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Eigenvalues of the Laplacian and extrinsic geometry

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Abstract We extend the results given by Colbois, Dryden and El Soufi on the relationships between the eigenvalues of the Laplacian and an extrinsic invariant called *intersection index*, in two directions. First, we replace this intersection index by invariants of the same nature which are stable under *small perturbations*. Second, we consider complex submanifolds of the complex projective space $\mathbb{C}P^N$ instead of submanifolds of $\mathbb{R}^N$ and we obtain an eigenvalue upper bound depending only on the dimension of the submanifold which is sharp for the first non-zero eigenvalue.

Keywords Laplacian · Eigenvalue · Upper bound · Intersection index

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1 Introduction and statement of the results

For a compact manifold without boundary, the spectrum of the Laplace-Beltrami operator Δ consists of an unbounded non-decreasing sequence of non-negative real numbers

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \nearrow \infty,$$

where each eigenvalue λ_k has finite multiplicity. The study of the relationship between the extrinsic geometry of submanifolds and the spectrum of the Laplace-Beltrami operator is an important topic of spectral geometry. One of the well-known extrinsic invariants is the mean curvature vector field of a submanifold. In this regard, we can mention the Reilly inequality [11] for an immersed m-dimensional submanifold M of $\mathbb{R}^N$

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$$\lambda_2(M) \leq \frac{m}{\mathrm{Vol}(M)} \|H(M)\|_2^2,$$

where $\|H(M)\|_2$ is the L^2-norm of the mean curvature vector field of M. For higher eigenvalues, it follows from results of El Soufi et al. [9] that for every $k \in \mathbb{N}^*$,

$$\lambda_k(M) \leq R(m)\|H(M)\|_\infty^2 \, k^{2/m},$$

where $\|H(M)\|_\infty$ is the L^∞-norm of $H(M)$ and $R(m)$ is a constant depending only on m. Since the variational characterization of eigenvalues do not depend on derivatives of the metric, we are interested in extrinsic invariants which do not depend on metric derivatives, excluding for instance curvature. The *intersection index* (see below for the definition) is an important example of such intrinsic invariants. Colbois et al. [6] studied the relationship between the intersection index, and the eigenvalues of the Laplace-Beltrami operator. In this paper, we review and extend their results.

For a compact m-dimensional immersed submanifold M of $\mathbb{R}^N = \mathbb{R}^{m+p}$, $p > 0$, the *intersection index* is given by

$$i(M) = \sup_{\Pi} \sharp(M \cap \Pi),$$

where Π runs over the set of all p-planes that are transverse to M; if M is not embedded, we count multiple points of M according to their multiplicity. We remark that the intersection index was also investigated by Thom [3] where it was called the *degree* of M. Colbois et al. [6] show that there is a positive constant $c(m)$, depending only on m, such that for every compact m-dimensional immersed submanifold M of $\mathbb{R}^{m+p}$, we have the following inequality:

$$\lambda_k(M)\mathrm{Vol}(M)^{2/m} \leq c(m) i(M)^{2/m} k^{2/m}. \tag{1}$$

Moreover, the intersection index in the above inequality is not replaceable with a constant depending only on the dimension m. Even for hypersurfaces, the first positive eigenvalue cannot be controlled only in terms of the volume and the dimension (see [6, Theorem 1.4]). As an immediate consequence of Inequality (1), the normalized eigenvalues on convex hypersurfaces are bounded above only in terms of the dimension. Another remarkable consequence of Inequality (1) concerns algebraic submanifolds [6, Corollary 4.1]: Let M be a compact real algebraic manifold, i.e. M is a zero locus of p real polynomials in $m + p$ variables of degrees $N_1, \ldots, N_p$. Then,

$$\lambda_k(M)\mathrm{Vol}(M)^{2/m} \leq c(m) N_1^{2/m} \cdots N_p^{2/m} k^{2/m}. \tag{2}$$

Note that Inequalities (1) and (2) are not stable under "*small*" perturbations, since the intersection index might dramatically change.

We extend the work of Colbois, Dryden and El Soufi in two directions. The first one consists in replacing the intersection index $i(M)$ by invariants of the same nature which are stable under *small perturbations*. The second direction concerns complex submanifolds of the complex projective space $\mathbb{C}P^N$. Here we obtain an eigenvalue upper bound for submanifolds of $\mathbb{C}P^N$ depending only on the dimension. Below we describe the main results of this paper.

