Geometriae Dedicata (2005) 114: 189–207
DOI 10.1007/s10711-005-5538-9

Anti-tori in Square Complex Groups

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(Received: 2 September 2004; accepted in final form: 15 April 2005)

Abstract. An anti-torus is a subgroup $\langle a, b\rangle$ in the fundamental group of a compact non-positively curved space X, acting in a specific way on the universal covering space $\tilde{X}$ such that a and b do not have any commuting nontrivial powers. We construct and investigate anti-tori in a class of commutative transitive fundamental groups of finite square complexes, in particular for the groups $\Gamma_{p,l}$ originally studied by Mozes [*Israel J. Math.* **90**(1–3) (1995), 253–294]. It turns out that anti-tori in $\Gamma_{p,l}$ directly correspond to noncommuting pairs of Hamilton quaternions. Moreover, free anti-tori in $\Gamma_{p,l}$ are related to free groups generated by two integer quaternions, and also to free subgroups of $SO_3(\mathbb{Q})$. As an application, we prove that the multiplicative group generated by the two quaternions $1+2i$ and $1+4k$ is not free.

Mathematics Subject Classifications (2000). primary: 11R52, 20E05, 20F67; secondary: 20E07, 20E08.

Key words. anti-torus, square complex, quaternion, free subgroup, commutative transitive.

1. Introduction

Bridson and Wise have given the following definition of an anti-torus [3, Definition 9.1]: Let X be a compact nonpositively curved space with universal cover p: $\tilde{X} \to X$. It is well-known that the fundamental group $\pi_1(X, x)$ acts on $\tilde{X}$, and that each element $\gamma \in \pi_1(X, x)$ leaves invariant in this action at least one isometrically embedded copy of the real line, a so-called axis for γ. Let $a, b \in \pi_1(X, x)$ and suppose that there is an isometrically embedded plane in $\tilde{X}$ which contains an axis for each of a, b and that these axes intersect in $p^{-1}x$. If a and b do not have powers that commute, then $\langle a, b\rangle$ is called an *anti-torus* in $\pi_1(X, x)$. If $\langle a, b\rangle$ is free then it is called a free *anti-torus*.

We will restrict to a class where $\tilde{X} = \mathcal{T}_{2m} \times \mathcal{T}_{2n}$, the product of two regular trees of degree $2m$ and $2n$, respectively, and X is a certain finite square complex having a single vertex x. The fundamental group $\pi_1(X, x) < \mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$ is then called a $(2m, 2n)$-group (see Section 2 for the precise definition).

Wise [20] has constructed an anti-torus in a (4, 6)-group to produce the first examples of nonresidually finite groups in the following three important classes: finitely presented small cancellation groups, automatic groups, and groups acting properly discontinuously and cocompactly on CAT(0)-spaces. Another application of anti-tori is the generation of aperiodic tilings of the Euclidean plane by unit squares (see [20, 16]).

In general, it seems to be very difficult to decide whether a subgroup $\langle a, b\rangle$ is an anti-torus, or to decide whether a group $\pi_1(X, x)$ has an anti-torus or not. In Section 3, we further restrict to *commutative transitive* $(2m, 2n)$-groups, i.e. to groups G where commutativity is a transitive relation on $G\backslash\{1\}$. In this context, we prove a dichotomy that $\langle a, b\rangle$ either is an anti-torus, or isomorphic to the abelian group $\mathbb{Z}\times\mathbb{Z}$. Moreover, it turns out that any commutative transitive $(2m, 2n)$-group has an anti-torus, if $(m, n)\neq(1, 1)$. In Section 4, we define for any pair (p, l) of distinct odd prime numbers a commutative transitive $(p+1, l+1)$-group $\Gamma_{p,l}$ and apply the results of Section 3. Anti-tori in $\Gamma_{p,l}$ are directly related to noncommuting Hamilton quaternions $x, y\in\mathbb{H}(\mathbb{Z})$ of norm a power of p and l, respectively. Although these considerations provide a very easy method to construct anti-tori in $\Gamma_{p,l}$, it is not clear at all if there are *free* anti-tori in $(2m, 2n)$-groups. We give in Section 5 a criterion for the construction of free anti-tori in terms of free groups generated by two quaternions, but do not know if such quaternions exist. Nevertheless, this criterion can be applied to prove that certain pairs of quaternions, for example $1+2i$ and $1+4k$, do not generate a free group, and we establish an explicit (long) relation in this example. Finally, we relate in Section 6 free subgroups of $\Gamma_{p,l}$ to free subgroups of $SO_3(\mathbb{Q})$, using an explicit embedding $\Gamma_{p,l}\to SO_3(\mathbb{Q})$.

Most results of this work are taken from the authors PhD thesis [16].

2. Preliminaries

Let $m, n\in\mathbb{N}$ and $E_h := \{a_1, \ldots, a_m\}^{\pm 1}$, $E_v := \{b_1, \ldots, b_n\}^{\pm 1}$. A $(2m, 2n)$-*group* is the fundamental group $\Gamma = \pi_1(X, x)$ of a finite two-dimensional cell complex X satisfying the following conditions:

- The one-skeleton $X^{(1)}$ consists of a single vertex x and $m+n$ oriented loops $a_1, \ldots, a_m, b_1, \ldots, b_n$, whose inverses are denoted by $a_1^{-1}, \ldots, a_m^{-1}, b_1^{-1}, \ldots, b_n^{-1}$. In other words, $X^{(1)}$ is the graph with vertex set $\{x\}$ and edge set $E_h \sqcup E_v$.
- To build X, exactly mn squares are attached to $X^{(1)}$, such that the boundary of each square is of the form $aba'b'$, where $a, a'\in E_h, b, b'\in E_v$. In particular, the four corners of each square are identified with the vertex x. We denote such a set of mn squares by $R_{m\cdot n}$.
- The link $\mathrm{Lk}(X, x)$ of the vertex x in X has to be isomorphic to the complete bipartite graph on $2m+2n$ vertices, where the bipartite structure is induced by the decomposition of the edges into the two classes $E_h\sqcup E_v$. Informally speaking, this condition means that for any $a\in E_h, b\in E_v$, the complex X must have a unique corner in a unique square with adjoining edges a and b.

As a consequence, the universal covering space $\tilde{X}$ of X is the product of two regular trees $\mathcal{T}_{2m}\times\mathcal{T}_{2n}$, see [5, Proposition 1.1] or [20, Theorem II.1.10]. By construction, $\Gamma < \mathrm{Aut}(\mathcal{T}_{2m})\times\mathrm{Aut}(\mathcal{T}_{2n})$ acts freely and transitively on the vertices of $\tilde{X}$, and for some purposes it is convenient to see Γ as a cocompact lattice in $\mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$, equipped with its usual topology. Indeed, the main motivation for Burger, Mozes and Zimmer to define and study such groups Γ were expected (super-)rigidity and arithmeticity phenomena analogous to the famous results for lattices in higher rank semisimple Lie groups (in particular by Margulis [12]). We will not treat this aspect, but refer to [5, 6] for interesting developments in this direction.

In the remaining parts of this section we want to discuss several group theoretic properties of $(2m, 2n)$-groups Γ needed in the subsequent sections.

A finite presentation of Γ with $m+n$ generators and mn relations can be directly read off from X:

$$\begin{aligned}\Gamma &= \langle a_1, \ldots, a_m, b_1, \ldots, b_n | aba'b' = 1, \text{ for each attached square } aba'b' \rangle \\ &= \langle a_1, \ldots, a_m, b_1, \ldots, b_n | R_{m \cdot n} \rangle.\end{aligned}$$

If the 2-cells of X are metrized as Euclidean squares, then X is nonpositively curved and Γ is a torsion-free CAT(0)-group by [2, Theorem 4.13(2)].

Due to the link condition in X, every element $\gamma \in \Gamma$ can be brought in a unique normal form, where 'the a's are followed by the b's'. The idea is to successively replace length 2 subwords of γ of the form ba by $a'b'$, if $a'b'a^{-1}b^{-1} = 1$ in Γ, or in other words if (exactly) one of the four squares $a'b'a^{-1}b^{-1}, ab'^{-1}a'^{-1}b, a'^{-1}bab'^{-1}, a^{-1}b^{-1}a'b'$ is in $R_{m \cdot n}$. Analogously, there is a unique normal form, where 'the b's are followed by the a's'. Here is the precise statement of Bridson and Wise:

PROPOSITION 1 (Bridson–Wise [3, Normal Form Lemma 4.3]). *Let γ be any element in a $(2m, 2n)$-group $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R_{m \cdot n} \rangle$. Then γ can be written as*

$$\gamma = \sigma_a \sigma_b = \sigma'_b \sigma'_a$$

where σ_a, σ'_a are freely reduced words in the subgroup $\langle a_1, \ldots, a_m \rangle_\Gamma$ and σ_b, σ'_b are freely reduced words in $\langle b_1, \ldots, b_n \rangle_\Gamma$. The words $\sigma_a, \sigma'_a, \sigma_b, \sigma'_b$ are uniquely determined by γ. Moreover, $|\sigma_a| = |\sigma'_a|$ and $|\sigma_b| = |\sigma'_b|$ where $|\cdot|$ is the word length with respect to the symmetric set of standard generators $\{a_1, \ldots, a_m, b_1, \ldots, b_n\}^{\pm 1}$ of Γ.

