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Mathematical and numerical analysis of a simplified time-dependent viscoelastic flow

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Abstract A time-dependent model corresponding to an Oldroyd-B viscoelastic fluid is considered, the convective terms being disregarded. Global existence in time is proved in Banach spaces provided the data are small enough, using the implicit function theorem and a maximum regularity property for a three fields Stokes problem. A finite element discretization in space is then proposed. Existence of the numerical solution is proved for small data, so as a priori error estimates, using again an implicit function theorem.

Keywords Viscoelastic fluids · Galerkin Least Square finite elements · Semigroup · Perturbed abstract Cauchy problem

Mathematics Subject Classification (2000) 76M10 · 65N15 · 65N30 · 76A10 · 35L05

1 Introduction

Numerical modeling of viscoelastic flows is of great importance for complex engineering applications involving foodstuff, blood, paints or adhesives. When considering viscoelastic flows, the velocity, pressure and stress must satisfy the mass and momen-

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tum equation, supplemented with a constitutive equation involving the velocity and stress. The simplest model is the so-called Oldroyd-B constitutive relation which can be derived from the kinetic theory of polymer dilute solutions, see for instance [6,40]. The unknowns of the Oldroyd-B model are the velocity u, the pressure p, the extra-stress σ (the non Newtonian part of the stress due to polymer chains for instance) which must satisfy :

$$\rho \frac{\partial u}{\partial t} + \rho(u \cdot \nabla)u - 2\eta_s \text{ div } \epsilon(u) + \nabla p - \text{div } \sigma = f,$$
$$\text{div } u = 0,$$
$$\sigma + \lambda \left(\frac{\partial \sigma}{\partial t} + (u \cdot \nabla)\sigma - (\nabla u)\sigma - \sigma(\nabla u)^T \right) - 2\eta_p \epsilon(u) = 0.$$

Here ρ is the density, f a force term, η_s and η_p are the solvent and polymer viscosities, λ the relaxation time, $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ the strain rate tensor, $(\nabla u)\sigma$ denotes the matrix-matrix product between ∇u and σ.

Obviously, when $\lambda = 0$, the Oldroyd-B model reduces to the incompressible Navier–Stokes equations, for which we refer to [33].

Although the Oldroyd-B model is too simple to describe complex experiments such as shear thinning for instance, it already contains some mathematical and numerical difficulties. Indeed, when solving numerically Oldroyd-B fluids, one is faced to the "high Deborah (or Weissenberg) number problem", that is to say a breakdown in convergence of algorithms when the Deborah number (the relaxation time λ times a characteristic velocity divided by a characteristic length) increases. The sources of this problem are due to :

(i)   the presence of the quadratic term $(\nabla u)\sigma + \sigma(\nabla u)^T$ which prevents a priori estimates to be obtained and therefore existence to be proved for any data;
(ii) the presence of a convective term $(u \cdot \nabla)\sigma$ which requires the use of numerical schemes suited to transport dominated problems;
(iii) the case $\eta_s = 0$ which requires either a compatibility condition between the finite element spaces for u and σ or the use of adequate stabilization procedures such as EVSS for instance.

For a description of numerical procedures used for solving viscoelastic flows in the engineering community, we refer for instance to [3,41].

Concerning mathematical analysis, the existence of slow steady viscoelastic flow has been proved in [45]. For the time-dependent case, existence of solutions locally in time and, for small data, globally in time has been proved in [34] in Hilbert spaces. Extensions to Banach spaces and a review can be found in [25]. Finally, existence for any data has been proved in [36] for a corotational Oldroyd model only.

From the numerical analysis viewpoint, convergence of finite element methods for the linear three fields Stokes problem have been studied for instance in [11,26,27,46]. Convergence of continuous and discontinuous finite element methods for steady state viscoelastic fluids have been presented in [4,24,39,47], provided the solution of the continuous problem is smooth and small enough. Extension to time-dependent problems have been proposed in [5,22,23,38].

In this paper, the mathematical and numerical analysis is proposed for a simplified time-dependent Oldroyd-B problem. More precisely, we focus on item i) above, thus remove the convective terms and assume $\eta_s > 0$. The reason for considering the time-dependent Oldroyd-B problem without convection is motivated by the fact that this simplified problem corresponds to the correction step in the splitting algorithm described in [10] for solving 3D viscoelastic flows with complex free surfaces. The consequence when removing convective terms is that the implicit function theorem can be used to prove an existence result, whenever the data are small enough, in accordance with the results of [25]. Our existence result is obtained using the semi-group framework and a maximum regularity property for the three fields Stokes problem. Moreover, the regularity of the solution is sufficient to prove convergence of a finite element discretization in space. Finally, it should be noted that the analysis remains valid for more realistic fluids such as Giesekus or Phan-Thien-Tanner.

Throughout the paper, implicit function theorems are used to prove mathematical existence and also numerical convergence of the finite element method, thus the techniques presented in [43] are extended to the time-dependent framework. Finally, we would like to mention that the technics presented here has also been successfully extended to the simplest kinetic model, namely the so-called Hookean dumbbells model for dilute polymer liquids [7–9]. In the latter, the constitutive equation is replaced by a stochastic differential equation for the dumbbells elongation. We refer to [35] for a review concerning numerical methods for kinetic theories of liquid polymers.

The outline of the paper is as follows. The simplified Oldroyd-B problem and its finite element approximation in space are introduced in the next section. Then, in Sect. 3, mathematical existence of a solution is proved in Banach spaces. Finally, existence and a priori error estimates are proposed for the finite element approximation in Sect. 4.

2 The simplified Oldroyd-B problem and its finite element approximation in space

Let Ω be a bounded, connected open set of $\mathbb{R}^d$, $d \geq 2$ with boundary $\partial \Omega$ of class $\mathcal{C}^2$, and let $T > 0$. We consider the following problem. Given initial conditions $u_0 : \Omega \to \mathbb{R}^d$, $\sigma_0 : \Omega \to \mathbb{R}^{d \times d}_{sym}$, a force term f, constant solvent and polymer viscosities $\eta_s > 0$, $\eta_p > 0$, a constant relaxation time $\lambda > 0$, find the velocity $u : \Omega \times (0, T) \to \mathbb{R}^d$, pressure $p : \Omega \times (0, T) \to \mathbb{R}$ and extra-stress $\sigma : \Omega \times (0, T) \to \mathbb{R}^{d \times d}_{sym}$ such that

$$\rho \frac{\partial u}{\partial t} - 2\eta_s \operatorname{div} \epsilon(u) + \nabla p - \operatorname{div} \sigma = f \quad \text{in } \Omega \times (0, T), \tag{1}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega \times (0, T), \tag{2}$$

$$\frac{1}{2\eta_p}\sigma + \frac{\lambda}{2\eta_p}\left(\frac{\partial \sigma}{\partial t} - (\nabla u)\sigma - \sigma(\nabla u)^T\right)$$
$$- \epsilon(u) = 0 \quad \text{in } \Omega \times (0, T), \tag{3}$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega, \tag{4}$$

$$\sigma(\cdot, 0) = \sigma_0 \quad \text{in } \Omega, \tag{5}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T). \tag{6}$$

For simplicity, the notation will be abridged as follow whenever there is no possible confusion. For $1 < r < +\infty$, the space L^r denotes $L^r(\Omega; \mathbb{R})$, $L^r(\Omega; \mathbb{R}^d)$ or $L^r(\Omega; \mathbb{R}_{sym}^{d \times d})$. Also, for $1 < q < +\infty$, $1 < r < +\infty$, $L^q(L^r)$ stands for $L^q(0, T; L^r(\Omega; \mathbb{R}))$, $L^q(0, T; L^r(\Omega; \mathbb{R}^d))$ or $L^q(0, T; L^r(D; \mathbb{R}_{sym}^{d \times d}))$ and, for $0 < \mu < 1$, $1 < r < +\infty$, $\mathcal{C}^\mu(L^r)$ stands for the classical Hölder space $\mathcal{C}^\mu([0, T]; L^r(D; \mathbb{R}))$. Similarly, $h^\mu(L^r)$ stands for the little Hölder space $h^\mu([0, T]; L^r(D; \mathbb{R}))$. Here, the little Hölder space $h^\mu([0, T]; B)$ is defined for all Banach space B and for all $0 < \mu < 1$ by

$$h^\mu([0, T]; B) := \left\{ f \in \mathcal{C}^\mu([0, T]; B); \right.$$

$$\left. \lim_{\delta \to 0} \sup_{t, s \in [0, T], |t-s| < \delta} \frac{\|f(t) - f(s)\|_B}{|t - s|^\mu} = 0 \right\}.$$

Assuming B is a separable Banach space, the space $h^k([0, T]; B)$ provided with the norm of $\mathcal{C}^\mu(0, T; B)$ is a separable Banach space and for all $0 < \mu < \mu' < 1$ we have $\mathcal{C}^{\mu'} \subset h^\mu$, see for instance [37]. We also denote by $h_0^\mu([0, T]; B)$ the restriction of functions of $h^\mu([0, T]; B)$ vanishing at the origin. The above notations apply for higher order spaces such as $W^{1,q}(W^{1,r})$ and $h^{1+\mu}(W^{1,r})$.

The implicit function theorem will be used to prove that (1)–(6) admits a unique solution

$$u \in W^{1,q}(L^r) \cap L^q(W^{2,r} \cap H_0^1), \quad p \in L^q(W^{1,r} \cap L_0^2),$$
$$\sigma \in W^{1,q}(W^{1,r}), \tag{7}$$

or

$$u \in h^{1+\mu}(L^r) \cap h^\mu(W^{2,r} \cap H_0^1), \quad p \in h^\mu(W^{1,r} \cap L_0^2),$$
$$\sigma \in h^{1+\mu}(W^{1,r}), \tag{8}$$

with $1 < q < \infty$, $d < r < \infty$ and $0 < \mu < 1$ for any data f, u_0, σ_0 small enough in appropriate spaces. Moreover, assuming more regularity on the data, we will also prove that

$$u \in W^{2,q}(L^r) \cap W^{1,q}(W^{2,r} \cap H_0^1), \quad p \in W^{1,q}(W^{1,r} \cap L_0^2),$$
$$\sigma \in W^{2,q}(W^{1,r}), \tag{9}$$

or

$$u \in h^{2+\mu}(L^r) \cap h^{1+\mu}(W^{2,r} \cap H_0^1), \quad p \in h^{1+\mu}(W^{1,r} \cap L_0^2),$$
$$\sigma \in h^{2+\mu}(W^{1,r}), \tag{10}$$

for any data f, u_0, σ_0 again small enough in appropriate spaces. The regularity (7) is sufficient to prove convergence of a finite element discretization in space, see Sect. 4. On the other hand, the regularity (9) will be needed to prove convergence of a space and time discretization, this being the subject of a forthcoming paper. Finally, the regularities (8) and (10) will be used when considering the Hookean dumbbells model, which is formally equivalent to the Oldroyd-B model, see for instance [40]. Analysis and numerical analysis of this stochastic model will be considered elsewhere [7].

Alternatively, local existence in time is proved for arbitrarily large data, using an abstract theorem for fully nonlinear parabolic equations, namely Theorem 8.1.1 of [37]. More precisely, we will prove that there exists $0 < T_* \leq T$ such that (1)–(6) admits a solution

$$
\begin{aligned}
&u \in \mathcal{C}^1([0, T_*], L^r) \cap \mathcal{C}^0([0, T_*], W^{2,r} \cap H_0^1), \\
&\sigma \in \mathcal{C}^1([0, T_*], W^{1,r}), \\
&p \in \mathcal{C}^0([0, T_*]; W^{1,r} \cap L_0^2),
\end{aligned}
\tag{11}
$$

with $d < r < \infty$ and for any data f, u_0 and σ_0 in appropriate spaces.

The finite element approximation in space is now introduced. For any $h > 0$, let $\mathcal{T}_h$ be a decomposition of Ω into triangles K with diameter h_K less than h, regular in the sense of [17]. We consider V_h, M_h and Q_h the finite element spaces for the velocity, extra-stress and pressure, respectively defined by :

$$
\begin{aligned}
V_h &:= \{v_h \in \mathcal{C}^0(\overline{\Omega}; \mathbb{R}^d); \, v_h \mid_K \in (\mathbb{P}_1)^d, \forall K \in \mathcal{T}_h\} \cap H_0^1(\Omega; \mathbb{R}^d), \\
M_h &:= \{\tau_h \in \mathcal{C}^0(\overline{\Omega}; \mathbb{R}^{d \times d}_{sym}); \, \tau_h \mid_K \in (\mathbb{P}_1)^{d \times d}_{sym}, \forall K \in \mathcal{T}_h\}, \\
Q_h &:= \{q_h \in \mathcal{C}^0(\overline{\Omega}; \mathbb{R}); \, q_h \mid_K \in \mathbb{P}_1, \forall K \in \mathcal{T}_h\} \cap L_0^2(\Omega; \mathbb{R}).
\end{aligned}
$$

We denote i_h the $L^2(\Omega)$ projection onto V_h, M_h or Q_h and introduce the following stabilized finite element discretization in space of (1)–(6). Given f, u_0, σ_0 find

$$
(u_h, \sigma_h, p_h) : t \rightarrow (u_h(t), \sigma_h(t), p_h(t)) \in V_h \times M_h \times Q_h
$$

such that $u_h(0) = i_h u_0$, $\sigma_h(0) = i_h \sigma_0$ and such that the following weak formulation holds in $]0, T[$:

$$
\begin{aligned}
&\rho\left(\frac{\partial u_h}{\partial t}, v_h\right) + 2\eta_s\left(\epsilon(u_h), \epsilon(v_h)\right) - \left(p_h, \operatorname{div} v_h\right) + \left(\sigma_h, \epsilon(v_h)\right) \\
&- \left(f, v_h\right) + \left(\operatorname{div} u_h, q_h\right) + \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\left(\nabla p_h, \nabla q_h\right)_K \\
&+ \frac{1}{2\eta_p}\left(\sigma_h, \tau_h\right) + \frac{\lambda}{2\eta_p}\left(\frac{\partial \sigma_h}{\partial t} - (\nabla u_h)\sigma_h - \sigma_h(\nabla u_h)^T, \tau_h\right) \\
&- \left(\epsilon(u_h), \tau_h\right) = 0,
\end{aligned}
\tag{12}
$$

for all $(v_h, \tau_h, q_h) \in V_h \times M_h \times Q_h$. Here $\alpha > 0$ is a dimensionless stabilization parameter and $(\cdot, \cdot)$ (respectively $(\cdot, \cdot)_K$) denotes the $L^2(\Omega)$ (resp. $L^2(K)$) scalar product for scalars, vectors and tensors.

The above nonlinear finite element scheme has already being studied in the stationary case [43]. Indeed, using the convergence result of [11] for the linear three fields Stokes problem and an implicit function theorem taken from [12,13,15], existence and convergence could be proved for small λ, the difficulty being again due to the fact that no a priori estimates can be obtained because of the presence of the quadratic terms $(\nabla u_h)\sigma_h + \sigma_h(\nabla u_h)^T$.

We will proceed in an analogous manner for the time dependent case and prove existence and convergence of a solution to (12) for a given λ but for small data f, u_0, σ_0. It should be noted that in this paper the case $\eta_s = 0$ is not considered, therefore some of the stabilization terms present in [11,43] are not included in the finite element formulation (12).

