

CHARACTERIZATION OF AN ALGEBRAIC STRUCTURE AND ITS
FUZZIFICATIONS

MUHAMMAD AKRAM

A thesis submitted in
fulfillment of the requirement for the award of the
Doctor of Philosophy in Science

Faculty of Science, Technology and Human Development
Universiti Tun Hussein Onn Malaysia

JANUARY 2017

To my mother: the signs of love, support and encouragement

ACKNOWLEDGEMENT

In the name of Allah, the Beneficent, the Merciful.

First and above all, I praise Allah SWT, the Almighty for providing me this opportunity and granting me the ability to proceed successfully. In completing this study, I met many people who have contributed a lot towards my understanding. I am extremely grateful to my Supervisor Associate Professor Dr. Kavikumar Jacob, Co-Supervisor Associate Professor Dr. Azme Bin Khamis and Dr. Nor Shamsidah for supporting me throughout the years and for providing technical guidance, supervision, encouragement and showing a great deal of patience. May Allah SWT keep all of them happy forever! I am also deeply grateful to my co-supervisor Assistant Professor Dr. Zaffar Iqbal, University of Gujrat, Pakistan, a real friend and adviser throughout the course of study. Also, I would like to express my special thanks to Universiti Tun Hussein Onn Malaysia and Office for Research, Innovation, Commercialization, Consultancy, and Management (ORICC), Center for Graduate Studies (CGS) and Faculty of Science, Technology and Human Development (FSTPi), for providing such a wonderful and research-friendly environment at the University. Furthermore, I would like to express my sincere gratitude other UTHM staff including all postgraduate students for providing a healthy research atmosphere.

I am also thankful to my family for believing in me and supporting me in all my endeavors. In the end, I would like to thank my wife "Alia" my daughters "Amna, Hira" and my son "Abdullah" who have brought a lot of joy and colors to my life.

Muhammad Akram

ABSTRACT

Semigroups are very important in many areas of mathematics and play a very significant role in the development of mathematics. The concept of Γ -semigroup was given as a generalization of semigroup and ternary semigroup. Fuzzy set and its extensions are equally important and have a lot of applications in every field of life. Some very important extensions of algebraic structures are their fuzzifications and intuitionistic fuzzifications. In this research, some work on different aspects of an algebraic structure and its fuzzifications are given. Here the concept of bi Γ -ternary semigroup is given as a generalization of ternary semigroup and Γ -semigroup. The notions of bi Γ -ternary subsemigroup, bi Γ -ternary left (right, lateral, bi and quasi) ideals of bi Γ -ternary semigroup are defined and characterized. Some properties of these ideals and relationships between them are discussed. Moreover, the regular bi Γ -ternary semigroup is defined and characterized in terms of its ideals. Furthermore, the concept of intuitionistic N-fuzzy set as a new extension of fuzzy set and intuitionistic fuzzy set is defined by considering the negative meanings of information. The notions of level set and characteristic functions of intuitionistic N-fuzzy set is also defined here. As the applications of intuitionistic N-fuzzy set, the notion of this set is applied to the ideal theory of some algebraic structures such as semigroup, Γ -semigroup and the bi Γ -ternary semigroup to get the corresponding intuitionistic N-fuzzy structures and substructures and characterized them. Finally, the concept of fuzzy soft set and bipolar fuzzy soft set is applied to the ideals of Γ -semigroups and bi Γ -ternary semigroups. Some special operations, OR, AND, extended union, extended intersection, restricted union and restricted intersection between the fuzzy soft and bipolar fuzzy soft ideals of Γ -semigroup and bi Γ -ternary semigroup are discussed. It is also proved that these operations also induced same type fuzzy soft and bipolar fuzzy soft ideals of Γ -semigroup and bi Γ -ternary semigroup.