If $\gamma \in \Gamma$ has the form $\sigma_a \sigma_b$ as in Proposition 1, then we say that γ is in *ab-normal form*. Proposition 1 has some immediate consequences on the structure of Γ.

COROLLARY 2. *Let $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R_{m \cdot n} \rangle$ be a $(2m, 2n)$-group. Then*

(1) *The two groups $\langle a_1, \ldots, a_m \rangle_\Gamma$ and $\langle b_1, \ldots, b_n \rangle_\Gamma$ are free subgroups of Γ of rank m and n, respectively.*
(2) *The center $Z\Gamma$ of Γ is trivial if $m, n \geqslant 2$.*

Proof. (1) This follows directly from the uniqueness of the normal forms described in Proposition 1.

(2) Assume that there is an element $\gamma \in Z\Gamma \backslash \{1\}$ and let

$$\gamma = a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)}$$

be its ab-normal form, $a^{(1)}, \dots, a^{(k)} \in E_h, b^{(1)}, \dots, b^{(l)} \in E_v$, where we can assume without loss of generality that $k \geqslant 1$ and $l \geqslant 0$. Take any element

$$a \in E_h \backslash \{a^{(1)}, a^{(1)^{-1}}\} \neq \emptyset.$$

(Here, we use $m \geqslant 2$. Under the assumption $k \geqslant 0, l \geqslant 1$, we would have used $n \geqslant 2$.) Then, we have $a\gamma = \gamma a$, i.e.

$$aa^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} = a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} a.$$

The left-hand side of this equation is already in ab-normal form, since $a \neq a^{(1)^{-1}}$. By uniqueness of the ab-normal form, we can conclude from the right-hand side that $a = a^{(1)}$, but this is a contradiction to the choice of a, and it follows $Z\Gamma = 1$. $\square$

For a $(2m, 2n)$-group Γ we define the homomorphism $\rho_v \colon \langle b_1, \dots, b_n \rangle_\Gamma \to \mathrm{Sym}(E_h)$ as follows. Let $b \in E_v$ and $a \in E_h$, then $\rho_v(b)(a) := a'$ is the uniquely determined element in E_h such that $a^{-1} b a' = \tilde{b}$ for some $\tilde{b} \in E_v$. For a geometric interpretation of ρ_v, just draw the square $a\tilde{b}a'^{-1}b^{-1}$.

Another application of Proposition 1 is the following sufficient criterion to show that the centralizer $Z_\Gamma(b) = \{\gamma \in \Gamma : \gamma b = b\gamma\}$ of $b \in E_v$ is as small as possible. This will be useful in some results of Sections 3 and 4.

LEMMA 3. *Let $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_{m \cdot n} \rangle$ be a $(2m, 2n)$-group. Assume that there is an element $b \in E_v$ such that $\rho_v(b)(a) \neq a$ for all $a \in E_h$. Then $Z_\Gamma(b) = \langle b \rangle_\Gamma \cong \mathbb{Z}$.*

Proof. Obviously, $\langle b \rangle_\Gamma < Z_\Gamma(b)$. We therefore have to show $Z_\Gamma(b) < \langle b \rangle_\Gamma$. Let

$$\gamma = a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} \in Z_\Gamma(b)$$

be in ab-normal form, $a^{(1)}, \dots, a^{(k)} \in E_h, b^{(1)}, \dots, b^{(l)} \in E_v, k, l \geqslant 0$. Then

$$a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} b = ba^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)}.$$

First assume that $k \geqslant 1$. The ab-normal form of γb starts with $a^{(1)} \dots a^{(k)}$. Bringing also $ba^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)}$ to its ab-normal form, we must have in a first step $ba^{(1)} = a^{(1)} \tilde{b}$ for some $\tilde{b} \in E_v$, i.e. $a^{(1)^{-1}} ba^{(1)} = \tilde{b} \in E_v$ and therefore $\rho_v(b)(a^{(1)}) = a^{(1)}$, which is impossible by assumption, hence $k = 0$. This means $\gamma = b^{(1)} \dots b^{(l)}$ and

$$b^{(1)} \dots b^{(l)} b = bb^{(1)} \dots b^{(l)}.$$

By uniqueness of the ab-normal form of

$$b = b^{(l)^{-1}} \dots b^{(1)-1} b b^{(1)} \dots b^{(l)}$$

we either have $l = 0$, or $b^{(1)}, \dots, b^{(l)} \in \{b, b^{-1}\}$ and, hence, $\gamma = b^{(1)} \dots b^{(l)} \in \langle b \rangle_\Gamma$. □

Observe that it is very easy to verify for a given set $R_{m \cdot n}$ and $b \in E_v$ if the condition $\rho_v(b)(a) \neq a$ of Lemma 3 holds or not.

We recall the definition of an anti-torus in the context we will use it.

DEFINITION 4. Let $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n | R_{m \cdot n} \rangle$ be a $(2m, 2n)$-group, and $a \in \langle a_1, \dots, a_m \rangle_\Gamma$, $b \in \langle b_1, \dots, b_n \rangle_\Gamma$ two elements. The subgroup $\langle a, b \rangle_\Gamma$ is called an *anti-torus* in Γ if a, b have no commuting nontrivial powers, i.e. if $a^r b^s \neq b^s a^r$ for all $r, s \in \mathbb{Z} \backslash \{0\}$.

3. Anti-tori in Commutative Transitive (2m, 2n)-Groups

A group G is called *commutative transitive*, if the relation of commutativity is transitive on the set $G \backslash \{1\}$ (i.e. $g_1 g_2 = g_2 g_1$, $g_2 g_3 = g_3 g_2$ always implies $g_1 g_3 = g_3 g_1$, if $g_1, g_2, g_3 \neq 1$). Restricting to commutative transitive $(2m, 2n)$-groups allows us to give a very easy criterion to construct anti-tori. The results stated in this section will be applied to an interesting subclass of commutative transitive $(2m, 2n)$-groups in Section 4.

PROPOSITION 5. *Let* $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_{m \cdot n} \rangle$ *be a commutative transitive* $(2m, 2n)$*-group and let* $a \in \langle a_1, \dots, a_m \rangle_\Gamma$, $b \in \langle b_1, \dots, b_n \rangle_\Gamma$ *be two elements. Then* $\langle a, b \rangle_\Gamma$ *is an anti-torus in* Γ *if and only if* a *and* b *do not commute in* Γ.

Proof. Assume first that $\langle a, b \rangle_\Gamma$ is no anti-torus in Γ, i.e. $a^r b^s = b^s a^r$ for some $r, s \in \mathbb{Z} \backslash \{0\}$. Obviously, a commutes with a^r, and b commutes with b^s. Using the assumption that Γ is commutative transitive, we conclude that a and b commute in Γ. The other direction follows immediately from the definition of an anti-torus. □

This gives a dichotomy for subgroups $\langle a, b \rangle_\Gamma$, where $a, b \neq 1$.

COROLLARY 6. *Let* $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_{m \cdot n} \rangle$ *be a commutative transitive* $(2m, 2n)$*-group and let* $a \in \langle a_1, \dots, a_m \rangle_\Gamma$, $b \in \langle b_1, \dots, b_n \rangle_\Gamma$ *be two nontrivial elements. Then either* $\langle a, b \rangle_\Gamma \cong \mathbb{Z} \times \mathbb{Z}$ *or* $\langle a, b \rangle_\Gamma$ *is an anti-torus in* Γ.

Proof. If a and b do not commute, then $\langle a, b \rangle_\Gamma$ is an anti-torus in Γ by Proposition 5. If a and b commute, then $\langle a, b \rangle_\Gamma$ is a finitely generated Abelian torsion-free quotient of $\mathbb{Z} \times \mathbb{Z}$, hence either 1, $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. The first two cases can be excluded by the assumption $a, b \neq 1$, and using the uniqueness of the normal forms of powers of a and b. □

COROLLARY 7. *Let* $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n | R_{m\cdot n}\rangle$ *be a commutative transitive* $(2m, 2n)$*-group. Then* Γ *has an anti-torus if and only if* $(m, n) \neq (1, 1)$.

