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COUNTING HYPERBOLIC MANIFOLDS

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Introduction

A classical theorem of Wang [W] implies that for a fixed dimension $n \geq 4$, and any $V \in \mathbb{R}$, there are only finitely many complete hyperbolic manifolds without boundary of volume at most V up to isometries. Let $\rho_n(V)$ be the number of these manifolds. In this note we establish the following estimate for $\rho_n(V)$:

Theorem. *For every $n \geq 4$, there are two constants $a = a(n) > 0$ and $b = b(n) > 0$ such that for all sufficiently large V*

$$aV \log V \leq \log \rho_n(V) \leq bV \log V \,.$$

This estimate answers a question asked by S. Carlip. Carlip has shown (cf. [C1,2]) that the lower bound estimate has some applications in theoretical physics.

Of course the theorem is not true for dimension $n = 2$ or 3. If $n = 2$ there is a continuum of different hyperbolic surfaces of bounded (even the same) area. When $n = 3$, there may be countably many hyperbolic 3-manifolds of bounded volume. In the last section we discuss some recent results and some problems concerning other locally symmetric spaces.

The Lower Bound

For every $n \geq 2$, Gromov and Piatetski-Shapiro [GrP] constructed a non-arithmetic cocompact lattice $\Gamma = \Gamma_n$ in $PO(n, 1)$ the group of isometries of the n-dimensional hyperbolic space $\mathbb{H}^n$. In [L3], Lubotzky showed that Γ has a finite index subgroup Δ, which is mapped onto a non-abelian free group F on 2 generators. By Selberg's lemma Δ can be arranged

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to be torsion free. Thus Δ defines a hyperbolic n-dimensional manifold $M = \mathbb{H}^n/\Delta$ whose volume is, say, v_0. Now, every finite index subgroup of F of index r defines an index r subgroup of Δ, which in turn gives an r-sheeted covering of M. The free group F has at least $r \cdot r!$ subgroups of index $\leq r$ (see [H] or [L1]). Thus, M has at least this number of coverings of volume $\leq rv_0$. Some of these covering spaces may be isometric, but if, say, M_1 and M_2 are isometric manifolds which correspond to subgroups Δ_1 and Δ_2 of Δ, respectively, then there exists an element $g \in PO(n,1)$ with $g^{-1}\Delta_1 g = \Delta_2$. Hence, by definition, g belongs to the commensurability group of Δ, $Comm(\Delta)=\{h\in PO(n,1):[\Delta:h^{-1}\Delta h\cap\Delta]<\infty, [h^{-1}\Delta h : h^{-1}\Delta h \cap \Delta] < \infty\}$. Since Δ is a non-arithmetic lattice, it follows from Margulis' Theorem ([M, Theorem 1, p. 2]) that $[Comm(\Delta) : \Delta] = m_1 < \infty$. Thus, the orbit of Δ_1 under conjugation by elements of $Comm(\Delta)$ consists of at most $m_1 \cdot r$ groups. This shows that there are at least $(r \cdot r!)/(m_1 \cdot r) = \frac{1}{m_1} r!$ non-isometric hyperbolic manifolds of volume at most $r \cdot v_0$, establishing the required lower bound. □

We remark that the constants may be explicitly estimated. This requires also an estimate of the index of the lattice Δ in its commensurator. This may be obtained using the lower bound, given by Kazhdan–Margulis to the covolume of the lattice(!) $Comm(\Delta) \leq PO(n,1)$.

The Upper Bound

Recall the Thick-Thin decomposition of a manifold M. For any $\epsilon > 0$ denote by $M_{>\epsilon}$ the subset of M consisting of those points for which the injectivity radius is larger than $\epsilon/2$. Let $M_{\leq\epsilon} = M \setminus M_{>\epsilon}$. We shall need the following:

Theorem (The thick-thin decomposition) (cf. [T, Theorem 4.5.6]). *For each $n \geq 2$ there exists $\epsilon(n) > 0$ such that for any $\epsilon < \epsilon(n)$ the thin part $M_{\leq\epsilon}$ of a complete hyperbolic manifold is a finite union of components of one of the following types: neighborhoods of short closed geodesics homeomorphic to ball bundles over the circle or neighborhoods of cusps, homeomorphic to products of a Euclidean manifolds with a half infinite interval. For $n \geq 3$ the thick part $M_{\geq\epsilon}$ is a connected compact manifold with boundary.* □

