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Approximating J-holomorphic curves by holomorphic ones

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Abstract. Given an almost complex structure J in a cylinder of $\mathbb{R}^{2p}$ $(p > 1)$ together with a compatible symplectic form ω and given an arbitrary $J-$holomorphic curve Σ without boundary in that cylinder, we construct an holomorphic perturbation of Σ, for the canonical complex structure J_0 of $\mathbb{R}^{2p}$, such that the distance between these two curves in $W^{1,2}$ and L^∞ norms, in a sub-cylinder, are controled by quantities depending on J, ω and by the area of Σ only. These estimates depend neither on the topology nor on the conformal class of Σ. They are key tools in the recent proof of the regularity of 1-1 integral currents in [RT].

1 Introduction

Let ω be a smooth symplectic form in $B^2_2(0) \times B^{2p-2}_2(0)$ $(p > 1)$ - ω is a closed 2-form satisfying $\omega^p > 0$ – and let J be a smooth compatible almost complex structure : $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is symmetric and therefore defines a scalar product in $B^2_2(0) \times B^{2p-2}_2(0)$ that we will denote by g. We assume that at the origin $\omega(0)$ coincides with the standard symplectic form of $\mathbb{R}^{2p}$, $\omega_0 = \sum_{i=1}^p dx_{2i-1} \wedge dx_{2i}$ and that $J(0)$ coincides with the standard almost-complex structure J_0 satisfying $J_0 \cdot e_{2i-1} = e_{2i}$ for $i = 1 \cdots p$ where e_k is the canonical basis of $\mathbb{R}^{2p}$.

We consider a J holomorphic curve $\Psi : \Sigma \to B^2_2(0) \times B^{2p-2}_2(0)$ (Σ is a smooth Riemann surface and Ψ a smooth $J-$holomorphic map from Σ into $(B^2_2(0) \times B^{2p-2}_2(0), J)$). we assume that the current $\Psi_*[\Sigma]$ satisfies

$$supp(\partial\,(\Psi_*[\Sigma])) \subset \partial B^2_2(0) \times B^{2p-2}_2(0) \quad . \tag{1.1}$$

We will adopt the following notation : for any $r < 2$

$$\Sigma_r := \Psi^{-1}(B^2_r(0) \times B^{2p-2}_2(0)) \quad . \tag{1.2}$$

(Under these notations one has for instance $\Sigma_2 = \Sigma$). We define now the "distortions" of $g(\cdot, J\cdot)$ relative to the canonical flat metric g_0 in $B^2_2(0) \times B^{2p-2}_2(0)$.

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These are the following quantities :

$$
\begin{aligned}
d_1(g) &:= \sup_x \sup_{X\in T_xR^{2p}} \frac{g(X,X)}{g_0(X,X)} + \frac{g_0(X,X)}{g(X,X)} \\
d_2(g) &:= \sup_x \sup_{X,Y\in T_xR^{2p}} \frac{g(X\wedge Y,X\wedge Y)}{g_0(X\wedge Y,X\wedge Y)} + \frac{g_0(X\wedge Y,X\wedge Y)}{g(X\wedge Y,X\wedge Y)} \quad .
\end{aligned}
\tag{1.3}
$$

Our main result in this paper is the following.

Theorem 1.1 *For any J holomorphic curve $\Psi : \Sigma \to B_2^2(0)\times B_2^{2p-2}(0)$ satisfying (1.1), there exists a map $\eta : \Sigma \to \mathbb{R}^{2p}$, such that $\Psi+\eta$ is J_0 holomorphic and η satisfies*

$$
\int_\Sigma |\nabla\eta|^2 \le 2\,\|J-J_0\|_\infty^2 \int_\Sigma \Psi^*\omega \quad , \tag{1.4}
$$

and

$$
\|\eta\|_{L^\infty(\Sigma_1)} \le K\,\|\nabla J\|_\infty \quad , \tag{1.5}
$$

where K is a constant depending only on $d_1(g)$, $d_2(g)$ and $\int_\Sigma \Psi^*\omega$.

The striking fact in this result is that the constants are independent of the conformal type and the topology of Σ. These estimates are key tools in the proof of the regularity of 1-1 integral currents in arbitrary dimension in [RT]. η is chosed to be the solution of the following elliptic problem – see Proposition A.3 -

$$
\begin{cases}
\overline{\partial}\eta = -\overline{\partial}\Psi & \text{in } \mathcal{D}'(\Sigma) \\[2ex]
\forall h\in H_0^+(\Sigma) & \displaystyle\int_{\partial\Sigma} \eta\, dh = 0 \quad . \\[2ex]
\forall \Sigma_k \quad \text{connected compo. of } \Sigma\ , & \displaystyle\int_{\partial\Sigma_k} \eta = 0
\end{cases}
\tag{1.6}
$$

where we are representing η and Ψ by their canonical complex coordinates in $(\mathbb{R}^{2p}, J_0)$ and where $H_0^+(\Sigma)$ denotes the space of $W^{1,2}(\Sigma)$ holomorphic functions on Σ. Observe that since Ψ is $J-$holomorphic, taking the ∂ of the first equation in (1.6) one gets for all $k = 1\cdots 2p$ (using the real coordinates this time)

$$
\Delta_\Sigma \eta^k = -*\left(\sum_{l=1}^{2p} d(J_l^k(\Psi))\wedge d\Psi^l\right) \qquad \text{in } \mathcal{D}'(\Sigma) \quad . \tag{1.7}
$$