1.1 First part

Let $\varepsilon < 1$ be a positive number. By an *ε-small perturbation*, we mean any perturbation in a region $D \subset M$ whose measure is at most equal to εVol(M). To avoid any technical complexity, we assume that $M \setminus D$ is a smooth manifold with smooth boundary. Here, we define new notions of intersection indices which are stable under any ε-small perturbation. Let

G be the Grassmannian of all m-vector spaces in $\mathbb{R}^{m+p}$ endowed with the $O(m+p)$- invariant Haar measure of total measure 1. Let $0 < \varepsilon < 1$ and D be any open subdomain of M such that $M \setminus D$ is a smooth manifold with smooth boundary and $\mathrm{Vol}(D) \leq \varepsilon \mathrm{Vol}(M)$. We denote $M \setminus D$ by M_ε^D. Let H be an m-vector space in G. We define $i_H(M_\varepsilon^D) := \sup_{P \perp H} \sharp(M_\varepsilon^D \cap P)$, where P runs over affine p-planes orthogonal to H. We now define the *ε-mean intersection index* as follows:

$$\bar{\imath}^\varepsilon(M) := \inf_D \int_G i_H(M_\varepsilon^D)dH,$$

where D runs over regions whose measure is smaller than $\varepsilon \mathrm{Vol}(M)$ and $M \setminus D$ is a smooth manifold with smooth boundary.

Similarly, for $r > 0$, we define the *(r, ε)-local intersection index* as:

$$\bar{\imath}_r^\varepsilon(M) := \inf_D \sup_{x \in M_\varepsilon^D} \int_G i_H(M_\varepsilon^D \cap B(x, r))dH,$$

where $B(x, r) \subset \mathbb{R}^{m+p}$ is an Euclidean ball centered at x and of radius r and D runs over regions whose measure is smaller than $\varepsilon \mathrm{Vol}(M)$ and $M \setminus D$ is a smooth manifold with smooth boundary.

We can now state our theorem.

Theorem 1.1 *There exist positive constants c_m, α_m and β_m depending only on m such that for every compact m-dimensional immersed submanifold M of $\mathbb{R}^{m+p}$, every $r > 0$, $k \in \mathbb{N}^*$, and $0 < \varepsilon < 1$, we have*

$$\lambda_k(M) Vol(M)^{2/m} \leq c_m \frac{\bar{\imath}^\varepsilon(M)^{2/m}}{(1-\varepsilon)^{1+2/m}} k^{2/m}, \tag{3}$$

and

$$\lambda_k(M) \leq \alpha_m \frac{1}{(1-\varepsilon)r^2} + \beta_m \frac{\bar{\imath}_r^\varepsilon(M)^{2/m}}{(1-\varepsilon)^{1+2/m}} \left(\frac{k}{Vol(M)}\right)^{2/m}. \tag{4}$$

The main feature of the inequalities (3) and (4) is that the upper bounds are not considerably affected by the presence of a large intersection index in a "small" part of M (i.e. a subdomain with small volume). In particular, for a compact hypersurface of $\mathbb{R}^{m+1}$ which is convex outside a region[1] D of measure at most εVol(M), one has $\bar{\imath}^\varepsilon(M) \leq i(M_\varepsilon^D)$ and then

$$\lambda_k(M) \mathrm{Vol}(M)^{2/m} \leq c_m \frac{2^{2/m}}{(1-\varepsilon)^{1+2/m}} k^{2/m}.$$

We also note that one has Inequality (2) not only for compact algebraic submanifolds of $\mathbb{R}^N$, but also for every ε-perturbation of those algebraic submanifolds, where the constant $c(m)$ in (2) depends only on m and on ε.

1.2 Second part

We study another natural context where algebraic submanifolds can be considered which is the complex projective space $\mathbb{C}P^N$. According to Chow's Theorem ([10]), every complex

[1] We say that M is convex outside of D, if after a perturbation of M which is the identity outside of D we get a convex compact hypersurface.

submanifold M of $\mathbb{C}P^N$ is a smooth algebraic variety, i.e. it is a zero locus of a family of complex polynomials. We obtain the following upper bound for complex submanifolds of $\mathbb{C}P^N$ endowed with Fubini-Study metric g_{FS}.