3 Existence of a solution to the simplified Oldroyd-B problem

We introduce, as in [25], the Helmholtz–Weyl projector [28–30] defined by

$$P_r : L^r(\Omega; \mathbb{R}^d) \to \mathcal{H}_r \qquad 1 < r < \infty,$$

where $\mathcal{H}_r$ is the completion of the divergence free $C_0^\infty(\Omega)$ vector fields with respect to the L^r norm. The space $\mathcal{H}_r$ can be characterized as follows (again see for instance [29])

$$\mathcal{H}_r = \left\{ v \in L^r(\Omega; \mathbb{R}^d) \; ; \; \text{div } v = 0, \; v \cdot n = 0 \text{ on } \partial\Omega \right\}.$$

Since Ω is of class C^2, there exists a constant C such that for all $f \in L^r$

$$\|P_r f\|_{L^r} \le C \|f\|_{L^r}.$$

We define $A_r := -P_r \Delta : \mathcal{D}_{A_r} \to \mathcal{H}_r$ the Stokes operator, where

$$\mathcal{D}_{A_r} := \left\{ v \in W^{2,r}(\Omega; \mathbb{R}^d) \cap W_0^{1,r}(\Omega; \mathbb{R}^d) \mid \text{div } v = 0 \right\}.$$

It is well known (see [32] for instance) that, for Ω of class C^2, the operator A_r equipped with the usual norm of $L^r(\Omega; \mathbb{R}^d)$ is closed and densely defined in $\mathcal{H}_r$. Moreover, the graph norm of A_r is equivalent to the $W^{2,r}$ norm.

With the above operators, (u, σ) is said to be a solution of (1)–(6) if

$$u \in W^{1,q}(\mathcal{H}_r) \cap L^q(\mathcal{D}_{A_r}), \qquad \sigma \in W^{1,q}(W^{1,r}),$$

with $1 < q < \infty, d < r < \infty$ and satisfies

$$\rho \frac{\partial u}{\partial t} + \eta_s A_r u - P_r \operatorname{div} \sigma = P_r f, \tag{13}$$

$$\frac{1}{2\eta_p}\sigma + \frac{\lambda}{2\eta_p}\left(\frac{\partial\sigma}{\partial t} - (\nabla u)\sigma - \sigma(\nabla u)^T\right) - \epsilon(u) = 0, \tag{14}$$

$$u(\cdot, 0) = u_0, \tag{15}$$

$$\sigma(\cdot, 0) = \sigma_0. \tag{16}$$

We will assume that the source term is $f \in L^q(L^r)$, the initial data are $u_0 \in E_{1-1/q,q}$ and $\sigma_0 \in W^{1,r}$. Here $E_{1-1/q,q} := \left(\mathcal{H}_r, \mathcal{D}_{A_r}\right)_{1-1/q,q}$ is a real interpolation space which can be defined as

$$E_{1-1/q,q} := \left\{ v \in \mathcal{H}_r; \int_0^{+\infty} \|A_r e^{-tA_r} v\|_{L_r}^q < \infty \right\}$$

and is a Banach space with norm

$$\|v\|_{E_{1-1/q,q}} := \|v\|_{L_r} + \left(\int_0^{+\infty} \|A_r e^{-tA_r} v\|_{L_r}^q\right)^{1/q}.$$

Moreover, when considering the little Hölder spaces $h^\mu(L^r)$ - see the regularities (8) and (10) - the space $E_{\mu,\infty} := \left(\mathcal{H}_r, \mathcal{D}_{A_r}\right)_{\mu,\infty}$ will also be needed. The space $E_{\mu,\infty}$ can be defined as

$$E_{\mu,\infty} = \left\{ v \in \mathcal{H}_r; \sup_{t>0} \|t^{1-\mu} A_r e^{-tA_r} v\|_{L_r} < +\infty \right\}$$

and is a Banach space endowed with the norm

$$\|v\|_{E_{\mu,\infty}} := \|v\|_{L_r} + \sup_{t>0} \|t^{1-\mu} A_r e^{-tA_r} v\|_{L_r}.$$

We refer to [18,19,49] for more details.

Remark 1 For $1 < q < +\infty, d < r < +\infty$, we have

$$W^{1,q}(\mathcal{H}_r) \cap L^q(\mathcal{D}_{A_r}) \subset L^2(W_0^{1,r}),$$

(see [21,32,51]). Thus a solution of (13)–(16) satisfies

$$\|u(T)\|_{L^2}^2 + \|\nabla u\|_{L^2(L^2)}^2 < \infty.$$

Uniqueness of a solution to problem (13)–(16) can be obtained proceeding as in [25], that is to say by proving an a priori estimate for the difference of two solutions when $q \geq 2$.

Lemma 1 *Let $d \geq 2$, let $\Omega \subset \mathbb{R}^d$ be a bounded, connected open set with boundary of class $\mathcal{C}^2$, let $T > 0$ and assume $2 \leq q < \infty$, $d < r < \infty$. Then, for any $f \in L^q(L^r)$, $u_0 \in E_{1-1/q,q}$, $\sigma_0 \in W^{1,r}$, there exists at most one solution (u, σ) of problem (13)–(16).*

Proof Let us start by noticing that for $1 \leq q < \infty$, $d < r < \infty$, $v \in L^q(W^{2,r})$ and $\tau \in W^{1,q}(W^{1,r})$ the nonlinearity $(\nabla v)\tau + \tau(\nabla v)^T \in L^q(W^{1,r})$. Indeed, $W^{1,r}$ is an algebra for $d < r$ and there exists $C > 0$ independent of v and τ such that

$$\|(\nabla v)\tau + \tau(\nabla v)^T\|_{W^{1,r}} \leq C\|\tau\|_{W^{1,r}}\|v\|_{W^{1,r}},$$

see for instance Theorem 5.23 of [1]. Moreover, using the same arguments as in [1], we have that $W^{1,q}(B) \subset_> L^\infty(B)$ for all Banach space B, thus

$$\|(\nabla v)\tau\|_{L^q(W^{1,r})}^q = \int_0^T \|(\nabla v)\tau\|_{W^{1,r}}^q \leq C \int_0^T \|\nabla v\|_{W^{1,r}}^q \|\tau\|_{W^{1,r}}^q \tag{17}$$

$$\leq C\|\tau\|_{L^\infty(W^{1,r})}^q \int_0^T \|\nabla v\|_{W^{1,r}}^q \tag{18}$$

$$\leq \tilde{C}\|\tau\|_{W^{1,q}(W^{1,r})}^q \|v\|_{L^q(W^{2,r})}^q, \tag{19}$$

where $\tilde{C}$ is independent of v and τ. Therefore, we can define the mapping $S : L^q(W^{2,r}) \times W^{1,q}(W^{1,r}) \to L^q(W^{1,r})$ by

$$S(v, \tau) := \frac{\lambda}{2\eta_p}\left((\nabla v)\tau + \tau(\nabla v)^T\right).$$

Let now $(u_i, \sigma_i) \in W^{1,q}(\mathcal{H}_r) \cap L^q(\mathcal{D}_{A_r}) \times W^{1,q}(W^{1,r})$ $i = 1, 2$, be two solutions of problem (13)–(16) and let $u = u_1 - u_2$, $\sigma = \sigma_1 - \sigma_2$. Using the well known properties of the Helmholtz-Weyl projector [28–30], there exists a unique pressure $p_i \in L^q(W^{1,r} \cap L_0^2)$, $i = 1, 2$, corresponding to each pair (u_i, σ_i) such that (u_i, σ_i, p_i) satisfies (1)–(6). When $2 \leq q < +\infty$, we can then take the weak formulation to obtain

$$\rho \int_0^t \left(\frac{\partial u_i}{\partial t}, u\right) + \frac{\lambda}{2\eta_p} \int_0^t \left(\frac{\partial \sigma_i}{\partial t}, \sigma\right) + \eta_s \int_0^t (\nabla u_i, \nabla u)$$

$$+ \int_0^t (\sigma_i, \epsilon(u)) + \frac{1}{2\eta_p} \int_0^t (\sigma_i, \sigma) - \int_0^t (\epsilon(u_i), \sigma) = \int_0^t (S(u_i, \sigma_i), \sigma), \tag{20}$$

for $i = 1, 2$. Hereabove we have used the fact that, since div $u_i = 0$, we have

$$2\text{div } \epsilon(u_i) = \Delta u_i.$$

All the terms in the previous equation are well defined because of the regularity of u_i and σ_i and since $u(0) = \sigma(0) = 0$ we have

$$\rho \int_0^t \left(\frac{\partial u_i}{\partial t}, u \right) + \frac{\lambda}{2\eta_p} \int_0^t \left(\frac{\partial \sigma_i}{\partial t}, \sigma \right) = \frac{\rho}{2} \| u(t) \|_{L^2}^2 + \frac{\lambda}{4\eta_p} \| \sigma(t) \|_{L^2}^2,$$

for $i = 1, 2$ and $t \in (0, T)$. Subtracting the two equalities (20), it follows that

$$\left(\frac{\rho}{2} \| u(t) \|_{L^2}^2 + \frac{\lambda}{4\eta_p} \| \sigma(t) \|_{L^2}^2 \right) + \eta_s \| \nabla u \|_{L^2(L^2)}^2 + \frac{1}{2\eta_p} \| \sigma \|_{L^2(L^2)}^2$$

$$= \int_0^t (S(u, \sigma_1), \sigma) + \int_0^t (S(u_2, \sigma), \sigma). \tag{21}$$

Then, using Cauchy–Schwarz and Young inequalities, we have for $t \in (0, T)$

$$\int_0^t (S(u, \sigma_1), \sigma) \leq \frac{2\lambda}{2\eta_p} \int_0^t \| \sigma_1 \|_{L^\infty} \| \nabla u \|_{L^2} \| \sigma \|_{L^2}$$

$$\leq \frac{\lambda^2}{2\eta_s \eta_p^2} \int_0^t \| \sigma_1 \|_{L^\infty}^2 \| \sigma \|_{L^2}^2 + \frac{\eta_s}{2} \int_0^t \| \nabla u \|_{L^2}^2$$

and

$$\int_0^t (S(u_2, \sigma), \sigma) \leq \frac{\lambda}{2\eta_p} \int_0^t \| \nabla u_2 \|_{L^\infty} \| \sigma \|_{L^2}^2.$$

Hence, with (21) and the continuous injection $W^{1,r} \subset_> L^\infty$ it follows that

$$\frac{\rho}{2} \| u(t) \|_{L^2}^2 + \frac{\lambda}{4\eta_p} \| \sigma(t) \|_{L^2}^2 \leq C \int_0^t \left(\| u_2 \|_{W^{2,r}} + \| \sigma_1 \|_{W^{1,r}}^2 \right) \| \sigma \|_{L^2}^2,$$

for $t \in (0, T)$. Here C is a constant independent of u_1, u_2, σ_1 and σ_2. Gronwall's Lemma is used to obtain for all $t \in (0, T)$

$$\rho \| u(t) \|_{L^2}^2 + \frac{\lambda}{2\eta_p} \| \sigma(t) \|_{L^2}^2 = 0,$$

so that $u \in W^{1,q}(\mathcal{H}_r) \cap L^q(\mathcal{D}_{A_r})$ and $\sigma \in W^{1,q}(W^{1,r})$ vanish.

We can now state the main results of this section.

Theorem 1 *Let $d \geq 2$, let $\Omega \subset \mathbb{R}^d$ be a bounded, connected open set with boundary of class $\mathcal{C}^2$, let $T > 0$ and assume $d < r < \infty$, $1 < q < \infty$, $0 < \mu < 1$. Then, there exists $\delta_0 > 0$ such that the following holds.*

(i) *If $f \in L^q(L^r)$, $u_0 \in E_{1-1/q,q}$, $\sigma_0 \in W^{1,r}$ satisfy*

$$\|P_r f\|_{L^q(L^r)} + \|u_0\|_{E_{1-1/q,q}} + \|\sigma_0\|_{W^{1,r}} \leq \delta_0,$$

then there exists a solution of (13)–(16).

(ii) *If $f \in W^{1,q}(L^r)$, $u_0 \in \mathcal{D}_{A_r}$, $\sigma_0 \in W^{1,r}$ satisfy the compatibility condition*

$$-\eta_s A_r u_0 + P_r f(0) + P_r \mathrm{div}\, \sigma_0 \in E_{1-1/q,q}$$

and are such that

$$\|P_r f\|_{W^{1,q}(L^r)} + \|u_0\|_{W^{2,r}} + \|\sigma_0\|_{W^{1,r}}$$
$$+ \| - \eta_s A_r u_0 + P_r f(0) + P_r \mathrm{div}\, \sigma_0\|_{E_{1-1/q,q}} \leq \delta_0,$$

then there exists a solution of (13)–(16) *with*

$$u \in W^{2,q}(\mathcal{H}_r) \cap W^{1,q}(\mathcal{D}_{A_r}), \quad \sigma \in W^{2,q}(W^{1,r}).$$

(iii) *If $f \in h^{\mu}(L^r)$, $u_0 \in \mathcal{D}_{A_r}$, $\sigma_0 \in W^{1,r}$ satisfying the compatibility condition*

$$-\eta_s A_r u_0 + P_r f(0) + P_r \mathrm{div}\, \sigma_0 \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}$$

and are such that

$$\|P_r(f - f(0))\|_{h^{\mu}(L^r)} + \|u_0\|_{W^{2,r}} + \|\sigma_0\|_{W^{1,r}}$$
$$+ \| - \eta_s A_r u_0 + P_r f(0) + P_r \mathrm{div}\, \sigma_0\|_{\overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}} \leq \delta_0,$$

then there exists a solution of (13)–(16) *with*

$$u \in h^{1+\mu}(\mathcal{H}_r) \cap h^{\mu}(\mathcal{D}_{A_r}), \quad \sigma \in h^{1+\mu}(W^{1,r}).$$

(iv) *If $f \in h^{1+\mu}(L^r)$, $u_0 \in \mathcal{D}_{A_r}$, $\sigma_0 \in W^{1,r}$ satisfying the compatibility conditions*

$$-\eta_s A_r u_0 + P_r f(0) + P_r \mathrm{div}\, \sigma_0 \in \mathcal{D}(A_r),$$

$$-\eta_s A_r(-\eta_s A_r u_0 + P_r f(0) + P_r \mathrm{div}\, \sigma_0) + \frac{\partial f}{\partial t}(0) \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}},$$

and are such that

$$\|P_r(f - f(0))\|_{h^\mu(L^r)} + \|P_r\left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial t}(0)\right)\|_{h^\mu(L^r)} + \|u_0\|_{W^{2,r}} + \|\sigma_0\|_{W^{1,r}}$$

$$+ \| - \eta_s A_r u_0 + P_r f(0) + P_r \text{div} \, \sigma_0\|_{W^{2,r}}$$

$$+ \| - \eta_s A_r(-\eta_s A_r u_0 + P_r f(0) + P_r \text{div} \, \sigma_0) + \frac{\partial f}{\partial t}(0)\|_{\overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}} \leq \delta_0,$$

then there exists a solution of (13)–(16) *with*

$$u \in h^{2+\mu}(\mathcal{H}_r) \cap h^{1+\mu}(\mathcal{D}_{A_r}), \quad \sigma \in h^{2+\mu}(W^{1,r}).$$

Moreover, in all cases, the mappings

$$(P_r f, u_0, \sigma_0) \mapsto \left(u(P_r f, u_0, \sigma_0), \sigma(P_r f, u_0, \sigma_0)\right)$$

are analytic in their respective spaces.

Using the well known properties of the Helmholtz–Weyl projector [28–30], we can then obtain the following result.

Corollary 1 *Under the assumptions of the above Theorem, there exists a unique* p *satisfying*

> (i) $p \in L^q(W^{1,r} \cap L_0^2)$,
> (ii) $p \in W^{1,q}(W^{1,r} \cap L_0^2)$,
> (iii) $p \in h^\mu(W^{1,r} \cap L_0^2)$,
> (iv) $p \in h^{1+\mu}(W^{1,r} \cap L_0^2)$,

such that (u, σ, p) *is solution of problem* (1)–(6). *Moreover, the mappings*

$$(f, u_0, \sigma_0) \mapsto \left(u(f, u_0, \sigma_0), \sigma(f, u_0, \sigma_0), p(f, u_0, \sigma_0)\right)$$

are analytic in their respective spaces.

