

Localization and colocalization in derived categories

Tsutomu Nakamura

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OKAYAMA UNIVERSITY

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Preface

In the 1960s, Grothendieck introduced the notion of local cohomology from the viewpoint of geometry. It has brought great benefits to commutative algebra, and has been an essential tool in such a field. Moreover, in the 1990s, Greenlees and May proved that the left derived functor of an ideal-adic completion functor is a right adjoint to a local cohomology functor. The left derived functor is called the local homology functor. In this thesis, we consider local (co)homology in terms of Bousfield (co)localization.

Bousfield localization, which is a natural generalization of classical localizations of commutative rings, appeared in the study of stable homotopy categories by Bousfield in the 1970s. We may simply explain a (Bousfield) localization functor on a triangulated category as follows; an idempotent triangulated functor endowed with some appropriate morphism from the identity functor. However, we need Brown representation theorem, to construct localization functors generally. Hence it is difficult to know concrete forms of them, except for special cases.

A key of this thesis is the classification of localizing subcategories of the derived category of a commutative Noetherian ring; it was given by Neeman in the 1990s. By this result, we can associate localization and colocalization functors with subsets of the spectrum of the ring, using the notion of support and cosupport. If a localization functor has a closed cosupport, then it coincides with a local homology functor. Moreover, if a colocalization functor has a closed support, then it coincides with a local cohomology functor. In this thesis, motivated by these facts, we establish various results about localization (resp. colocalization) functors with cosupport (resp. support) in general subsets of the spectrum.

The organization of this thesis is as follows; it consists of four chapters.

In Chapter 1, we study colocalization functors. We first prove that if a colocalization has a 0-dimensional support, then the colocalization is a direct sum of composition of known functors. This fact gives a new class of colocalization functors which are right derived functors of additive functors on the category of modules. After that, we give a method to compute

any colocalization functor inductively with respect to the dimension of its support. Finally we extend local duality theorem and Grothendieck type vanishing theorem of local cohomology to colocalization functors.

In Chapter 2, we establish several results about localization functors. As with the case of colocalization, we first prove that if a localization has a 0-dimensional cosupport, then the localization is a direct product of composition of known functors. Next, we generalize Mayer-Vietoris triangles of local (co)homology to (co)localization functors. Using such triangles, we obtain a simpler proof of a classical theorem by Gruson and Raynaud. The theorem states that the projective dimension of a flat module is at most the Krull dimension of the base ring. Moreover, we give an explicit way to compute localization functors via Čech complexes. As a result, it is possible to describe all localization functors concretely. In addition, this way yields a functorial method to construct pure-injective resolutions for complexes of flat modules and complexes of finitely generated modules.

In Chapter 3, we treat some problem about cosupport. We prove that a polynomial ring over a field has full-cosupport, that is, the cosupport of the ring is equal to its spectrum. This fact was expected by several researchers from a few years ago, but was not known, except for the case that variables are less than three, or the base field is countable. As a corollary, it follows that every affine ring over a field has full-cosupport. Using this result, we give a complete description of a minimal pure-injective resolution of an affine ring, provided the cardinality of the base field is \aleph_1 . Furthermore, we give a partial answer to a conjecture by Gruson.

In Chapter 4, we treat generalized local cohomology, which was introduced by Herzog in the 1970s. We extend some result by Saremi and Mafi (2013), and simplify their proof. Although this generalization of local cohomology is different from the one we consider in Chapter 1, we also observe there is some connection between generalized local cohomology and colocalization.

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1. Local duality principle in derived categories

1.1 Introduction

This chapter is based on the author's paper [33] with Yuji Yoshino. Let R be a commutative Noetherian ring. We denote by $\mathcal{D} = D(\text{Mod } R)$ the derived category of complexes of R -modules, by which we mean that \mathcal{D} is the unbounded derived category. Neeman [36] proved that there is a natural one-one correspondence between the set of subsets of $\text{Spec } R$ and the set of localizing subcategories of \mathcal{D} . We denote by \mathcal{L}_W the localizing subcategory corresponding to a subset W of $\text{Spec } R$. The localization theory of triangulated categories [27] yields a right adjoint γ_W to the inclusion functor $\mathcal{L}_W \hookrightarrow \mathcal{D}$, and such an adjoint is unique. This functor $\gamma_W : \mathcal{D} \rightarrow \mathcal{L}_W (\hookrightarrow \mathcal{D})$ is our main target of this chapter, and we call it *the colocalization functor with support in W* .

If V is a specialization-closed subset of $\text{Spec } R$, then γ_V is nothing but the right derived functor $\text{R}\Gamma_V$ of the section functor Γ_V with support in V , whose i th right derived functor $H_V^i(-) = H^i(\text{R}\Gamma_V(-))$ is known as the i th local cohomology functor. For a general subset W of $\text{Spec } R$, the colocalization functor γ_W is not necessarily a right derived functor of an additive functor defined on the category $\text{Mod } R$ of R -modules.

In this chapter, we establish several results concerning the colocalization functor γ_W , where W is an arbitrary subset of $\text{Spec } R$. Notable are extensions of the local duality theorem and Grothendieck type vanishing theorem of local cohomology. The local duality can be viewed as an isomorphism

$$\text{R}\Gamma_V \text{RHom}_R(X, Y) \cong \text{RHom}_R(X, \text{R}\Gamma_V Y),$$

where V is a specialization-closed subset of $\text{Spec } R$, $X \in \mathcal{D}_{\text{fg}}^-$ and $Y \in \mathcal{D}^+$; see [16, Proposition 6.1]. The following theorem generalizes this isomorphism to the case of colocalization functors γ_W .

Theorem 1.1.1 (Theorem 1.4.1). *Let W be a subset of $\text{Spec } R$ and let $X, Y \in \mathcal{D}$. We denote by $\dim W$ the supremum of the lengths of chains of prime ideals in W . Suppose that one of the following conditions holds:*

- (1) $X \in \mathcal{D}_{\text{fg}}^-, Y \in \mathcal{D}^+$ and $\dim W$ is finite;
- (2) $X \in \mathcal{D}_{\text{fg}}, Y$ is a bounded complex of injective R -modules and $\dim W$ is finite;
- (3) W is generalization-closed.

Then there exists a natural isomorphism

$$\gamma_W \text{RHom}_R(X, Y) \cong \text{RHom}_R(X, \gamma_W Y).$$

We shall call Theorem 1.1.1 the *Local Duality Principle*, which naturally implies the following corollary.

Corollary 1.1.2 (Corollary 1.4.5). *Assume that R admits a dualizing complex D_R . Let W be an arbitrary subset of $\text{Spec } R$ and $X \in \mathcal{D}_{\text{fg}}$. We write $X^\dagger = \text{RHom}_R(X, D_R)$. Then we have a natural isomorphism*

$$\gamma_W X \cong \text{RHom}_R(X^\dagger, \gamma_W D_R).$$

The local duality theorem states the validity of this isomorphism in the case that W is specialization-closed, see [22, Chapter V; Theorem 6.2] and [16, Corollary 6.2].

As an application of the Local Duality Principle, we can prove the vanishing theorem of Grothendieck type for the colocalization functor γ_W with support in an arbitrary subset W . Let \mathfrak{a} be an ideal of R and $X \in \mathcal{D}$. The \mathfrak{a} -depth of X , which we denote by $\text{depth}(\mathfrak{a}, X)$, is the infimum of the set $\{i \in \mathbb{Z} \mid \text{Ext}_R^i(R/\mathfrak{a}, X) \neq 0\}$. More generally, for a specialization-closed subset W , the W -depth of X , which we denote by $\text{depth}(W, X)$, is defined as the infimum of the set of values $\text{depth}(\mathfrak{a}, X)$ for all ideals \mathfrak{a} with $V(\mathfrak{a}) \subseteq W$. When $X \in \mathcal{D}_{\text{fg}}$, we denote by $\dim X$ the supremum of the set $\{\dim H^i(X) + i \mid i \in \mathbb{Z}\}$.

For a finitely generated R -module M , the Grothendieck vanishing theorem says that the i th local cohomology module $H_W^i(M) = H^i(\text{R}\Gamma_W M)$ of M with support in W is zero for $i < \text{depth}(W, M)$ and $i > \dim M$.

We are able to generalize this theorem to the following result in §6.

Theorem 1.1.3 (Theorem 1.6.5). *Assume that R admits a dualizing complex. Let W be an arbitrary subset of $\text{Spec } R$ with the specialization closure \overline{W}^s . If $X \in \mathcal{D}_{\text{fg}}$, then $H^i(\gamma_W X) = 0$ unless $\text{depth}(\overline{W}^s, X) \leq i \leq \dim X$.*

In §3, we give an explicit description of γ_W for subsets W of certain special type, see Theorem 1.3.12. For example, if W is a one-point set $\{\mathfrak{p}\}$, then it is proved the colocalization functor $\gamma_{\{\mathfrak{p}\}}$ equals $R\Gamma_{V(\mathfrak{p})} \mathrm{RHom}_R(R_{\mathfrak{p}}, -)$, see Corollary 1.3.3. This is one of the rare cases that we know the explicit form of γ_W , while for a general subset W we give in Theorem 4.2.4 the way how we calculate γ_W by the induction on $\dim W$.

In §4, we give a complete proof of the Local Duality Principle (Theorem 1.1.1).

The subsequent section §5 is devoted to the relationship between γ_W and left derived functors of completion functors. In particular, we see that there is a subset W such that $H^i(\gamma_W I) \neq 0$ for an injective module I and some $i < 0$. This observation shows that γ_W is not a right derived functor of an additive functor defined on $\mathrm{Mod} R$ in general.

In the last section §6, we present a precise and complete proof for Theorem 1.1.3 above.

1.2 Colocalization functors

In this section, we summarize some notions and basic facts used later in this chapter. As in the introduction, R denotes a commutative Noetherian ring and we work in the derived category $\mathcal{D} = D(\mathrm{Mod} R)$. Note that complexes X are cohomologically indexed;

$$X = (\cdots \rightarrow X^{i-1} \rightarrow X^i \rightarrow X^{i+1} \rightarrow \cdots).$$

We denote by \mathcal{D}^+ (resp. \mathcal{D}^-) the full subcategory of \mathcal{D} consisting of complexes X such that $H^i(X) = 0$ for $i \ll 0$ (resp. $i \gg 0$). We write $\mathcal{D}_{\mathrm{fg}}$ for the full subcategory of \mathcal{D} consisting of complexes with finitely generated cohomology modules. Furthermore we write $\mathcal{D}_{\mathrm{fg}}^- = \mathcal{D}^- \cap \mathcal{D}_{\mathrm{fg}}$.

For a complex X in \mathcal{D} , the (small) support of X is a subset of $\mathrm{Spec} R$ defined as

$$\mathrm{supp} X = \{ \mathfrak{p} \in \mathrm{Spec} R \mid X \otimes_R^{\mathbb{L}} \kappa(\mathfrak{p}) \neq 0 \},$$

where $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. It is well-known that for $X \in \mathcal{D}$, $\mathrm{supp} X \neq \emptyset$ if and only if $X \neq 0$, see [16, Lemma 2.6] or [36, Lemma 2.12]. In order to compare with the ordinary support, recall that the (big) support $\mathrm{Supp} X$ is the set of primes \mathfrak{p} of R satisfying $X_{\mathfrak{p}} \neq 0$ in \mathcal{D} . In general, we have $\mathrm{supp} X \subseteq \mathrm{Supp} X$ and equality holds if $X \in \mathcal{D}_{\mathrm{fg}}^-$, see [16, p. 158].

A full subcategory \mathcal{L} of \mathcal{D} is said to be localizing if \mathcal{L} is triangulated and closed under arbitrary direct sums. If we are given a subset W of $\mathrm{Spec} R$, then, since the tensor product commutes with taking direct sums, it is easy to

see that the full subcategory $\mathcal{L}_W = \{ X \in \mathcal{D} \mid \text{supp } X \subseteq W \}$ is localizing. A theorem of Neeman [36, Theorem 2.8] ensures that any localizing subcategory of \mathcal{D} is obtained in this way from a subset W of $\text{Spec } R$.

If A is a set of objects in \mathcal{D} , then $\text{Loc } A$ denotes the smallest localizing subcategory of \mathcal{D} containing all objects of A . We write $E_R(R/\mathfrak{p})$ for the injective envelope of the R -module R/\mathfrak{p} for $\mathfrak{p} \in \text{Spec } R$. It is easy to see $\text{supp } \kappa(\mathfrak{p}) = \text{supp } E_R(R/\mathfrak{p}) = \{\mathfrak{p}\}$. Moreover, Neeman [36, Theorem 2.8] shows the equalities

$$\mathcal{L}_W = \text{Loc} \{ \kappa(\mathfrak{p}) \mid \mathfrak{p} \in W \} = \text{Loc} \{ E_R(R/\mathfrak{p}) \mid \mathfrak{p} \in W \}.$$

For a localizing subcategory \mathcal{L} of \mathcal{D} , its right orthogonal subcategory is defined as

$$\mathcal{L}^\perp = \{ Y \in \mathcal{D} \mid \text{Hom}_{\mathcal{D}}(X, Y) = 0 \text{ for all } X \in \mathcal{L} \}.$$

Note that \mathcal{L}^\perp is a triangulated subcategory of \mathcal{D} that is closed under arbitrary direct products. In other words, \mathcal{L}^\perp is a colocalizing subcategory of \mathcal{D} . The following equalities hold for any subset W of $\text{Spec } R$;

$$\begin{aligned} \mathcal{L}_W^\perp &= \{ Y \in \mathcal{D} \mid \text{Hom}_{\mathcal{D}}(\kappa(\mathfrak{p})[i], Y) = 0 \text{ for all } \mathfrak{p} \in W \text{ and } i \in \mathbb{Z} \} \\ &= \{ Y \in \mathcal{D} \mid \text{RHom}_R(\kappa(\mathfrak{p}), Y) = 0 \text{ for all } \mathfrak{p} \in W \}. \end{aligned}$$

Let W be an arbitrary subset of $\text{Spec } R$. We denote by i_W (resp. j_W) the natural inclusion functor $\mathcal{L}_W \hookrightarrow \mathcal{D}$ (resp. $\mathcal{L}_W^\perp \hookrightarrow \mathcal{D}$). Then there exist a couple of adjoint pairs (i_W, γ_W) and (λ_W, j_W) as it is indicated in the following diagram:

$$\mathcal{L}_W \begin{array}{c} \xrightarrow{i_W} \\ \xleftarrow{\gamma_W} \end{array} \mathcal{D} \begin{array}{c} \xrightarrow{\lambda_W} \\ \xleftarrow{j_W} \end{array} \mathcal{L}_W^\perp$$

Moreover, it holds that $\text{Ker } \gamma_W = \mathcal{L}_W^\perp$ and $\text{Ker } \lambda_W = \mathcal{L}_W$. For details, see [27, §4.9, §5.1, §7.2]. See also Remark 1.3.14 (ii).

In the following lemma, we identify γ_W and λ_W with $i_W \cdot \gamma_W$ and $j_W \cdot \lambda_W$ respectively.

Lemma 1.2.1. *Let W be any subset of $\text{Spec } R$. For any object X of \mathcal{D} , there is a triangle*

$$\gamma_W X \longrightarrow X \longrightarrow \lambda_W X \longrightarrow \gamma_W X[1],$$

where $\gamma_W X \rightarrow X$ and $X \rightarrow \lambda_W X$ are the natural morphisms. Furthermore, if

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow X'[1]$$

is a triangle with $X' \in \mathcal{L}_W$ and $X'' \in \mathcal{L}_W^\perp$, then there exist unique isomorphisms $a : \gamma_W X \rightarrow X'$ and $b : \lambda_W X \rightarrow X''$ such that the following diagram is commutative:

$$\begin{array}{ccccccc} \gamma_W X & \longrightarrow & X & \longrightarrow & \lambda_W X & \longrightarrow & \gamma_W X[1] \\ \downarrow a & & \parallel & & \downarrow b & & \downarrow a[1] \\ X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & X'[1] \end{array}$$

See [27, §4.11] for the proof of this lemma.

Let W be a subset of $\text{Spec } R$. Then λ_W is a localization functor. In other words, writing $\eta : \text{id}_{\mathcal{D}} \rightarrow \lambda_W$ for the natural morphism, it follows that $\lambda_W \eta : \lambda_W \rightarrow \lambda_W^2$ is invertible and $\lambda_W \eta = \eta \lambda_W$. Conversely, γ_W is a colocalization functor, that is, for the natural morphism $\varepsilon : \gamma_W \rightarrow \text{id}_{\mathcal{D}}$, it follows that $\gamma_W \varepsilon : \gamma_W^2 \rightarrow \gamma_W$ is invertible and $\gamma_W \varepsilon = \varepsilon \gamma_W$. Note that we uniquely obtain the localization (resp. colocalization) functor λ_W (resp. γ_W) for a subset W of $\text{Spec } R$.

Definition 1.2.2. Let W be a subset of $\text{Spec } R$. We call γ_W the colocalization functor with support in W .

Recall that a subset W of $\text{Spec } R$ is called specialization-closed (resp. generalization-closed) if the following condition holds:

(*) Let $\mathfrak{p}, \mathfrak{q} \in \text{Spec } R$. If $\mathfrak{p} \in W$ and $\mathfrak{p} \subseteq \mathfrak{q}$ (resp. $\mathfrak{p} \supseteq \mathfrak{q}$), then \mathfrak{q} belongs to W .

If V is a specialization-closed subset, then the colocalization functor γ_V coincides with the right derived functor $\text{R}\Gamma_V$ of the section functor Γ_V with support in V , see [28, Appendix 3.5].

1.3 Auxiliary results on colocalization functors

Let W be a subset of $\text{Spec } R$ and let γ_W be the colocalization functor with support in W . In general, it is hard to describe the functor γ_W explicitly. However there are some cases in which the colocalization functor γ_W is the form of composition of known functors.

Let S be a multiplicatively closed subset of R . We denote by U_S the generalization-closed subset $\{ \mathfrak{q} \in \text{Spec } R \mid \mathfrak{q} \cap S = \emptyset \}$. Note that U_S is naturally identified with $\text{Spec } S^{-1}R$. We also write $U(\mathfrak{p}) = \{ \mathfrak{q} \in \text{Spec } R \mid \mathfrak{q} \subseteq \mathfrak{p} \}$ for a prime ideal \mathfrak{p} of R (cf. [3]). Then, setting $S = R \setminus \mathfrak{p}$, we have $U(\mathfrak{p}) = U_S$.

Proposition 1.3.1. *Let S be a multiplicatively closed subset of R and V be a specialization-closed subset of $\text{Spec } R$. We set $W = V \cap U_S$. Then we have an isomorphism*

$$\gamma_W \cong \text{R}\Gamma_V \text{RHom}_R(S^{-1}R, -).$$

Proof. The ring homomorphism $R \rightarrow S^{-1}R$ induces a morphism

$$\text{RHom}_R(S^{-1}R, X) \rightarrow X$$

for $X \in \mathcal{D}$. Write $f : \text{R}\Gamma_V \text{RHom}_R(S^{-1}R, X) \rightarrow X$ for the composition of the natural morphism $\text{R}\Gamma_V \text{RHom}_R(S^{-1}R, X) \rightarrow \text{RHom}_R(S^{-1}R, X)$ with this morphism, and we consider the triangle

$$\text{R}\Gamma_V \text{RHom}_R(S^{-1}R, X) \xrightarrow{f} X \longrightarrow C \longrightarrow \text{R}\Gamma_V \text{RHom}_R(S^{-1}R, X)[1].$$

Since the complex $\text{R}\Gamma_V \text{RHom}_R(S^{-1}R, X)$ can be regarded as a complex of $S^{-1}R$ -modules, it follows that $\text{supp } \text{R}\Gamma_V \text{RHom}_R(S^{-1}R, X) \subseteq U_S$. At the same time, it follows from the definition that $\text{supp } \text{R}\Gamma_V \text{RHom}_R(S^{-1}R, X) \subseteq V$. Therefore $\text{supp } \text{R}\Gamma_V \text{RHom}_R(S^{-1}R, X)$ must be contained in W .

On the other hand, if $\mathfrak{p} \in W$, then there are isomorphisms

$$\begin{aligned} \text{RHom}_R(\kappa(\mathfrak{p}), \text{R}\Gamma_V \text{RHom}_R(S^{-1}R, X)) &\cong \text{RHom}_R(\kappa(\mathfrak{p}), \text{RHom}_R(S^{-1}R, X)) \\ &\cong \text{RHom}_R(\kappa(\mathfrak{p}), X). \end{aligned}$$

Hence it holds that $\text{RHom}_R(\kappa(\mathfrak{p}), f)$ is an isomorphism. Thus we have $\text{RHom}_R(\kappa(\mathfrak{p}), C) = 0$.

Since we have shown that $\text{R}\Gamma_V \text{RHom}_R(S^{-1}R, X) \in \mathcal{L}_W$ and $C \in \mathcal{L}_W^\perp$, we can use Lemma 1.2.1 to deduce $\gamma_W X \cong \text{R}\Gamma_V \text{RHom}_R(S^{-1}R, X)$. \square

In the following lemma we show that the colocalization functor considered in Proposition 1.3.1 is a right derived functor of a left exact functor defined on $\text{Mod } R$. We say that a complex I of R -modules is K -injective if $\text{Hom}_R(-, I)$ preserves quasi-isomorphisms.

Lemma 1.3.2. *Let S, V and W be the same as in Proposition 1.3.1. Then the colocalization functor γ_W is the right derived functor of the functor*

$$\Gamma_V \text{Hom}_R(S^{-1}R, -)$$

defined on $\text{Mod } R$.

Proof. Let $X \in \mathcal{D}$ and take a K -injective resolution I of X that consists of injective R -modules. Then $\text{RHom}_R(S^{-1}R, X) \cong \text{Hom}_R(S^{-1}R, I)$, and the right-hand complex consists of injective R -modules, too. It is known by [28,

Lemma 3.5.1] that for any complex J of injective R -modules (that is not necessarily K -injective), $R\Gamma_V J$ is naturally isomorphic to $\Gamma_V J$. Therefore we have

$$R\Gamma_V \mathrm{RHom}_R(S^{-1}R, X) \cong \Gamma_V \mathrm{Hom}_R(S^{-1}R, I) \cong R(\Gamma_V \mathrm{Hom}_R(S^{-1}R, -))(X).$$

□

Henceforth, for a subset W of $\mathrm{Spec} R$, we write $W^c = \mathrm{Spec} R \setminus W$. By Proposition 1.3.1, we have an isomorphism

$$\gamma_{U_S} \cong \mathrm{RHom}_R(S^{-1}R, -).$$

We should mention that the isomorphism $\gamma_{U(\mathfrak{p})} \cong \mathrm{RHom}_R(R_{\mathfrak{p}}, -)$ already appeared in [4, §4; P175], in which the authors use the notation $V^{Z(\mathfrak{p})}$, where $Z(\mathfrak{p}) = U(\mathfrak{p})^c$, see also Remark 1.5.2.

Corollary 1.3.3. *Let $\mathfrak{p} \in \mathrm{Spec} R$. Then we have an isomorphism*

$$\gamma_{\{\mathfrak{p}\}} \cong R\Gamma_{V(\mathfrak{p})} \mathrm{RHom}_R(R_{\mathfrak{p}}, -),$$

the right-hand side of which is the right derived functor of

$$\Gamma_{V(\mathfrak{p})} \mathrm{Hom}_R(R_{\mathfrak{p}}, -)$$

defined on $\mathrm{Mod} R$.

If I is an injective R -module, then $\gamma_{\{\mathfrak{p}\}} I \cong \Gamma_{V(\mathfrak{p})} \mathrm{Hom}_R(R_{\mathfrak{p}}, I)$ is also an injective R -module by the corollary. We can describe how this injective R -module is decomposed into a sum of indecomposable ones.

Corollary 1.3.4. *Let \mathfrak{p} be a prime ideal of R and I be an injective R -module. Then $\gamma_{\{\mathfrak{p}\}} I$ is isomorphic to the direct sum $\bigoplus_B E_R(R/\mathfrak{p})$ of B -copies of $E_R(R/\mathfrak{p})$, where $B = \dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_R(\kappa(\mathfrak{p}), I)$.*

Proof. Since $\gamma_{\{\mathfrak{p}\}} I$ is an injective R -module with support in $\{\mathfrak{p}\}$, there is a cardinal number B with $\gamma_{\{\mathfrak{p}\}} I \cong \bigoplus_B E_R(R/\mathfrak{p})$. On the other hand, $\mathrm{Hom}_R(\kappa(\mathfrak{p}), I) \cong \mathrm{Hom}_R(\kappa(\mathfrak{p}), \gamma_{\{\mathfrak{p}\}} I) \cong \bigoplus_B \kappa(\mathfrak{p})$. Therefore we have $B = \dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_R(\kappa(\mathfrak{p}), I)$. □

Remark 1.3.5. Let I be an injective R -module such that $\mathrm{supp} I = \{\mathfrak{q}\}$ for $\mathfrak{q} \in \mathrm{Spec} R$, that is, I is of the form $\bigoplus_A E_R(R/\mathfrak{q})$ for some index set A . Then it is easily seen that $\mathrm{Hom}_R(\kappa(\mathfrak{p}), I) \neq 0$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$. Therefore, it follows from Corollary 1.3.4 that $\gamma_{\{\mathfrak{p}\}} I \neq 0$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$.

If \mathfrak{p} is a prime ideal of R which is not maximal, then the colocalization functor $\gamma_{\{\mathfrak{p}\}}$ is distinct from $\mathrm{R}\Gamma_{V(\mathfrak{p})}((-)\otimes_R R_{\mathfrak{p}})$, which is written as $\Gamma_{\mathfrak{p}}$ by Benson, Iyengar and Krause in [3]. In fact, for a prime ideal \mathfrak{q} such that $\mathfrak{p} \subsetneq \mathfrak{q}$, it follows that $\Gamma_{\mathfrak{p}}E_R(R/\mathfrak{q}) = 0$, while $\gamma_{\{\mathfrak{p}\}}E_R(R/\mathfrak{q}) \neq 0$ by Remark 1.3.5.

Definition 1.3.6. For a subset W of $\mathrm{Spec} R$, we denote by $\dim W$ the supremum of the lengths of chains of prime ideals belonging to W , *i.e.*,

$$\dim W = \sup \{ n \mid \text{there are } \mathfrak{p}_0, \dots, \mathfrak{p}_n \text{ in } W \text{ with } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_n \}.$$

Thus $\dim W = 0$ means that two distinct prime ideals taken from W have no inclusion relation. Moreover, if $W = \emptyset$, then $\dim W = -\infty$ by the definition.

We now want to extend Corollary 1.3.3 to the case where $\dim W = 0$. For this purpose we need some preparatory observations. Compare the next remark with [3, Lemma 3.4 (1)].

Remark 1.3.7. (i) Let W_0 and W be subsets of $\mathrm{Spec} R$ with inclusion relation $W_0 \subseteq W$. In this case, we should note that $\mathcal{L}_{W_0} \subseteq \mathcal{L}_W$ and $\mathcal{L}_W^\perp \subseteq \mathcal{L}_{W_0}^\perp$. Then it is clear from the uniqueness of adjoint functors that

$$\gamma_{W_0}\gamma_W \cong \gamma_{W_0} \cong \gamma_W\gamma_{W_0}, \quad \lambda_W\lambda_{W_0} \cong \lambda_W \cong \lambda_{W_0}\lambda_W.$$

(ii) Let W_1 and W_2 be subsets of $\mathrm{Spec} R$. In general, $\gamma_{W_1}\gamma_{W_2}$ need not be isomorphic to $\gamma_{W_2}\gamma_{W_1}$. For example, if $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec} R$ with $\mathfrak{p} \subsetneq \mathfrak{q}$, then it is seen from Corollary 1.3.4 and Remark 1.3.5 that $\gamma_{\{\mathfrak{p}\}}\gamma_{\{\mathfrak{q}\}}E_R(R/\mathfrak{q}) \neq 0$ and $\gamma_{\{\mathfrak{q}\}}\gamma_{\{\mathfrak{p}\}}E_R(R/\mathfrak{q}) = 0$. Similarly, $\lambda_{W_1}\lambda_{W_2}$ need not be isomorphic to $\lambda_{W_2}\lambda_{W_1}$. Moreover, for a general subset W , γ_W does not necessarily commute with the localization $(-)\otimes_R S^{-1}R$ with respect to a multiplicatively closed subset S .

The following lemma will be used in the later sections.

Lemma 1.3.8 (Foxby-Iyengar [17]). *Let (R, \mathfrak{m}, k) be a commutative Noetherian local ring. Then the following conditions are equivalent for any $X \in \mathcal{D}$:*

- (1) $X \otimes_R^L k \neq 0$;
- (2) $\mathrm{RHom}_R(k, X) \neq 0$;
- (3) $\mathrm{R}\Gamma_{V(\mathfrak{m})} X \neq 0$.

Proof. See [17, Theorem 2.1, Theorem 4.1]. □

This is implicitly used by Benson, Iyengar and Krause [3] to prove the following lemma.

Lemma 1.3.9 ([3, Theorem 5.6]). *Let V be a specialization-closed subset of $\text{Spec } R$. Then, for each X in \mathcal{D} , the following equalities hold;*

$$\text{supp } \gamma_V X = V \cap \text{supp } X, \quad \text{supp } \lambda_V X = V^c \cap \text{supp } X.$$

Notice that Lemma 1.3.9 applies only to specialization-closed subsets, and the equalities do not necessarily hold for general subsets, see Corollary 1.3.4 and Remark 1.3.5.

