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# CLASSIFICATION(S) OF DANIELEWSKI HYPERSURFACES

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**Abstract.** The Danielewski hypersurfaces are the hypersurfaces $X_{Q,n}$ in $\mathbb{C}^3$ defined by an equation of the form $x^n y = Q(x, z)$ where $n \geqslant 1$ and $Q(x, z)$ is a polynomial such that $Q(0, z)$ is of degree at least two. They were studied by many authors during the last twenty years. In the present article, we give their classification as algebraic varieties. We also give their classification up to automorphism of the ambient space. As a corollary, we obtain that every Danielewski hypersurface $X_{Q,n}$ with $n \geqslant 2$ admits at least two nonequivalent embeddings into $\mathbb{C}^3$.

## Introduction

The history of Danielewski hypersurfaces goes back to 1989, when Danielewski [Dan] showed that, if $W_n$ denotes the hypersurface in $\mathbb{C}^3$ defined by the equation $x^n y - z(z-1) = 0$, then $W_n \times \mathbb{C}$ and $W_m \times \mathbb{C}$ are isomorphic algebraic varieties for all $n, m \geqslant 1$, whereas the surfaces $W_1$ and $W_2$ are not isomorphic. He discovered the first counterexamples to the Cancellation Problem over the complex numbers. Then, Fieseler [F] proved that $W_n$ and $W_m$ are not isomorphic if $n \neq m$.

Since these results appeared, complex algebraic surfaces defined by equations of the form $x^n y - Q(x, z) = 0$ (now called *Danielewski hypersurfaces*) have been studied by many different authors (see [W], [ML], [Dai], [C], [FMJ], [MJP]), leading to new interesting examples as byproducts. Let us mention two of them.

In their work on embeddings of Danielewski hypersurfaces given by $x^n y = p(z)$, Freudenburg and Moser-Jauslin [FMJ] discovered an example of two smooth algebraic surfaces which are algebraically nonisomorphic but holomorphically isomorphic.

More recently, the study of Danielewski hypersurfaces of equations $x^2 y - z^2 - xq(z) = 0$ produced the first counterexamples to the Stable Equivalence Problem [MJP]; that is, two polynomials of $\mathbb{C}[X_1, X_2, X_3]$ which are not equivalent (i.e., such that there exists no algebraic automorphism of $\mathbb{C}[X_1, X_2, X_3]$ which maps one to the other one) but, when considered as polynomials of $\mathbb{C}[X_1, X_2, X_3, X_4]$, become equivalent.

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The purpose of the present paper is to classify all Danielewski hypersurfaces, both as algebraic varieties, and also as hypersurfaces in $\mathbb{C}^3$. More precisely, we will give necessary and sufficient conditions for isomorphism of two Danielewski hypersurfaces; and, on the other hand, we will give necessary and sufficient conditions for equivalence of two isomorphic Danielewski hypersurfaces. Recall that two isomorphic hypersurfaces $H_1, H_2 \subset \mathbb{C}^n$ are said to be *equivalent* if there exists an algebraic automorphism $\Phi$ of $\mathbb{C}^n$ which maps one to the other one, i.e., such that $\Phi(H_1) = H_2$.

We know indeed that isomorphic classes and equivalence classes are distinct for Danielewski hypersurfaces. This was first observed by Freudenburg and Moser-Jauslin, who showed in [FMJ] that the Danielewski hypersurfaces defined respectively by the equations $f = x^2y - (1+x)(z^2-1) = 0$ and $g = x^2y - z^2 + 1 = 0$ are isomorphic but nonequivalent. (One way to see that they are not equivalent is to remark that the level surfaces $f^{-1}(c)$ are smooth for every constant $c \in \mathbb{C}$, whereas the surface $g^{-1}(1)$ is singular along the line $\{x = z = 0\}$.)

Several papers already contain the classification, up to isomorphism, of Danielewski hypersurfaces of a certain form. Makar-Limanov proved in [ML] that two Danielewski hypersurfaces of equations $x^{n_1}y - p_1(z) = 0$ and $x^{n_2}y - p_2(z) = 0$ with $n_1, n_2 \geqslant 2$ and $p_1, p_2 \in \mathbb{C}[z]$ are isomorphic if and only if they are equivalent via an affine automorphism of the form $(x, y, z) \mapsto (ax, by, cz + d)$ with $a, b, c \in \mathbb{C}^*$ and $d \in \mathbb{C}$. Then, Daigle generalized in [Dai] this result to the case $n_1, n_2 \geqslant 1$. Next, Wilkens has given in [W] the classification of Danielewski hypersurfaces of equations $x^ny - z^2 - h(x)z = 0$ with $n \geqslant 2$ and $h(x) \in \mathbb{C}[x]$.

Finally, Dubouloz and the author showed in [DP] that every Danielewski hypersurface $X_{Q,n}$ of equation $x^ny = Q(x, z)$, where $Q(x, z)$ is such that $Q(0, z)$ has simple roots, is isomorphic to one defined by an equation of the form $x^ny = \prod_{i=1}^{d}(z - \sigma_i(x))$, where $\{\sigma_1(x), \ldots, \sigma_d(x)\}$ is a collection of polynomials in $\mathbb{C}[x]$ so that $\sigma_i(0) \neq \sigma_j(0)$ if $i \neq j$. In the same paper, we classified these last ones and called them *standard forms*. This effectively classifies, up to isomorphism, all Danielewski hypersurfaces of equations $x^ny = Q(x, z)$, where $Q(0, z)$ has simple roots.

In the present paper, we generalize the notion of the Danielewski hypersurface in standard form and we prove that every Danielewski hypersurface is isomorphic to one in standard form (which can be found by an algorithmic procedure). Then, we are able to classify all Danielewski hypersurfaces. The terminology *standard form* is relevant since every isomorphism between two Danielewski hypersurfaces in standard form — and every automorphism of such a Danielewski hypersurface — extends to a triangular automorphism of $\mathbb{C}^3$.

We also give a criterion (Theorem 11) to distinguish isomorphic but not equivalent Danielewski hypersurfaces.

As a corollary, we obtain that every Danielewski hypersurface defined by an equation of the form $x^ny - Q(x, z) = 0$ with $n \geqslant 2$ admits at least two nonequivalent embeddings into $\mathbb{C}^3$.

Most of these results are based on a precise picture of the sets of locally nilpotent derivations of coordinate rings of Danielewski hypersurfaces, obtained using techniques which were mainly developed by Makar-Limanov in [ML].

The paper is organized as follows. In Section 1, we fix some notations and definitions. Section 2 is devoted to the Danielewski hypersurfaces in standard form. In Section 3, we study the locally nilpotent derivations on the Danielewski hypersurfaces in order to get information on what an isomorphism between two Danielewski hypersurfaces looks like. In Section 4, we classify the Danielewski hypersurfaces up to isomorphism, whereas we give their classification up to equivalence in Section 5.

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## 1. Definitions and notations

In this paper, our base field is $\mathbb{C}$, the field of complex numbers. If $n \geqslant 1$, then $\mathbb{C}^{[n]}$ will denote a polynomial ring in $n$ variables over $\mathbb{C}$.

**Definition 1.** Two hypersurfaces $X_1$ and $X_2$ of $\mathbb{C}^n$ are said to be *equivalent* if there exists a (polynomial) automorphism $\Phi$ of $\mathbb{C}^n$ such that $\Phi(X_1) = X_2$.

This notion is related to the notion of *equivalent embeddings* in the following sense. If $X_1$ and $X_2$ are two isomorphic hypersurfaces of $\mathbb{C}^n$ which are not equivalent, then $X_1$ admits two nonequivalent embeddings into $\mathbb{C}^n$. More precisely, let $\varphi : X_1 \to X_2$ be an isomorphism and denote $i_1 : X_1 \to \mathbb{C}^n$ and $i_2 : X_2 \to \mathbb{C}^n$ the inclusion maps. Then, $i_1$ and $i_2 \circ \varphi$ are two nonequivalent embeddings of $X_1$ into $\mathbb{C}^n$, since $\varphi$ does not extend to an automorphism of $\mathbb{C}^n$.

**Definition 2.** A *Danielewski hypersurface* is a hypersurface $X_{Q,n} \subset \mathbb{C}^3$ defined by an equation of the form $x^n y - Q(x, z) = 0$, where $n \in \mathbb{N}$ and $Q(x, z) \in \mathbb{C}[x, z]$ is such that $\deg(Q(0, z)) \geqslant 2$.

We will denote by $S_{Q,n}$ the coordinate ring of a Danielewski hypersurface $X_{Q,n}$, i.e., $S_{Q,n} = \mathbb{C}[X_{Q,n}] = \mathbb{C}[x, y, z]/(x^n y - Q(x, z))$.

It can be easily seen that every Danielewski hypersurface is equivalent to one of the form $X_{Q,n}$ with $\deg_x Q(x, z) < n$.

**Lemma 1.** *Let $X_{Q,n}$ be a Danielewski hypersurface and $R(x, z) \in \mathbb{C}[x, z]$ be a polynomial. Then $X_{Q,n}$ is equivalent to the Danielewski hypersurface of equation $x^n y = Q(x, z) + x^n R(x, z)$.*

*Proof.* It suffices to consider the triangular automorphism of $\mathbb{C}^3$ defined by the formula $(x, y, z) \mapsto (x, y + R(x, z), z)$. $\square$

## 2. Standard forms

In [DP], A. Dubouloz and the author proved that every Danielewski hypersurface $X_{Q,n}$ where $Q(x, z)$ is a polynomial such that $Q(0, z)$ has $d \geqslant 2$ simple roots is isomorphic to a hypersurface of a certain type (called standard form) and then classified all these standard forms up to isomorphism.