Local existence in time can be proved for arbitrarily large data, using an abstract theorem for fully nonlinear parabolic equations, namely Theorem 8.1.1 of [37].

Theorem 2 *Let* $d \geq 2$, *let* $\Omega \subset \mathbb{R}^d$ *be a bounded, connected open set with boundary of class* $\mathcal{C}^2$ *and assume* $d < r < \infty$, $0 < \mu < 1$, $T > 0$. *If*

$$f \in \mathcal{C}^\mu(L^r), \quad u_0 \in \mathcal{D}_{A_r}, \quad \sigma_0 \in W^{1,r},$$

then there exists $T_* \in (0, T]$ *such that problem* (13)–(16) *possesses a solution*

$$u \in \mathcal{C}^1([0, T_*], \mathcal{H}_r) \cap \mathcal{C}^0([0, T_*], \mathcal{D}_{A_r}), \quad \sigma \in \mathcal{C}^1([0, T_*], W^{1,r}).$$

As for Corollary 1 we can deduce the following result.

Corollary 2 *Under the assumptions of the above Theorem, there exists a unique*

$$p \in \mathcal{C}^1([0, T_*], W^{1,r} \cap L_0^2)$$

such that (u, σ, p) is a solution of problem (1)–(6).

Remark 2 Part (i) of Theorem 1 is compatible with Theorem 9.2 of [25], in which the convective terms have been taken into account. Moreover, if $2/q + d/r < 1$, we have $L^q(\mathcal{D}_{A_r}) \cap W^{1,q}(\mathcal{H}_r) \subset_> \mathcal{C}^0(\mathcal{C}^1)$ and thus $\nabla u \in \mathcal{C}^0([0, T] \times \overline{\Omega})$ which implies $(u \cdot \nabla)u \in L^q(L^r)$. Therefore, Theorem 1 part (i) still holds when the convective term $(u \cdot \nabla)u$ is added to the momentum equation (1) or (13). However, since $(u \cdot \nabla)\sigma \notin W^{1,q}(W^{1,r})$, the convective term $(u \cdot \nabla)\sigma$ can not be added to (14) in the present analysis.

Remark 3 Since

$$\|(u \cdot \nabla)u\|_{W^{1,q}(L^r)} \leq C\Big(\|u\|_{W^{1,q}(\mathcal{D}_{A_r})} + \|u\|_{W^{2,q}(\mathcal{H}_r)}\Big),$$

then Theorem 1 part (ii) still holds if the convective term $(u \cdot \nabla)u$ is added to (13). However, since $(u \cdot \nabla)\sigma \notin W^{2,q}(W^{1,r})$, the convective term $(u \cdot \nabla)\sigma$ can not be added to (14) in the present analysis.

Remark 4 Parts (iii) and (iv) of Theorem 1 still hold when replacing little Hölder spaces by the classical Hölder spaces. Indeed, the only difference is that the trace space is not $\overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}$ anymore but $E_{\mu,\infty}$.

Remark 5 The trace spaces $E_{1-1/q,q}$ or $\overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}$ are abstract space but they both contain $\mathcal{D}_{A_r}$. For instance in part (i), if $u_0 \in W^{2,r} \cap W_0^{1,r}$ then $u_0 \in E_{1-1/q,q}$. Also, the condition "$\|P_r f\|_{L^q(L^r)}$ small" is satisfied whenever $\|f\|_{L^q(L^r)}$ is small.

Remark 6 The existence results presented in this section still hold when considering more realistic constitutive equations for the extra-stress tensor σ. This is for instance the case of the simplified Giesekus [31] and Phan-Thien Tanner [42] models, respectively defined by

$$\sigma + \lambda\left(\frac{\partial\sigma}{\partial t} - (\nabla u)\sigma - \sigma(\nabla u)^T\right) + \alpha\frac{\lambda}{\eta_p}\sigma\sigma = 2\eta_p\epsilon(u),$$

$$\sigma + \lambda\left(\frac{\partial\sigma}{\partial t} - (\nabla u)\sigma - \sigma(\nabla u)^T\right) + \epsilon\frac{\lambda}{\eta_p}tr(\sigma)\sigma = 2\eta_p\epsilon(u),$$

where α and ϵ are given positive parameters.

Remark 7 If we assume that the operator A_r satisfies the maximal regularity property (see Definition (1) in Appendix A) when Ω is a convex polygon, then Theorem 1 still holds. We did not find such a result in the literature, therefore we will make this

assumption and prove convergence of the finite element scheme. It should be noted that the corresponding property is true in stationary case for some $r > 2$ depending on the angles of the polygon, see [43].

3.1 Proof of Theorem 1

The proof is detailed for part (ii) only, which contains the major mathematical difficulties. Then, we will briefly explain how the same arguments can be used to prove parts (i), (iii) and (iv).

In order to prove part (ii) of Theorem 1, we shall introduce the mapping F : $Y \times X \to Z$, where

$$Y = \big\{ (P_r f, u_0, \sigma_0), \text{ such that } (f, u_0, \sigma_0) \in W^{1,q}(L^r) \times \mathcal{D}_{A_r} \times W^{1,r}$$
$$\text{and } -\eta_s A_r u_0 + P_r f(0) + P_r \text{div } \sigma_0 \in E_{1-1/q,q} \big\},$$
$$X = W^{2,q}(\mathcal{H}_r) \cap W^{1,q}(\mathcal{D}_{A_r}) \times W^{2,q}(W^{1,r}),$$
$$Z = W^{1,q}(W^{1,r}) \times Y.$$

The mapping F is defined for all $y = (P_r f, u_0, \sigma_0) \in Y$ and $x = (u, \sigma) \in X$ by

$$F(y, x) = \begin{pmatrix} \dfrac{\lambda}{2\eta_p} \dfrac{\partial \sigma}{\partial t} + \dfrac{1}{2\eta_p} \sigma - \epsilon(u) - S(u, \sigma) \\ \rho \dfrac{\partial u}{\partial t} + \eta_s A_r u - P_r \text{div } \sigma - P_r f \\ u(., 0) - u_0 \\ \sigma(., 0) - \sigma_0 \end{pmatrix},$$

with

$$S(u, \sigma) = \frac{\lambda}{2\eta_p} \big((\nabla u)\sigma + \sigma(\nabla u)^T \big). \tag{22}$$

Then problem (13)–(16) can be reformulated as follows. Given $y \in Y$, find $x \in X$ such that

$$F(y, x) = 0 \quad \text{in } Z. \tag{23}$$

The aim is to use the implicit function theorem, hence noticing that $F(0, 0) = 0$, we will prove that

- the spaces X, Y and Z equipped with appropriate norms are Banach spaces,
- F is a well defined, real analytic mapping,
- the Fréchet derivative $D_x F(0, 0)$ is an isomorphism from X to Z.

This will establish existence for part ii) of Theorem 1. Uniqueness follows from Lemma 1 for $2 \le q < \infty$.

The space X is equipped with the norm $\| \cdot \|_X$ defined for $x = (u, \sigma) \in X$ by

$$\|x\|_X = \|u, \sigma\|_X = \|u\|_{W^{2,q}(L^r)} + \|u\|_{W^{1,q}(W^{2,r})} + \|\sigma\|_{W^{2,q}(W^{1,r})}.$$

Obviously, $(X, \|.\|_X)$ becomes a Banach space. The space Y is equipped with the norm $\| \cdot \|_Y$ defined for $y = (P_r f, u_0, \sigma_0) \in Y$ by

$$\|y\|_Y = \|P_r f, u_0, \sigma_0\|_Y$$
$$= \|P_r f\|_{W^{1,q}(L^r)} + \|u_0\|_{W^{2,r}} + \|\sigma_0\|_{W^{1,r}}$$
$$+ \| - \eta_s A_r u_0 + P_r f(0) + P_r \mathrm{div}\, \sigma_0\|_{E_{1-1/q,q}}.$$

As a consequence of the continuity of the linear mapping

$$(P_r f, u_0, \sigma_0) \longmapsto -\eta_s A_r u_0 + P_r f(0) + P_r \mathrm{div}\, \sigma_0$$

from $W^{1,q}(\mathcal{H}_r) \times \mathcal{D}_{A_r} \times W^{1,r}$ (equipped with the product norm) to L^r and due to the completeness of $E_{1-1/q,q}$, the space $(Y, \|.\|_Y)$ is a closed subspace of $W^{1,q}(\mathcal{H}_r) \times \mathcal{D}_{A_r} \times W^{1,r}$ and thus a Banach space. The space Z is equipped with the product norm and becomes a Banach space.

In order to prove that F is well defined and analytic we need to prove that $S : X \to W^{1,q}(W^{1,r})$ is well defined and analytic. For this purpose, will use the following Lemma.

Lemma 2 *For every pair $x_1 = (u_1, \sigma_1)$, $x_2 = (u_2, \sigma_2) \in X$,*

$$b(x_1, x_2) := \nabla u_1 \sigma_2 + \sigma_1 (\nabla u_2)^T \in W^{1,q}(W^{1,r}).$$

Moreover, the corresponding bilinear mapping $b : X \times X \to W^{1,q}(W^{1,r})$ is continuous, that is, there exists a constant C such that for all $x_1, x_2 \in X$ we have

$$\|b(x_1, x_2)\|_{W^{1,q}(W^{1,r})} \le C \|x_1\|_X \|x_2\|_X. \tag{24}$$

Proof Let $x_1 = (u_1, \sigma_1)$, $x_2 = (u_2, \sigma_2) \in X$. Since $r > d$, $W^{1,r}(\Omega) \subset_> L^{\infty}(\Omega)$ so that $W^{1,r}(\Omega)$ is an algebra (see [1]) and there exists a constant C depending only on Ω such that

$$\|b(x_1, x_2)\|_{W^{1,r}} \le C \|u_1\|_{W^{2,r}} \|\sigma_2\|_{W^{1,r}}.$$

Then we have

$$\|b(x_1, x_2)\|_{L^q(W^{1,r})}^q = \int_0^T \|b(x_1, x_2)\|_{W^{1,r}}^q$$

$$\le C \int_0^T \|u_1\|_{W^{2,r}}^q \|\sigma_2\|_{W^{1,r}}^q$$

$$\le C \|u_1\|_{L^{\infty}(W^{2,r})}^q \int_0^T \|\sigma_2\|_{W^{1,r}}^q.$$

Since $q > 1$, $W^{1,q}(0, T) \subset_> L^\infty(0, T)$ and we also have that $W^{1,q}(B) \subset_> L^\infty(B)$ for any Banach space B, thus

$$\|b(x_1, x_2)\|^q_{L^q(W^{1,r})} \leq C\|u_1\|^q_{W^{1,q}(W^{2,r})}\|\sigma_2\|^q_{L^q(W^{1,r})},$$

which proves that

$$\|b(x_1, x_2)\|_{L^q(W^{1,r})} \leq C\|u_1, \sigma_1\|_X\|u_2, \sigma_2\|_X. \tag{25}$$

Similarly, there exists a constant C depending only on Ω, λ and η_p such that

$$\|\frac{\partial b}{\partial t}(x_1, x_2)\|^q_{L^q(W^{1,r})} = \int_0^T \|\frac{\partial b}{\partial t}(x_1, x_2)\|^q_{W^{1,r}}$$

$$\leq C \int_0^T \left(\|\frac{\partial u_1}{\partial t}\|^q_{W^{2,r}}\|\sigma_2\|^q_{W^{1,r}} + \|u_1\|^q_{W^{2,r}}\|\frac{\partial \sigma_2}{\partial t}\|^q_{W^{1,r}}\right)$$

$$\leq C \left(\|\sigma_2\|^q_{L^\infty(W^{1,r})}\int_0^T \|\frac{\partial u_1}{\partial t}\|^q_{W^{2,r}} + \|u_1\|^q_{L^\infty(W^{2,r})}\int_0^T \|\frac{\partial \sigma_2}{\partial t}\|^q_{W^{1,r}}\right)$$

$$\leq C \left(\|\sigma_2\|^q_{W^{1,q}(W^{1,r})}\int_0^T \|\frac{\partial u_1}{\partial t}\|^q_{W^{2,r}} + \|u_1\|^q_{W^{1,q}(W^{2,r})}\int_0^T \|\frac{\partial \sigma_2}{\partial t}\|^q_{W^{1,r}}\right),$$

which proves that

$$\|\frac{\partial b}{\partial t}(x_1, x_2)\|_{L^q(W^{1,r})} \leq C\|u_1, \sigma_1\|_X\|u_2, \sigma_2\|_X. \tag{26}$$

The estimates (25) and (26) prove that $b(x_1, x_2) \in W^{1,q}(W^{1,r})$ and (24).

Remark 8 In fact we also have proved that $W^{1,q}(W^{1,r})$ is an algebra for $1 < q < \infty$ and $d < r < \infty$.

Remark 9 For $x \in X$ we have $S(x) = \dfrac{\lambda}{2\eta_p}b(x, x)$, where $S : X \rightarrow W^{1,q}(W^{1,r})$ is introduced in (22). Thus, in virtue of [16], S is well defined and analytic.

Corollary 3 *The mapping $F : Y \times X \rightarrow Z$ is well defined and analytic. Moreover, for $x = (v, \tau) \in X$ its Fréchet derivative $D_x F(0, 0)x$ is given by*

$$D_x F(0, 0)x = \begin{pmatrix} \dfrac{\lambda}{2\eta_p}\dfrac{\partial \tau}{\partial t} + \dfrac{1}{2\eta_p}\tau - \epsilon(v) \\ \rho\dfrac{\partial v}{\partial t} + \eta_s A_r v - P_r \mathrm{div}\,\tau \\ v(.,0) \\ \tau(.,0) \end{pmatrix}.$$

Proof In order to study the property of the mapping $F : Y \times X \to Z$ we rewrite it as follows

$$F(y, x) = L_1 y + L_2 x + \begin{pmatrix} \dfrac{\lambda}{2\eta_p} b(x, x) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad (27)$$

where $L_1 : Y \to Z$, $L_2 : X \to Z$ are bounded linear operator defined for $y = (P_r f, u_0, \sigma_0) \in Y$ and $x = (u, \sigma) \in X$ by

$$L_1 y = \begin{pmatrix} 0 \\ -y \end{pmatrix}, \qquad L_2 x = \begin{pmatrix} \dfrac{\lambda}{2\eta_p} \dfrac{\partial \sigma}{\partial t} + \dfrac{1}{2\eta_p} \sigma - \epsilon(u) \\ \rho \dfrac{\partial u}{\partial t} + \eta_s A_r u - P_r \mathrm{div}\, \sigma \\ u(., 0) \\ \sigma(., 0) \end{pmatrix}$$

and $b : X \times X \to W^{1,q}(W^{1,r})$ is defined in Lemma 2. Clearly, the first two terms in (27) are analytic. The last term is also analytic in virtue of [16], which proves that F is analytic. Moreover $D_x F(0, 0) = L_2$ which completes the proof.

