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ORIGINAL ARTICLE

A polynomial time approximation algorithm for the two-commodity splittable flow problem

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Abstract We consider a generalization of the unsplittable maximum two-commodity flow problem on undirected graphs where each commodity $i \in \{1, 2\}$ can be split into a bounded number k_i of equally-sized chunks that can be routed on different paths. We show that in contrast to the single-commodity case this problem is NP-hard, and hard to approximate to within a factor of $\alpha > 1/2$. We present a polynomial time $1/2$ -approximation algorithm for the case of uniform chunk size over both commodities and show that for even k_i and a mild cut condition it can be modified to yield an exact method. The uniform case can be used to derive a $1/4$ -approximation for the maximum concurrent (k_1, k_2) -splittable flow without chunk size restrictions for fixed demand ratios.

Keywords Splittable flow · 2-commodity flow · Approximation algorithm

1 Introduction

We consider a generalization of the unsplittable maximum two-commodity flow problem defined by [Kleinberg \(1996\)](#) on an undirected capacitated graph $G = (V, E)$ introduced by [Baier et al. \(2002\)](#) where each commodity i can be split into a bounded

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number k_i of chunks (of potentially different size) which can be routed on different paths (k -splittable flow problem). This problem is NP-hard even for one commodity and $k = 2$, see [Baier et al. \(2005\)](#), unless extra restrictions are imposed.

In the following we will always work with an undirected graph $G = (V, E)$, with $s_1, s_2 \in V$ the sources, and $t_1, t_2 \in V$ the sinks of two commodities of flow.

Definition 1 (*splittable flow*) Let $G = (V, E)$ be an undirected graph with edge capacities u_e ($e \in E$), and let $s_1, s_2 \in V$ be the sources and $t_1, t_2 \in V$ be the sinks of two commodities of flow, and k_1, k_2 two nonnegative integers. A (k_1, k_2) -splittable flow is a two-commodity flow respecting the edge capacities using k_1 s_1 - t_1 -paths for commodity 1 and k_2 s_2 - t_2 -paths for commodity 2.

Since we allow that a path can be used multiple times and flow on certain paths can be equal to 0, the notion of k_1, k_2 -splittability includes the case where ‘at most k_i paths’ may be used for commodity i .

However, in many applications commodities cannot be split into arbitrarily sized chunks, which puts restrictions on the allowable flow values of the splittable flow. One reasonable restriction is to require that for each commodity the individual flows need to have the same flow value. The paths in a splittable flow do not need to be different, therefore integral multiples of such ‘chunk-sized’ transport can be accommodated on the same path.

Definition 2 (*bi-uniform splittable flow*) A k_1, k_2 -splittable flow is called *bi-uniform* if the flow values of the paths for each commodity are the same.

Note that with uniformity restrictions, a 0-flow on some path will force all flows for the respective commodity to be 0. Thus the problem reduces to a problem with one commodity less.

In the single-commodity case [Baier et al. \(2005\)](#) show that assuming uniformity makes the problem solvable in polynomial time. We will show that this is not the case for two commodities, not even if we ask for uniformity across both commodities. The latter restriction is also not artificial: Imagine that each commodity models a different service level, but the underlying good is divisible only in the same fashion, e.g., into packet size or base channel bandwidth in a telecommunication network.

Definition 3 (*totally uniform splittable flow*) A k_1, k_2 -splittable flow is called *totally uniform* if the flow values of all paths for all commodities are the same.

There are various notions of maximality for splittable flows that in general do not yield the same solutions.

Definition 4 (*maximality notions*) Let $(f_1^1, \dots, f_{k_1}^1, f_1^2, \dots, f_{k_2}^2)$ be a k_1, k_2 -splittable two-commodity flow in a graph G . It is called

– *maximal total flow* if it is optimal for

$$\max \sum_{i=1}^{k_1} f_i^1 + \sum_{i=1}^{k_2} f_i^2,$$

- *maximal concurrent flow* if for some given demand parameters $d_1, d_2 \in \mathbf{R}_{\geq 0}$ it is optimal for

$$\max_{f \text{ a } k_1, k_2\text{-splittable 2-c-f}} \min_{i \in \{1, 2\}} \frac{1}{d_i} \sum_{j=1}^{k_i} f_j^i,$$

- *maximal flow* if it is optimal for

$$\sum_{i=1}^2 \max_{j \in \{1, \dots, k_i\}} f_i^j$$

among all feasible k_1, k_2 -splittable two-commodity flows of G .