Definition 1.3.10. Let $W_0 \subseteq W$ be subsets of $\text{Spec } R$. We say that W_0 is *specialization-closed in W* if $V(\mathfrak{p}) \cap W \subseteq W_0$ for any $\mathfrak{p} \in W_0$.

Moreover we denote by \overline{W}^s the specialization closure of W , which is defined to be the smallest specialization-closed subset of $\text{Spec } R$ containing W .

Lemma 1.3.11. *Let $W_0 \subseteq W \subseteq \text{Spec } R$ be sets. Suppose W_0 is specialization-closed in W . Setting $W_1 = W \setminus W_0$, we have $\mathcal{L}_{W_1} \subseteq \mathcal{L}_{W_0}^\perp$.*

Proof. It is obvious from the definition that $\overline{W_0}^s \cap W_1 = \emptyset$. Assume that $X \in \mathcal{L}_{W_1}$. Then $\text{supp } \gamma_{\overline{W_0}^s} X = \overline{W_0}^s \cap \text{supp } X = \emptyset$ by Lemma 1.3.9. Therefore we have $\gamma_{\overline{W_0}^s} X = 0$. It then follows from Remark 1.3.7 (i) that $\gamma_{W_0} X \cong \gamma_{W_0} \gamma_{\overline{W_0}^s} X = 0$. Thus we have $X \cong \lambda_{W_0} X \in \mathcal{L}_{W_0}^\perp$ as desired. \square

The following theorem is one of the main results in this section; it extends Corollary 1.3.3.

Theorem 1.3.12. *Let W be a subset of $\text{Spec } R$ with $\dim W = 0$. Then we have the following isomorphisms of functors*

$$\gamma_W \cong \bigoplus_{\mathfrak{p} \in W} \gamma_{\{\mathfrak{p}\}} \cong \bigoplus_{\mathfrak{p} \in W} \text{R}\Gamma_{V(\mathfrak{p})} \text{RHom}_R(R_{\mathfrak{p}}, -).$$

Furthermore, γ_W is the right derived functor of the left exact functor

$$\bigoplus_{\mathfrak{p} \in W} \Gamma_{V(\mathfrak{p})} \text{Hom}_R(R_{\mathfrak{p}}, -)$$

defined on $\text{Mod } R$.

Proof. Let $X \in \mathcal{D}$. Summing up all the natural morphisms $\gamma_{\{\mathfrak{p}\}}X \rightarrow X$ for $\mathfrak{p} \in W$, we obtain a morphism $f : \bigoplus_{\mathfrak{p} \in W} \gamma_{\{\mathfrak{p}\}}X \rightarrow X$, from which we obtain a triangle

$$\bigoplus_{\mathfrak{p} \in W} \gamma_{\{\mathfrak{p}\}}X \xrightarrow{f} X \longrightarrow C \longrightarrow \bigoplus_{\mathfrak{p} \in W} \gamma_{\{\mathfrak{p}\}}X[1].$$

It is clear that $\bigoplus_{\mathfrak{p} \in W} \gamma_{\{\mathfrak{p}\}}X \in \mathcal{L}_W$.

Now let \mathfrak{p} be a prime in W . Since $\{\mathfrak{p}\}$ is specialization-closed in W , it follows from Lemma 1.3.11 that $\bigoplus_{\mathfrak{q} \in W \setminus \{\mathfrak{p}\}} \gamma_{\{\mathfrak{q}\}}X \in \mathcal{L}_{\{\mathfrak{p}\}}^\perp$. Hence we have

$$\mathrm{RHom}_R(\kappa(\mathfrak{p}), \bigoplus_{\mathfrak{q} \in W} \gamma_{\{\mathfrak{q}\}}X) \cong \mathrm{RHom}_R(\kappa(\mathfrak{p}), \gamma_{\{\mathfrak{p}\}}X) \cong \mathrm{RHom}_R(\kappa(\mathfrak{p}), X).$$

This implies that $\mathrm{RHom}_R(\kappa(\mathfrak{p}), f)$ is an isomorphism for $\mathfrak{p} \in W$. Therefore it follows that $\mathrm{RHom}_R(\kappa(\mathfrak{p}), C) = 0$ for all $\mathfrak{p} \in W$, equivalently $C \in \mathcal{L}_W^\perp$. Hence we conclude by Lemma 1.2.1 that $\gamma_W X \cong \bigoplus_{\mathfrak{p} \in W} \gamma_{\{\mathfrak{p}\}}X$ as desired. The rest of the theorem follows from Lemma 1.3.2 and Corollary 1.3.3. \square

We denote by $\mathrm{Inj} R$ the full subcategory of $\mathrm{Mod} R$ consisting of all injective R -modules. When $\dim W > 0$, even for $I \in \mathrm{Inj} R$, it may happen that there is a negative integer i with $H^i(\gamma_W I) \neq 0$, see Example 1.5.3. Therefore, for a general subset W of $\mathrm{Spec} R$, γ_W is not necessarily a right derived functor of an additive functors defined on $\mathrm{Mod} R$.

The following theorem enables us to compute $\gamma_W X$ by using induction on $\dim W$.

Theorem 1.3.13. *Let $W_0 \subseteq W \subseteq \mathrm{Spec} R$ be sets. Assume that W_0 is specialization-closed in W , and set $W_1 = W \setminus W_0$. For any $X \in \mathcal{D}$, there is a triangle of the following form;*

$$\gamma_{W_0} X \longrightarrow \gamma_W X \longrightarrow \gamma_{W_1} \lambda_{W_0} X \longrightarrow \gamma_{W_0} X[1].$$

Proof. By virtue of Lemma 1.2.1, we have triangles;

$$\begin{aligned} \gamma_{W_0} X &\longrightarrow X \longrightarrow \lambda_{W_0} X \longrightarrow \gamma_{W_0} X[1], \\ \gamma_{W_1} \lambda_{W_0} X &\longrightarrow \lambda_{W_0} X \longrightarrow \lambda_{W_1} \lambda_{W_0} X \longrightarrow \gamma_{W_1} \lambda_{W_0} X[1]. \end{aligned}$$

It then follows from the octahedron axiom that there is a commutative diagram whose rows and columns are triangles:

$$\begin{array}{ccccccc}
& & X[-1] & \xlongequal{\quad} & X[-1] & & \\
& & \downarrow & & \downarrow & & \\
\gamma_{W_1}\lambda_{W_0}X[-1] & \longrightarrow & \lambda_{W_0}X[-1] & \longrightarrow & \lambda_{W_1}\lambda_{W_0}X[-1] & \longrightarrow & \gamma_{W_1}\lambda_{W_0}X \\
\parallel & & \downarrow & & \downarrow & & \parallel \\
\gamma_{W_1}\lambda_{W_0}X[-1] & \longrightarrow & \gamma_{W_0}X & \longrightarrow & C & \longrightarrow & \gamma_{W_1}\lambda_{W_0}X \\
& & \downarrow & & \downarrow & & \\
& & X & \xlongequal{\quad} & X & &
\end{array}$$

Focusing on the triangle in the second row, we notice from Lemma 1.3.11 that both $\gamma_{W_1}\lambda_{W_0}X$ and $\lambda_{W_0}X$ belong to $\mathcal{L}_{W_0}^\perp$. Hence we have $\lambda_{W_1}\lambda_{W_0}X \in \mathcal{L}_{W_0}^\perp$. Then it follows that $\lambda_{W_1}\lambda_{W_0}X \in \mathcal{L}_{W_1}^\perp \cap \mathcal{L}_{W_0}^\perp = \mathcal{L}_W^\perp$.

On the other hand, in the third row above, we know $\gamma_{W_1}\lambda_{W_0}X \in \mathcal{L}_{W_1} \subseteq \mathcal{L}_W$ and $\gamma_{W_0}X \in \mathcal{L}_{W_0} \subseteq \mathcal{L}_W$. Consequently we have $C \in \mathcal{L}_W$.

Taking a look at the third column above, since $C \in \mathcal{L}_W$ and $\lambda_{W_1}\lambda_{W_0}X \in \mathcal{L}_W^\perp$, we deduce from Lemma 1.2.1 that $\gamma_W X \cong C$. Thus the third row is a required triangle. \square

The reader should compare Theorem 4.2.4 with [3, Lemma 3.4 (4)].

Remark 1.3.14. (i) Let W , W_0 and W_1 be as in the theorem. In its proof, we have shown an isomorphism $\lambda_{W_1}\lambda_{W_0} \cong \lambda_W$.

(ii) Let W be a subset of $\text{Spec } R$, and assume that $\dim W = n$ is finite. Then we can give an alternative proof of the existence of γ_W and λ_W . In fact, in the case that $n = 0$, γ_W can be given explicitly by Theorem 1.3.12. Note that λ_W exists at the same time. Furthermore, if $n > 0$, we can show the existence of γ_W and λ_W inductively by the formula in (i).

Let X be a complex of R -modules. We say that X is left (resp. right) bounded if $X^i = 0$ for $i \ll 0$ (resp. $i \gg 0$). When X is left and right bounded, X is called bounded. We denote by $\mathcal{K} = K(\text{Inj } R)$ the homotopy category of complexes of injective R -modules. Moreover, we write \mathcal{K}^+ for the full subcategory of \mathcal{K} consisting of left bounded complexes. Let a and b be taken from $\mathbb{Z} \cup \{+\infty\}$, and assume that $a \leq b$. We denote by $\mathcal{K}^{[a,b]}$ the full subcategory of \mathcal{K}^+ consisting of complexes I such that $I^i = 0$ for $i \notin [a,b]$ (cf. [26, Notations 11.3.7 (ii)]).

Recall that the canonical functor $\mathcal{K} \rightarrow \mathcal{D}$ induces an equivalence $\mathcal{K}^+ \xrightarrow{\sim} \mathcal{D}^+$ of triangulated categories, whose quasi-inverse sends a complex $X \in \mathcal{D}^+$

to its minimal injective resolution $I \in \mathcal{K}^+$. In the following corollary, we identify \mathcal{K}^+ with \mathcal{D}^+ in this way.

Corollary 1.3.15. *Let W be a subset of $\text{Spec } R$, and assume that $n = \dim W$ is finite. Let $a, b \in \mathbb{Z} \cup \{+\infty\}$ with $a \leq b$ and $I \in \mathcal{K}^{[a,b]}$. Then $\gamma_W I$ belongs to $\mathcal{K}^{[a-n,b]}$ under the equivalence $\mathcal{K}^+ \cong \mathcal{D}^+$. Therefore, γ_W maps objects of \mathcal{D}^+ to objects of \mathcal{D}^+ .*

Proof. We prove the corollary by induction on n . If $n = 0$, then it follows from Theorem 1.3.12 that $\gamma_W I \in \mathcal{K}^{[a,b]}$.

Suppose that $n > 0$. Let W_0 be the set of all prime ideals in W that are maximal with respect to the inclusion relation in W , and we set $W_1 = W \setminus W_0$. Notice that $\dim W_0 = 0$ and $\dim W_1 = n - 1$. Since there is a triangle $\gamma_{W_0} I \rightarrow I \rightarrow \lambda_{W_0} I \rightarrow \gamma_{W_0} I[1]$, and since both I and $\gamma_{W_0} I$ belong to $\mathcal{K}^{[a,b]}$, we see that $\lambda_{W_0} I \in \mathcal{K}^{[a-1,b]}$. Hence $\gamma_{W_1} \lambda_{W_0} I \in \mathcal{K}^{[a-n,b]}$ by the inductive hypothesis.

On the other hand, we have from Theorem 4.2.4 a triangle

$$\gamma_{W_0} I \rightarrow \gamma_W I \rightarrow \gamma_{W_1} \lambda_{W_0} I \rightarrow \gamma_{W_0} I[1].$$

Since $\gamma_{W_0} I \in \mathcal{K}^{[a,b]}$ by Theorem 1.3.12, and since $\gamma_{W_1} \lambda_{W_0} I \in \mathcal{K}^{[a-n,b]}$ as shown in above, it follows that $\gamma_W I \in \mathcal{K}^{[a-n,b]}$ as desired. \square

If $\dim W$ is infinite, then it may happen that $\gamma_W X \notin \mathcal{D}^+$ for a complex $X \in \mathcal{D}^+$, see Example 1.5.5.

1.4 Local Duality Principle

Local duality theorem is a duality concerning local cohomology modules with supports in closed subsets in schemes, which was presented in [22] and [23]. Dualizing complexes or dualizing modules play a significant role there. However, Foxby [16, Proposition 6.1] discovered a general principle that underlies local duality, which does not require dualizing complexes. He considered such duality only for the right derived functor $\text{R}\Gamma_V$ of the section functor Γ_V with support in a specialization-closed subset V of $\text{Spec } R$. We propose the local duality principle as generalization of Foxby's theorem.

Theorem 1.4.1 (Local Duality Principle). *Let W be a subset of $\text{Spec } R$ and let $X, Y \in \mathcal{D}$. Suppose that one of the following conditions holds:*

- (1) $X \in \mathcal{D}_{\text{fg}}^-, Y \in \mathcal{D}^+$ and $\dim W < +\infty$;

- (2) $X \in \mathcal{D}_{\text{fg}}$, Y is a bounded complex of injective R -modules and $\dim W < +\infty$;
- (3) W is generalization-closed.

Then there exist natural isomorphisms

$$\gamma_W \text{RHom}_R(X, Y) \cong \text{RHom}_R(X, \gamma_W Y),$$

$$\lambda_W \text{RHom}_R(X, Y) \cong \text{RHom}_R(X, \lambda_W Y).$$

Note that Foxby's theorem states the validity of the first isomorphism when, added to the condition (1), W is a specialization-closed subset of $\text{Spec } R$.

Lemma 1.4.2. *Let $W \subseteq \text{Spec } R$. If $Z \in \mathcal{L}_W^\perp$, then $\text{RHom}_R(X, Z) \in \mathcal{L}_W^\perp$ for any $X \in \mathcal{D}$.*

Proof. The lemma is clear from

$$\text{RHom}_R(Y, \text{RHom}_R(X, Z)) \cong \text{RHom}_R(X, \text{RHom}_R(Y, Z)) = 0$$

for $Y \in \mathcal{L}_W$. □

Lemma 1.4.3. *Let W be a generalization-closed subset of $\text{Spec } R$. Then it holds that*

$$\mathcal{L}_W = \mathcal{L}_{W^c}^\perp.$$

Proof. We note that W^c is specialization-closed. The equality $\mathcal{L}_W = \mathcal{L}_{W^c}^\perp$ is deduced from Lemma 1.3.9 as follows: If $X \in \mathcal{L}_W$, then $\text{supp } \gamma_{W^c} X = W^c \cap \text{supp } X = \emptyset$ hence $\gamma_{W^c} X = 0$ equivalently $X = \lambda_{W^c} X \in \mathcal{L}_{W^c}^\perp$. On the contrary, if $X \in \mathcal{L}_{W^c}^\perp$ then $\text{supp } \lambda_{W^c} X = W \cap \text{supp } X$ therefore $X = \lambda_{W^c} X$ has small support in W thus $X \in \mathcal{L}_W$. □

Let X be a complex of R -modules and n be an integer. The cohomological truncations $\sigma_{\leq n} X$ and $\sigma_{> n} X$ are defined as follows:

$$\begin{aligned} \sigma_{\leq n} X &= (\cdots \rightarrow X^{n-2} \rightarrow X^{n-1} \rightarrow \text{Ker } d_X^n \rightarrow 0 \rightarrow \cdots) \\ \sigma_{> n} X &= (\cdots \rightarrow 0 \rightarrow \text{Im } d_X^n \rightarrow X^{n+1} \rightarrow X^{n+2} \rightarrow \cdots) \end{aligned}$$

See [22, Chapter I; §7] for details.

Proof of Theorem 1.4.1. Applying the functor $\mathrm{RHom}_R(X, -)$ to the triangle $\gamma_W Y \rightarrow Y \rightarrow \lambda_W Y \rightarrow \gamma_W Y[1]$, we obtain a triangle of the form;

$$\mathrm{RHom}_R(X, \gamma_W Y) \rightarrow \mathrm{RHom}_R(X, Y) \rightarrow \mathrm{RHom}_R(X, \lambda_W Y) \rightarrow .$$

It follows from Lemma 1.2.1 that, to prove the desired isomorphisms, we only have to show that

$$\mathrm{RHom}_R(X, \gamma_W Y) \in \mathcal{L}_W \quad \text{and} \quad \mathrm{RHom}_R(X, \lambda_W Y) \in \mathcal{L}_W^\perp.$$

Since $\lambda_W Y \in \mathcal{L}_W^\perp$, we see from Lemma 1.4.2 that $\mathrm{RHom}_R(X, \lambda_W Y) \in \mathcal{L}_W^\perp$. Thus it remains to show that $\mathrm{RHom}_R(X, \gamma_W Y) \in \mathcal{L}_W$.

Case (1): Let $\mathfrak{p} \in W^c$. We want to show that $\mathrm{RHom}_R(X, \gamma_W Y) \otimes_R^L \kappa(\mathfrak{p}) = 0$. Since $X \in \mathcal{D}_{\mathrm{fg}}^-$ and $Y \in \mathcal{D}^+$, Corollary 1.3.15 implies $\gamma_W Y \in \mathcal{D}^+$, so it follows that

$$\mathrm{RHom}_R(X, \gamma_W Y) \otimes_R^L \kappa(\mathfrak{p}) \cong \mathrm{RHom}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}, (\gamma_W Y)_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}}^L \kappa(\mathfrak{p}).$$

Now thanks to Lemma 1.3.8 together with this isomorphism, it is sufficient to show that $\mathrm{RHom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), \mathrm{RHom}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}, (\gamma_W Y)_{\mathfrak{p}})) = 0$. Noting that

$$\begin{aligned} \mathrm{RHom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), \mathrm{RHom}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}, (\gamma_W Y)_{\mathfrak{p}})) \\ \cong \mathrm{RHom}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}, \mathrm{RHom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), (\gamma_W Y)_{\mathfrak{p}})), \end{aligned}$$

we have sufficiently to show that $\mathrm{RHom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), (\gamma_W Y)_{\mathfrak{p}}) = 0$. However, by using Lemma 1.3.8 again, we see that this is equivalent to show that $(\gamma_W Y) \otimes_R^L \kappa(\mathfrak{p}) = 0$, *i.e.*, $\mathfrak{p} \notin \mathrm{supp} \gamma_W Y$. The last is clear, since $\mathrm{supp} \gamma_W Y \subseteq W$ and $\mathfrak{p} \notin W$.

Case (2): As in the case (1), taking $\mathfrak{p} \in W^c$, we show

$$\mathrm{RHom}_R(X, \gamma_W Y) \otimes_R^L \kappa(\mathfrak{p}) = 0.$$

We consider the triangle $\sigma_{\leq n} X \rightarrow X \rightarrow \sigma_{> n} X \rightarrow (\sigma_{\leq n} X)[1]$ for an integer n . Since $\sigma_{\leq n} X \in \mathcal{D}_{\mathrm{fg}}^-$, we have $\mathrm{RHom}_R(\sigma_{\leq n} X, \gamma_W Y) \in \mathcal{L}_W$ by the case (1). Hence, applying $\mathrm{RHom}_R(-, \gamma_W Y) \otimes_R^L \kappa(\mathfrak{p})$ to the triangle, we obtain an isomorphism

$$\mathrm{RHom}_R(X, \gamma_W Y) \otimes_R^L \kappa(\mathfrak{p}) \cong \mathrm{RHom}_R(\sigma_{> n} X, \gamma_W Y) \otimes_R^L \kappa(\mathfrak{p}).$$

Let i be any integer. It suffices to show that

$$H^0(\mathrm{RHom}_R(\sigma_{> n} X, \gamma_W Y[i]) \otimes_R^L \kappa(\mathfrak{p})) = 0$$

for some n . By Corollary 1.3.15, $\gamma_W Y$ is isomorphic to a bounded complex I of injective R -modules, so that there is an integer m with $I^j = 0$ for $j > m$.

Moreover, any element of $H^0(\mathrm{RHom}_R(\sigma_{>n}X, \gamma_W Y[i])) \cong \mathrm{Hom}_{\mathcal{D}}(\sigma_{>n}X, I[i])$ is represented by a chain map $\sigma_{>n}X \rightarrow I[i]$. Thus, taking n with $n > m - i$, we see that $H^j(\mathrm{RHom}_R(\sigma_{>n}X, \gamma_W Y[i])) \cong \mathrm{Hom}_{\mathcal{D}}(\sigma_{>n}X, I[i+j]) = 0$ for all $j \geq 0$. Then it is easily seen that $H^0(\mathrm{RHom}_R(\sigma_{>n}X, \gamma_W Y[i]) \otimes_R^L \kappa(\mathfrak{p})) = 0$.

Case (3): By Lemma 1.4.3, we have $\gamma_W Y \in \mathcal{L}_W = \mathcal{L}_{W^c}^\perp$, thus it follows from Lemma 1.4.2 that $\mathrm{RHom}_R(X, \gamma_W Y) \in \mathcal{L}_{W^c}^\perp = \mathcal{L}_W$ as desired. \square

Remark 1.4.4. In the case (3), the isomorphisms in the theorem are also proved by [2, Theorem 5.14].

When R admits a dualizing complex D_R , we write $X^\dagger = \mathrm{RHom}_R(X, D_R)$ for $X \in \mathcal{D}$. Then we have $X \cong X^{\dagger\dagger}$ for $X \in \mathcal{D}_{\mathrm{fg}}$, see [22, Chapter V; §2]. The following result is the generalized form of the local duality theorem [22, Chapter V; Theorem 6.2].

Corollary 1.4.5. *Assume that R admits a dualizing complex D_R . Let W be a subset of $\mathrm{Spec} R$ and $X \in \mathcal{D}_{\mathrm{fg}}$. Then we have a natural isomorphism*

$$\gamma_W X \cong \mathrm{RHom}_R(X^\dagger, \gamma_W D_R).$$

The second author and Maiko Ono had proved the following theorem when W is specialization-closed, and had asked if it holds for an arbitrary subset W , see [38, Theorem 2.3, Question 2.5].

Theorem 1.4.6 (AR Principle). *Let $X, I \in \mathcal{D}$. Suppose that a subset W of $\mathrm{Spec} R$ satisfies the condition $\dim W < +\infty$. We assume furthermore that the following conditions hold for an integer n :*

- (1) *I is a bounded complex of injective R -modules with $I^i = 0$ for $i > n$;*
- (2) *$\sigma_{\leq -1}X \in \mathcal{L}_W$.*

Then there exists a natural isomorphism

$$\sigma_{>n} \mathrm{RHom}_R(X, \gamma_W I) \cong \sigma_{>n} \mathrm{RHom}_R(X, I).$$

We are now able to prove that this theorem holds in general. As we have shown in Corollary 1.3.15, $\gamma_W I$ is isomorphic to a bounded complex J of injective R -modules with $J^i = 0$ for $i > n$. Then one can observe that the proof of [38, Theorem 2.3] works well.

The AR Principle is a version of classical Auslander-Reiten duality theorem in terms of complexes, see [38, Corollary 3.2].

1.5 Relation with completion

In this section, we shall explain the relationship between λ_W and left derived functors of completion functors. Furthermore, we give some nontrivial examples of colocalization functors γ_W , for which $\gamma_W I$ has a non-zero negative cohomology module even for an injective R -module I .

Let \mathfrak{a} be an ideal of R which defines a closed subset $V(\mathfrak{a})$ of $\text{Spec } R$. We denote by $\Lambda^{V(\mathfrak{a})}$ the \mathfrak{a} -adic completion functor $\varprojlim(- \otimes_R R/\mathfrak{a}^n)$ defined on $\text{Mod } R$. It is known that the left derived functor $\text{L}\Lambda^{V(\mathfrak{a})}$ of $\Lambda^{V(\mathfrak{a})}$ is a right adjoint to $\text{R}\Gamma_{V(\mathfrak{a})}$ by Greenlees and May [21] and Alonso Tarrío, Jeremías López and Lipman [1]. One also finds an outline of the proof of this fact in [28, §4; p. 69].

Recall that $\lambda_{V(\mathfrak{a})^c}$ is a left adjoint to the inclusion functor $j_{V(\mathfrak{a})^c} : \mathcal{L}_{V(\mathfrak{a})^c} \hookrightarrow \mathcal{D}$. Now we shall prove that $\text{L}\Lambda^{V(\mathfrak{a})}$ coincides with $\lambda_{V(\mathfrak{a})^c}$. By the universality of derived functors, there is a natural morphism $X \rightarrow \text{L}\Lambda^{V(\mathfrak{a})} X$ for any $X \in \mathcal{D}$, from which we have a triangle of the form

$$C \longrightarrow X \longrightarrow \text{L}\Lambda^{V(\mathfrak{a})} X \longrightarrow C[1].$$

Applying $\text{R}\Gamma_{V(\mathfrak{a})}$ to this triangle, we have the following triangle

$$\text{R}\Gamma_{V(\mathfrak{a})} C \longrightarrow \text{R}\Gamma_{V(\mathfrak{a})} X \longrightarrow \text{R}\Gamma_{V(\mathfrak{a})} \text{L}\Lambda^{V(\mathfrak{a})} X \longrightarrow \text{R}\Gamma_{V(\mathfrak{a})} C[1].$$

By [1, Corollary 5.1.1 (ii)], $\text{R}\Gamma_{V(\mathfrak{a})} X \rightarrow \text{R}\Gamma_{V(\mathfrak{a})} \text{L}\Lambda^{V(\mathfrak{a})} X$ is an isomorphism. Hence $\text{R}\Gamma_{V(\mathfrak{a})} C = 0$, and thus we have $C \in \mathcal{L}_{V(\mathfrak{a})}^\perp = \mathcal{L}_{V(\mathfrak{a})^c}$ by Lemma 1.4.3. On the other hand, since $\text{L}\Lambda^{V(\mathfrak{a})}$ is a right adjoint to $\text{R}\Gamma_{V(\mathfrak{a})}$, we have

$$\text{RHom}_R(\kappa(\mathfrak{p}), \text{L}\Lambda^{V(\mathfrak{a})} X) \cong \text{RHom}_R(\text{R}\Gamma_{V(\mathfrak{a})} \kappa(\mathfrak{p}), X) = 0$$

for any $\mathfrak{p} \in V(\mathfrak{a})^c$. This implies $\text{L}\Lambda^{V(\mathfrak{a})} X \in \mathcal{L}_{V(\mathfrak{a})^c}^\perp$. Thus it follows that $\gamma_{V(\mathfrak{a})^c} X \cong C$ and $\lambda_{V(\mathfrak{a})^c} X \cong \text{L}\Lambda^{V(\mathfrak{a})} X$ by Lemma 1.2.1.

We summarize this fact in the following.

Proposition 1.5.1. *Let \mathfrak{a} be an ideal of R . Then $\lambda_{V(\mathfrak{a})^c}$ is isomorphic to the left derived functor $\text{L}\Lambda^{V(\mathfrak{a})}$ of the \mathfrak{a} -adic completion functor $\Lambda^{V(\mathfrak{a})}$.*

Remark 1.5.2. Let \mathfrak{a} be an ideal of R and W be a specialization-closed subset of $\text{Spec } R$. Note that it is also proved that $\text{L}\Lambda^{V(\mathfrak{a})}$ is isomorphic to $\text{RHom}_R(\text{R}\Gamma_{V(\mathfrak{a})} R, -)$ in [21] and [1]. More generally, $\text{RHom}_R(\text{R}\Gamma_W R, -)$ is a right adjoint to $\text{R}\Gamma_W$. Furthermore, by using Lemma 1.4.3, it is not hard

to see that λ_{W^c} is a right adjoint to γ_W . Thus it follows from the uniqueness of adjoint functors that there is an isomorphism

$$\lambda_{W^c} \cong \mathrm{RHom}_R(\mathrm{R}\Gamma_W R, -).$$

This fact and Proposition 1.5.1 is essentially stated in [4], where γ_{W^c} and λ_{W^c} appear as V^W and Λ^W respectively.

Now we are ready to give an example as we have previously announced.

Example 1.5.3. Let (R, \mathfrak{m}, k) be a local ring of dimension d and \widehat{R} be the \mathfrak{m} -adic completion of R . Let $D_{\widehat{R}}$ be a dualizing complex of \widehat{R} with $\mathrm{Ext}_{\widehat{R}}^d(k, D_{\widehat{R}}) \cong k$. We regard $D_{\widehat{R}}$ as a complex of R -modules in a natural way. Using the isomorphism $\mathrm{L}\Lambda^{V(\mathfrak{m})} \mathrm{R}\Gamma_{V(\mathfrak{m})} \cong \mathrm{L}\Lambda^{V(\mathfrak{m})}$ by [1, Corollary 5.1.1 (i)], we can show that $\mathrm{L}\Lambda^{V(\mathfrak{m})} E_R(k) \cong D_{\widehat{R}}[d]$. Hence it follows from Proposition 1.5.1 that $\lambda_{V(\mathfrak{m})^c} E_R(k) \cong D_{\widehat{R}}[d]$. Now we suppose that $d > 1$. Then, considering the triangle

$$\gamma_{V(\mathfrak{m})^c} E_R(k) \longrightarrow E_R(k) \longrightarrow \lambda_{V(\mathfrak{m})^c} E_R(k) \longrightarrow \gamma_{V(\mathfrak{m})^c} E_R(k)[1],$$

we have $H^{-d+1}(\gamma_{V(\mathfrak{m})^c} E_R(k)) \neq 0$ and $-d+1 < 0$. \square

Remark 1.5.4. Let $X \in \mathcal{D}$. If W is specialization-closed, then it holds that $\mathrm{supp} H^i(\gamma_W X) \subseteq W$ for all $i \in \mathbb{Z}$, since $\gamma_W \cong \mathrm{R}\Gamma_W$. However we see from Example 1.5.3 that the inclusion relation $\mathrm{supp} H^i(\gamma_W X) \subseteq W$ does not necessarily hold in general.