In order to use the implicit function theorem, it remains to check that $D_x F(0, 0)$ is an isomorphism from X to Z. Therefore, we have to check that, for $g \in W^{1,q}(W^{1,r})$ and $(h, v_0, \tau_0) \in Y$ there exists a unique $(v, \tau) \in X$ such that

$$\begin{cases} \dfrac{\lambda}{2\eta_p} \dfrac{\partial \tau}{\partial t} + \dfrac{1}{2\eta_p} \tau - \epsilon(v) = g, \\ \rho \dfrac{\partial v}{\partial t} + \eta_s A_r v - P_r \mathrm{div}\, \tau = h, \\ v(., 0) = v_0, \\ \tau(., 0) = \tau_0. \end{cases} \qquad (28)$$

Lemma 3 *Let $d \geq 2$, let $\Omega \subset \mathbb{R}^d$ be a bounded, connected open set with boundary of class $\mathcal{C}^2$, let $T > 0$ and assume $d < r < \infty$, $1 < q < \infty$. Given $g \in W^{1,q}(W^{1,r})$ and $(h, v_0, \tau_0) \in Y$, there exists a unique $(v, \tau) \in X$ solution of (28). Moreover, there exists a constant C such that for $g \in W^{1,q}(W^{1,r})$ and $(h, v_0, \tau_0) \in Y$*

$$\|v, \tau\|_X \leq C \left(\|g\|_{W^{1,q}(W^{1,r})} + \|h, v_0, \tau_0\|_Y \right). \qquad (29)$$

Proof Solving the first equation of (28) we obtain for the extra-stress

$$\tau = k\tau_0 + \frac{2\eta_p}{\lambda} k * (\epsilon(v) + g), \qquad (30)$$

with $k \in C^\infty([0, T])$ defined by $k(t) := e^{-\frac{t}{\lambda}}$ and the convolution operator $*$ by

$$(f * g)(t) := \int_0^t f(t - s)g(s) \, ds \quad \forall t \in [0, T], \forall f, g \in L^1(0, T).$$

Introducing (30) in the second equation of (28), yields

$$\begin{cases} \rho \dfrac{\partial v}{\partial t} + \eta_s A_r v + \dfrac{\eta_p}{\lambda} k * A_r v = \tilde{h}, \\ v(., 0) = v_0. \end{cases} \tag{31}$$

where $\tilde{h} := h + P_r \text{div}\,(k\tau_0) + \frac{2\eta_p}{\lambda} P_r \text{div}\,(k * g) \in W^{1,q}(\mathcal{H}_r)$. Since Ω is of class $\mathcal{C}^2$, $-A_r$ satisfies the maximal regularity property (see Theorem 15 p. 102 in Sect. 17 of [52] for the $L^r(L^r)$ estimate and Remark 13 in Appendix (A) for the $L^q(L^r)$ estimate). Moreover, $\overline{\mathcal{D}_{A_r}} = L^r(\Omega)$ and since $v_0 \in \mathcal{D}_{A_r}$, $-A_r v_0 + \tilde{h}(0) \in E_{1-1/q,q}$, Corollary 5 and Lemma 14 both in Appendix (A) prove the existence and uniqueness of the solution $v \in W^{2,q}(\mathcal{H}_r) \cap W^{1,q}(\mathcal{D}_{A_r})$. The estimates of Corollary 5, Lemma 14 and Remark 12 (in Appendix A) ensure the existence of a constant C such that for $(h, v_0, \tau_0) \in Y, g \in W^{1,q}(W^{1,r})$ <

$$\|v\|_{W^{2,q}(L^r)} + \|v\|_{W^{1,q}(\mathcal{D}_{A_r})} \le C \left(\|h, v_0, \tau_0\|_Y + \|g\|_{W^{1,q}(W^{1,r})} \right).$$

Because of the regularity of Ω, the graph norm $\|.\|_{\mathcal{D}_{A_r}}$ is equivalent to the whole norm $\|.\|_{W^{2,r}}$, thus there exists a constant C such that

$$\|v\|_{W^{2,q}(L^r)} + \|v\|_{W^{1,q}(W^{2,r})} \le C \left(\|h, v_0, \tau_0\|_Y + \|g\|_{W^{1,q}(W^{1,r})} \right). \tag{32}$$

Going back to the extra-stress, Eq. (30), since $g + \epsilon(v) \in W^{1,q}(W^{1,r})$, Remark 12 in Appendix (A) ensures that $k * (g + \epsilon(v)) \in W^{2,q}(W^{1,r})$ and there exists a constant C such that

$$\|k * (g + \epsilon(v))\|_{W^{2,q}(W^{1,r})} \le C \left(\|g\|_{W^{1,q}(W^{1,r})} + \|v\|_{W^{1,q}(W^{2,r})} \right).$$

It remains to use (30) to obtain the existence and uniqueness of $\tau \in W^{2,q}(W^{1,r})$. Moreover there exists a constant C such that

$$\|\tau\|_{W^{2,q}(W^{1,r})} \le C \left(\|h, v_0, \tau_0\|_Y + \|g\|_{W^{1,q}(W^{1,r})} \right). \tag{33}$$

Collecting the estimations (32) and (33) we obtain (29).

Proof (of Theorem 1, part ii) We apply the implicit function theorem to (23). From Corollary 3, F is well defined and analytic, $F(0, 0) = 0$. Moreover, from Lemma 3 $D_x F(0, 0)$ is an isomorphism from X to Z. Therefore, we can apply the implicit function theorem (see for instance Theorem 4.5.4 chapter 4 p. 56 of [14]). Thus there

exists $\delta_0 > 0$ and $\varphi : Y \to X$ analytic such that for all $y := (P_r f, u_0, \sigma_0) \in Y$ with $\|y\|_Y < \delta_0$ we have $F(y, \varphi(y)) = 0$.

We will now briefly explain how same arguments can be used to prove parts (i), (iii) and (iv) of Theorem 1.

The proof of part (i) is very similar to the one presented hereabove. Indeed, it suffices to use the spaces

$$Y = \left\{ (P_r f, u_0, \sigma_0), \text{ such that } (f, u_0, \sigma_0) \in L^q(L^r) \times E_{1-1/q,q} \times W^{1,r} \right\},$$
$$X = W^{1,q}(\mathcal{H}_r) \cap L^q(\mathcal{D}_{A_r}) \times W^{1,q}(W^{1,r}),$$
$$Z = L^q(W^{1,r}) \times Y$$

and to use Corollary 5 (Appendix A) in order to prove the existence and uniqueness of the function v solution of (31). Concerning part (iii), we shall use the spaces

$$Y = \left\{ (P_r f, u_0, \sigma_0), \text{ such that } (f, u_0, \sigma_0) \in h^\mu(L^r) \times \mathcal{D}_{A_r} \times W^{1,r} \right.$$
$$\left. \text{and } -\eta_s A_r u_0 + P_r f(0) + P_r \text{div} \, \sigma_0 \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}} \right\},$$
$$X = h^{1+\mu}(\mathcal{H}_r) \cap h^\mu(\mathcal{D}_{A_r}) \times h^{1+\mu}(W^{1,r}),$$
$$Z = h^\mu(W^{1,r}) \times Y.$$

and Lemma 13 (Appendix A) in order to prove the existence and uniqueness of the function v solution of (31). Finally, the link between parts (i) and (ii) is the same as between parts (iii) and (iv). Thus we can extend the arguments presented in part (ii) to little Hölder spaces in order to obtain more regularity in time.

3.2 Proof of Theorem 2

This result is obtained using the fully nonlinear theory for parabolic problems which can be found in [37]. More precisely, Theorem 8.1.1 pp. 290 will be used on problem (13)–(16) that can be rewritten as follows

$$\dot{x}(t) = G(t, x(t)), \quad t > 0, \quad x(0) = x_0,$$

where $x := (u, \sigma)$, $x_0 := (u_0, \sigma_0)$ and $G : [0, T] \times \mathcal{D}_{A_r} \times W^{1,r} \to \mathcal{H}_r \times W^{1,r}$ is defined by

$$G(t, x) := \begin{pmatrix} -\dfrac{\eta_s}{\rho} A_r & G_1 \\ G_2 & -\dfrac{1}{\lambda} I_d \end{pmatrix} x + \begin{pmatrix} \dfrac{1}{\rho} P_r f(t) \\ \hat{S}(x) \end{pmatrix}.$$

Hereabove, $G_1 \in \mathcal{L}(W^{1,r}; \mathcal{H}_r)$ and $G_2 \in \mathcal{L}(\mathcal{D}_{A_r}, W^{1,r})$ are defined by

$$G_1 \sigma = \frac{1}{\rho} P_r \text{div} \, \sigma, \quad G_2 u = \frac{2\eta_p}{\lambda} \epsilon(u)$$

whilst $\hat{S} : \mathcal{D}_{A_r} \times W^{1,r} \to W^{1,r}$ is defined by

$$\hat{S}(u, \sigma) := (\nabla u)\sigma + \sigma(\nabla u)^T.$$

Lemma 4 *The application $\hat{S} : \mathcal{D}_{A_r} \times W^{1,r} \to W^{1,r}$ is well defined and analytic. Moreover, $G : [0, T] \times \mathcal{D}_{A_r} \times W^{1,r} \to \mathcal{H}_r \times W^{1,r}$ is continuous with respect to (t, x).*

Proof Same arguments as provided in Lemma 2 and Remark 9 in the previous subsection can be used to ensure $\hat{S} : \mathcal{D}_{A_r} \times W^{1,r} \to W^{1,r}$ is well defined and analytic. Let us prove now the continuity of G. In order to simplify the notations, let us introduce de linear part of G, namely $L \in \mathcal{L}(\mathcal{D}_{A_r} \times W^{1,r}, \mathcal{H}_r \times W^{1,r})$ defined by

$$L := \begin{pmatrix} -\dfrac{\eta_s}{\rho} A_r & G_1 \\ G_2 & -\dfrac{1}{\lambda} Id \end{pmatrix}.$$

Fix $(t, x) \in [0, T] \times \mathcal{D}_{A_r} \times W^{1,r}$ and let $\{t_n\}_{n \geq 0} \subset [0, T]$, $\{x_n\}_{n \geq 0} \subset \mathcal{D}_{A_r} \times W^{1,r}$ such that $t_n \to t$ and $x_n \to x$ when t goes to infinity. Therefore,

$$\|G(t, x) - G(t_n, x_n)\|_{L^r \times W^{1,r}}$$
$$\leq \|L(x - x_n)\|_{L^r \times W^{1,r}} + \|\frac{1}{\rho}(P_r f(t) - P_r f(t_n))\|_{L^r} + \|\hat{S}(x) - \hat{S}(x_n)\|_{W^{1,r}}.$$

Thus, since $f \in \mathcal{C}^\mu(L^r)$ and $\hat{S}$ is continuous from $\mathcal{D}_{A_r} \times W^{1,r}$ to $W^{1,r}$, it follows

$$\|G(t, x) - G(t_n, x_n)\|_{L^r \times W^{1,r}}$$
$$\leq \|L\|_{\mathcal{L}(\mathcal{D}_{A_r} \times W^{1,r}, \mathcal{H}_r \times W^{1,r})} \|x - x_n\|_{W^{2,r} \times W^{1,r}}$$
$$+ C \left(\|P_r f\|_{\mathcal{C}^\mu(L^r)} |t - t_n|^\mu + \|x - x_n\|_{W^{2,r} \times W^{1,r}} \right).$$

Hence

$$\|G(t, x) - G(t_n, x_n)\|_{L^r \times W^{1,r}} \to 0 \quad \text{when} \quad n \to \infty.$$

The crucial point in order to prove Theorem 2 is

$$\begin{cases} \text{for } t \in [0, T] \text{and } x \in \mathcal{D}_{A_r} \times W^{1,r} \text{the Fréchet derivative } D_x G(t, x) \\ \text{is the generator of an analytic semigroup.} \end{cases} \quad (34)$$

The above property will be a consequence of a result by S. B. Angenent [2].

Lemma 5 *For $t \in [0, T]$ and $x \in \mathcal{D}_{A_r} \times W^{1,r}$ the Fréchet derivative $D_x G(t, x)$ is the generator of an analytic semigroup.*

Proof Let $x := (u, \sigma) \in \mathcal{D}_{A_r} \times W^{1,r}$. In order to characterize the Fréchet deriv-ative $D_x G(t, x)$, $t \in [0, T]$, let us define the operators $S_u \in \mathcal{L}(W^{1,r}, W^{1,r})$ and $S_\sigma \in \mathcal{L}(\mathcal{D}_{A_r}, W^{1,r})$ by

$$S_u \tau := (\nabla u)\tau + \tau \nabla u, \quad \forall \tau \in W^{1,r}$$

and

$$S_\sigma v := (\nabla v)\sigma + \sigma \nabla v, \quad \forall v \in \mathcal{D}_{A_r}.$$

Lemma 2 ensures $S_\sigma \in \mathcal{L}(\mathcal{D}_{A_r}, W^{1,r})$, $S_u \in \mathcal{L}(W^{1,r}, W^{1,r})$. Using this notations we obtain for $t \in [0, T]$ and $x \in \mathcal{D}_{A_r} \times W^{1,r}$

$$D_x G(t, x) = \begin{pmatrix} -\dfrac{\eta_s}{\rho} A_r & G_1 \\ G_2 + S_\sigma & -\dfrac{1}{\lambda} I_d + S_u \end{pmatrix}, \tag{35}$$

and since $G_1 \in \mathcal{L}(W^{1,r}, \mathcal{H}_r)$, $G_2 \in \mathcal{L}(\mathcal{D}_{A_r}, W^{1,r})$, we have

$$D_x G(t, x) \in \mathcal{L}(\mathcal{D}_{A_r} \otimes W^{1,r}, \mathcal{H}_r \otimes W^{1,r}).$$

Finally, since $-A_r : \mathcal{D}_{A_r} \to \mathcal{H}_r$ is generator of an analytic semigroup (see [32]), Lemma 2.6 p. 98 (part a) of [2] concludes the proof. ∎

Let us go back to the proof of Theorem 2.

Proof (of Theorem 2) We apply Theorem 8.1.1 p. 290 of [37] with $\overline{u} = x_0 = (u_0, \sigma_0)$, $t_0 = 0$, $\overline{t} = 0$ and $\mathcal{O} = \mathcal{D}_{A_r} \times W^{1,r}$. Since $\overline{\mathcal{D}_{A_r} \times W^{1,r}} = \mathcal{H}_r \times W^{1,r}$, the condition $G(0, x_0) \in \overline{\mathcal{D}_{A_r} \times W^{1,r}}$ is satisfied for all $x_0 \in \mathcal{D}_{A_r} \times W^{1,r}$. Thus it remains to check

(i)   property (34) is satisfied,
(ii) for $t \in [0, T]$ and $x \in \mathcal{D}_{A_r} \times W^{1,r}$, the graph norm of the operator $D_x G(t, x)$ is equivalent to the norm $\|.\|_{W^{2,r} \times W^{1,r}}$,
(iii) $(t, x) \mapsto G(t, x)$ is continuous with respect to (t, x), and it is Fréchet differen-tiable with respect to x,
(iv) for all $\overline{x} := (\overline{u}, \overline{\sigma}) \in \mathcal{D}_{A_r} \times W^{1,r}$ there are $R = R(\overline{x})$, $L = L(\overline{x})$, $K = K(\overline{x}) > 0$ verifying

$$\|D_x G(t, x) - D_x G(t, z)\|_{\mathcal{L}(\mathcal{D}_{A_r} \times W^{1,r}, \mathcal{H}_r \times W^{1,r})} \le L \|x - z\|_{W^{2,r} \times W^{1,r}},$$

$$\begin{aligned} &\|G(t, x) - G(s, x)\|_{L^r \times W^{1,r}} \\ &\quad + \|D_x G(t, x) - D_x G(s, x)\|_{\mathcal{L}(\mathcal{D}_{A_r} \times W^{1,r}, \mathcal{H}_r \times W^{1,r})} \le K |t - s|^\mu, \end{aligned}$$

for all $t, s \in [0, T]$, $x, z \in B(\overline{x}, R) \subset \mathcal{D}_{A_r} \times W^{1,r}$.