In the present paper, we generalize these results by dropping the assumption that $Q(0, z)$ has simple roots. In order to do this, we first generalize the definition of standard form given in [DP].

**Definition 3.** We say that a Danielewski hypersurface $X_{Q,n}$ is in *standard form* if the polynomial $Q$ can be written as follows:

$$Q(x,z) = p(z) + xq(x,z), \quad \text{with } \deg_z(q(x,z)) < \deg(p).$$

We also introduce a notion of reduced standard form.

**Definition 4.** A Danielewski hypersurface $X_{Q,n}$ is in *reduced standard form* if $\deg_x(Q(x,z)) < n$ and

$$Q(x,z) = p(z) + xq(x,z), \quad \text{with } \deg_z(q(x,z)) < \deg(p) - 1.$$

When $X_1$ and $X_2$ are two isomorphic Danielewski hypersurfaces with $X_2$ in (reduced) standard form, we say that $X_2$ is a (reduced) standard form of $X_1$.

One can find many Danielewski hypersurfaces in standard form in the literature. Let us give some examples.

**Example 1.**

(1) Danielewski hypersurfaces defined by equations of the form $x^n y - p(z) = 0$ are in reduced standard form. These hypersurfaces were studied by Makar-Limanov in [ML].

(2) The Danielewski hypersurfaces, studied by Danielewski [Dan] and Wilkens [W], defined by $x^2y - z^2 - h(x)z = 0$ are in standard form.

(3) Danielewski hypersurfaces $X_{\sigma,n}$ defined by equations $x^n y = \prod_{i=1}^{d}(z - \sigma_i(x))$ where $\sigma = \{\sigma_i(x)\}_{i=1\cdots d}$ is a collection of $d \geqslant 2$ polynomials, are in standard form. They are those we have called standard form in [DP].

(4) If $r(x) \in \mathbb{C}[x]$ is a nonconstant polynomial, then a Danielewski hypersurface defined by $x^n y - r(x)p(z) = 0$ is not in standard form. Notice that Freudenburg and Moser-Jauslin showed in [FMJ] that a reduced standard form of such a hypersurface is given by the equation $x^n y - p(z) = 0$.

*Remark 1.* It turns out that every Danielewski hypersurface in standard form is equivalent to one in reduced standard form (see Lemma 2 below). Therefore, the notion of reduced standard form is, in some sense, more relevant than the notion of standard form if we are interested in the classification of Danielewski hypersurfaces. Nevertheless, the notion of standard form is of an interest too. Indeed, nice properties are true for all Danielewski hypersurfaces in (not necessarily reduced) standard form. For example, we will see that all their automorphisms extend to automorphisms of the ambient space. Recall that this does not hold for general Danielewski hypersurfaces (see [DP]).

**Lemma 2.** *Every Danielewski hypersurface in standard form is equivalent to one in reduced standard form.*

*Proof.* Let $X_{Q,n}$ be a Danielewski hypersurface in standard form. We let

$$Q(x,z) = p(z) + xq(x,z) = \sum_{i=0}^{d} a_i z^i + x \sum_{i=0}^{d-1} z^i \alpha_i(x)$$

where $a_0, \ldots, a_{d-1} \in \mathbb{C}$, $a_d \in \mathbb{C}^*$ and $\alpha_0(x), \ldots, \alpha_{d-1}(x) \in \mathbb{C}[x]$ and we consider the automorphism of $\mathbb{C}^3$ defined by $\Phi : (x, y, z) \mapsto (x, y, z - x(da_d)^{-1}\alpha_{d-1}(x))$.

One checks that $\Phi^*(x^n y - p(z) - xq(x, z)) = x^n y - p(z) - x\tilde{q}(x, z)$ for a suitable polynomial $\tilde{q}(x, z) \in \mathbb{C}[x, z]$ with $\deg_z(\tilde{q}(x, z)) < \deg(p) - 1$. Then, we can conclude by applying Lemma 1. □

We will now prove that every Danielewski hypersurface is isomorphic to one in reduced standard form. Our proof consists of the following two lemmas. We took the first one in [FMJ].

**Lemma 3.** *Let $n \geqslant 1$ be a natural number and $Q_1(x, z)$ and $Q_2(x, z)$ be two polynomials of $\mathbb{C}[x, z]$ such that*

$$Q_2(x, z) = (1 + x\pi(x, z))Q_1(x, z) + x^n R(x, z)$$

*for some polynomials $\pi(x, z), R(x, z) \in \mathbb{C}[x, z]$. Then the endomorphism of $\mathbb{C}^3$ defined by*

$$\Phi(x, y, z) = (x, (1 + x\pi(x, z))y + R(x, z), z)$$

*induces an isomorphism $\varphi : X_{Q_1,n} \to X_{Q_2,n}$.*

*Proof.* Note that, since $\Phi^*(x^n y - Q_2(x, z)) = (1 + x\pi(x, z))(x^n y - Q_1(x, z))$, $\Phi$ induces a morphism $\varphi : X_{Q_1,n} \to X_{Q_2,n}$.

Let $f(x, z)$ and $g(x, z)$ be two polynomials in $\mathbb{C}[x, z]$ so that $(1 + x\pi(x, z))f(x, z) + x^n g(x, z) = 1$ and define $\Psi$, an endomorphism of $\mathbb{C}^3$, by setting

$$\begin{cases} \Psi^*(x) = x, \\ \Psi^*(y) = f(x, z)y + g(x, z)Q_1(x, z) - f(x, z)R(x, z), \\ \Psi^*(z) = z. \end{cases}$$

We check easily that

$$\Psi^*(x^n y - Q_1(x, z)) = f(x, z)(x^n y - Q_2(x, z))$$

and that

$$\begin{aligned} &\Psi^* \circ \Phi^*(x) = x, \\ &\Psi^* \circ \Phi^*(y) = y - g(x, y)(x^n y - Q_2(x, z)), \\ &\Psi^* \circ \Phi^*(z) = z. \end{aligned}$$

Therefore, $\Psi^* \circ \Phi^*$ is the identity map on $S_{Q_2,n} = \mathbb{C}[X_{Q_2,n}]$. Hence, $\Psi$ induces the inverse morphism of $\varphi$, and $X_{Q_1,n} \cong X_{Q_2,n}$. □

**Lemma 4.** *Let $p(z) \in \mathbb{C}[z] \setminus \{0\}$ and $q(x, z) \in \mathbb{C}[x, z]$. For every $n \geqslant 1$ there exist polynomials $\pi(x, z), q_s(x, z), R(x, z) \in \mathbb{C}[x, z]$ such that $\deg_z(q_s(x, z)) < \deg(p)$ and*

$$p(z) + xq(x, z) = (1 + x\pi(x, z))(p(z) + xq_s(x, z)) + x^n R(x, z).$$

*Proof.* The proof goes by induction on $n$. The assertion is obvious for $n = 1$. Let $n \geqslant 1$ and suppose that there exist polynomials $\pi_n(x,z), q_{s,n}(x,z), R_n(x,z) \in \mathbb{C}[x,z]$ such that $\deg_z(q_{s,n}(x,z)) < \deg(p)$ and

$$p(z) + xq(x,z) = (1 + x\pi_n(x,z))(p(z) + xq_{s,n}(x,z)) + x^n R_n(x,z).$$

Let $R_n(0,z) = p(z)\tilde{\pi}_{n+1}(z) + r_{n+1}(z)$ be the Euclidean division in $\mathbb{C}[z]$ of $R_n(0,z)$ by $p$. Then we obtain:

$$\begin{aligned} p(z) + xq(x,z) &= (1 + x\pi_n(x,z))(p(z) + xq_{s,n}(x,z)) + x^n R_n(x,z) \\ &\equiv (1 + x\pi_n(x,z) + x^n\tilde{\pi}_{n+1}(z)) \\ &\quad \cdot (p(z) + xq_{s,n}(x,z) + x^n r_{n+1}(z)) \mod (x^{n+1}) \\ &= (1 + x\pi_{n+1}(x,z))(p(z) + xq_{s,n+1}(x,z)) + x^{n+1}R_{n+1}(x,z), \end{aligned}$$

where

$$\begin{aligned} \pi_{n+1}(x,z) &= \pi_n(x,z) + x^{n-1}\tilde{\pi}_{n+1}(z) \quad \text{and} \\ q_{s,n+1}(x,z) &= q_{s,n}(x,z) + x^{n-1}r_{n+1}(z). \end{aligned}$$

This allows us to conclude, since $\deg(r_{n+1}(z)) < \deg(p(z))$ by construction. □

**Theorem 5.** *Every Danielewski hypersurface is isomorphic to a Danielewski hypersurface in reduced standard form. Furthermore, there is an algorithmic procedure which computes one of the reduced standard forms of a given Danielewski hypersurface.*

*Proof.* The theorem follows directly from Lemmas 4, 3 and 2. Since the proofs of these lemmas are algorithmic, they give an algorithmic procedure for finding a (reduced) standard form of a given Danielewski hypersurface. □

## 3. Using locally nilpotent derivations

One important property of Danielewski hypersurfaces is that they admit nontrivial actions of the additive group $\mathbb{C}_+$. For instance, we can define a $\mathbb{C}_+$-action $\delta_{Q,n} : \mathbb{C} \times X_{Q,n} \to X_{Q,n}$ on a hypersurface $X_{Q,n}$ by setting

$$\delta_{Q,n}(t,(x,y,z)) = (x, y + x^{-n}(Q(x,z+tx^n) - Q(x,z)), z + tx^n).$$

Since a $\mathbb{C}_+$-actions on an affine complex surface $S$ induce a $\mathbb{C}$-fibration over an affine curve, affine complex surfaces with $\mathbb{C}_+$-actions split into two cases. Either there is only one $\mathbb{C}$-fibration on $S$ up to an isomorphism of the base, or there exists a second one. In other words, either the surface has a Makar-Limanov invariant of transcendence degree one, or its Makar-Limanov invariant is trivial.