We now give an example such that $\gamma_W I \notin \mathcal{D}^+$ even for an injective R -module I .

Example 1.5.5. We assume that $\dim R = +\infty$. Let W be the set of maximal ideals of R . Then we can show that $\gamma_{W^c}(\bigoplus_{\mathfrak{m} \in W} E_R(R/\mathfrak{m})) \notin \mathcal{D}^+$. In fact, it is clear that $\mathrm{R}\Gamma_W \cong \bigoplus_{\mathfrak{m} \in W} \mathrm{R}\Gamma_{V(\mathfrak{m})}$. Thus it follows from Remark 1.5.2 that

$$\lambda_{W^c} \cong \mathrm{RHom}_R(\mathrm{R}\Gamma_W R, -) \cong \prod_{\mathfrak{m} \in W} \mathrm{RHom}_R(\mathrm{R}\Gamma_{V(\mathfrak{m})} R, -) \cong \prod_{\mathfrak{m} \in W} \mathrm{L}\Lambda^{V(\mathfrak{m})}.$$

Then, by Example 1.5.3, we see that $\lambda_{W^c}(\bigoplus_{\mathfrak{m} \in W} E_R(R/\mathfrak{m})) \notin \mathcal{D}^+$. Hence we have $\gamma_{W^c}(\bigoplus_{\mathfrak{m} \in W} E_R(R/\mathfrak{m})) \notin \mathcal{D}^+$. \square

Remark 1.5.6. More generally, we can prove that λ_{W^c} is isomorphic to

$$\prod_{\mathfrak{p} \in W} \mathrm{L}\Lambda^{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}})$$

for an arbitrary subset W of $\mathrm{Spec} R$ with $\dim W = 0$, see Theorem 2.3.10.

1.6 Vanishing for cohomology modules

Let V be a specialization-closed subset of $\text{Spec } R$. In such a classical case, Grothendieck showed that if M is a finitely generated R -module, then

$$H_V^i(M) = H^i(R\Gamma_V M) = 0$$

for $i < 0$ and $i > \dim M$.

Our aim in this section is to prove the same type of vanishing theorem holds for the colocalization functor γ_W with support in W , where W is not necessarily specialization-closed.

Notice from Example 1.5.3 that it is truly nontrivial even to prove that $H^i(\gamma_W M) = 0$ for $i < 0$.

Proposition 1.6.1. *Let W be a subset of $\text{Spec } R$ and suppose that $\dim W$ is finite. Then we have $H^i(\gamma_W M) = 0$ for any $i < 0$ and for any finitely generated R -module M .*

Proof. First of all we note the following: Let $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ be an exact sequence of finitely generated R -modules. If N is a counterexample to the theorem, then so is one of N' and N'' .

Secondly we note that a finitely generated R -module M has a filtration

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

such that $M_{i+1}/M_i \cong R/\mathfrak{p}_i$ and $\mathfrak{p}_i \in \text{Spec } R$ for $0 \leq i < n$. It thus follows that we have sufficiently to prove that $H^i(\gamma_W R/\mathfrak{p}) = 0$ for $i < 0$ and $\mathfrak{p} \in \text{Spec } R$.

Now supposed that there would exist $\mathfrak{p} \in \text{Spec } R$ with $H^i(\gamma_W R/\mathfrak{p}) \neq 0$ for some $i < 0$, and take a maximal \mathfrak{p} among such prime ideals. Then it is easy to see from the remark made in the first part of this proof that the theorem is true for $M = R/I$ for all $I \supsetneq \mathfrak{p}$.

The key for the proof is Corollary 1.3.15 which states that $\gamma_W R/\mathfrak{p} \in \mathcal{D}^+$. This exactly means there is the least integer $\ell < 0$ with $H^\ell(\gamma_W R/\mathfrak{p}) \neq 0$. Let x be an arbitrary element of R with $x \notin \mathfrak{p}$. Apply the functor γ_W to the exact sequence

$$0 \rightarrow R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \rightarrow R/(\mathfrak{p} + (x)) \rightarrow 0,$$

and we obtain an isomorphism $H^\ell(\gamma_W R/\mathfrak{p}) \xrightarrow{x} H^\ell(\gamma_W R/\mathfrak{p})$. Since every element from $R \setminus \mathfrak{p}$ acts bijectively on $H^\ell(\gamma_W R/\mathfrak{p})$, consequently $H^\ell(\gamma_W R/\mathfrak{p})$ is a $\kappa(\mathfrak{p})$ -vector space. Now let I be a minimal injective resolution of $\gamma_W R/\mathfrak{p}$. By our choice of integer ℓ , we see that I starts from the ℓ th term, being of the form

$$0 \longrightarrow I^\ell \xrightarrow{\partial^\ell} I^{\ell+1} \xrightarrow{\partial^{\ell+1}} I^{\ell+2} \xrightarrow{\partial^{\ell+2}} \cdots$$

Note that $\text{Ker } \partial^\ell = H^\ell(\gamma_W R/\mathfrak{p})$. Since I is a minimal injective resolution, I^ℓ is an essential extension of $H^\ell(\gamma_W R/\mathfrak{p})$ that is a $\kappa(\mathfrak{p})$ -module. Therefore I^ℓ must be isomorphic to a direct sum of copies of $E_R(R/\mathfrak{p})$. This forces that $\Gamma_{V(\mathfrak{p})}(I_\mathfrak{p})$ has a nontrivial cohomology in degree ℓ , thus it follows from Lemma 1.3.8 that $\mathfrak{p} \in \text{supp } I = \text{supp } \gamma_W R/\mathfrak{p} \subseteq W$. Since we have shown that $\mathfrak{p} \in W$, the following isomorphisms hold;

$$0 \neq \text{Hom}_{\mathcal{D}}(\kappa(\mathfrak{p}), \gamma_W R/\mathfrak{p}[\ell]) \cong \text{Hom}_{\mathcal{D}}(\kappa(\mathfrak{p}), R/\mathfrak{p}[\ell]) \cong \text{Ext}_R^\ell(\kappa(\mathfrak{p}), R/\mathfrak{p}),$$

the last of which must be zero since $\ell < 0$. By this contradiction we complete the proof. \square

Let $a, b \in \mathbb{Z} \cup \{\pm\infty\}$ with $a \leq b$. We denote by $\mathcal{D}^{[a,b]}$ for the full subcategory of \mathcal{D} consisting of complexes X such that $H^i(X) = 0$ for $i \notin [a, b]$ (cf. [26, Notation 13.1.11]). Moreover, we write $\mathcal{D}_{\text{fg}}^{[a,b]} = \mathcal{D}^{[a,b]} \cap \mathcal{D}_{\text{fg}}$. For $X \in \mathcal{D}$, we denote by $\inf X$ the infimum of the set $\{i \in \mathbb{Z} \mid H^i(X) \neq 0\}$.

Corollary 1.6.2. *Let $X \in \mathcal{D}_{\text{fg}}$ and W be a subset of $\text{Spec } R$. We assume that $\dim W$ is finite. Then we have $\inf X \leq \inf \gamma_W X$.*

Proof. We may assume that $\inf X > -\infty$. Thus it is enough to show that if $X \in \mathcal{D}_{\text{fg}}^{[0,+\infty]}$, then $\gamma_W X \in \mathcal{D}^{[0,+\infty]}$. To see this, setting $n = \dim W$, we consider the triangle $\sigma_{\leq n} X \rightarrow X \rightarrow \sigma_{> n} X \rightarrow (\sigma_{\leq n} X)[1]$. Since $\sigma_{\leq n} X \in \mathcal{D}_{\text{fg}}^{[0,n]}$, by Proposition 1.6.1, we can show that $\gamma_W(\sigma_{\leq n} X) \in \mathcal{D}^{[0,+\infty]}$. Furthermore, it follows from Corollary 1.3.15 that $\gamma_W(\sigma_{> n} X) \in \mathcal{D}^{[0,+\infty]}$. Thus we can conclude that $\gamma_W X \in \mathcal{D}^{[0,+\infty]}$ by the triangle. \square

Let $X \in \mathcal{D}$. For an ideal \mathfrak{a} of R , we define the \mathfrak{a} -depth of X as

$$\text{depth}(\mathfrak{a}, X) = \inf \text{RHom}_R(R/\mathfrak{a}, X).$$

Let $\mathbf{x} = \{x_1, \dots, x_n\}$ be a system of generators of \mathfrak{a} . For each x_i , $K(x_i)$ denotes the complex $(0 \rightarrow R \xrightarrow{x_i} R \rightarrow 0)$ concentrated in degrees -1 and 0 . The Koszul complex with respect to \mathbf{x} is the complex $K(\mathbf{x}) = K(x_1) \otimes_R \cdots \otimes_R K(x_n)$. By [17, Theorem 2.1], it holds that $\text{depth}(\mathfrak{a}, X) = \inf \text{Hom}_R(K(\mathbf{x}), X) = \inf \text{R}\Gamma_{V(\mathfrak{a})} X$.

If W is a specialization-closed subset of $\text{Spec } R$, then we define the W -depth of X , which we denote by $\text{depth}(W, X)$, as the infimum of the set of values $\text{depth}(\mathfrak{a}, X)$ for all ideals \mathfrak{a} with $V(\mathfrak{a}) \subseteq W$. It is easily seen that $\text{depth}(W, X) = \inf \text{R}\Gamma_W X$.

Proposition 1.6.3. *Let W be a subset of $\text{Spec } R$, and assume that $\dim W$ is finite. Let $X \in \mathcal{D}_{\text{fg}}$. Then we have $\text{depth}(\overline{W}^s, X) \leq \inf \gamma_W X$.*

Proof. By Remark 1.3.7 (i), it holds that $\gamma_W X \cong \mathrm{R}\Gamma_{\overline{W}^s}(\gamma_W X)$. Therefore we have

$$\inf \gamma_W X = \inf \mathrm{R}\Gamma_{\overline{W}^s}(\gamma_W X) = \mathrm{depth}(\overline{W}^s, \gamma_W X).$$

Hence it remains to show that $\mathrm{depth}(\overline{W}^s, X) \leq \mathrm{depth}(\overline{W}^s, \gamma_W X)$. Let \mathfrak{a} be an ideal of R with $V(\mathfrak{a}) \subseteq \overline{W}^s$ and $\mathbf{x} = \{x_1, \dots, x_n\}$ be a system of generators of \mathfrak{a} . Since $K(\mathbf{x})$ is a bounded complex of finitely generated free R -modules, it follows from Corollary 1.6.2 that

$$\inf \mathrm{Hom}_R(K(\mathbf{x}), X) \leq \inf \gamma_W \mathrm{Hom}_R(K(\mathbf{x}), X) = \inf \mathrm{Hom}_R(K(\mathbf{x}), \gamma_W X).$$

Thus we have $\mathrm{depth}(\mathfrak{a}, X) \leq \mathrm{depth}(\mathfrak{a}, \gamma_W X)$. Hence it holds that

$$\mathrm{depth}(\overline{W}^s, X) \leq \mathrm{depth}(\overline{W}^s, \gamma_W X)$$

by definition. □

This proposition states that an inequality

$$\inf \mathrm{R}\Gamma_{\overline{W}^s} X \leq \inf \gamma_W X$$

holds for a subset W of $\mathrm{Spec} R$ with $\dim W < +\infty$ and $X \in \mathcal{D}_{\mathrm{fg}}$.

Let $X \in \mathcal{D}_{\mathrm{fg}}$. Write $\dim X$ for the supremum of $\{\dim H^i(X) + i \mid i \in \mathbb{Z}\}$, see [16, §3]. Note that $\dim X[1] = \dim X - 1$. Let D_R be a dualizing complex of R with $D_R \in \mathcal{D}_{\mathrm{fg}}^{[0, d]}$, where $d = \dim R$. In the proof of the next proposition, we use a basic fact that $X^\dagger \in \mathcal{D}_{\mathrm{fg}}^{[d-n, +\infty]}$, where $n = \dim X$. To show this, we first suppose that X is a finitely generated R -module. Then it is straightforward to see that $X^\dagger \in \mathcal{D}_{\mathrm{fg}}^{[d-n, d]} \subset \mathcal{D}_{\mathrm{fg}}^{[d-n, +\infty]}$. Next, suppose that X is any complex of $\mathcal{D}_{\mathrm{fg}}$. Notice that it suffices to treat the case that $n = \dim X = 0$. Then, using the triangle $\sigma_{\leq -d} X \rightarrow X \rightarrow \sigma_{> -d} X \rightarrow (\sigma_{\leq -d} X)[1]$, one can deduce that $X^\dagger \in \mathcal{D}_{\mathrm{fg}}^{[d, +\infty]}$ as desired. This fact is essentially proved in [16, Proposition 3.14 (d)].

Proposition 1.6.4. *Assume that R admits a dualizing complex. Let W be a subset of $\mathrm{Spec} R$ and $X \in \mathcal{D}_{\mathrm{fg}}$. Then $H^i(\gamma_W X) = 0$ for all $i > \dim X$.*

Proof. Let D_R be a dualizing complex with $D_R \in \mathcal{D}_{\mathrm{fg}}^{[0, d]}$, where $d = \dim R$. By Corollary 1.3.15, $\gamma_W D_R$ is isomorphic to a bounded complex I of injective R -modules with $I^i = 0$ for $i > d$. Then, by Corollary 1.4.5, there are isomorphisms

$$\gamma_W X \cong \mathrm{RHom}_R(X^\dagger, \gamma_W D_R) \cong \mathrm{Hom}_R(X^\dagger, I).$$

Therefore each element of $H^i(\gamma_W X) \cong \text{Hom}_{\mathcal{D}}(X^\dagger, I[i])$ is represented by a chain map from X^\dagger to $I[i]$. Moreover, setting $n = \dim X$, we have $X^\dagger \in \mathcal{D}_{\text{fg}}^{[d-n, +\infty]}$ by the above-mentioned fact. Hence it holds that $H^i(\gamma_W X) = 0$ for all $i > n = \dim X$. \square

We sum up Proposition 1.6.3 and Proposition 1.6.4 in the following theorem.

Theorem 1.6.5. *Assume that R admits a dualizing complex. Let W be a subset of $\text{Spec } R$. If $X \in \mathcal{D}_{\text{fg}}$, then $H^i(\gamma_W X) = 0$ unless $\text{depth}(\overline{W}^s, X) \leq i \leq \dim X$.*

2. Localization functors and cosupport in derived categories

2.1 Introduction

This chapter is based on the author's paper [34] with Yuji Yoshino. Throughout this chapter, we assume that R is a commutative Noetherian ring. We denote by $\mathcal{D} = D(\text{Mod } R)$ the derived category of all complexes of R -modules, by which we mean that \mathcal{D} is the unbounded derived category. For a triangulated subcategory \mathcal{T} of \mathcal{D} , its left (resp. right) orthogonal subcategory is defined as ${}^{\perp}\mathcal{T} = \{ X \in \mathcal{D} \mid \text{Hom}_{\mathcal{D}}(X, \mathcal{T}) = 0 \}$ (resp. $\mathcal{T}^{\perp} = \{ Y \in \mathcal{D} \mid \text{Hom}_{\mathcal{D}}(\mathcal{T}, Y) = 0 \}$). Moreover, \mathcal{T} is called localizing (resp. colocalizing) if \mathcal{T} is closed under arbitrary direct sums (resp. direct products).

Recall that the support of a complex $X \in \mathcal{D}$ is defined as follows;

$$\text{supp } X = \{ \mathfrak{p} \in \text{Spec } R \mid X \otimes_R^{\mathbb{L}} \kappa(\mathfrak{p}) \neq 0 \},$$

where $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. We write $\mathcal{L}_W = \{ X \in \mathcal{D} \mid \text{supp } X \subseteq W \}$ for a subset W of $\text{Spec } R$. Then \mathcal{L}_W is a localizing subcategory of \mathcal{D} . Neeman [36] proved that any localizing subcategory of \mathcal{D} is obtained in this way. The localization theory of triangulated categories [27] yields a couple of adjoint pairs (i_W, γ_W) and (λ_W, j_W) as it is indicated in the following diagram:

$$(2.1.1) \quad \mathcal{L}_W \begin{array}{c} \xrightarrow{i_W} \\ \xleftarrow{\gamma_W} \end{array} \mathcal{D} \begin{array}{c} \xrightarrow{\lambda_W} \\ \xleftarrow{j_W} \end{array} \mathcal{L}_W^{\perp}$$

Here, i_W and j_W are the inclusion functors $\mathcal{L}_W \hookrightarrow \mathcal{D}$ and $\mathcal{L}_W^{\perp} \hookrightarrow \mathcal{D}$ respectively. In the previous chapter, we introduced the colocalization functor with support in W as the functor γ_W . If V is a specialization-closed subset of $\text{Spec } R$, then γ_V coincides with the right derived functor $\text{R}\Gamma_V$ of the section functor Γ_V with support in V ; it induces the local cohomology functors $H_V^i(-) = H^i(\text{R}\Gamma_V(-))$. In *loc. cit.*, we established some methods to compute γ_W for general subsets W of $\text{Spec } R$. Furthermore, the local duality theorem

and Grothendieck type vanishing theorem of local cohomology were extended to the case of γ_W .

On the other hand, in this chapter, we introduce the notion of localization functors with cosupports in arbitrary subsets W of $\text{Spec } R$. Recall that the cosupport of a complex $X \in \mathcal{D}$ is defined as follows;

$$\text{cosupp } X = \{ \mathfrak{p} \in \text{Spec } R \mid \text{RHom}_R(\kappa(\mathfrak{p}), X) \neq 0 \}.$$

We write $\mathcal{C}^W = \{ X \in \mathcal{D} \mid \text{cosupp } X \subseteq W \}$ for a subset W of $\text{Spec } R$. Then \mathcal{C}^W is a colocalizing subcategory of \mathcal{D} . Neeman [37] proved that any colocalizing subcategory of \mathcal{D} is obtained in this way.

We remark that there are equalities

$$(2.1.2) \quad {}^\perp \mathcal{C}^W = \mathcal{L}_{W^c}, \quad \mathcal{C}^W = \mathcal{L}_{W^c}^\perp,$$

where $W^c = \text{Spec } R \setminus W$. The second equality follows from Neeman's theorem [36, Theorem 2.8], which states that \mathcal{L}_{W^c} is equal to the smallest localizing subcategory of \mathcal{D} containing the set $\{ \kappa(\mathfrak{p}) \mid \mathfrak{p} \in W^c \}$. Then it is seen that the first one holds, since ${}^\perp(\mathcal{L}_{W^c}^\perp) = \mathcal{L}_{W^c}$ (cf. [27, §4.9]).

Now we write $\lambda^W = \lambda_{W^c}$ and $j^W = j_{W^c}$. By (2.1.1) and (2.1.2), there is a diagram of adjoint pairs:

$${}^\perp \mathcal{C}^W = \mathcal{L}_{W^c} \begin{array}{c} \xrightarrow{i_{W^c}} \\ \xleftarrow{\gamma_{W^c}} \end{array} \mathcal{D} \begin{array}{c} \xrightarrow{\lambda^W} \\ \xleftarrow{j^W} \end{array} \mathcal{C}^W = \mathcal{L}_{W^c}^\perp$$

We call λ^W *the localization functor with cosupport in W* .

For a multiplicatively closed subset S of R , the localization functor λ^{U_S} with cosupport in U_S is nothing but $(-) \otimes_R S^{-1}R$. Moreover, for an ideal \mathfrak{a} of R , the localization functor $\lambda^{V(\mathfrak{a})}$ with cosupport in $V(\mathfrak{a})$ is isomorphic to the left derived functor $\text{L}\Lambda^{V(\mathfrak{a})}$ of the \mathfrak{a} -adic completion functor $\Lambda^{V(\mathfrak{a})} = \varprojlim (- \otimes_R R/\mathfrak{a}^n)$ defined on $\text{Mod } R$. See Section 2 for details.

In this chapter, we establish several results about the localization functor λ^W with cosupport in a general subset W of $\text{Spec } R$.

In Section 3, we prove that λ^W is isomorphic to $\prod_{\mathfrak{p} \in W} \text{L}\Lambda^{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}})$ if there is no inclusion relation between two distinct prime ideals in W . Furthermore, we give a method to compute λ^W for a general subset W . We write $\eta^W : \text{id}_{\mathcal{D}} \rightarrow \lambda^W (= j^W \lambda^W)$ for the natural morphism given by the adjointness of (λ^W, j^W) . In addition, note that when $W_0 \subseteq W$, there is a morphism $\eta^{W_0} \lambda^W : \lambda^W \rightarrow \lambda^{W_0} \lambda^W \cong \lambda^{W_0}$. The following theorem is one of the main results of this chapter.

Theorem 2.1.3 (Theorem 2.3.15). *Let W , W_0 and W_1 be subsets of $\text{Spec } R$ with $W = W_0 \cup W_1$. We denote by $\overline{W_0}^s$ (resp. $\overline{W_1}^g$) is the specialization (resp. generalization) closure of W . Suppose that one of the following conditions holds:*

- (1) $W_0 = \overline{W_0}^s \cap W$;
- (2) $W_1 = W \cap \overline{W_1}^g$.

Then, for any $X \in \mathcal{D}$, there is a triangle

$$\lambda^W X \xrightarrow{f} \lambda^{W_1} X \oplus \lambda^{W_0} X \xrightarrow{g} \lambda^{W_1} \lambda^{W_0} X \longrightarrow \lambda^W X[1],$$

where

$$f = \begin{pmatrix} \eta^{W_1} \lambda^W X \\ \eta^{W_0} \lambda^W X \end{pmatrix}, \quad g = \begin{pmatrix} \lambda^{W_1} \eta^{W_0} X & (-1) \cdot \eta^{W_1} \lambda^{W_0} X \end{pmatrix}.$$

This theorem enables us to compute λ^W by using λ^{W_0} and λ^{W_1} for smaller subsets W_0 and W_1 . Furthermore, as long as we consider the derived category \mathcal{D} , the theorem and Theorem 2.3.22 in Section 3 generalize Mayer-Vietoris triangles by Benson, Iyengar and Krause [3, Theorem 7.5].

In Section 4, as an application, we give a simpler proof of a classical theorem due to Gruson and Raynaud. The theorem states that the projective dimension of a flat R -module is at most the Krull dimension of R .

Section 5 contains some basic facts about cotorsion flat R -modules.

Section 6 is devoted to study the cosupport of a complex X consisting of cotorsion flat R -modules. As a consequence, we can calculate $\gamma_{V^c} X$ and $\lambda^V X$ explicitly for a specialization-closed subset V of $\text{Spec } R$.

In Section 7, using Theorem 2.1.3 above, we give a new way to get λ^W . In fact, provided that $d = \dim R$ is finite, we are able to calculate λ^W by a Čech complex of functors of the following form;

$$\prod_{0 \leq i \leq d} \bar{\lambda}^{W_i} \longrightarrow \prod_{0 \leq i < j \leq d} \bar{\lambda}^{W_j} \bar{\lambda}^{W_i} \longrightarrow \dots \longrightarrow \bar{\lambda}^{W_d} \dots \bar{\lambda}^{W_0},$$

where $W_i = \{\mathfrak{p} \in W \mid \dim R/\mathfrak{p} = i\}$ and $\bar{\lambda}^{W_i} = \prod_{\mathfrak{p} \in W_i} \Lambda^{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}})$ for $0 \leq i \leq d$. This Čech complex sends a complex X of R -module to a double complex in a natural way. We shall prove that $\lambda^W X$ is isomorphic to the total complex of the double complex if X consists of flat R -modules.

Section 8 treats commutativity of λ^W with tensor products. Consequently, we show that $\lambda^W Y$ can be computed by using the Čech complex above if Y is a complex of finitely generated R -modules.

In Section 9, as an application, we give a functorial way to construct quasi-isomorphisms from complexes of flat R -modules or complexes of finitely generated R -modules to complexes of pure-injective R -modules.

2.2 Localization functors

In this section, we summarize some notions and basic facts used in the later sections.

We write $\text{Mod } R$ for the category of all modules over a commutative Noetherian ring R . For an ideal \mathfrak{a} of R , $\Lambda^{V(\mathfrak{a})}$ denotes the \mathfrak{a} -adic completion functor $\varprojlim(- \otimes_R R/\mathfrak{a}^n)$ defined on $\text{Mod } R$. Moreover, we also denote by $M_{\mathfrak{a}}^{\wedge}$ the \mathfrak{a} -adic completion $\Lambda^{V(\mathfrak{a})}M = \varprojlim M/\mathfrak{a}^n M$ of an R -module M . If the natural map $M \rightarrow M_{\mathfrak{a}}^{\wedge}$ is an isomorphism, then M is called \mathfrak{a} -adically complete. In addition, when R is a local ring with maximal ideal \mathfrak{m} , we simply write \widehat{M} for the \mathfrak{m} -adic completion of M .

We start with the following proposition.

Proposition 2.2.1. *Let \mathfrak{a} be an ideal of R . If F is a flat R -module, then so is $F_{\mathfrak{a}}^{\wedge}$.*

As stated in [42, 2.4], this fact is known. For the reader's convenience, we mention that this proposition follows from the two lemmas below.

Lemma 2.2.2. *Let \mathfrak{a} be an ideal of R and F be a flat R -module. We consider a short exact sequence of finitely generated R -modules*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Then

$$0 \longrightarrow (F \otimes_R L)_{\mathfrak{a}}^{\wedge} \longrightarrow (F \otimes_R M)_{\mathfrak{a}}^{\wedge} \longrightarrow (F \otimes_R N)_{\mathfrak{a}}^{\wedge} \longrightarrow 0$$

is exact.

Lemma 2.2.3. *Let \mathfrak{a} and F be as above. Then we have a natural isomorphism*

$$(F \otimes_R M)_{\mathfrak{a}}^{\wedge} \cong F_{\mathfrak{a}}^{\wedge} \otimes_R M$$

for any finitely generated R -module M .

Using Artin-Rees Lemma and [6, I; §2; Proposition 6], we can prove Lemma 2.2.2, from which we obtain Lemma 2.2.3. Furthermore, Lemma 2.2.2 and Lemma 2.2.3 imply that $F_{\mathfrak{a}}^{\wedge} \otimes_R (-)$ is an exact functor from the category of finitely generated R -modules to $\text{Mod } R$. Therefore Proposition 2.2.1 holds. \square

It is also possible to show that $F_{\mathfrak{a}}^{\wedge}$ is flat over $R_{\mathfrak{a}}^{\wedge}$ by the same argument as above.

If R is a local ring with maximal ideal \mathfrak{m} , then \mathfrak{m} -adically complete flat R -modules are characterized as follows:

Lemma 2.2.4. *Let (R, \mathfrak{m}, k) be a local ring and F a flat R -module. Set $B = \dim_k F/\mathfrak{m}F$. Then there is an isomorphism*

$$\widehat{F} \cong \widehat{\bigoplus_B R},$$

where $\bigoplus_B R$ is the direct sum of B -copies of R .

This lemma is proved in [19, II; Proposition 2.4.3.1]. See also [15, Lemma 6.7.4].

As in the introduction, we denote by $\mathcal{D} = D(\text{Mod } R)$ the derived category of all complexes of R -modules. We write complexes X cohomologically;

$$X = (\cdots \rightarrow X^{i-1} \rightarrow X^i \rightarrow X^{i+1} \rightarrow \cdots).$$

For a complex P (resp. F) of R -modules, we say that P (resp. F) is K -projective (resp. K -flat) if $\text{Hom}_R(P, -)$ (resp. $(-) \otimes_R F$) preserves acyclicity of complexes, where a complex is called acyclic if all its cohomology modules are zero.

Let \mathfrak{a} be an ideal of R and $X \in \mathcal{D}$. If P is a K -projective resolution of X , then we have $\text{L}\Lambda^{V(\mathfrak{a})} X \cong \Lambda^{V(\mathfrak{a})} P$. Moreover, $\text{L}\Lambda^{V(\mathfrak{a})} X$ is also isomorphic to $\Lambda^{V(\mathfrak{a})} F$ if F is a K -flat resolution of X . In addition, it is known that the following proposition holds.

Proposition 2.2.5. *Let \mathfrak{a} be an ideal of R and X be a complex of flat R -modules. Then $\text{L}\Lambda^{V(\mathfrak{a})} X$ is isomorphic to $\Lambda^{V(\mathfrak{a})} X$.*

To prove this proposition, we remark that there is an integer $n \geq 0$ such that $H^i(\text{L}\Lambda^{V(\mathfrak{a})} M) = 0$ for all $i > n$ and all R -modules M , see [21, Theorem 1.9] or [1, p. 15]. Using this fact, we can show that $\Lambda^{V(\mathfrak{a})}$ preserves

acyclicity of complexes of flat R -modules. Then it is straightforward to see that $\mathrm{L}\Lambda^{V(a)} X$ is isomorphic to $\Lambda^{V(a)} X$. \square

Let W be any subset of $\mathrm{Spec} R$. Recall that γ_W denotes a right adjoint to the inclusion functor $i_W : \mathcal{L}_W \hookrightarrow \mathcal{D}$, and λ^W denotes a left adjoint to the inclusion functor $j^W : \mathcal{C}^W \hookrightarrow \mathcal{D}$. Moreover, γ_W and λ^W are identified with $i_W \gamma_W$ and $j^W \lambda^W$ respectively. We write $\varepsilon_W : \gamma_W \rightarrow \mathrm{id}_{\mathcal{D}}$ and $\eta^W : \mathrm{id}_{\mathcal{D}} \rightarrow \lambda^W$ for the natural morphisms induced by the adjointness of (i_W, γ_W) and (λ^W, j^W) respectively.