Corollary. *Let M be a complete hyperbolic manifold of dimension $n \geq 4$ and $\epsilon = \epsilon(n)$ as above. Then $\pi_1(M) = \pi_1(M_{>\epsilon})$ where $M_{>\epsilon}$ is the thick part of M.*

Proof. Let Y be a connected component of the thin part $M_{\leq\epsilon}$. Note that when the dimension is at least 4 we have $\pi_1(Y) = \pi_1(\partial Y)$. Indeed when Y is a ball bundle over a circle its fundamental group coincides with that of its boundary (as $n \geq 4$, note that this fails for $n \leq 3$). Similarly when Y is a product of a Euclidean manifold with a half infinite interval (in which case there is no need for a restriction on the dimension).

By a successive use of Van Kampen's theorem we can remove the component of the thin part $M_{\leq\epsilon}$ one by one, and obtain the desired statement. □

REMARK. As noted in the proof, the corollary holds for $n = 3$ if the manifold has no "short" closed geodesics. In fact the main theorem has a version which is still true for dimension 3: Given $\delta > 0$, the number of 3-dimensional hyperbolic manifolds without closed geodesic of length $\leq \delta$ is at most $V^{b_\delta V}$, and if $\delta > \delta_0$ then this number is at least $v^{a_\delta v}$.

In the sequel we shall need to look more closely at the geometric structure, rather than just the topology, of the connected components of the thin part. It is convenient to look at the preimage of such a component in the universal covering $\mathbb{H}^n$ of the manifold. We shall use the "Upper half space" model of the hyperbolic space, which we denote by $\mathbb{H}^n_+$. In this model the preimage of a cusp thin component is a half space consists of a union of convex neighborhood of a unique point, say ∞, of the ideal boundary $\partial(\mathbb{H}^n_+)$. The preimage of a compact connected thin component is (up to isometry) a cone, around the line $(0, \infty)$, over a finite union of concentric coaxial ellipsoids. We refer to the book [BP, section D.3] for a detailed discussion.

By Mostow rigidity a hyperbolic manifold of dimension at least 3 is determined by its fundamental group. Thus to bound the number of hyperbolic manifolds of given volume it suffices to bound the number of possible fundamental groups. The basic idea in counting the number of possible fundamental groups of hyperbolic manifolds of a fixed dimension whose volume is bounded by V is to associate with each of them a two dimensional complex with the same fundamental group and count these complexes. For the purpose of clarity let us first give a WRONG argument which has the advantage of avoiding some technical difficulties and then give the correct argument.

Fix $n \geq 4$ and some $\epsilon, \epsilon_0, \epsilon_1$ ($\epsilon_1 \leq \epsilon_0 \leq \epsilon = \epsilon(n)/10$). Given a complete hyperbolic n-manifold M of volume at most V we can choose a finite cover $\mathcal{C}_M$ of $M_{\geq\epsilon}$ by open balls of radius ϵ_0 such that the balls having the same centers and of radius $\epsilon_1/2$ are pairwise disjoint. (Considering a maximal collection of points which are at least ϵ_1 apart from one another in $M_{\geq\epsilon}$ yields such a collection of balls.) Notice that the number of balls in $\mathcal{C}_M$ is bounded by c_1V where $c_1 = c_1(n, \epsilon_1)$ is some fixed constant, namely 1 over the volume of an $\epsilon_1/2$-ball. Observe also that the intersection of any of these balls is either empty or convex and hence diffeomorphic to $\mathbb{R}^n$. Thus in the terminology of [BotT] it is a "good cover". It follows (cf. [BotT, Theorem 13.4]) that $\pi_1(\cup\mathcal{C}_M) = \pi_1(\mathfrak{N})$ where $\mathfrak{N} = \mathfrak{N}(\mathcal{C}_M)$ is the simplicial complex corresponding to the "nerve" of the cover $\mathcal{C}_M$. I.e. the vertices of $\mathfrak{N}$ correspond to the open balls in the cover $\mathcal{C}_M$ and a set of vertices forms a simplex when the intersection of the corresponding balls is non empty. Here lies the problem in this argument – we would have liked to be able to claim that actually $\cup\mathcal{C}_M$ and $M_{\geq\epsilon}$ have the same fundamental group. However, note that some of the balls in $\mathcal{C}_M$ may "extend" out of $M_{\geq\epsilon}$, alternatively if one tries to restrict each of the balls to $M_{\geq\epsilon}$ we encounter the problem that the truncated balls are no longer convex and we do not know that intersections of balls are contractible. As said above, let us first ignore this problem and complete the argument. We will show afterwards how to correct this argument by proving that one can choose $\epsilon_0(n), \epsilon_1(n)$, and the cover so that $\cup\mathcal{C}_M$ and $M_{\geq\epsilon}$ are homotopic to one another.