Theorem 1.2 *Let M^m be an m-dimensional complex manifold admitting a holomorphic immersion $\phi : M \to \mathbb{C}P^N$. Then for every $k \in \mathbb{N}^*$ we have*

$$\lambda_{k+1}(M, \phi^* g_{FS}) \le 2(m+1)(m+2)k^{\frac{1}{m}} - 2m(m+1). \tag{5}$$

In particular, one has Inequality (5) for every complex submanifold of $\mathbb{C}P^N$. Note that the power of k is compatible with the Weyl law. Under the assumption of Theorem 1.2, for $k = 1$, one has

$$\lambda_2(M, \phi^* g_{FS}) \le 4(m+1), \tag{6}$$

which is a sharp inequality since the equality holds for $\mathbb{C}P^m$. Inequality (6) is obtained by Bourguignon et al. [2, page 200], and also by Arezzo et al. [1]. Note that the results in [1] and [2] are for the first non-zero eigenvalue of Laplacian on a larger family of complex manifolds (see page 527). However, Theorem 1.2 gives an upper bound for higher eigenvalues in addition to a sharp upper bound for λ_2, when we consider the complex submanifolds of $\mathbb{C}P^N$ endowed with the Fubini-Study metric. For a complex submanifold M of $\mathbb{C}P^{m+p}$ of the complex dimension m, we have

$$\mathrm{Vol(M)} = \deg(M)\mathrm{Vol}(\mathbb{C}P^m), \tag{7}$$

where $\deg(M)$ is the intersection number of M with a projective p-plane in a generic position (see for example [10, pages 171–172]). Multiplying Inequality (5) by (7), we get

$$\lambda_{k+1}(M, g_{FS})\mathrm{Vol}(M)^{\frac{1}{m}} \le C(m)\deg(M)^{\frac{1}{m}} k^{\frac{1}{m}}. \tag{8}$$

Moreover, one can describe M as a zero locus of a family of irreducible homogenous polynomials and then $\deg(M)$ is bounded by the multiplication of degrees of the irreducible polynomials describing M. One can now compare Inequality (8) with Inequality (2).

This paper is organized as follows. In Sect. 2, we recall one of the main methods to estimate the eigenvalues in the abstract setting of metric measure spaces introduced by Colbois and Maerten [8]. We use this method to prove Theorem 1.1 in Sect. 3. In Sect. 4, we consider algebraic submanifolds of $\mathbb{C}P^N$ and we prove Theorem 1.2. The method which is used in Sect. 4 to show Theorem 1.2 is independent from what we introduce in Sects. 2 and 3.

2 A general preliminary result

A classical way to estimate the eigenvalues of the Laplacian is to construct a family of disjoint domains and then, to estimate the Rayleigh quotients of the test functions supported on these domains. Colbois and Maerten [8] introduce a method to construct an elaborated family of disjoint domains in the general setting of metric-measure $(m - m)$ spaces. This method shows that eigenvalue upper bounds and controlling the local volume concentration of balls are linked. Here, for an m-dimensional Riemannian submanifold M of $\mathbb{R}^N$, controlling the local volume concentration of balls means to control the constant C in the following inequality for some $\rho > 0$

$$\mathrm{Vol}(M \cap B(x, r)) \le Cr^m \quad \forall x \in M, \quad 0 < r \le \rho,$$

where $B(x, r)$ is a ball of radius r centered at x in $\mathbb{R}^N$.

This section is devoted to recall this construction for metric measure spaces. Throughout this section the triple (X, d, μ) will designate a complete locally compact $m-m$ space with a distance d and a finite, positive, non-atomic Borel measure μ. We also assume that balls in (X, d) are pre-compact. Each pair (F, G) of Borel sets in X such that $F \subset G$ is called a capacitor. For $F \subseteq X$ and $r > 0$, we denote the *r-neighborhood of F* by F^r, that is

$$F^r = \{x \in X : d(x, F) \leq r\}.$$

Definition 2.1 Given $\kappa > 1$, $\rho > 0$ and $N \in \mathbb{N}^*$, we say that a metric space (X, d) satisfies the $(\kappa, N; \rho)$-*covering property* if each ball of radius $0 < r \leq \rho$ can be covered by N balls of radius $\frac{r}{\kappa}$.

Note that when $\rho = \infty$, we simply say that the metric space (X, d) satisfies the (κ, N)-covering property. It is clear that $(\kappa, N; \rho)$-covering property implies $(\kappa, N; \lambda)$-covering property for any $0 < \lambda \leq \rho$.