Recall that algebraic $\mathbb{C}_+$-actions on an affine variety $\operatorname{Spec}(A)$ correspond to locally nilpotent derivations on the $\mathbb{C}$-algebra $A$, and that the Makar-Limanov invariant $\mathrm{ML}(A)$ of an algebra $A$ is defined as the intersection of all kernels of locally nilpotent derivations of $A$. If $X = \mathrm{Spec}\,(A)$ is an affine variety, one defines the Makar-Limanov invariant of $X$ as $\mathrm{ML}(X) = \mathrm{ML}(A)$. Equivalently, $\mathrm{ML}(X)$ is the intersection of all invariant rings of algebraic $\mathbb{C}_+$-actions on $X$.

Note that the action $\delta_{Q,n}$ on a surface $X_{Q,n}$ corresponds to the locally nilpotent derivation $\Delta_{Q,n} = x^n(\partial/\partial z) + (\partial Q(x,z)/\partial z)(\partial/\partial y)$ on its coordinate ring $S_{Q,n}$. More precisely, $\Delta_{Q,n} = x^n(\partial/\partial z) + (\partial Q(x,z)/\partial z)(\partial/\partial y)$ is a locally nilpotent derivation on the polynomial ring $\mathbb{C}[x,y,z]$ which annihilates $x^n y - Q(x,z)$ and induces therefore a locally nilpotent derivation on $S_{Q,n}$. By an abuse of notation, we still denote by $\Delta_{Q,n} = x^n(\partial/\partial z) + (\partial Q(x,z)/\partial z)(\partial/\partial y)$ the induced derivation on $S_{Q,n}$.

Applying techniques developed by Kaliman and Makar-Limanov in [KML], one can obtain an important result concerning Danielewski hypersurfaces: the Makar-Limanov invariant of a Danielewski hypersurface $X_{Q,n}$ is nontrivial if $n \geqslant 2$.

**Theorem 6.** *Let $X_{Q,n}$ be a Danielewski hypersurface. Then* $\mathrm{ML}(X_{Q,n}) = \mathbb{C}$ *if* $n = 1$ *and* $\mathrm{ML}(X_{Q,n}) = \mathbb{C}[x]$ *if* $n \geqslant 2$. *Moreover, if* $n \geqslant 2$ *then* $\mathrm{Ker}(\delta) = \mathbb{C}[x]$ *and* $\mathrm{Ker}(\delta^2) = \mathbb{C}[x]z + \mathbb{C}[x]$ *for any nonzero locally nilpotent derivation $\delta$ on $S_{Q,n}$.*

*Proof.* Let $X_{Q,n}$ be a Danielewski hypersurface.

If $n = 1$, the result is easy. Indeed, by Lemma 1, we can suppose that $Q(x,z) = p(z) \in \mathbb{C}[z]$. Then, it suffices to check that $\mathrm{Ker}(\delta_1) \cap \mathrm{Ker}(\delta_2) = \mathbb{C}$, where $\delta_1, \delta_2$ are the locally nilpotent derivations on the coordinate ring $S_{p,1} = \mathbb{C}[x,y,z]/(xy - p(z))$ induced by the following derivations on $\mathbb{C}[x,y,z]$:

$$\delta_1 = x\frac{\partial}{\partial z} + p'(z)\frac{\partial}{\partial y} \qquad \text{and} \qquad \delta_2 = y\frac{\partial}{\partial z} + p'(z)\frac{\partial}{\partial x}.$$

Suppose now that $n \geqslant 2$. Without loss of generality, we can suppose that the leading term of $Q(0,z)$ is $z^d$ with $d \geqslant 2$ and that $Q(0,0) = 0$.

Let us consider first the case where $X_{Q,n}$ is in standard form. In this case, we can imitate the proof given by Makar-Limanov in [ML] for hypersurfaces of equation $x^n y = p(z)$. This proof goes as follows.

The main idea is to construct an ascending $\mathbb{Z}$-filtration on $S_{Q,n}$ such that the corresponding graded algebra $\mathrm{Gr}(S_{Q,n})$ is isomorphic to the algebra $\mathbb{C}[x,y,z]/(x^n y - z^d)$. Let us recall briefly this construction and refer to [KML] for more details.

First, we define a $\mathbb{Z}$-grading on $\mathbb{C}[x,y,z]$ by declaring that $x$, $y$ and $z$ are homogeneous of degrees $-1$, $n + dN$ and $N$, respectively, where $N \geqslant 1$. Different values of $N$ will be considered in the course of proving the theorem. This grading determines a weight degree function $w$ on $\mathbb{C}[x,y,z]$ such that $w(x) = -1$, $w(y) = n + dN$ and $w(z) = N$. Throughout the proof, we will denote by $\bar{p}$ the *principal component* of a polynomial $p \in \mathbb{C}[x,y,z]$. Recall that it is defined as the homogeneous part of $p$ of highest $w$-degree.

Let $I = (x^n y - Q(x,z))$ be the ideal of $\mathbb{C}[x,y,z]$ defining $X_{Q,n}$. Since $X_{Q,n}$ is in standard form, the ideal $\hat{I}$ generated by the principal components of the elements of $I$ is $\hat{I} = (x^n y - z^d)$.

Consider the natural projection $\pi : \mathbb{C}[x,y,z] \to S_{Q,n}$ and set $d_A(f) = \inf\{w(p) \mid p \in \pi^{-1}(f)\}$ for any $f \in S_{Q,n} \setminus \{0\}$. By Lemma 3.2 in [KML], $d_A$ is a degree function on $S_{Q,n}$ and we have $d_A(f) = w(p)$ for a polynomial $p \in \mathbb{C}[x,y,z]$ if and only if $p \in \pi^{-1}(f)$ is such that $\bar{p} \notin \hat{I}$. In particular, $d_A(x) = w(x) = -1$, $d_A(y) = w(y) = n + dN$ and $d_A(z) = w(z) = N$. (Here, $x, y, z$ denote also, by an abuse of notation, the images of $x, y, z$ in $S_{Q,n}$.)

Using the degree function $d_A$, we can define a filtration $\mathcal{F} = (F_i)_{i \in \mathbb{Z}}$ on $S_{Q,n}$ by setting

$$F_i = \{f \in S_{Q,n} \mid d_A(f) \leqslant i\}$$

for each $i \in \mathbb{Z}$. Now, we consider the graded algebra $\mathrm{Gr}(S_{Q,n})$ associated to the filtration $\mathcal{F}$. It is defined as

$$\mathrm{Gr}(S_{Q,n}) = \bigoplus_{i \in \mathbb{Z}} F_i / F_{i-1}.$$

By [KML, Prop. 4.1], we know that

$$\mathrm{Gr}(S_{Q,n}) \cong \mathbb{C}[x, y, z]/\hat{I} = \mathbb{C}[x, y, z]/(x^n y - z^d).$$

Let $\mathrm{gr} : S_{Q,n} \to \mathrm{Gr}(S_{Q,n})$ denote the natural function from $S_{Q,n}$ to $\mathrm{Gr}(S_{Q,n})$ and set $\hat{x} = \mathrm{gr}(x)$, $\hat{y} = \mathrm{gr}(y)$ and $\hat{z} = \mathrm{gr}(z)$. The map gr can be described as follows. Let $p \in \pi^{-1}(f)$ for an element $f \in S_{Q,n}$. If $\bar{p} \notin \hat{I}$, then $\mathrm{gr}(f) = \bar{p}(\hat{x}, \hat{y}, \hat{z})$ (see [KML, Rem. 4.1]). Moreover, we will now prove that we can choose, for any element $f \in S_{Q,n} \setminus \{0\}$, a sufficiently large weight $w(z) = N \geqslant 1$ such that $\mathrm{gr}(f)$ is a monomial of the form $\lambda \hat{x}^i \hat{y}^j \hat{z}^k$ with $\lambda \in \mathbb{C}^*$, $i, j, k \in \mathbb{N}$, $i < N$ and $k < d$.

Let $f \in S_{Q,n} \setminus \{0\}$. Since $X_{Q,n}$ is in standard form and since $x^n y = Q(x, z)$ is the unique relation in $S_{Q,n}$, there exists a unique polynomial $p \in \mathbb{C}[x, y, z]$ such that $\deg_z(p(x, y, z))) < d$ and $f = \pi(p)$. To choose a weight $w(z) = N > \deg_x(p(x, y, z))$ greater than the degree of $p$ in $x$ ensures then that all monomials of $p$ have distinct $w$-degree. To see this, assume, by contradiction, that $w(x^{i_1} y^{j_1} z^{k_1}) = w(x^{i_2} y^{j_2} z^{k_2})$ for distinct triples $(i_1, j_1, k_1), (i_2, j_2, k_2) \in \mathbb{N}^3$ with $i_1, i_2 < N$ and $k_1, k_2 < d$. Recall that $w(x^i y^j z^k) = -i + j(n + Nd) + kN$ for any nonzero triple $(i, j, k) \in \mathbb{N}^3$ by definition of the weight degree function $w$. It follows

$$|j_1 - j_2|(n + Nd) = |(-i_2 + i_1) + (k_2 - k_1)N| \leqslant |i_2 - i_1| + |k_2 - k_1|N < N + (d-1)N.$$

Thus $j_1 = j_2$ and $(-i_2 + i_1) + (k_2 - k_1)N = 0$. Since $i_1, i_2 < N$, the last equality implies that $i_1 = i_2$ and $k_1 = k_2$, a contradiction. Therefore, $\bar{p}$ is a monomial $\lambda x^i y^j z^k$ with $\lambda \in \mathbb{C}^*$, $(i, j, k) \in \mathbb{N}^3$, $i < N$ and $k < d$. So $\mathrm{gr}(f) = \lambda \hat{x}^i \hat{y}^j \hat{z}^k$ as desired.