Since the fundamental group of a simplicial complex is the same as that of its 2-skeleton (cf. [S]) it is enough to consider the 2-skeleton of $\mathfrak{N}$ which we shall denote by $\mathfrak{N}^{(2)}$. Note that the 1-skeleton, $\mathfrak{N}^{(1)}$, is a finite graph such that the degree of each vertex is at most $d = d(n, \epsilon_0, \epsilon_1)$. This bound may be deduced by considering the ratio of the volume of a ball of radius $2\epsilon_0 + \epsilon_1/2$ to that of a ball of radius $\epsilon_1/2$ in the hyperbolic n-space. Thus we have the following estimates:

Proposition. *(1) The number of graphs obtained as the* 1*-skeleton* $\mathfrak{N}^{(1)}$ *of a simplicial complex associated via the above process with a complete hyperbolic manifold of volume at most* V *is at most* $e^{c_2 V\log V}$ *for some constant* $c_2 = c_2(d)$.

(2) The number of 2*-dimensional simplicial complexes* $\mathfrak{N}^{(2)}$ *obtained via the above process for manifolds of volume bounded by* V *is at most* $e^{c_3 V\log V}$ *for some constant* $c_3 = c_3(d)$.

Proof. Part (1) is just a crude estimate on the number of graphs having c_1V vertices and of degree bounded by d. Going through the vertices one by one and for each one choosing at each step neighboring vertices from the available vertices at that stage yields the required estimate.

Part (2) follows from part (1) combined with the observation that in each graph of degree at most d, the number of triangles, i.e. closed paths of length 3, is at most d^2 times the number of vertices. Thus for each graph as in (1) we have at most $2^{d^2 \# \text{vertices}} = 2^{c'V}$ possible 2-dimensional simplicial complexes having it as their 1-skeleton. $\square$

We can thus deduce that if M is a complete hyperbolic n-manifold whose volume is at most V then its fundamental group is isomorphic to the fundamental group of one of at most $e^{c_3 V \log V}$ 2-dimensional simplicial complexes. Now, as by Mostow rigidity theorem, $\pi_1(M)$ determines M, we conclude that the number of hyperbolic manifolds of a fixed dimension $n \geq 4$ having volume $\leq V$ is at most $e^{c_3 V \log V}$.

Let us now show how to modify the above construction so that we would get coverings such that $\pi_1(\cup \mathcal{C}_M) = \pi_1(M_{\geq \epsilon}) = \pi_1(M)$.

We shall need some notation. Let $M = \mathbb{H}^n/\Gamma$ be a hyperbolic manifold of dimension n with fundamental group Γ. Let $\epsilon(n)$ be the constant from the thick-thin decomposition theorem, and let $\epsilon = \epsilon(n)/10$. For $\gamma \in \Gamma$, the set

$$T(\gamma) = \big\{x \in \mathbb{H}^n : d\big(\gamma(x), x\big) \leq \epsilon\big\}$$

is convex. The preimage in $\mathbb{H}^n = \tilde{M}$ of the ϵ thin part $M_{\leq \epsilon}$ is a union of convex sets

$$\tilde{M}_{\leq \epsilon} = \cup_{\gamma \in \Gamma \setminus \{1\}} T(\gamma)\,.$$

For a set A and $t > 0$ we denote by $(A)_t$ its $t-$neighborhood

$$(A)_t = \big\{x \in \mathbb{H}^n : d(x, A) < t\big\}\,.$$

If A is convex then $(A)_t$ is convex with smooth boundary. For $x_t \in \partial(\tilde{M}_{\leq \epsilon})_t$ we denote by $\{\hat{n}_i(x_t)\}$ the finite set of unit length external normals to $(M_{\leq \epsilon})_t$, i.e. to the convex sets $(T(\gamma_i))_t$ which contains x_t on their boundary.