Lemma 2.1 *([8, Corollary 2.3] and [7, Lemma 2.1]) Let (X, d, μ) be an $m-m$ space satisfying the $(4, N; \rho)$-covering property. For every $n \in \mathbb{N}^*$, let $0 < r \leq \rho$ be such that for each $x \in X$, $\mu(B(x, r)) \leq \frac{\mu(X)}{4N^2 n}$. Then there exists a family $\mathcal{A} = \{(A_i, A_i^r)\}_{i=1}^n$ of capacitors in X such that*

(a) for each i, $\mu(A_i) \geq \frac{\mu(X)}{2Nn}$, and
(b) the subsets $\{A_i^r\}_{i=1}^n$ are mutually disjoint.

We define the *dilatation* of a function $f : (X, d) \to \mathbb{R}$ as

$$\operatorname{dil}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)},$$

and the *local dilatation* at $x \in X$ as

$$\operatorname{dil}_x(f) = \lim_{\varepsilon \to 0} \operatorname{dil}(f|_{B(x,\varepsilon)}).$$

When different distance functions are considered, $\operatorname{dil}_d(f)$ and $\operatorname{dil}_{d,x}(f)$ stand for the dilatation and local dilatation at x associated with the distance d, respectively. A map f is called Lipschitz if $\operatorname{dil}(f) < \infty$. Let (M, g) be a Riemannian manifold and d_g be the distance associated to the Riemannian metric g. A Lipschitz function on a Riemannian manifold M is differentiable almost everywhere and $|\nabla_g f(x)|$ coincides with $\operatorname{dil}_x(f)$ almost everywhere. Hence, $|\nabla_g f(x)| \leq \operatorname{dil}(f)$ almost everywhere.

The following theorem relies on the construction given in the above lemma. It gives a construction of a family of disjointly supported functions with a nice control on their dilatations. Before stating the theorem we need to define the following notation. Given a capacitor (F, G), let $\mathcal{T}(F, G)$ be the set of all compactly supported real valued functions on X such that for every $\varphi \in \mathcal{T}(F, G)$ we have $\operatorname{supp} \varphi \subset \mathrm{G}^\circ = \mathrm{G} \setminus \partial \mathrm{G}$ and $\varphi \equiv 1$ in a neighborhood of F.

Theorem 2.1 *Let positive constants p, ρ, L and N be given and (X, d, μ) be an $m-m$ space satisfying the $(4, N; \rho)$-covering property and*

$$\mu(B(x, r)) \leq Lr^p, \quad \textit{for every} \quad x \in X \quad \textit{and} \quad 0 < r \leq \rho.$$

Then for every $n \in \mathbb{N}^*$ and every $r \leq \min\{\rho, \left(\frac{\mu(X)}{4N^2 Ln}\right)^{1/p}\}$, there is a family of n mutually disjoint bounded capacitors $\{(A_i, A_i^r)\}_{i=1}^n$, of X and a family $\{f_i\}$ of n Lipschitz functions with $f_i \in \mathcal{T}(A_i, A_i^r)$ such that $\mu(A_i) \geq \frac{\mu(X)}{2Nn}$ and

$$\mathrm{dil}_d(f_i) \leq \frac{1}{\rho} + (4N^2L)^{1/p}\left(\frac{n}{\mu(X)}\right)^{1/p}. \tag{9}$$

If the condition $\mu(B(x,r)) \leq Lr^p$ is satisfied for every $r > 0$ then we take $\rho = \infty$. Hence, the first term on the right-hand side of the above inequality vanishes.

Proof of Theorem 2.1 According to Lemma 2.1, if the $m - m$ space (X, d, μ) satisfies $(4, N; \rho)$-covering property, then for every $r \leq \rho$ such that

$$\mu(B(x,r)) \leq \frac{\mu(X)}{4N^2n}, \quad \forall x \in X, \tag{10}$$

we have a family $\{(A_i, A_i^r)\}$ of mutually disjoint capacitors of X with the desired property mentioned in the theorem. We claim that when $r \leq \min\{\rho, \left(\frac{\mu(X)}{4N^2Ln}\right)^{1/p}\}$, the Inequality (10) is automatically satisfied. Indeed, according to the assumptions we have

$$\mu(B(x,r)) \leq Lr^p \leq \min\{L\rho^p, \frac{\mu(X)}{4N^2n}\} \leq \frac{\mu(X)}{4N^2n}.$$