Note that $\lambda^W \eta^W$ (resp. $\gamma_W \varepsilon_W$) is invertible, and the equality $\lambda^W \eta^W = \eta^W \lambda^W$ (resp. $\gamma_W \varepsilon_W = \varepsilon_W \gamma_W$) holds, *i.e.*, λ^W (resp. γ_W) is a localization (resp. colocalization) functor on \mathcal{D} . See [27] for more details. In this chapter, we call λ^W the localization functor with cosupport in W .

Using (2.1.2), we restate Lemma 1.2.1 as follows.

Lemma 2.2.6. *Let W be a subset of $\mathrm{Spec} R$. For any $X \in \mathcal{D}$, there is a triangle of the following form;*

$$\gamma_{W^c} X \xrightarrow{\varepsilon_{W^c} X} X \xrightarrow{\eta^W X} \lambda^W X \longrightarrow \gamma_{W^c} X[1].$$

Furthermore, if

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow X'[1]$$

is a triangle with $X' \in {}^\perp \mathcal{C}^W = \mathcal{L}_{W^c}$ and $X'' \in \mathcal{C}^W = \mathcal{L}_{W^c}^\perp$, then there exist unique isomorphisms $a : \gamma_{W^c} X \rightarrow X'$ and $b : \lambda^W X \rightarrow X''$ such that the following diagram is commutative:

$$\begin{array}{ccccccc} \gamma_{W^c} X & \xrightarrow{\varepsilon_{W^c} X} & X & \xrightarrow{\eta^W X} & \lambda^W X & \longrightarrow & \gamma_{W^c} X[1] \\ \downarrow a & & \parallel & & \downarrow b & & \downarrow a[1] \\ X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & X'[1] \end{array}$$

Remark 2.2.7. (i) Let $X \in \mathcal{D}$ and W be a subset of $\mathrm{Spec} R$. By Lemma 2.2.6, X belongs to ${}^\perp \mathcal{C}^W = \mathcal{L}_{W^c}$ if and only if $\lambda^W X = 0$. This is equivalent to saying that $\lambda^{\{\mathfrak{p}\}} X = 0$ for all $\mathfrak{p} \in W$, since ${}^\perp \mathcal{C}^W = \mathcal{L}_{W^c} = \bigcap_{\mathfrak{p} \in W} \mathcal{L}_{\{\mathfrak{p}\}^c} = \bigcap_{\mathfrak{p} \in W} {}^\perp \mathcal{C}^{\{\mathfrak{p}\}}$.

(ii) Let $W_0 \subseteq W$ be subsets of $\mathrm{Spec} R$ with $W_0 \subseteq W$. It follows from the uniqueness of adjoint functors that

$$\lambda^{W_0} \lambda^W \cong \lambda^{W_0} \cong \lambda^W \lambda^{W_0},$$

see also Remark 1.3.7 (i).

Now we give a typical example of localization functors. Let S be a multiplicatively closed subset of R , and set $U_S = \{ \mathfrak{p} \in \text{Spec } R \mid \mathfrak{p} \cap S = \emptyset \}$. It is known that the localization functor λ^{U_S} with cosupport in U_S is nothing but $(-) \otimes_R S^{-1}R$. For the reader's convenience, we give a proof of this fact. Let $X \in \mathcal{D}$. It is clear that $\text{cosupp } X \otimes_R S^{-1}R \subseteq U_S$, or equivalently, $X \otimes_R S^{-1}R \in \mathcal{C}^{U_S}$. Moreover, embedding the natural morphism $X \rightarrow X \otimes_R S^{-1}R$ into a triangle

$$C \longrightarrow X \longrightarrow X \otimes_R S^{-1}R \longrightarrow C[1],$$

we have $C \otimes_R S^{-1}R = 0$. This yields an inclusion relation $\text{supp } C \subseteq (U_S)^c$. Hence it holds that $C \in \mathcal{L}_{(U_S)^c}$. Since we have shown that $C \in \mathcal{L}_{(U_S)^c}$ and $X \otimes_R S^{-1}R \in \mathcal{C}^{U_S}$, it follows from Lemma 2.2.6 that $\lambda^{U_S} X \cong X \otimes_R S^{-1}R$. Therefore we obtain the isomorphism

$$(2.2.8) \quad \lambda^{U_S} \cong (-) \otimes_R S^{-1}R.$$

For $\mathfrak{p} \in \text{Spec } R$, we write $U(\mathfrak{p}) = \{ \mathfrak{q} \in \text{Spec } R \mid \mathfrak{q} \subseteq \mathfrak{p} \}$. If $S = R \setminus \mathfrak{p}$, then $U(\mathfrak{p})$ is equal to U_S , so that $\lambda^{U(\mathfrak{p})} \cong (-) \otimes_R R_{\mathfrak{p}}$ by (2.2.8). We remark that $\lambda^{U(\mathfrak{p})} = \lambda_{U(\mathfrak{p})^c}$ is written as $L_{Z(\mathfrak{p})}$ in [3], where $Z(\mathfrak{p}) = U(\mathfrak{p})^c$.

There is another important example of localization functors. Let \mathfrak{a} be an ideal of R . It was proved by [21] and [1] that $\text{L}\Lambda^{V(\mathfrak{a})} : \mathcal{D} \rightarrow \mathcal{D}$ is a right adjoint to $\text{R}\Gamma_{V(\mathfrak{a})} : \mathcal{D} \rightarrow \mathcal{D}$. In Proposition 1.5.1, using the adjointness property of $(\text{R}\Gamma_{V(\mathfrak{a})}, \text{L}\Lambda^{V(\mathfrak{a})})$, we proved that $\lambda^{V(\mathfrak{a})} = \lambda_{V(\mathfrak{a})^c}$ coincides with $\text{L}\Lambda^{V(\mathfrak{a})}$. Hence there is an isomorphism

$$(2.2.9) \quad \lambda^{V(\mathfrak{a})} \cong \text{L}\Lambda^{V(\mathfrak{a})}.$$

The functor $H_i^{\mathfrak{a}}(-) = H^{-i}(\text{L}\Lambda^{V(\mathfrak{a})}(-))$ is called the i th local homology functor with respect to \mathfrak{a} .

A subset W of $\text{Spec } R$ is called specialization-closed (resp. generalization-closed) provided that the following condition holds; if $\mathfrak{p} \in W$ and $\mathfrak{q} \in \text{Spec } R$ with $\mathfrak{p} \subseteq \mathfrak{q}$ (resp. $\mathfrak{p} \supseteq \mathfrak{q}$), then $\mathfrak{q} \in W$.

If V is a specialization-closed subset, then we have

$$(2.2.10) \quad \gamma_V \cong \text{R}\Gamma_V,$$

see [28, Appendix 3.5].

2.3 Auxiliary results on localization functors

In this section, we give several results to compute localization functors λ^W with cosupports in arbitrary subsets W of $\text{Spec } R$.

We first give the following lemma.

Lemma 2.3.1. *Let V be a specialization-closed subset of $\text{Spec } R$. Then we have the following equalities;*

$${}^\perp\mathcal{C}^V = \mathcal{L}_{V^c} = \mathcal{L}_V^\perp = \mathcal{C}^{V^c}.$$

Proof. This follows from Lemma 1.4.3 and (2.1.2). \square

Let W_0 and W be subsets of $\text{Spec } R$ with $W_0 \subseteq W$. We remark that W_0 is specialization-closed in W if and only if $\overline{W_0^s} \cap W = W_0$.

Corollary 2.3.2. *Let $W_0 \subseteq W \subseteq \text{Spec } R$ be sets. Suppose that W_0 is specialization-closed in W . Setting $W_1 = W \setminus W_0$, we have $\mathcal{C}^{W_1} \subseteq {}^\perp\mathcal{C}^{W_0}$.*

Proof. Notice that $W_1 \subseteq (\overline{W_0^s})^c$. Furthermore, we have ${}^\perp\mathcal{C}^{\overline{W_0^s}} = \mathcal{C}^{(\overline{W_0^s})^c}$ by Lemma 2.3.1. Hence it holds that $\mathcal{C}^{W_1} \subseteq \mathcal{C}^{(\overline{W_0^s})^c} = {}^\perp\mathcal{C}^{\overline{W_0^s}} \subseteq {}^\perp\mathcal{C}^{W_0}$. \square

Remark 2.3.3. For an ideal \mathfrak{a} of R , $\lambda^{V(\mathfrak{a})}$ is a right adjoint to $\gamma_{V(\mathfrak{a})}$ by (2.2.9) and (2.2.10). More generally, it is known that for any specialization-closed subset V , $\lambda^V : \mathcal{D} \rightarrow \mathcal{D}$ is a right adjoint to $\gamma_V : \mathcal{D} \rightarrow \mathcal{D}$. We now prove this fact, which will be used in the next proposition. Let $X, Y \in \mathcal{D}$, and consider the following triangles;

$$\begin{array}{ccccccc} \gamma_V X & \longrightarrow & X & \longrightarrow & \lambda^{V^c} X & \longrightarrow & \gamma_V X[1], \\ \gamma_{V^c} Y & \longrightarrow & Y & \longrightarrow & \lambda^V Y & \longrightarrow & \gamma_{V^c} Y[1]. \end{array}$$

Since $\lambda^{V^c} X \in \mathcal{C}^{V^c} = {}^\perp\mathcal{C}^V$ by Lemma 2.3.1, applying $\text{Hom}_{\mathcal{D}}(-, \lambda^V Y)$ to the first triangle, we have $\text{Hom}_{\mathcal{D}}(\gamma_V X, \lambda^V Y) \cong \text{Hom}_{\mathcal{D}}(X, \lambda^V Y)$. Moreover, Lemma 2.3.1 implies that $\gamma_{V^c} Y \in \mathcal{L}_{V^c} = \mathcal{L}_V^\perp$. Thus, applying $\text{Hom}_{\mathcal{D}}(\gamma_V X, -)$ to the second triangle, we have $\text{Hom}_{\mathcal{D}}(\gamma_V X, Y) \cong \text{Hom}_{\mathcal{D}}(\gamma_V X, \lambda^V Y)$. Thus there is a natural isomorphism $\text{Hom}_{\mathcal{D}}(\gamma_V X, Y) \cong \text{Hom}_{\mathcal{D}}(X, \lambda^V Y)$, so that (γ_V, λ^V) is an adjoint pair. See also Remark 1.5.2.

Proposition 2.3.4. *Let V and U be arbitrary subsets of $\text{Spec } R$. Suppose that one of the following conditions holds:*

- (1) V is specialization-closed;

(2) U is generalization-closed.

Then we have an isomorphism

$$\lambda^V \lambda^U \cong \lambda^{V \cap U}.$$

Proof. Let $X \in \mathcal{D}$ and $Y \in \mathcal{C}^{V \cap U} = \mathcal{C}^V \cap \mathcal{C}^U$. Then there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(\lambda^V \lambda^U X, Y) \cong \mathrm{Hom}_{\mathcal{D}}(\lambda^U X, Y) \cong \mathrm{Hom}_{\mathcal{D}}(X, Y).$$

Recall that $\lambda^{V \cap U}$ is a left adjoint to the inclusion functor $\mathcal{C}^{V \cap U} \hookrightarrow \mathcal{D}$. Hence, by the uniqueness of adjoint functors, we only have to verify that $\lambda^V \lambda^U X \in \mathcal{C}^{V \cap U}$. Since $\lambda^V \lambda^U X \in \mathcal{C}^V$, it remains to show that $\lambda^V \lambda^U X \in \mathcal{C}^U$.

Case (1): Let $\mathfrak{p} \in U^c$. Since $\mathrm{supp} \gamma_V \kappa(\mathfrak{p}) \subseteq \{\mathfrak{p}\}$, it follows from (2.1.2) that $\gamma_V \kappa(\mathfrak{p}) \in \mathcal{L}_{U^c} = {}^\perp \mathcal{C}^U$. Thus, by the adjointness of (γ_V, λ^V) , we have

$$\mathrm{RHom}_R(\kappa(\mathfrak{p}), \lambda^V \lambda^U X) \cong \mathrm{RHom}_R(\gamma_V \kappa(\mathfrak{p}), \lambda^U X) = 0.$$

This implies that $\mathrm{cosupp} \lambda^V \lambda^U X \subseteq U$, *i.e.*, $\lambda^V \lambda^U X \in \mathcal{C}^U$.

Case (2): Since U^c is specialization-closed, the case (1) yields an isomorphism $\lambda^{U^c} \lambda^V \cong \lambda^{U^c \cap V}$. Furthermore, setting $W = (U^c \cap V) \cup U$, we see that $U^c \cap V$ is specialization-closed in W , and $W \setminus (U^c \cap V) = U$. Hence we have $\lambda^{U^c}(\lambda^V \lambda^U X) \cong \lambda^{U^c \cap V} \lambda^U X = 0$, by Corollary 2.3.2. It then follows from Lemma 2.3.1 that $\lambda^V \lambda^U X \in {}^\perp \mathcal{C}^{U^c} = \mathcal{C}^U$. \square

Remark 2.3.5. For arbitrary subsets W_0 and W_1 of $\mathrm{Spec} R$, Remark 2.2.7 (ii) and Proposition 2.3.4 yield the following isomorphisms;

$$\begin{aligned} \lambda^{W_0} \lambda^{W_1} &\cong \lambda^{W_0} \lambda^{\overline{W_0^s}} \lambda^{W_1} \cong \lambda^{W_0} \lambda^{\overline{W_0^s} \cap W_1}, \\ \lambda^{W_0} \lambda^{W_1} &\cong \lambda^{W_0} \lambda^{\overline{W_1^g}} \lambda^{W_1} \cong \lambda^{W_0 \cap \overline{W_1^g}} \lambda^{W_1}. \end{aligned}$$

The next result is a corollary of (2.2.8), (2.2.9) and Proposition 2.3.4.

Corollary 2.3.6. *Let S be a multiplicatively closed subset of R and \mathfrak{a} be an ideal of R . We set $W = V(\mathfrak{a}) \cap U_S$. Then we have*

$$\lambda^W \cong \mathrm{L}\Lambda^{V(\mathfrak{a})}(- \otimes_R S^{-1}R).$$

Since $V(\mathfrak{p}) \cap U(\mathfrak{p}) = \{\mathfrak{p}\}$ for $\mathfrak{p} \in \mathrm{Spec} R$, as a special case of this corollary, we have the following result.

Corollary 2.3.7. *Let \mathfrak{p} be a prime ideal of R . Then we have*

$$\lambda^{\{\mathfrak{p}\}} \cong \mathrm{L}\Lambda^{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}}).$$

The next lemma follows from this corollary and Lemma 2.2.4.

Lemma 2.3.8. *Let \mathfrak{p} be a prime ideal of R and F be a flat R -module. Then $\lambda^{\{\mathfrak{p}\}}F$ is isomorphic to $(\bigoplus_B R_{\mathfrak{p}})_{\mathfrak{p}}^{\wedge}$, where $\bigoplus_B R_{\mathfrak{p}}$ is the direct sum of B -copies of $R_{\mathfrak{p}}$ and $B = \dim_{\kappa(\mathfrak{p})} F \otimes_R \kappa(\mathfrak{p})$.*

Remark 2.3.9. Let W_1 and W_2 be subsets of $\mathrm{Spec} R$. In general, $\lambda^{W_1}\lambda^{W_2}$ need not be isomorphic to $\lambda^{W_2}\lambda^{W_1}$. For example, let $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec} R$ with $\mathfrak{p} \subsetneq \mathfrak{q}$. Then $(\lambda^{\{\mathfrak{p}\}}R) \otimes_R \kappa(\mathfrak{q}) = \widehat{R}_{\mathfrak{p}} \otimes_R \kappa(\mathfrak{q}) = 0$ and $(\lambda^{\{\mathfrak{q}\}}R) \otimes_R \kappa(\mathfrak{p}) = \widehat{R}_{\mathfrak{q}} \otimes_R \kappa(\mathfrak{p}) \neq 0$. Then we see from Lemma 2.3.8 that $\lambda^{\{\mathfrak{q}\}}\lambda^{\{\mathfrak{p}\}}R = 0$ and $\lambda^{\{\mathfrak{p}\}}\lambda^{\{\mathfrak{q}\}}R \neq 0$.

Compare this remark with [3, Example 3.5]. See also Remark 1.3.7 (ii).

Let \mathfrak{p} be a prime ideal which is not maximal. Then $\lambda^{\{\mathfrak{p}\}}$ is distinct from $\Lambda^{\mathfrak{p}} = \mathrm{L}\Lambda^{V(\mathfrak{p})} \mathrm{RHom}_R(R_{\mathfrak{p}}, -)$, which is introduced in [4]. To see this, let \mathfrak{q} be a prime ideal with $\mathfrak{p} \subsetneq \mathfrak{q}$. Then it holds that $\mathrm{cosupp} \widehat{R}_{\mathfrak{q}} = \{\mathfrak{q}\} \subseteq U(\mathfrak{p})^c$. Hence $\widehat{R}_{\mathfrak{q}}$ belongs to $\mathcal{C}^{U(\mathfrak{p})^c}$. Then we have $\mathrm{RHom}_R(R_{\mathfrak{p}}, \widehat{R}_{\mathfrak{q}}) = 0$ since $R_{\mathfrak{p}} \in \mathcal{L}_{U(\mathfrak{p})} = {}^{\perp}\mathcal{C}^{U(\mathfrak{p})^c}$ by (2.1.2). This implies that $\Lambda^{\mathfrak{p}}\widehat{R}_{\mathfrak{q}} = \mathrm{L}\Lambda^{V(\mathfrak{p})} \mathrm{RHom}_R(R_{\mathfrak{p}}, \widehat{R}_{\mathfrak{q}}) = 0$, while $\lambda^{\{\mathfrak{p}\}}\widehat{R}_{\mathfrak{q}} \cong \lambda^{\{\mathfrak{p}\}}\lambda^{\{\mathfrak{q}\}}R \neq 0$ by Remark 2.3.9.

Theorem 2.3.10. *Let W be a subset of $\mathrm{Spec} R$. We assume that $\dim W = 0$. Then there are isomorphisms*

$$\lambda^W \cong \prod_{\mathfrak{p} \in W} \lambda^{\{\mathfrak{p}\}} \cong \prod_{\mathfrak{p} \in W} \mathrm{L}\Lambda^{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}}).$$

Proof. Let $X \in \mathcal{D}$, and consider the natural morphisms $\eta^{\{\mathfrak{p}\}}X : X \rightarrow \lambda^{\{\mathfrak{p}\}}X$ for $\mathfrak{p} \in W$. Take the product of the morphisms, and we obtain a morphism $f : X \rightarrow \prod_{\mathfrak{p} \in W} \lambda^{\{\mathfrak{p}\}}X$. Embed f into a triangle

$$C \longrightarrow X \xrightarrow{f} \prod_{\mathfrak{p} \in W} \lambda^{\{\mathfrak{p}\}}X \longrightarrow C[1].$$

Note that $\prod_{\mathfrak{p} \in W} \lambda^{\{\mathfrak{p}\}}X \in \mathcal{C}^W$. We have to prove that $C \in {}^{\perp}\mathcal{C}^W$. For this purpose, take any prime ideal $\mathfrak{q} \in W$. Then $\{\mathfrak{q}\}$ is specialization-closed in W , because $\dim W = 0$. Hence we have $\prod_{\mathfrak{p} \in W \setminus \{\mathfrak{q}\}} \lambda^{\{\mathfrak{p}\}}X \in \mathcal{C}^{W \setminus \{\mathfrak{q}\}} \subseteq {}^{\perp}\mathcal{C}^{\{\mathfrak{q}\}}$, by Corollary 2.3.2. Thus an isomorphism $\lambda^{\{\mathfrak{q}\}}(\prod_{\mathfrak{p} \in W} \lambda^{\{\mathfrak{p}\}}X) \cong \lambda^{\{\mathfrak{q}\}}X$ holds. Then it is seen from the triangle above that $\lambda^{\{\mathfrak{q}\}}C = 0$ for all $\mathfrak{q} \in W$, so that $C \in {}^{\perp}\mathcal{C}^W$, see Remark 2.2.7 (i). Therefore Lemma 2.2.6 yields $\lambda^W X \cong \prod_{\mathfrak{p} \in W} \lambda^{\{\mathfrak{p}\}}X$. The second isomorphism in the theorem follows from Corollary 2.3.7. \square

Example 2.3.11. Let W be a subset of $\text{Spec } R$ such that W is an infinite set with $\dim W = 0$. Let $X^{\{\mathfrak{p}\}}$ be a complex with $\text{cosupp } X^{\{\mathfrak{p}\}} = \{\mathfrak{p}\}$ for each $\mathfrak{p} \in W$. We take $\mathfrak{p} \in W$. Since $\dim W = 0$, it holds that $X^{\{\mathfrak{q}\}} \in \mathcal{C}^{V(\mathfrak{p})^c}$ for any $\mathfrak{q} \in W \setminus \{\mathfrak{p}\}$. Furthermore, Lemma 2.3.1 implies that $\mathcal{C}^{V(\mathfrak{p})^c}$ is equal to ${}^\perp\mathcal{C}^{V(\mathfrak{p})}$, which is closed under arbitrary direct sums. Thus it holds that $\bigoplus_{\mathfrak{q} \in W \setminus \{\mathfrak{p}\}} X^{\{\mathfrak{q}\}} \in \mathcal{C}^{V(\mathfrak{p})^c} = {}^\perp\mathcal{C}^{V(\mathfrak{p})} \subseteq {}^\perp\mathcal{C}^{\{\mathfrak{p}\}}$. Therefore, setting $Y = \bigoplus_{\mathfrak{p} \in W} X^{\{\mathfrak{p}\}}$, we have $\lambda^{\{\mathfrak{p}\}} Y \cong X^{\{\mathfrak{p}\}}$. It then follows from Theorem 2.3.10 that

$$\lambda^W Y \cong \prod_{\mathfrak{p} \in W} \lambda^{\{\mathfrak{p}\}} Y \cong \prod_{\mathfrak{p} \in W} X^{\{\mathfrak{p}\}}.$$

Under this identification, the natural morphism $Y \rightarrow \lambda^W Y$ coincides with the canonical morphism $\bigoplus_{\mathfrak{p} \in W} X^{\{\mathfrak{p}\}} \rightarrow \prod_{\mathfrak{p} \in W} X^{\{\mathfrak{p}\}}$.

Remark 2.3.12. Let $W, X^{\{\mathfrak{p}\}}$ be as in Example 2.3.11, and suppose that each $X^{\{\mathfrak{p}\}}$ is an R -module. Then $\bigoplus_{\mathfrak{p} \in W} X^{\{\mathfrak{p}\}}$ is not in \mathcal{C}^W , because the natural morphism $\bigoplus_{\mathfrak{p} \in W} X^{\{\mathfrak{p}\}} \rightarrow \lambda^W(\bigoplus_{\mathfrak{p} \in W} X^{\{\mathfrak{p}\}})$ is not an isomorphism. Hence the cosupport of $\bigoplus_{\mathfrak{p} \in W} X^{\{\mathfrak{p}\}}$ properly contains W . In particular, setting $X^{\{\mathfrak{p}\}} = \kappa(\mathfrak{p})$, we have $W \subsetneq \text{cosupp } \bigoplus_{\mathfrak{p} \in W} \kappa(\mathfrak{p})$. Similarly, we can prove that $W \subsetneq \text{supp } \prod_{\mathfrak{p} \in W} \kappa(\mathfrak{p})$. The first author noticed these facts through discussion with Srikanth Iyengar.

It is possible to give another type of examples, by which we also see that a colocalizing subcategory of \mathcal{D} is not necessarily closed under arbitrary direct sums. Suppose that (R, \mathfrak{m}) is a complete local ring with $\dim R \geq 1$. Then we have $R \cong \widehat{R} \in \mathcal{C}^{V(\mathfrak{m})}$. However the free module $\bigoplus_{\mathbb{N}} R$ is never \mathfrak{m} -adically complete, so that $\bigoplus_{\mathbb{N}} R$ is not isomorphic to $\lambda^{V(\mathfrak{m})}(\bigoplus_{\mathbb{N}} R)$. Hence $\bigoplus_{\mathbb{N}} R$ is not in $\mathcal{C}^{V(\mathfrak{m})}$.

For a subset W of $\text{Spec } R$, \overline{W}^g denotes the generalization closure of W , which is the smallest generalization-closed subset of $\text{Spec } R$ containing W . In addition, for a subset $W_1 \subseteq W$, we say that W_1 is generalization-closed in W if $W \cap U(\mathfrak{q}) \subseteq W_1$ for any $\mathfrak{q} \in W_1$. This is equivalent to saying that $W \cap \overline{W_1}^g = W_1$.

We extend Proposition 2.3.4 to the following corollary, which will be used in a main theorem of this section.

Corollary 2.3.13. *Let W_0 and W_1 be arbitrary subsets of $\text{Spec } R$. Suppose that one of the following conditions hold:*

- (1) W_0 is specialization-closed in $W_0 \cup W_1$;
- (2) W_1 is generalization-closed in $W_0 \cup W_1$.

Then we have an isomorphism

$$\lambda^{W_0}\lambda^{W_1} \cong \lambda^{W_0 \cap W_1}.$$

Proof. Set $W = W_0 \cup W_1$. By the assumption, we have

$$\overline{W_0}^s \cap W = W_0 \quad \text{or} \quad W \cap \overline{W_1}^g = W_1.$$

Therefore, it holds that

$$\overline{W_0}^s \cap W_1 = W_0 \cap W_1 \quad \text{or} \quad W_0 \cap \overline{W_1}^g = W_0 \cap W_1.$$

Hence this proposition follows from Remark 2.3.5 and Remark 2.2.7 (ii). \square

Remark 2.3.14. (i) Let W_0 and W be subsets of $\text{Spec } R$ with $W_0 \subseteq W$. Under the isomorphism $\lambda^{W_0}\lambda^W \cong \lambda^{W_0}$ by Remark 2.2.7 (ii), there is a morphism $\eta^{W_0}\lambda^W : \lambda^W \rightarrow \lambda^{W_0}$.

(ii) Let W_0 and W_1 be subsets of $\text{Spec } R$. Let $X \in \mathcal{D}$. Since $\eta^{W_1} : \text{id}_{\mathcal{D}} \rightarrow \lambda^{W_1}$ is a morphism of functors, there is a commutative diagram of the following form:

$$\begin{array}{ccc} X & \xrightarrow{\eta^{W_0}X} & \lambda^{W_0}X \\ \downarrow \eta^{W_1}X & & \downarrow \eta^{W_1}\lambda^{W_0}X \\ \lambda^{W_1}X & \xrightarrow{\lambda^{W_1}\eta^{W_0}X} & \lambda^{W_1}\lambda^{W_0}X \end{array}$$

Now we prove the following result, which is the main theorem of this section.

Theorem 2.3.15. *Let W , W_0 and W_1 be subsets of $\text{Spec } R$ with $W = W_0 \cup W_1$. Suppose that one of the following conditions holds:*

- (1) W_0 is specialization-closed in W ;
- (2) W_1 is generalization-closed in W .

Then, for any $X \in \mathcal{D}$, there is a triangle of the following form;

$$\lambda^W X \xrightarrow{f} \lambda^{W_1} X \oplus \lambda^{W_0} X \xrightarrow{g} \lambda^{W_1}\lambda^{W_0} X \longrightarrow \lambda^W X[1],$$

where f and g are morphisms represented by the following matrices;

$$f = \begin{pmatrix} \eta^{W_1}\lambda^W X \\ \eta^{W_0}\lambda^W X \end{pmatrix}, \quad g = \left(\lambda^{W_1}\eta^{W_0}X \quad (-1) \cdot \eta^{W_1}\lambda^{W_0}X \right).$$

Proof. We embed the morphism g into a triangle

$$C \xrightarrow{a} \lambda^{W_1} X \oplus \lambda^{W_0} X \xrightarrow{g} \lambda^{W_1} \lambda^{W_0} X \longrightarrow C[1].$$

Notice that $C \in \mathcal{C}^W$ since $\mathcal{C}^{W_0}, \mathcal{C}^{W_1} \subseteq \mathcal{C}^W$. By Remark 2.3.14, it is easily seen that $g \cdot f = 0$. Thus there is a morphism $b : \lambda^W X \rightarrow C$ making the following diagram commutative:

$$(2.3.16) \quad \begin{array}{ccccccc} \lambda^W X & \xlongequal{\quad} & \lambda^W X & \longrightarrow & 0 & \longrightarrow & \lambda^W X[1] \\ \downarrow b & & \downarrow f & & \downarrow & & \downarrow b[1] \\ C & \xrightarrow{a} & \lambda^{W_1} X \oplus \lambda^{W_0} X & \xrightarrow{g} & \lambda^{W_1} \lambda^{W_0} X & \longrightarrow & C[1] \end{array}$$

We only have to show that b is an isomorphism. To do this, embedding the morphism b into a triangle

$$(2.3.17) \quad Z \longrightarrow \lambda^W X \xrightarrow{b} C \longrightarrow Z[1],$$

we prove that $Z = 0$. Since $\lambda^W X, C \in \mathcal{C}^W$, Z belongs to \mathcal{C}^W . Hence it suffices to show that $Z \in {}^\perp \mathcal{C}^W$.