The following lemma is what we need

Lemma (Constructing a good cover). *There exist constants $\eta, \delta > 0$, $b > 1$, depending only on n, such that:*

1. *For a maximal δ-discrete subset $\mathcal{F} \subset M \setminus (M_{\leq \epsilon})_{\eta+\delta}$ the union of the $(b+1)\delta$-balls $\cup_{y \in \mathcal{F}} B(y, (b+1)\delta)$ covers $M \setminus (M_{\leq \epsilon})_\eta$. We fix $\mathcal{F}$ and denote this union by $U = \cup_{y \in \mathcal{F}} B(y, (b+1)\delta)$.*
2. *There is a deformation retract from the intersection $U \cap (M_{\leq \epsilon})_\eta$ to the boundary of $(M_{\leq \epsilon})_\eta$.*
3. *There is a homotopy equivalence between $(M_{\leq \epsilon}, \partial M_{\leq \epsilon})$ and $((M_{\leq \epsilon})_\eta, \partial (M_{\leq \epsilon})_\eta)$.*

Assuming this lemma, our theorem follows by taking $\epsilon_1 = \delta$, $\epsilon_0 = (b+1)\delta$ and $\mathcal{C}_M$ to be the cover of U by $\epsilon_0 = (b+1)\delta$ balls whose centers form $\mathcal{F}$.

Indeed, (1) implies that $U = \cup\mathcal{C}_M$ contains $M \setminus (M_{\leq\epsilon})_\eta$, (2) implies that U is diffeomorphic to $M \setminus (M_{\leq\epsilon})_\eta$, and (3) implies that $M \setminus (M_{\leq\epsilon})_\eta$ is homotopically equivalent to the thick part $M_{\geq\epsilon}$.

Remark. We will look at points outside the ϵ-thin part, while our relevant sizes are much smaller then ϵ (namely $\eta, (b+1)\delta$). Therefore we may lift the picture to the universal covering $\tilde{M} = \mathbb{H}^n$ of M without distorting it, and prove some of our claims there. We work with the upper half space model $\mathbb{H}^n_+$. Notice that every hyperbolic ball is also an Euclidean ball (in the standard metric induced from $\mathbb{R}^n \supset \mathbb{H}^n_+$) with different center and radius. We identify the tangent space $T_x(\mathbb{H}^n_+)$ at each $x \in \mathbb{H}^n_+$ with $\mathbb{R}^n$ (equipped with its inner product and coordinates) in the obvious way. For a subset $X \subset M$ we denote by $\tilde{X}$ its pre-image in the universal covering $\mathbb{H}^n_+$.

Proof. The proof, which will be carried out in a few steps, is based on the existence of a nice vector field which is, in a weak sense, transversal to the boundary of the thin part.

Step A (Constructing the vector field and determining the constant b)**:** Fix

$$b = \max\left\{n^{1/2}, 1/\cos(\arctan(2/\epsilon))\right\}.$$

We will show that there is a normalized vector field F, defined on $(\tilde{M}_{\leq\epsilon})_{2\epsilon} \setminus \tilde{M}_{<\epsilon}$, and continuous on its integral curves, such that for any $\tilde{x}_t \in \partial(\tilde{M}_{\leq\epsilon})_t$, $0 \leq t \leq \epsilon$, the inner product of $F(\tilde{x}_t)$ with each of the normals $\hat{n}_i(\tilde{x}_t)$ to $\partial(\tilde{M}_{\leq\epsilon})_t$ at $\tilde{x}_t$ is $\geq 1/b$.

Let $\tilde{M}^0_{\leq\epsilon}$ be a connected component of $\tilde{M}_{\leq\epsilon}$. If $\tilde{M}^0_{\leq\epsilon}$ is a hyperbolic component, i.e. one which corresponds to a hyperbolic isometry of $\mathbb{H}^n$, then it is a neighborhood of a geodesic line. Taking this line to be the one connecting the origin to ∞, this neighborhood is a cone over a finite union of concentric coaxial ellipsoids (note that the horospheres through ∞ inherit an $(n-1)$-Euclidean structure from $\mathbb{R}^n$ in which our model $\mathbb{H}^n_+$ sits). We may assume that the axes of these ellipsoids are the standard coordinates of $R^{n-1} \subset \mathbb{H}^n_+$. In this case, at a point $x = (x^0, x^1, ..., x^{n-1}) \in \mathbb{H}^n_+ = \{x \in \mathbb{R}^n : x^0 > 0\}$, we take

$$\tilde{F}(x) = \big(-1, sign(x^1), sign(x^2), \dots, sign(x^{n-1})\big) \in \mathbb{R}^n = T_x(\mathbb{H}^n_+),$$

and normalize it

$$F(x) = \frac{\tilde{F}(x)}{\|\tilde{F}(x)\|}.$$

It is then easy to see that $F(x_t) \cdot \hat{n}_i(x_t) \geq n^{-1/2}$ for any $x_t \in (\tilde{M}^0_{\leq\epsilon})_t$, $0 \leq t \leq \epsilon$.