We are now ready to prove the theorem. Let $\delta$ be a nonzero locally nilpotent derivation on $S_{Q,n}$.

By [KML, Sect. 5], $\delta$ induces a nonzero locally nilpotent derivation $\mathrm{gr}(\delta)$ on $\mathrm{Gr}(S_{Q,n})$. Moreover, $\deg_{\mathrm{gr}(\delta)}(\mathrm{gr}(f)) \leqslant \deg_\delta(f)$ for any element $f \in S_{Q,n}$. (Recall that one associates, to each locally nilpotent derivation $D$ on an algebra $A$, a degree function $\deg_D$ by letting $\deg(0) = -\infty$ and $\deg_D(a) := \max\{n \in \mathbb{N} \mid D^n(a) \neq 0\}$ if $a \in A \setminus \{0\}$.)

On the other hand, Makar-Limanov proved in [ML] that each nonzero locally nilpotent derivation on $\mathbb{C}[x, y, z]/(x^n y - z^d)$ has kernel equal to $\mathbb{C}[x]$. Since $\mathrm{Gr}(S_{Q,n}) \cong \mathbb{C}[x, y, z]/(x^n y - z^d)$, it follows that $\mathrm{Ker}(\mathrm{gr}(\delta)) = \mathbb{C}[\hat{x}]$.

Then $\deg_{\mathrm{gr}(\delta)}(\hat{z}) \geqslant 1$ and $\deg_{\mathrm{gr}(\delta)}(\hat{y}) = d\deg_{\mathrm{gr}(\delta)}(\hat{z}) \geqslant d \geqslant 2$, since $\hat{x}^n\hat{y} = \hat{z}^d$. Note that we also have

$$\deg_{\mathrm{gr}(\delta)}(\hat{x}^i\hat{y}^j\hat{z}^k) = i\deg_{\mathrm{gr}(\delta)}(\hat{x}) + j\deg_{\mathrm{gr}(\delta)}(\hat{y}) + k\deg_{\mathrm{gr}(\delta)}(\hat{z}) = (dj+k)\deg_{\mathrm{gr}(\delta)}(\hat{z}),$$

for all triples $(i,j,k) \in \mathbb{N}^3$.

This implies that $\mathrm{Ker}(\delta) = \mathbb{C}[x]$. Indeed, let $f \in S_{Q,n} \setminus \mathbb{C}[x]$ and let $p \in \pi^{-1}(f)$ with $\deg_z(p(x,y,z)) < d$. Let us choose a weight $w(z) = N > \deg_x(p(x,y,z))$. Then $w(\lambda x^i y^j z^k) > 0$ for any monomial of $p$ of the form $\lambda x^i y^j z^k$ where $\lambda \in \mathbb{C}^*$ and $(i,j,k) \in \mathbb{N}^3$ with $j > 0$ or $k > 0$. Since $p$ contains at least one such monomial, we obtain $\mathrm{gr}(f) = \bar{p}(\hat{x},\hat{y},\hat{z}) \in \mathrm{Gr}(S_{Q,n}) \setminus \mathbb{C}[\hat{x}]$. Then $\mathrm{Ker}(\delta) \subset \mathbb{C}[x]$ follows from the inequalities $1 \leqslant \deg_{\mathrm{gr}(\delta)}(\mathrm{gr}(f)) \leqslant \deg_\delta(f)$. Since $\mathrm{Ker}(\delta)$ is of transcendence degree one over $\mathbb{C}$ and is algebraically closed in $S_{Q,n}$, we obtain that $x \in \mathrm{Ker}(\delta)$. Thus, $\mathrm{Ker}(\delta) = \mathbb{C}[x]$ and therefore $\mathrm{ML}(X_{Q,n}) = \mathbb{C}[x]$.

We will now prove $\mathrm{Ker}(\delta^2) = \mathbb{C}[x]z + \mathbb{C}[x]$. Let $f \in S_{Q,n} \setminus (\mathbb{C}[x]z + \mathbb{C}[x])$ and let $p \in \pi^{-1}(f)$ with $\deg_z(p(x,y,z)) < d$. Choose $N > \deg_x(p)$. We have seen that $\mathrm{gr}(f) = \lambda\hat{x}^i\hat{y}^j\hat{z}^k$ with $\lambda \in \mathbb{C}^*$, $(i,j,k) \in \mathbb{N}^3$. By assumption, $p$ contains at least one monomial of the form $\mu x^{i'} y^{j'} z^{k'}$ with $\mu \in \mathbb{C}^*$, $(i',j',k') \in \mathbb{N}^3$, $i' < N$ and $j' \geqslant 1$ or $k' \geqslant 2$. Since $w(\lambda x^{i'} y^{j'} z^{k'}) = -i' + j'(n+Nd) + k'N > N \geqslant w(a(x)z + b(x))$ for any polynomial $a(x)z + b(x) \in \mathbb{C}[x]z + \mathbb{C}[x]$, it follows $j \geqslant 1$ or $k \geqslant 2$. Hence

$$\deg_{\mathrm{gr}(\delta)}(\mathrm{gr}(f)) = \deg_{\mathrm{gr}(\delta)}(\lambda\hat{x}^i\hat{y}^j\hat{z}^k) = (dj+k)\deg_{\mathrm{gr}(\delta)}(\hat{z}) \geqslant 2.$$

This proves $\mathrm{Ker}(\delta^2) \subset \mathbb{C}[x]z + \mathbb{C}[x]$ as $\deg_\delta(f) \geqslant \deg_{\mathrm{gr}(\delta)}(\mathrm{gr}(f)) \geqslant 2$.

Conversely, let $f \in \mathrm{Ker}(\delta^2) \setminus \mathrm{Ker}(\delta)$. We have shown that $f = a(x)z + b(x)$ with $a(x) \in \mathbb{C}[x] \setminus \{0\}$ and $b(x) \in \mathbb{C}[x]$. Then $\delta(f) = a(x)\delta(z) \in \mathrm{Ker}(\delta)$ and $\delta(z) \in \mathrm{Ker}(\delta)$ since $\mathrm{Ker}(\delta)$ is factorially closed in $S_{Q,n}$. Hence $z \in \mathrm{Ker}(\delta^2)$, $\mathbb{C}[x]z + \mathbb{C}[x] \subset \mathrm{Ker}(\delta^2)$ and $\mathrm{Ker}(\delta^2) = \mathbb{C}[x]z + \mathbb{C}[x]$ as desired.

This proves the theorem for Danielewski hypersurfaces in standard form. The general case will follow easily.

Let $X_{Q,n}$ be a general Danielewski hypersurface with $n \geqslant 2$ and let $\varphi : X_{Q,n} \to X_{Q_s,n}$ be an isomorphism between $X_{Q,n}$ and one of its standard form $X_{Q_s,n}$.

Let $\delta$ be a nonzero locally nilpotent derivation on $S_{Q,n}$. Then $\delta_s = (\varphi^*)^{-1} \circ \delta \circ \varphi^*$ is a nonzero locally nilpotent derivation on $S_{Q_s,n}$ and the equalities $\mathrm{Ker}(\delta) = \varphi^*(\mathrm{Ker}(\delta_s))$ and $\mathrm{Ker}(\delta^2) = \varphi^*(\mathrm{Ker}(\delta_s^2))$ hold. Moreover, by Lemma 3 and Lemma 4, we can assume that $\varphi$ is induced by an endomorphism of $\mathbb{C}^3$ of the form $(x,y,z) \mapsto (x, (1 + x\pi(x,z))y + R(x,z), z)$ where $\pi(x,z)$ and $R(x,z)$ are polynomials such that $Q_s(x,z) = (1 + x\pi(x,z))Q(x,z) + x^n R(x,z)$; then $\varphi^*$ maps $\mathbb{C}[x]$ onto $\mathbb{C}[x]$ and $\mathbb{C}[x]z + \mathbb{C}[x]$ onto $\mathbb{C}[x]z + \mathbb{C}[x]$.

This allows us to conclude that $\mathrm{Ker}(\delta) = \mathbb{C}[x]$ and $\mathrm{Ker}(\delta^2) = \mathbb{C}[x]z + \mathbb{C}[x]$ since $\mathrm{Ker}(\delta_s) = \mathbb{C}[x]$ and $\mathrm{Ker}(\delta_s^2) = \mathbb{C}[x]z + \mathbb{C}[x]$ as $X_{Q_s,n}$ is in standard form and $n \geqslant 2$. The theorem is proved. □

Using this result, we can obtain a precise picture of the set of locally nilpotent derivations on rings $S_{Q,n}$ when $n \geqslant 2$.

**Theorem 7.** *Let $X_{Q,n}$ be a Danielewski hypersurface with $n \geqslant 2$ and let $S_{Q,n}$ denote its coordinate ring. Consider the element*

$$\Delta_{Q,n} = x^n \frac{\partial}{\partial z} + \frac{\partial Q(x,z)}{\partial z}\frac{\partial}{\partial y}$$

*of* $\operatorname{LND}(S_{Q,n})$. *Then*

$$\operatorname{LND}(S_{Q,n}) = \{h(x)\Delta_{Q,n} \mid h(x) \in \mathbb{C}[x]\}.$$

*Proof.* Let $\delta$ be a nonzero locally nilpotent derivation on an algebra $S_{Q,n}$ with $n \geqslant 2$.

By Theorem 6, we know that $\delta(x) = 0$ and $\delta(z) = a(x)$ for a nonzero polynomial $a(x) \in \mathbb{C}[x] \setminus \{0\}$. Let $\delta(y) = f(x,y,z) \in \mathbb{C}[x,y,z]$.