Since $\int_\Sigma |\nabla\psi|^2 = 2\int_\Sigma \Psi^*\omega$ which is one of the variable of the problem one is led to a first order formulation of Wente's Problem : Let u be a function on Σ satisfying

$$
\begin{cases}
\overline{\partial}u = f & \text{in } \mathcal{D}'(\Sigma) \\[2ex]
\forall h\in H_0^+(\Sigma) & \displaystyle\int_{\partial\Sigma} u\, dh = 0 \quad , \\[2ex]
\forall \Sigma_k \quad \text{connected compo. of } \Sigma\ , & \displaystyle\int_{\partial\Sigma_k} u = 0
\end{cases}
\tag{1.8}
$$

where f is a L^2 $\overline{\partial}$ exact $(0,1)$ form $f = \overline{\partial}\phi$ satisfying

$$*\partial f = da \wedge db \quad , \tag{1.9}$$

where a and b are $W^{1,2}$ functions in Σ. Assuming f is $L^2(\Sigma)$ perpendicular to $\overline{\partial}H_0^-(\Sigma)\oplus\overline{\partial}V$, where $H_0^-(\Sigma)$ is the space of anti-holomorphic functions in Σ and V is the finite dimensional space of harmonic functions in Σ which are constant on each connected component of $\partial\Sigma$, then one easily verifies, see the appendix, that the harmonic extension $\tilde{u}$ is perpendicular to $H_0^+(\Sigma)\oplus H_0^-(\Sigma)\oplus V$ and therefore is equal to 0. Thus u satisfies

$$\begin{cases} *\Delta u = da \wedge db & \text{in } \mathcal{D}'(\Sigma) \\ \\ u = 0 & \text{on } \partial\Sigma \end{cases} \tag{1.10}$$

and from P.Topping's result [To] one has

$$\|u\|_{L^\infty(\Sigma)} \leq \frac{1}{2\pi}\|\nabla a\|_{L^2(\Sigma)}\ \|\nabla b\|_{L^2(\Sigma)} \quad , \tag{1.11}$$

(see more on the second order Wente Problem in [Ge] and [He]). Therefore if one would know that Ψ is perpendicular to $H_0^-(\Sigma)\oplus V$ we would directly have obtained

$$\|\eta\|_{L^\infty(\Sigma)} \leq \frac{4p}{\pi}\int_\Sigma \Psi^*\omega\ \|\nabla J\|_\infty \quad . \tag{1.12}$$

Of course there is no reason for Ψ to satisfy this assumption and the difficulty comes then from the L^2 projection of Ψ over $H_0^-(\Sigma)\oplus V$. A solution η to the problem (1.8) and (1.9) can even not be bounded in L^∞ on the whole Σ. Take for instance $\Sigma = D^2$ and

$$f = \overline{\partial}\left(\sum_{n=1}^{\infty}\frac{1}{n\ \log\ n}\ e^{-i\,n\theta}\right) \quad .$$

Therefore there is a real need to restrict to a subdomain of Σ as we do in (1.5). In this sense our result is optimal.

In the proof below we were influenced by the proofs in [Ch] and [To] .

2 Proof of Theorem 1.1.

Before to prove Theorem 1.1 we need an intermediate result.

Lemma 2.1 *Let $\Psi : \Sigma \to B_2^2(0)\times B_2^{2p-2}(0)$ be a $J-$holomorphic curve satisfying (1.1). For any smooth function u whose average on each connected component of Σ is zero, or any function u in $C_0^\infty(\Sigma_2)$, the following inequality holds*

$$\left(\int_{\Sigma_1}|u|^2\right)^{\frac{1}{2}} \leq K\int_\Sigma |\nabla u| \quad , \tag{2.13}$$

where Σ_1 is defined in (1.2), the metric on Σ is the pull-back by Ψ of $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ and K is a constant depending only on $d_1(g)$, $d_2(g)$ and $\int\Psi^*\omega$.

Proof of Lemma 2.1. We present the proof in the case where the average of u vanishes on each connected component of Σ (the other case $u \in C_0^\infty(\Sigma)$ being more easy). Let Σ^k be a connected component of Σ intersecting Σ_1. We divide Σ^k into 2 subsurfaces $\Sigma^{k,+}$ (res. $\Sigma^{k,-}$) being the subset of Σ^k where u is positive (resp. negative). Using the coarea formula (see [Fe]) we have

$$\int_{\Sigma^{k,+}} |\nabla u| = \int_0^{+\infty} \mathcal{H}^1(u^{-1}(s) \cap \Sigma^k)\, ds \tag{2.14}$$

Since u is smooth and Σ^k is connected and since 0 is a value of u in Σ^k, for every regular value $s \in \mathbb{R}^+$ of u 2 cases may happen.

Case 1:

$$\exists r \in [1,2] \quad \text{such that} \quad \partial\{x \in \Sigma^k_r\ ;\ u(x) \geq s\} = u^{-1}(s) \cap \Sigma^k_r \quad .$$

In that case, since Σ_r is an area minimizing surface in $\mathbb{R}^{2p}$ for the metric $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$, we have

$$\begin{aligned} \left(\mathcal{H}^1\left(u^{-1}(s) \cap \Sigma^k_r\right)\right)^2 &\geq K_0^{-1} \mathcal{H}^2\left(u^{-1}([s,+\infty)) \cap \Sigma^k_r\right) \\ &\geq K_0^{-1} \mathcal{H}^2\left(u^{-1}([s,+\infty)) \cap \Sigma^k_1\right) \quad . \end{aligned} \tag{2.15}$$

where K_0 is the isoperimetric constant of $(\mathbb{R}^{2p}, g)$.