Proof. Up to isomorphism, there are only two (2, 2)-groups: the Abelian group $\mathbb{Z} \times \mathbb{Z}$, and the (noncommutative transitive) group $\langle a_1, b_1 | a_1 b_1 a_1 = b_1\rangle$, where a_1 commutes with b_1^2. Both groups obviously have no anti-torus.

For the other direction, assume that $(m, n) \neq (1, 1)$. Then there are elements $a \in E_h$ and $b \in E_v$ which do not commute; otherwise the $(2m, 2n)$-group Γ would be a direct product of free groups

$$\langle a_1, \dots, a_m\rangle_\Gamma \times \langle b_1, \dots, b_n\rangle_\Gamma \cong F_m \times F_n,$$

which is not commutative transitive if $(m, n) \neq (1, 1)$. By Proposition 5, $\langle a, b\rangle_\Gamma$ is an anti-torus in Γ. □

The following corollary gives infinitely many anti-tori in Γ, provided the centralizer of some b is cyclic. By Lemma 3, this is for example satisfied for elements $b \in E_v$ such that $\rho_v(b)(a) \neq a$ for all $a \in E_h$.

COROLLARY 8. *Let* $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n | R_{m\cdot n}\rangle$ *be a commutative transitive* $(2m, 2n)$*-group and let* $b \in \langle b_1, \dots, b_n\rangle_\Gamma$ *be an element such that* $Z_\Gamma(b) = \langle b\rangle_\Gamma$. *Then* $\langle a, b\rangle_\Gamma$ *is an anti-torus in* Γ *for each* $a \in \langle a_1, \dots, a_m\rangle_\Gamma \setminus \{1\}$.

Proof. The assumption $Z_\Gamma(b) = \langle b\rangle_\Gamma$ implies that $b \neq 1$ and that b does not commute with any element $a \in \langle a_1, \dots, a_m\rangle_\Gamma \setminus \{1\}$. Now apply Proposition 5. □

Similar as for lattices in higher rank semisimple Lie groups, there is also the important notion of 'reducibility' and 'irreducibility' for lattices acting on a product of trees, see [5, Chapter 1]: A lattice $\Gamma < \mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$ is *reducible* if it is commensurable to a direct product $\Gamma_1 \times \Gamma_2$ of lattices $\Gamma_1 < \mathrm{Aut}(\mathcal{T}_{2m}), \Gamma_2 < \mathrm{Aut}(\mathcal{T}_{2n})$. Otherwise, Γ is called *irreducible*. Many $(2m, 2n)$-groups with interesting group theoretic properties, like non-residually finite groups or virtually simple groups [5, 16], are irreducible, since reducible $(2m, 2n)$-groups contain a subgroup of finite index which is a direct product of two free groups of finite rank. There is no known algorithm in general to decide whether a given $(2m, 2n)$-group is irreducible. However, $(2m, 2n)$-groups having an anti-torus are always irreducible.

PROPOSITION 9 (Wise [20, Section II.4]). *Let* $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n | R_{m\cdot n}\rangle$ *be a* $(2m, 2n)$*-group. If* Γ *has an anti-torus, then it is irreducible.*

Proof. For $\Gamma < \mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$ let $pr_1 \colon \Gamma \to \mathrm{Aut}(\mathcal{T}_{2m})$ and $pr_2 \colon \Gamma \to \mathrm{Aut}(\mathcal{T}_{2n})$ be the two canonical projections. Define $\Lambda_1 = \mathrm{pr}_1(\ker(\mathrm{pr}_2)) < \mathrm{Aut}(\mathcal{T}_{2m})$ and $\Lambda_2 = \mathrm{pr}_2(\ker(\mathrm{pr}_1)) < \mathrm{Aut}(\mathcal{T}_{2n})$. Let $\langle a, b\rangle_\Gamma$ be an anti-torus in Γ, where $a \in \langle a_1, \dots, a_m\rangle_\Gamma, b \in \langle b_1, \dots, b_n\rangle_\Gamma$, and suppose that Γ is reducible. Then by [5, Proposition 1.2], the group $\Lambda_1 \times \Lambda_2$ is a subgroup of finite index in Γ, in particular the indices $[\langle a_1,\dots,a_m\rangle_\Gamma\colon\Lambda_1]$ and $[\langle b_1,\dots,b_n\rangle_\Gamma\colon\Lambda_2]$ are finite. It follows that $a^r\in\Lambda_1, b^s\in\Lambda_2$ for some $r,s\in\mathbb{N}$. But then $a^rb^s=b^sa^r$, a contradiction. □

COROLLARY 10 *A commutative transitive* $(2m,2n)$*-group is irreducible if and only if* $(m,n)\neq(1,1)$.

Proof. Any (2, 2)-group is reducible. If $(m,n)\neq(1,1)$, then we combine Corollary 7 and Proposition 9.

4. Illustration for the Quaternion Groups $\Gamma_{p,l}$

For any pair of distinct odd prime numbers p,l, we define in this section a commutative transitive $(p+1,l+1)$-group $\Gamma_{p,l}$, and can therefore apply the results of Section 3. With the restriction $p,l\equiv 1 \pmod 4$, the groups $\Gamma_{p,l}$ were originally used by Mozes [13–15] to define certain tiling systems, so-called two-dimensional subshifts of finite type, and to study a resulting dynamical system. Later, Burger–Mozes [5] used the residually finite group $\Gamma_{13,17}$ as a building block in the construction of a *non*residually finite (196, 324)-group and in a construction of an infinite family of finitely presented torsion-free virtually simple groups. Kimberley–Robertson [9] made explicit computations for many small values of p,l, for example on the Abelianization of $\Gamma_{p,l}$. The condition $p,l\equiv 1 \pmod 4$ was dropped in [16], and it was shown in [17] that these generalized groups $\Gamma_{p,l}$ are CSA (i.e. all maximal Abelian subgroups are malnormal), in particular they are commutative transitive.

We need some preparation to define the groups $\Gamma_{p,l}$. For a commutative ring R with unit, let

$$\mathbb{H}(R)=\{x_0+x_1i+x_2j+x_3k : x_0,x_1,x_2,x_3\in R\}$$

be the ring of Hamilton quaternions over R, i.e. $1,i,j,k$ is a free basis, and the multiplication is determined by $i^2=j^2=k^2=-1$ and $ij=-ji=k$. Let $\overline{x}:=x_0-x_1i-x_2j-x_3k\in\mathbb{H}(R)$ be the *conjugate* of $x=x_0+x_1i+x_2j+x_3k\in\mathbb{H}(R)$, and $|x|^2:=x\overline{x}=\overline{x}x=x_0^2+x_1^2+x_2^2+x_3^2\in R$ its *norm*. We write $\Re(x):=x_0$ for the 'real part' of x.

If R is any ring, we denote by $U(R)$ the group of invertible elements (with respect to the multiplication) in R.

From now on, let p,l be any pair of distinct odd prime numbers. Let $\mathbb{Q}_p,\mathbb{Q}_l$ be the p-adic and l-adic numbers, respectively. If K is a field, let as usual $\mathrm{PGL}_2(K)=\mathrm{GL}_2(K)/\mathrm{ZGL}_2(K)$, and write brackets [A] to denote the image of the matrix $A\in\mathrm{GL}_2(K)$ under the quotient homomorphism $\mathrm{GL}_2(K)\to\mathrm{PGL}_2(K)$. We define the homomorphism of groups

$$\psi_{p,l}\colon U(\mathbb{H}(\mathbb{Q}))\to\mathrm{PGL}_2(\mathbb{Q}_p)\times\mathrm{PGL}_2(\mathbb{Q}_l)$$

by

$$\psi_{p,l}(x_0+x_1i+x_2j+x_3k)=\left(\left[\begin{pmatrix} x_0+x_1c_p+x_3d_p & -x_1d_p+x_2+x_3c_p \\ -x_1d_p-x_2+x_3c_p & x_0-x_1c_p-x_3d_p \end{pmatrix}\right],\right.$$
$$\left.\left[\begin{pmatrix} x_0+x_1c_l+x_3d_l & -x_1d_l+x_2+x_3c_l \\ -x_1d_l-x_2+x_3c_l & x_0-x_1c_l-x_3d_l \end{pmatrix}\right]\right),$$

where $c_p, d_p \in \mathbb{Q}_p$ and $c_l, d_l \in \mathbb{Q}_l$ are elements such that $c_p^2 + d_p^2 + 1 = 0 \in \mathbb{Q}_p$ and $c_l^2 + d_l^2 + 1 = 0 \in \mathbb{Q}_l$. This definition is motivated by the following well-known isomorphism:

PROPOSITION 11 (see [7, Proposition 2.5.2]). *Let K be a field of characteristic different from 2, and assume that there exist $c, d \in K$ such that $c^2+d^2+1=0$. Then $\mathbb{H}(K)$ is isomorphic to the algebra $M_2(K)$ of (2×2)-matrices over K. An isomorphism of algebras is given by the map*

$$\mathbb{H}(K) \to M_2(K)$$

$$x = x_0+x_1i+x_2j+x_3k \mapsto \begin{pmatrix} x_0+x_1c+x_3d & -x_1d+x_2+x_3c \\ -x_1d-x_2+x_3c & x_0-x_1c-x_3d \end{pmatrix}$$

and we have

$$\det\begin{pmatrix} x_0+x_1c+x_3d & -x_1d+x_2+x_3c \\ -x_1d-x_2+x_3c & x_0-x_1c-x_3d \end{pmatrix} = |x|^2.$$

If $p, l \equiv 1 \pmod 4$, we can choose $d_p = 0$ and $d_l = 0$ in the definition of $\psi_{p,l}$, as in the original definition of Mozes [13]. Note that

$$U(\mathbb{H}(\mathbb{Q})) = \{x \in \mathbb{H}(\mathbb{Q}) : |x|^2 \in U(\mathbb{Q})\} = \mathbb{H}(\mathbb{Q})\setminus\{0\}.$$

The homomorphism $\psi_{p,l}$ is not injective, in fact

$$\ker(\psi_{p,l}) = ZU(\mathbb{H}(\mathbb{Q})) = \{x \in U(\mathbb{H}(\mathbb{Q})) : x = \overline{x}\} \cong U(\mathbb{Q}) = \mathbb{Q}\setminus\{0\},$$

and $\psi_{p,l}(x) = \psi_{p,l}(y)$ if and only if $y = \lambda x$ for some $\lambda \in U(\mathbb{Q})$. Observe that

$$\psi_{p,l}(x)^{-1} = \psi_{p,l}(x^{-1}) = \psi_{p,l}\left(\frac{\overline{x}}{|x|^2}\right) = \psi_{p,l}(\overline{x}).$$

For an odd prime number q, let X_q be the set

$$\begin{aligned} X_q := \{x = x_0+x_1i+x_2j+x_3k \in \mathbb{H}(\mathbb{Z});\ |x|^2 = q;\\ x_0 \text{ odd}, x_1, x_2, x_3 \text{ even, if } q \equiv 1 \pmod 4;\\ x_1 \text{ even}, x_0, x_2, x_3 \text{ odd, if } q \equiv 3 \pmod 4\}. \end{aligned}$$

By Jacobi's Theorem (see for example [11, Theorem 2.1.8]), X_q has $2(q+1)$ elements. Let $Q_{p,l}$ be the subgroup of $U(\mathbb{H}(\mathbb{Q}))$ generated by $X_p \cup X_l \subset \mathbb{H}(\mathbb{Z})$ and $\Gamma_{p,l}$ be its image $\psi_{p,l}(Q_{p,l})$. Observe that

$$\ker(\psi_{p,l} \mid_{Q_{p,l}}) = \ker(\psi_{p,l}) \cap Q_{p,l} \cong \{\pm p^r l^s : r, s \in \mathbb{Z}\} < U(\mathbb{Q}).$$

Equivalently, $\Gamma_{p,l}$ can be defined as

$$\begin{aligned}&\psi_{p,l}(\{x \in \mathbb{H}(\mathbb{Z});\, |x|^2 = p^r l^s, r, s \geqslant 0;\\ &\qquad x_0 \text{ odd}, x_1, x_2, x_3 \text{ even, if } |x|^2 \equiv 1 (\text{mod } 4);\\ &\qquad x_1 \text{ even}, x_0, x_2, x_3 \text{ odd, if } |x|^2 \equiv 3 (\text{mod } 4)\}).\end{aligned}$$

Note that the set $\psi_{p,l}(X_p)$ has $p+1$ elements, since $|X_p| = 2(p+1)$ and $\psi_{p,l}(x) = \psi_{p,l}(-x)$. These elements generate a free subgroup $\psi_{p,l}(\langle X_p \rangle_{Q_{p,l}}) = \langle a_1, \dots, a_{\frac{p+1}{2}} \rangle_{\Gamma_{p,l}}$ of $\Gamma_{p,l}$ of rank $(p+1)/2$, since $\psi_{p,l}(x)^{-1} = \psi_{p,l}(\overline{x})$. Similarly, $\psi_{p,l}(X_l)$ generates a free subgroup $\psi_{p,l}(\langle X_l \rangle_{Q_{p,l}}) = \langle b_1, \dots, b_{\frac{p+1}{2}} \rangle_{\Gamma_{p,l}}$ of $\Gamma_{p,l}$ of rank $(l+1)/2$.

We summarize the definitions in the following commutative diagram, where $\psi_{p,l}|$ denotes the restriction of $\psi_{p,l}$ to the respective domain:

$$\begin{tikzcd}
\{\pm p^r : r \in \mathbb{Z}\} \arrow[r, hook] \arrow[d, hook] & \{\pm p^r l^s : r, s \in \mathbb{Z}\} \arrow[d, hook] & \{\pm l^s : s \in \mathbb{Z}\} \arrow[l, hook'] \arrow[d, hook] \\
\langle X_p \rangle_{Q_{p,l}} \arrow[r, hook] \arrow[d, "\psi_{p,l}|"'] & Q_{p,l} \arrow[d, "\psi_{p,l}|"] & \langle X_l \rangle_{Q_{p,l}} \arrow[l, hook'] \arrow[d, "\psi_{p,l}|"] \\
\langle a_1, \dots, a_{\frac{p+1}{2}} \rangle_{\Gamma_{p,l}} \arrow[r, hook] & \Gamma_{p,l} & \langle b_1, \dots, b_{\frac{p+1}{2}} \rangle_{\Gamma_{p,l}} \arrow[l, hook']
\end{tikzcd}$$

Our basic general philosophy is to transfer properties of the quaternions to the group $\Gamma_{p,l}$, and vice versa. For example, the fact that $U(\mathbb{H}(\mathbb{Q}))$ is commutative transitive on noncentral elements (cf. Lemma 12) is transferred by Lemma 13 to the fact (Proposition 14) that the group $\Gamma_{p,l}$ is commutative transitive. To simplify the proofs, we introduce the following notation: If $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Q})$, let $\tau(x) := \mathbb{Q}(x_1, x_2, x_3)^T \in \mathbb{Q}^3$.

LEMMA 12. *Two quaternions* $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Q})$ *and* $y = y_0 + y_1 i + y_2 j + y_3 k \in \mathbb{H}(\mathbb{Q})$ *commute, if and only if* $(x_1, x_2, x_3)^T$ *and* $(y_1, y_2, y_3)^T$ *are linearly dependent over* $\mathbb{Q}$. *In particular, if* $x, y \in \mathbb{H}(\mathbb{Q})$ *are two quaternions such that* $x \neq \bar{x}$ *and* $y \neq \bar{y}$, *then* $xy = yx$ *if and only if* $\tau(x) = \tau(y)$.

Proof. The first part follows from the elementary computation

$$xy - yx = 2(x_2 y_3 - x_3 y_2) i + 2(x_3 y_1 - x_1 y_3) j + 2(x_1 y_2 - x_2 y_1) k = 2 \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

This implies the second part, observing that the condition $x \neq \bar{x}$ is equivalent to the condition $(x_1, x_2, x_3)^T \neq (0,0,0)^T$. □

LEMMA 13. *Two quaternions $x, y \in Q_{p,l}$ commute if and only if their images $\psi_{p,l}(x), \psi_{p,l}(y) \in \Gamma_{p,l}$ commute.*

Proof. Obviously $xy = yx$ implies $\psi_{p,l}(x)\psi_{p,l}(y) = \psi_{p,l}(y)\psi_{p,l}(x)$.