We will mostly be concerned with maximal totally uniform or bi-uniform flows, except for Sect. 3, where we study maximal concurrent flow. In the former case the objective function simplifies to $\max x + y$ where x and y are the flow values per path for the two commodities (and $x = y$ for totally uniform flows).

Lemma 1 *The following problems are NP-hard:*

- Find the maximal flow per path of a totally uniform k_1, k_2 -splittable flow,
- Maximize the sum $x + y$ where x (y) is the maximal flow per path of commodity 1 (of commodity 2) of a bi-uniform k_1, k_2 -splittable flow,
- Maximize the total non-uniform k_1, k_2 -splittable flow.

Proof The variant without any uniformity constraints was shown to be NP-hard by Baier et al. (2005), as noted above.

We will show that the integral 2-commodity flow problem with unit capacities is reducible to both the totally uniform and the bi-uniform k_1, k_2 -splittable flow problem.

Let $G = (V, E)$ with sources s_1, s_2 and sinks t_1, t_2 , identical capacities of 1 on each edge $e \in E$, and demands $d_1, d_2 \in \mathbf{Z}_{\geq 0}$ be given. Even et al. (1976) show that asking whether there exists an integral 2-commodity flow satisfying the demands for such a graph is NP-hard (even though the capacities are all 1).

Let $k_1 = d_1$ and $k_2 = d_2$. Solving the totally uniform (respectively, the bi-uniform) k_1, k_2 -splittable flow problem on G yields a solution composed of k_1 paths for commodity 1 and k_2 paths of commodity 2. All paths have the same flow value x (resp.: x and y) for the commodities. If $x = 1$ (resp.: $x + y = 2$) then we have found an integral two-commodity flow satisfying the demands. If $x < 1$ (resp.: $x + y < 2$) then there exists no integral two-commodity flow satisfying the demands: Assume there were an integral two-commodity flow satisfying d_1 and d_2 , then without loss of generality we can assume that it exactly satisfies the demands. Then it is, however, also a k_1, k_2 -splittable flow – since each edge carries an integral flow, i.e. a value of 0 or 1, we can split it into exactly k_1 and k_2 paths for commodity 1 and 2, respectively. In particular, the flow value of each of the paths is 1, contradicting $x < 1$ (resp.: $x + y < 2$). \square

Re-reading the proof we can see that the flow value x (resp.: x and y) on the paths of an optimal k_1, k_2 -splittable totally uniform (resp.: bi-uniform) flow solution on the class of instances considered can never lie in the open interval $(1/2, 1)$, since such a flow can always be increased to 1. A flow value of $1/2$ could be possible, if some edge is used by two paths [this corresponds to fractional, and therefore half-integral, solvability of the 2-commodity integral flow problem, see Hu (1963)]. Hence, any

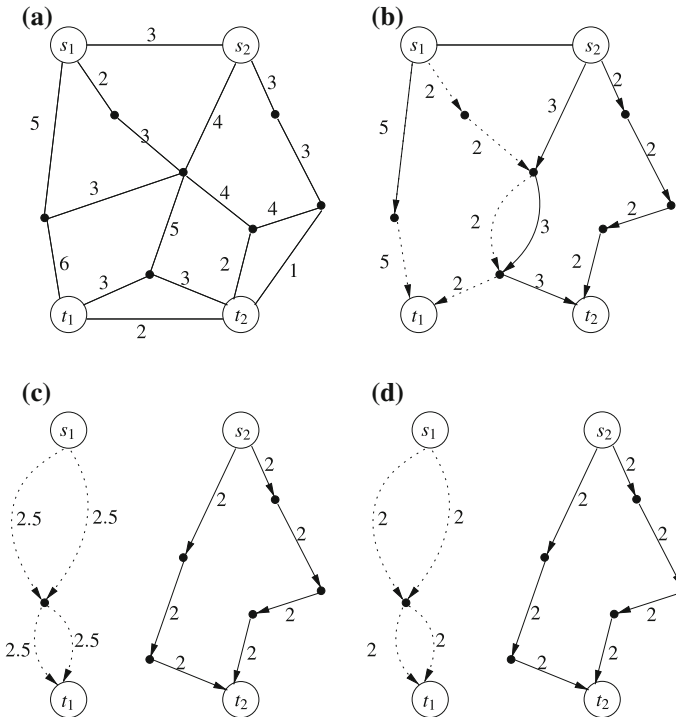
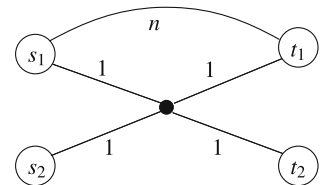