First, we prove that $\lambda^{W_1} b$ is an isomorphism. We employ a similar argument to [3, Theorem 7.5]. Consider the following sequence

$$(2.3.18) \quad \lambda^W X \xrightarrow{f} \lambda^{W_1} X \oplus \lambda^{W_0} X \xrightarrow{g} \lambda^{W_1} \lambda^{W_0} X,$$

and apply λ^{W_1} to it. Then we obtain a sequence which can be completed to a split triangle. The triangle appears in the first row of the diagram below. Moreover, λ^{W_1} sends the second row of the diagram (2.3.16) to a split triangle, which appears in the second row of the the diagram below.

$$\begin{array}{ccccccc} \lambda^{W_1} X & \xrightarrow{\lambda^{W_1} f} & \lambda^{W_1} X \oplus \lambda^{W_1} \lambda^{W_0} X & \xrightarrow{\lambda^{W_1} g} & \lambda^{W_1} \lambda^{W_0} X & \xrightarrow{0} & \lambda^{W_1} X[1] \\ \downarrow \lambda^{W_1} b & & \parallel & & \parallel & & \downarrow \lambda^{W_1} b[1] \\ \lambda^{W_1} C & \xrightarrow{\lambda^{W_1} a} & \lambda^{W_1} X \oplus \lambda^{W_1} \lambda^{W_0} X & \xrightarrow{\lambda^{W_1} g} & \lambda^{W_1} \lambda^{W_0} X & \xrightarrow{0} & \lambda^{W_1} C[1] \end{array}$$

Since this diagram is commutative, we conclude that $\lambda^{W_1} b$ is an isomorphism.

Next, we prove that $\lambda^{W_0} b$ is an isomorphism. Thanks to Corollary 2.3.13, we are able to follow the same process as above. In fact, the corollary implies that $\lambda^{W_0} \lambda^{W_1} \cong \lambda^{W_0 \cap W_1}$. Thus, applying λ^{W_0} to the sequence (2.3.18), we obtain a sequence which can be completed into a split triangle. Furthermore, λ^{W_0} sends the second row of the diagram (2.3.16) to a split triangle.

Consequently we see that there is a morphism of triangles:

$$\begin{array}{ccccccc}
\lambda^{W_0} X & \xrightarrow{\lambda^{W_0} f} & \lambda^{W_0 \cap W_1} X \oplus \lambda^{W_0} X & \xrightarrow{\lambda^{W_0} g} & \lambda^{W_0 \cap W_1} X & \xrightarrow{0} & \lambda^{W_0} X[1] \\
\downarrow \lambda^{W_0} b & & \parallel & & \parallel & & \downarrow \lambda^{W_0} b[1] \\
\lambda^{W_0} C & \xrightarrow{\lambda^{W_0} a} & \lambda^{W_0 \cap W_1} X \oplus \lambda^{W_0} X & \xrightarrow{\lambda^{W_0} g} & \lambda^{W_0 \cap W_1} X & \xrightarrow{0} & \lambda^{W_0} C[1]
\end{array}$$

Therefore $\lambda^{W_0} b$ is an isomorphism.

Since we have shown that $\lambda^{W_0} b$ and $\lambda^{W_1} b$ are isomorphisms, it follows from the triangle (2.3.17) that $\lambda^{W_0} Z = \lambda^{W_1} Z = 0$. Thus we have $Z \in {}^\perp \mathcal{C}^W$ by Remark 2.2.7 (i). \square

Remark 2.3.19. Let f, g and a as above. Let $h : X \rightarrow \lambda^{W_1} X \oplus \lambda^{W_0} X$ be a morphism induced by $\eta^{W_1} X$ and $\eta^{W_0} X$. Then $g \cdot h = 0$ by Remark 2.3.14 (ii). Hence there is a morphism $b' : X \rightarrow C$ such that the following diagram commutative:

$$\begin{array}{ccccccc}
X & \xlongequal{\quad} & X & \longrightarrow & 0 & \longrightarrow & X[1] \\
\downarrow b' & & \downarrow h & & \downarrow & & \downarrow b'[1] \\
C & \xrightarrow{a} & \lambda^{W_1} X \oplus \lambda^{W_0} X & \xrightarrow{g} & \lambda^{W_1} \lambda^{W_0} X & \longrightarrow & C[1].
\end{array}$$

We can regard any morphism b' making this diagram commutative as the natural morphism $\eta^W X$. In fact, since $\lambda^W h = f$, applying λ^W to this diagram, and setting $\lambda^W b' = b$, we obtain the diagram (2.3.16). Note that $b \cdot \eta^W X = b'$. Moreover, the above proof implies that $b : \lambda^W X \rightarrow C$ is an isomorphism. Thus we can identify b' with $\eta^W X$ under the isomorphism b .

We give some examples of Theorem 2.3.15.

Example 2.3.20. (1) Let x be an element of R . Recall that $\lambda^{V(x)} \cong \text{L}\Lambda^{V(x)}$ by (2.2.9). We put $S = \{x^n \mid n \geq 0\}$. Since $V(x)^c = U_S$, it holds that $\lambda^{V(x)^c} = \lambda^{U_S} \cong (-) \otimes_R R_x$ by (2.2.8). Set $W = \text{Spec } R$, $W_0 = V(x)$ and $W_1 = V(x)^c$. Then the theorem yields the following triangle

$$R \longrightarrow R_x \oplus R_{(x)}^\wedge \longrightarrow (R_{(x)}^\wedge)_x \longrightarrow R[1].$$

(2) Suppose that (R, \mathfrak{m}) is a local ring with $\mathfrak{p} \in \text{Spec } R$ and having $\dim R/\mathfrak{p} = 1$. Setting $W = V(\mathfrak{p})$, $W_0 = V(\mathfrak{m})$ and $W_1 = \{\mathfrak{p}\}$, we see from the theorem and Corollary 2.3.7 that there is a short exact sequence

$$0 \longrightarrow R_{\mathfrak{p}}^\wedge \longrightarrow \widehat{R}_{\mathfrak{p}} \oplus \widehat{R} \longrightarrow (\widehat{R})_{\mathfrak{p}} \longrightarrow 0.$$

Actually, this gives a pure-injective resolution of $R_{\mathfrak{p}}^{\wedge}$, see Section 9. Moreover, if R is a 1-dimensional local domain with quotient field Q , then this short exact sequence is of the form

$$0 \longrightarrow R \longrightarrow Q \oplus \widehat{R} \longrightarrow \widehat{R} \otimes_R Q \longrightarrow 0.$$

By similar arguments to Proposition 2.3.4 and Corollary 2.3.13, one can prove the following proposition, which is a generalized form of Proposition 1.3.1.

Proposition 2.3.21. *Let W_0 and W_1 be arbitrary subsets of $\text{Spec } R$. Suppose that one of the following conditions hold:*

- (1) W_0 is specialization-closed in $W_0 \cup W_1$;
- (2) W_1 is generalization-closed in $W_0 \cup W_1$.

Then we have an isomorphism

$$\gamma_{W_0} \gamma_{W_1} \cong \gamma_{W_0 \cap W_1}.$$

As with Theorem 2.3.15, it is possible to prove the following theorem, in which we implicitly use the fact that $\gamma_{W_0} \gamma_W \cong \gamma_{W_0}$ if $W_0 \subseteq W$ (cf. Remark 1.3.7 (i)).

Theorem 2.3.22. *Let W , W_0 and W_1 be subsets of $\text{Spec } R$ with $W = W_0 \cup W_1$. Suppose that one of the following conditions holds:*

- (1) W_0 is specialization-closed in W ;
- (2) W_1 is generalization-closed in W .

Then, for any $X \in \mathcal{D}$, there is a triangle of the following form;

$$\gamma_{W_1} \gamma_{W_0} X \xrightarrow{f} \gamma_{W_1} X \oplus \gamma_{W_0} X \xrightarrow{g} \gamma_W X \longrightarrow \gamma_{W_1} \gamma_{W_0} X[1],$$

where f and g are the morphisms represented by the following matrices;

$$f = \begin{pmatrix} \gamma_{W_1} \varepsilon_{W_0} X \\ (-1) \cdot \varepsilon_{W_1} \gamma_{W_0} X \end{pmatrix}, \quad g = \begin{pmatrix} \varepsilon_{W_1} \gamma_W X & \varepsilon_{W_0} \gamma_W X \end{pmatrix}.$$

Remark 2.3.23. As long as we work on the derived category \mathcal{D} , Theorem 2.3.15 and Theorem 2.3.22 generalize Mayer-Vietoris triangles in the sense of Benson, Iyengar and Krause [3, Theorem 7.5], in which γ_V and λ_V are written as Γ_V and L_V respectively for a specialization-closed subset V of $\text{Spec } R$.

2.4 Projective dimension of flat modules

As an application of results in Section 3, we give a simpler proof of a classical theorem due to Gruson and Raynaud.

Theorem 2.4.1 ([19, II; Corollary 3.2.7]). *Let F be a flat R -module. Then the projective dimension of F is at most $\dim R$.*

We start by showing the following lemma.

Lemma 2.4.2. *Let F be a flat R -module and \mathfrak{p} be a prime ideal of R . Suppose that $X \in \mathcal{C}^{\{\mathfrak{p}\}}$. Then there is an isomorphism*

$$\mathrm{RHom}_R(F, X) \cong \prod_B X,$$

where $B = \dim_{\kappa(\mathfrak{p})} F \otimes_R \kappa(\mathfrak{p})$.

Proof. Since $\lambda^{\{\mathfrak{p}\}} : \mathcal{D} \rightarrow \mathcal{C}^{\{\mathfrak{p}\}}$ is a left adjoint to the inclusion functor $\mathcal{C}^{\{\mathfrak{p}\}} \hookrightarrow \mathcal{D}$, we have $\mathrm{RHom}_R(F, X) \cong \mathrm{RHom}_R(\lambda^{\{\mathfrak{p}\}} F, X)$. Moreover it follows from Lemma 2.3.8 that $\lambda^{\{\mathfrak{p}\}} F \cong (\bigoplus_B R_{\mathfrak{p}})_{\mathfrak{p}}^{\wedge} \cong \lambda^{\{\mathfrak{p}\}}(\bigoplus_B R)$, where $B = \dim_{\kappa(\mathfrak{p})} F \otimes_R \kappa(\mathfrak{p})$. Therefore we obtain isomorphisms $\mathrm{RHom}_R(F, X) \cong \mathrm{RHom}_R(\lambda^{\{\mathfrak{p}\}}(\bigoplus_B R), X) \cong \mathrm{RHom}_R(\bigoplus_B R, X) \cong \prod_B X$. \square

Let $a, b \in \mathbb{Z} \cup \{\pm\infty\}$ with $a \leq b$. We write $\mathcal{D}^{[a, b]}$ for the full subcategory of \mathcal{D} consisting of all complexes X of R -modules such that $H^i(X) = 0$ for $i \notin [a, b]$ (cf. [26, Notation 13.1.11]). For a subset W of $\mathrm{Spec} R$, $\max W$ denotes the set of prime ideals $\mathfrak{p} \in W$ which are maximal with respect to inclusion in W .

Proposition 2.4.3. *Let F be a flat R -module and $X \in \mathcal{D}^{[-\infty, 0]}$. Suppose that W is a subset of $\mathrm{Spec} R$ such that $n = \dim W$ is finite. Then we have $\mathrm{Ext}_R^i(F, \lambda^W X) = 0$ for $i > n$.*

Proof. We use induction on n . First, we suppose that $n = 0$. It then holds that $\lambda^W X \cong \prod_{\mathfrak{p} \in W} \lambda^{\{\mathfrak{p}\}} X \cong \prod_{\mathfrak{p} \in W} \mathrm{L}\Lambda^{V(\mathfrak{p})} X_{\mathfrak{p}} \in \mathcal{D}^{[-\infty, 0]}$, by Theorem 2.3.10. Hence, noting that $\mathrm{RHom}_R(F, \lambda^W X) \cong \prod_{\mathfrak{p} \in W} \mathrm{RHom}_R(F, \lambda^{\{\mathfrak{p}\}} X)$, we have $\mathrm{Ext}_R^i(F, \lambda^W X) = 0$ for $i > 0$, by Lemma 2.4.2.

Next, we suppose $n > 0$. Set $W_0 = \max W$ and $W_1 = W \setminus W_0$. By Theorem 2.3.15, there is a triangle

$$\lambda^W X \longrightarrow \lambda^{W_1} X \oplus \lambda^{W_0} X \longrightarrow \lambda^{W_1} \lambda^{W_0} X \longrightarrow \lambda^W X[1].$$

Note that $\dim W_0 = 0$ and $\dim W_1 = n - 1$. By the argument above, it holds that $\text{Ext}_R^i(F, \lambda^{W_0} X) = 0$ for $i > 0$. Furthermore, since $X, \lambda^{W_0} X \in \mathcal{D}^{[-\infty, 0]}$, we have $\text{Ext}_R^i(F, \lambda^{W_1} X) = \text{Ext}_R^i(F, \lambda^{W_1} \lambda^{W_0} X) = 0$ for $i > n - 1$, by the inductive hypothesis. Hence it is seen from the triangle that $\text{Ext}_R^i(F, \lambda^W X) = 0$ for $i > n$. \square

Proof of Theorem 2.4.1. We may assume that $d = \dim R$ is finite. Let M be any R -module. We only have to show that $\text{Ext}_R^i(F, M) = 0$ for $i > d$. Setting $W = \text{Spec } R$, we have $\dim W = d$ and $M \cong \lambda^W M$. It then follows from Proposition 2.4.3 that $\text{Ext}_R^i(F, M) \cong \text{Ext}_R^i(F, \lambda^W M) = 0$ for $i > d$. \square

2.5 Cotorsion flat modules and cosupport

In this section, we summarize some basic facts about cotorsion flat R -modules.

Recall that an R -module M is called cotorsion if $\text{Ext}_R^1(F, M) = 0$ for any flat R -module F . This is equivalent to saying that $\text{Ext}_R^i(F, M) = 0$ for any flat R -module F and any $i > 0$. Clearly, all injective R -modules are cotorsion.

A cotorsion flat R -module means an R -module which is cotorsion and flat. If F is a flat R -module and $\mathfrak{p} \in \text{Spec } R$, then Corollary 2.3.7 implies that $\lambda^{\{\mathfrak{p}\}} F$ is isomorphic to $\widehat{F}_{\mathfrak{p}}$, which is a cotorsion flat R -module by Lemma 2.4.2 and Proposition 2.2.1. Moreover, recall that $\widehat{F}_{\mathfrak{p}}$ is isomorphic to the \mathfrak{p} -adic completion of a free $R_{\mathfrak{p}}$ -module by Lemma 2.3.8.

We remark that arbitrary direct products of flat R -modules are flat, since R is Noetherian. Hence, if $T_{\mathfrak{p}}$ is the \mathfrak{p} -adic completion of a free $R_{\mathfrak{p}}$ module for each $\mathfrak{p} \in \text{Spec } R$, then $\prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}}$ is a cotorsion flat R -module. Conversely, the following fact holds.

Proposition 2.5.1 (Enochs [13]). *Let F be a cotorsion flat R -module. Then there is an isomorphism*

$$F \cong \prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}},$$

where $T_{\mathfrak{p}}$ is the \mathfrak{p} -adic completion of a free $R_{\mathfrak{p}}$ module.

Proof. See [13, Theorem] or [15, Theorem 5.3.28]. \square

Let S be a multiplicatively closed subset of R and \mathfrak{a} be an ideal of R . For a cotorsion flat R -module F , we have $\text{RHom}_R(S^{-1}R, F) \cong \text{Hom}_R(S^{-1}R, F)$

and $L\Lambda^{V(\mathfrak{a})} F \cong \Lambda^{V(\mathfrak{a})} F$. Moreover, by Proposition 2.5.1, we may regard F as an R -module of the form $\prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}}$. Then it holds that

$$(2.5.2) \quad \text{RHom}_R(S^{-1}R, \prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}}) \cong \text{Hom}_R(S^{-1}R, \prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}}) \cong \prod_{\mathfrak{p} \in U_S} T_{\mathfrak{p}}.$$

This fact appears implicitly in [47, §5.2]. Furthermore we have

$$(2.5.3) \quad L\Lambda^{V(\mathfrak{a})} \prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}} \cong \Lambda^{V(\mathfrak{a})} \prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}} \cong \prod_{\mathfrak{p} \in V(\mathfrak{a})} T_{\mathfrak{p}}.$$

One can show (2.5.2) and (2.5.3) by Lemma 2.3.1 and (2.2.9). See also the recent paper [44, Lemma 2.2] of Thompson.

Let F be a cotorsion flat R -module with $\text{cosupp } F \subseteq W$ for a subset W of $\text{Spec } R$. Then it follows from Proposition 2.5.1 that F is isomorphic to an R -module of the form $\prod_{\mathfrak{p} \in W} T_{\mathfrak{p}}$. More precisely, using Lemma 2.2.4, (2.5.2) and (2.5.3), one can show the following corollary, which is essentially proved in [15, Lemma 8.5.25].

Corollary 2.5.4. *Let F be a cotorsion flat R -module, and set $W = \text{cosupp } F$. Then we have an isomorphism*

$$F \cong \prod_{\mathfrak{p} \in W} T_{\mathfrak{p}},$$

where $T_{\mathfrak{p}}$ is of the form $(\bigoplus_{B_{\mathfrak{p}}} R_{\mathfrak{p}})_{\mathfrak{p}}^{\wedge}$ with $B_{\mathfrak{p}} = \dim_{\kappa(\mathfrak{p})} \text{Hom}_R(R_{\mathfrak{p}}, F) \otimes_R \kappa(\mathfrak{p})$.

2.6 Complexes of cotorsion flat modules and cosupport

In this section, we study the cosupport of a complex X consisting of cotorsion flat R -modules. As a consequence, we obtain an explicit way to calculate $\gamma_{V^c} X$ and $\lambda^V X$ for a specialization-closed subset V of $\text{Spec } R$.

Notation 2.6.1. Let W be a subset of $\text{Spec } R$. Let X be a complex of cotorsion flat R -modules such that $\text{cosupp } X^i \subseteq W$ for all $i \in \mathbb{Z}$. Under Corollary 2.5.4, we use a presentation of the following form;

$$X = (\cdots \rightarrow \prod_{\mathfrak{p} \in W} T_{\mathfrak{p}}^i \rightarrow \prod_{\mathfrak{p} \in W} T_{\mathfrak{p}}^{i+1} \rightarrow \cdots),$$

where $X^i = \prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}}^i$ and $T_{\mathfrak{p}}^i$ is the \mathfrak{p} -adic completion of a free $R_{\mathfrak{p}}$ -module.

Remark 2.6.2. Let $X = (\cdots \rightarrow \prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}}^i \rightarrow \prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}}^{i+1} \rightarrow \cdots)$ be a complex of cotorsion flat R -modules. Let V be a specialization-closed subset of $\text{Spec } R$. By Lemma 2.3.1, we have $\text{Hom}_R(\prod_{\mathfrak{p} \in V^c} T_{\mathfrak{p}}^i, \prod_{\mathfrak{p} \in V} T_{\mathfrak{p}}^{i+1}) = 0$ for all $i \in \mathbb{Z}$. Therefore $Y = (\cdots \rightarrow \prod_{\mathfrak{p} \in V^c} T_{\mathfrak{p}}^i \rightarrow \prod_{\mathfrak{p} \in V^c} T_{\mathfrak{p}}^{i+1} \rightarrow \cdots)$ is a subcomplex of X , where the differentials in Y are the restrictions of ones in X .

We say that a complex X of R -modules is left (resp. right) bounded if $X^i = 0$ for $i \ll 0$ (resp. $i \gg 0$). When X is left and right bounded, X is called bounded.

Proposition 2.6.3. *Let W be a subset of $\text{Spec } R$ and X be a complex of cotorsion flat R -modules such that $\text{cosupp } X^i \subseteq W$ for all $i \in \mathbb{Z}$. Suppose that one of the following conditions holds;*

- (1) X is left bounded;
- (2) W is equal to $V(\mathfrak{a})$ for an ideal \mathfrak{a} of R ;
- (3) W is generalization-closed;
- (4) $\dim W$ is finite.

Then it holds that $\text{cosupp } X \subseteq W$, i.e., $X \in \mathcal{C}^W$.

To prove this proposition, we use the next elementary lemma. In the lemma, for a complex X and $n \in \mathbb{Z}$, we define the truncations $\tau_{\leq n} X$ and $\tau_{> n} X$ as follows (cf. [22, Chapter I, §7]):

$$\begin{aligned} \tau_{\leq n} X &= (\cdots \rightarrow X^{n-1} \rightarrow X^n \rightarrow 0 \rightarrow \cdots) \\ \tau_{> n} X &= (\cdots \rightarrow 0 \rightarrow X^{n+1} \rightarrow X^{n+2} \rightarrow \cdots) \end{aligned}$$

Lemma 2.6.4. *Let W be a subset of $\text{Spec } R$. We assume that $\tau_{\leq n} X \in \mathcal{C}^W$ (resp. $\tau_{> n} X \in \mathcal{L}_W$) for all $n \geq 0$ (resp. $n < 0$). Then we have $X \in \mathcal{C}^W$ (resp. $X \in \mathcal{L}_W$).*

Recall that \mathcal{C}^W (resp. \mathcal{L}_W) is closed under arbitrary direct products (resp. sums). Then one can show this lemma by using homotopy limit (resp. colimit), see [5, Remark 2.2, Remark 2.3]. \square

Proof of Proposition 2.6.3. Case (1): We have $\tau_{\leq n} X \in \mathcal{C}^W$ for all $n \geq 0$, since $\tau_{\leq n} X$ are bounded. Thus Lemma 2.6.4 implies that $X \in \mathcal{C}^W$.

Case (2): By (2.2.9), Proposition 2.2.5 and (2.5.3), it holds that $\lambda^{V(\mathfrak{a})} X \cong \Lambda^{V(\mathfrak{a})} X \cong \Lambda^{V(\mathfrak{a})} X \cong X$. Hence X belongs to $\mathcal{C}^{V(\mathfrak{a})}$.

Case (3): It follows from the case (1) that $\tau_{>n}X \in \mathcal{C}^W$ for all $n < 0$. Moreover, we have $\mathcal{C}^W = \mathcal{L}_W$ by Lemma 2.3.1. Thus Lemma 2.6.4 implies that $X \in \mathcal{L}_W = \mathcal{C}^W$.

Case (4): Under Notation 2.6.1, we write $X^i = \prod_{\mathfrak{p} \in W} T_{\mathfrak{p}}^i$ for $i \in \mathbb{Z}$. Set $n = \dim W$, and use induction on n . First, suppose that $n = 0$. It is seen from Remark 2.6.2 that X is the direct product of complexes of the form $Y^{\{\mathfrak{p}\}} = (\cdots \rightarrow T_{\mathfrak{p}}^i \rightarrow T_{\mathfrak{p}}^{i+1} \rightarrow \cdots)$ for $\mathfrak{p} \in W$. Furthermore, by the cases (2) and (3), we have $\text{cosupp} Y^{\{\mathfrak{p}\}} \subseteq V(\mathfrak{p}) \cap U(\mathfrak{p}) = \{\mathfrak{p}\}$. Thus it holds that $X \cong \prod_{\mathfrak{p} \in W} Y^{\{\mathfrak{p}\}} \in \mathcal{C}^W$.

Next, suppose that $n > 0$. Set $W_0 = \max W$ and $W_1 = W \setminus W_0$. We write $Y = (\cdots \rightarrow \prod_{\mathfrak{p} \in W_1} T_{\mathfrak{p}}^i \rightarrow \prod_{\mathfrak{p} \in W_1} T_{\mathfrak{p}}^{i+1} \rightarrow \cdots)$, which is a subcomplex of X by Remark 2.6.2. Hence there is a short exact sequence of complexes;

$$0 \longrightarrow Y \longrightarrow X \longrightarrow X/Y \longrightarrow 0,$$

where $X/Y = (\cdots \rightarrow \prod_{\mathfrak{p} \in W_0} T_{\mathfrak{p}}^i \rightarrow \prod_{\mathfrak{p} \in W_0} T_{\mathfrak{p}}^{i+1} \rightarrow \cdots)$. Note that $\dim W_0 = 0$ and $\dim W_1 = n - 1$. Then we have $\text{cosupp} X/Y \subseteq W_0$, by the argument above. Moreover the inductive hypothesis implies that $\text{cosupp} Y \subseteq W_1$. Hence it holds that $\text{cosupp} X \subseteq W_0 \cup W_1 = W$. \square

Under some assumption, it is possible to extend the condition (4) in Proposition 2.6.3 to the case where $\dim W$ is infinite, see Remark 2.7.15.

Corollary 2.6.5. *Let X be a complex of cotorsion flat R -modules and W be a specialization-closed subset of $\text{Spec } R$. Under Notation 2.6.1, we write*

$$X = (\cdots \rightarrow \prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}}^i \rightarrow \prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}}^{i+1} \rightarrow \cdots).$$

Suppose that one of the conditions in Proposition 2.6.3 holds. Then it holds that

$$\begin{aligned} \gamma_{W^c} X &\cong (\cdots \rightarrow \prod_{\mathfrak{p} \in W^c} T_{\mathfrak{p}}^i \rightarrow \prod_{\mathfrak{p} \in W^c} T_{\mathfrak{p}}^{i+1} \rightarrow \cdots), \\ \lambda^W X &\cong (\cdots \rightarrow \prod_{\mathfrak{p} \in W} T_{\mathfrak{p}}^i \rightarrow \prod_{\mathfrak{p} \in W} T_{\mathfrak{p}}^{i+1} \rightarrow \cdots). \end{aligned}$$

Proof. Since $Y = (\cdots \rightarrow \prod_{\mathfrak{p} \in W^c} T_{\mathfrak{p}}^i \rightarrow \prod_{\mathfrak{p} \in W^c} T_{\mathfrak{p}}^{i+1} \rightarrow \cdots)$ is a subcomplex of X by Remark 2.6.2, there is a triangle

$$Y \longrightarrow X \longrightarrow X/Y \longrightarrow Y[1],$$

where $X/Y = (\cdots \rightarrow \prod_{\mathfrak{p} \in W} T_{\mathfrak{p}}^i \rightarrow \prod_{\mathfrak{p} \in W} T_{\mathfrak{p}}^{i+1} \rightarrow \cdots)$. By Proposition 2.6.3, we have $X/Y \in \mathcal{C}^W$. Moreover, since W^c is generalization-closed, it holds that $Y \in \mathcal{C}^{W^c} = {}^{\perp} \mathcal{C}^W$ by Proposition 2.6.3 and Lemma 2.3.1. Therefore we conclude that $\gamma_{W^c} X \cong Y$ and $\lambda^W X \cong X/Y$ by Lemma 2.2.6. \square

Let X be a complex of cotorsion flat R -modules and S be a multiplicatively closed subset of R . We assume that X is left bounded, or $\dim R$ is finite. It then follows from the corollary and (2.5.2) that

$$\gamma_{U_S} X \cong (\cdots \rightarrow \prod_{\mathfrak{p} \in U_S} T_{\mathfrak{p}}^i \rightarrow \prod_{\mathfrak{p} \in U_S} T_{\mathfrak{p}}^{i+1} \rightarrow \cdots) \cong \mathrm{Hom}_R(S^{-1}R, X).$$

We now recall that $\gamma_{U_S} \cong \mathrm{RHom}_R(S^{-1}R, -)$, see Proposition 1.3.1. Hence it holds that $\mathrm{RHom}_R(S^{-1}R, X) \cong \mathrm{Hom}_R(S^{-1}R, X)$. This fact also follows from Lemma 2.9.1.

2.7 Localization functors via Čech complexes

In this section, we introduce a new notion of Čech complexes to calculate $\lambda^W X$, where W is a general subset W of $\mathrm{Spec} R$ and X is a complex of flat R -modules.

We first make the following notations.

Notation 2.7.1. Let W be a subset of $\mathrm{Spec} R$ with $\dim W = 0$. We define a functor $\bar{\lambda}^W : \mathrm{Mod} R \rightarrow \mathrm{Mod} R$ by

$$\bar{\lambda}^W = \prod_{\mathfrak{p} \in W} \Lambda^{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}}).$$

For a prime ideal \mathfrak{p} in W , we write $\bar{\eta}^{\{\mathfrak{p}\}} : \mathrm{id}_{\mathrm{Mod} R} \rightarrow \bar{\lambda}^{\{\mathfrak{p}\}} = \Lambda^{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}})$ for the composition of the natural morphisms $\mathrm{id}_{\mathrm{Mod} R} \rightarrow (-) \otimes_R R_{\mathfrak{p}}$ and $(-) \otimes_R R_{\mathfrak{p}} \rightarrow \Lambda^{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}})$. Moreover, $\bar{\eta}^W : \mathrm{id}_{\mathrm{Mod} R} \rightarrow \bar{\lambda}^W = \prod_{\mathfrak{p} \in W} \bar{\lambda}^{\{\mathfrak{p}\}}$ denotes the product of the morphisms $\bar{\eta}^{\{\mathfrak{p}\}}$ for $\mathfrak{p} \in W$.