For the converse, write as usual $x = x_0 + x_1 i + x_2 j + x_3 k \in Q_{p,l} < U(\mathbb{H}(\mathbb{Q}))$, $y = y_0 + y_1 i + y_2 j + y_3 k \in Q_{p,l}$, and assume that $\psi_{p,l}(x)\psi_{p,l}(y) = \psi_{p,l}(y)\psi_{p,l}(x)$. Then $\psi_{p,l}(xy) = \psi_{p,l}(yx)$, hence $xy = \lambda yx$ for some $\lambda \in U(\mathbb{Q})$. Taking the norm of $xy = \lambda yx$ and using the rule $|xy|^2 = |x|^2|y|^2$, we conclude that $1 = |\lambda|^2 = \lambda^2$ and therefore $\lambda = \pm 1$. If $\lambda = -1$, then $xy = -yx$ and the two general rules $\Re(xy) = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3 = \Re(yx)$ and $\Re(-x) = -x_0 = -\Re(x)$ imply that $\Re(xy) = \Re(-yx) = -\Re(yx) = -\Re(xy)$, hence $\Re(xy) = 0$. This is impossible, since $\Re(xy) = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3$ is always an *odd* integer (divided by $p^r l^s$ for some $r, s \geqslant 0$), using the parity conditions in the definition of X_p and X_l. Consequently, $\lambda = 1$ and $xy = yx$. □

PROPOSITION 14. *The group $\Gamma_{p,l}$ is a commutative transitive $(p+1, l+1)$-group.*

Proof. Mozes showed in [13, Section 3] that $\Gamma_{p,l}$ is a $(p+1, l+1)$-group in the case $p, l \equiv 1 \pmod 4$. It is not difficult to adapt this proof to the general case (see [16, Theorem 3.30(5)]).

To show that the group $\Gamma_{p,l}$ is commutative transitive, let $\psi_{p,l}(x), \psi_{p,l}(y), \psi_{p,l}(z) \in \Gamma_{p,l} \setminus \{1\}$ such that $\psi_{p,l}(x)\psi_{p,l}(y) = \psi_{p,l}(y)\psi_{p,l}(x)$ and $\psi_{p,l}(y)\psi_{p,l}(z) = \psi_{p,l}(z)\psi_{p,l}(y)$, where $x, y, z \in Q_{p,l}$ satisfy $x \neq \bar{x}$, $y \neq \bar{y}$ and $z \neq \bar{z}$. It follows by Lemma 13 that $xy = yx$ and $yz = zy$, hence $\tau(x) = \tau(y) = \tau(z)$ and $xz = zx$ by Lemma 12. Thus, we get $\psi_{p,l}(x)\psi_{p,l}(z) = \psi_{p,l}(z)\psi_{p,l}(x)$. □

We can therefore apply the results of Section 3 to the groups $\Gamma_{p,l}$.

PROPOSITION 15. *Let $\Gamma = \Gamma_{p,l} = \langle a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}} \mid R_{\frac{p+1}{2}\cdot\frac{l+1}{2}} \rangle$ and let $a \in \langle a_1, \ldots, a_{\frac{p+1}{2}} \rangle_\Gamma$, $b \in \langle b_1, \ldots, b_{\frac{l+1}{2}} \rangle_\Gamma$ be two elements. Then*

(1) *$\langle a, b \rangle_\Gamma$ is an anti-torus in Γ if and only if a and b do not commute in Γ.*
(2) *If $a, b \neq 1$, then either $\langle a, b \rangle_\Gamma \cong \mathbb{Z} \times \mathbb{Z}$ or $\langle a, b \rangle_\Gamma$ is an anti-torus in Γ.*
(3) *The group Γ has an anti-torus and is irreducible.*
(4) *If $Z_\Gamma(b) = \langle b \rangle_\Gamma$, then $\langle a, b \rangle_\Gamma$ is an anti-torus in Γ for each $a \neq 1$.*

Proof.

(1) Combine Propositions 5 and 14.
(2) Use Corollary 6 and Proposition 14.
(3) Combine Corollary 7, Propositions 14 and 9.
(4) This follows from Corollary 8 and Proposition 14. □

Remark 16. For the group $\Gamma = \Gamma_{p,l}$ there is the following easy sufficient criterion [17, Corollary 3.7] to generate elements $b \in \langle b_1, \dots, b_{\frac{l+1}{2}} \rangle_\Gamma$ such that $Z_\Gamma(b) = \langle b \rangle_\Gamma$. Take $b \in \langle b_1, \dots, b_{\frac{l+1}{2}} \rangle \setminus \{1\}$, write $b = \psi_{p,l}(x_0 + z_0(c_1 i + c_2 j + c_3 k))$, such that $c_1, c_2, c_3 \in \mathbb{Z}$ are relatively prime, $x_0, z_0 \in \mathbb{Z} \setminus \{0\}$, and define $n(b) = c_1^2 + c_2^2 + c_3^2 \in \mathbb{N}$. If

$$-\left(\frac{-n(b)}{p}\right) = 1 = \left(\frac{-n(b)}{l}\right),$$

then $Z_\Gamma(b) = \langle b \rangle_\Gamma$, where the *Legendre symbol* is defined as

$$\left(\frac{n}{p}\right) = \begin{cases} 0 & \text{if } p \mid n, \\ 1 & \text{if } p \nmid n \text{ and } n \text{ is a square mod } p, \\ -1 & \text{if } p \nmid n \text{ and } n \text{ is not a square mod } p. \end{cases}$$

Using Lemma 13, we also directly get the following result:

PROPOSITION 17. *Let $x \in \langle X_p \rangle_{Q_{p,l}}$, $y \in \langle X_l \rangle_{Q_{p,l}}$ be two noncommuting quaternions. Then $\langle \psi_{p,l}(x), \psi_{p,l}(y) \rangle_{\Gamma_{p,l}}$ is an anti-torus in $\Gamma_{p,l}$.*

Proof. Combine Lemma 13 and Proposition 15(1). □

In the other direction, we can use the structure of $\Gamma_{p,l}$ to get a statement on quaternions.

PROPOSITION 18. *Let*

$$\Gamma = \Gamma_{p,l} = \langle a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}} \mid R_{\frac{p+1}{2} \cdot \frac{l+1}{2}} \rangle.$$

Assume that there is an element $b \in E_v$ such that $\rho_v(b)(a) \neq a$ for all $a \in E_h$. Let $y \in X_l$ such that $\psi_{p,l}(y) = b$. Then there is no $x \in \langle X_p \rangle_{Q_{p,l}}$ such that $x \neq \bar{x}$ and $xy = yx$.

Proof. By Lemma 3, we have $Z_\Gamma(b) = \langle b \rangle_\Gamma$. If $x \in \langle X_p \rangle_{Q_{p,l}}$ such that $xy = yx$, then $\psi_{p,l}(x) \in \langle a_1, \dots, a_{\frac{p+1}{2}} \rangle_\Gamma$ commutes with $\psi_{p,l}(y) = b$, hence $\psi_{p,l}(x) = 1$ and $x = \bar{x}$. □

We also give an application to number theory in Corollary 20, using the following result of Mozes:

PROPOSITION 19 (Mozes [15, Proposition 3.15]). *Let $p, l \equiv 1$ (mod 4) be two distinct prime numbers,*

$$\Gamma_{p,l} = \langle a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}} \mid R_{\frac{p+1}{2} \cdot \frac{l+1}{2}} \rangle$$

and let $x \in \langle X_l \rangle_{Q_{p,l}}$ such that $x \neq \bar{x}$. Take $c_1, c_2, c_3 \in \mathbb{Z}$ relatively prime such that $c := c_1 i + c_2 j + c_3 k \in \mathbb{H}(\mathbb{Z})$ commutes with x. Then there exists a nontrivial element

$a \in \langle a_1, \ldots, a_{\frac{p+1}{2}} \rangle$ commuting with $\psi_{p,l}(x)$ if and only if there are integers $t, u \in \mathbb{Z}$ such that

$$\gcd(t, u) = \gcd(t, pl) = \gcd(u, pl) = 1$$

and $t^2 + 4|c|^2 u^2 \in \{p^r l^s : r, s \in \mathbb{N}\}$.

COROLLARY 20. *Let $p, l \equiv 1 \pmod 4$ be two distinct prime numbers and $\Gamma = \Gamma_{p,l}$. Let $b = \psi_{p,l}(x_0 + x_1 i + x_2 j + x_3 k) \in \langle b_1, \ldots, b_{\frac{l+1}{2}} \rangle_\Gamma \backslash \{1\}$ such that $x_1, x_2, x_3 \in \mathbb{Z}$ are relatively prime. Moreover, assume that $Z_\Gamma(b) = \langle b \rangle_\Gamma$. Then there are no integers $t, u \in \mathbb{Z}$ such that*

$$\gcd(t, u) = \gcd(t, pl) = \gcd(u, pl) = 1$$

and $t^2 + 4(x_1^2 + x_2^2 + x_3^2)u^2 \in \{p^r l^s : r, s \in \mathbb{N}\}$.