Relation (i) is satisfied by using Lemma 5. Property (ii) is satisfied since $W^{1,r} \subset_> L^\infty$ (see [1]). The application G is continuous by Lemma 4. The Fréchet derivative is given by (35) and is well defined. Finally, (iv) may be proved as follow. Let $x = (u, \sigma)$, $z := (v, \tau)$ and $\tilde{z} := (w, \xi)$ all belonging to $\mathcal{D}_{A_r} \times W^{1,r}$, using again the continuous embedding $W^{1,r} \subset_> L^\infty$ we have

$$
\begin{aligned}
\|D_x G(t, x)\tilde{z} - D_x G(t, z)\tilde{z}\|_{L^r \times W^{1,r}} &= \|D_x \hat{S}(x)\tilde{z} - D_x \hat{S}(z)\tilde{z}\|_{L^r \times W^{1,r}} \\
&= \|\nabla(u - v)\xi + \xi(\nabla(u - v))^T + \nabla w(\sigma - \tau) + (\sigma - \tau)(\nabla w)^T\|_{W^{1,r}} \\
&\leq C\|\tilde{z}\|_{W^{2,r} \times W^{1,r}}\|x - z\|_{W^{2,r} \times W^{1,r}},
\end{aligned}
$$

where C is independent of u and σ. Moreover, for $t, s \in [0, T]$ and $x \in \mathcal{D}_{A_r} \times W^{1,r}$

$$
D_x G(t, x) = D_x G(s, x).
$$

Hence, since $f \in C^\mu(L^r)$, we have for $t, s \in [0, T]$ and $x \in \mathcal{D}_{A_r} \times W^{1,r}$

$$
\|G(t, x) - G(s, x)\|_{L^r \times W^{1,r}} = \|\frac{1}{\rho}(P_r f(t) - P_r f(s))\|_{L^r} \leq C |t - s|^\mu,
$$

where C is independent of t, s and x. Relations (i)–(iv) ensure the existence of $0 < T_* < T$ such that there exists a solution

$$
x \in C^0([0, T_*], \mathcal{D}_{A_r} \times W^{1,r}) \cap C^1([0, T_*], \mathcal{H}_r \times W^{1,r})
$$

of (13)–(16).

4 Existence of the finite element approximation and a priori error estimates

In this section we assume that Ω is a convex polygon and that

$$
2 \leq q < +\infty, \qquad 2 = d \leq r < +\infty.
$$

We set

$$
\begin{aligned}
Y &= L^q(L^r) \times E_{1-1/q,q} \times W^{1,r}, \\
X &= W^{1,q}(L^r) \cap L^q(W^{2,r}) \times W^{1,q}(W^{1,r}),
\end{aligned}
$$

the data and solution spaces, respectively. According to Theorem 1 part (i), Corollary 1 and Remark 7 we know that, if $y = (f, u_0, \sigma_0) \in Y$ is sufficiently small, then there exists a unique solution

$$
\left(u(y), \sigma(y), p(y) \right)
$$

of (1)–(6), the mapping

$$y \to \Big(u(y), \sigma(y), p(y)\Big)$$

being analytic (therefore continuous).

In order to prove that the solution of the nonlinear finite element discretization (12) exists and converges to that of (1)–(6), we introduce $X_h \subset X$ defined by

$$X_h = L^2(V_h) \times L^\infty(M_h)$$

equipped with the norm $|| \cdot ||_{X_h}$ defined for all $x_h = (u_h, \sigma_h) \in X_h$ by

$$\|x_h\|_{X_h}^2 := 2\eta_s \int_0^T \|\epsilon(u_h(t))\|_{L^2(\Omega)}^2 dt + \frac{\lambda}{4\eta_p} \sup_{t\in[0,T]} \|\sigma_h(t)\|_{L^2(\Omega)}^2.$$

Then, we rewrite the solution of (12) as the following fixed point problem. Given $y = (f, u_0, \sigma_0) \in Y$, find $x_h = (u_h, \sigma_h) \in X_h$ such that

$$x_h = \mathsf{T}_h\Big(y, S(x_h)\Big), \tag{36}$$

where S is still defined as in (22) but has been extended to the larger space

$$S : L^2(H^1) \times L^\infty(L^2) \to L^2(L^2).$$

The operator T_h is the semi-discrete time-dependent three fields Stokes problem defined by

$$\mathsf{T}_h : Y \times L^2(L^2) \to X_h$$
$$(f, u_0, \sigma_0, g) \to \mathsf{T}_h(f, u_0, \sigma_0, g) \underset{\text{def.}}{=} (\tilde{u}_h, \tilde{\sigma}_h)$$

where for $t \in (0, T)$

$$(\tilde{u}_h, \tilde{\sigma}_h, \tilde{p}_h) : t \longmapsto (\tilde{u}_h(t), \tilde{\sigma}_h(t), \tilde{p}_h(t)) \in V_h \times M_h \times Q_h$$

satisfies $\tilde{u}_h(0) = i_h u_0$, $\tilde{\sigma}_h(0) = i_h \sigma_0$ and

$$\rho\Big(\frac{\partial \tilde{u}_h}{\partial t}, v_h\Big) + 2\eta_s\Big(\epsilon(\tilde{u}_h), \epsilon(v_h)\Big) - \Big(\tilde{p}_h, \operatorname{div} v_h\Big) + \Big(\tilde{\sigma}_h, \epsilon(v_h)\Big) - \Big(f, v_h\Big)$$
$$+ \Big(\operatorname{div} \tilde{u}_h, q_h\Big) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\Big(\nabla \tilde{p}_h, \nabla q_h\Big)_K$$
$$+ \frac{1}{2\eta_p}\Big(\tilde{\sigma}_h, \tau_h\Big) + \frac{\lambda}{2\eta_p}\Big(\frac{\partial \tilde{\sigma}_h}{\partial t}, \tau_h\Big) - \Big(\epsilon(\tilde{u}_h), \tau_h\Big) - \frac{\lambda}{2\eta_p}\Big(g, \tau_h\Big) = 0 \tag{37}$$

for all $(v_h, \tau_h, q_h) \in V_h \times M_h \times Q_h$, a.e in $(0, T)$.

It should be noticed that, given $y = (f, u_0, \sigma_0) \in Y$ sufficiently small, the solution $x(y) = (u(y), \sigma(y)) \in X$ of the continuous Oldroyd-B problem (1)–(6) also satisfies a fixed point problem, namely

$$x(y) = \mathsf{T}\Big(y, S(x(y))\Big). \tag{38}$$

Here the operator T is the time-dependent three fields Stokes problem defined by

$$\mathsf{T} : Y \times L^q(W^{1,r}) \to X$$
$$(f, u_0, \sigma_0, g) \to \mathsf{T}(f, u_0, \sigma_0, g) \underset{\text{def.}}{=} (\tilde{u}, \tilde{\sigma}),$$

where $(\tilde{u}, \tilde{\sigma}, \tilde{p})$ satisfy

$$\rho \frac{\partial \tilde{u}}{\partial t} - 2\eta_s \, \mathrm{div}\, \epsilon(\tilde{u}) + \nabla \tilde{p} - \mathrm{div}\, \tilde{\sigma} = f \quad \text{in } \Omega \times (0, T), \tag{39}$$

$$\mathrm{div}\, \tilde{u} = 0 \quad \text{in } \Omega \times (0, T), \tag{40}$$

$$\frac{1}{2\eta_p} \tilde{\sigma} + \frac{\lambda}{2\eta_p} \Big(\frac{\partial \tilde{\sigma}}{\partial t}\Big) - \epsilon(\tilde{u}) = g \quad \text{in } \Omega \times (0, T), \tag{41}$$

$$\tilde{u}(\cdot, 0) = u_0 \quad \text{in } \Omega, \tag{42}$$

$$\tilde{\sigma}(\cdot, 0) = \sigma_0 \quad \text{in } \Omega, \tag{43}$$

$$\tilde{u} = 0 \quad \text{on } \partial\Omega \times (0, T). \tag{44}$$

We then have the following stability and convergence result, which proof can be found in Appendix (B).

Lemma 6 *The operator* T_h *is well defined and uniformly bounded with respect to* h : *there exists* $C_1 > 0$ *such that for all* $h > 0$ *and for all* $(f, u_0, \sigma_0, g) \in Y \times L^2(L^2)$ *we have*

$$||\mathsf{T}_h(f, u_0, \sigma_0, g)||_{X_h} \leq C_1 \Big(||f, u_0, \sigma_0||_Y + ||g||_{L^2(L^2)}\Big). \tag{45}$$

Moreover, there exists $C_2 > 0$ *such that for all* $h > 0$ *and for all* $(f, u_0, \sigma_0, g) \in Y \times L^q(W^{1,r})$ *we have*

$$||(\mathsf{T} - \mathsf{T}_h)(f, u_0, \sigma_0, g)||_{X_h} \leq C_2 h \Big(||f, u_0, \sigma_0||_Y + ||g||_{L^q(W^{1,r})}\Big). \tag{46}$$

Our goal is now to prove that (36) has a unique solution converging to that of (38). For this purpose, we use, as in [43], an abstract framework and write (36) as the following problem : given $y = (f, u_0, \sigma_0) \in Y$, find $x_h = (u_h, \sigma_h) \in X_h$ such that

$$F_h(y, x_h) = 0, \tag{47}$$

where $F_h : Y \times X_h \to X_h$ is defined by

$$F_h(y, x_h) = x_h - \mathsf{T}_h\Big(y, S(x_h)\Big). \tag{48}$$

In order to prove existence and convergence of a solution to (47), we use Theorem 2.1 of [15]. The mapping $F_h : Y \times X_h \to X_h$ is $\mathcal{C}^1$. Moreover, we need to prove that the scheme is consistent, that $D_x F_h$ has bounded inverse at $i_h x$ - recall that i_h is the $L^2(\Omega)$ projection onto the finite element space, x is the solution of (38) - and that $D_x F_h$ is locally Lipschitz at $i_h x$.

Lemma 7 *Let δ_0 be as in Theorem 1 part i), 1, let $y := (f, u_0, \sigma_0) \in Y$ with $\|y\|_Y \le \delta_0$ and let $x(y) = (u(y), \sigma(y)) \in X$ be the solution of (38). Then, there exists a constant C_1 such that for all $y \in Y$ with $\|y\|_Y \le \delta_0$, for all $0 < h \le 1$, we have*

$$\|F_h(y, i_h x(y))\|_{X_h} \le C_1 h \Big(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2 \Big). \tag{49}$$

Moreover, there exists a constant C_2 such that for all $y \in Y$ with $\|y\|_Y \le \delta_0$, for all $0 < h \le 1$, for all $z \in X_h$ we have

$$\|D_x F_h(y, i_h x(y)) - D_x F_h(y, z)\|_{\mathcal{L}(X_h)} \le \frac{C_2}{h} \|i_h x(y) - z\|_{X_h}. \tag{50}$$

Proof Using (38) and (48), we have

$$\begin{aligned} F_h(y, i_h x) &= i_h x - x - \mathsf{T}_h(y, S(i_h x)) + \mathsf{T}(y, S(x)) \\ &= i_h x - x + \mathsf{T}_h(0, S(x) - S(i_h x)) + (\mathsf{T} - \mathsf{T}_h)(y, S(x)), \end{aligned}$$

so that,

$$\begin{aligned} \frac{1}{3} \|F_h(y, i_h x)\|_{X_h}^2 &\le \|i_h x - x\|_{X_h}^2 \\ &\quad + \|\mathsf{T}_h(0, S(x) - S(i_h x))\|_{X_h}^2 + \|(\mathsf{T} - \mathsf{T}_h)(y, S(x))\|_{X_h}^2. \end{aligned}$$

Using standard interpolation results for the first term of the right hand side, Lemma 6 for the second and third terms, it follows that

$$\|F_h(y, i_h x)\|_{X_h}^2 \le C \Big(h^2 \|x\|_X^2 + \|S(x) - S(i_h x)\|_{L^2(L^2)}^2 + h^2 \|y\|_Y^2 + h^2 \|S(x)\|_{L^q(W^{1,r})}^2 \Big), \tag{51}$$

C being independent of h and y. Proceeding as in Lemma 1, we have

$$\|S(x)\|_{L^q(W^{1,r})}^2 \le C \|x\|_X^4, \tag{52}$$

C being independent of h and y. On the other hand, we also have

$$
\begin{aligned}
\frac{2\eta_p}{\lambda}\Big(S(x) &- S(i_h x)\Big) \\
&= \nabla u\, \sigma + \sigma (\nabla u)^T - (\nabla i_h u) i_h \sigma - i_h \sigma\, (\nabla i_h u)^T \\
&= \nabla(u - i_h u)\sigma + (\nabla i_h u)(\sigma - i_h \sigma) \\
&\quad + \sigma\, (\nabla(u - i_h u))^T + (\sigma - i_h \sigma)(\nabla i_h u)^T,
\end{aligned}
$$

so that, using a Cauchy–Schwarz inequality

$$
\|S(x) - S(i_h x)\|^2_{L^2(L^2)} \le C\|x - i_h x\|^2_{X_h}\left(\|\sigma\|^2_{L^\infty(L^\infty)} + \|\nabla i_h u\|^2_{L^2(L^\infty)}\right),
$$

C being independent of h and y. Standard interpolation results lead to

$$
\|\nabla i_h u\|_{L^\infty} \le \|\nabla u\|_{L^\infty} + \|\nabla(i_h u - u)\|_{L^\infty} \le C\|u\|_{W^{2,r}},
$$

C being independent of h and y. Thus, using again standard interpolation, we have

$$
\|S(x) - S(i_h x)\|^2_{L^2(L^2)} \le Ch^2\|x\|^4_X, \tag{53}
$$

C being independent of h and y. Finally, (52) and (53) in (51) yields (49).

Let us now prove (50). Let $z = (v, \tau) \in X_h$, let $\tilde{z} := (\tilde{v}, \tilde{\tau}) \in X_h$, we have

$$
\Big(D_x F_h(y, i_h x) - D_x F_h(y, z)\Big)\tilde{z} = -\mathsf{T}_h\Big(0, (DS(i_h x) - DS(z))\tilde{z}\Big).
$$

Using Lemma 6 we obtain

$$
\begin{aligned}
\|\, (D_x F_h(y, i_h x) &- D_x F_h(y, z))\, \tilde{z}\|_{X_h} \\
&\le C\|(DS(i_h x) - DS(z))\tilde{z}\|_{L^2(L^2)}, \tag{54}
\end{aligned}
$$

C being independent of h and y. We have

$$
\begin{aligned}
\frac{2\eta_p}{\lambda}\Big(DS(i_h x) &- DS(z)\Big)\tilde{z} = (\nabla(i_h u - v))\tilde{\tau} + \tilde{\tau}(\nabla(i_h u - v))^T \\
&+ \nabla\tilde{v}(i_h \sigma - \tau) + (i_h \sigma - \tau)(\nabla\tilde{v})^T.
\end{aligned}
$$

Then, using Cauchy–Schwarz inequality, there exists a constant C independent of h and y such that

$$
\begin{aligned}
\|(DS(i_h x) - DS(z))\tilde{z}\|_{L^2(L^2)} &\le C\Big(\|\nabla(i_h u - v)\|_{L^2(L^\infty)}\|\tilde{\tau}\|_{L^\infty(L^2)} \\
&+ \|\nabla\tilde{v}\|_{L^2(L^\infty)}\|i_h \sigma - \tau\|_{L^\infty(L^2)}\Big).
\end{aligned}
$$

A classical inverse inequality yields

$$\|(DS(i_hx) - DS(z))\tilde{z}\|_{L^2(L^2)} \leq \frac{C}{h}\Big(\|\nabla(i_hu - v)\|_{L^2(L^2)}\|\tilde{\tau}\|_{L^\infty(L^2)}$$
$$+ \|\nabla\tilde{v}\|_{L^2(L^2)}\|i_h\sigma - \tau\|_{L^\infty(L^2)}\Big),$$

so that we finally have

$$\|(DS(i_hx) - DS(z))\tilde{z}\|_{L^2(L^2)} \leq \frac{C}{h}\|i_hx - z\|_{X_h}\|\tilde{z}\|_{X_h}.$$

This last inequality in (54) yields (50).