ABSTRAK

Semikumpulan adalah sangat penting di dalam kebanyakan bahagian matematik dan ia memainkan satu peranan yang signifikan dalam perkembangan matematik. Konsep Γ -semikumpulan diberi sebagai semikumpulan generalisasi dan semikumpulan pertigaan. Set kabur dan sambungannya adalah sama penting dan mempunyai banyak aplikasi di dalam semua bidang kehidupan. Sebahagian sambungan struktur aljabar yang sangat penting ialah 'fuzzifications' dan 'intuitionistic fuzzifications'. Di dalam kajian ini diberi sedikit kerja tentang aspek yang berbeza tentang struktur aljabar dan fuzzifications. Disini konsep $bi\Gamma$ -pertigaan semikumpulan diberi sebagai generalisasi semikumpulan pertigaan dan Γ -semikumpulan. Fahaman tentang $bi\Gamma$ -subsemikumpulan pertigaan, $bi\Gamma$ -pertigaan kiri (kanan, rusuk, bi dan kuasi) ideal $bi\Gamma$ -semikumpulan pertigaan didefinisikan dan dicirikan. Sebahagian ciri-ciri ideal ini dan perhubungan diantara mereka dibincangkan. Lebih-lebih lagi, $bi\Gamma$ -semikumpulan pertigaan yang biasa ditakrifkan dan disifatkan berdasarkan terma idealnya. Tambahan pula, konsep intuitionistic set N-kabur sebagai satu sambungan set kabur yang baru dan intuitionistic set kabur ditakrifkan dengan pertimbangan maksud negatif informasi. Fahaman tentang kategori set dan ciri-ciri fungsi intuitionistic set N-kabur juga didefinisikan disini. Seperti aplikasi intuitionistic set N-kabur, fahaman mengenai set ini diaplikasikan keatas teori ideal kemas sebahagian struktur aljabar seperti semikumpulan, Γ -semikumpulan dan $bi\Gamma$ -semikumpulan pertigaan untuk mendapatkan struktur intuitionistic set N-kabur dan substruktur yang sepadan dan dicirikannya. Akhirnya, konsep set kabur lembut dan set bipolar kabur lembut diaplikasikan kepada ideal Γ -semikumpulan dan $bi\Gamma$ -semikumpulan pertigaan. Sedikit operasi khas, ATAU, DAN, lanjutan kesatuan, lanjutan intersekyen, kesatuan terhadap dan intersekyen terhadap diantara kabur lembut dan ideal bipolar kabur soft Γ -semikumpulan dan $bi\Gamma$ -semikumpulan pertigaan dibincangkan. Ia juga membuktikan

yang operasioperasi ini juga didorong oleh set kabur dan bipolar kabur lembut ideal Γ -semikumpulan dan $\text{bi}\Gamma$ -semikumpulan pertigaan yang sama jenis.

CONTENTS

DECLARATION	ii
DEDICATION	iii
ACKNOWLEDGEMENT	iv
ABSTRACT	v
ABSTRAK	vi
CONTENTS	viii
LIST OF FIGURES	xi
LIST OF SYMBOLS AND ABBREVIATIONS	xii
LIST OF APPENDICES	xiv
 CHAPTER 1 INTRODUCTION	 1
1.1 Research Background	1
1.2 Motivation	2
1.3 Problem Statement	2
1.4 Aim of the Research	3
1.5 Research Objective	4
1.6 Scope and Limitation	4
1.7 Research Contributions	5
1.8 Thesis Organization	5
 CHAPTER 2 LITERATURE REVIEW	 7
2.1 Introduction	7
2.2 Preliminaries	9
2.2.1 Semigroup	10
2.2.2 Ternary semigroup	10
2.2.3 Γ -semigroup	11

2.2.4	Fuzzy set and intuitionistic fuzzy set	12
CHAPTER 3 BI Γ-TERNARY SEMIGROUP AND ITS BI Γ-IDEALS		14
3.1	Bi Γ -ternary semigroup	14
3.1.1	Basic definitions and examples	14
3.1.2	Bi Γ -ideals of bi Γ -ternary semigroup	17
3.1.3	Bi Γ -quasi and bi Γ -bi-ideals of bi Γ -ternary semigroup	22
3.1.4	Regular bi Γ -ternary semigroups	28
3.2	Chapter summary	34
CHAPTER 4 FUZZY AND N-FUZZY BIΓ-TERNARY SEMIGROUPS		35
4.1	Fuzzy bi Γ -ternary semigroup	35
4.1.1	Fuzzy bi Γ -ideals of bi Γ -ternary semigroup	35
4.1.2	Fuzzy bi Γ -quasi and fuzzy bi Γ -bi ideals of bi Γ -ternary semigroup	42
4.2	N-fuzzy bi Γ -ternary semigroups	54
4.2.1	N-fuzzy set	54
4.2.2	N-fuzzy bi Γ -ideals of bi Γ -ternary semigroup	56
4.2.3	N-fuzzy bi Γ -quasi and bi Γ -bi-ideals of bi Γ -ternary semigroup	63
4.3	Chapter summary	75
CHAPTER 5 INTUITIONISTIC N-FUZZY SET AND ITS APPLICATIONS		76
5.1	Intuitionistic N-fuzzy set	76
5.1.1	Basic definitions and examples	77
5.2	Intuitionistic N-fuzzy semigroup	79
5.2.1	Intuitionistic N-fuzzy ideals of semigroup	79
5.2.2	Intuitionistic N-fuzzy quasi and bi-ideals of semigroup	86
5.3	Intuitionistic N-fuzzy Γ -semigroups	97
5.3.1	Intuitionistic N-fuzzy Γ -ideals of Γ -semigroup	97
5.3.2	Intuitionistic N-fuzzy Γ -quasi and Γ -bi-ideals of Γ -semigroups	105