We illustrate some previous results for the group $\Gamma_{5,17}$:

COROLLARY 21. *Let $\Gamma = \Gamma_{5,17}$, $\psi = \psi_{5,17}$ and $b = \psi(3 + 2i + 2j) \in E_v$. Then*

(1) *The subgroup $\langle \psi(1 + 2i), \psi(1 + 4k) \rangle_\Gamma$ is an anti-torus in Γ.*
(2) *The subgroup $\langle \psi(1 + 2i), \psi(1 + 4i) \rangle_\Gamma \cong \mathbb{Z} \times \mathbb{Z}$ is no anti-torus in Γ.*
(3) *$Z_\Gamma(b) = \langle b \rangle_\Gamma$.*
(4) *If $a \in \langle a_1, a_2, a_3 \rangle_\Gamma \backslash \{1\}$, then $\langle a, b \rangle_\Gamma$ is an anti-torus in Γ.*
(5) *There are no integers $t, u \in \mathbb{Z}$ such that*

$$\gcd(t, u) = \gcd(t, 85) = \gcd(u, 85) = 1$$

and $t^2 + 8u^2 \in \{5^r 17^s : r, s \in \mathbb{N}\}$.

Proof. (1) We apply Proposition 17, using the obvious fact that $1 + 2i$ and $1 + 4k$ do not commute.

(2) The two quaternions $1 + 2i$ and $1 + 4i$ commute, hence $\psi(1 + 2i) \in \langle a_1, a_2, a_3 \rangle_\Gamma \backslash \{1\}$ commutes with $\psi(1 + 4i) \in \langle b_1, \ldots, b_9 \rangle_\Gamma \backslash \{1\}$. Now apply Proposition 15(1),(2).

(3) We check that $\rho_v(b)(a) \neq a$ for all $a \in E_h$ and apply Lemma 3. Alternatively, we can use [17, Corollary 3.7] (see Remark 16), since we have $n(b) = 2$ and

$$-\left(\frac{-2}{5}\right) = 1 = \left(\frac{-2}{17}\right).$$

(4) Apply Proposition 15(4), using part (3) of this corollary.

(5) This follows from Corollary 20, using part (3) of this corollary and $b = \psi(3 + 2i + 2j) = \psi(\frac{3}{2} + i + j)$. $\square$

Let $b \in \langle b_1, \dots, b_{\frac{l+1}{2}} \rangle_\Gamma \setminus \{1\}$ be a fixed element. It may happen that $\langle a, b \rangle_{\Gamma_{p,l}}$ is an anti-torus for all $a \in E_h$, but not for all $a \in \langle a_1, \dots, a_{\frac{p+1}{2}} \rangle_{\Gamma_{p,l}} \setminus \{1\}$.

COROLLARY 22. *Let* $\Gamma = \Gamma_{5,7}, \psi = \psi_{5,7}, b = \psi(1+2i+j+k), a_1 = \psi(1+2i), a_2 = \psi(1+2j)$ *and* $a_3 = \psi(1+2k)$. *Then* $\langle a, b \rangle_\Gamma$ *is an anti-torus in* Γ *for all* $a \in \{a_1, a_2, a_3\}^{\pm 1} = E_h$. *However,* $\langle a_2 a_3, b \rangle_\Gamma$ *is no anti-torus in* Γ.

Proof. This follows, since $a_2 a_3 = \psi(1+4i+2j+2k)$, and $1+4i+2j+2k$ commutes with $1+2i+j+k$. □

5. Free Anti-tori

An anti-torus $\langle a, b \rangle_\Gamma$ isomorphic to the free group F_2 of rank 2 is called a *free anti-torus* in Γ. It is not known whether there are free anti-tori in $(2m, 2n)$-groups, but we will give in Proposition 24 a sufficient criterion to construct free anti-tori in $\Gamma_{p,l}$, using certain free subgroups in $U(\mathbb{H}(\mathbb{Q}))$. An existence theorem for free anti-tori in a class of fundamental groups of nonpositively curved 2-complexes not including $(2m, 2n)$-groups, appears in [3, Proposition 9.2], but no explicit example of a free anti-torus is given there. To state our criterion for free anti-tori in $\Gamma_{p,l}$, we need the following general lemma.

LEMMA 23. *Let* $\phi : G \to H$ *be a homomorphism of groups such that* $\ker(\phi) = ZG$ *and let* $g_1, \dots, g_t \in G, t \geqslant 2$. *Then* $\langle \phi(g_1), \dots, \phi(g_t) \rangle_H \cong F_t$ *if and only if* $\langle g_1, \dots, g_t \rangle_G \cong F_t$.

Proof. First suppose that $\langle g_1, \dots, g_t \rangle_G \cong F_t$. The restriction

$$\phi|_{\langle g_1, \dots, g_t \rangle_G} : \langle g_1, \dots, g_t \rangle_G \to \langle \phi(g_1), \dots, \phi(g_t) \rangle_H$$

is surjective. It is also injective, since

$$\begin{aligned}\ker(\phi|_{\langle g_1, \dots, g_t \rangle_G}) &= \langle g_1, \dots, g_t \rangle_G \cap \ker(\phi) \\ &= \langle g_1, \dots, g_t \rangle_G \cap ZG < Z\langle g_1, \dots, g_t \rangle_G \cong ZF_t = \{1\},\end{aligned}$$

hence $\langle \phi(g_1), \dots, \phi(g_t) \rangle_H \cong \langle g_1, \dots, g_t \rangle_G \cong F_t$.

The other direction is clear, since any relation in $\langle g_1, \dots, g_t \rangle_G \cong F_t$ induces a relation in $\langle \phi(g_1), \dots, \phi(g_t) \rangle_H$. □

PROPOSITION 24. *Let* $x \in \langle X_p \rangle_{Q_{p,l}}, y \in \langle X_l \rangle_{Q_{p,l}}$. *Then* $\langle \psi_{p,l}(x), \psi_{p,l}(y) \rangle_{\Gamma_{p,l}}$ *is a free anti-torus in* $\Gamma_{p,l}$ *if and only if* $\langle x, y \rangle_{Q_{p,l}} \cong F_2$.

Proof. If $\langle x, y \rangle_{Q_{p,l}} \cong F_2$, then $\langle \psi_{p,l}(x), \psi_{p,l}(y) \rangle_{\Gamma_{p,l}}$ is an anti-torus in $\Gamma_{p,l}$ by Proposition 17. The claim follows now from Lemma 23 applied to the homomorphism $\psi_{p,l}|_{Q_{p,l}} : Q_{p,l} \twoheadrightarrow \Gamma_{p,l}$, where

$$\ker(\psi_{p,l}|_{Q_{p,l}}) = \ker(\psi_{p,l}) \cap Q_{p,l} = ZQ_{p,l}. \qquad \square$$

We do not know how to apply Proposition 24 to generate explicit free anti-tori. Therefore, we pose the following problems:

PROBLEM 25. (1) Construct a pair $x \in \langle X_p \rangle_{Q_{p,l}}$, $y \in \langle X_l \rangle_{Q_{p,l}}$ such that $\langle x, y \rangle_{Q_{p,l}} \cong F_2$.

(2) Construct a pair $x, y \in \mathbb{H}(\mathbb{Z})$ such that $|x|^2 = p^r$, $|y|^2 = l^s$ for some $r, s \in \mathbb{N}$ and $\langle x, y \rangle_{U(\mathbb{H}(\mathbb{Q}))} \cong F_2$.

Nevertheless, we can apply Proposition 24 in the other direction to show that certain 2-generator groups of quaternions are *not* free. We first give a general lemma:

LEMMA 26. *Let* $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n | R_{m\cdot n} \rangle$ *be a* $(2m, 2n)$*-group and let* $a \in \langle a_1, \ldots, a_m \rangle_\Gamma$, $b \in \langle b_1, \ldots, b_n \rangle_\Gamma$ *be two elements. If the subgroup* $\langle a, b \rangle_\Gamma$ *has finite index in* Γ *then* $\langle a, b \rangle_\Gamma \ncong F_2$.

Proof. By [18], finitely generated, torsion-free, virtually free groups are free, but Γ is clearly not free. □

This gives an explicit application of Proposition 24:

PROPOSITION 27. *Let* $x = 1 + 2i$ *and* $y = 1 + 4k$. *Then* $\langle x, y \rangle_{U(\mathbb{H}(\mathbb{Q}))} \ncong F_2$.