Fig. 1 Variants of splittable flows: **a** the graph, **b** a maximal unconstrained $(2, 2)$ -splittable flow, **c** a maximal bi-uniform $(2, 2)$ -splittable flow, **d** a maximal totally uniform $(2, 2)$ -splittable flow

Fig. 2 A graph with maximal 1, 1-splittable flows of different values depending on the version of uniformity: The maximal totally uniform 1, 1-splittable flow has a value $x + x = 2$; the maximal bi-uniform 1, 1-splittable flow has a value $x + y = n + 1$



α -approximation algorithm of the totally uniform k_1, k_2 -splittable flow problem with $\alpha > 1/2$ will also answer solve the integral 2-commodity flow problem: approximate solutions with flow $x > 1/2$ must correspond to “YES”-instances of the 2-commodity integral multicommodity flow problem, and approximate solutions with flow $x \leq 1/2$ to “NO”-instances. This yields the following:

Corollary 1 *It is NP-hard to approximate the maximum totally uniform k_1, k_2 -splittable flow problem to within a factor of $\alpha > 1/2$, even for graphs with unit capacities.*

It would be tempting to try and use totally uniform splittable flows to approximate bi-uniform splittable flows, but, as Fig. 2 shows, this is not possible.

2 Bi-uniform and totally uniform splittable flows

From classical multicommodity flow theory we know that the maximum multicommodity flow is bounded by the minimum multicommodity cut. In the single-commodity case this bound is tight, as asserted by the max-flow min-cut theorem. In [Baier et al. \(2005\)](#) this was extended to the case of single-commodity uniform k -splittable s - t -flows:

Definition 5 (*minimum k -cut*) Let $S \subseteq V$ with $s \in S$ and $t \in V \setminus S$ be a cut in $G = (V, E)$, and define

$$c_k(S) := \max\{x \in \mathbf{R}_{\geq 0} : \sum_{e \in \delta(S)} n(e) = k, n(e) \in \mathbf{Z}_{\geq 0} \text{ and } n(e)x \leq u_e \text{ for all } e \in \delta(S)\} \quad (1)$$

as the maximum item size such that k elements of equal size fractionally fit into the bins created by the edge capacities of $\delta(S) := \{(u, v) \in E : (u \in S \wedge v \notin S) \text{ or } (v \in S \wedge u \notin S)\}$. Then

$$c_k(G) = \min\{c_k(S) : S \subseteq V, s \in S, t \in V \setminus S\} \quad (2)$$

is called *minimum k -cut value* of G .

[Baier et al. \(2005\)](#) show that the value of the maximum uniform k -splittable s - t -flow in G equals the minimum k -cut value $c_k(G)$.

One can consider a similar approach for the two-commodity flow problem, i.e. consider a similar packing problem for two different items:

$$\begin{aligned} \max \quad & x + y \\ \text{s.t.} \quad & n_1(e)x + n_2(e)y \leq u_e \quad \forall e \in \delta(S) \\ & \sum_{e \in \delta(S)} n_1(e) \geq k_1 \text{ if } (s_1 \in S, t_1 \in V \setminus S) \text{ or } (t_1 \in S, s_1 \in V \setminus S) \\ & \sum_{e \in \delta(S)} n_2(e) \geq k_2 \text{ if } (s_2 \in S, t_2 \in V \setminus S) \text{ or } (t_2 \in S, s_2 \in V \setminus S) \\ & n_1(e), n_2(e) \in \mathbf{Z}_{\geq 0} \quad \forall e \in \delta(S) \\ & x, y \in \mathbf{R}_{\geq 0} \end{aligned} \quad (3)$$