5.4	Intuitionistic N-fuzzy $\text{bi}\Gamma$ -ternary semigroups	115
5.4.1	Product of three INFSs	116
5.4.2	Intuitionistic N-fuzzy $\text{bi}\Gamma$ -ideals of $\text{bi}\Gamma$ -ternary semigroups	116
5.4.3	Intuitionistic N-fuzzy $\text{bi}\Gamma$ -quasi and $\text{bi}\Gamma$ -bi-ideals of $\text{bi}\Gamma$ -ternary semigroups	125
5.5	Chapter summery	140
CHAPTER 6 FUZZY SOFT AND BIPOLAR FUZZY SOFT BI Γ-TERNARY SEMIGROUP		142
6.1	Fuzzy soft Γ -semigroup and $\text{bi}\Gamma$ -ternary semigroup	142
6.1.1	Fuzzy soft set	142
6.1.2	Fuzzy soft Γ -ideals over Γ -semigroup	144
6.1.3	Fuzzy soft $\text{bi}\Gamma$ -ideals over $\text{bi}\Gamma$ -ternary semigroup	151
6.2	Bipolar fuzzy soft Γ -semigroup and $\text{bi}\Gamma$ -ternary semigroup	164
6.2.1	Bipolar fuzzy soft set	164
6.2.2	Bipolar fuzzy soft Γ -ideals over Γ -semigroup	168
6.2.3	Bipolar fuzzy soft $\text{bi}\Gamma$ -ideals over $\text{bi}\Gamma$ -ternary semigroup	182
6.3	Chapter summary	194
CHAPTER 7 CONCLUSIONS AND FUTURE WORK		195
7.1	Conclusions	195
7.2	Future Work	196
REFERENCES		198
APPENDIX A		207

LIST OF FIGURES

6.1	Fuzzy Soft Set	143
6.2	Bipolar Fuzzy Set	165
6.3	Bipolar Fuzzy Soft Set	166

LIST OF SYMBOLS AND ABBREVIATIONS

A	–	Ideal
B	–	Bi-ideal
BF^U	–	Set of all bipolar fuzzy subsets of U .
E	–	Set of parameters
$F(U)$	–	Set of all fuzzy subsets of U
I	–	Indexing set
i	–	Element of I
L	–	Left ideal
M	–	Lateral ideal
N	–	Set of natural numbers
Q	–	Quasi ideal
R	–	Right ideal
S	–	Semigroup or Γ -semigroup as stated
T	–	Bi Γ -ternary semigroup
U	–	Initial universe set
Z	–	Set of integers
Z^-	–	Set of negative integers
Z_0^-	–	Set of non-positive integers
μ	–	Fuzzy set
ν	–	Fuzzy set
\bigvee	–	Join operation
\bigwedge	–	Meet operation
\circ	–	Special product
$\bar{\cap}$	–	Intersection of intuitionistic N-fuzzy sets
μ^t	–	Level set of μ
$\bar{\mu}$	–	N-fuzzy set
$\bar{\cup}$	–	Union of intuitionistic N-fuzzy sets
$\bar{\chi}_C$	–	N-fuzzy characteristic function of C
χ_C	–	Characteristic function of C
\cup_E	–	Extended union
\cap_E	–	Extended intersection
$(t)_m$	–	Principal bi Γ -lateral ideal
$(t)_l$	–	Principal bi Γ -left ideal
$(t)_r$	–	Principal bi Γ -right ideal
\cap_R	–	Restricted intersection
\cup_R	–	Restricted union
\circ_Γ	–	Special product of Γ -structures

$\langle \mu_A^+, \mu_A^- \rangle$	–	Bipolar fuzzy set
$(\widehat{\mu}, A)$	–	Fuzzy soft set
(μ_A, γ_A)	–	Intuitionistic fuzzy set
$(\overline{\mu}_{\overline{X}_C}, \overline{\gamma}_{\overline{X}_C})$	–	Intuitionistic N-fuzzy characteristic function of C
$(\overline{\mu}_A, \overline{\gamma}_A)$	–	Intuitionistic N-fuzzy set
$N_A(t, s)$	–	Level set of intuitionistic N-fuzzy set A
$N(\overline{\mu}, t)$	–	Level set of N-fuzzy set $\overline{\mu}$
BCK algebra	–	Algebra of combinators of B,C and K
BCI algebra	–	Algebra of combinators of B,C and I
$\text{Bi}\Gamma\text{BI}(s)$	–	$\text{Bi}\Gamma$ -bi-ideal(s)
$\text{Bi}\Gamma\text{I}(s)$	–	$\text{Bi}\Gamma$ -ideal(s)
$\text{Bi}\Gamma\text{LI}(s)$	–	$\text{Bi}\Gamma$ -left ideals(s)
$\text{Bi}\Gamma\text{MI}(s)$	–	$\text{Bi}\Gamma$ -lateral ideal(s)
$\text{Bi}\Gamma\text{QI}(s)$	–	$\text{Bi}\Gamma$ -quasi-ideal(s)
$\text{Bi}\Gamma\text{RI}(s)$	–	$\text{Bi}\Gamma$ -right ideals(s)
$\text{Bi}\Gamma\text{T}(s)$	–	$\text{Bi}\Gamma$ -ternary semigroup(s)
$\text{Bi}\Gamma\text{TSS}(s)$	–	$\text{Bi}\Gamma$ -ternary subsemigroup(s)
INFS	–	Intuitionistic N-fuzzy set
INFS(s)	–	Intuitionistic N-fuzzy set(s)
INFSS	–	Intuitionistic N-fuzzy subset
	–	