Proof. Let $\Gamma = \Gamma_{5,17}$, $a = \psi_{5,17}(x)$ and $b = \psi_{5,17}(y)$. Using GAP [8], we check that $\langle a, b \rangle_\Gamma$ has index 32 in Γ. By Lemma 26, we have $\langle a, b \rangle_\Gamma \ncong F_2$, and Proposition 24 implies $\langle x, y \rangle_{Q_{5,17}} \ncong F_2$. In fact, using the GAP-command PresentationSubgroupMtc, we have found for example the relation

$$
\begin{aligned}
&x^3y^2xy^{-1}x^2y^{-1}x^2y^{-1}x^{-4}y^{-2}x^{-1}yx^{-2}y^{-1}x^{-8}y^{-1}xy^2\\
&xy^{-1}x^{-2}yx^{-1}y^{-2}x^{-2}y^{-2}x^3yx^{-2}y^2x^2y^2xy^{-1}x^2yx^{-1}y^{-2}\\
&x^{-1}yx^8yx^2y^{-1}xy^2x^4yx^{-2}yx^{-2}yx^{-1}y^{-2}x^{-5}y^{-1}x = 1
\end{aligned}
$$

of length 106 in $U(\mathbb{H}(\mathbb{Q}))$. We do not know if there is a shorter relation. □

We give another example:

EXAMPLE 28. Let $\psi = \psi_{3,5}$ and $\Gamma = \Gamma_{3,5} = \langle a_1, a_2, b_1, b_2, b_3 | a_1b_1a_2b_2, a_1b_2a_2b_1^{-1}, a_1b_3a_2^{-1}b_1, a_1b_3^{-1}a_1b_2^{-1}, a_1b_1^{-1}a_2^{-1}b_3, a_2b_3a_2b_2^{-1} \rangle$, where

$$
\begin{aligned}
a_1 &= \psi(1 + j + k), & a_1^{-1} &= \psi(1 - j - k),\\
a_2 &= \psi(1 + j - k), & a_2^{-1} &= \psi(1 - j + k),\\
b_1 &= \psi(1 + 2i), & b_1^{-1} &= \psi(1 - 2i),\\
b_2 &= \psi(1 + 2j), & b_2^{-1} &= \psi(1 - 2j),\\
b_3 &= \psi(1 + 2k), & b_3^{-1} &= \psi(1 - 2k).
\end{aligned}
$$

Then $\langle a_1, b_1\rangle_\Gamma$ has index 4 in Γ and $\langle a_1^2, b_1^2\rangle_\Gamma$ has index 896 in Γ, in particular $\langle 1+j+k, 1+2i\rangle_{U(\mathbb{H}(\mathbb{Q}))} \not\cong F_2$ and

$$\langle (1+j+k)^2, (1+2i)^2\rangle_{U(\mathbb{H}(\mathbb{Q}))} = \langle -1+2j+2k, -3+4i\rangle_{U(\mathbb{H}(\mathbb{Q}))} \not\cong F_2.$$

There is for example the relation $yx^3y^2xy^{-1}x^{-3}y^{-2}x^{-1}=1$, where $x=1+j+k$ and $y=1+2i$.

We do not know what happens for increasing powers of a_1 and b_1:

QUESTION 29. Let $\Gamma=\Gamma_{3,5}, a_1=\psi_{3,5}(1+j+k)$ and $b_1=\psi_{3,5}(1+2i)$.

(1) Is the index of $\langle a_1^3, b_1^3\rangle_\Gamma$ infinite in Γ?
(2) Is $\langle a_1^3, b_1^3\rangle_\Gamma$ a free anti-torus in Γ? Equivalently, is

$$\langle (1+j+k)^3, (1+2i)^3\rangle_{U(\mathbb{H}(\mathbb{Q}))} = \langle -5+j+k, -11-2i\rangle_{U(\mathbb{H}(\mathbb{Q}))} \cong F_2?$$

There is a more general question of Wise:

QUESTION 30 ([1, Question 2.7]). Let G act properly discontinuously and cocompactly on a CAT(0) space (or let G be automatic). Consider two elements a, b of G. Does there exist $n>0$ such that either the subgroup $\langle a^n, b^n\rangle_G$ is free or $\langle a^n, b^n\rangle_G$ is Abelian?

Observe that if $\langle a, b\rangle_G$ is an anti-torus, then $\langle a^n, b^n\rangle_G$ is never Abelian, and therefore Wise's question in this context is whether there exists a number $n>0$ such that $\langle a^n, b^n\rangle_G$ is a free anti-torus.

6. Free Subgroups of $\mathrm{SO}_3(\mathbb{Q})$

The construction of free subgroups of $\mathrm{SO}_3(\mathbb{R})$ has been studied for example in the context of the Banach–Tarski paradox (see e.g. [19]). We relate free subgroups of $\mathrm{SO}_3(\mathbb{Q})$ (hence of $\mathrm{SO}_3(\mathbb{R})$) to free subgroups of $\Gamma_{p,l}$ and to certain free subgroups of $U(\mathbb{H}(\mathbb{Q}))$.

Define $\vartheta\colon U(\mathbb{H}(\mathbb{Q}))\to \mathrm{SO}_3(\mathbb{Q})$ by mapping $x=x_0+x_1i+x_2j+x_3k\in U(\mathbb{H}(\mathbb{Q}))$ to the (3×3)-matrix

$$\frac{1}{|x|^2}\begin{pmatrix} x_0^2+x_1^2-x_2^2-x_3^2 & 2(x_1x_2-x_0x_3) & 2(x_1x_3+x_0x_2) \\ 2(x_1x_2+x_0x_3) & x_0^2-x_1^2+x_2^2-x_3^2 & 2(x_2x_3-x_0x_1) \\ 2(x_1x_3-x_0x_2) & 2(x_2x_3+x_0x_1) & x_0^2-x_1^2-x_2^2+x_3^2 \end{pmatrix}.$$

Note that this is the matrix which represents the $\mathbb{Q}$-linear map $\mathbb{Q}^3\to\mathbb{Q}^3$, $y\mapsto xyx^{-1}$ with respect to the standard basis of $\mathbb{Q}^3$, where the vector $y=(y_1,y_2,y_3)^T\in\mathbb{Q}^3$ is identified with the 'purely imaginary' quaternion $y_1i+y_2j+y_3k\in\mathbb{H}(\mathbb{Q})$. It is well-known that ϑ is a surjective homomorphism of groups. Even the restricted map

$$\vartheta|_{\mathbb{H}(\mathbb{Z})\setminus\{0\}}: \mathbb{H}(\mathbb{Z})\setminus\{0\} \to \mathrm{SO}_3(\mathbb{Q})$$

is surjective, since $\vartheta(\lambda x) = \vartheta(x)$, if $\lambda \in U(\mathbb{Q})$ and $x \in U(\mathbb{H}(\mathbb{Q}))$. See [10] for an elementary proof of the surjectivity of $\vartheta|_{\mathbb{H}(\mathbb{Z})\setminus\{0\}}$. Moreover, it is easy to check by solving a system of equations that

$$\ker(\vartheta) = \{x \in U(\mathbb{H}(\mathbb{Q})) : x = \bar{x}\} = ZU(\mathbb{H}(\mathbb{Q})).$$

Alternatively, seeing $\vartheta(x)$ as $\mathbb{Q}$-linear map $y \mapsto xyx^{-1}$ as described above, we can easily determine the kernel of ϑ as follows:

$$\begin{aligned}\ker(\vartheta) &= \{x \in U(\mathbb{H}(\mathbb{Q})) : xyx^{-1} = y, \forall y \in \mathbb{H}(\mathbb{Q}) \text{ such that } \Re(y) = 0\} \\ &= \{x \in U(\mathbb{H}(\mathbb{Q})) : xy = yx, \forall y \in \mathbb{H}(\mathbb{Q}) \text{ such that } \Re(y) = 0\} \\ &= \{x \in U(\mathbb{H}(\mathbb{Q})) : x = \bar{x}\} \cong U(\mathbb{Q}).\end{aligned}$$

Observe that if $x \in U(\mathbb{H}(\mathbb{Q}))\setminus ZU(\mathbb{H}(\mathbb{Q}))$, then the axis of the rotation $\vartheta(x)$ is the line $(x_1, x_2, x_3)^T \cdot \mathbb{Q}$, and the rotation angle ω satisfies

$$\cos\omega = \frac{x_0^2 - x_1^2 - x_2^2 - x_3^2}{|x|^2},$$

or equivalently

$$\cos\frac{\omega}{2} = \frac{x_0}{\sqrt{|x|^2}}.$$

Now, we realize $\Gamma_{p,l}$ as a subgroup of $\mathrm{SO}_3(\mathbb{Q})$, using the homomorphism ϑ:

PROPOSITION 31. *If $\gamma \in \Gamma_{p,l}$, let $x \in Q_{p,l}$ be any quaternion such that $\psi_{p,l}(x) = \gamma$, and define $\eta_{p,l}(\gamma) := \vartheta(x)$. Then $\eta_{p,l} : \Gamma_{p,l} \to \mathrm{SO}_3(\mathbb{Q})$ is an injective homomorphism of groups.*

Proof. We first show that $\eta_{p,l}$ is well-defined, i.e. it does not depend on the choice of $x \in Q_{p,l}$. Let $x, y \in Q_{p,l}$ such that $\psi_{p,l}(x) = \psi_{p,l}(y) = \gamma$. Then $y = \lambda x$ for some $\lambda \in U(\mathbb{Q})$, hence $\vartheta(y) = \vartheta(x)$.

