that $v_n = 0$. Since $\mathbb{1}$ is in the class of zero in M, we may and will assume $e_n = -1$. Then, the condition $v_j(e) \geq v_n(e) + 1$ yields $e_j \geq 0$, $\forall j \in \{1, \ldots, n-1\}$. Furthermore, $\partial(x_k) = \partial(\chi^{\mu_k}) = 0$, $k \in \{1, \ldots, n-1\}$, and

$$\partial(x_n) = \partial(t\chi^{\mu_n}) = \lambda \chi^{e+\mu_n} = \lambda \prod_{j \neq n} x_j^{e_j} = \lambda \mathbf{x}^{\boldsymbol{\alpha}},$$

where $\alpha_n = 0$, and $\alpha_j = e_j \geq 0$, for all $j \in \{1, \ldots, n-1\}$. Hence, $\partial = \lambda \cdot \mathbf{x}^{\boldsymbol{\alpha}} \cdot (\partial/\partial x_n)$, for some $\lambda \in \mathbf{k}^*$ and some $\boldsymbol{\alpha} \in \mathbb{Z}^n_{\geq 0}$ such that $\alpha_n = 0$.

The last assertion of the theorem follows easily from the fact that the root corresponding to the homogeneous LND ∂ is the character $\chi^{\deg \partial}$. □

Finally, we describe the characters that appear as a root of $\mathrm{Aut}^*(\mathbb{A}^n)$.

Corollary 6. *The character $\chi \in \chi(\mathbf{T})$ given by* $\mathrm{diag}(\gamma_1, \ldots, \gamma_n) \mapsto \gamma_1^{\beta_1} \cdots \gamma_n^{\beta_n}$ *is a root of* $\mathrm{Aut}^*(\mathbb{A}^n)$ *with respect to the diagonal torus* $\mathbf{T}$ *if and only if the minimum of the set $\{\beta_1, \ldots, \beta_n\}$ is achieved by one and only one of the β_i.*

Proof. By Theorem 1, the roots of $\mathrm{Aut}^*(\mathbb{A}^n)$ are the characters $\mathrm{diag}(\gamma_1, \ldots, \gamma_n) \mapsto \gamma_1^{\beta_1} \cdots \gamma_n^{\beta_n}$, where $\beta_i = -1$ for some $i \in \{1, \ldots, n\}$ and $\beta_j \geq 0$ $\forall j \neq i$. The corollary follows from the fact that $\gamma_1 \cdots \gamma_n = 1$. □

References

[AH06] K. Altmann, J. Hausen, *Polyhedral divisors and algebraic torus actions*, Math. Ann. **334** (2006), no. 3, 557–607.

[BB66] A. Białynicki-Birula, *Remarks on the action of an algebraic torus on k^n*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **14** (1966), 177–181.

[BB67] A. Białynicki-Birula, *Remarks on the action of an algebraic torus on k^n*, II, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **15** (1967), 123–125.

[Fre06] G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*, Encyclopaedia of Mathematical Sciences, Vol. 136, Series *Invariant Theory and Algebraic Transformation Groups*, Vol. VII, Springer, Berlin, 2006.

[Kam79] T. Kambayashi, *Automorphism group of a polynomial ring and algebraic group action on an affine space*, J. Algebra **60** (1979), no. 2, 439–451.

[Lie10] A. Liendo, *Affine $\mathbb{T}$-varieties of complexity one and locally nilpotent derivations* Transform. Groups **15** (2010), no. 2, 389–425.

[$\text{Pop}_1$05] V. L. Popov, *Problems for problem session*, `arXiv:math.AG/0504517v1` 25 Apr 2005.

[$\text{Pop}_2$05] V. L. Popov, *Roots of the affine Cremona group*, in: *Affine Algebraic Geometry*, Seville, Spain, June 18–21, 2003, Contemporary Mathematics, Vol. 369, American Mathematical Society, Providence, RI, 2005, pp. 12–13.

[Sha66] I. R. Shafarevich, *On some infinite-dimensional groups*, Rend. Mat. e Appl. (5) **25** (1966), nos. 1–2, 208–212.

[Spr98] T. A. Springer, *Linear Algebraic Groups*, 2nd ed., Progress in Mathematics, Vol. 9, Birkhäuser Boston, Boston, MA, 1998.