We now consider Lipschitz functions f_i supported on A_i^r with $f_i(x) = 1 - \frac{d(x,A_i)}{r}$ on $A_i^r \setminus A_i$, $f_i(x) = 1$ on A_i and zero outside of A_i^r. One can easily check that $\mathrm{dil}_d(f_i) \leq \frac{1}{r}$. Hence, we obtain

$$\mathrm{dil}_d(f_i) \leq \frac{1}{\rho} + (4N^2L)^{1/p}\left(\frac{n}{\mu(X)}\right)^{1/p}.$$

This completes the proof. □

Let (M, g, μ) be a Riemannian manifold endowed with a finite non-atomic Borel measure μ. We define the following quantity that coincides with the eigenvalues of the Laplace-Beltrami operator when μ coincides with the Riemannian measure μ_g.

$$\lambda_k(M, g, \mu) := \inf_L \sup\{R(f) : f \in L\},$$

where L is a k-dimensional vector space of Lipschitz functions and

$$R(f) = \frac{\int_M |\nabla_g f|^2 d\mu}{\int_M f^2 d\mu}$$

The following corollary is a straightforward consequence of Theorem 2.1 and it is the key result that we use in the next section.

Corollary 2.1 *Let (M, g, μ) be a Riemannian manifold with a finite non-atomic Borel measure μ and the distance d_g associated to the Riemannian metric g. If there exists a measure ν and a distance d so that*

$$d(x, y) \leq d_g(x, y), \quad \forall x, y \in M; \tag{11}$$

$$\nu(A) \leq \mu(A) \quad \textit{for all measurable subsets } A \textit{ of } M, \tag{12}$$

and moreover, there exist positive constants p, ρ, N and L so that (M, d, ν) satisfies the assumptions of Theorem 2.1, then, for every $k \in \mathbb{N}^*$ we have

$$\lambda_k(M, g, \mu) \leq \frac{16N}{\rho^2}\frac{\mu(M)}{\nu(M)} + 16N(8N^2L)^{2/p}\left(\frac{\mu(M)}{\nu(M)}\right)^{1+2/p}\left(\frac{k}{\mu(M)}\right)^{2/p}. \tag{13}$$

Proof Take (M, d, ν) as an $m - m$ space. According to Theorem 2.1, for every $2k \in \mathbb{N}^*$ and every $r \leq \min\{\rho, \left(\frac{\nu(X)}{4N^2Ln}\right)^{1/p}\}$, we have a family of $2k$ mutually disjoint capacitors $\{(A_i, A_i^r)\}_{i=1}^{2k}$ and $2k$ Lipschitz functions f_i such that for every $1 \leq i \leq 2k$, $\nu(A_i) \geq \frac{\nu(M)}{4Nk}$ and the following inequality satisfies almost everywhere.

$$|\nabla_g f_i| \leq \mathrm{dil}_{d_g}(f_i) \leq \mathrm{dil}_d(f_i) \leq \frac{1}{\rho} + (4N^2L)^{1/p}\left(\frac{2k}{\nu(M)}\right)^{1/p},$$

where the last inequality comes form Inequality (9). Since $\mu \geq \nu$, one has

$$\mu(A_i) \geq \nu(A_i) \geq \frac{\nu(M)}{4Nk}. \tag{14}$$

Supports of the f_i are disjoint and $\sum_{i=1}^{2k} \mu(A_i^r) \leq \mu(M)$; therefore, at least k of them have measure smaller than $\frac{\mu(M)}{k}$. Up to re-ordering, we assume that for the first k of the A_i^r, we have

$$\mu(A_i^r) \leq \frac{\mu(M)}{k}. \tag{15}$$

Therefore,

$$\begin{aligned}\lambda_k(M, g, \mu) \leq \max_i R(f_i) &\leq \max_i \left(\frac{1}{\rho} + (4N^2L)^{1/p}\left(\frac{2k}{\nu(M)}\right)^{1/p}\right)^2 \frac{\mu(A_i^r)}{\mu(A_i)}\\ &\leq 16N\left(\frac{1}{\rho^2} + (4N^2L)^{2/p}\left(\frac{2k}{\nu(M)}\right)^{2/p}\right)\frac{\mu(M)}{\nu(M)}.\end{aligned}$$

The last inequality comes from applying Inequalities 14 and 15, together with using the following inequality.