Now we prove that $\eta_{p,l}$ is a homomorphism. Let $\gamma_1, \gamma_2 \in \Gamma_{p,l}$ and $x, y \in Q_{p,l}$ such that $\psi_{p,l}(x) = \gamma_1, \psi_{p,l}(y) = \gamma_2$. Then $\psi_{p,l}(xy) = \psi_{p,l}(x)\psi_{p,l}(y) = \gamma_1\gamma_2$ and $\eta_{p,l}(\gamma_1\gamma_2) = \vartheta(xy) = \vartheta(x)\vartheta(y) = \eta_{p,l}(\gamma_1)\eta_{p,l}(\gamma_2)$.

Finally, we show that $\eta_{p,l}$ is injective. Let $\gamma \in \Gamma_{p,l}$ such that $\eta_{p,l}(\gamma) = 1_{\mathrm{SO}_3(\mathbb{Q})}$. Then $\vartheta(x) = 1_{\mathrm{SO}_3(\mathbb{Q})}$, where $x \in Q_{p,l}$ such that $\psi_{p,l}(x) = \gamma$. It follows that $x \in U(\mathbb{Q})$, hence $\gamma = \psi_{p,l}(x) = 1_{\Gamma_{p,l}}$. □

We therefore have a commutative diagram

$$\begin{tikzcd}
U(\mathbb{Q}) & \{\pm p^r l^s : r, s \in \mathbb{Z}\} \arrow[l, hook'] \arrow[d, hook] & & \\
U(\mathbb{H}(\mathbb{Q})) \arrow[from=u, hook] \arrow[d, "\psi_{p,l}"'] & Q_{p,l} \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\psi_{p,l}|_{Q_{p,l}}"'] \arrow[dr, "\vartheta|_{Q_{p,l}}"] & U(\mathbb{H}(\mathbb{Q})) \arrow[d, "\vartheta"] & \\
\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l) & \Gamma_{p,l} \arrow[l, hook'] \arrow[r, hook, "\eta_{p,l}"'] & \mathrm{SO}_3(\mathbb{Q}) \arrow[r, hook] & \mathrm{SO}_3(\mathbb{R})
\end{tikzcd}$$

Free subgroups of $Q_{p,l}, \Gamma_{p,l}$ and $\mathrm{SO}_3(\mathbb{Q})$ are related as follows:

PROPOSITION 32. *Let $x^{(1)}, \ldots, x^{(t)}$ be $t \geqslant 2$ quaternions in $Q_{p,l}$. Then the following three statements are equivalent*

(1) $\langle x^{(1)}, \ldots, x^{(t)} \rangle_{Q_{p,l}} \cong F_t$,

(2) $\langle \psi_{p,l}(x^{(1)}), \ldots, \psi_{p,l}(x^{(t)}) \rangle_{\Gamma_{p,l}} \cong F_t$,

(3) $\langle \vartheta(x^{(1)}), \ldots, \vartheta(x^{(t)}) \rangle_{\mathrm{SO}_3(\mathbb{Q})} \cong F_t$.

Proof. To show that (1) and (2) are equivalent, we apply Lemma 23 to the homomorphism $\psi_{p,l}|_{Q_{p,l}} \colon Q_{p,l} \twoheadrightarrow \Gamma_{p,l}$, where $\ker(\psi_{p,l}|_{Q_{p,l}}) = ZQ_{p,l}$.

The equivalence between (2) and (3) again follows from Lemma 23, now applied to the homomorphism $\eta_{p,l} \colon \Gamma_{p,l} \to \mathrm{SO}_3(\mathbb{Q})$, using $\eta_{p,l}(\psi_{p,l}(x)) = \vartheta(x)$ and $\ker(\eta_{p,l}) = \{1\} = Z\Gamma_{p,l}$. Note that $Z\Gamma_{p,l} = \{1\}$ holds, since $\Gamma_{p,l}$ is commutative transitive and non-Abelian. In fact, $Z\Gamma = \{1\}$ holds for any $(2m, 2n)$-group Γ such that $m, n \geqslant 2$, as seen in Corollary 2(2).

We know some free subgroups of $\Gamma_{p,l}$ and can therefore apply Proposition 32.

COROLLARY 33. *Let*

$$\Gamma_{p,l} = \langle a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}} \mid R_{\frac{p+1}{2} \cdot \frac{l+1}{2}} \rangle$$

and $x^{(1)}, \ldots, x^{(\frac{p+1}{2})} \in X_p$ *such that* $\psi_{p,l}(x^{(1)}) = a_1, \ldots, \psi_{p,l}(x^{(\frac{p+1}{2})}) = a_{\frac{p+1}{2}}$. *Then*

$$\langle x^{(1)}, \ldots, x^{(\frac{p+1}{2})} \rangle_{Q_{p,l}} \cong F_{\frac{p+1}{2}}$$

and

$$\langle \vartheta(x^{(1)}), \ldots, \vartheta(x^{(\frac{p+1}{2})}) \rangle_{\mathrm{SO}_3(\mathbb{Q})} \cong F_{\frac{p+1}{2}}.$$

Proof. This follows from Proposition 32, using

$$\langle a_1, \ldots, a_{\frac{p+1}{2}} \rangle_{\Gamma_{p,l}} \cong F_{\frac{p+1}{2}}$$

which holds by Corollary 2(1). □

This gives many examples of free groups.

EXAMPLE 34. Taking the group $\Gamma_{3,5}$, Proposition 32 implies that

$$\begin{aligned}
&F_2 \cong \langle 1+j+k, 1+j-k\rangle_{Q_{3,5}},\\
&F_2 \cong \langle \vartheta_{3,5}(1+j+k), \vartheta_{3,5}(1+j-k)\rangle_{\mathrm{SO}_3(\mathbb{Q})}\\
&\quad = \left\langle \frac{1}{3}\begin{pmatrix} -1 & -2 & 2\\ 2 & 1 & 2\\ -2 & 2 & 1\end{pmatrix}, \frac{1}{3}\begin{pmatrix} -1 & 2 & 2\\ -2 & 1 & -2\\ -2 & -2 & 1\end{pmatrix}\right\rangle_{\mathrm{SO}_3(\mathbb{Q})},
\end{aligned}$$

and

$$\begin{aligned}
&F_3 \cong \langle 1+2i, 1+2j, 1+2k\rangle_{Q_{3,5}},\\
&F_3 \cong \langle \vartheta_{3,5}(1+2i), \vartheta_{3,5}(1+2j), \vartheta_{3,5}(1+2k)\rangle_{\mathrm{SO}_3(\mathbb{Q})}\\
&\quad = \left\langle \begin{pmatrix} 1 & 0 & 0\\ 0 & -3/5 & -4/5\\ 0 & 4/5 & -3/5\end{pmatrix}, \begin{pmatrix} -3/5 & 0 & 4/5\\ 0 & 1 & 0\\ -4/5 & 0 & -3/5\end{pmatrix}, \begin{pmatrix} -3/5 & -4/5 & 0\\ 4/5 & -3/5 & 0\\ 0 & 0 & 1\end{pmatrix}\right\rangle_{\mathrm{SO}_3(\mathbb{Q})}.
\end{aligned}$$

On the other hand, we also get examples of nonfree groups:

EXAMPLE 35. Using Propositions 32 and 27, we see that

$$F_2 \ncong \langle 1+2i, 1+4k\rangle_{Q_{5,17}}$$

and

$$\begin{aligned}
&F_2 \ncong \langle \vartheta_{5,17}(1+2i), \vartheta_{5,17}(1+4k)\rangle_{\mathrm{SO}_3(\mathbb{Q})}\\
&\quad = \left\langle \begin{pmatrix} 1 & 0 & 0\\ 0 & -3/5 & -4/5\\ 0 & 4/5 & -3/5\end{pmatrix}, \begin{pmatrix} -15/17 & -8/17 & 0\\ 8/17 & -15/17 & 0\\ 0 & 0 & 1\end{pmatrix}\right\rangle_{\mathrm{SO}_3(\mathbb{Q})}.
\end{aligned}$$

In fact, the long relation in $x^{\pm 1}, y^{\pm 1}$ given in the proof of Proposition 27 also holds in $\mathrm{SO}_3(\mathbb{Q})$ for the matrices $x = \vartheta_{5,17}(1+2i)$, $y = \vartheta_{5,17}(1+4k)$.

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