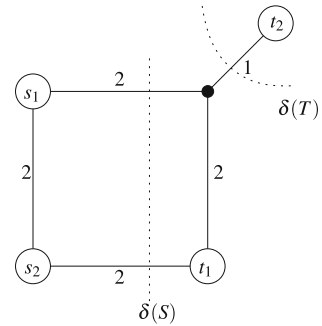
Proposition 1 (cut bound) *For a graph $G = (V, E)$ and each cut $S \subseteq V$ with $s_1, s_2 \in S$ and $t_1, t_2 \in V \setminus S$ the two-commodity bin packing problem (3) provides an upper bound for the value of a bi-uniform k_1, k_2 -splittable flow on G , but this minimum cut bound need not be tight.*

Proof Clearly, the flow values (x^*, y^*) of a valid bi-flow which is split according to n_1^*, n_2^* have to satisfy the conditions of (3), hence the optimum of (3) provides an upper bound.

The graph in Fig. 3 for $k_1 = 1$ and $k_2 = 1$ allows a maximal bi-uniform flow of value $x + y = 2$, but minimizing (3) over all cuts only yields a bound of 4. \square

One might consider adding two independent sets of cut constraints to the system (3), in an attempt to allow one cut to bound x well, and the other to bound y well, and thus

Fig. 3 A two-commodity digraph with maximum 1, 1-splittable flow of value 2 but best 1-cut packing bound of 4 (realized by S) and best 2-cut packing bound of 3 (realized by S and T)



obtain a stronger cut bound. Clearly, such a formulation will not be weaker than (3), but it still does not yield a tight cut bound in general, as we also illustrate in Figure 3: All possible cuts have values of either 1, 4, 5 or more. The cuts S and T in the Figure are therefore exemplary best cuts, and yield only a bound of 1 for y (cut T), and 2 for x (cut S), giving a joint bound of $x + y \leq 3$. We therefore only consider system (3) with one cut.

Note that (3) is a mixed-integer nonlinear optimization program which we cannot expect to directly use for solving the problem. If, however, one assumes uniformity across commodities, the bin-packing problem (3) turns out to be useful even in the two-commodity case. Let $\{s_1, s_2, t_1, t_2\}$ be the sources and destinations of the k_1 , k_2 -splittable totally uniform two-commodity flow problem and consider a set of nodes $S \subseteq V$. We define

$$\text{dem}(S) = \begin{cases} k_1 & (s_1 \in S \wedge \{s_2, t_1, t_2\} \not\subseteq S) \text{ or } (t_1 \in S \wedge \{s_1, s_2, t_2\} \not\subseteq S) \\ k_2 & (s_2 \in S \wedge \{s_1, t_1, t_2\} \not\subseteq S) \text{ or } (t_2 \in S \wedge \{s_1, s_2, t_1\} \not\subseteq S) \\ k_1 + k_2 & (s_1, s_2 \in S \wedge \{t_1, t_2\} \not\subseteq S) \text{ or } (t_1, t_2 \in S \wedge \{s_1, s_2\} \not\subseteq S) \\ k_1 + k_2 & (s_1, t_2 \in S \wedge \{s_2, t_1\} \not\subseteq S) \text{ or } (s_2, t_1 \in S \wedge \{s_1, t_2\} \not\subseteq S) \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

the demand necessarily crossing $\delta(S)$ in a feasible flow.

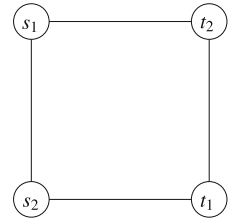
Then (3) can be rewritten as

$$\begin{aligned} c_{k_1, k_2}(S) := \max x \\ \text{s.t. } n(e)x &\leq u_e \quad \forall e \in \delta(S) \\ \sum_{e \in \delta(S)} n(e) &\geq \text{dem}(S) \\ n(e) &\in \mathbf{Z}_{\geq 0} \quad \forall e \in \delta(S) \\ x &\in \mathbf{R}_{\geq 0} \end{aligned} \quad (5)$$