$$(a+b)^2 \leq 4(a^2+b^2) \quad \forall a, b \in \mathbb{R}.$$

In conclusion, we obtain Inequality (13). □

3 Eigenvalues of immersed submanifolds of $\mathbb{R}^N$

In this section, we prove Theorem 1.1. Let S be an m-dimensional immersed submanifold of $\mathbb{R}^{m+p}$ (with or without boundary). We recall that G is the Grassmannian of all m-vector spaces in $\mathbb{R}^{m+p}$ endowed with the $O(m+p)$- invariant Haar measure with total measure 1. Let H be an m-vector space in G and $i_H(S) := \sup_{P \perp H} \sharp(S \cap P)$, where P runs over affine p-planes orthogonal to H. We define the *mean intersection index* of S as follows:

$$\bar{\imath}(S) := \int_G i_H(S)\,dH.$$

Similarly, for every $r > 0$, we define the *r-local intersection index* of S by

$$\bar{\imath}_r(S) := \sup_{x \in S} \int_G i_H(S \cap B(x, r))\,dH,$$

where $B(x, r) \subset \mathbb{R}^{m+p}$ is an Euclidean ball of radius r centered at x.

Let $H \in G$ and $\pi_H : S \to H$ be the orthogonal projection of S on H. The following lemma is an extension of [6, Lemma 2.1].

Lemma 3.1 *Let S be an m-dimensional immersed submanifold of $\mathbb{R}^{m+p}$, (not necessarily without boundary). Then there exists $H_0 \in G$ such that the following inequality satisfies*

$$\mathrm{Vol}(S) \le C_m \bar{\imath}(S)\mathrm{Vol}(\pi_{\mathrm{H}_0}(\mathrm{S})), \tag{16}$$

where C_m is a constant depending only on m.

Proof Since for almost all $H \in G$, a point in $\pi_H(S)$ has finite number of preimages, one can take a generic H and get

$$\int_S \pi_H^* v_H = \int_S |\theta_H(x)| v_S \le \int_{\pi_H(S)} i_H(S) v_H = i_H(S)\mathrm{Vol}(\pi_H(S)),$$

where v_S and v_H are volume elements of S and H, respectively, and

$$|\theta_H(x)| v_S = \pi_H^* v_H.$$

Now, by integrating over G we get

$$\begin{aligned} \int_G i_H(S)\mathrm{Vol}(\pi_{\mathrm{H}}(\mathrm{S}))\mathrm{dH} &\ge \int_G dH \int_S |\theta_H(x)| v_S \\ &= \int_S \left(\int_G |\theta_H(x)| dH \right) v_S \\ &= I(G)\mathrm{Vol}(S), \end{aligned} \tag{17}$$

where $I(G) := \int_G |\theta_H(x)| dH$. The last equality comes from the fact that $I(G)$ does not depend on the point x (see [6, page 101]). We also have

$$\begin{aligned} \int_G i_H(S)\mathrm{Vol}(\pi_{\mathrm{H}}(\mathrm{S}))\mathrm{dH} &\le \sup_H \mathrm{Vol}(\pi_H(S))\bar{\imath}(S) \\ &\le 2\mathrm{Vol}(\pi_{H_0}(S))\bar{\imath}(S), \end{aligned} \tag{18}$$

where H_0 is an m-plane such that $2\mathrm{Vol}(\pi_{H_0}(S)) \ge \sup_H \mathrm{Vol}(\pi_H(S))$. By Inequalities (17) and (18), we get the following inequality:

$$\mathrm{Vol}(\pi_{H_0}(S)) \ge \frac{I(G)\mathrm{Vol}(S)}{2\bar{\imath}(S)}.$$

This proves Inequality (16) with $C_m = \frac{2}{I(G)}$. □

Let M be an m-dimensional immersed submanifold of $\mathbb{R}^{m+p}$. Throughout the rest of this section, for every $\varepsilon \ge 0$, M_ε^D stands for $M \setminus D$, where D is any open subdomain of M such that $M \setminus D$ is a smooth manifold with smooth boundary and $\mathrm{Vol}(D) \le \varepsilon \mathrm{Vol}(M)$.

Corollary 3.1 *For all $x \in \mathbb{R}^{m+p}$ and $\varepsilon \ge 0$, we have*

$$\mathrm{Vol}\big(M_\varepsilon^D \cap B(x,s)\big) \le \frac{2Vol(B^m)}{I(G)} \bar{\imath}_r(M_\varepsilon^D)s^m, \quad \forall\, 0 < s \le r; \tag{19}$$

$$Vol\big(M_\varepsilon^D \cap B(x,r)\big) \le \frac{2Vol(B^m)}{I(G)} \bar{\imath}(M_\varepsilon^D)r^m, \quad \forall r > 0, \tag{20}$$

where B^m is the m-dimensional Euclidean unit ball.