Notation 2.7.2. Let $\{W_i\}_{0 \leq i \leq n}$ be a family of subsets of $\mathrm{Spec} R$, and suppose that $\dim W_i = 0$ for $0 \leq i \leq n$. For a sequence (i_m, \dots, i_1, i_0) of integers with $0 \leq i_0 < i_1 < \cdots < i_m \leq n$, we write

$$\bar{\lambda}^{(i_m, \dots, i_1, i_0)} = \bar{\lambda}^{W_{i_m}} \cdots \bar{\lambda}^{W_{i_1}} \bar{\lambda}^{W_{i_0}}.$$

If the sequence is empty, then we use the general convention that $\lambda^{(\)} = \mathrm{id}_{\mathrm{Mod} R}$. For an integer s with $0 \leq s \leq m$, $\bar{\eta}^{W_{i_s}} : \mathrm{id}_{\mathrm{Mod} R} \rightarrow \bar{\lambda}^{(i_s)}$ induces a morphism

$$\bar{\lambda}^{(i_m, \dots, i_{s+1})} \bar{\eta}^{W_{i_s}} \bar{\lambda}^{(i_{s-1}, \dots, i_0)} : \bar{\lambda}^{(i_m, \dots, \widehat{i_s}, \dots, i_0)} \longrightarrow \bar{\lambda}^{(i_m, \dots, i_0)},$$

where we mean by \widehat{i}_s that i_s is omitted. We set

$$\partial^{m-1} : \prod_{0 \leq i_0 < \dots < i_{m-1} \leq n} \bar{\lambda}^{(i_{m-1}, \dots, i_0)} \longrightarrow \prod_{0 \leq i_0 < \dots < i_m \leq n} \bar{\lambda}^{(i_m, \dots, i_0)}$$

to be the product of the morphisms $\bar{\lambda}^{(i_m, \dots, \widehat{i}_s, \dots, i_0)} \rightarrow \bar{\lambda}^{(i_m, \dots, i_0)}$ multiplied by $(-1)^s$.

Remark 2.7.3. Let $W_0, W_1 \subseteq \text{Spec } R$ be subsets such that $\dim W_0 = \dim W_1 = 0$. As with Remark 2.3.14 (ii), the following diagram is commutative:

$$\begin{array}{ccc} \text{id}_{\text{Mod } R} & \xrightarrow{\bar{\eta}^{W_0}} & \bar{\lambda}^{W_0} \\ \downarrow \bar{\eta}^{W_1} & & \downarrow \bar{\eta}^{W_1} \bar{\lambda}^{W_0} \\ \bar{\lambda}^{W_1} & \xrightarrow{\bar{\lambda}^{W_1} \bar{\eta}^{W_0}} & \bar{\lambda}^{W_1} \bar{\lambda}^{W_0} \end{array}$$

Definition 2.7.4. Let $\mathbb{W} = \{W_i\}_{0 \leq i \leq n}$ be a family of subsets of $\text{Spec } R$, and suppose that $\dim W_i = 0$ for $0 \leq i \leq n$. By Remark 2.7.3, it is possible to construct a Čech complex of functors of the following form;

$$\prod_{0 \leq i_0 \leq n} \bar{\lambda}^{(i_0)} \xrightarrow{\partial^0} \prod_{0 \leq i_0 < i_1 \leq n} \bar{\lambda}^{(i_1, i_0)} \rightarrow \dots \rightarrow \prod_{0 \leq i_0 < \dots < i_{n-1} \leq n} \bar{\lambda}^{(i_{n-1}, \dots, i_0)} \xrightarrow{\partial^{n-1}} \bar{\lambda}^{(n, \dots, 0)},$$

which we denote by $L^{\mathbb{W}}$ and call it *the Čech complex with respect to \mathbb{W}* .

For an R -module M , $L^{\mathbb{W}}M$ denotes the complex of R -modules obtained by $L^{\mathbb{W}}$ in a natural way, where it is concentrated in degrees from 0 to n . We call $L^{\mathbb{W}}M$ *the Čech complex of M with respect to \mathbb{W}* . Note that there is a chain map $\ell^{\mathbb{W}}M : M \rightarrow L^{\mathbb{W}}M$ induced by the map $M \rightarrow \prod_{0 \leq i_0 \leq n} \bar{\lambda}^{(i_0)}M$ in degree 0, which is the product of $\bar{\eta}^{W_{i_0}}M : M \rightarrow \bar{\lambda}^{(i_0)}M$ for $0 \leq i_0 \leq n$.

More generally, we regard every term of $L^{\mathbb{W}}$ as a functor $C(\text{Mod } R) \rightarrow C(\text{Mod } R)$, where $C(\text{Mod } R)$ denotes the category of complexes of R -modules. Then $L^{\mathbb{W}}$ naturally sends a complex X to a double complex, which we denote by $L^{\mathbb{W}}X$. Furthermore, we write $\text{tot } L^{\mathbb{W}}X$ for the total complex of $L^{\mathbb{W}}X$. The family of chain maps $\ell^{\mathbb{W}}X^j : X^j \rightarrow L^{\mathbb{W}}X^j$ for $j \in \mathbb{Z}$ induces a morphism $X \rightarrow L^{\mathbb{W}}X$ as double complexes, from which we obtain a chain map $\ell^{\mathbb{W}}X : X \rightarrow \text{tot } L^{\mathbb{W}}X$.

Remark 2.7.5. (i) We regard $\text{tot } L^{\mathbb{W}}$ as a functor $C(\text{Mod } R) \rightarrow C(\text{Mod } R)$. Then $\ell^{\mathbb{W}}$ is a morphism $\text{id}_{C(\text{Mod } R)} \rightarrow \text{tot } L^{\mathbb{W}}$ of functors. Moreover, if M is an R -module, then $\text{tot } L^{\mathbb{W}}M = L^{\mathbb{W}}M$.

(ii) Let $a, b \in \mathbb{Z} \cup \{\pm\infty\}$ with $a \leq b$ and X be a complex of R -modules such that $X^i = 0$ for $i \notin [a, b]$. Then it holds that $(\text{tot } L^{\mathbb{W}} X)^i = 0$ for $i \notin [a, b + n]$, where n is the number given to $\mathbb{W} = \{W_i\}_{0 \leq i \leq n}$.

(iii) Let X be a complex of flat R -modules. Then we see that $\text{tot } L^{\mathbb{W}} X$ consists of cotorsion flat R -modules with cosupports in $\bigcup_{0 \leq i \leq n} W_i$.

Definition 2.7.6. Let W be a non-empty subset of $\text{Spec } R$ and $\{W_i\}_{0 \leq i \leq n}$ be a family of subsets of W . We say that $\{W_i\}_{0 \leq i \leq n}$ is a *system of slices of W* if the following conditions hold:

1. $W = \bigcup_{0 \leq i \leq n} W_i$;
2. $W_i \cap W_j = \emptyset$ if $i \neq j$;
3. $\dim W_i = 0$ for $0 \leq i \leq n$;
4. W_i is specialization-closed in $\bigcup_{i \leq j \leq n} W_j$ for each $0 \leq i \leq n$.

Compare this definition with the filtrations discussed in [22, Chapter IV; §3].

If $\dim W$ is finite, then there exists at least one system of slices of W . Conversely, if there is a system of slices of W , then $\dim W$ is finite.

Proposition 2.7.7. *Let W be a subset of $\text{Spec } R$ and $\mathbb{W} = \{W_i\}_{0 \leq i \leq n}$ be a system of slices of W . Then, for any flat R -module F , there is an isomorphism in \mathcal{D} ;*

$$\lambda^W F \cong L^{\mathbb{W}} F.$$

Under this isomorphism, $\ell^{\mathbb{W}} F : F \rightarrow L^{\mathbb{W}} F$ coincides with $\eta^W F : F \rightarrow \lambda^W F$ in \mathcal{D} .

Proof. We use induction on n , which is the number given to $\mathbb{W} = \{W_i\}_{0 \leq i \leq n}$. Suppose that $n = 0$. It then holds that $L^{\mathbb{W}} F = \bar{\lambda}^{W_0} F = \bar{\lambda}^W F$ and $\ell^{\mathbb{W}} F = \bar{\eta}^{W_0} F = \bar{\eta}^W F$. Hence this proposition follows from Theorem 2.3.10.

Next, suppose that $n > 0$, and write $U = \bigcup_{1 \leq i \leq n} W_i$. Setting $U_{i-1} = W_i$, we obtain a system of slices $\mathbb{U} = \{U_i\}_{0 \leq i \leq n-1}$ of U . Consider the following two squares, where the first (resp. second) one is in $C(\text{Mod } R)$ (resp. \mathcal{D}):

$$\begin{array}{ccc} F & \xrightarrow{\bar{\eta}^{W_0} F} & \bar{\lambda}^{W_0} F \\ \downarrow \ell^U F & & \downarrow \ell^U \bar{\lambda}^{W_0} F \\ L^U F & \xrightarrow{L^U \bar{\eta}^{W_0} F} & L^U \bar{\lambda}^{W_0} F \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\eta^{W_0} F} & \lambda^{W_0} F \\ \downarrow \eta^U F & & \downarrow \eta^U \lambda^{W_0} F \\ \lambda^U F & \xrightarrow{\lambda^U \eta^{W_0} F} & \lambda^U \lambda^{W_0} F \end{array}$$

By Remark 2.7.5 (i) and Remark 2.3.14 (ii), both of them are commutative. Moreover, $\lambda^U \eta^{W_0} F$ is the unique morphism which makes the right square commutative, because λ^U is a left adjoint to the inclusion functor $\mathcal{C}^U \hookrightarrow \mathcal{D}$. Then, regarding the left one as being in \mathcal{D} , we see from the inductive hypothesis that the left one coincides with the right one in \mathcal{D} .

Let $\bar{g} : L^U F \oplus \bar{\lambda}^{W_0} F \rightarrow L^U \bar{\lambda}^{W_0} F$ and $\bar{h} : F \rightarrow L^U F \oplus \bar{\lambda}^{W_0} F$ be chain maps represented by the following matrices;

$$\bar{g} = \begin{pmatrix} L^U \bar{\eta}^{W_0} F & (-1) \cdot \ell^U \bar{\lambda}^{W_0} X \end{pmatrix}, \quad \bar{h} = \begin{pmatrix} \ell^U F \\ \bar{\eta}^{W_0} F \end{pmatrix}.$$

Notice that the mapping cone of $\bar{g}[-1]$ is nothing but $L^W F$. Then we can obtain the following morphism of triangles, where it is regarded as being in \mathcal{D} :

$$(2.7.8) \quad \begin{array}{ccccccc} F[-1] & \longrightarrow & F[-1] & \longrightarrow & 0 & \longrightarrow & F \\ \downarrow \ell^{W F[-1]} & & \downarrow \bar{h}[-1] & & \downarrow & & \downarrow \ell^{W F} \\ L^W F[-1] & \longrightarrow & (L^U F \oplus \bar{\lambda}^{W_0} F)[-1] & \xrightarrow{\bar{g}[-1]} & L^U \bar{\lambda}^{W_0} F[-1] & \longrightarrow & L^W F \end{array}$$

Therefore, by Theorem 2.3.15 and Remark 2.3.19, there is an isomorphism $\lambda^W F \cong L^W F$ such that $\ell^{W F}$ coincides with $\eta^W F$ under this isomorphism. \square

The following corollary is one of the main results of this chapter.

Corollary 2.7.9. *Let W and $\mathbb{W} = \{W_i\}_{0 \leq i \leq n}$ be as above. Let X be a complex of flat R -modules. Then there is an isomorphism in \mathcal{D} ;*

$$\lambda^W X \cong \text{tot } L^W X.$$

Under this isomorphism, $\ell^W X : X \rightarrow \text{tot } L^W X$ coincides with $\eta^W X : X \rightarrow \lambda^W X$ in \mathcal{D} .

Proof. We embed $\ell^W X : X \rightarrow \text{tot } L^W X$ into a triangle

$$C \longrightarrow X \xrightarrow{\ell^W X} \text{tot } L^W X \longrightarrow C[1].$$

Proposition 2.6.3 and Remark 2.7.5 (iii) imply that $\text{tot } L^W X \in \mathcal{C}^W$. Thus it suffices to show that $\lambda^{W_i} C = 0$ for each i , by Lemma 2.2.6 and Remark 2.2.7 (i). For this purpose, we prove that $\lambda^{W_i} \ell^W X$ is an isomorphism in \mathcal{D} . This is equivalent to showing that $\bar{\lambda}^{W_i} \ell^W X$ is a quasi-isomorphism, since X and $\text{tot } L^W X$ consist of flat R -modules.

Consider the natural morphism $X \rightarrow L^{\mathbb{W}}X$ as double complexes, which is induced by the chain maps $\ell^{\mathbb{W}}X^j : X^j \rightarrow L^{\mathbb{W}}X^j$ for $j \in \mathbb{Z}$. To prove that $\bar{\lambda}^{W_i}\ell^{\mathbb{W}}X$ is a quasi-isomorphism, it is enough to show that $\bar{\lambda}^{W_i}\ell^{\mathbb{W}}X^j$ is a quasi-isomorphism for each $j \in \mathbb{Z}$, see [26, Theorem 12.5.4]. Furthermore, by Proposition 2.7.7, each $\ell^{\mathbb{W}}X^j$ coincides with $\eta^W X^j : X^j \rightarrow \lambda^W X^j$ in \mathcal{D} . Since $W_i \subseteq W$, it follows from Remark 2.2.7 (ii) that $\lambda^{W_i}\eta^W X^j$ is an isomorphism in \mathcal{D} . This means that $\bar{\lambda}^{W_i}\ell^{\mathbb{W}}X^j$ is a quasi-isomorphism. \square

Let W be a subset of $\text{Spec } R$, and suppose that $n = \dim W$ is finite. Then Corollary 2.7.9 implies $\lambda^W R \in \mathcal{D}^{[0,n]}$. We give an example such that $H^n(\lambda^W R) \neq 0$.

Example 2.7.10. Let (R, \mathfrak{m}) be a local ring of dimension $d \geq 1$. Then we have $\dim V(\mathfrak{m})^c = d - 1$. By Lemma 2.2.6, there is a triangle

$$\gamma_{V(\mathfrak{m})}R \longrightarrow R \longrightarrow \lambda^{V(\mathfrak{m})^c}R \longrightarrow \gamma_{V(\mathfrak{m})}R[1].$$

Since $\text{R}\Gamma_{V(\mathfrak{m})} \cong \gamma_{V(\mathfrak{m})}$ by (2.2.10), Grothendieck's non-vanishing theorem implies that $H^d(\gamma_{V(\mathfrak{m})}R)$ is non-zero. Then we see from the triangle that $H^{d-1}(\lambda^{V(\mathfrak{m})^c}R) \neq 0$.

We denote by \mathcal{D}^- the full subcategory of \mathcal{D} consisting of complexes X such that $H^i(X) = 0$ for $i \gg 0$. Let W be a subset of $\text{Spec } R$ and $X \in \mathcal{D}^-$. If $\dim W$ is finite, then we have $\lambda^W R \in \mathcal{D}^-$ by Corollary 2.7.9. However, as shown in the following example, it can happen that $\lambda^W R \notin \mathcal{D}^-$ when $\dim W$ is infinite.

Example 2.7.11. Assume that $\dim R = +\infty$, and set $W = \max(\text{Spec } R)$. Then it holds that $\dim W = 0$ and $\dim W^c = +\infty$. Since each $\mathfrak{m} \in W$ is maximal, there are isomorphisms

$$\gamma_W \cong \text{R}\Gamma_W \cong \bigoplus_{\mathfrak{m} \in W} \text{R}\Gamma_{V(\mathfrak{m})}.$$

Thus we see from Example 2.7.10 that $\gamma_W R \notin \mathcal{D}^-$. Then, considering the triangle

$$\gamma_W R \longrightarrow R \longrightarrow \lambda^{W^c}R \longrightarrow \gamma_W R[1],$$

we have $\lambda^{W^c}R \notin \mathcal{D}^-$.

Let W be a subset of $\text{Spec } R$ and $X \in \mathcal{C}^W$. Then $\eta^W X : X \rightarrow \lambda^W X$ is an isomorphism in \mathcal{D} . Thus Remark 2.7.5 (iii) and Corollary 2.7.9 yield the following result.

Corollary 2.7.12. *Let W be a subset of $\text{Spec } R$, and $\mathbb{W} = \{W_i\}_{0 \leq i \leq n}$ be a system of slices of W . Let X be a complex of flat R -modules with $\text{cosupp } X \subseteq W$. Then the chain map $\ell^{\mathbb{W}} X : X \rightarrow \text{tot } L^{\mathbb{W}} X$ is a quasi-isomorphism, where $\text{tot } L^{\mathbb{W}} X$ consists of cotorsion flat R -modules with cosupports in W .*

Remark 2.7.13. If $d = \dim R$ is finite, then any complex Y is quasi-isomorphic to a K -flat complex consisting of cotorsion flat R -modules. To see this, set $W_i = \{\mathfrak{p} \in \text{Spec } R \mid \dim R/\mathfrak{p} = i\}$ for $0 \leq i \leq d$. Then $\mathbb{W} = \{W_i\}_{0 \leq i \leq d}$ is a system of slices of $\text{Spec } R$. We take a K -flat resolution X of Y such that X consists of flat R -modules. Corollary 2.7.12 implies that $\ell^{\mathbb{W}} X : X \rightarrow \text{tot } L^{\mathbb{W}} X$ is a quasi-isomorphism, and $\text{tot } L^{\mathbb{W}} X$ consists of cotorsion flat R -modules. At the same time, the chain maps $\ell^{\mathbb{W}} X^i : X^i \rightarrow L^{\mathbb{W}} X^i$ are quasi-isomorphisms for all $i \in \mathbb{Z}$. Then it is not hard to see that the mapping cone of $\ell^{\mathbb{W}} X$ is K -flat. Thus $\text{tot } L^{\mathbb{W}} X$ is K -flat.

By Proposition 2.6.3 and Corollary 2.7.12, we have the next result.

Corollary 2.7.14. *Let W be a subset of $\text{Spec } R$ such that $\dim W$ is finite. Then a complex $X \in \mathcal{D}$ belongs to \mathcal{C}^W if and only if X is isomorphic to a complex Z of cotorsion flat R -modules such that $\text{cosupp } Z^i \subseteq W$ for all $i \in \mathbb{Z}$.*

Remark 2.7.15. If $\dim W$ is infinite, it is possible to construct a similar family to systems of slices. We first put $W_0 = \max W$. Let $i > 0$ be an ordinal, and suppose that subsets W_j of W are defined for all $j < i$. Then we put $W_i = \max(W \setminus \bigcup_{j < i} W_j)$. In this way, we obtain the smallest ordinal $o(W)$ satisfying the following conditions: (1) $W = \bigcup_{0 \leq i < o(W)} W_i$; (2) $W_i \cap W_j = \emptyset$ if $i \neq j$; (3) $\dim W_i \leq 0$ for $0 \leq i < o(W)$; (4) W_i is specialization-closed in $\bigcup_{i \leq j < o(W)} W_j$ for each $0 \leq i < o(W)$.

One should remark that the ordinal $o(W)$ can be uncountable in general, see [18, p. 48, Theorem 9.8]. However, if R is an infinite dimensional commutative Noetherian ring given by Nagata [31, Appendix A1; Example 1], then $o(W)$ is at most countable. Moreover, using transfinite induction, it is possible to extend the condition (4) in Proposition 2.6.3 and Corollary 2.6.5 to the case where $o(W)$ is countable. One can also extend Corollary 2.7.14 to the case where $o(W)$ is countable.

Using Theorem 2.3.22 and results in §1.3, it is possible to give a similar result to Corollary 2.7.9, for colocalization functors γ_W and complexes of injective R -modules.

2.8 Čech complexes and complexes of finitely generated modules

Let W be a subset of $\text{Spec } R$ and $\mathbb{W} = \{W_i\}_{0 \leq i \leq n}$ be a system of slices of W . In this section, we prove that $\lambda^W Y$ is isomorphic to $\text{tot } L^{\mathbb{W}} Y$ if Y is a complex of finitely generated R -modules.

We denote by \mathcal{D}_{fg} the full subcategory of \mathcal{D} consisting of all complexes with finitely generated cohomology modules, and set $\mathcal{D}_{\text{fg}}^- = \mathcal{D}^- \cap \mathcal{D}_{\text{fg}}$. We first prove the following proposition.

Proposition 2.8.1. *Let W be a subset of $\text{Spec } R$ such that $\dim W$ is finite. Let $X, Y \in \mathcal{D}$. We suppose that one of the following conditions holds;*

- (1) $X \in \mathcal{D}^-$ and $Y \in \mathcal{D}_{\text{fg}}^-$;
- (2) X is a bounded complex of flat R -modules and $Y \in \mathcal{D}_{\text{fg}}$.

Then there are natural isomorphisms

$$(\gamma_{W^c} X) \otimes_R^L Y \cong \gamma_{W^c}(X \otimes_R^L Y), \quad (\lambda^W X) \otimes_R^L Y \cong \lambda^W(X \otimes_R^L Y).$$

For $X \in \mathcal{D}$ and $n \in \mathbb{Z}$, we define the cohomological truncations $\sigma_{\leq n} X$ and $\sigma_{> n} X$ as follows (cf. [22, Chapter I; §7]):

$$\begin{aligned} \sigma_{\leq n} X &= (\cdots \rightarrow X^{n-2} \rightarrow X^{n-1} \rightarrow \text{Ker } d_X^n \rightarrow 0 \rightarrow \cdots) \\ \sigma_{> n} X &= (\cdots \rightarrow 0 \rightarrow \text{Im } d_X^n \rightarrow X^{n+1} \rightarrow X^{n+2} \rightarrow \cdots) \end{aligned}$$

Proof of Proposition 2.8.1. Apply $(-)\otimes_R^L Y$ to the triangle $\gamma_{W^c} X \rightarrow X \rightarrow \lambda^W X \rightarrow \gamma_{W^c} X[1]$, and we obtain the following triangle;

$$(\gamma_{W^c} X) \otimes_R^L Y \longrightarrow X \otimes_R^L Y \longrightarrow (\lambda^W X) \otimes_R^L Y \longrightarrow (\gamma_{W^c} X) \otimes_R^L Y[1].$$

Since $\text{supp } \gamma_{W^c} X \subseteq W^c$, it holds that $\text{supp}((\gamma_{W^c} X) \otimes_R^L Y) \subseteq W^c$, that is, $(\gamma_{W^c} X) \otimes_R^L Y \in \mathcal{L}_{W^c}$. Hence it remains to show that $(\lambda^W X) \otimes_R^L Y \in \mathcal{C}^W$, see Lemma 2.2.6.

Case (1): We remark that X is isomorphic to a right bounded complex of flat R -modules. Then it is seen from Corollary 2.7.9 that $\lambda^W X$ is isomorphic to a right bounded complex Z of cotorsion flat R -modules such that $\text{cosupp } Z^i \subseteq W$ for all $i \in \mathbb{Z}$. Furthermore, Y is isomorphic to a right bounded complex P of finite free R -modules. Hence it follows that $X \otimes_R^L Y \cong Z \otimes_R P$, where the second one consists of cotorsion flat R -modules

with cosupports in W . Then we have $X \otimes_R^L Y \cong Z \otimes_R P \in \mathcal{C}^W$ by Proposition 2.6.3.

Case (2): By Corollary 2.7.9, $\lambda^W X$ is isomorphic to a bounded complex consisting of cotorsion flat R -modules with cosupports in W . Thus it is enough to prove that $Z \otimes_R Y \in \mathcal{C}^W$ for a cotorsion flat R -module Z with $\text{cosupp } Z \subseteq W$.

We consider the triangle $\sigma_{\leq n} Y \rightarrow Y \rightarrow \sigma_{> n} Y \rightarrow \sigma_{\leq n} Y[1]$ for an integer n . Applying $Z \otimes_R (-)$ to this triangle, we obtain the following one;

$$Z \otimes_R \sigma_{\leq n} Y \longrightarrow Z \otimes_R Y \longrightarrow Z \otimes_R \sigma_{> n} Y \longrightarrow Z \otimes_R \sigma_{\leq n} Y[1].$$

Let $\mathfrak{p} \in W^c$. The case (1) implies that $Z \otimes_R \sigma_{\leq n} Y \in \mathcal{C}^W$ for any $n \in \mathbb{Z}$, since $\lambda^W Z \cong Z$. Thus, applying $\text{RHom}_R(\kappa(\mathfrak{p}), -)$ to the triangle above, we have

$$\text{RHom}_R(\kappa(\mathfrak{p}), Z \otimes_R Y) \cong \text{RHom}_R(\kappa(\mathfrak{p}), Z \otimes_R \sigma_{> n} Y).$$

Furthermore, taking a projective resolution P of $\kappa(\mathfrak{p})$, we have

$$\text{RHom}_R(\kappa(\mathfrak{p}), Z \otimes_R \sigma_{> n} Y) \cong \text{Hom}_R(P, Z \otimes_R \sigma_{> n} Y).$$

Let j be any integer. To see that $\text{RHom}_R(\kappa(\mathfrak{p}), Z \otimes_R Y) = 0$, it suffices to show that there exists an integer n such that $H^0(\text{Hom}_R(P[j], Z \otimes_R \sigma_{> n} Y)) = 0$. Note that $P^i = 0$ for $i > 0$. Moreover, each element of $H^0(\text{Hom}_R(P[j], Z \otimes_R \sigma_{> n} Y)) \cong \text{Hom}_{\mathcal{D}}(P[j], Z \otimes_R \sigma_{> n} Y)$ is represented by a chain map $P[j] \rightarrow Z \otimes_R \sigma_{> n} Y$. Therefore it holds that $H^0(\text{Hom}_R(P[j], Z \otimes_R \sigma_{> n} Y)) = 0$ if $n > -j$. \square

Remark 2.8.2. (i) In the proposition, we can remove the finiteness condition on $\dim W$ if $W = V(\mathfrak{a})$ for an ideal \mathfrak{a} . In such case, we need only use \mathfrak{a} -adic completions of free R -modules instead of cotorsion flat R -modules.

(ii) If W is a generalization-closed subset of $\text{Spec } R$, then the isomorphisms in the proposition hold for any $X, Y \in \mathcal{D}$ because γ_{W^c} is isomorphic to $\text{R}\Gamma_{W^c}$.

Let W be a subset of $\text{Spec } R$ and $\mathbb{W} = \{W_i\}_{0 \leq i \leq n}$ be a system of slices of W . Let $Y \in \mathcal{D}_{\text{fg}}$. By Proposition 2.8.1 and Proposition 2.7.7, we have

$$(2.8.3) \quad \lambda^W Y \cong (\lambda^W R) \otimes_R^L Y \cong (L^{\mathbb{W}} R) \otimes_R Y.$$

Let F be a flat R -module and M be a finitely generated R -module. Then we see from Lemma 2.2.3 that

$$(\bar{\lambda}^{W_i} F) \otimes_R M \cong \bar{\lambda}^{W_i} (F \otimes_R M).$$

This fact ensures that $(\bar{\lambda}^{(i_m, \dots, i_1, i_0)} R) \otimes_R M \cong \bar{\lambda}^{(i_m, \dots, i_1, i_0)} M$. Thus, if Y is a complex of finitely generated R -modules, then there is a natural isomorphism

$$(2.8.4) \quad (L^{\mathbb{W}} R) \otimes_R Y \cong \text{tot } L^{\mathbb{W}} Y$$

in $C(\text{Mod } R)$. By (2.8.3) and (2.8.4), we have shown the following proposition.

Proposition 2.8.5. *Let W be a subset of $\text{Spec } R$ and $\mathbb{W} = \{W_i\}_{0 \leq i \leq n}$ be a system of slices of W . Let Y be a complex of finitely generated R -modules. Then there is an isomorphism in \mathcal{D} :*

$$\lambda^W Y \cong \text{tot } L^{\mathbb{W}} Y.$$

Under this identification, $\ell^{\mathbb{W}} Y : Y \rightarrow \text{tot } L^{\mathbb{W}} Y$ coincides with $\eta^{\mathbb{W}} Y : Y \rightarrow \lambda^W Y$ in \mathcal{D} .

We see from (2.8.4) and the remark below that it is also possible to give a quick proof of this proposition, provided that Y is a right bounded complex of finitely generated R -modules.

Remark 2.8.6. Let W be a subset of $\text{Spec } R$ and $\mathbb{W} = \{W_i\}_{0 \leq i \leq n}$ be a system of slices of W . We denote by $K(\text{Mod } R)$ the homotopy category of complexes of R -modules. Note that $\text{tot } L^{\mathbb{W}}$ induces a triangulated functor $K(\text{Mod } R) \rightarrow K(\text{Mod } R)$, which we also write $\text{tot } L^{\mathbb{W}}$. Then it is seen from Corollary 2.7.9 that $\lambda^W : \mathcal{D} \rightarrow \mathcal{D}$ is isomorphic to the left derived functor of $\text{tot } L^{\mathbb{W}} : K(\text{Mod } R) \rightarrow K(\text{Mod } R)$.

Let W be a subset of $\text{Spec } R$ such that $n = \dim W$ is finite. By Corollary 2.8.5, if an R -module M is finitely generated, then $\lambda^W M \in \mathcal{D}^{[0, n]}$. On the other hand, since $\lambda^{V(\mathfrak{a})} \cong L\Lambda^{V(\mathfrak{a})}$ for an ideal \mathfrak{a} , it can happen that $H^i(\lambda^W M) \neq 0$ for some $i < 0$ when M is not finitely generated, see Example 1.5.3.