Before proving existence of a solution to (47) we still need to check that $D_x F_h$ (y, i_hx) is invertible.

Lemma 8 *Let δ_0 be as in Theorem 1 part i), 1, let $y := (f, u_0, \sigma_0) \in Y$ with $\|y\|_Y \leq \delta_0$ and let $x(y) = (u(y), \sigma(y)) \in X$ be the solution of (38). Then, there exists $0 < \delta_1 \leq \delta_0$ such that for all $y \in Y$ with $\|y\|_Y \leq \delta_1$, for all $0 < h \leq 1$, we have*

$$\|D_x F_h(y, i_hx(y))^{-1}\|_{\mathcal{L}(X_h)} \leq 2.$$

Proof By definition of F_h, we have

$$D_x F_h(y, i_hx) = I - \mathsf{T}_h(0, DS(i_hx)),$$

so that we can write

$$D_x F_h(y, i_hx) = I - G_h \quad \text{with} \quad G_h := \mathsf{T}_h(0, DS(i_hx)).$$

If we prove that $\|G_h\|_{\mathcal{L}(X_h)} \leq 1/2$ for y sufficiently small, then $D_x F_h(y, i_hx)$ is invertible and $\|D_x F_h(y, i_hx)^{-1}\|_{\mathcal{L}(X_h)} \leq 2$.

Let $z := (v, \tau) \in X_h$. Using Lemma 6 we have

$$\|G_h(z)\|_{X_h} \leq C_1\|DS(i_hx)z\|_{L^2(L^2)},$$

C_1 being independent of y and h. Using the same arguments as in the proof of Lemma 7, we have

$$\frac{2\eta_p}{\lambda}\|DS(i_hx)z\|_{L^2(L^2)}$$
$$= \|(\nabla i_hu)\tau + \tau(\nabla i_hu)^T + (\nabla v)i_h\sigma + i_h\sigma(\nabla v)^T\|_{L^2(L^2)}$$
$$\leq 2\Big(\|\nabla i_hu\|_{L^2(L^\infty)}\|\tau\|_{L^\infty(L^2)} + \|\nabla i_h\sigma\|_{L^\infty(L^\infty)}\|\nabla v\|_{L^2(L^2)}\Big)$$
$$\leq C_2\Big(\|u\|_{L^2(W^{2,r})}\|\tau\|_{L^\infty(L^2)} + \|\nabla v\|_{L^2(L^2)}\|\sigma\|_{W^{1,q}(W^{1,r})}\Big),$$

C_2 being independent of y and h. Hence,

$$\|G_h(z)\|_{X_h} \le C_3 \|x\|_X \|z\|_{X_h},$$

where C_3 is independent of y and h. From Corollary 1, the mapping $y \to x(y)$ is continuous, thus if $||y||_Y$ is sufficiently small we have $\|x\|_X \le 1/(2C_3)$ so that

$$\|G_h(z)\|_{X_h} \le \frac{1}{2} \|z\|_{X_h}.$$

We can now prove existence of a solution to the finite element scheme (12) and convergence to the solution of (1)–(6).

Theorem 3 *Let δ_0 be as in Theorem 1 part i), 1, let $y := (f, u_0, \sigma_0) \in Y$ with $\|y\|_Y \le \delta_0$ and let $x(y) = (u(y), \sigma(y)) \in X$ be the solution of (38). Then, there exists $0 < \delta_2 \le \delta_0$ and $\zeta > 0$ such that for all $y \in Y$ with $\|y\|_Y \le \delta_2$, for all $0 < h \le 1$, there exists a unique $x_h(y) = (u_h(y), \sigma_h(y))$ in the ball of X_h centered at $i_h x(y)$ with radius ζh, satisfying*

$$F_h(y, x_h(y)) = 0.$$

Moreover, the mapping $y \to x_h(y)$ is continuous and there exists $C > 0$ independent of h and y such that the following a priori error estimate holds

$$\|x(y) - x_h(y)\|_{X_h} \le Ch. \tag{55}$$

Remark 10 The above Theorem still holds when the stabilization term in (12) is replaced by

$$\sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p} \left(-\mathrm{div}\,(2\eta_s \epsilon(u_h) + \sigma_h) + \nabla p_h - f, \nabla q_h \right)_K,$$

provided $0 < \alpha \le C_I$. Here C_I is the largest constant satisfying the following inverse estimate

$$C_I \sum_{K \in \mathcal{T}_h} h_K^2 \|\mathrm{div}\,\tau_h\|_{L^2(K)}^2 \le \|\tau_h\|_{L^2(\Omega)}^2 \qquad \forall \tau_h \in M_h.$$

Remark 11 Theorem 3 also holds when $y = (f, u_0, \sigma_0) \in Y$ is sufficiently small, with Y corresponding to Theorem 1 part (iii) thus defined by

$$Y = \{(f, u_0, \sigma_0) \in h^\mu(L^r) \times \mathcal{D}_{A_r} \times W^{1,r}$$
$$\text{such that} - \eta_s A_r u_0 + P_r f(0) + P_r \mathrm{div}\,\sigma_0 \in E_{1-1/q,q}\}.$$

This convergence result in little Hölder spaces rather than Sobolev (with respect to the time variable) will be used in a forthcoming paper [7] when considering a finite element scheme for the Hookean dumbbells model which is a stochastic model formally equivalent to Oldroyd-B.

In order to prove the above Theorem, we will use the following abstract result.

Lemma 9 (Theorem 2.1 of [15]) *Let Y and Z be two real Banach spaces with norms $\|.\|_Y$ and $\|.\|_Z$ respectively. Let $G : Y \to Z$ be a C^1 mapping and $v \in Y$ be such that $DG(v) \in \mathcal{L}(Y; Z)$ is an isomorphism. We introduce the notations*

$$\epsilon = \|G(v)\|_Z,$$

$$\gamma = \|DG(v)^{-1}\|_{\mathcal{L}(Y;Z)},$$

$$L(\alpha) = \sup_{x \in \overline{B}(v,\alpha)} \|DG(v) - DG(x)\|_{\mathcal{L}(Y;Z)},$$

with $\overline{B}(v, \alpha) = \{y \in Y; \|v - y\|_Y \leq \alpha\}$, and we are interested in finding $u \in Y$ such that

$$G(u) = 0. \tag{56}$$

We assume that $2\gamma L(2\gamma\epsilon) \leq 1$. Then Problem (56) has a unique solution u in the ball $\overline{B}(v, 2\gamma\epsilon)$ and, for all $x \in \overline{B}(v, 2\gamma\epsilon)$, we have

$$\|x - u\|_Y \leq 2\gamma \|G(x)\|_Z.$$

Proof (Proof of Theorem 3) We apply Lemma 9 with $Y = X_h$, $Z = X_h$, $G = F_h$ and $v = i_h x(y)$. According to Lemma 7 there exists a constant C_1 independent of y and h such that

$$\epsilon = \|F_h(y, i_h x(y))\|_{X_h} \leq C_1 h \left(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2 \right).$$

According to Lemma 8, for $\|y\|_Y$ sufficiently small

$$\gamma = \|D_x F_h(y, i_h x(y))\|_{\mathcal{L}(X_h)} \leq 2.$$

According to Lemma 7, there is a constant C_2 independent of y and h such that

$$L(\beta) = \sup_{z \in \overline{B}(i_h x(y), \beta)} \|DF_h(i_h x(y)) - DF_h(z)\|_{\mathcal{L}(X_h)} \leq \frac{C_2}{h} \beta.$$

Hence, we have

$$2\gamma L(2\gamma\epsilon) \leq 2.2 \frac{C_2}{h} \left(2.2 C_1 h \left(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2 \right) \right)$$

$$= 16 C_1 C_2 \left(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2 \right).$$

Using the continuity of the mapping $y \to x(y)$, there exists $0 < \delta_2 \leq \delta_0$ such that for all $y \in Y$ with $\|y\|_Y \leq \delta_2$, then

$$\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2 \leq \frac{1}{32 C_1 C_2}$$

so that $2\gamma L(2\gamma\epsilon) \leq 1/2 < 1$ and Lemma 9 applies. There exists a unique $x_h(y)$ in the ball $\overline{B}(i_h x(y), 2\gamma\epsilon)$ such that

$$F_h(y, x_h(y)) = 0$$

and we have

$$\|i_h x(y) - x_h(y)\|_{X_h} \leq 4C_1 h\Big(\|y\|_Y + \|x(y)\|_X + \|x(y)\|_X^2\Big)$$
$$\leq \frac{4C_1 h}{32C_1 C_2} = \frac{1}{8C_2} h.$$

It suffices to use the triangle inequality

$$\|x(y) - x_h(y)\|_{X_h} \leq \|x(y) - i_h x(y)\|_{X_h} + \|i_h x(y) - x_h(y)\|_{X_h},$$

and standard interpolation results to obtain (55). The fact that the mapping $y \to x_h(y)$ is continuous is a direct consequence of the implicit function theorem used to prove Lemma 9.

A Appendix: pertubated abstract Cauchy problem

The aim of this appendix is to state some existence results taken from [18–20,48–50] and used in this paper on the abstract Cauchy problem

$$\dot{u}(t) = Au(t) + f(t) \quad t \in (0, T), \quad u(0) = u_0. \tag{57}$$

where $f : [0, T] \to E$, $u_0 \in E$ and $(E, \|.\|)$ is a real Banach space. Then, these results will be extended to the case when a convolution term $k * Au$ is added to (57). Here the convolution product $*$ is defined for $f, g \in L^1(0, T)$ by

$$(f * g)(t) = \int_0^t f(t - s)g(s)ds.$$

Remark 12 From [48] we have: if $f \in L^q(0, T; E)$, $g \in L^1(0, T)$ then $f * g \in L^q(0, T; E)$ and

$$\|f * g\|_{L^q(E)} \leq \|g\|_{L^1(0,T)}\|f\|_{L^q(E)}.$$

We have the following result in Hölder spaces (see [18,19]).

Theorem 4 *Let $(E, \|.\|)$ be a Banach space. Let A be a closed, densely defined operator in E with domain $\mathcal{D}_{A_r}$, generator of an analytic semigroup on E. Let $0 < \mu < 1$ and let $f \in \mathcal{C}^\mu(E)$. Assume that the compatibility conditions $u_0 \in \mathcal{D}_{A_r}$ and $Au_0 + f(0) \in E_{\mu,\infty} := \left(E, \mathcal{D}_{A_r}\right)_{\mu,\infty}$ holds. Then there exists a unique solution u of problem (57) in $\mathcal{C}^{1+\mu}(E) \cap \mathcal{C}^\mu(\mathcal{D}_{A_r})$ and u satisfies*

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}f(s)ds, \quad t \in (0, T).$$

Moreover there exists a constant C such that

$$\|\dot{u}\|_{C^\mu(E)} + \|Au\|_{C^\mu(E)} \leq C\left(\|f - f(0)\|_{C^\mu(E)} + \|Au_0 + f(0)\|_{E_{\mu,\infty}}\right).$$

A similar result also holds in little Hölder spaces (see again [18,19]).

Theorem 5 *Let $(E, \|.\|_E)$ be a Banach space. Let A be a closed, densely defined operator in E with domain $\mathcal{D}_{A_r}$, generator of an analytic semigroup on E. Let $0 < \mu < 1$ and let $f \in h^\mu(E)$. Assume the compatibility conditions $u_0 \in \mathcal{D}_{A_r}$ and $Au_0 + f(0) \in \overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}$ where $E_{\mu,\infty} := \left(E, \mathcal{D}_{A_r}\right)_{\mu,\infty}$. Then there exists a unique solution u of problem (57) in $h^{1+\mu}(E) \cap h^\mu(\mathcal{D}_{A_r})$ and u satisfies*

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}f(s)ds, \quad t \in (0, T).$$

Moreover there exists a constant C such that

$$\|\dot{u}\|_{h^\mu(E)} + \|Au\|_{h^\mu(E)} \leq C\left(\|f - f(0)\|_{h^\mu(E)} + \|Au_0 + f(0)\|_{\overline{\mathcal{D}_{A_r}}^{E_{\mu,\infty}}}\right).$$

Before stating some existence results for Sobolev spaces, we have to introduce the maximal regularity property (MRp).

Definition 1 Let $1 < q < \infty$. The operator A possesses the maximal L^q-regularity property (MRp) if for $u_0 = 0$ and any $f \in L^q(E)$, there exists a unique solution u of (57) in $W^{1,q}(E) \cap L^q(\mathcal{D}_A)$ and there exists a constant C such that

$$\|\dot{u}\|_{L^q(E)} + \|Au\|_{L^q(E)} \leq C\|f\|_{L^q(E)}.$$

Remark 13 Let $1 < q_0 < \infty$. If A possesses the maximal L^{q_0}-regularity property, then $\beta A + \omega I$ possesses the maximal L^q-regularity property for all $1 < q < \infty$, for all $\beta > 0$ and for all $w \in \mathbb{R}$ (see [20,50]).

In general, a maximal regularity result does not hold for Sobolev spaces. It has to be assumed. The following result can be found in [19].

Lemma 10 *Let $(E, \|.\|_E)$ be a Banach space. Let A be a closed, densely defined operator in E with domain $\mathcal{D}_A$, generator of an analytic semigroup e^{tA} on E. Let $1 < q < \infty$, assume A satisfies MRp, $f \in L^q(E)$ and $u_0 \in E_{1-1/q,q} := (E, \mathcal{D}_A)_{1-\frac{1}{q},q}$. Then there exists a unique solution u of problem (57) in $W^{1,q}(E) \cap L^q(\mathcal{D}_A)$ and u satisfies*

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s)ds, \quad t \in (0, T).$$

Moreover, there exists a constant C such that

$$\|\dot{u}\|_{L^q(E)} + \|Au\|_{L^q(E)} \le C \left(\|u_0\|_{E_{1-1/q,q}} + \|f\|_{L^q(E)} \right).$$

Assuming more regularity of the data with respect to the time variable, and compatibility conditions at initial time, the following regularity result can be obtained.

Corollary 4 *Let $(E, \|.\|_E)$ be a Banach space. Let A be a closed, densely defined operator in E with domain $\mathcal{D}_A$, generator of an analytic semigroup on E. Let $1 < q < \infty$, assume A satisfies MRp, let $f \in W^{1,q}(E)$ and $u_0 \in E$. Assume that the compatibility conditions $u_0 \in \mathcal{D}_A$ and $Au_0 + f(0) \in E_{1-1/q,q} := (E, \mathcal{D}_A)_{1-\frac{1}{q},q}$ hold. Then the solution u of (57) satisfies $\dot{u} \in W^{1,q}(E) \cap L^q(\mathcal{D}_A)$. Moreover, there exists a constant C such that*

$$\|\ddot{u}\|_{L^q(E)} + \|A\dot{u}\|_E \le C \left(\|Au_0 + f(0)\|_{E_{1-1/q,q}} + \|\dot{f}\|_{L^q(E)} \right).$$

We now perturb the first equation of (57) by adding a term of the form $k * Au$. In order to obtain an existence result, this following technical Lemma will be useful.