Proof Replacing S by $M_\varepsilon^D \cap B(x,s)$ in Lemma 3.1, we obtain

$$\begin{aligned}\mathrm{Vol}\left(M_\varepsilon^D \cap B(x,s)\right) &\leq \frac{2}{I(G)}\bar{\iota}\left(M_\varepsilon^D \cap B(x,s)\right)\mathrm{Vol}\left(\pi_{H_0}\left(M_\varepsilon^D \cap B(x,s)\right)\right)\\ &\leq \frac{2\mathrm{Vol}(B^m)}{I(G)}\bar{\iota}_s\left(M_\varepsilon^D\right)s^m,\end{aligned}$$

where B^m is the m-dimensional Euclidean unit ball. The last inequality comes from

$$\mathrm{Vol}\left(\pi_{H_0}\left(M_\varepsilon^D \cap B(x,s)\right)\right) \leq \mathrm{Vol}\left(\pi_{H_0}\left(B(x,s)\right)\right) \leq \mathrm{Vol}(B^m)s^m$$

Since $\bar{\iota}_s\left(M_\varepsilon^D\right) \leq \bar{\iota}_r\left(M_\varepsilon^D\right)$ for all $0 < s \leq r$ and $\bar{\iota}_s\left(M_\varepsilon^D\right) \leq \bar{\iota}\left(M_\varepsilon^D\right)$ for all $s > 0$, therefore, we derive Inequalities (19) and (20). □

Remark 3.1 For $\varepsilon = 0$, we have $M_\varepsilon^D = M$. Hence, we have the Inequalities (19) and (20) for M_ε^D replaced by M.

Proof of Theorem 1.1 This theorem is a straightforward consequence of Corollary 2.1. Here, M with the induced metric from $\mathbb{R}^{m+p}$ and the riemannian measure associated to this metric is our metric measure space. We begin with giving candidates for the distance d and the measure ν appeared in the statement of Corollary 2.1, such that the assumptions of Corollary 2.1 are satisfied. Let $d = d_{eu}$ be the Euclidean distance in $\mathbb{R}^{m+p}$ and $\nu = \mu_\epsilon^D$, where $\mu_\epsilon^D(A)$ is the Riemannian volume of $A \cap M_\varepsilon^D$. One can easily check that (M, d_{eu}) has the $(4, N)$-covering property where N depends only on the dimension of the ambient space $\mathbb{R}^{m+p}$. Moreover, one can consider N as a function depending only on the dimension m according to the Nash embedding theorem (see [6, page 106]). There also exists $L > 0$ such that $\mu_\varepsilon^D(B(x,s)) \leq Ls^m$ for $s \leq \rho$. We now consider the two following cases:

- Take $\rho = r$. According to Corollary 3.1, one can take $L = \frac{2\mathrm{Vol}(B^m)}{I(G)}\bar{\iota}_r(M_\varepsilon^D)$. Therefore, Corollary 2.1 implies

$$\lambda_k(M) \leq \alpha_m \frac{1}{(1-\varepsilon)r^2} + \beta_m \frac{\bar{\iota}_r(M_\varepsilon^D)^{2/m}}{(1-\varepsilon)^{1+2/m}}\left(\frac{k}{\mathrm{Vol}(M)}\right)^{2/m}. \tag{21}$$

- Take $\rho = \infty$. According to Corollary 3.1, one can take $L = \frac{2\mathrm{Vol}(B^m)}{I(G)}\bar{\iota}(M_\varepsilon^D)$. Therefore, Corollary 2.1 implies

$$\lambda_k(M) \leq \beta_m \frac{\bar{\iota}(M_\varepsilon^D)^{2/m}}{(1-\varepsilon)^{1+2/m}}\left(\frac{k}{\mathrm{Vol}(M)}\right)^{2/m}. \tag{22}$$

Note that here we replace $\nu(M)$ and $\mu(M)$ in Corollary 2.1 by $\mu_\epsilon^D(M)$ and $\mathrm{Vol}(M)$, respectively. The left hand-sides of Inequalities (21) and (22) do not depend on D. Hence, taking the infimum over D, we get Inequalities (3) and (4). □

4 Eigenvalues of complex submanifolds of $\mathbb{C}P^N$

In this section, we provide the proof of Theorem 1.2. Before going into the proof, we need to recall the universal inequality proved by El Soufi, Harrell and Ilias which is the key idea of the proof. The following lemma is a special case of that universal inequality [9, Theorem 3.1] (see also [5]):

Lemma 4.1 *Let M^m be a compact complex manifold of complex dimension m and $\phi : M \to \mathbb{C}P^N$ be a holomorphic immersion. Then the eigenvalues of the Laplace-Beltrami operator on $(M, \phi^* g_{FS})$ satisfy the following inequality:*

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \le \frac{2}{m} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i + c_m), \tag{23}$$

where $c_m = 2m(m+1)$.