Remark 2.8.7. Let $n \geq 0$ be an integer. Let \mathfrak{a}_i be ideals of R and S_i be multiplicatively closed subsets of R for $0 \leq i \leq n$. In Notation 2.7.2 and Definition 2.7.4, one can replace $\bar{\lambda}^{(i)} = \bar{\lambda}^{W_i}$ by $\Lambda^{V(\mathfrak{a}_i)}(- \otimes_R S_i^{-1} R)$, and construct a kind of Čech complexes. For the Čech complex and λ^W with $W = \bigcup_{0 \leq i \leq n} (V(\mathfrak{a}_i) \cap U_{S_i})$, it is possible to show similar results to Corollary 2.7.9 and Proposition 2.8.5, provided that one of the following conditions holds: (1) $V(\mathfrak{a}_i) \cap U_{S_i}$ is specialization-closed in $\bigcup_{i \leq j \leq n} (V(\mathfrak{a}_j) \cap U_{S_j})$ for each $0 \leq i \leq n$; (2) $V(\mathfrak{a}_i) \cap U_{S_i}$ is generalization-closed in $\bigcup_{0 \leq j \leq i} (V(\mathfrak{a}_j) \cap U_{S_j})$ for each $0 \leq i \leq n$.

2.9 Čech complexes and complexes of pure-injective modules

In this section, as an application, we give a functorial way to construct a quasi-isomorphism from a complex of flat R -modules or a complex of finitely generated R -modules to a complex of pure-injective R -modules.

We start with the following well-known fact.

Lemma 2.9.1. *Let X be a complex of flat R -modules and Y be a complex of cotorsion R -modules. We assume that one of the following conditions holds:*

- (1) X is a right bounded and Y is left bounded;
- (2) X is bounded and $\dim R$ is finite.

Then we have $\mathrm{RHom}_R(X, Y) \cong \mathrm{Hom}_R(X, Y)$.

One can prove this lemma by [26, Theorem 12.5.4] and Theorem 2.4.1.

Next, we recall the notion of pure-injective modules and resolutions. We say that a morphism $f : M \rightarrow N$ of R -modules is pure if $f \otimes_R L$ is a monomorphism in $\mathrm{Mod} R$ for any R -module L . Moreover an R -module P is called pure-injective if $\mathrm{Hom}_R(f, P)$ is an epimorphism in $\mathrm{Mod} R$ for any pure morphism $f : M \rightarrow N$ of R -modules. Clearly, all injective R -modules are pure-injective. Furthermore, all pure-injective R -modules are cotorsion, see [15, Lemma 5.3.23].

Let M be an R -module. A complex P together with a quasi-isomorphism $M \rightarrow P$ is called a pure-injective resolution of M if P consists of pure-injective R -modules and $P^i = 0$ for $i < 0$. It is known that any R -module has a minimal pure-injective resolution, which is constructed by using pure-injective envelopes, see [14] and [15, Example 6.6.5, Definition 8.1.4]. Moreover, if F is a flat R -module and P is a pure-injective resolution of M , then we have $\mathrm{RHom}_R(F, M) \cong \mathrm{Hom}_R(F, P)$ by Lemma 2.9.1.

Now we observe that any cotorsion flat R -module is pure-injective. Consider an R -module of the form $(\bigoplus_B R_{\mathfrak{p}})_{\mathfrak{p}}^{\wedge}$ with some index set B and a prime ideal \mathfrak{p} , which is a cotorsion flat R -module. Writing $E_R(R/\mathfrak{p})$ for the injective hull of R/\mathfrak{p} , we have

$$\left(\bigoplus_B R_{\mathfrak{p}}\right)_{\mathfrak{p}}^{\wedge} \cong \mathrm{Hom}_R(E_R(R/\mathfrak{p}), \bigoplus_B E_R(R/\mathfrak{p})),$$

see [15, Theorem 3.4.1]. Tensor-hom adjunction implies that $\mathrm{Hom}_R(M, I)$ is pure-injective for any R -module M and any injective R -module I . Hence

$(\bigoplus_B R_{\mathfrak{p}})_{\mathfrak{p}}^{\wedge}$ is pure-injective. Thus we see that any cotorsion flat R -module is pure-injective, see Proposition 2.5.1.

There is another example of pure-injective R -modules. Let M be a finitely generated R -module. Using Five Lemma, we are able to prove an isomorphism

$$\begin{aligned} \mathrm{Hom}_R \left(E_R(R/\mathfrak{p}), \bigoplus_B E_R(R/\mathfrak{p}) \right) \otimes_R M \\ \cong \mathrm{Hom}_R \left(\mathrm{Hom}_R(M, E_R(R/\mathfrak{p})), \bigoplus_B E_R(R/\mathfrak{p}) \right). \end{aligned}$$

Therefore $(\bigoplus_B R_{\mathfrak{p}})_{\mathfrak{p}}^{\wedge} \otimes_R M$ is pure-injective; it is also isomorphic to $(\bigoplus_B M_{\mathfrak{p}})_{\mathfrak{p}}^{\wedge}$ by Lemma 2.2.3. Moreover, Proposition 2.8.1 implies that

$$\mathrm{cosupp} \left(\bigoplus_B M_{\mathfrak{p}} \right)_{\mathfrak{p}}^{\wedge} \subseteq \{\mathfrak{p}\}.$$

By the above observation, we see that Corollary 2.7.12, (2.8.4) and Proposition 2.8.5 yield the following theorem, which is one of the main results of this chapter.

Theorem 2.9.2. *Let W be a subset of $\mathrm{Spec} R$ and $\mathbb{W} = \{W_i\}_{0 \leq i \leq n}$ be a system of slices of W . Let Z be a complex of flat R -modules or a complex of finitely generated R -modules. We assume that $\mathrm{cosupp} Z \subseteq W$. Then $\ell^{\mathbb{W}} Z : Z \rightarrow \mathrm{tot} L^{\mathbb{W}} Z$ is a quasi-isomorphism, where $\mathrm{tot} L^{\mathbb{W}} Z$ consists of pure-injective R -modules with cosupports in W .*

Remark 2.9.3. Let N be a flat or finitely generated R -module. Suppose that $d = \dim R$ is finite. Set $W_i = \{\mathfrak{p} \in \mathrm{Spec} R \mid \dim R/\mathfrak{p} = i\}$ and $\mathbb{W} = \{W_i\}_{0 \leq i \leq d}$. By Theorem 2.9.2, we obtain a pure-injective resolution $\ell^{\mathbb{W}} N : N \rightarrow L^{\mathbb{W}} N$ of N , that is, there is an exact sequence of R -modules of the following form:

$$0 \rightarrow N \rightarrow \prod_{0 \leq i_0 \leq d} \bar{\lambda}^{(i_0)} N \rightarrow \prod_{0 \leq i_0 < i_1 \leq d} \bar{\lambda}^{(i_1, i_0)} N \rightarrow \dots \rightarrow \bar{\lambda}^{(d, \dots, 0)} N \rightarrow 0$$

We remark that, in $C(\mathrm{Mod} R)$, $L^{\mathbb{W}} N$ need not be isomorphic to a minimal pure-injective resolution P of N . In fact, when N is a projective or finitely generated R -module, it holds that $P^0 \cong \prod_{\mathfrak{m} \in W_0} \widehat{N}_{\mathfrak{m}} = \bar{\lambda}^{(0)} N$ (cf. [46, Theorem 3] and [15, Remark 6.7.12]), while $(L^{\mathbb{W}} N)^0 = \prod_{0 \leq i_0 \leq d} \bar{\lambda}^{(i_0)} N$. Furthermore, Enochs [14, Theorem 2.1] proved that if N is flat R -module, then P^i is of the form $\prod_{\mathfrak{p} \in W_{\geq i}} T_{\mathfrak{p}}^i$ for $0 \leq i \leq d$ (cf. Notation 2.6.1), where $W_{\geq i} = \{\mathfrak{p} \in \mathrm{Spec} R \mid \dim R/\mathfrak{p} \geq i\}$.

On the other hand, for a flat or finitely generated R -module N , the differential maps in the pure-injective resolution $L^{\mathbb{W}}N$ are concretely described. In addition, our approach based on the localization functor $\lambda^{\mathbb{W}}$ and the Čech complex $L^{\mathbb{W}}$ provide a natural morphism $\ell^{\mathbb{W}} : \text{id}_{C(\text{Mod } R)} \rightarrow \text{tot } L^{\mathbb{W}}$ which induces isomorphisms in \mathcal{D} for all complexes of flat R -modules and complexes of finitely generated R -modules. The reader should also compare Theorem 2.9.2 with [44, Proposition 5.9].

We close this chapter with the following example of Theorem 2.9.2.

Example 2.9.4. Let R be a 2-dimensional local domain with quotient field Q . Let $\mathbb{W} = \{W_i\}_{0 \leq i \leq 2}$ be as in Remark 2.9.3. Then $L^{\mathbb{W}}R$ is a pure-injective resolution of R , and $\widehat{L}^{\mathbb{W}}R$ is of the following form:

$$0 \rightarrow Q \oplus \left(\prod_{\mathfrak{p} \in W_1} \widehat{R}_{\mathfrak{p}} \right) \oplus \widehat{R} \rightarrow \left(\prod_{\mathfrak{p} \in W_1} \widehat{R}_{\mathfrak{p}} \right)_{(0)} \oplus (\widehat{R})_{(0)} \oplus \prod_{\mathfrak{p} \in W_1} (\widehat{R})_{\mathfrak{p}} \rightarrow \left(\prod_{\mathfrak{p} \in W_1} (\widehat{R})_{\mathfrak{p}} \right)_{(0)} \rightarrow 0$$

3. Cosupports of affine rings

3.1 Introduction

This chapter is based on the author's paper [35]. Let R be a commutative Noetherian ring. We denote by $D(R)$ the unbounded derived category of R . Note that the objects of $D(R)$ are complexes of R -modules, which are cohomologically indexed;

$$X = (\dots \rightarrow X^{i-1} \rightarrow X^i \rightarrow X^{i+1} \rightarrow \dots).$$

The (small) support of $X \in D(R)$ is defined as

$$\operatorname{supp}_R X = \{ \mathfrak{p} \in \operatorname{Spec} R \mid X \otimes_R^L \kappa(\mathfrak{p}) \neq 0 \},$$

where $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Let \mathfrak{a} be an ideal of R , and write $\Gamma_{\mathfrak{a}}$ for the \mathfrak{a} -torsion functor $\varinjlim \operatorname{Hom}_R(R/\mathfrak{a}^n, -)$ on the category $\operatorname{Mod} R$ of R -modules. Then we have $\operatorname{supp}_R R\Gamma_{\mathfrak{a}} X \subseteq V(\mathfrak{a})$ for any $X \in D(R)$.

The cosupport of $X \in D(R)$ is defined as

$$\operatorname{cosupp}_R X = \{ \mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{RHom}_R(\kappa(\mathfrak{p}), X) \neq 0 \}.$$

We write $\Lambda^{\mathfrak{a}}$ for the \mathfrak{a} -adic completion functor $\varprojlim(- \otimes_R R/\mathfrak{a}^n)$ on $\operatorname{Mod} R$. Greenlees and May [21] proved that the left derived functor $L\Lambda^{\mathfrak{a}} : D(R) \rightarrow D(R)$ is a right adjoint to $R\Gamma_{\mathfrak{a}} : D(R) \rightarrow D(R)$, see also [28, §4; p.69]. It follows from the adjointness property that $\operatorname{cosupp}_R L\Lambda^{\mathfrak{a}} X \subseteq V(\mathfrak{a})$ for any $X \in D(R)$.

If M is a finitely generated R -module, Nakayama's lemma implies that $\operatorname{supp}_R M = \operatorname{Supp}_R M = \{ \mathfrak{p} \in \operatorname{Spec} R \mid M_{\mathfrak{p}} \neq 0 \}$. In particular, $\operatorname{supp}_R R$ is nothing but $\operatorname{Spec} R$. However, it is not easy to compute $\operatorname{cosupp}_R R$ in general.

Since $L\Lambda^{\mathfrak{a}} R \cong \Lambda^{\mathfrak{a}} R$, the cosupport of $\Lambda^{\mathfrak{a}} R$ is contained in $V(\mathfrak{a})$. Hence we have $\operatorname{cosupp}_R R \subseteq V(\mathfrak{c}_R)$, where \mathfrak{c}_R denotes the sum of all ideals \mathfrak{a} such that R is \mathfrak{a} -adically complete. In [40, Question 6.13], Sather-Wagstaff and Wicklein questioned whether the equality $\operatorname{cosupp}_R R = V(\mathfrak{c}_R)$ holds for any commutative Noetherian ring R or not. Thompson [43, Example 5.6] gave

a negative answer to this question. He proved that if k is a field and $R = k[[X]][Y]$, then $\text{cosupp}_R R$ is strictly contained in $V(\mathfrak{c}_R)$. Moreover, it was also shown that the cosupport of this ring is not Zariski closed.

Following [43], we say that a commutative Noetherian ring R has *full-cosupport* if $\text{cosupp}_R R$ is equal to $\text{Spec } R$. Let k be a field and R be the polynomial ring $k[X_1, \dots, X_n]$ in n variables over k . Then, since $\mathfrak{c}_R = (0)$, we expect that $\text{cosupp}_R R = \text{Spec } R$. Indeed, this is true.

Theorem 3.1.1. *Let k be a field and n be a non-negative integer. The polynomial ring $k[X_1, \dots, X_n]$ has full-cosupport.*

Surprisingly, this fact was known only in the case where $n \leq 2$ or k is countable, see [43, Theorem 4.11]. See also [40, Question 6.16], [4, Proposition 4.18] and [20, Proposition 3.2].

We say that R is an affine ring over a field k if R is finitely generated as a k -algebra. By Theorem 3.1.1, we can show the following corollary, which is the main result of this chapter.

Corollary 3.1.2. *Any affine ring over a field has full-cosupport.*

In Section 2, we prove the two results above. Section 3 is devoted to explain some relationship between cosupport and minimal pure-injective resolutions. Section 4 contains applications of the main theorem. Let k be a field with $|k| = \aleph_1$ and R be an affine ring over k such that $\dim R \geq 2$. We specify all terms of a minimal pure-injective resolution of R . As a corollary, it is possible to give a partial answer to Gruson's conjecture. Suppose that R is a polynomial ring $k[X_1, \dots, X_n]$ over a field k . The conjecture states that $\text{Ext}_R^i(R_{(0)}, R) \neq 0$ if and only if $i = \inf\{s + 1, n\}$, where s is defined by the equation $|k| = \aleph_s$ if k is infinite, and $s = 0$ otherwise. We prove that this conjecture is true when $s = 1$.

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3.2 Proof of main result

We start with the following lemma.

Lemma 3.2.1 (Thompson [43, Lemma 4.4]). *Let $\varphi : R \rightarrow S$ be a morphism of commutative Noetherian rings. Suppose that φ is finite, i.e., S is finitely*

generated as an R -module. Let $f : \text{Spec } S \rightarrow \text{Spec } R$ be the canonical map defined by $f(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})$ for $\mathfrak{q} \in \text{Spec } S$. Then, there is an equality

$$\text{cosupp}_S S = f^{-1}(\text{cosupp}_R R).$$

In other words, for $\mathfrak{q} \in \text{Spec } S$, we have $\mathfrak{q} \in \text{cosupp}_S S$ if and only if $f(\mathfrak{q}) \in \text{cosupp}_R R$.

In the next section, we will give an outline of the proof of this lemma, provided that R has finite Krull dimension.

Remark 3.2.2. Let R be a commutative Noetherian ring. We denote by 0_R the zero element of R . The following statements hold by Lemma 3.2.1.

(i) Let \mathfrak{p} be a prime ideal of R . Then we have $(0_{R/\mathfrak{p}}) \in \text{cosupp}_{R/\mathfrak{p}} R/\mathfrak{p}$ if and only if $\mathfrak{p} \in \text{cosupp}_R R$.

(ii) Let S be an integral domain. Suppose that there is a finite morphism $\varphi : R \rightarrow S$ of rings such that φ is an injection. Then we have $(0_S) \in \text{cosupp}_S S$ if and only if $(0_R) \in \text{cosupp}_R R$.

We next recall a well-known description of local cohomology via Čech complexes. Let R be a commutative Noetherian ring and \mathfrak{a} be an ideal of R . Let $\mathbf{x} = \{x_1, \dots, x_n\}$ be a system of generators of \mathfrak{a} . In $D(R)$, $R\Gamma_{\mathfrak{a}} R$ is isomorphism to the (extended) Čech complex with respect to \mathbf{x} ;

$$0 \longrightarrow R \longrightarrow \bigoplus_{1 \leq i \leq n} R_{x_i} \longrightarrow \bigoplus_{1 \leq i < j \leq n} R_{x_i x_j} \longrightarrow \cdots \longrightarrow R_{x_1 \cdots x_n} \longrightarrow 0.$$

Here, for an element $y \in R \setminus \{0_R\}$, R_y denotes the localization of R with respect to the multiplicatively closed set $\{y^n \mid n \geq 0\}$. See [25, Lecture 7; §4] for details.

Remark 3.2.3. Let k be a field and n be a non-negative integer. Set $R = k[X_1, \dots, X_n]$ and $S = k[X_1, \dots, X_{n+1}]$. Take $y \in R \setminus \{0_R\}$. Then $S/(1 - yX_{n+1})$ is isomorphic to R_y as a k -algebra.

Let R be a commutative Noetherian ring. When R is an integral domain, we denote by $Q(R)$ the quotient field of R . Moreover, for $\mathfrak{p} \in \text{Spec } R$, $E_R(R/\mathfrak{p})$ denotes the injective envelope of R/\mathfrak{p} .

Proof of Theorem 3.1.1. Set $R = k[X_1, \dots, X_n]$ and take $\mathfrak{p} \in \text{Spec } R$. By Remark 3.2.2 (i), $(0_{R/\mathfrak{p}}) \in \text{cosupp}_{R/\mathfrak{p}} R/\mathfrak{p}$ if and only if $\mathfrak{p} \in \text{cosupp}_R R$. Moreover, Noether normalization theorem yields a finite map $\varphi : k[X_1, \dots, X_m] \rightarrow R/\mathfrak{p}$ of rings such that φ is an injection, where $\dim R/\mathfrak{p} = m$. Therefore, by

Remark 3.2.2 (ii), it is sufficient to show that $(0_R) \in \text{cosupp}_R R$ for any $n \geq 0$.

We suppose that $(0_R) \notin \text{cosupp}_R R$ for some $n \geq 0$, and deduce a contradiction. Let \mathfrak{p} be a prime ideal of $S = k[X_1, \dots, X_{n+1}]$ with $\dim S/\mathfrak{p} = n$. By Noether normalization theorem, there is a finite morphism $\psi : R \rightarrow S/\mathfrak{p}$ of rings such that ψ is an injection. Hence we have $(0_{S/\mathfrak{p}}) \notin \text{cosupp}_{S/\mathfrak{p}} S/\mathfrak{p}$ by Remark 3.2.2 (ii). Consequently, for any $y \in R \setminus \{0_R\}$, it follows from Remark 3.2.3 that $(0_{R_y}) \notin \text{cosupp}_{R_y} R_y$. In other words, it holds that $\text{RHom}_{R_y}(Q(R_y), R_y) = 0$ in $D(R_y)$. This implies that $(0_R) \notin \text{cosupp}_R R_y$ for any $y \in R \setminus \{0_R\}$, since $\text{RHom}_{R_y}(Q(R_y), R_y) \cong \text{RHom}_R(Q(R), R_y)$ in $D(R)$.

Now set $\mathfrak{m} = (X_1, \dots, X_n) \subseteq R$. Then $\text{R}\Gamma_{\mathfrak{m}} R$ is isomorphic to the Čech complex with respect to $\mathbf{x} = \{X_1, \dots, X_n\}$. Hence, by the above argument, we have $\text{RHom}_R(Q(R), \text{R}\Gamma_{\mathfrak{m}} R) = 0$. However, there is an isomorphism $\text{R}\Gamma_{\mathfrak{m}} R \cong E_R(R/\mathfrak{m})[-n]$ in $D(R)$, see [25, Theorem 11.26]. Moreover, the canonical map $R \rightarrow R/\mathfrak{m}$ induces a non-trivial map $Q(R) \rightarrow E_R(R/\mathfrak{m})$, since $E_R(R/\mathfrak{m})$ is injective. Therefore $\text{RHom}_R(Q(R), \text{R}\Gamma_{\mathfrak{m}} R)$ must be non-zero in $D(R)$. This is a contradiction. \square

Proof of Corollary 3.1.2. When a commutative Noetherian ring R has full-cosupport, Lemma 3.2.1 implies that R/\mathfrak{a} has full-cosupport for any ideal \mathfrak{a} of R . Therefore, this corollary follows from Theorem 3.1.1. \square

Remark 3.2.4. Let k be a field and R be an affine ring over k . Let $y \in R \setminus \{0_R\}$. We see from Corollary 3.1.2 and Remark 3.2.3 that R_y has full-cosupport.

Question 3.2.5. Let k be a field and n be a non-negative integer. Set $R = k[X_1, \dots, X_n]$, and let U be a multiplicatively closed subset of R . Does the ring $U^{-1}R$ have full-cosupport?

A commutative ring R is said to be essentially of finite type over a field k when R is a localization of an affine ring over k . If the question above is true, then any ring essentially of finite type over k has full-cosupport, by Corollary 3.1.2.

Remark 3.2.6. Let R be a commutative Noetherian ring with finite Krull dimension and $X \in D(R)$ be a complex with finitely generated cohomology modules. As shown in [43, Corollary 4.3], there is an equality $\text{cosupp}_R X = \text{supp}_R X \cap \text{cosupp}_R R$, see also [40, Theorem 6.6]. Hence, if R has full-cosupport, then we have $\text{cosupp}_R X = \text{supp}_R X$, so that the cosupport of any finitely generated R -module M is Zariski-closed, since $\text{supp}_R M = \text{Supp}_R M$.

Thompson [43, Example 5.7] showed that the cosupport of $k[[X]][Y_1, \dots, Y_n]$ is not Zariski-closed for $n \geq 0$, where k is any field. Hence the following question naturally arises.

Question 3.2.7. Let R be any commutative Noetherian ring. Is the cosupport of R specialization-closed?

3.3 Minimal pure-injective resolutions

In this section, we summarize some known facts about cosupport and minimal pure-injective resolutions.

Let R be a commutative Noetherian ring. For $\mathfrak{p} \in \text{Spec } R$ and $X \in D(R)$, it is well-known that the following bi-implications hold;

$$(3.3.1) \quad \mathfrak{p} \in \text{supp}_R X \Leftrightarrow \text{R}\Gamma_{\mathfrak{p}} X_{\mathfrak{p}} \neq 0 \Leftrightarrow \text{RHom}_R(\kappa(\mathfrak{p}), X_{\mathfrak{p}}) \neq 0.$$

See [17, Theorem 2.1, Theorem 4.1]. In *loc. cit.*, it is also shown that these conditions are equivalent to saying that $\text{L}\Lambda^{\mathfrak{p}} X_{\mathfrak{p}} \neq 0$. Moreover, the last condition of (3.3.1) means that $\mathfrak{p} \in \text{cosupp}_R X_{\mathfrak{p}}$. Therefore, setting $X = \text{RHom}_R(R_{\mathfrak{p}}, Y)$ for $Y \in D(R)$, one can deduce the following lemma, see also [40, Proposition 4.4].

Lemma 3.3.2. For $\mathfrak{p} \in \text{Spec } R$ and $Y \in D(R)$, the following bi-implications hold;

$$\mathfrak{p} \in \text{cosupp}_R Y \Leftrightarrow \text{L}\Lambda^{\mathfrak{p}} \text{RHom}_R(R_{\mathfrak{p}}, Y) \neq 0 \Leftrightarrow \text{RHom}_R(R_{\mathfrak{p}}, Y) \otimes_R^{\text{L}} \kappa(\mathfrak{p}) \neq 0.$$

For any $X \in D(R)$, $\text{supp}_R X = \emptyset$ if and only if $X = 0$, see [16, Lemma 2.6] or [36, Lemma 2.12]. Similarly, $\text{cosupp}_R X = \emptyset$ if and only if $X = 0$. This is a direct consequence of [36, Theorem 2.8]. See also [4, Theorem 4.5] or [8, Corollary 3.3].

For an R -module M and an ideal \mathfrak{a} of R , we denote by $M_{\mathfrak{a}}^{\wedge}$ the \mathfrak{a} -adic completion $\Lambda^{\mathfrak{a}} M = \varprojlim M/\mathfrak{a}^n$. In addition, for the localization $M_{\mathfrak{p}}$ at a prime ideal \mathfrak{p} , we also write $\widehat{M}_{\mathfrak{p}} = \Lambda^{\mathfrak{p}} M_{\mathfrak{p}}$.

Let $B \neq 0$ be some cardinality and $\bigoplus_B E_R(R/\mathfrak{p})$ be the direct sum of B -copies of $E_R(R/\mathfrak{p})$. We remark that $\text{supp}_R \bigoplus_B E_R(R/\mathfrak{p}) = \{\mathfrak{p}\}$. Furthermore there is an isomorphism

$$(3.3.3) \quad \text{Hom}_R(E_R(R/\mathfrak{p}), \bigoplus_B E_R(R/\mathfrak{p})) \cong (\bigoplus_B R_{\mathfrak{p}})_{\mathfrak{p}}^{\wedge},$$

see [15, Theorem 3.4.1; (7)]. Then, tensor-hom adjunction in $D(R)$ implies that $\text{cosupp}_R(\bigoplus_B R_{\mathfrak{p}})_{\mathfrak{p}}^{\wedge} = \{\mathfrak{p}\}$. In addition, we see that $(\bigoplus_B R_{\mathfrak{p}})_{\mathfrak{p}}^{\wedge}$ is cotorsion, where an R -module M is said to be cotorsion if $\text{Ext}_R^i(F, M) = 0$ for any flat R -module F and any $i > 0$. By [15, Theorem 5.3.28], an R -module M is cotorsion and flat iff M is of the form $\prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}}$, where $T_{\mathfrak{p}}$ is the \mathfrak{p} -adic completion of a free $R_{\mathfrak{p}}$ -module.

There are two known formulas for such an R -module $\prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}}$. Take $\mathfrak{q} \in \text{Spec } R$, and write $U(\mathfrak{q}) = \{\mathfrak{p} \in \text{Spec } R \mid \mathfrak{p} \subseteq \mathfrak{q}\}$. It then holds that

$$\text{Hom}_R(R_{\mathfrak{q}}, \prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}}) \cong \prod_{\mathfrak{p} \in U(\mathfrak{q})} T_{\mathfrak{p}}, \quad \Lambda^{\mathfrak{q}}(\prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}}) \cong \prod_{\mathfrak{p} \in V(\mathfrak{q})} T_{\mathfrak{p}}$$

One can deduce the first one from (3.3.3) and tensor-hom adjunction in $\text{Mod } R$. Furthermore, the other one holds since $\Lambda^{\mathfrak{q}} = \varprojlim(- \otimes_R R/\mathfrak{q}^n)$ commutes with arbitrary direct products in $\text{Mod } R$.

A morphism $f : M \rightarrow N$ of R -modules is called pure if $f \otimes_R L$ is an injective map for any R -module L . We say that an R -module P is pure-injective if $\text{Hom}_R(f, P)$ is a surjection for any pure morphism $f : M \rightarrow N$. It is easily seen that any pure-injective module is cotorsion.

Let $(0 \rightarrow M \rightarrow P^0 \rightarrow P^1 \rightarrow \dots)$ be an exact sequence of R -modules. If every P^i is pure-injective, then we call the complex $P = (0 \rightarrow P^0 \rightarrow P^1 \rightarrow \dots)$ a pure-injective resolution of M . Since each P^i is cotorsion, we have $\text{RHom}_R(F, M) \cong \text{Hom}_R(F, X)$ in $D(R)$ for a flat R -module F . Any R -module M has a pure-injective envelope, so that there exists a minimal pure-injective resolution of M ; $(0 \rightarrow PE^0(M) \rightarrow PE^1(M) \rightarrow \dots)$. Moreover, if M is flat, then each $PE^i(M)$ is isomorphic to the direct product of the \mathfrak{p} -adic completion of a free $R_{\mathfrak{p}}$ -module for $\mathfrak{p} \in \text{Spec } R$. See [15, §6.7, §8.5] for more details.

Remark 3.3.4. Let F be a flat R -module. As mentioned above, we may write $PE^i(F) = \prod_{\mathfrak{p} \in \text{Spec } R} T_{\mathfrak{p}}^i$, where $T_{\mathfrak{p}}^i = (\bigoplus_{B_{\mathfrak{p}}^i} R_{\mathfrak{p}})_{\mathfrak{p}}^{\wedge}$ for some cardinality $B_{\mathfrak{p}}^i$. We write

$$W_{\geq i} = \{\mathfrak{p} \in \text{Spec } R \mid \dim R/\mathfrak{p} \geq i\}, \quad W_i = \{\mathfrak{p} \in \text{Spec } R \mid \dim R/\mathfrak{p} = i\}.$$

It follows from [15, Corollary 8.5.10] that $PE^i(F) = \prod_{\mathfrak{p} \in W_{\geq i}} T_{\mathfrak{p}}^i$. In addition, if $F = R$, then $PE^0(R) = \prod_{\mathfrak{m} \in W_0} \widehat{R}_{\mathfrak{m}}$, see [15, Proposition 6.7.3].

Lemma 3.3.5. *Suppose that $\dim R$ is finite. Let F be a flat R -module. Under Remark 3.3.4, we denote by $(0 \rightarrow \prod_{\mathfrak{p} \in W_{\geq 0}} T_{\mathfrak{p}}^0 \rightarrow \prod_{\mathfrak{p} \in W_{\geq 1}} T_{\mathfrak{p}}^1 \rightarrow \dots)$ a minimal pure-injective resolution of F . Then $\mathfrak{p} \in \text{cosupp}_R F$ if and only if $T_{\mathfrak{p}}^i \neq 0$ for some i .*

To see this lemma, we make a remark.