Case 2:

$$\forall r \in [1,2] \qquad \partial\{x \in \Sigma^k_r\ ;\ u(x) \geq s\} \neq u^{-1}(s) \cap \Sigma^k_r \quad .$$

This means that, in such a case, $\forall r \in [1,2]$ $u^{-1}(s) \cap \partial\Sigma^k_r \neq \emptyset$. Since the distance for g in $\mathbb{R}^{2p}$ between $\partial\Sigma^k_1$ and $\partial\Sigma^k$ is larger than $K_1 > 0$, where K_1 only depends on g, we get

$$\mathcal{H}^1\left(u^{-1}(s) \cap \Sigma^k_2\right) \geq K_1 \quad . \tag{2.16}$$

Let us denote $K_2 = \int_\Sigma \Psi^*\omega = \mathcal{H}^2(\Sigma)$, where the Hausdorff distance in Σ is computed acording to the pull-back metric by Ψ of $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$. We then have in that case

$$\begin{aligned} \left(\mathcal{H}^1\left(u^{-1}(s) \cap \Sigma^k_2\right)\right)^2 &\geq K_1^2 \geq K_1^2 K_2^{-1} \mathcal{H}^2(\Sigma) \\ &\geq K_1^2 K_2^{-1} \mathcal{H}^2\left(u^{-1}([s,+\infty)) \cap \Sigma^k_1\right) \quad . \end{aligned} \tag{2.17}$$

Combining (2.14), (2.15) and (2.17), we obtain the existence of K depending only of $d_1(g)$, $d_2(g)$ and $\int_\Sigma \Psi^*\omega$ such that

$$\int_{\Sigma^{k,+}} |\nabla u| \geq K^{-1} \int_0^{+\infty} ds \left[\mathcal{H}^2\left(u^{-1}([s,+\infty)) \cap \Sigma^k_1\right)\right]^{\frac{1}{2}} \tag{2.18}$$

Observe that the right-hand-side of this last inequality is a multiple of the Lorentz $L^{2,1}-$norm of u in $\Sigma_1^{k,+}$. We claim that the L^2-weak norm of $u^+ = \max\{u, 0\}$, $L^{2,\infty}$ on Σ_1^k can be bounded by $\|u\|_{L^2(\Sigma_1^{k,+})}$

$$\|u\|_{L^{2,\infty}(\Sigma_1^{k,+})} = \sup_{s\geq 0} s\left[\mathcal{H}^2\left(u^{-1}([s,+\infty))\cap\Sigma_1^k\right)\right]^{\frac{1}{2}} \leq K_3\|u\|_{L^2(\Sigma_1^{k,+})} \tag{2.19}$$

where K_3 only depends on $d_1(g)$, $d_2(g)$ and $\int_\Sigma \Psi^*\omega$. Indeed, we consider in Σ^k the pseudo-distance d_g which is given by the distance in $(B_2^2(0)\times B_2^{2p-2}(0), g)$ – Since Σ^k is not necessarily embedded, it may happens that $d_g(x,y) = 0$ and $x \neq y$. For this pseudo-distance in Σ^k, we consider the balls $B_r^{d_g}(x) := \Psi^{-1}(B_r^{2p}(x)) \cap \Sigma^k$. Since the current $\Psi_*[\Sigma_1^k]$ is area minimizing in $(B_2^2(0)\times B_2^{2p-2}(0), g)$ – it is calibrated by ω -, using the monotonicity formula, we obtain that for every $x \in \Sigma_1^k$ and $r < 1/2$

$$\pi r^2 \leq \mathcal{H}^2(B_r^{d_g}(x)) \leq r^2\int_\Sigma \Psi^*\omega = r^2 K_2$$

Therefore these balls satisfy the doubling property

$$4\mathcal{H}^2(B_r^{d_g}(x))\pi^{-1}K_2 \geq \mathcal{H}^2(B_{2r}^{d_g}(x))$$

We then adapt to our case the proof of the covering lemma page 9 of [St] for m being the 2 Hausdorff measure restricted to Σ_2^k and the balls being balls for the pseudo-distance d_g to get the corresponding statement to that lemma. We can now obtain (2.19) by following the first part of the proof of Theorem 1 page 5 of [St], taking for the covering of pseudo-balls $B_j^{d_g}$ given by the covering lemma but considering this time the metric $\Psi^* g$ on Σ^k. From (2.19) we deduce

$$\begin{aligned}&\|u\|_{L^2(\Sigma_1^{k,+})}\int_0^{+\infty} ds\left[\mathcal{H}^2\left(u^{-1}([s,+\infty))\cap\Sigma_1^k\right)\right]^{\frac{1}{2}} \geq \\ &K_3^{-1}\int_0^{+\infty} s\, ds\left[\mathcal{H}^2\left(u^{-1}([s,+\infty))\cap\Sigma_1^k\right)\right] = \|u\|^2_{L^2(\Sigma_1^{k,+})}\quad .\end{aligned} \tag{2.20}$$

Combining now (2.18) and (2.20) we obtain the desired inequality (2.13) for $\Sigma^k_.$ instead of $\Sigma_. = \cup_k \Sigma^k_.$. Observing that the number of components Σ^k having some non empty intersection with Σ_1 is bounded by $\int_\Sigma \Psi^*\omega$ times a constant depending only of $d_1(g)$, $d_2(g)$ (this is a consequence of the monotonicity formula coming from fact that Σ^k are area minimizing), then we get (2.13) for $\Sigma_.$ this time and Lemma 2.1 is proved. □

Proof of Theorem 1.1. Using local conformal coordinates ξ_1 ξ_2 in Σ, we have for all $k = 1\cdots 2p$

$$\frac{\partial\Psi^k}{\partial\xi_1} = -\sum_{l=1}^{2p} J_l^k(\Psi)\frac{\partial\Psi^l}{\partial\xi_2} \qquad\text{and}\qquad \frac{\partial\Psi^k}{\partial\xi_2} = \sum_{l=1}^{2p} J_l^k(\Psi)\frac{\partial\Psi^l}{\partial\xi_1}$$