Since $0 = \delta(x^n y - Q(x,z)) = x^n f(x,y,z) - a(x)(\partial Q(x,z)/\partial z)$, there exists a polynomial $R \in \mathbb{C}[X,Y,Z]$ such that

$$X^n f(X,Y,Z) - a(X)\frac{\partial Q(X,Z)}{\partial Z} = R(X,Y,Z)(X^n Y - Q(X,Z)).$$

Letting $X=0$ gives $a(0)(\partial Q(0,Z)/\partial Z) = R(0,Y,Z)Q(0,Z)$ and so $a(0) = R(0,Y,Z) = 0$ since $\deg Q(0,Z) \geqslant 2$.

Thus, there exist polynomials $\tilde{a} \in \mathbb{C}[X]$ and $\tilde{R} \in \mathbb{C}[X,Y,Z]$ such that $a(X) = X\tilde{a}(X)$ and $R(X,Y,Z) = X\tilde{R}(X,Y,Z)$, and we have

$$X^{n-1} f(X,Y,Z) - \tilde{a}(X)\frac{\partial Q(X,Z)}{\partial Z} = \tilde{R}(X,Y,Z)(X^n Y - Q(X,Z)).$$

Then $\tilde{a}(0) = \tilde{R}(0,Y,Z) = 0$ follows as above, and we obtain $a(X) = X^n h(X)$ with $h(X) \in \mathbb{C}[X]$ by induction.

The theorem is actually proved as $\delta(x) = 0$, $\delta(z) = a(x) = x^n h(x)$ and $\delta(y) = h(x)(\partial Q(x,z)/\partial z)$. $\square$

This theorem gives us a very powerful tool for classifying Danielewski hypersurfaces. Indeed, note that an isomorphism $\varphi : A \to B$, between two algebras $A$ and $B$, conjugates the sets $\operatorname{LND}(A)$ and $\operatorname{LND}(B)$ of locally nilpotent derivations on $A$ and $B$, i.e., $\operatorname{LND}(A) = \varphi^{-1}\operatorname{LND}(B)\varphi$ if $\varphi : A \to B$ is an isomorphism. In turn, we obtain the following result.

**Corollary 8.**

(1) *Let $\varphi : X_{Q_1,n_1} \to X_{Q_2,n_2}$ be an isomorphism between two Danielewski hypersurfaces with $n_1, n_2 \geqslant 2$ and denote by $x_i, y_i, z_i$ the images of $x, y, z$ in the coordinate ring $\mathbb{C}[X_{Q_i,n_i}]$ for $i = 1,2$. Then there exist constants $a, \alpha, \mu \in \mathbb{C}^*$ and a polynomial $\beta(x) \in \mathbb{C}[x]$ such that $\varphi^*(x_2) = ax_1$, $\varphi^*(z_2) = \alpha z_1 + \beta(x_1)$ and $Q_2(0, \alpha z + \beta(0)) = \mu Q_1(0,z)$.*

(2) *If $X_{Q_1,n_1}$ and $X_{Q_2,n_2}$ are two isomorphic Danielewski hypersurfaces, then $n_1 = n_2$ and* $\deg(Q_1(0,z)) = \deg(Q_2(0,z))$.

(3) *Suppose that $X_{Q_1,n}$ and $X_{Q_2,n}$ are two equivalent Danielewski hypersurfaces with $n \geqslant 2$, and let $\Phi : \mathbb{C}^3 \to \mathbb{C}^3$ be an algebraic automorphism such that $\Phi(X_{Q_1,n}) = X_{Q_2,n}$. Then there exist constants $a, \alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$ and a polynomial $B \in \mathbb{C}^{[2]}$ such that $\Phi^*(x) = ax$ and $\Phi^*(z) = \alpha z + \beta + xB(x, x^n y - Q_1(x,z))$.*

*Proof.* For (1) and (2), we follow the ideas of a proof given by Makar-Limanov in [ML].

Let $\varphi : X_{Q_1,n_1} \to X_{Q_2,n_2}$ be an isomorphism between two Danielewski hypersurfaces with $n_1, n_2 \geqslant 2$. Let $x_i, y_i, z_i$ denote the images of $x, y, z$ in the coordinate ring $S_i = S_{Q_i,n_i} = \mathbb{C}[X_{Q_i,n_i}]$.

If $\delta \in \mathrm{LND}(S_1)$, then $(\varphi^*)^{-1} \circ \delta \circ \varphi^* \in \mathrm{LND}(S_2)$. Thus, Theorem 7 implies $\delta^2(\varphi^*(z_2)) = 0$ for any nonzero locally nilpotent derivation $\delta \in \mathrm{LND}(S_1)$ and therefore $\varphi^*(z_2) = \alpha(x_1)z_1 + \beta(x_1)$ for some polynomials $\alpha$ and $\beta$ as $\mathrm{Ker}(\delta^2) = \mathbb{C}[x_1]z_1 + \mathbb{C}[x_1]$.

Since $\mathbb{C}[x_i, z_i] \cong \mathbb{C}^{[2]}$ is the subalgebra of $S_i$ generated by $\mathrm{Ker}(\delta^2)$ for any nonzero derivation $\delta \in \mathrm{LND}(S_i) \setminus \{0\}$, $\varphi^*$ restricts to an isomorphism between $\mathbb{C}[x_2, z_2]$ and $\mathbb{C}[x_1, z_1]$. This implies that the polynomial $\alpha$ is a nonzero constant $\alpha \in \mathbb{C}^*$. On the other hand, $\varphi^*$ restricts to an isomorphism $\mathrm{ML}(S_2) = \mathbb{C}[x_2] \to \mathrm{ML}(S_1) = \mathbb{C}[x_1]$. Consequently, $\varphi^*(x_2) = ax_1 + b$ for some constants $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$.

In order to prove $b = 0$, consider the locally nilpotent derivation $\delta_0 \in \mathrm{LND}(S_2)$ defined by $\delta_0 = (\varphi^*)^{-1} \circ \big(x_1^{n_1}(\partial/\partial z_1) + (\partial Q_1(x_1, z_1)/\partial z_1)(\partial/\partial y_1)\big) \circ \varphi^*$. Now Theorem 7 implies that $\delta_0(z_2)$ is divisible by $x_2^{n_2}$. Since $\delta_0(z_2) = a^{-n_1}\alpha(x_2 - b)^{n_1}$, we must have $b = 0$ and $n_1 \geqslant n_2$. This proves the first part of the corollary.

Moreover, repeating this analysis with $\varphi^{-1}$ instead of $\varphi$, we also obtain $n_2 \geqslant n_1$ and so $n_1 = n_2 = n$.

As $a^n x_1^n \varphi^*(y_2) - Q_2(ax_1, \alpha z_1 + \beta(x_1)) = \varphi^*(x_2^n y_2 - Q_2(x_2, z_2)) = \varphi^*(0) = 0$ in $S_1$, it follows that $x^n y - Q_1(x, z)$ divides $a^n x^n H(x, y, z) - Q_2(ax, \alpha z + \beta(x))$ in $\mathbb{C}[x, y, z]$, where $H(x, y, z)$ is any polynomial such that $H(x_1, y_1, z_1) = \varphi^*(y_2)$. Consequently we have that the polynomial $Q_2(0, \alpha z_1 + \beta(0))$ belongs to the ideal $(Q_1(0, z_1))$ of $\mathbb{C}[z]$. Thus $\deg(Q_2(0, z)) \geqslant \deg(Q_1(0, z))$.

Working with $\varphi^{-1}$, the same analysis allows us to conclude that $\deg(Q_1(0, z)) \geqslant \deg(Q_2(0, z))$. Moreover, this implies also that $Q_2(0, \alpha z + \beta(0)) = \mu Q_1(0, z)$ for a certain constant $\mu \in \mathbb{C}^*$.

Since the case $n_1 = n_2 = 1$ was already done by Daigle [Dai], this suffices to prove the second part of the corollary.

It remains to prove the third part.

Let $X_{Q_1,n}$ and $X_{Q_2,n}$ be two equivalent Danielewski hypersurfaces with $n \geqslant 2$, and let $\Phi$ be an algebraic automorphism of $\mathbb{C}^3$ such that $\Phi(X_{Q_1,n}) = X_{Q_2,n}$.

Since the polynomial $x^n y - Q_1(x, z)$ is irreducible, there exists a nonzero constant $\mu \in \mathbb{C}^*$ so that $\Phi^*(x^n y - Q_2(x, z)) = \mu(x^n y - Q_1(x, z))$.

Thus, $\Phi$ induces an isomorphism $\Phi_c$ between the Danielewski hypersurfaces of equation $x^n y - Q_2(x, z) = \mu c$ and $x^n y - Q_1(x, z) = c$ for every $c \in \mathbb{C}$.

Since $n \geqslant 2$, the Makar-Limanov invariant of these hypersurfaces is $\mathbb{C}[x]$. By (1), we obtain now that the image by $\Phi^*$ of the ideal $(x, x^n y - Q_2(x, z) - \mu c)$ is included in the ideal $(x, x^n y - Q_1(x, z) - c) = (x, Q_1(0, z) + c)$ for each $c \in \mathbb{C}$. It follows that

$$\Phi^*(x) \in \bigcap_{c \in \mathbb{C}} (x, Q_1(0, z) + c) = (x).$$

Since $\Phi$ is invertible, this implies that $\Phi^*(x) = ax$ for a certain constant $a \in \mathbb{C}^*$.