Another useful result is the following recursion formula given by Cheng and Yang:

Lemma 4.2 [4, Corollary 2.1] *If a positive sequence of numbers $\mu_1 \le \mu_2 \le \cdots \le \mu_{k+1}$ satisfies the following inequality*

$$\sum_{i=1}^{k} (\mu_{k+1} - \mu_i)^2 \le \frac{4}{n} \sum_{i=1}^{k} \mu_i (\mu_{k+1} - \mu_i), \tag{24}$$

then,

$$\mu_{k+1} \le \left(1 + \frac{4}{n}\right) k^{2/n} \mu_1.$$

Theorem 4.1 *Let M^m be a compact complex manifold of complex dimension m admitting a holomorphic immersion $\phi : M \to \mathbb{C}P^N$. Then for every $k \in \mathbb{N}^*$ we have*

$$\lambda_{k+1}(M, \phi^* g_{FS}) \le 2(m+1)(m+2)k^{\frac{1}{m}} - 2m(m+1). \tag{25}$$

Proof of Theorem 4.1 According to Lemma 4.1, the eigenvalues of the Laplace operator on M satisfy universal Inequality (23). We replace λ_i by $\mu_i := \lambda_i + c_m$ in Inequality (23) and we obtain,

$$\sum_{i=1}^{k} (\mu_{k+1} - \mu_i)^2 \le \frac{2}{m} \sum_{i=1}^{k} \mu_i (\mu_{k+1} - \mu_i).$$

One now has a positive sequence of numbers $\mu_1 \le \mu_2 \le \cdots \le \mu_{k+1}$ that satisfies Inequality (24) with $n = 2m$. Applying the recursion formula of Cheng and Yang, we get the following inequality:

$$\mu_{k+1} \le \left(1 + \frac{4}{2m}\right) k^{2/2m} \mu_1. \tag{26}$$

By replacing μ_i by $\lambda_i + c_m$ in Inequality (26), we obtain:

$$\lambda_{k+1}(M, \phi^* g_{FS}) \le (1 + \frac{2}{m})(\lambda_1(M, \phi^* g_{FS}) + c_m)k^{1/m} - c_m.$$

Since M is a compact manifold, $\lambda_1(M, \phi^*(g_{FS}) = 0$. Therefore,

$$\lambda_{k+1}(M, \phi^* g_{FS}) \le \left(1 + \frac{2}{m}\right) c_m k^{1/m} - c_m = 2(m+1)(m+2)k^{1/m} - 2m(m+1),$$

which completes the proof. □

As we mentioned in the introduction, for $k = 1$ we get a sharp upper bound:

$$\lambda_2(M, \phi^* g_{FS}) \leq \lambda_2(\mathbb{C}P^m, g_{FS}) = 4(m+1). \tag{27}$$

Bourguignon et al. [2] obtained an upper bound for the first non-zero eigenvalue of a complex manifold (M, ω) which admits a *full* holomorphic immersion (i.e. $\Phi(M)$ is not contained in any hyperplane of $\mathbb{C}P^N$) into $\mathbb{C}P^N$.

$$\lambda_2(M, \omega) \leq 4m \frac{N+1}{N} d([\Phi], [\omega]). \tag{28}$$

Here, $d([\Phi], [\omega])$ is the *holomorphic immersion degree*—a homological invariant—defined as

$$d([\Phi], [\omega]) = \frac{\int_M \Phi^*(\omega_{FS}) \wedge \omega^{m-1}}{\int_M \omega^m},$$

where ω_{FS} is the Kähler form of $\mathbb{C}P^N$ with respect to the Fubini-Study metric and ω is Kähler form on M. If one takes $\omega = \Phi^*(\omega_{FS})$, then $d([\Phi], [\omega]) = 1$ and Inequality (27) becomes a corollary of Inequality (28). Theorem 4.1 gives another proof of this sharp inequality. Moreover, it gives upper bounds for higher eigenvalues of complex submanifolds of $\mathbb{C}P^N$ endowed with the Fubini-Study metric.

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