Taking respectively the ξ_1 derivative and the ξ_2 derivative of these two equations we obtain

$$\forall k = 1 \cdots 2p \quad * (\partial\overline{\partial}\Psi^k) = \Delta_\Sigma \Psi^k = * \left(\sum_{l=1}^{2p} d(J_l^k(\Psi)) \wedge d\Psi^l \right) \quad . \tag{2.21}$$

Since $\Psi + \eta$ is J_0-holomorphic, using the canonical complex coordinates in $\mathbb{R}^{2p}$, we have $\overline{\partial}(\Psi + \eta) = 0$ from which we deduce $\Delta_\Sigma(\Psi + \eta) = 0$ and therefore this yields

$$\forall k = 1 \cdots 2p \qquad \Delta_\Sigma \eta^k = - * \left(\sum_{l=1}^{2p} d(J_l^k(\Psi)) \wedge d\Psi^l \right) \quad . \tag{2.22}$$

Since Ψ is an isometry for the induced metric, we then deduce from (2.22) that

$$\forall k = 1 \cdots 2p \quad \|\Delta_\Sigma \eta^k\|_{L^\infty(\Sigma)} \leq 4p \, \|\nabla J\|_\infty \int_\Sigma \Psi^* \omega \quad . \tag{2.23}$$

Let $\chi(t)$ be a smooth cut-off function equal to 1 in $[0, 1]$ and equal to zero for $t \geq 2$ with $\|\chi^l\|_\infty \leq K_l$. We define in $B_2^2(0) \times B_2^{2p-2}(0)$ the cut-off function – that we also denote χ – $\chi(x) := \chi(x_1^2 + x_2^2)$. For any function ϕ in $B_2^2(0) \times B_2^{2p-2}(0)$ we denote by $\nabla^C \phi$ the tangent vector field to $\Psi(\Sigma)$ obtained by taking the orthogonal projection of the gradient of ϕ for the metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$. For any vector field Y in $B_2^2(0) \times B_2^{2p-2}(0)$ we denote by $div^\Sigma Y$ the divergence along $\Psi(\Sigma)$ of that vector-field (taking normal coordinates $(y_1, \cdots, y_{2p})$ for g in a neighborhood of $x_0 \in supp\, \Psi_*[\Sigma]$ we have $div^\Sigma Y(x_0) = \sum_{l=1}^{2p} \nabla^\Sigma Y_l \cdot \frac{\partial}{\partial y_l}(x_0)$. It is a classical fact that, for a vector field Y normal to $\Psi(\Sigma)$ one has $div^\Sigma Y = H \cdot Y$ where H is the mean curvature vector of $\Psi(\Sigma)$ which is zero in our case. Therefore we have in particular $\Delta_\Sigma \chi := div^\Sigma \nabla^\Sigma \chi(x_0) = div^\Sigma \nabla \chi(x_0) = \sum_{l=1}^{2p} \nabla^\Sigma (\frac{\partial \chi}{\partial y_l}) \cdot \frac{\partial}{\partial y_l}(x_0)$, still using the normal coordinates in $(\mathbb{R}^{2p}, g)$ about x_0. Since $|\nabla \frac{\partial \chi}{\partial y_l}|(x) \leq \|\chi\|_{C^2}$, we then deduce that there exists a constant K independent of the variables of our problem such that

$$|\Delta_\Sigma \chi|_\infty \leq K \quad . \tag{2.24}$$

Finally we have, using (2.23) and (2.24)

$$|\Delta_\Sigma(\chi\eta^k)| \leq K \mathbf{1}_{\Sigma_{\sqrt{2}}} \left[|\eta^k| + |\nabla \eta^k| + \|\nabla J\|_\infty \int_\Sigma \Psi^* \omega \right] \tag{2.25}$$

where K only depends on p and where $\mathbf{1}_{\Sigma_{\sqrt{2}}}$ is the characteristic function equal to 1 on $\Sigma_{\sqrt{2}}$ and 0 outside. Using (2.25) and Lemma 2.1 (for $\Sigma_{\sqrt{2}}$ instead of Σ_1) – having chosed η^k with average 0 on each connected component of Σ – we finally have

$$\int_{\Sigma_{\sqrt{2}}} |\Delta_\Sigma(\chi\eta^k)|^2 \leq K \, \|\nabla J\|_\infty^2. \tag{2.26}$$

where K only depend on $d_1(g)$, $d_2(g)$ and $\int_\Sigma \Psi^*\omega$.

We denote by G_a the Green Function of Δ_Σ on $\Sigma_{\sqrt{2}}$ for the zero boundary condition on $\partial\Sigma_{\sqrt{2}}$ (recall that each connected component of $\Sigma_{\sqrt{2}}$ has a boundary since it is an area minimizing surface and therefore posses a Green function – see [FK]). Precisely G_a solves

$$\begin{cases} \Delta_\Sigma G_a = \delta_a & \text{in } \Sigma_{\sqrt{2}} \\ \\ G_a = 0 & \text{on } \partial\Sigma_{\sqrt{2}} \end{cases} \quad , \tag{2.27}$$

where δ_a denotes the Dirac mass at a. From the strong maximum principle $G_a > 0$ on the connected component of $\Sigma_{\sqrt{2}}$ containing a whereas $G_a \equiv 0$ elsewhere. Since $supp(\chi\eta^k) \subset \Sigma_{\sqrt{2}}$, we have

$$\forall a \in \Sigma_{\sqrt{2}} \qquad \chi\eta^k(a) = \int_{\Sigma_{\sqrt{2}}} G_a(x)\, \Delta_\Sigma(\chi\eta^k)(x)\, dx \quad . \tag{2.28}$$

For $0 \le s_1 \le s_2 \le +\infty$, we denote

$$\mathcal{G}_a^{s_1,s_2} := \{x \in \Sigma_{\sqrt{2}}\ ;\ s_1 \le G_a(x) \le s_2\} \quad .$$