If $\tilde{M}^0_{\leq\epsilon}$ is a component of the pre-image of a cusp, i.e. it corresponds to a group of parabolic isometries. We assume that ∞ is the fixed point of the fundamental group of this component, and take

$$F(x) = (-1, 0, 0, \dots, 0)\,.$$

Notice that if γ is a parabolic isometry (which stabilize ∞ in $\mathbb{H}^n_+$) then the angle between the external normal to the boundary of $T(\gamma)$ and the vector $(-1, 0, 0, \dots, 0)$ is at most $\arctan(2/\epsilon)$: If $d(\gamma(x), x) = \epsilon$ and y is a point at the same altitude (the same horosphere through ∞) at infinitesimal distance τ from x, then $d(\gamma(y), y) \leq \epsilon + 2\tau$. It follows that the point at distance $2\tau/\epsilon$ above y is in $T(\gamma)$.

Moreover in the parabolic case the inner product of F with each the normals of $(\tilde{M}_{\leq\epsilon})_t$ is increasing with t, and thus $> \cos(\arctan 2/\epsilon)$ for any $t \geq 0$.

This completes Step A.

Remark. The axes of the concentric ellipsoids described above are not necessarily uniquely defined, and hence, the vector field F is not necessarily Γ-invariant, and does not project to a vector field on M. It is possible to define a Γ-invariant vector field with the same properties. Alternatively, we shall use F only when estimating distances and angles between things which are lifted to $\tilde{M}$.

Step B (Proving condition 1)**:** The existence of the vector field F, constructed above, implies that for any $\eta \leq \epsilon$ and $\delta < \frac{\epsilon-\eta}{b}$ we have the following:

1′) $M \setminus (M_{\leq\epsilon})_\eta \subset \big(M \setminus (M_{\leq\epsilon})_{(\eta+\delta)}\big)_{b\delta}$, i.e. each point outside $(M_{<\epsilon})_\eta$ is at distance at most $b\delta$ from the complement of $(M_{\leq\epsilon})_{(\eta+\delta)}$.

Indeed, for $x_{t_0} \in \partial(M_{\leq\epsilon})_{t_0}$ (for $t_0 \geq \eta$) we take a lift $\tilde{x}_{t_0} \in \partial(\tilde{M}_{\leq\epsilon})_{t_0}$ and let it flow $b\delta$ seconds on F to the point $\tilde{x}_{t_0+b\delta} \in \tilde{M} \setminus (\tilde{M}_{\leq\epsilon})_{\eta+\delta}$.

Now, it follows from the definition of $\mathcal{F}$ that its δ-neighborhood $(\mathcal{F})_\delta$ contains $M \setminus (M_{\leq\epsilon})_{\eta+\delta}$. It follows from $1'$ that

$$M \setminus (M_{\leq\epsilon})_\eta \subset \big(M \setminus (M_{\leq\epsilon})_{(\eta+\delta)}\big)_{b\delta} \subset ((\mathcal{F})_\delta)_{b\delta} = (\mathcal{F})_{(b+1)\delta}\,,$$

which is exactly the statement of condition 1.

We turn now to proving condition 2. We will do this in a few steps. The following is easily verified:

Step C (Small curvature of the boundary)**:** Let $A \subset \mathbb{H}^n$ be a convex set (below, we shall take $A = (T(\gamma))_\epsilon$). Then for any boundary point

$x \in \partial(A)_\eta$ the η-ball, tangent to $\partial(A)_\eta$ at x, with the same external normal, is contained in $(A)_\eta$.

Indeed, if we denote by $P_A(x)$ the projection of x to A then this ball is no other then $B(P_A(x), \eta)$.