We denote by $c_{k_1, k_2}(G)$ the minimum such cut value:

$$c_{k_1, k_2}(G) := \min_{S \subseteq V, \text{dem}(S) \neq 0} c_{k_1, k_2}(S) \quad (6)$$

Fig. 4 Forbidden minor for integrality of two-commodity flow problems



Lemma 2 Let $G = (V, E)$ be an undirected graph with edge capacities $u \in \mathbb{Z}_{\geq 0}^{|E|}$ and let $k_1, k_2 \in \mathbb{Z}_{\geq 0} \setminus \{0\}$. Then there exists a $2k_1, 2k_2$ -splittable totally uniform flow with value $(k_1 + k_2)c_{k_1, k_2}(G)$. Furthermore, if the graph in Fig. 4 is not a minor of G , there exists a k_1, k_2 -splittable totally uniform flow with this value.

Proof Let $x = c_{k_1, k_2}(G)$ be the minimum k_1, k_2 -cut value as defined in (6) and let $n \in \mathbb{Z}_{\geq 0}^{|E|}$ be the corresponding feasible solution. We construct an auxiliary graph $G' = (V, E)$ with edge capacities $u'_e = \lfloor \frac{u_e}{x} \rfloor$.

Now consider the two-commodity flow problem on G' with demands $d_1 = k_1$ and $d_2 = k_2$. As $n(e)x \leq u_e$ for all $e \in E$, we have $n(e) \leq \frac{u_e}{x}$. As $n(e) \in \mathbb{Z}_{\geq 0}$ we can round the right-hand side of this inequality. Therefore, $n(e) \leq \lfloor \frac{u_e}{x} \rfloor = u'_e$.

In particular for every $S \subseteq V$, $\sum_{e \in \delta(S)} u'_e \geq \sum_{e \in \delta(S)} n(e) \geq \text{dem}(S)$. According to Hu's two-commodity flow theorem, Hu (1963), there exists a half-integral solution for demands $d_1 = k_1, d_2 = k_2$. This half-integral solution can be constructed in polynomial time, see e.g. Schrijver (2003, Theorem 71.1b). Regular flow-decomposition techniques yield a solution with $2k_1$ paths for commodity 1 and $2k_2$ paths for commodity 2, each carrying a flow of $1/2$.

On the original graph G we assign these paths a flow of $\frac{1}{2}x$. We thus obtain a feasible two-commodity flow on G with total flow of $(k_1 + k_2)x$.

If the graph in Fig. 4 is not a minor of G there even exists an integral two-commodity flow solution instead of a half-integral one [see e.g. Schrijver (2003, Theorem 71.2)], which directly yields a k_1, k_2 -splittable solution with the same value. \square

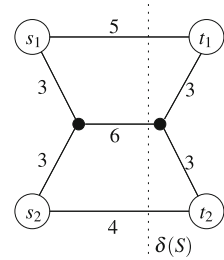
The factor of 2 for the number of paths in Lemma 2 is sometimes best possible, as the following example shows.

Example 1 Consider the graph in Fig. 4 with edge capacities $u_e = 1$ for all edges, and $k_1 = 1 = k_2$. Then clearly $c_{1,1}(G) = 1$, but there is no 1, 1-splittable totally uniform flow with a value of $(k_1 + k_2)c_{k_1, k_2} = 2 \cdot 1 = 2$. However, there exists a 2, 2-splittable totally uniform flow with the value $(2 + 2) \cdot (1/2) = 2$.

For even k_1 and k_2 dividing these parameters by 2 and applying Lemma 2 obviously always yields a feasible solution of the k_1, k_2 -splittable totally uniform flow problem. One could hope that it would be possible to use Lemma 2 for $\bar{k}_1 = k_1/2$ and $\bar{k}_2 = k_2/2$ when k_1 and k_2 are even to compute a maximum k_1, k_2 -splittable flow. The next example shows, however, that this is not possible in general.

Example 2 Let $k_1 = k_2 = 2$ and consider the graph in Fig. 5. Here $c_{1,1}(G) = c_{1,1}(S) = 4$ and the corresponding auxiliary graph has precisely one integral solution.