Lemma 11 *Given $\beta \neq 0$, $m \ge 1$ and $k \in W^{m,1}(0, T)$, there exists an unique $b \in W^{m+1,1}(0, T)$ such that*

$$\beta b + k * b = 1.$$

Proof We recall a result given in [44], Theorem 1.4 p.46. For all $p \ge 1$, there exists an unique $r : W^{m,p}(0, T) \to W^{m,p}(0, T)$ such that for all $a \in W^{m,p}(0, T)$

$$r(a) + a * r(a) = a.$$

Then, taking $a = \beta^{-1}k$ in the equation above, the unique solution $S(b_0) \in W^{m,1}(0, T)$ such that $S(b_0) + \beta^{-1}k * S(b_0) = 1$ is given by

$$S(b_0) = 1 - r(\beta^{-1}k) * 1.$$

Thus we obtain

$$b(t) = \beta^{-1}S(b_0) = \beta^{-1} - \beta^{-1}r(\beta^{-1}k) * 1$$

and

$$\dot{b}(t) = -\beta^{-1} r(\beta^{-1}k) \quad \text{in } L^1(0, T).$$

Since $r(\beta^{-1}k) \in W^{m,1}(0, T)$, it follows $b \in W^{m+1,1}(0, T)$.

Lemma 12 *Let $(E, \|.\|_E)$ be a Banach space. Let A be a closed, densely defined operator in E with domain $\mathcal{D}_A$, generator of an analytic semigroup on E. Let $1 < q < \infty$, assume A satisfies MRp, let $\beta > 0$, $\gamma \in \mathbb{R}$ and $a \in L^1(0, T)$. Let $f \in L^q(E)$, $u_0 \in E_{1-1/q,q} := (E, \mathcal{D}_A)_{1-\frac{1}{q},q}$, then there exists a unique $u \in W^{1,q}(E) \cap L^q(\mathcal{D}_A)$ satisfying*

$$\dot{u} = \beta A u + \gamma u + a * u + f, \quad u(0) = u_0.$$

Moreover, there exists a constant C such that

$$\|\dot{u}\|_{L^q(E)} + \|Au\|_{L^q(E)} \le C \left(\|f\|_{L^q(E)} + \|u_0\|_{E_{1-1/q,q}} \right). \tag{58}$$

Proof Let $B := \beta A + \gamma$, since A satisfies the MRp using Remark 13 it follows B satisfies the MRp and there exists a constant such that

$$\|Au\|_{L^q(E)} + \|u\|_{L^q(E)} \le C \left(\|Bu\|_{L^q(E)} + \|u\|_{L^q(E)} \right), \tag{59}$$

for all $u \in \mathcal{D}_A$. Therefore, it remains to prove for given $f \in L^q(E)$ and $u_0 \in E_{1-1/q,q}$ there exists an unique $z \in W^{1,q}(E) \cap L^q(\mathcal{D}_A)$ such that

$$z = e^{tB} u_0 + \int_0^t e^{(t-s)B} f(s) ds + \int_0^t e^{(t-s)B} a * z(s) ds. \tag{60}$$

Lemma 10 ensures $z_0 := e^{tB} u_0 + \int_0^t e^{(t-s)B} f(s) ds \in W^{1,q}(E) \cap L^q(\mathcal{D}_A)$. We rewrite (60) as a fixed point problem. Given $z_0 \in Z := L^q(E)$ and let $F : Z \to Z$ defined for all $z \in Z$ by

$$F(z) := v,$$

where, $v \in Z$ satisfies

$$\dot{v} = Bv + a * z, \quad v(0) = 0.$$

Let us notice F is well defined using Remark 12 and Lemma 10. Then (60) becomes

$$z = z_0 + F(z).$$

We will show there exists $n > 0$ such that

$$\|F^n\|_{\mathcal{L}(Z)} < 1. \tag{61}$$

Lemma 10 again ensures there exists a constant C such that

$$\|v\|_{L^q(E)} \leq C\|a * u\|_{L^q(E)} \leq C\|a\|_{L^1(0,T)}\|u\|_{L^q(E)}.$$

Denoting by $c^{(n)} := \underbrace{c * \ldots * c}_{n \text{ times}}$ for all $c \in L^1(0, T)$, it follows

$$\|F^n\|_{\mathcal{L}(Z)} \leq C^n\|a^{(n)}\|_{L^1(0,T)}. \tag{62}$$

Since

$$\|c * c\|_{L^1(0,T)} \leq \|c\|_{L^\infty(0,T)} * \|c\|_{L^\infty(0,T)} = \|c\|_{L^\infty(0,T)}^2 1 * 1 = \|c\|_{L^\infty(0,T)}^2 T,$$

for all $c \in L^\infty(0, T)$ we find

$$\|c^{(n)}\|_{L^1(0,T)} \leq \|c\|_{L^\infty(0,T)}^n \frac{T^{n-1}}{(n - 1)!}, \quad \forall c \in L^\infty(0, T).$$

Using the above inequality in (62), it follows

$$\|F^n\|_{\mathcal{L}(Z)} \leq \frac{T^{n-1}}{(n - 1)!} C^n \|a\|_{L^\infty(0,T)}^n$$

which tends to 0 when n goes to infinity. Thus (61) is proved and a fixed point theorem (see Theorem 4.4.1 of [16]) ensures the existence of an unique $z \in Z$ satisfying (60) and there exists a constant constant C such that

$$\|z\|_{L^q(E)} \leq C\|z_0\|_{L^q(E)}. \tag{63}$$

The fact that $z \in W^{1,q}(E) \cap L^q(E)$ is a direct consequence of (60) since $z_0, a * z \in W^{1,q}(E) \cap L^q(E)$. It remains to prove the estimation (58). Going back to (60) and using Lemma 10 again, there exists a constant C such that

$$\|\dot{z}\|_{L^q(E)} + \|Bu\|_{L^q(E)}$$
$$\leq C \left(\|u_0\|_{E_{1-1/q,q}} + \|f\|_{L^q(E)} + \|a * u\|_{L^q(E)}\right). \tag{64}$$

Using (63) we obtain

$$\|a * z\|_{L^q(E)} \leq \|a\|_{L^1(0,T)}\|z\|_{L^q(E)} \leq C\|a\|_{L^1(0,T)}\|z_0\|_{L^q(E)},$$

which coupled with (59) and (64) proves (70).

Corollary 5 *Let $(E, \|.\|_E)$ be a Banach space. Let A be a closed, densely defined operator in E with domain $\mathcal{D}_A$, generator of an analytic semigroup on E. Let $1 < q < \infty$, assume A satisfies MRp, let $\beta > 0$, $k \in W^{1,1}(0, T)$, $f \in L^q(E)$ and*

$u_0 \in E_{1-1/q,q} := (E, \mathcal{D}_A)_{1-\frac{1}{q},q}$, *then there exists a unique* $u \in W^{1,q}(E) \cap L^q(\mathcal{D}_A)$
satisfying

$$\dot{u} = \beta A u + k * A u + f, \quad u(0) = u_0. \tag{65}$$

Moreover, there exists a constant C such that

$$\|\dot{u}\|_{L^q(E)} + \|Au\|_{L^q(E)} \leq C \left(\|f\|_{L^q(E)} + \|u_0\|_E \right). \tag{66}$$

Proof Since $k \in W^{1,1}(0, T)$, Lemma 11 ensures the existence of a $b \in W^{2,1}(0, T)$
such that

$$\beta b + k * b = 1. \tag{67}$$

Moreover $b(0) = \beta^{-1}$. Convolving the equation for u in (65) and using the equation
above, we have

$$b * \dot{u} = 1 * Au + b * f.$$

Differentiating with respect to time the equation above, using $b(0) = \beta^{-1}$, we obtain

$$\beta^{-1}\dot{u} + \dot{b} * \dot{u} = Au + \beta^{-1}f + \dot{b} * f. \tag{68}$$

Noticing that

$$\dot{b} * \dot{u} + \dot{b}u_0 = \frac{d}{dt}\left(\dot{b} * u\right) = \dot{b}(0)u + \ddot{b} * u,$$

the equation (68) becomes

$$\dot{u} = \beta Au - \beta \dot{b}(0)u + f + \beta \dot{b} * f + \beta \dot{b}u_0 - \beta \ddot{b} * u.$$

Differentiating equation (67) and since $\dot{b} \in \mathcal{C}^0([0, T])$, $k \in W^{1,1}(0, T)$ we find
$\dot{b}(0) = -\beta^{-2}k(0)$. Finally, (65) reduce to

$$\dot{u} = \beta Au + f + \beta \dot{b} * f + \beta \dot{b}u_0 - \beta \ddot{b} * u + \beta^{-1}k(0)u, \quad u(0) = u_0.$$

The Lemma 12 completes the proof. $\qquad \square$

The previous Corollary also holds in little Hölder spaces.

Lemma 13 *Let* $(E, \|.\|_E)$ *be a Banach space. Let A be a closed, densely defined oper-*
ator in E with domain $\mathcal{D}_{A_r}$, *generator of an analytic semigroup on E. Let* $0 < \mu < 1$.
Let $k \in W^{1, \frac{1}{1-\mu}}(0, T)$, $f \in h^\mu([0, T]; E)$ *and* $u_0 \in \mathcal{D}(A)$ *satisfying* $Au_0 + f(0) \in$
$\overline{\mathcal{D}(A)}^{(E, \mathcal{D}_A)_{\mu,\infty}}$, *then there exists a unique* $u \in h^{1+\mu}([0, T]; E) \cap h^\mu([0, T]; \mathcal{D}(A))$
satisfying

$$\dot{u} = \beta Au + k * Au + f, \quad u(0) = u_0.$$

Proof The proof use same arguments of the proof for Corollary 5. It has to be slightly modified in the two following senses. Remark 12, has be replace by the affirmation: let $0 < \mu < 1$, for all $g \in L^{\frac{1}{1-\mu}}(0, T)$, for all $f \in h^\mu([0, T]; E)$ there exists a constant C such that

$$\|f * g\|_{h^\mu(E)} \leq C\|g\|_{L^{\frac{1}{1-\mu}}(0,T)} \|f\|_{h^\mu(E)}. \tag{69}$$

In the proof for Lemma 12, relation (62) does not holds in $h^\kappa(E)$ but using some properties of the operator $L \in \mathcal{L}(h^\mu(E))$ defined for $v \in h^\mu(E)$ by $Lv := a * v$ and Theorem 3 p.211 and Theorem 4 p. 212 in [53] the same conclusion follows.

Lemma 14 *Let $(E, \|.\|_E)$ be a Banach space. Let A be a closed, densely defined operator in E with domain $\mathcal{D}_A$, generator of an analytic semigroup on E. Let $1 < q < \infty$, assume A satisfies MRp, let $\beta > 0$ and $k \in W^{1,1}(0, T)$. Let $f \in W^{1,q}(E)$, $u_0 \in \mathcal{D}_A$ such that $\beta A u_0 + f(0) \in E_{1-1/q,q} := (E, \mathcal{D}_A)_{1-\frac{1}{q},q}$, then there exists a unique $u \in W^{2,q}(E) \cap W^{1,q}(\mathcal{D}_A)$ satisfying*

$$\dot{u} = \beta A u + k * A u + f, \quad u(0) = u_0.$$

Moreover, there exists a constant C such that

$$\|\ddot{u}\|_{L^q(E)} + \|A\dot{u}\|_{L^q(E)}$$
$$\leq C \left(\|\dot{f}\|_{L^q(E)} + \|Au_0\|_E + \|Au_0 + f(0)\|_{E_{1-1/q,q}} \right). \tag{70}$$

Proof Let u be the unique solution in $W^{1,q}(E) \cap L^q(\mathcal{D}_A)$ satisfying

$$\dot{u} = \beta A u + k * A u + f, \quad u(0) = u_0.$$

Let us define $z \in W^{1,q}(E) \cap L^q(\mathcal{D}_A)$ such that

$$\dot{z} = \beta A z + k A u_0 + k * A z + \dot{f} \quad z(0) = \beta A u_0 + f(0). \tag{71}$$

Corollary 5 ensures z is well defined since $\dot{f} \in L^q(E)$ and since $\beta A u_0 + f(0) \in E_{1-1/q,q}$. Moreover there exists a constant C such that

$$\|\dot{z}\|_{L^q(E)} + \|Az\|_{L^q(E)}$$
$$\leq C \left(\|Au_0 + f(0)\|_{E_{1-1/q,q}} + \|\dot{f}\|_{L^q(E)} + \|kAu_0\|_{L^q(E)} \right). \tag{72}$$

Let $v \in W^{2,q}(E) \cap W^{1,q}(\mathcal{D}_A)$ defined by

$$v(t) := u_0 + \int_0^t z(s)ds. \tag{73}$$

We will show $v = u$. Let us recall that if A is a bounded operator we have

$$\int_0^t Ay(s)ds = A \int_0^t y(s)ds, \quad \forall y \in L^1(0, t'; \mathcal{D}_A), \; \forall t' \in [0, t].$$

From (71) and since A is bounded, it follows

$$z(t) = \beta Au_0 + f(0) + \beta A \int_0^t z(s)ds + f(t) - f(0)$$

$$+ 1 * kAu_0 + k * A \int_0^T z(s)ds.$$

Using the definition of v (73) we have

$$\begin{cases} \dot{v} = \beta A(v - u_0) + \beta Au_0 + f(t) + (1 * k)Au_0 + k * A(v - u_0), \\ v_0 = u_0. \end{cases}$$

Thus the uniqueness of the solution ensured by Corollary 5 proves $v = u$ or $z = \dot{u}$. The estimate (70) is a direct consequence of (72).

B Appendix: Proof of Lemma 6

A priori, (37) has to be understood in a weak sense with respect to the time variable: find $(\tilde{u}_h, \tilde{\sigma}_h, \tilde{q}_h) \in X_h$ such that

$$-\rho \int_0^T \left(\tilde{u}_h, \frac{\partial v_h}{\partial t} \right) + \rho \left(u_0, v_h(0) \right) + 2\eta_s \int_0^T \left(\epsilon(\tilde{u}_h), \epsilon(v_h) \right)$$

$$- \int_0^T \left(\tilde{p}_h, \operatorname{div} v_h \right) + \int_0^T \left(\tilde{\sigma}_h, \epsilon(v_h) \right) - \int_0^T \left(f, v_h \right)$$

$$+ \int_0^T \left(\operatorname{div} \tilde{u}_h, q_h \right) + \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p} \int_0^T \left(\nabla \tilde{p}_h, \nabla q_h \right)_K$$

$$+ \frac{1}{2\eta_p} \int_0^T \left(\tilde{\sigma}_h, \tau_h \right) - \frac{\lambda}{2\eta_p} \int_0^T \left(\tilde{\sigma}_h, \frac{\partial \tau_h}{\partial t} \right) + \frac{\lambda}{2\eta_p} \left(\sigma_0, \tau_h(0) \right)$$

$$- \left(\epsilon(\tilde{u}_h), \tau_h \right) - \frac{\lambda}{2\eta_p} \int_0^T \left(g, \tau_h \right) = 0,$$

for all $v_h \in H^1(V_h)$ such that $v_h(T) = 0$, for all $\tau_h \in H^1(M_h)$ such that $\tau_h(T) = 0$ and for all $q_h \in L^2(Q_h)$. A solution of the above equation satisfies $u_h \in H^1(V_h)$ and $\sigma_h \in H^1(M_h)$.

We then claim that problem (37) has a unique solution $(\tilde{u}_h, \tilde{\sigma}_h, \tilde{p}_h) \in X_h$. Indeed, when writing $(\tilde{u}_h, \tilde{p}_h, \tilde{\sigma}_h)$ with respect to a finite basis of $V_h \times Q_h \times M_h$, problem (37) can be expressed as a linear differential system. The degrees of freedom corresponding to the pressure can be eliminated. By a classical result of ODE, the resulting differential system has a unique solution, each components being in $H^1(0, T)$.