Step D (Existence of "large" ball tangent to any boundary point)**:** If η and δ are sufficiently small (so that $(*)$ and $(**)$ hold) then we have:

2′) For any point x in $(M_{\le\epsilon})_\eta \cap U$ there is a unique closest point $\pi(x)$ on the boundary $\partial(M_{\le\epsilon})_\eta$, and $(M_{\le\epsilon})_\eta$ contains the ball of radius $\eta/4b$ which contains $\pi(x)$ on its boundary sphere and the normal at $\pi(x)$ to this sphere is tangent to the geodesic line $\overline{x\pi(x)}$.

To prove $2'$ take a closest point to x in the boundary of $(M_{\le\epsilon})_\eta$, and denote it by $\pi(x)$. We may assume that δ is small enough so that

$$(b+1)\delta < \eta/4b \tag{$*$}$$

and thus uniqueness of $\pi(x)$ follows from the existence of this $\eta/4b$-ball that we will show now. As noted above, since η, δ are much smaller the ϵ, we may lift the picture to $\tilde{M}$ without distorting it. We shall lift x and $\pi(x)$ to $\tilde{M}$ without changing their names.

Sublemma. *The tangent $\hat{n}$ to the geodesic line $\overline{x\pi(x)}$ at $\pi(x)$ must be inside the convex cone of the external normals $\{\hat{n}_i\}$ to $(\tilde{M}_{\le\epsilon})_\eta$ at $\pi(x)$.*

Proof. The point $\pi(x)$ is on the boundary of a finite union of convex sets of the form $(T(\gamma_i))_\eta$. The finite set $\{\hat{n}_i\}$ consists of the normals to the smooth boundaries of these sets. As $\pi(x)$ is closest to x the intersections of the half spaces

$$\cap_i\{v \in T_{\pi(x)}\tilde{M} : \hat{n}_i \cdot v \ge 0\} \cap \{v \in T_{\pi(x)}\tilde{M} : \hat{n} \cdot v \le 0\}$$

has empty interior. Helley's theorem implies that we may consider the case where only $k+1$ half spaces involved (one of them must be the one defined by $\hat{n}$) where $k = \dim \mathrm{span}\{\hat{n}_i\}$. Thus there is a unique expression $\hat{n} = \Sigma_{i=1}^k \alpha_i \hat{n}_i$ (Lagrange's multipliers theorem implies that $\hat{n} \in \mathrm{span}\{\hat{n}_i\}$). Pass to the relevant subspace $\mathrm{span}\{\hat{n}_i\}$ of dimension k. We need to show that $\alpha_i \ge 0$ for all i. Assume the contrary, say, $\alpha_1 < 0$. Let N denote the matrix with rows corresponding to $\hat{n}_i$ and let $\overline{\alpha}$ be the row vector $(\alpha_1, \dots, \alpha_k)$. We have $\hat{n} = \overline{\alpha}N$. Our condition that "the interiors have no intersection" reads "$Nv \ge 0$ implies $\hat{n} \cdot v \ge 0$", but taking v which satisfies $Nv = (1, 0, 0, \dots, 0)^t$ gives us a contradiction:

$$\hat{n} \cdot v = \overline{\alpha}Nv = \overline{\alpha}(1, 0, 0, \dots, 0)^t = \alpha_1 < 0\,.$$

Thus $\hat{n} = \Sigma_{i=1}^k \alpha_i \hat{n}_i$ with $\alpha_i \ge 0$. □

We next claim

SUBLEMMA. $\Sigma_{i=1}^{k}\alpha_i \leq b$.

Proof. $1 \geq \hat{n}\cdot F\big(\pi(x)\big) = \Sigma_{i=1}^{k}\alpha_i\hat{n}_i \cdot F\big(\pi(x)\big) \geq 1/b\Sigma_{i=1}^{k}\alpha_i$. □

This implies that if $v \in \mathbb{R}^n = T_{\pi(x)}\tilde{M}$ satisfies $v\cdot\hat{n} \geq b$ then $v\cdot\hat{n}_i \geq 1$ for some $1 \leq i \leq k$. In other words, the half space $\{v : \hat{n}\cdot v \geq b\}$ is contained in the union of the half spaces $\{v : \hat{n}_i \cdot v \geq 1\}$. Applying inversion by the sphere of radius $2^{1/2}$ around $0 \in \mathbb{R}^n$ we obtain:

COROLLARY. *The Euclidean ball of radius $1/b$ with normal $\hat{n}$ at $\pi(x)$ is contained in the union of the unit Euclidean balls tangent to $\pi(x)$ with normals $\hat{n}_i$'s.*