Fig. 5 A graph with optimal 2, 2-splittable totally uniform flow of value 12, and optimal 1, 1-splittable totally uniform flow of value 8



However, $c_{2,2}(G) = c_{2,2}(S) = 3$, and there is indeed a 2, 2-splittable totally uniform solution yielding a total flow of 12.

It is easy to obtain the necessary condition for the flow obtained by Lemma 2 to be maximal though. We start with the following observation.

Observation 1 For a graph $G = (V, E)$ with edge capacities $u_e \in \mathbb{Z}_{\geq 0}$ for all $e \in E$ and nonnegative integers k_1, k_2 it holds that

$$2c_{2k_1, 2k_2}(G) \geq c_{k_1, k_2}(G). \quad (7)$$

This follows from the fact that a feasible flow x for c_{k_1, k_2} in (5) always yields a feasible flow $x/2$ for $c_{2k_1, 2k_2}$ in (5). Hence $2c_{2k_1, 2k_2}(G)$ can not be smaller than c_{k_1, k_2} . Tightness in (7) is the necessary condition for applicability of the following Lemma:

Lemma 3 Let $k_1, k_2 \in 2\mathbb{Z}_{\geq 0}$ be even integers, $G = (V, E)$ with edge capacities $u_e \in \mathbb{Z}_{\geq 0}$ and assume $2c_{k_1, k_2}(G) = c_{k_1/2, k_2/2}(G)$. Then an optimal solution of the k_1, k_2 -splittable totally uniform flow problem can be obtained by applying Lemma 2 to $G, k_1/2$, and $k_2/2$.

Proof Using Lemma 2 for $k_1/2$ and $k_2/2$ yields a k_1, k_2 -splittable totally uniform flow where each path carries a flow of $1/2c_{k_1/2, k_2/2}(G)$. Since $2c_{k_1, k_2}(G) = c_{k_1/2, k_2/2}(G)$ by assumption and $c_{k_1, k_2}(G)$ is an upper bound by Proposition 1, the claim follows. \square

We will now show that the value of $c_{k_1, k_2}(G)$ can be computed in polynomial time, allowing us to check whether (7) is satisfied. Furthermore, knowing the value of $c_{k_1, k_2}(G)$ allows us to compute a factor $1/2$ -approximation for the maximum totally uniform flow problem in the general case.

Lemma 4 The value $c_{k_1, k_2}(G)$ can be computed in polynomial time $\mathcal{O}((k_1 + k_2)|E| \log |E|)$.

Proof To compute $c_{k_1, k_2}(G)$ we have to find the minimum of $c_{k_1, k_2}(S)$ over all cuts S in G with $\text{dem}(S) \neq 0$.

We can distinguish four cases according to (4), depending on which subset of $\{s_1, s_2, t_1, t_2\}$ is contained in S , yielding four relevant values of $\text{dem}(S)$:

1. $\text{dem}(S) = k_1$. Then $c_{k_1, k_2}(S) = c_{k_1}(S)$.