Thus

$$-\mu Q_1(0,z) \equiv \mu(x^n y - Q_1(x,z)) \equiv \Phi^*(x^n y - Q_2(x,z)) \equiv -Q_2(0,\Phi^*(z)) \mod (x).$$

Since $\deg Q_1(0,z) = \deg Q_2(0,z)$ (by the second part of the corollary), this implies that $\Phi^*(z) \equiv \alpha z + \beta \mod (x)$ for certain constants $\alpha$ and $\beta$ such that $Q_2(0, \alpha z + \beta) = \mu Q_1(0,z)$.

Thus, we can write $\Phi^*(z) = \alpha z + \beta + xB(x,y,z)$ with $B \in \mathbb{C}[x,y,z]$.

Now, we use again the first part of the corollary. For every $c \in \mathbb{C}$, there exist a constant $\alpha_c \in \mathbb{C}^*$ and a polynomial $\beta_c \in \mathbb{C}^{[1]}$ such that

$$\Phi^*(z) = \alpha z + \beta + xB(x,y,z) \equiv \alpha_c z + \beta_c(x) \mod (x^n y - Q_1(x,z) - c).$$

Therefore, for every $c \in \mathbb{C}$, we have $\alpha_c = \alpha$, $\beta_c(0) = \beta$ and

$$B(x,y,z) \equiv x^{-1}(\beta_c(x) - \beta) \mod (x^n y - Q_1(x,z) - c).$$

In particular, $B$ has the following property: For infinitely many constants $c \in \mathbb{C}$, there exist polynomials $r_c(x) \in \mathbb{C}[x]$ and $s_c(x,y,z) \in \mathbb{C}[x,y,z]$ such that

$$B(x,y,z) = r_c(x) + s_c(x,y,z)(x^n y - Q_1(x,z) - c).$$

We will show that any polynomial with this property must belong to $\mathbb{C}[x, x^n y - Q_1(x,z)]$. Note that it suffices to show that at least one polynomial $s_c$ belongs to $\mathbb{C}[x, x^n y - Q_1(x,z)]$.

In order to see this, we consider the degree function relative to $y$ and $z$, i.e., the degree function $d$ on $\mathbb{C}[x,y,z]$ defined, for every $f \in \mathbb{C}[x,y,z]$, by $d(f) := \deg(\tilde{f}(y,z))$, with $\tilde{f}(y,z) = f(x,y,z) \in \mathbb{C}[x][y,z]$.

If $B(x,y,z) = r_{c_0}(x) + s_{c_0}(x,y,z)(x^n y - Q_1(x,z) - c_0)$ for one $c_0 \in \mathbb{C}$, then $s_{c_0}$ satisfies also the above property. Indeed, as $B(x,y,z) = r_c(x) + s_c(x,y,z)(x^n y - Q_1(x,z) - c)$ for infinitely many $c \in \mathbb{C}$, we can write

$$(c - c_0)s_{c_0}(x,y,z) = r_c(x) - r_{c_0}(x) + (s_c(x,y,z) - s_{c_0}(x,y,z))(x^n y - Q_1(x,z) - c),$$

for infinitely many constants $c \in \mathbb{C}$. Therefore, since the degree $d(s_0)$ of $s_0$ is strictly less than $d(B)$, the desired result can be obtained by induction on the degree $d$. □

## 4. Classification up to isomorphism

In this section we give the classification of Danielewski hypersurfaces in standard form. Together with Theorem 5, this effectively classifies all the Danielewski hypersurfaces up to isomorphism.

**Theorem 9.**

(1) *Two Danielewski hypersurfaces $X_{Q_1,n_1}$ and $X_{Q_2,n_2}$ in standard form are isomorphic if and only if the two following conditions are satisfied*:

(a) $n_1 = n_2 = n$;
(b) $\exists a, \alpha, \mu \in \mathbb{C}^*$, $\exists \beta(x) \in \mathbb{C}[x]$ *such that*

$$Q_2(ax, \alpha z + \beta(x)) \equiv \mu Q_1(x, z) \mod (x^n).$$

(2) *Two Danielewski hypersurfaces* $X_{Q_1,n_1}$ *and* $X_{Q_2,n_2}$ *in reduced standard form are isomorphic if and only if the two following conditions are satisfied*:

(a) $n_1 = n_2$;
(b) $\exists a, \alpha, \mu \in \mathbb{C}^*$, $\exists \beta \in \mathbb{C}$ *such that*

$$Q_2(ax, \alpha z + \beta) = \mu Q_1(x, z).$$

*Proof.* Let $X_1 = X_{Q_1,n_1}$ and $X_2 = X_{Q_2,n_2}$ be two isomorphic Danielewski hypersurfaces in standard form and let $\varphi : X_1 \to X_2$ be an isomorphism. Then Corollary 8 implies that $n_1 = n_2 = n$. Since the case $n = 1$ was already done by Daigle [Dai], we can suppose that $n \geqslant 2$.

Denote by $x_i, y_i, z_i$ the images of $x, y, z$ in the coordinate ring $\mathbb{C}[X_i]$ for $i = 1, 2$. Then, due to Corollary 8, there exist constants $a, \alpha, \mu \in \mathbb{C}^*$ and a polynomial $\beta(x) \in \mathbb{C}[x]$ such that $\varphi^*(x_2) = ax_1$, $\varphi^*(z_2) = \alpha z_1 + \beta(x_1)$ and

$$Q_2(0, \alpha z + \beta(0)) = \mu Q_1(0, z). \tag{1}$$

It follows that

$$a^n x_1^n \varphi^*(y_2) = \varphi^*(x_2^n y_2) = \varphi^*(Q_2(x_2, z_2)) = Q_2(ax_1, \alpha z_1 + \beta(x_1)) = \mu x_1^n y_1 + \Delta(x_1, z_1)$$

where $\Delta(x, z) = Q_2(ax, \alpha z + \beta(x)) - \mu Q_1(x, z)$. Thus, the following equality holds in $\mathbb{C}[X_1]$:

$$x_1^n(a^n \varphi^*(y_2) - \mu y_1) = \Delta(x_1, z_1). \tag{2}$$

Then (1) and the fact that $X_1$ and $X_2$ are in standard form imply $\deg_z \Delta(x, z) < d$, where $d = \deg_z Q_1(0, z) = \deg_z Q_2(0, z)$.

This implies that $a^n \varphi^*(y_2) - \mu y_1 = p(x_1, z_1)$ for a polynomial $p \in \mathbb{C}[x, z]$. To see this, use equation (2) and the fact that, since $X_1$ is in standard form and since $x_1^n y_1 = Q_1(x_1, z_1)$ is the unique relation in $\mathbb{C}[X_1]$, any element $f \in \mathbb{C}[X_1]$ admits a unique expression of the form $f = p(x_1, y_1, z_1)$ where $p \in \mathbb{C}[x, y, z]$ is a polynomial with $\deg_z(p) < d$.

Therefore $x^n$ divides $\Delta(x, z)$ in $\mathbb{C}[x, z]$ and $X_1$ and $X_2$ fulfill conditions (1)(a) and (1)(b).

If $X_1$ and $X_2$ are in reduced standard form, then we see easily that $\Delta(x, z) \equiv 0 \mod (x^n)$ is possible only if $\beta(x) \equiv \beta(0) \mod (x^n)$. If so, $Q_2(ax_1, \alpha z_1 + \beta(0)) = \mu Q_1(x_1, z_1)$ and $X_1$ and $X_2$ fulfill conditions (2)(a) and (2)(b).

Conversely, suppose that $X_1 = X_{Q_1,n}$ $X_2 = X_{Q_2,n}$ are two Danielewski hypersurfaces which satisfy conditions (a) and (b) of part (1). Then the following triangular automorphism of $\mathbb{C}^3$ induces an isomorphism between $X_1$ and $X_2$:

$$(x, y, z) \mapsto (ax, \mu a^{-n} y + (ax)^{-n}(Q_2(ax, \alpha z + \beta(x)) - \mu Q_1(x, z)), \alpha z + \beta(x)). \quad \square$$

As a corollary, we observe that two isomorphic Danielewski hypersurfaces in standard form are equivalent via a triangular automorphism of $\mathbb{C}^3$, and that two isomorphic Danielewski hypersurfaces in reduced standard form are equivalent via an affine one. In fact, we have even proven a stronger result in the proof of Theorem 9.

**Proposition 10.** *Every isomorphism between two isomorphic Danielewski hypersurfaces in standard form can be lifted to a triangular automorphism of* $\mathbb{C}^3$.

## 5. Equivalence classes

In this section we prove the following result.

**Theorem 11.** *Two Danielewski hypersurfaces* $X_{Q_1,n_1}$ *and* $X_{Q_2,n_2}$ *are equivalent if and only if* $n_1 = n_2 = n$ *and there exist* $a, \alpha, \mu \in \mathbb{C}^*$, $\beta \in \mathbb{C}$ *and* $B \in \mathbb{C}^{[2]}$ *such that*

$$Q_2\big(ax, \alpha z + \beta + xB(x, -Q_1(x,z))\big) \equiv \mu Q_1(x,z) \mod (x^n).$$

Before proving Theorem 11, let us give another result. Given two Danielewski hypersurfaces, it is not easy to check if the second condition in Theorem 11 is fulfilled. Therefore, we also show that any Danielewski hypersurface is equivalent to another one which is unique up to an affine automorphism.

**Theorem 12.**

(1) *Every Danielewski hypersurface is equivalent to a Danielewski hypersurface* $X(p, \{q_{k,i}\}, n)$ *defined by an equation of the form*

$$x^n y - p(z) - \sum_{k=1}^{n-1} \sum_{i=2}^{\deg(p)} x^k p^{(i)}(z) q_{k,i}(p(z)) = 0,$$

*where* $p(z) \in \mathbb{C}[z]$, $q_{k,i} \in \mathbb{C}^{[1]}$ *and where* $p^{(i)}(z)$ *denotes the ith derivative of* $p$. *Moreover, there is an algorithmic procedure which computes, given a Danielewski hypersurface* $X$, *a hypersurface* $X(p, \{q_{k,i}\}, n)$ *which is equivalent to* $X$.