In order to prove (45), we choose $v_h = \tilde{u}_h(t)$, $\tau_h = \tilde{\sigma}_h(t)$, $q_h = \tilde{p}_h(t)$ in (37) and integrate from $t = 0$ to s, with $0 \le s \le T$. We obtain

$$\frac{\rho}{2}\|\tilde{u}_h(s)\|^2_{L^2(\Omega)} + \frac{\lambda}{4\eta_p}\|\tilde{\sigma}_h(s)\|^2_{L^2(\Omega)} + 2\eta_s \int_0^s \|\epsilon(\tilde{u}_h)\|^2_{L^2(\Omega)}$$

$$+ \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p} \int_0^s \|\nabla \tilde{p}_h\|^2_{L^2(K)} + \frac{1}{2\eta_p} \int_0^s \|\tilde{\sigma}_h\|^2_{L^2(\Omega)}$$

$$= \frac{\rho}{2}\|\tilde{u}_h(0)\|^2_{L^2(\Omega)} + \frac{\lambda}{4\eta_p}\|\tilde{\sigma}_h(0)\|^2_{L^2(\Omega)} + \int_0^s (f, \tilde{u}_h) + \int_0^s (g, \tilde{\sigma}_h).$$

Using Young and Poincaré inequalities, there exists a constant C such that

$$\frac{\rho}{2}\|\tilde{u}_h(s)\|^2_{L^2(\Omega)} + \frac{\lambda}{4\eta_p}\|\tilde{\sigma}_h(s)\|^2_{L^2(\Omega)} + \eta_s \int_0^s \|\epsilon(\tilde{u}_h)\|^2_{L^2(\Omega)}$$

$$+ \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p} \int_0^s \|\nabla \tilde{p}_h\|^2_{L^2(K)} + \frac{\lambda}{2\eta_p} \int_0^s \|\tilde{\sigma}_h\|^2_{L^2(\Omega)}$$

$$\le \frac{\rho}{2}\|\tilde{u}_h(0)\|^2_{L^2(\Omega)} + \frac{\lambda}{4\eta_p}\|\tilde{\sigma}_h(0)\|^2_{L^2(\Omega)} + C \left(\int_0^s \|f\|^2_{L^2(\Omega)} + \int_0^s \|g\|^2_{L^2(\Omega)} \right).$$

It suffices to note that

$$\|\tilde{u}_h(0)\|_{L^2(\Omega)} = \|i_h u_0\|_{L^2(\Omega)} \le \|u_0\|_{L^2(\Omega)} \le C\|u_0\|_{E_{1-1/q,q}},$$

$$\|\tilde{\sigma}_h(0)\|_{L^2(\Omega)} = \|i_h \sigma_0\|_{L^2(\Omega)} \le \|\sigma_0\|_{L^2(\Omega)} \le C\|\sigma_0\|_{W^{1,r}},$$

to obtain (45).

We now prove the convergence result (46). Let

$$e_u = \tilde{u} - \tilde{u}_h = \Pi_u + C_u, \qquad \Pi_u = \tilde{u} - i_h \tilde{u}, \qquad C_u = i_h \tilde{u} - \tilde{u}_h,$$

$$e_\sigma = \tilde{\sigma} - \tilde{\sigma}_h = \Pi_\sigma + C_\sigma, \qquad \Pi_\sigma = \tilde{\sigma} - i_h \tilde{\sigma}, \qquad C_\sigma = i_h \tilde{\sigma} - \tilde{\sigma}_h,$$

$$e_p = \tilde{p} - \tilde{p}_h = \Pi_p + C_p, \qquad \Pi_p = \tilde{p} - i_h \tilde{p}, \qquad C_p = i_h \tilde{p} - \tilde{p}_h,$$

where $(\tilde{u}_h, \tilde{p}_h, \tilde{\sigma}_h)$ solve (37) and $(\tilde{u}, \tilde{p}, \tilde{\sigma})$ solve (39)–(44). Using the triangle inequality we have

$$\|e_u, e_\sigma\|_{X_h} \leq \|\Pi_u, \Pi_\sigma\|_{X_h} + \|C_u, C_\sigma\|_{X_h}.$$

Using classical interpolation results, we obtain

$$\|\Pi_u, \Pi_\sigma\|_{X_h} \leq Ch\|u, \sigma\|_X.$$

We now estimate $\|C_u, C_\sigma\|_{X_h}$. The solution of (39)–(44) satisfies

$$\rho\left(\frac{\partial \tilde{u}}{\partial t}, v_h\right) + 2\eta_s\left(\epsilon(\tilde{u}), \epsilon(v_h)\right) - \left(\tilde{p}, \text{div } v_h\right) + \left(\tilde{\sigma}, \epsilon(v_h)\right) - \left(f, v_h\right)$$

$$+ \left(\text{div } \tilde{u}, q_h\right) + \frac{1}{2\eta_p}\left(\tilde{\sigma}, \tau_h\right) + \frac{\lambda}{2\eta_p}\left(\frac{\partial \tilde{\sigma}}{\partial t}, \tau_h\right) - \left(\epsilon(\tilde{u}), \tau_h\right) - \left(g, \tau_h\right) = 0$$

for all $(v_h, \tau_h, q_h) \in V_h \times M_h \times Q_h$. Subtracting (37) to the above equation, it follows that

$$\rho\left(\frac{\partial e_u}{\partial t}, v_h\right) + 2\eta_s\left(\epsilon(e_u), \epsilon(v_h)\right) - \left(e_p, \text{div } v_h\right) + \left(e_\sigma, \epsilon(v_h)\right)$$

$$+ \left(\text{div } e_u, q_h\right) + \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\left(\nabla e_p - \nabla \tilde{p}, \nabla q_h\right)_K$$

$$+ \frac{1}{2\eta_p}\left(e_\sigma, \tau_h\right) + \frac{\lambda}{2\eta_p}\left(\frac{\partial e_\sigma}{\partial t}, \tau_h\right) - \left(\epsilon(e_u), \tau_h\right) = 0, \tag{74}$$

for all $(v_h, \tau_h, q_h) \in V_h \times M_h \times Q_h$. On the other hand, from the definition of C_u, C_σ and C_p, we have

$$\rho\left(\frac{\partial C_u}{\partial t}, C_u\right) + 2\eta_s\left(\epsilon(C_u), \epsilon(C_u)\right) - \left(C_p, \text{div } C_u\right) + \left(C_\sigma, \epsilon(C_u)\right)$$

$$+ \left(\text{div } C_u, C_p\right) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\left(\nabla C_p, \nabla C_p\right)_K$$

$$+ \frac{1}{2\eta_p}\left(C_\sigma, C_\sigma\right) + \frac{\lambda}{2\eta_p}\left(\frac{\partial C_\sigma}{\partial t}, C_\sigma\right) - \left(\epsilon(C_u), C_\sigma\right)$$

$$= \rho\left(\frac{\partial(e_u - \Pi_u)}{\partial t}, C_u\right) + 2\eta_s\left(\epsilon(e_u - \Pi_u), \epsilon(C_u)\right) - \left(e_p - \Pi_p, \text{div } C_u\right)$$

$$+ \left(e_\sigma - \Pi_\sigma, \epsilon(C_u)\right) + \left(\text{div }(e_u - \Pi_u), C_p\right) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\left(\nabla(e_p - \Pi_p), \nabla C_p\right)_K$$

$$+ \frac{1}{2\eta_p}\left((e_\sigma - \Pi_\sigma), C_\sigma\right) + \frac{\lambda}{2\eta_p}\left(\frac{\partial(e_\sigma - \Pi_\sigma)}{\partial t}, C_\sigma\right) - \left(\epsilon(e_u - \Pi_u), C_\sigma\right).$$

$$(75)$$

From the definition of i_h (the L^2 projection onto the finite element spaces), we obviously have

$$\left(\frac{\partial \Pi_u}{\partial t}, C_u\right) = 0, \qquad \left(\Pi_\sigma, C_\sigma\right) = 0, \qquad \left(\frac{\partial \Pi_\sigma}{\partial t}, C_\sigma\right) = 0,$$

so that, using (74), (75) yields

$$\rho\left(\frac{\partial C_u}{\partial t}, C_u\right) + 2\eta_s\left(\epsilon(C_u), \epsilon(C_u)\right) + \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\left(\nabla C_p, \nabla C_p\right)_K$$

$$+ \frac{1}{2\eta_p}\left(C_\sigma, C_\sigma\right) + \frac{\lambda}{2\eta_p}\left(\frac{\partial C_\sigma}{\partial t}, C_\sigma\right)$$

$$= 2\eta_s\left(\epsilon(\Pi_u), \epsilon(C_u)\right) + \left(\Pi_p, \text{div } C_u\right) - \left(\Pi_\sigma, \epsilon(C_u)\right)$$

$$- \left(\text{div }(\Pi_u), C_p\right) - \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\left(\nabla \Pi_p, \nabla C_p\right)_K + \left(\epsilon(\Pi_u), C_\sigma\right)$$

$$+ \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p}\left(\nabla \tilde{p}, \nabla C_p\right)_K$$

$$= I_1 + \cdots + I_7.$$

$$(76)$$

It now remains to bound the terms $I_1, \ldots, I_7$ in the above equality. Using Cauchy–Schwarz and Young's inequalities, we have

$$
\begin{aligned}
I_1 &= 2\eta_s \left(\epsilon(\Pi_u), \epsilon(C_u) \right) \\
&\leq 2\eta_s \|\epsilon(\Pi_u)\|_{L^2(\Omega)} \|\epsilon(C_u)\|_{L^2(\Omega)} \\
&\leq 3\eta_s \|\epsilon(\Pi_u)\|_{L^2(\Omega)}^2 + \frac{\eta_s}{3} \|\epsilon(C_u)\|_{L^2(\Omega)}^2 .
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
I_2 &= \left(\Pi_p, \operatorname{div} C_u \right) \leq \frac{3}{4\eta_s} \|\Pi_p\|_{L^2(\Omega)}^2 + \frac{\eta_s}{3} \|\operatorname{div} C_u\|_{L^2(\Omega)}^2 \\
&\leq \frac{3}{4\eta_s} \|\Pi_p\|_{L^2(\Omega)}^2 + \frac{\eta_s}{3} \|\epsilon(C_u)\|_{L^2(\Omega)}^2 ,
\end{aligned}
$$

and

$$
I_3 = -\left(\Pi_\sigma, \epsilon(C_u) \right) \leq \frac{3}{4\eta_s} \|\Pi_\sigma\|_{L^2(\Omega)}^2 + \frac{\eta_s}{2} \|\epsilon(C_u)\|_{L^2(\Omega)}^2 .
$$

An integration by parts yields, since $\Pi_u = 0$ on $\partial\Omega$

$$
\begin{aligned}
I_4 &= \left(\operatorname{div}(\Pi_u), C_p \right) = -\left(\Pi_u, \nabla C_p \right) = -\sum_{K \in \mathcal{T}_h} \left(\Pi_u, \nabla C_p \right)_K \\
&\leq \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{12\eta_p} \|\nabla C_p\|_{L^2(K)}^2 + \sum_{K \in \mathcal{T}_h} \frac{3\eta_p}{\alpha h_K^2} \|\Pi_u\|_{L^2(K)}^2 .
\end{aligned}
$$

Again, Cauchy–Schwarz and Young's inequalities yield

$$
\begin{aligned}
I_5 &= -\sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p} \left(\nabla \Pi_p, \nabla C_p \right)_K \\
&\leq \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{12\eta_p} \|\nabla C_p\|_{L^2(K)}^2 + \frac{3\alpha h^2}{4\eta_p} \|\nabla \Pi_p\|_{L^2(\Omega)}^2 ,
\end{aligned}
$$

and

$$
I_6 = \left(\epsilon(\Pi_u), C_\sigma \right) \leq \eta_p \|\epsilon(\Pi_u)\|_{L^2(\Omega)}^2 + \frac{1}{4\eta_p} \|C_\sigma\|_{L^2(\Omega)}^2 .
$$

Finally, we have

$$
\begin{aligned}
I_8 &= \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p} \left(\nabla \tilde{p}, \nabla C_p \right)_K \\
&\leq \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{12\eta_p} \|\nabla C_p\|_{L^2(K)}^2 + \frac{3\alpha h^2}{4\eta_p} \|\nabla \tilde{p}\|_{L^2(\Omega)}^2 .
\end{aligned}
$$

The above estimates of $I_1, \ldots, I_7$ in (76) yield

$$
\begin{aligned}
&\rho\left(\frac{\partial C_u}{\partial t}, C_u\right) + \frac{1}{2}2\eta_s\left(\epsilon(C_u), \epsilon(C_u)\right) + \frac{1}{2}\sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\left(\nabla C_p, \nabla C_p\right)_K \\
&\quad + \frac{1}{4\eta_p}\left(C_\sigma, C_\sigma\right) + \frac{\lambda}{2\eta_p}\left(\frac{\partial C_\sigma}{\partial t}, C_\sigma\right) \\
&\leq C\Big(||\epsilon(\Pi_u)||^2_{L^2(\Omega)} + ||\Pi_p||^2_{L^2(\Omega)} + ||\Pi_\sigma||^2_{L^2(\Omega)} \\
&\quad + \sum_{K\in\mathcal{T}_h}\frac{1}{h_K^2}||\Pi_u||^2_{L^2(K)} + h^2||\nabla\Pi_p||^2_{L^2(\Omega)} + h^2||\nabla\tilde{p}||^2_{L^2(\Omega)}\Big)
\end{aligned}
$$

where C depends only on ρ, η_s, η_p and α. Time integration for $0 \leq s \leq T$ yields

$$
\begin{aligned}
&\frac{\rho}{2}||C_u(s)||^2_{L^2(\Omega)} + \eta_s\int_0^s ||\epsilon(C_u)||^2_{L^2(\Omega)} + \frac{\lambda}{4\eta_p}||C_\sigma(s)||^2_{L^2(\Omega)} \\
&\leq \frac{\rho}{2}||C_u(0)||^2_{L^2(\Omega)} + \frac{\lambda}{4\eta_p}||C_\sigma(0)||^2_{L^2(\Omega)} \\
&\quad + C\int_0^s\Big(||\epsilon(\Pi_u)||^2_{L^2(\Omega)} + ||\Pi_p||^2_{L^2(\Omega)} + ||\Pi_\sigma||^2_{L^2(\Omega)} \\
&\quad + \sum_{K\in\mathcal{T}_h}\frac{1}{h_K^2}||\Pi_u||^2_{L^2(K)} + h^2||\nabla\Pi_p||^2_{L^2(\Omega)} + h^2||\nabla\tilde{p}||^2_{L^2(\Omega)}.
\end{aligned}
$$

Using standard interpolation results, we finally obtain

$$
||C_u, C_\sigma||^2_{X_h} \leq Ch^2\Big(||\tilde{u}, \tilde{\sigma}||^2_X + ||\tilde{p}||^2_{L^q(W^{1,r})} + ||\nabla u_0||^2_{L^2(\Omega)} + ||\nabla\sigma_0||^2_{L^2(\Omega)}\Big),
$$

where C does not depend on h, f, u_0, σ_0 and g. Then, using continuous embeddings between interpolation spaces (see [37]), we have

$$
||\nabla u_0||^2_{L^2(\Omega)} \leq C||u_0||^2_{E_{1-1/q,q}},
$$

where C does not depend on h, f, u_0, σ_0 and g. Moreover, using the fact that the mapping

$$
(f, u_0, \sigma_0, g) \rightarrow (\tilde{u}, \tilde{\sigma}, \tilde{p})
$$

is continuous from

$$
Y \times L^q(W^{1,r}) \rightarrow L^q(W^{2,r}) \times L^q(W^{1,r}) \times L^q(W^{1,r}),
$$

we obtain

$$||C_u, C_\sigma||_{X_h} \leq Ch\Big(||f, u_0, \sigma_0||_Y + ||g||_{L^q(W^{1,r})}\Big),$$

which concludes the proof.

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