Now move $\pi(x)$ (or, more precisely, a pre-image of $\pi(x)$ in $\mathbb{H}^n_+$) to the point $(1,0,0,\dots,0) \in \mathbb{H}^n_+$. If η is small enough, the Euclidean radius r_E, of a ball containing $\pi(x)$ on its sphere, of hyperbolic radius $r_h \leq \eta$, satisfies

$$r_h/2 < r_E < 2r_h\,. \tag{$**$}$$

Now it follows from Step C that the hyperbolic balls of radius η tangent to the relevant $T(\gamma)$'s at $\pi(x)$ are contained in $(M_{\leq\epsilon})_\eta$. By the choice of η we get that these balls have Euclidean radiuses $> \eta/2$, hence the $\eta/2b$ Euclidean ball, whose boundary sphere passes through $\pi(x)$ with normal $\hat{n}$, is contained in $(M_{\leq\epsilon})_\eta$. This ball has hyperbolic radius $> \eta/4b$. This finishes the proof of $2'$.

Step E (Positive direction)**:** One can easily verify that, after b and η are fixed, for any small enough δ the following is satisfied:

$2''$) $\beta < \pi/2$, see Figure 1.

To see this, think of the following. Instead of shrinking δ (until it is small enough), keep it as the fixed parameter, and let η tends to infinity (by rescaling the Riemannian metric each time). We get, in the limit, two parallel lines at distance δ and two points, one on each line, at distance $2(b+1)\delta$. Thus, the limit angle is certainly $< \pi/2$. □

This together with $2'$ implies:

SUBLEMMA. *If $x \in U \cap (M_{\leq\epsilon})_\eta$ and $y \in M \setminus (M_{\leq\epsilon})_{\eta+\delta}$ is a point for which $d(y,x) \leq (b+1)\delta$ then (since $d\big(x,\pi(x)\big) \leq d(x,y) \leq (b+1)\delta$) the angle between the tangents at $\pi(x)$ to $[x,\pi(x)]$ and $[\pi(x),y]$ is at most $\beta < \pi/2$.* □

Step F (Proving condition 2)**:** We have to show that there is a deformation retract from $U \cap (M_{\leq\epsilon})_\eta$ to $\partial(M_{\leq\epsilon})_\eta$. For this we let any $x \in U \cap (M_{\leq\epsilon})_\eta$ to flow at constant rate $d(x,\pi(x))$ in the direction of $\pi(x)$.

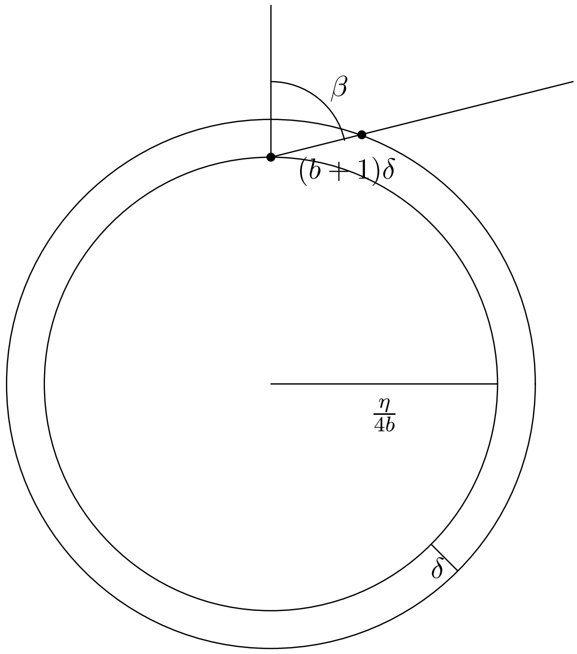

Figure 1: The figure shows two concentric circles of radiuses $\eta/4b$ and $\eta/4b+\delta$ and two points, one on each circle, at distance $2(b+1)\delta$, and the segment between them. β is the angle between this segment and the external normal to the smaller circle.