(2) *Two such Danielewski hypersurfaces* $X(p_1, \{q_{1,k,i}\}, n_1)$ *and* $X(p_2, \{q_{2,k,i}\}, n_2)$ *are equivalent if and only if* $n = n_1 = n_2$, $\deg(p_1) = \deg(p_2) = d$ *and there exist some constants* $a, \alpha, \mu \in \mathbb{C}^*$, $\beta \in \mathbb{C}$ *such that* $p_2(\alpha z + \beta) = \mu p_1(z)$ *and* $a^k \alpha^{-i} q_{2,k,i}(\mu t) = q_{1,k,i}(t)$ *for every* $1 \leqslant k \leqslant n-1$ *and* $2 \leqslant i \leqslant d$.

*Remark 2.* This result generalizes the classification of Danielewski hypersurfaces of the form $x^2y - z^2 - xq(z) = 0$ given by Moser-Jauslin and the author in [MJP].

*Proof of Theorem* 11. Let $X_{Q_1,n_1}$ and $X_{Q_2,n_2}$ be two equivalent Danielewski hypersurfaces. Then, the second part of Corollary 8 implies $n_1 = n_2 = n$.

If $n = 1$, the result is already known. Indeed, by Lemma 1, every Danielewski hypersurface $X_{Q,1}$ with $n = 1$ is equivalent to one of the form $X_{p,1}$ with $p(x,z) = p(z) \in \mathbb{C}[z]$ and D. Daigle [Dai] has proven that two such hypersurfaces $X_{p_1,1}$ and $X_{p_2,1}$ are isomorphic if and only if $p_2(az+b) = \mu p_1(z)$ for some constants $a, \mu \in \mathbb{C}^*$ and $b \in \mathbb{C}$.

We now assume $n \geqslant 2$ and we let $\Phi$ be an automorphism of $\mathbb{C}^3$ such that $\Phi(X_{Q_1,n}) = X_{Q_2,n}$. Corollary 8 gives us constants $a, \alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$ and a polynomial $B \in \mathbb{C}^{[2]}$ such that $\Phi^*(x) = ax$ and $\Phi^*(z) = \alpha z + \beta + xB(x, x^n y - Q_1(x,z))$. Since the polynomial $x^n y - Q_1(x,z)$ is irreducible, there exists a nonzero constant $\mu \in \mathbb{C}^*$ such that

$$\Phi^*\big(x^n y - Q_2(x,z)\big) = \mu(x^n y - Q_1(x,z)).$$

It follows that $Q_2\big(ax, \alpha z + \beta + xB(x, -Q_1(x,z))\big) \equiv \mu Q_1(x,z) \mod (x^n)$, as desired.

Conversely, let $X_{Q_1,n}$ and $X_{Q_2,n}$ be two Danielewski hypersurfaces with

$$Q_2\big(ax, \alpha z + \beta + xB(x, -Q_1(x,z))\big) \equiv \mu Q_1(x,z) \mod (x^n)$$

for some $a, \alpha, \mu \in \mathbb{C}^*$, $\beta \in \mathbb{C}$ and $B \in \mathbb{C}^{[2]}$. We let

$$R(x,y,z) = x^{-n}\big[Q_2\big(ax, \alpha z + \beta + xB(x, x^n y - Q_1(x,z))\big) - \mu Q_1(x,z)\big] \in \mathbb{C}[x,y,z]$$

and define an endomorphism of $\mathbb{C}^3$ by

$$\Phi(x,y,z) = (ax, a^{-n}\mu y + a^{-n}R(x,y,z), \alpha z + \beta + xB(x, x^n y - Q_1(x,z))).$$

Note that $\Phi^*(x^n y - Q_2(x,z)) = \mu(x^n y - Q_1(x,z))$. Therefore, the theorem will be proved if we show that $\Phi$ is invertible. It is enough to prove that $\Phi^*$ is surjective, i.e.,

$$\mathbb{C}[x,y,z] \subset \Phi^*(\mathbb{C}[x,y,z]) = \mathbb{C}[\Phi^*(x), \Phi^*(y), \Phi^*(z)].$$

We know already that $x$ and $P_1 := x^n y - Q_1(x,z)$ are in the image of $\Phi^*$. Then, since $z = \alpha^{-1}(\Phi^*(z) - \beta - xB(x, P_1))$, we obtain that $z$ belongs to $\Phi^*(\mathbb{C}[x,y,z])$.

Thus, $\Phi^*$ is a birational map, i.e., $\mathbb{C}(\Phi^*(x), \Phi^*(y), \Phi^*(z)) = \mathbb{C}(x,y,z)$. On the other hand, one checks that the determinant of the Jacobian of $\Phi^*$ equals $a^{-n+1}\mu\alpha \in \mathbb{C}^*$. This allows us to conclude that $\Phi^*$ is surjective (see, for example, [vdE, Cor. 1.1.34]). The theorem is proved. $\square$

*Proof of Theorem* 12. (1) Let $Q(x,z) \in \mathbb{C}[x,z]$ be such that $p(z) = Q(0,z)$ is nonconstant. We begin by proving that the following statement is true for all positive integers $m$:

**Claim.** *There exist $B_m \in \mathbb{C}^{[2]}$, $q_{k,i} \in \mathbb{C}^{[1]}$ such that*

$$Q\big(x, z + xB_m(x, -Q_m(x,z))\big) \equiv Q_m(x,z) \mod (x^m),$$

*where*

$$Q_m(x,z) = p(z) + \sum_{k=1}^{m-1} \sum_{i=2}^{\deg(p)} x^k p^{(i)}(z) q_{k,i}(p(z)).$$

If the above statement is called "$P(m)$", then it is obvious that $P(1)$ is true. Assume now that $P(m)$ is true. We prove $P(m+1)$. Let $q_m(z) \in \mathbb{C}[z]$ be such that

$$Q\big(x, z + xB_m(x, -Q_m(x,z))\big) \equiv Q_m(x,z) + x^m q_m(z) \mod (x^{m+1})$$

and define, for $1 \leqslant i \leqslant \deg(p)$, the polynomials $q_{m,i} \in \mathbb{C}^{[1]}$ by expressing $q_m$ as the following sum:

$$q_m(z) = \sum_{i=1}^{\deg(p)} p^{(i)}(z) q_{m,i}(p(z)).$$

Then we let

$$Q_{m+1}(x,z) = Q_m(x,z) + x^m \sum_{i=2}^{\deg(p)} p^{(i)}(z) q_{m,i}(p(z)) \in \mathbb{C}[x,z]$$

and

$$B_{m+1}(x,t) = B_m(x,t) - x^{m-1} q_{m,1}(-t) \in \mathbb{C}[x,t].$$

By Taylor's Formula, it follows that

$$\begin{aligned}
&p\big(z+xB_{m+1}(x,-Q_{m+1}(x,z))\big)\\
&\quad = p\big(z + xB_m(x,-Q_{m+1}(x,z)) - x^m q_{m,1}(Q_{m+1}(x,z))\big)\\
&\quad = \sum_{i=0}^{\deg(p)} \frac{1}{i!} p^{(i)}\big(z + xB_m(x,-Q_{m+1}(x,z))\big)\big(-x^m q_{m,1}(Q_{m+1}(x,z))\big)^i\\
&\quad \equiv p\big(z + xB_m(x,-Q_{m+1}(x,z))\big) - x^m p'(z) q_{m,1}(Q_{m+1}(x,z)) \mod (x^{m+1})\\
&\quad \equiv p\big(z + xB_m(x,-Q_m(x,z))\big) - x^m p'(z) q_{m,1}(p(z)) \mod (x^{m+1}).
\end{aligned}$$

If we denote $Q(x,z) = p(z) + xq(x,z)$, we finally obtain

$$\begin{aligned}
&Q\big(x, z + xB_{m+1}(x,-Q_{m+1}(x,z))\big)\\
&\quad = p\big(z + xB_{m+1}(x,-Q_{m+1}(x,z))\big) + xq\big(x, z + xB_{m+1}(x,-Q_{m+1}(x,z))\big)\\
&\quad \equiv p\big(z + xB_m(x,-Q_m(x,z))\big) - x^m p'(z) q_{m,1}(p(z)) +\\
&\qquad\quad xq\big(x, z + xB_m(x,-Q_m(x,z))\big) \mod (x^{m+1})\\
&\quad \equiv Q\big(x, z + xB_m(x,-Q_m(x,z))\big) - x^m p'(z) q_{m,1}(p(z)) \mod (x^{m+1})\\
&\quad \equiv Q_m(x,z) + x^m q_m(z) - x^m p'(z) q_{m,1}(p(z)) \mod (x^{m+1})\\
&\quad \equiv Q_{m+1}(x,z) \mod (x^{m+1}).
\end{aligned}$$

So $P(m)$ is true for all $m \geq 1$. To complete the proof of assertion (1), consider an arbitrary Danielewski hypersurface $X_{Q,n}$. As $P(n)$ is true, there exist $B \in \mathbb{C}^{[2]}$, $q_{k,i} \in \mathbb{C}^{[1]}$ such that

$$Q\big(x, z + xB(x,-\tilde{Q}(x,z))\big) \equiv \tilde{Q}(x,z) \mod (x^n),$$

where $\tilde{Q}(x,z) = p(z) + \sum_{k=1}^{n-1} \sum_{i=2}^{\deg(p)} x^k p^{(i)}(z) q_{k,i}(p(z))$. By Theorem 11, it follows that $X_{Q,n}$ is equivalent to $X_{\tilde{Q},n} = X(p, \{q_{k,i}\}, n)$, as desired.