Using the coarea formula (see [Fe]), we have

$$\int_{\mathcal{G}_a^{s_1,s_2}} |\nabla G_a|^2 = \int_{s_1}^{s_2} ds \int_{G_a^{-1}(s)} |\nabla G_a|(x)\, d\mathcal{H}^1 \quad . \tag{2.29}$$

Using the fact that for regular values s of G_a, for $x \in G_a^{-1}(s)$ $|\nabla G_a|(x) = -\frac{\partial G_a}{\partial\nu}(x)$ where ν is the outward unit normal to $\mathcal{G}_a^{s,\infty}$ and the fact that

$$\int_{G_a^{-1}(s)} -\frac{\partial G_a}{\partial\nu}(x)\, d\mathcal{H}^1 = \int_{\mathcal{G}_a^{s,\infty}} \Delta_\Sigma G_a = 1 \quad , \tag{2.30}$$

we finally obtain

$$\int_{\mathcal{G}_a^{s_1,s_2}} |\nabla G_a|^2 = s_2 - s_1 \quad . \tag{2.31}$$

Let $\delta > 0$, we deduce from (2.31)

$$\begin{aligned} \frac{1}{\delta} &= \int_{s=1}^{+\infty} s^{-2-\delta} \int_{\mathcal{G}_a^{0,s}} |\nabla G_a|^2 \\ &= \frac{1}{1+\delta} + \frac{1}{1+\delta} \int_1^{+\infty} s^{-1-\delta} \int_{G_a^{-1}(s)} |\nabla G_a| \\ &= \frac{1}{1+\delta} + \frac{1}{1+\delta} \frac{1}{\left(\frac{1}{2}-\frac{\delta}{2}\right)^2} \int_{\mathcal{G}_a^{1,\infty}} |\nabla G_a^{\frac{1}{2}-\frac{\delta}{2}}|^2 \end{aligned} \tag{2.32}$$

Thus, taking $\delta = \frac{1}{2}$, we have

$$\int_{\mathcal{G}_a^{1,\infty}} |\nabla G_a^{\frac{1}{4}}|^2 \leq 48 \quad . \tag{2.33}$$

Let f be a smooth function equal to t on $[0, \frac{1}{2}]$ and equal to $t^{\frac{1}{4}}$ on $[1, +\infty]$. Since $f(G_a) = 0$ on $\partial\Sigma_{\sqrt{2}}$, we can use one by one the arguments of Lemma 2.1 to obtain

$$\begin{aligned} &\left[\int_{\Sigma_{\sqrt{2}}} |f(G_a)|^8\right]^{\frac{3}{8}} \left[\int_{\Sigma_{\sqrt{2}}} |\nabla f(G_a)|^{\frac{8}{5}}\right]^{\frac{5}{8}} \geq \\ &\int_{\Sigma_{\sqrt{2}}} |f(G_a)|^3 |\nabla f(G_a)| \geq \frac{1}{4}\int_{\Sigma_{\sqrt{2}}} |\nabla f^4(G_a)| \geq \\ &\int_0^{+\infty} \mathcal{H}^1(f^4(G_a)^{-1}(s))\, ds \geq K \int_0^{+\infty} ds \left[|x\ ;\ f^4(G_a)(x) \geq s|\right]^{\frac{1}{2}} \\ &= K\|f(G_a)^4\|_{L^{2,1}(\Sigma_{\sqrt{2}})} \geq K \left[\int_{\Sigma_{\sqrt{2}}} f(G_a)^8\right]^{\frac{1}{2}} \end{aligned} \tag{2.34}$$

Combining (2.33 and (2.34) we obtain that

$$\int_{\Sigma_{\mathcal{G}_a^{1,+\infty}}} |G_a|^2 \leq K \quad . \tag{2.35}$$

where K has the usual dependence in $d_1(g)$, $d_2(g)$ and $\int_\Sigma \Psi^*\omega$. Using the coarea formula again (2.28) becomes

$$\chi\eta^k(a) = \int_0^{+\infty} s\, ds \int_{G_a^{-1}(s)} \frac{\Delta_\Sigma(\chi\eta^k)}{|\nabla G_a|} \quad . \tag{2.36}$$

Since G_a is harmonic aside from a the zeros of ∇G_a are isolated points and then for every $s \in \mathbb{R}_*^+$ $G_a^{-1}(s)$ is a union of finitely many smooth closed curves aside eventually from isolated points. Therefore, since also $\chi\eta^k$ is smooth, we have that $s \to \int_{\mathcal{G}_a^{s,\infty}} \Delta_\Sigma(\chi\eta^k)$ is continuous everywhere and smooth aside from finitely many s corresponding to the values of the finitely many critical points of G_a. Moreover, aside from these points, it's derivative is the function $s \to -\int_{G_a^{-1}(s)} \frac{\Delta_\Sigma(\chi\eta^k)}{|\nabla G_a|}$. For all these reasons we have a BV function without jump points and without Cantor parts in the derivative and the following holds in a distributional sense

$$\frac{d}{ds}\left[\int_{\mathcal{G}_a^{s,+\infty}} \Delta_\Sigma(\chi\eta^k)\right] = -\int_{G_a^{-1}(s)} \frac{\Delta_\Sigma(\chi\eta^k)}{|\nabla G_a|} \qquad \text{in } \mathcal{D}'(\mathbb{R}_+^*) \quad . \tag{2.37}$$

Using a Taylor expansion of the smooth function $\chi\eta^k$ at a it is not difficult to justify the following integration by parts

$$\int_0^{+\infty} s\, ds \int_{G_a^{-1}(s)} \frac{\Delta_\Sigma(\chi\eta^k)}{|\nabla G_a|} = \int_0^{+\infty} \int_{\mathcal{G}_a^{s,+\infty}} \Delta_\Sigma(\chi\eta^k) \quad . \tag{2.38}$$