Uniqueness of $\pi(x)$ implies continuity. We only need to show that the segment $[x,\pi(x)]$ is contained in $U\cap(M_{\leq\epsilon})_\eta$. Clearly $[x,\pi(x)]\subset(M_{\leq\epsilon})_\eta$. Since $x\in U$ there is $y\in\mathcal{F}$ with $d(x,y)\leq(b+1)\delta$. Let $c(t)$ be the geodesic line which contains the segment $[x,\pi(x)]$ with $c(0)=x$. The negative curvature implies that $\varphi(t=d(c(t),y)$ is a convex function of t. The above sublemma implies that the derivative

$$\dot\varphi\big(d(x,\pi(x))\big)\leq-\cos\beta<0\,.$$

Since the derivative of a convex function is non-decreasing, it follows that $\dot\varphi(t)<0$ for any $t<d(x,\pi(x))$, i.e. $\varphi(t)$ is monotonically decreasing and

$$\big[x,\pi(x)\big]\subset B\big(y,(b+1)\delta\big)\subset U\,.$$

Step G (Proving condition 3)**:** Condition 3 follows from the "star-shape" structure of every connected component of $M_{\leq\epsilon}$, that enables us to define an appropriate vector field which induces a deformation retract which proves the homotopy equivalence. The hyperbolic components are "star-shaped" with respect to the line $(0,\infty)$ where the displacement function $d(x,\gamma(x))$ attains its minimum, i.e. the axis of γ. Denote this line by L and let $D(x) = d(x,L)$ be the distance function from this line. The corresponding vector field is its gradient $\nabla(D)(x)$. The parabolic components are "star-shaped" with respect to ∞, so we can use the "constant" vector field $(-1,0,\dots,0)$ which is the gradient of the associated Busemann function. □

Remark. All the constants in the above proof may be estimated effectively yielding an explicit constant depending on the dimension.

Some Concluding Remarks

A general theorem of Wang (see [W] and [Bo]) asserts

Theorem. *Let G be a semisimple Lie group without compact factors, not locally isomorphic to $SL_2(\mathbb{R})$ or $SL_2(\mathbb{C})$. Then for any $V > 0$, there are only finitely many conjugacy classes of irreducible lattices in G of covolume at most V.*

The result of Wang quoted at the introduction is just the very special case when $G = PO(n,1)$, $n \geq 4$ and only torsion-free lattices are considered. Our work, thus, can be viewed as a first attempt towards a quantitative version of Wang's theorem, whose original proof is non-effective and gives no estimate.

Let $\rho_G(V)$ denote the number of conjugacy classes of irreducible lattices in G of covolume at most V. Denote by $\rho_G^\circ(V)$ the number of those which are torsion-free. An interesting problem is to estimate the growth of $\rho_G(V)$ as a function of V. Our theorem actually says that for $G = PO(n,1)$, $n \geq 4$, $\log \rho_G(V) \approx V \log V$.

One may expect that the growth of $\rho_G(V)$ and $\rho_G^\circ(V)$ is essentially the same, but this has not been verified yet even for $PO(n,1)$.

Recently Gelander extended the result of this paper by proving upper bounds for $\rho_G^\circ(V)$ (and for some classes of lattices with torsion) for almost any (rank one and higher rank) semisimple Lie group G (see [G]). As above, the upper bounds obtained in [G] are of the form $V^{c(G)V}$.

However, it is hard to believe that the estimates in [G] are tight in the general case, in the sense proved here. In fact, one may tend to believe that if G is a higher rank semisimple, then $\log \rho_G^\circ(V)$ and even $\log \rho_G(V)$ grows like $\log^2 V/\log\log V$ (which is much smaller then the bound $V \log V$). Let us explain the difference: The proof of the lower bound presented above is based on the non-arithmeticity of the lattice Δ and, more crucially, on the fact that it has a large subgroup growth. There is not much hope for a similar argument in the higher rank case. If $rank(G) \geq 2$ then all lattices are arithmetic. Given an arithmetic lattice Γ it has congruence subgroups. The growth of the number of congruence subgroups was determined by Lubotzky in [L2] where he showed that the number of index n congruence subgroups grows like $n^{c\frac{\log n}{\log\log n}}$. It was conjectured by Serre (and proved in most cases, cf. [PR]) that the congruence subgroup problem has an affirmative solution for higher rank arithmetic groups. If one expects "few" maximal arithmetic groups, and the existence of a uniform bound on the constant c (independent of the lattice) in the last expression (see [GoLP]), then $\log \rho_G(V)$ should grow at most like $c(G)\frac{(\log V)^2}{\log\log V}$. At this point we do not know a better bound then $O(V\log V)$ for any G. We even do not know a better bound if one counts only the (conjugacy classes of) lattices in $SL_3(\mathbb{R})$ commensurable to $SL_3(\mathbb{Z})$.

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