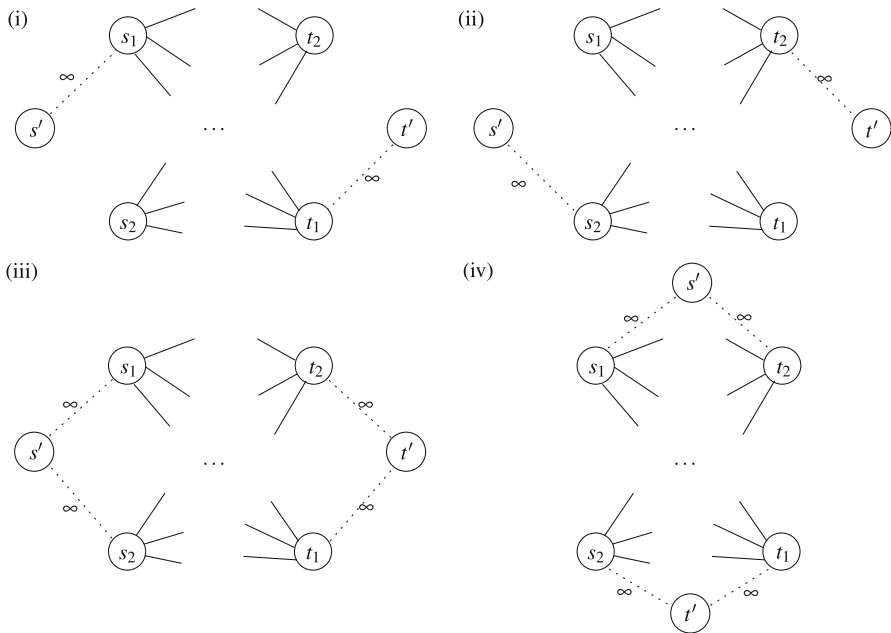


Fig. 6 The auxiliary graphs for determining a minimum k_1, k_2 -cut on G . The auxiliary edges are displayed as *dotted lines* and have capacity ∞

2. $\text{dem}(S) = k_2$. Then $c_{k_1, k_2}(S) = c_{k_2}(S)$.
3. $\text{dem}(S) = k_1 + k_2$ because $s_1, s_2 \in S$ and $t_1, t_2 \in V \setminus S$ (or symmetrically $t_1, t_2 \in S$ and $s_1, s_2 \in V \setminus S$). Then $c_{k_1, k_2}(S) = c_{k_1 + k_2}(S)$.
4. $\text{dem}(S) = k_1 + k_2$ because $s_1, t_2 \in S$ and $s_2, t_1 \in V \setminus S$ (or symmetrically $s_2, t_1 \in S$ and $s_1, t_2 \in V \setminus S$). Then $c_{k_1, k_2}(S) = c_{k_1 + k_2}(S)$.

Determining the value $c_{k_1, k_2}(G)$ thus amounts to determining the minimum of three single-commodity l -cut values (for $l \in \{k_1, k_2, k_1 + k_2\}$) w.r.t. certain auxiliary graphs. The auxiliary graphs are presented in Fig. 6. In each case we have to determine a l -cut value for a $s' - t'$ -flow.

Computing an individual value c_k can be done in time $O(k|E| \log |E|)$ using the algorithm of Baier et al. (2005). \square

So far we have shown that in the special case where the graph in Fig. 4 is not a minor of G and equality holds in (7) we can solve the maximum totally uniform flow problem exactly using two calls to a single-commodity integral flow algorithm.

In the general case a factor $1/2$ approximation is achievable in polynomial time. Given Corollary 1 this is best possible unless $P = NP$.

Theorem 1 Consider the k_1, k_2 -splittable totally uniform 2-commodity flow problem on an undirected graph $G = (V, E)$ with edge capacities $u_e \in \mathbb{Z}_{\geq 0}$ for $e \in E$. Then a $1/2$ -approximation for the maximal totally uniform flow can be computed in polynomial time.

Proof This is a direct consequence of Lemma 2: It yields a feasible two-commodity flow composed of $2k_1$ and $2k_2$ paths with total flow value of $(k_1 + k_2)c_{k_1, k_2}$. Dropping k_1 paths carrying commodity 1 and dropping k_2 paths carrying commodity 2 we obtain a k_1, k_2 -splittable solution with totally uniform path-flow across commodities and a total flow of $\frac{1}{2}(k_1 + k_2)c_{k_1, k_2}$. This is at least a $1/2$ approximation since c_{k_1, k_2} is an upper bound on the path flow. \square

3 Approximating nonuniform concurrent flow

Finally we will show that a general k_1, k_2 -splittable two-commodity flow can be approximated with the help of uniform flows.

Theorem 2 *Let $G = (V, E)$ be an undirected graph with edge capacities $u_e \in \mathbf{Z}_{\geq 0}$ for all $e \in E$. Let $k_1, k_2 \in \mathbf{Z}_{\geq 0}$ be integral parameters. A maximal totally uniform k_1, k_2 -splittable flow provides a $\frac{1}{2}$ -approximation of a maximal concurrent k_1, k_2 -splittable flow for a demand ratio $d_1/d_2 = k_1/k_2$.*

Proof Theorem 13 in Baier et al. (2005) states that every maximal bi-uniform k_1, k_2 -splittable flow is a $\frac{1}{2}$ -approximation of a maximal k_1, k_2 -splittable flow. We will show that for $d_1/d_2 = k_1/k_2$, a maximal bi-uniform flow is in fact totally uniform.