We then have, using also (2.26)

$$\begin{aligned}
|\chi\eta^k|(a) &\leq \int_0^{+\infty} ds\,\left[\mathcal{H}^2(\mathcal{G}_a^{s,+\infty})\right]^{\frac{1}{2}}\ \left[\int_{\Sigma_{\sqrt{2}}} |\Delta_\Sigma(\chi\eta^k)|^2\right]^{\frac{1}{2}} \\
&\leq K\,\|\nabla J\|_\infty \int_0^{+\infty} ds\ \left[\mathcal{H}^2(\mathcal{G}_a^{s,+\infty})\right]^{\frac{1}{2}} \qquad (2.39)\\
&\leq K\,\|\nabla J\|_\infty \int_0^{+\infty} ds\,\mathcal{H}^1(G_a^{-1}(s)) \quad ,
\end{aligned}$$

where we have used the fact that, for $s > 0$, $G_a^{-1}(s)$ is contained in the interior of $\Sigma_{\sqrt{2}}$, is the boundary of $\mathcal{G}_a^{s,+\infty}$ and the fact that, $\Sigma_{\sqrt{2}}$ being an area minimizing surface, it inerhits the isoperimetric constant of the ambiant space $(\mathbb{R}^{2p}, g)$ depending only on $d_1(g)$ and $d_2(g)$. We have, using (2.31)

$$\begin{aligned}
\int_0^{+\infty} ds\,\mathcal{H}^1(G_a^{-1}(s)) &= \int_{\mathcal{G}_a^{0,1}} |\nabla G_a| + \int_1^{+\infty} ds\,\mathcal{H}^1(G_a^{-1}(s)) \\
&\leq K\ + \int_1^{+\infty} \left[\int_{G_a^{-1}(s)} |\nabla G_a|\right]^{\frac{1}{2}}\ \left[\int_{G_a^{-1}(s)} \frac{1}{|\nabla G_a|}\right]^{\frac{1}{2}} \\
&\leq K\ + \int_1^{+\infty} \left[\int_{G_a^{-1}(s)} \frac{\partial G_a}{\partial \nu}\right]^{\frac{1}{2}} \left[\int_{G_a^{-1}(s)} \frac{1}{|\nabla G_a|}\right]^{\frac{1}{2}} \qquad (2.40)\\
&\leq K\ + \int_1^{+\infty} \frac{1}{s} \left[\int_{G_a^{-1}(s)} \frac{s^2}{|\nabla G_a|}\right]^{\frac{1}{2}} \\
&\leq K\ + K\,\left[\int_{\mathcal{G}_a^{1,+\infty}} G_a^2\right]^{\frac{1}{2}} \leq K
\end{aligned}$$

where K is controlled by the usual quantities. Combining (2.39) and (2.40) we obtain (1.5) and Theorem 1.1 is proved. □

A Appendix

Definition A.1 *A Riemann surface Σ is said to be finite if each connected component Σ^k of Σ is an open subset of a closed Riemann surface $\tilde{\Sigma}^k$ and $\partial\Sigma^k$ is a non-empty finite union of closed regular curves embedded in $\tilde{\Sigma}^k$.*

We have the following classical proposition (see for instance [FK]).

Proposition A.1 *Let Σ be a finite Riemann surface , then Σ is hyperbolic (admits a Green function) and for every $\phi \in W^{\frac{1}{2},2}(\partial Sigma, \mathbb{C})$, there exists a unique $u \in W^{1,2}(\Sigma, \mathbb{C})$ such that*

$$\begin{cases} \Delta u = 0 & \text{in } \Sigma \\ \\ u = \phi & \text{on } \partial\Sigma \end{cases}$$

Observe that spaces and equations above are independant of the metric chosen compatible with the complex structure on Σ. We consider the following Hermitian scalar product on $L^2(\Sigma,\mathbb{C})$

$$< \omega_1,\omega_2 >:= \int_\Sigma \omega_1 \wedge *\overline{\omega}_2 \quad , \tag{A.1}$$

and the following antihermitian sesquilinear form on $L^2(\Sigma,\mathbb{C})$

$$(\omega_1,\omega_2) := \int_\Sigma \omega_1 \wedge \overline{\omega}_2 \quad . \tag{A.2}$$

(recall that if ξ are local complex coordinates in Σ $*d\xi = id\xi$ and $*d\overline{\xi} = -id\overline{\xi}$). Recall that $H^0_\pm(\Sigma)$ denote the sets of holomorphic and antiholomorphic functions in $W^{1,2}(\Sigma,\mathbb{C})$. Let $\partial H^0_+(\Sigma)$ and $\overline{\partial} H^0_-(\Sigma)$ be the sets of exact holomorphic and antiholomorphic 1-forms. Let Γ_k for $k = 1\cdots q$ be the connected components of Σ. Denote $\mathcal{H}^0(\Sigma)$ the set of harmonic functions in $W^{1,2}(\Sigma,\mathbb{C})$. Let v_i be the solution of

$$\begin{cases} \Delta v_i = 0 & \text{in } \Sigma \\ \\ v_i = \delta_{ik} & \text{on } \Gamma_k \quad \text{for } k = 1\cdots q \end{cases} \tag{A.3}$$

(δ_{ik} are the Kronecker Symbols). Finally we introduce the following notation

$$V := \text{Vect}_{\mathbb{C}}\,\{v_1,\cdots,v_q\} \quad . \tag{A.4}$$

The following proposition holds.