(2) Let $X_1 = X(p_1, \{q_{1,k,i}\}, n_1)$ and $X_2 = X(p_2, \{q_{2,k,i}\}, n_2)$ and set, for $j = 1, 2$,

$$Q_j(x,z) = p_j(z) + \sum_{k=1}^{n_j-1} \sum_{i=2}^{\deg(p_j)} x^k p_j^{(i)}(z) q_{j,k,i}(p_j(z)).$$

If $X_1$ and $X_2$ are equivalent, then, by Theorem 11, $n_1 = n_2 = n$ and there exist $a, \alpha, \mu \in \mathbb{C}^*$, $\beta \in \mathbb{C}$ and $B \in \mathbb{C}^{[2]}$ such that

$$Q_2(ax, \alpha z + \beta + xB(x,-Q_1(x,z))) \equiv \mu Q_1(x,z) \mod (x^n).$$

This implies $p_2(\alpha z+\beta)=\mu p_1(z)$. Thus, $\deg(p_1)=\deg(p_2)=d$.

We will now prove that $B(x,t)\equiv 0 \mod (x^{n-1})$. In order to do this, suppose that we can write $B(x,t)\equiv b_l(t)x^l \mod (x^{l+1})$ for some $0\leqslant l\leqslant n-2$ and $b_l(t)\in\mathbb{C}[t]\setminus\{0\}$. Then, we obtain the following congruences modulo $(x^{l+2})$.

$$\begin{aligned}
\mu Q_1(x,z) &\equiv Q_2\big(ax,\alpha z+\beta+xB(x,-Q_1(x,z))\big) \mod (x^n)\\
&\equiv Q_2\big(ax,\alpha z+\beta+x^{l+1}b_l(-p_1(z))\big) \mod (x^{l+2})\\
&\equiv p_2\big(\alpha z+\beta+x^{l+1}b_l(-p_1(z))\big)+\\
&\quad \sum_{k=1}^{l+1}\sum_{i=2}^{d}(ax)^k p_2^{(i)}(\alpha z+\beta)q_{2,k,i}(p_2(\alpha z+\beta)) \mod (x^{l+2})\\
&\equiv p_2(\alpha z+\beta)+x^{l+1}b_l(-p_1(z))p_2'(\alpha z+\beta)+\\
&\quad \sum_{k=1}^{l+1}\sum_{i=2}^{d} a^k x^k \alpha^{-i}\mu p_1^{(i)}(z)q_{2,k,i}(\mu p_1(z)) \mod (x^{l+2})\\
&\equiv \mu p_1(z)+x^{l+1}b_l(-p_1(z))\alpha^{-1}\mu p_1'(z)+\\
&\quad \mu\sum_{k=1}^{l+1}\sum_{i=2}^{d} a^k\alpha^{-i}x^k p_1^{(i)}(z)q_{2,k,i}(\mu p_1(z)) \mod (x^{l+2}).
\end{aligned}$$

Since $\mathbb{C}[x,z]$ is a free $\mathbb{C}[p_1(z)]$-module with basis $\{x^k p_1^{(i)}(z)\mid k\in\mathbb{N},1\leqslant i\leqslant d\}$, the polynomial $Q_1(x,z)$ has a unique expression as a finite sum

$$Q_1(x,z)=\sum_{k\in\mathbb{N}}\sum_{i=2}^{d}x^k p_1^{(i)}(z)f_{k,i}(p_1(z))$$

where $f_{k,i}(p_1(z))\in\mathbb{C}[p_1(z)]$ for all $k,i$. Note that the assumption on $Q_1$ implies that $f_{k,1}=0$ for all $k$. But, on the other hand, the above calculation gives $f_{l+1,1}(p_1(z))=\alpha^{-1}b_l(-p_1(z))\neq 0$, a contradiction.

Therefore, $B(x,t)\equiv 0 \mod (x^{n-1})$ and it follows, since $\deg_x(Q_j(x,z))<n$ for $j=1,2$ by hypothesis, that $Q_2(ax,\alpha z+\beta)=\mu Q_1(x,z)$. Then, we can easily check that this last equality implies $a^k\alpha^{-i}q_{2,k,i}(\mu t)=q_{1,k,i}(t)$ for every $1\leqslant k\leqslant n-1$ and $2\leqslant i\leqslant d$, as desired.

Finally note that under these conditions, $X_1$ and $X_2$ are equivalent via the affine automorphism of $\mathbb{C}^3$ given by $(x,y,z)\mapsto(ax,a^{-n}\mu y,\alpha z+\beta)$. This concludes the proof. □

It should be noticed that a Danielewski hypersurface is in general not equivalent to its (reduced) standard form given by Theorem 5. Moreover, one can use this fact to construct nonequivalent embeddings for every Danielewski hypersurface with nontrivial Makar-Limanov invariant.

**Proposition 13.** *Every Danielewski hypersurface $X_{Q,n}$ with $n\geqslant 2$ admits at least two nonequivalent embeddings into $\mathbb{C}^3$.*

*Proof.* Since, by Theorem 5, every Danielewski hypersurface is isomorphic to one in standard form, it suffices to show that every Danielewski hypersurface in standard form $X_{Q,n}$ with $n \geqslant 2$ admits at least two nonequivalent embeddings in $\mathbb{C}^3$.

Let $X = X_{Q,n}$ be a Danielewski hypersurface in standard form with $n \geqslant 2$. Due to Lemma 3, $X$ is isomorphic to the hypersurface $Y = X_{(1+x)Q(x,z),n}$. Nevertheless, it turns out that $X$ and $Y$ are nonequivalent hypersurfaces of $\mathbb{C}^3$. Indeed, if they were, Theorem 11 would give us constants $a, \alpha, \mu \in \mathbb{C}^*$, $\beta \in \mathbb{C}$ and a polynomial $B \in \mathbb{C}^{[2]}$ such that

$$(1+ax)Q\big(ax, \alpha z + \beta + xB(x, -Q(x,z))\big) \equiv \mu Q(x,z) \mod (x^n).$$

If we denote $Q(x,z) = p(z) + xq(x,z)$, it would give the following congruences modulo $(x^2)$:

$$\begin{aligned}
\mu Q(x,z) &\equiv (1+ax)Q\big(ax, \alpha z + \beta + xB(0, -Q(0,z))\big) \mod (x^2) \\
&\equiv (1+ax)\big[p\big(\alpha z + \beta + xB(0, -p(z))\big) + axq(0, \alpha z + \beta)\big] \mod (x^2) \\
&\equiv (1+ax)p\big(\alpha z + \beta + xB(0, -p(z))\big) + axq(0, \alpha z + \beta) \mod (x^2) \\
&\equiv p(\alpha z + \beta) + xB(0, -p(z))p'(\alpha z + \beta) + axp(\alpha z + \beta) + \\
&\quad\ axq(0, \alpha z + \beta) \mod (x^2).
\end{aligned}$$

Thus,

$$B(0, -p(z))p'(\alpha z + \beta) + ap(\alpha z + \beta) + aq(0, \alpha z + \beta) = \mu q(0, z),$$

which is impossible since $\deg(q(0,z)) < \deg(p)$ by definition of a standard form. $\square$

*Remark 3.* This proof is similar to the proof of Freudenburg and Moser-Jauslin in [FMJ] for hypersurfaces of equation $x^n y = p(z)$ with $n \geqslant 2$. In their article, they also have constructed nonequivalent embeddings into $\mathbb{C}^3$ for Danielewski hypersurfaces of the form $xy - z^d - 1 = 0$ for some $d \in \mathbb{N}$. Nevertheless, we do not know if every Danielewski hypersurface $X_{Q,1}$ admits nonequivalent embeddings into $\mathbb{C}^3$. For instance, the following question, which they posed in [FMJ], is still open.

**Question 1.** Does the hypersurface of equation $xy + z^2 = 0$ admit a unique embedding into $\mathbb{C}^3$?

Note also that the two nonequivalent embeddings of a Danielewski hypersurface $X_{Q,n}$ with $n \geqslant 2$ which we construct in Proposition 13 are *analytically* equivalent. Indeed, it can be easily seen, as in [FMJ] and [DP], that a Danielewski hypersurface is analytically equivalent to its standard form given by Theorem 5. Then we obtain the following result.

**Proposition 14.** *If $X_1$ and $X_2$ are two isomorphic Danielewski hypersurfaces, then there is an analytic automorphism $\Psi$ of $\mathbb{C}^3$ such that $\Psi(X_1) = X_2$.*

*Proof.* Let $X = X_{Q,n}$ be a Danielewski hypersurface and let $X_{Q_s,n}$ be its standard form given by Theorem 5. By Lemma 4, we can let $Q(x,z) = (1+x\pi(x,z))Q_s(x,z) + x^n R(x,z)$ for certain polynomials $\pi(x,z), R(x,z) \in \mathbb{C}[x,z]$. Consider the analytic automorphism of $\mathbb{C}^3$ defined by

$$\begin{aligned}\Psi(x,y,z) = (x, \exp(xf(x,z))y - x^{-n}(&\exp(xf(x,z)) \\ &- 1 - x\pi(x,z))Q_s(x,z) + R(x,z), z),\end{aligned}$$

where $f(x,z) \in \mathbb{C}[x,z]$ is a polynomial so that $\exp(xf(x,z)) \equiv 1 + x\pi(x,z) \mod (x^n)$. One checks that $\Psi^*(x^n y - Q(x,z)) = \exp(xf(x,z))(x^n y - Q_s(x,z))$. Thus, $\Psi$ maps $X_{Q_s,n}$ onto $X_{Q,n}$. In other words, every Danielewski hypersurface is analytically equivalent to one in standard form. The proposition follows since two isomorphic Danielewski hypersurfaces in standard form are equivalent by Proposition 10. □

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