Remark 3.3.6. Let P be the minimal pure-injective resolution of F in Lemma 3.3.5. Notice that $P^i = 0$ for $i > n = \dim R$. Moreover, recall that $\mathrm{L}\Lambda^{\mathfrak{p}} X \cong \Lambda^{\mathfrak{p}} X$ if X is a complex of flat R -modules with $X^i = 0$ for $i \gg 0$. Then we see that

$$\begin{aligned} \mathrm{L}\Lambda^{\mathfrak{p}} \mathrm{RHom}_R(R_{\mathfrak{p}}, F) &\cong \Lambda^{\mathfrak{p}} \mathrm{Hom}_R(R_{\mathfrak{p}}, P) \\ &\cong (0 \rightarrow T_{\mathfrak{p}}^0 \rightarrow \cdots \rightarrow T_{\mathfrak{p}}^n \rightarrow 0), \\ \mathrm{RHom}_R(R_{\mathfrak{p}}, F) \otimes_R^{\mathrm{L}} \kappa(\mathfrak{p}) &\cong \mathrm{Hom}_R(R_{\mathfrak{p}}, P) \otimes_R \kappa(\mathfrak{p}) \\ &\cong (0 \rightarrow \bigoplus_{B_{\mathfrak{p}}^0} \kappa(\mathfrak{p}) \rightarrow \cdots \rightarrow \bigoplus_{B_{\mathfrak{p}}^n} \kappa(\mathfrak{p}) \rightarrow 0), \end{aligned}$$

where $T_{\mathfrak{p}}^i = (\bigoplus_{B_{\mathfrak{p}}^i} R_{\mathfrak{p}})_{\mathfrak{p}}^{\wedge}$.

Lemma 3.3.5 follows from Lemma 3.3.2, Remark 3.3.6 and the following fact.

Lemma 3.3.7. *Let F be a flat R -module and P be a minimal pure-injective resolution of F . Then the differentials of $\mathrm{Hom}_R(R_{\mathfrak{p}}, P) \otimes_R \kappa(\mathfrak{p})$ are zero.*

See [15, Proposition 8.5.26] for the proof.

Lemma 3.3.5 and Lemma 3.3.7 were formulated by Thompson in more general setting. See [43, Theorem 2.5.] and [44, Theorem 3.5].

Remark 3.3.8. By Remark 3.3.6 and Lemma 3.3.7, the cardinality $B_{\mathfrak{p}}^i$ in Remark 3.3.4 is nothing but $\dim_{\kappa(\mathfrak{p})} H^i(\mathrm{RHom}_R(R_{\mathfrak{p}}, F) \otimes_R^{\mathrm{L}} \kappa(\mathfrak{p}))$.

Proposition 3.3.9. *Suppose that $\varphi : R \rightarrow S$ is a finite morphism of commutative Noetherian rings. Let P be a minimal pure-injective resolution of a flat R -module F . Then $P \otimes_R S$ is a minimal pure-injective resolution of $F \otimes_R S$ in $\mathrm{Mod} S$.*

See [15, Theorem 8.5.1] for the proof.

Finally, we give an outline of the proof of Lemma 3.2.1. Let $\varphi : R \rightarrow S$ and $f : \mathrm{Spec} S \rightarrow \mathrm{Spec} R$ be as in the lemma. For simplicity, we assume that the Krull dimension of R is finite. Let $B_{\mathfrak{p}}$ be some cardinality for $\mathfrak{p} \in \mathrm{Spec} R$. Then it holds that

$$(3.3.10) \quad \left(\prod_{\mathfrak{p} \in \mathrm{Spec} R} \left(\bigoplus_{B_{\mathfrak{p}}} R_{\mathfrak{p}} \right)_{\mathfrak{p}}^{\wedge} \right) \otimes_R S \cong \prod_{\mathfrak{q} \in \mathrm{Spec} S} \left(\bigoplus_{B_{\mathfrak{q}}} S_{\mathfrak{q}} \right)_{\mathfrak{q}}^{\wedge},$$

where $B_{\mathfrak{q}} = B_{\mathfrak{p}}$ if $f(\mathfrak{q}) = \mathfrak{p}$. This follows from (3.3.3) and the following isomorphism; $\mathrm{Hom}_R(S, E_R(R/\mathfrak{p})) \cong \bigoplus_{\mathfrak{q} \in f^{-1}(\mathfrak{p})} E_S(S/\mathfrak{q})$ (cf. [39, Theorem 1.1]). Thus, Lemma 3.2.1 is a consequence of Lemma 3.3.5 and Proposition 3.3.9. See Thompson's paper [43, Lemma 4.4] for more details.

3.4 Applications

In this section, we give a complete description of all terms in a minimal pure-injective resolution of an affine ring R over a field k , where $|k| = \aleph_1$ and $\dim R \geq 2$. Consequently we obtain a partial answer to the following conjecture by Gruson, which is stated in the recent paper [45] of Thorup.

Conjecture 3.4.1. Let k be a field. We define s by the equation $|k| = \aleph_s$ of cardinalities if k is infinite, and $s = 0$ otherwise. Let n be a non-negative integer, and set $R = k[X_1, \dots, X_n]$. Then $\text{Ext}_R^i(Q(R), R) \neq 0$ if and only if $i = \inf\{s + 1, n\}$.

In the classical paper [20, Proposition 3.2], Gruson verified this conjecture under the assumption that $s \geq 1$ and $n = 2$.

In relation to this problem, Thorup proved the following result.

Proposition 3.4.2 ([45, Theorem 13]). *Let k be a field with $|k| \geq \aleph_1$ and n be an integer with $n \geq 2$. Set $R = k[X_1, \dots, X_n]$. Then $\text{Ext}_R^1(Q(R), R) = 0$.*

We first extend this to the following corollary.

Corollary 3.4.3. *Let k be a field with $|k| \geq \aleph_1$. Suppose that S is an affine domain over k with $\dim S \geq 2$. Then $\text{Ext}_S^1(Q(S), S) = 0$.*

Proof. Set $n = \dim S$ and $R = k[X_1, \dots, X_n]$. By Noether normalization theorem, there is a finite morphism $\varphi : R \rightarrow S$ of rings such that φ is an injection. Write $f : \text{Spec } S \rightarrow \text{Spec } R$ for the canonical map induced by φ . Let P be a minimal pure-injective resolution of R . Under Remark 3.3.4 and Remark 3.3.8, we write $P^i = \prod_{\mathfrak{p} \in W_{\geq i}} T_{\mathfrak{p}}^i$ for $0 \leq i \leq n$, where $T_{\mathfrak{p}}^i = (\bigoplus_{B_{\mathfrak{p}}^i} R_{\mathfrak{p}})_{\mathfrak{p}}^{\wedge}$ and $B_{\mathfrak{p}}^i = \dim_{\kappa(\mathfrak{p})} H^i(\kappa(\mathfrak{p}) \otimes_R^L \text{RHom}_R(R_{\mathfrak{p}}, R))$. By Proposition 3.4.2, we have $\text{Ext}_R^1(Q(R), R) = 0$, so that $B_{(0_R)}^1 = 0$. In other words, it holds that

$$P^1 = \prod_{\mathfrak{p} \in W_{\geq 1}} T_{\mathfrak{p}}^1 = \prod_{\mathfrak{p} \in W_{\geq 1} \setminus \{(0_R)\}} T_{\mathfrak{p}}^1.$$

Recall that $P \otimes_R S$ is a minimal pure-injective resolution of S by Proposition 3.3.9. Moreover, since $f(0_S) = (0_R)$, it follows from (3.3.10) that $\text{Hom}_S(Q(S), P^1 \otimes_R S) = 0$. Hence we have

$$\text{Ext}_S^1(Q(S), S) \cong H^1(\text{Hom}_S(Q(S), P \otimes_R S)) = 0.$$

□

Using this corollary, we can show the next result.

Corollary 3.4.4. *Let k be a field with $|k| \geq \aleph_1$. Let R be an affine ring over k . Suppose that \mathfrak{p} is a prime ideal of R with $\dim R/\mathfrak{p} \geq 2$. Then we have*

$$H^1(\mathrm{RHom}_R(R_{\mathfrak{p}}, R) \otimes_R^L \kappa(\mathfrak{p})) = 0.$$

Proof. Notice that $\mathrm{RHom}_R(R_{\mathfrak{p}}, R) \otimes_R^L \kappa(\mathfrak{p}) \cong \mathrm{RHom}_R(R_{\mathfrak{p}}, R) \otimes_R^L R/\mathfrak{p}$. Let P be a minimal pure-injective resolution of R . By Remark 3.3.6 and (3.3.10), we see that there is an isomorphism of complexes;

$$\mathrm{Hom}_R(R_{\mathfrak{p}}, P) \otimes_R R/\mathfrak{p} \cong \mathrm{Hom}_{R/\mathfrak{p}}(Q(R/\mathfrak{p}), P \otimes_R R/\mathfrak{p}).$$

Since $P \otimes_R R/\mathfrak{p}$ is a minimal pure-injective resolution of R/\mathfrak{p} by Proposition 3.3.9, it holds that

$$H^i(\mathrm{RHom}_R(R_{\mathfrak{p}}, R) \otimes_R^L R/\mathfrak{p}) \cong \mathrm{Ext}_{R/\mathfrak{p}}^i(Q(R/\mathfrak{p}), R/\mathfrak{p}).$$

The right-hand side vanishes if $i = 1$, by Corollary 3.4.3. \square

Remark 3.4.5. There is a more general form of the last isomorphism in the proof above. Let $\varphi : R \rightarrow S$ be a finite morphism of commutative Noetherian rings, where R has finite Krull dimension. Let $\mathfrak{p} \in \mathrm{Spec} R$ and $X \in D(R)$ with $H^i(X) = 0$ for $i \gg 0$. Then it is possible to prove the following isomorphism in $D(S)$;

$$\mathrm{RHom}_R(R_{\mathfrak{p}}, X) \otimes_R^L S \cong \mathrm{RHom}_S(R_{\mathfrak{p}} \otimes_R S, X \otimes_R^L S).$$

Let k and s as in Conjecture 3.4.1. Let R be an affine ring over k such that $n = \dim R$ and P be a minimal pure-injective resolution of R . Under Remark 3.3.4, we write $P^0 = \prod_{\mathfrak{m} \in W_0} \widehat{R}_{\mathfrak{m}}$ and $P^i = \prod_{\mathfrak{p} \in W_{\geq i}} T_{\mathfrak{p}}^i$ for $1 \leq i \leq n$. By [18, II; Corollary 3.2.7, Corollary 3.3.2], the projective dimension of any flat R -module is at most $\inf\{s+1, n\}$. Hence the pure-injective dimension of R is at most $\inf\{s+1, n\}$, that is, $P^i = 0$ for $i > \inf\{s+1, n\}$. One can observe this fact from Remark 3.3.6 and Lemma 3.3.7. See also [15, Theorem 8.4.12].

Now suppose that $s = 0$ or $n = 1$. Then the minimal pure-injective resolution of R is of the form

$$(0 \rightarrow \prod_{\mathfrak{m} \in W_0} \widehat{R}_{\mathfrak{m}} \rightarrow \prod_{\mathfrak{p} \in W_{\geq 1}} T_{\mathfrak{p}}^1 \rightarrow 0).$$

In this case, it is known that $T_{\mathfrak{p}}^1 \neq 0$ for any $\mathfrak{p} \in W_{\geq 1}$. One can also prove this fact by Corollary 3.1.2 and Lemma 3.3.5.

Next, suppose that $s \geq 1$ and $n \geq 2$. By Remark 3.3.8 and Corollary 3.4.4, we have

$$(3.4.6) \quad \prod_{\mathfrak{p} \in W_{\geq 1}} T_{\mathfrak{p}}^1 = \prod_{\mathfrak{p} \in W_1} T_{\mathfrak{p}}^1.$$

This extends Enochs's result [12, Theorem 3.5], in which he proved the above equality for polynomial rings over the fields of complex numbers and real numbers.

Combining Corollary 3.1.2, Lemma 3.3.5, Remark 3.3.8 and (3.4.6), we have the following result.

Theorem 3.4.7. *Let k be a field with $|k| = \aleph_1$. Let R be an affine ring over k with $\dim R \geq 2$. Then, a minimal pure-injective resolution of R is of the following form;*

$$(0 \rightarrow \prod_{\mathfrak{m} \in W_0} \widehat{R}_{\mathfrak{m}} \rightarrow \prod_{\mathfrak{p} \in W_1} T_{\mathfrak{p}}^1 \rightarrow \prod_{\mathfrak{p} \in W_{\geq 2}} T_{\mathfrak{p}}^2 \rightarrow 0),$$

where $T_{\mathfrak{p}}^i = (\bigoplus_{B_{\mathfrak{p}}^i} R_{\mathfrak{p}})_{\mathfrak{p}}^{\wedge}$ and $B_{\mathfrak{p}}^i = \dim_{\kappa(\mathfrak{p})} H^i(\kappa(\mathfrak{p}) \otimes_R^L \mathrm{RHom}_R(R_{\mathfrak{p}}, R))$. In addition, $T_{\mathfrak{p}}^1 \neq 0$ for all $\mathfrak{p} \in W_1$, and $T_{\mathfrak{p}}^2 \neq 0$ for all $\mathfrak{p} \in W_{\geq 2}$.

By this theorem, we obtain a positive answer to Conjecture 3.4.1 in the case that $s = 1$ and $n \geq 2$.

Corollary 3.4.8. *Let k be a field with $|k| = \aleph_1$. Let R be an affine ring over k with $n = \dim R \geq 2$ and \mathfrak{p} be a prime ideal of R such that $\dim R/\mathfrak{p} \geq 2$. Then $\mathrm{Ext}_R^i(R_{\mathfrak{p}}, R) \neq 0$ if and only if $i = \inf\{2, n\}$.*

When R is an affine domain, this corollary yields a (non-)vanishing property on right derived functors of some inverse limits, see [45, Setup 3]. See also [20, Corollary 3.4].

There is an analogue to Conjecture 3.4.1; it claims that if \mathfrak{m} is a maximal ideal of $R = k[X_1, \dots, X_n]$, then $\mathrm{Ext}_{R_{\mathfrak{m}}}^i(Q(R), R_{\mathfrak{m}}) \neq 0$ if and only if $i = \inf\{s + 1, n\}$ (cf [45, §1; 1]). When $s = 0$ or $n = 1$, this is true. In [45, Theorem 13], Thorup also proved that $\mathrm{Ext}_{R_{\mathfrak{m}}}^1(Q(R), R_{\mathfrak{m}}) = 0$, provided that $s \geq 1$ and $n \geq 2$. Therefore, when $s = 1$ and $n \geq 2$, it is enough to solve Question 3.2.5.

4. On ideals preserving generalized local cohomology modules

4.1 Introduction

This chapter is based on the author's paper [32]. Let R be a commutative Noetherian ring. Let \mathfrak{a} be an ideal of R and M, N are two finitely generated R -modules. We denote by \mathbb{N}_0 the set of non-negative integers. For an R -module $L \neq 0$, $\dim \text{Supp } L$ is defined by $\sup_{\mathfrak{p} \in \text{Supp } L} \dim R/\mathfrak{p}$. If $L = 0$, we set $\dim \text{Supp } L = -1$ (cf. [7, Reminder 6.1.1]). For a subset T of $\text{Spec}(R)$, we denote by $\text{Min } T$ the set of ideals \mathfrak{q} such that \mathfrak{q} is minimal in T .

The generalized local cohomology was defined by Herzog in [24]. The i th generalized local cohomology module $H_{\mathfrak{a}}^i(M, N)$ of M and N with respect to \mathfrak{a} is defined by $H_{\mathfrak{a}}^i(M, N) = \varinjlim_n \text{Ext}_R^i(M/\mathfrak{a}^n M, N)$. Clearly, $H_{\mathfrak{a}}^i(R, N)$ is just the ordinary local cohomology module $H_{\mathfrak{a}}^i(N)$ of N with respect to \mathfrak{a} .

Following [10], we use the convenience that if $t = \infty$, then the set $\{i \in \mathbb{N}_0 \mid i < t\}$ means \mathbb{N}_0 . Let t be a positive integer or ∞ . We denote by Ω_t the set of ideals \mathfrak{c} such that $H_{\mathfrak{c}}^i(M, N) \cong H_{\mathfrak{a}}^i(M, N)$ for all $i < t$. The first purpose of this chapter is to show that there exists the ideal \mathfrak{b}_t such that \mathfrak{b}_t is the largest in Ω_t and $\dim R/\mathfrak{b}_t = \sup_{i < t} \dim \text{Supp } H_{\mathfrak{a}}^i(M, N)$. As a consequence, we obtain a short proof and a generalization of [41, Theorem 2.7] due to Saremi and Mafi. Next, we prove that if \mathfrak{d} is an ideal such that $\mathfrak{a} \subseteq \mathfrak{d} \subseteq \mathfrak{b}_t$, then $H_{\mathfrak{d}}^i(M, N) \cong H_{\mathfrak{a}}^i(M, N)$ for all $i < t$.

4.2 Results

Let t be a positive integer or ∞ . We set $\Sigma_t = \bigcup_{i < t} \text{Supp } H_{\mathfrak{a}}^i(M, N)$. An ideal \mathfrak{b}_t is defined to be $\bigcap_{\mathfrak{p} \in \text{Min } \Sigma_t} \mathfrak{p}$ if $\Sigma_t \neq \emptyset$, otherwise $\mathfrak{b}_t = R$.

We start this section by the following lemma.

Lemma 4.2.1. *Let t be a positive integer or ∞ . We denote by J_t the ideal $\bigcap_{i < t} \text{ann}(\text{Ext}_R^i(M/\mathfrak{a}M, N))$. Then $H_{J_t}^i(M, N) \cong H_{\mathfrak{a}}^i(M, N)$ for all $i < t$.*

Proof. See [10, Lemma 2.7] for the proof. \square

Lemma 4.2.2. *Let t be a positive integer or ∞ . Then we have*

$$\Sigma_t = \bigcup_{i < t} \text{Supp Ext}_R^i(M/\mathfrak{a}M, N).$$

Proof. See [10, Lemma 2.8] for the proof. \square

Remark 4.2.3. Let t be a positive integer or ∞ . Let J_t be as in Lemma 4.2.1. It follows that $V(J_t) = \bigcup_{i < t} \text{Supp Ext}_R^i(M/\mathfrak{a}M, N)$ (see [10, Lemma 2.8]). If t is a positive integer, this is clear. For the reader, we shall show that the equation holds in the case that $t = \infty$. It is obvious that $V(J_\infty) \supset \bigcup_{i \in \mathbb{N}_0} \text{Supp Ext}_R^i(M/\mathfrak{a}M, N)$. Conversely, let $\mathfrak{p} \in V(J_\infty)$. We set $\mathfrak{a}_M = \text{ann}(M/\mathfrak{a}M)$. Note that $\mathfrak{a}_M \subset J_\infty$ and $\text{ann}(N) \subset J_\infty$. Thus it follows that $N_{\mathfrak{p}} \neq 0$ and $N_{\mathfrak{p}} \neq (\mathfrak{a}_M)_{\mathfrak{p}} N_{\mathfrak{p}}$. Set $\nu = \text{depth}((\mathfrak{a}_M)_{\mathfrak{p}}, N_{\mathfrak{p}})$. Then $\nu < \infty$. It implies that $\mathfrak{p} \in \text{Supp Ext}_R^\nu(M/\mathfrak{a}M, N)$. Therefore it follows that $V(J_\infty) \subset \bigcup_{i \in \mathbb{N}_0} \text{Supp Ext}_R^i(M/\mathfrak{a}M, N)$.

Now we prove the first main result of this chapter.

Theorem 4.2.4. *Let t be a positive integer or ∞ . Then the following statements hold.*

- i) $\Sigma_t = V(\mathfrak{b}_t)$,
- ii) $H_{\mathfrak{b}_t}^i(M, N) \cong H_{\mathfrak{a}}^i(M, N)$ for all $i < t$,
- iii) \mathfrak{b}_t is the largest ideal in Ω_t ,
- iV) $\dim R/\mathfrak{b}_t = \sup_{i < t} \dim \text{Supp } H_{\mathfrak{a}}^i(M, N)$.

Proof. i) Set $J_t = \bigcap_{i < t} \text{ann}(\text{Ext}_R^i(M/\mathfrak{a}M, N))$. By Lemma 4.2.2, we have $\Sigma_t = V(J_t)$. Therefore the set $\text{Min } \Sigma_t$ is finite and $\Sigma_t = V(J_t) = V(\mathfrak{b}_t)$.

ii) By i), $\sqrt{J_t} = \mathfrak{b}_t$. Hence, by Lemma 4.2.1, it follows that $H_{\mathfrak{b}_t}^i(M, N) \cong H_{\mathfrak{a}}^i(M, N)$ for all $i < t$.

iii) Let $\mathfrak{c} \in \Omega_t$. Then $\Sigma_t = \bigcup_{i < t} \text{Supp } H_{\mathfrak{c}}^i(M, N)$. Therefore \mathfrak{c} is included in \mathfrak{b}_t .

iv) If $\Sigma_t = \emptyset$, there is nothing to prove. So we may assume that $\Sigma_t \neq \emptyset$. Let $\mathfrak{q} \in \text{Spec}(R)$ such that $\mathfrak{q} \supset \mathfrak{b}_t$. Since the set $\text{Min } \Sigma_t$ is finite by i), there exists an ideal $\mathfrak{p} \in \text{Min } \Sigma_t$ such that $\mathfrak{q} \supset \mathfrak{p}$. Therefore it follows that $\dim R/\mathfrak{b}_t = \sup_{\mathfrak{p} \in \text{Min } \Sigma_t} \dim R/\mathfrak{p} = \sup_{i < t} \dim \text{Supp } H_{\mathfrak{a}}^i(M, N)$. \square

By Theorem 4.2.4, we obtain a generalization of [41, Theorem 2.7] due to Saremi and Mafi. In fact, we have the following result.

Corollary 4.2.5. *Let s be a non-negative integer and t be a positive integer or ∞ . Then the following statements are equivalent.*

- i) $\dim \text{Supp } H_{\mathfrak{a}}^i(M, N) \leq s$ for all $i < t$,*
- ii) There exists an ideal \mathfrak{b} of R such that $\dim R/\mathfrak{b} \leq s$ and $H_{\mathfrak{b}}^i(M, N) \cong H_{\mathfrak{a}}^i(M, N)$ for all $i < t$.*

Proof. i) \implies ii) By assumption, $\sup_{i < t} \dim \text{Supp } H_{\mathfrak{a}}^i(M, N) \leq s$. Hence, by Theorem 4.2.4, we obtain the ideal $\mathfrak{b} = \mathfrak{b}_t$ such that $\dim R/\mathfrak{b}_t \leq s$ and $H_{\mathfrak{b}_t}^i(M, N) \cong H_{\mathfrak{a}}^i(M, N)$ for all $i < t$.

ii) \implies i) Since $\text{Supp } H_{\mathfrak{a}}^i(M, N) = \text{Supp } H_{\mathfrak{b}}^i(M, N) \subseteq V(\mathfrak{b})$ for all $i < t$, it follows that $\dim \text{Supp } H_{\mathfrak{a}}^i(M, N) \leq \dim R/\mathfrak{b} \leq s$ for all $i < t$. \square

Lemma 4.2.6. *Let x be an element of R . Then there exists a long exact sequence*

$$\cdots \rightarrow H_{\mathfrak{a}+(x)}^i(M, N) \rightarrow H_{\mathfrak{a}}^i(M, N) \rightarrow H_{\mathfrak{a}R_x}^i(M_x, N_x) \rightarrow H_{\mathfrak{a}+(x)}^{i+1}(M, N) \rightarrow \cdots$$

Proof. See [11, Lemma 3.1] for the proof. \square

To prove the following result, we employ the method similar to that followed in [41, Theorem 2.7].

Theorem 4.2.7. *Let t be a positive integer or ∞ . If \mathfrak{d} is an ideal such that $\mathfrak{a} \subseteq \mathfrak{d} \subseteq \mathfrak{b}_t$, then $H_{\mathfrak{d}}^i(M, N) \cong H_{\mathfrak{a}}^i(M, N)$ for all $i < t$.*

Proof. If $\mathfrak{a} = \mathfrak{d}$, there is nothing to prove. Hence we may assume that $\mathfrak{a} \subsetneq \mathfrak{d} \subseteq \mathfrak{b}_t$. Then there exists $x \in \mathfrak{d} \setminus \mathfrak{a}$. We shall show that $H_{\mathfrak{a}+(x)}^i(M, N) \cong H_{\mathfrak{a}}^i(M, N)$ for all $i < t$. By Lemma 4.2.6, there exists a long exact sequence

$$\cdots \rightarrow H_{\mathfrak{a}R_x}^{i-1}(M_x, N_x) \rightarrow H_{\mathfrak{a}+(x)}^i(M, N) \rightarrow H_{\mathfrak{a}}^i(M, N) \rightarrow H_{\mathfrak{a}R_x}^i(M_x, N_x) \rightarrow \cdots$$

Since $H_{\mathfrak{a}R_x}^i(M_x, N_x) \cong H_{\mathfrak{a}}^i(M, N)_x$, it follows that $\text{Ass}_{R_x} H_{\mathfrak{a}R_x}^i(M_x, N_x) = \{ \mathfrak{p}R_x \mid \mathfrak{p} \in \text{Ass}_R H_{\mathfrak{a}}^i(M, N), x \notin \mathfrak{p} \}$. We shall see $H_{\mathfrak{a}R_x}^i(M_x, N_x) = 0$ for all $i < t$. If $\Sigma_t = \emptyset$, then $\text{Ass}_R H_{\mathfrak{a}}^i(M, N) = \emptyset$ for all $i < t$. Hence $\text{Ass}_{R_x} H_{\mathfrak{a}R_x}^i(M_x, N_x) = \emptyset$ for all $i < t$. So we have $H_{\mathfrak{a}R_x}^i(M_x, N_x) = 0$ for all $i < t$. Assume that $\Sigma_t \neq \emptyset$. We suppose that $\text{Ass}_{R_x} H_{\mathfrak{a}R_x}^i(M_x, N_x) \neq \emptyset$ for some $i < t$ and look for a contradiction. Then there exists an ideal $\mathfrak{q} \in \text{Ass}_R H_{\mathfrak{a}}^i(M, N)$ which does not contains x . Since $x \in \mathfrak{b}_t = \bigcap_{\mathfrak{p} \in \text{Min } \Sigma_t} \mathfrak{p}$, \mathfrak{q} is in $\Sigma_t \setminus \text{Min } \Sigma_t$. Then there exists an ideal $\mathfrak{q}' \in \text{Min } \Sigma_t$ such that $\mathfrak{q}' \subset \mathfrak{q}$. On the other hand, x is included in \mathfrak{q}' . Then x is in \mathfrak{q} , which is a contradiction. Hence

$\text{Ass}_{R_x} H_{\mathfrak{a}R_x}^i(M_x, N_x) = \emptyset$ for all $i < t$. So we deduce that $H_{\mathfrak{a}R_x}^i(M_x, N_x) = 0$ for all $i < t$. Therefore, from the above long exact sequence, it follows that $H_{\mathfrak{a}+(x)}^i(M, N) \cong H_{\mathfrak{a}}^i(M, N)$ for all $i < t$.

If $\mathfrak{a} + (x) = \mathfrak{d}$, there is nothing to prove. In the case that $\mathfrak{a} + (x) \subsetneq \mathfrak{d}$, replace \mathfrak{a} with $\mathfrak{a} + (x)$ and continue the same process. Now R is Noetherian, thus we have $H_{\mathfrak{d}}^i(M, N) \cong H_{\mathfrak{a}}^i(M, N)$ for all $i < t$ in a finite number of steps. \square

We close this thesis with the following remark.

Remark 4.2.8. For arbitrary (not necessarily finitely generated) R -modules M and N , one can also define the i th generalized local cohomology module $H_{\mathfrak{a}}^i(M, N) = \varinjlim \text{Ext}_R^i(M/\mathfrak{a}^n M, N)$. Let S be a multiplicatively closed subset of R . By Proposition 1.3.1, the colocalization functor γ_W on the derived category $\mathcal{D} = D(\text{Mod } R)$ coincides with $\text{R}\Gamma_{V(\mathfrak{a})} \text{RHom}_R(S^{-1}R, -)$, where $W = V(\mathfrak{a}) \cap U_S$. Then it is seen that

$$H^i(\gamma_W N) \cong \varinjlim \text{Ext}_R^i(S^{-1}R/\mathfrak{a}^n S^{-1}R, N) = H_{\mathfrak{a}}^i(S^{-1}R, N).$$

In this sense, the colocalization functor γ_W has connection with generalized local cohomology. However, the infiniteness of $S^{-1}R$ causes some difficulties concerning computation of $H_{\mathfrak{a}}^i(S^{-1}R, N)$. For example, when R is an integral domain, we have $H^i(\gamma_{\{(0)\}} R) \cong \text{Ext}_R^i(Q(R), R) = H_{(0)}^i(Q(R), R)$. One may notice from Chapter 3 that it is hard work to compute $\text{Ext}_R^i(Q(R), R)$ in general. On the other hand, as shown in Theorem 1.6.5, Grothendieck type vanishing holds for colocalization functors with supports in arbitrary subsets; it does not necessarily hold for generalized local cohomology.

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