Proposition A.2 *Let Σ be a finite Riemann surface. Then the following orthogonal decomposition of $d\mathcal{H}^0(\Sigma)$ for $< \cdot,\cdot >$ holds*

$$d\mathcal{H}^0(\Sigma) = \partial H^0_+(\Sigma) \oplus \overline{\partial} H^0_-(\Sigma) \oplus dV \quad . \tag{A.5}$$

Remark A.1 Observe that the above decomposition (A.5) corresponds to a Sylvester decomposition of $d\mathcal{H}^0(\Sigma)$ for the Hermitian form $i^{-1}(\cdot,\cdot)$. Precisely on $\partial H^0_+(\Sigma)$ the sesquilinear form $i^{-1}(\cdot,\cdot)$ is definite positive, on $\partial H^0_-(\Sigma)$, $i^{-1}(\cdot,\cdot)$ is definite negative and, on dV, $(\cdot,\cdot)$ is identically zero.

Proof of Proposition A.2. First of all we construct a particular basis of the de Rham Group $H^1(\Sigma,\mathbb{R})$ by taking Poincaré-Lefschetz duals of some chosed basis of $H_1(\Sigma,\partial\Sigma)$. Since $\partial\Sigma$ is non-empty and has a finite topology it is homeomorphic (see [Ma]) to the disk D^2 to which $q-1$ disjoint subdisks, $D_1,\cdots,D_{q-1}$, that we may assume to be included in $D^2_- := D^2 \cap \{(x,y)\ ;\ x \le 0\}$, have been removed, to which $2p$ other disjoint subdisks $d_1, d_2 \cdots d_{2p}$, that we may assume to be included in $D^2_+ := D^2 \cap \{(x,y)\ ;\ x \ge 0\}$, have also been removed and to which, finally, p Handels $h_l = S^1 \times [0,1]$ $l = 1\cdots p$, have been glued by identifying the two connected components of ∂h_i with respectively d_{2i-1} and d_{2i}. We now chose the following basis for $H_1(\Sigma,\partial\Sigma)$. First for each $l = 1\cdots p$ we chose γ_l to be ∂d_{2l-1} and $\delta_l \subset D^2_+$ to be a closed curve in $(D^2_+ \setminus \cup_l d_l) \cup \overline{h}_l$

made of the meridian $\{(0,1)\} \times [0,1]$ and a curve in $(D^2_+ \setminus \cup_l d_l)$ connecting the two ends of this meridian. We can assume that the δ_l do not intersect each-other. We then complete the famillies $(\gamma_l)_{l=1\cdots p}$ and $(\delta_l)_{l=1\cdots p}$ by a collection of $q-1$ curves η_l in D^2_-, each curve η_l connecting the segment ∂D_l with the boundary ∂D^2. $(\gamma_l)_{l=1\cdots p}$, $(\delta_l)_{l=1\cdots p}$ and $(\eta_l)_{l=1\cdots q-1}$ form a basis of $H_1(\Sigma, \partial\Sigma)$. If we add this time the familly of circles $(\partial D_l)_{l=1\cdots q-1}$ to $(\gamma_l)_{l=1\cdots p}$, $(\delta_l)_{l=1\cdots p}$ we get a basis of $H_1(\Sigma)$. Consider one curve c taken among the two first types $c = \gamma_l$ or $c = \delta_l$. Since the intersection number of c with the η_l is zero, by the standard construction method (see [BT]), one gets the existence of a representant α_c of the Lefschetz-Poincaré dual of c in $H^1(\Sigma, \partial\Sigma)$ (see the relative to the boundary de Rham cohomology pages 78-79 of [BT], corresponding here also to the compactly supported de Rham cohomology) which is compactly supported in Σ :

$$\exists \alpha_c \in C^\infty_0(\wedge^1\Sigma) \cap \mathrm{Ker}d \qquad \text{s. t. } \forall \phi \in C^\infty(\wedge^1\Sigma) \int_\Sigma \alpha_c \wedge \phi = \int_c \phi \quad . \tag{A.6}$$

Among the representants of the class given by the Lefschetz-Poincaré we choose the Coulomb Gauge minimizing the following problem

$$\min \left\{ \int_\Sigma |d^*\alpha|^2 \quad \begin{array}{c} \alpha \in C^\infty(\Sigma)\ \mathrm{Ker}d \quad \iota^*_{\partial\Sigma} * \alpha = 0 \\ \\ \forall \phi \in C^\infty(\wedge^1\Sigma) \int_\Sigma \alpha_c \wedge \phi = \int_c \phi \quad . \end{array} \right\} \tag{A.7}$$

where $\iota_{\partial\Sigma}$ is the canonical embedding of $\partial\Sigma$ in $\overline{\Sigma}$. The minimizer α^0_c solves then

$$\begin{cases} d\alpha^0_c = 0 \qquad \text{in } \Sigma \\ \\ d^*\alpha^0_c \qquad \text{in } \Sigma \\ \\ \iota^*_{\partial\Sigma} * \alpha^0_c \\ \\ \forall \phi \in C^\infty(\wedge^1\Sigma) \cap \ \mathrm{Ker}d \qquad \displaystyle\int_\Sigma \alpha^0_c \wedge \phi = \int_c \phi \end{cases} \tag{A.8}$$

(the uniqueness of α^0_c comes from the following fact : if β solves $d\beta = 0$ in Σ, $d^*\beta = 0$ in Σ, $\iota^*_{\partial\Sigma}\beta = 0$ and $\int_\Sigma \beta \wedge \phi = 0$ for any $\phi \in C^\infty(\wedge^1\Sigma) \cap$ Ker$d$ then $\beta = dh$ where h solves $\Delta h = 0$ in Σ and $\frac{\partial h}{\partial \nu} = 0$ which clearly implies that h is constant and therefore that $\beta = 0$). Take now $f \in \mathcal{H}^0(\Sigma)$ and assume that

$$\forall k = 1, \cdots, q \qquad \int_{\Gamma_k} *df = 0 \quad . \tag{A.9}$$

Then we claim that $*df$ is exact in Σ.. For any $c \in \{\gamma_l\} \cup \{\delta_l\}$ we have

$$\int_\Sigma *df \wedge \alpha^0_c = -\int_\Sigma df \wedge *\alpha^0_c = \int_\Sigma f \wedge d * \alpha^0_c - \int_{\partial\Sigma} f * \alpha^0_c = 0 \quad . \tag{A.10}$$