Let \mathcal{P}_i denote the set of s_i - t_i paths of commodity i and consider the maximum concurrent bi-uniform k_1, k_2 -splittable flow problem for demands $d_1/d_2 = k_1/k_2$:

$$\begin{aligned}
 \max \quad & \lambda \\
 \text{s.t.} \quad & \sum_{\substack{p \in \mathcal{P}_1, \\ e \in p}} x \delta_p + \sum_{\substack{q \in \mathcal{P}_2, \\ e \in q}} y \delta_q \leq u_e & \forall e \in E \\
 & \sum_{p \in \mathcal{P}_1} \delta_p = k_1 \\
 & \sum_{q \in \mathcal{P}_2} \delta_q = k_2 \\
 & \lambda d_1 = k_1 x \\
 & \lambda d_2 = k_2 y \\
 & \delta_p, \delta_q \in \{0, 1\} & \forall p \in \mathcal{P}_1, \forall q \in \mathcal{P}_2 \\
 & x, y, \lambda \in \mathbf{R}_{\geq 0}
 \end{aligned} \tag{8}$$

The first set of inequalities ensures the edge capacities are respected. The second and third set of equalities ensures that k_1 paths for commodity 1 and k_2 paths for commodity 2 are used. The fourth and fifth set of inequalities finally relate the demands λd_i , of commodity i , to the flow for commodity i , $k_1 x$ and $k_2 y$, respectively. From these last two equalities (and from $d_1/d_2 = k_1/k_2$), we obtain that $x = y$ has to hold, and thus a feasible k_1, k_2 -splittable bi-uniform flow is in fact totally uniform.

Now we will show that a maximal totally uniform k_1, k_2 -splittable flow provides an optimal solution for the program (8). Let x be the flow value on the k_i paths of

commodity i . Then $\bar{d}_1 := k_1 x$ is the total flow of commodity 1 and $\bar{d}_2 := k_2 x$ is the total flow of commodity 2. We will show that $\bar{d}_1 = \lambda d_1$ and $\bar{d}_2 = \lambda d_2$ for maximal λ . We have $\bar{d}_1 = k_1 x = \frac{d_1 k_2}{d_2} x = \frac{\bar{d}_2}{d_2} d_1$ and thus $\bar{d}_2 = \frac{\bar{d}_1}{d_1} d_2$. Therefore, we have to show that $\frac{\bar{d}_2}{d_2} = \frac{\bar{d}_1}{d_1}$ holds. But this follows directly from

$$\frac{\bar{d}_2}{d_2} = \frac{k_2 x}{d_2} = \frac{k_1 x}{d_1} = \frac{\bar{d}_1}{d_1}.$$

Therefore, $\lambda = \frac{k_1 x}{d_1} = \frac{k_2 x}{d_2}$. As d_i and k_i are fix, it is clear that a maximum value of x yields a maximal value of λ .

This concludes our proof: as a maximal k_1, k_2 -splittable totally uniform flow is a maximal concurrent k_1, k_2 -splittable bi-uniform flow for demand ratios $d_1/d_2 = k_1/k_2$, it provides a $\frac{1}{2}$ approximation for the maximal concurrent k_1, k_2 -splittable flow for demand ratios $d_1/d_2 = k_1/k_2$. \square

As a direct consequence of applying both Theorems 2 and 1 consecutively we obtain

Corollary 2 *Let $G = (V, E)$ be an undirected graph with edge capacities $u_e \in \mathbf{Z}_{\geq 0}$ for all $e \in E$. Let $k_1, k_2 \in \mathbf{Z}_{\geq 0}$ be integral parameters. A $1/4$ -approximation of a maximal concurrent k_1, k_2 -splittable flow can be computed in polynomial time for demand-ratios $d_1/d_2 = k_1/k_2$.*

It would be interesting to see whether polynomiality results for computing the maximal concurrent splittable flow where the demand ratios are not fixed, or at least not fixed as a consequence of the splitting values k_1, k_2 . Furthermore, it would be desirable to identify suitable restrictions of multi-commodity flow problems that are still tractable.

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