Combining (A.9) and (A.10) we get that $*df$ is null-cohomologic and is therefore exact which proves the claim. Let then h such that $dh = *f$, we have

$$f = \frac{1}{2}(f + ih) + \frac{1}{2}(f - ih) \in H^0_+(\Sigma) \oplus H^0_-(\Sigma).$$

Thus, the codimension of $dH^0_+(\Sigma) \oplus dH^0_-(\Sigma)$ in $\mathcal{H}^0(\Sigma)$ is at most $q-1$ (because we have to substract the relation $0 = \int_\Sigma d * df = \sum_{k=1}^q \int_{\Gamma_k} *df$). Let v_j one of the function introduced in (A.3) and let $f \in H^0_+(\Sigma)$, then we have

$$\begin{aligned} < dv_j, df >= \int_\Sigma dv_j \wedge *d\overline{f} = \int_\Sigma dv_j \wedge *d\Re f - i \int_\Sigma dv_j \wedge d\Im f \\ = \int_\Sigma dv_j \wedge d\Im f + i \int_\Sigma dv_j \wedge d\Re f = \int_{\Gamma_j} d\Im f + i \int_{\Gamma_j} d\Re f = 0 \quad . \end{aligned} \tag{A.11}$$

Thus, we have that dV is perpendicular to $dH^0_+(\Sigma)$ and a similar argument shows that it is also perpendicular to $dH^0_-(\Sigma)$. Thus $dV \perp (dH^0_+ \oplus dH^0_-)$. It is also straightforward to check that the dimension of dV is $q-1$. Therefore $d\mathcal{H}^0 = dV \oplus dH^0_+ \oplus dH^0_-$ and Proposition A.2 is proved. □

Proposition A.3 *Let Σ be a finite Riemann surface whose connected components are denoted by Σ_k, $k = 1 \cdots n$. Let $\psi \in W^{1,2}(\Sigma, \mathbb{C})$, there exists a unique complex valued function $\eta \in W^{1,2}(\Omega, \mathbb{C})$*

$$\begin{cases} \overline{\partial}\eta = \overline{\partial}\psi \qquad \textit{in } \Sigma \\ \tilde{\eta} \in H^0_-(\Sigma) \oplus V \quad , \\ \forall \Sigma_k \quad \textit{connected compo. of } \Sigma \ , \quad \int_{\partial\Sigma_k} \eta = 0 \end{cases} \tag{A.12}$$

where $\tilde{\eta}$ is the harmonic extension of the restriction of η to $\partial\Sigma$ inside Σ. Moreover we have

$$\int_\Sigma |\nabla\eta|^2 \le 2 \int_\Sigma |\overline{\partial}\psi|^2 \quad . \tag{A.13}$$

Proof of Proposition A.3. Let $\tilde{\psi}$ be the harmonic extension of ψ restricted to $\partial\Sigma$ inside Σ. From Proposition A.2, $d\tilde{\psi}$ admits a unique decomposition $d\tilde{\psi} = d\psi_+ + d\psi_- + d\psi_V$ where $d\psi_\pm \in H^0_\pm(\Sigma)$ and $d\psi_V \in dV$. We chose η such that $d\eta = d\psi - d\psi_+$ with constant adjusted in such a way that $\forall k = 1 \cdots n \quad \int_{\partial\Sigma_k} \eta = 0$ where $\tilde{\eta}$ is the harmonic extension of η restricted to $\partial\Sigma$ inside Σ. η clearly solves (A.12). The uniqueness is given by the fact that a solution to

$$\begin{cases} \overline{\partial}\delta = 0 \qquad \text{in } \Sigma \\ \tilde{\delta} \in H^0_-(\Sigma) \oplus V \end{cases} \tag{A.14}$$

is constant on each connected component of Σ. This is a direct consequence of Proposition A.2 since $\overline{\partial}\delta = 0$ is equivalent to $\tilde{\eta} = \eta \in H^0_+(\Sigma)$ and $dH^0_+(\Sigma) \cap dH^0_-(\Sigma) \oplus dV = \{0\}$.

Integration by parts gives

$$\begin{aligned}
\int_\Sigma \left|\frac{\partial\eta}{\partial z}\right|^2 - \left|\frac{\partial\eta}{\partial\overline{z}}\right|^2 \; dz \wedge d\overline{z} &= \int_\Sigma d\eta \wedge d\overline{\eta} \\
&= \int_{\partial\Sigma} \eta \; d\overline{\eta} = \int_{\partial\Sigma} \eta_- + \eta_V \; d(\overline{\eta}_- + \overline{\eta}_V) \\
&= \int_{\partial\Sigma} \eta_- \; d\overline{\eta}_- = \int_\Sigma d\eta_- \wedge d\overline{\eta}_- \\
&= -\int_\Sigma \left|\frac{\partial\eta_-}{\partial\overline{z}}\right|^2 \; dz \wedge d\overline{z} \leq 0
\end{aligned} \tag{A.15}$$

Therefore we have

$$\int_\Sigma \left|\frac{\partial\eta}{\partial z}\right|^2 dz \wedge d\overline{z} \leq \int_\Sigma \left|\frac{\partial\eta}{\partial\overline{z}}\right|^2 \; dz \wedge d\overline{z} = \int_\Sigma \left|\frac{\partial\psi}{\partial\overline{z}}\right|^2 \; dz \wedge d\overline{z}$$

and Proposition A.3 follows. □

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