LIST OF APPENDICES

A	LIST OF PUBLICATIONS	207
---	----------------------	-----

CHAPTER 1

INTRODUCTION

The theory of semigroups is similar to group theory and ring theory. The earliest major contributions to the theory of semigroups are strongly motivated by comparisons with groups and rings. Semigroup theory can be considered as one of the most successful off-springs of ring theory in the sense that the ring theory gives a clue how to develop the ideal theory of semigroups. There are several algebraic structures introduced by many authors as algebraic extensions of the semigroup such as, ternary semigroup, LA-semigroup, Γ -semigroup and so on. A semigroup S is a nonempty set along with a closed and associative binary operation whereas a ternary semigroup is a nonempty set T along with a closed ternary operation satisfying a special associative law, $((abc)de) = (a(bcd)e) = (ab(cde))$. It is clear that any ordinary semigroup $(S; *)$ induces a ternary semigroup $(S; [\])$ by putting $[abc] = a*b*c$ but a ternary semigroup may not be a semigroup e.g. $T = \{-i, 0, i\}$ is a ternary semigroup which is not a semigroup under the multiplication of complex numbers. Also $Z^- = \{-1, -2, -3, \dots\}$, is a ternary semigroup but not a semigroup under the integers multiplication. The fuzzy algebraic structures and their extensions are very important. Nowadays, a lot of extensions of fuzzy algebraic structures have been introduced by many authors and have been applied to real life problems in different fields of science.

1.1 Research Background

In 1981, Sen [1] introduced the concept of Γ -semigroup as a generalization of semigroup and ternary semigroup. He combined a nonempty set Γ with another nonempty set S to form this structure. He defined an operation from $S \times \Gamma \times S \rightarrow S$ with extended associative law for Γ -semigroup. Another very important

extension of algebraic structures is their fuzzifications and intuitionistic fuzzifications. Many classical notions of algebraic structures have been extended to the fuzzy and intuitionistic fuzzy structures. The concept of fuzzy set was given by Zadeh [2] in 1965. The study of fuzzy algebraic structures was started by Rosenfeld [3], when he introduced the concepts of fuzzy subgroup in 1971. Kuroki [4, 5, 6, 7, 8] applied the concept of fuzziness to the semigroups and introduced the notion of fuzzy ideals of semigroups. The concept of intuitionistic fuzzy set was introduced by Atanassov [9] as a generalization of fuzzy set in 1986. Biswas [10] introduced the concept of intuitionistic fuzzy subgroupoids. Kim and Jun [11] applied the concept of intuitionistic fuzzification to the ideal of semigroups.

1.2 Motivation

The idea of construction of Γ -algebraic structures like Γ -semigroup, Γ -semiring, Γ -ring, Γ -near ring, motivated us to extend this work to form some further extensions of Γ -algebraic structure so we constructed $\text{bi}\Gamma$ -ternary semigroup. We also inspired from the different extensions of fuzzy set particularly L-fuzzy set, N-fuzzy set, intuitionistic fuzzy set and we constructed intuitionistic N-fuzzy set as an extensions of fuzzy set. The concept of fuzzy and intuitionistic fuzzy algebraic structures gives us the idea about the fuzzifications of this constructed structure. Finally the notions of fuzzy soft set and bipolar fuzzy soft set takes our attention towards the application of these notions on the Γ -algebraic structure which we have constructed.

1.3 Problem Statement

Construction of some new algebraic structure and its fuzzifications is not a straightforward work. It needs the understanding and application of several theories but when a new structure is formed it opens a new chapter for the researchers to write about it and soon a new theory is created. In the literature, there are many Γ -extensions of algebraic structure such as, Γ -groupoid, Γ -AG-groupoid, Γ -semigroup, Γ -semiring, Γ -ring, Γ -near-ring and so on. In all these structures researchers combined the nonempty set Γ with some other nonempty set S (say) with some binary operations to form these structures. Till now, no any further generalizations of Γ -structures have been given, so

in the first part of this research we are planing to extend the concept of Γ -extensions to Γ - Γ -extensions or $\text{bi}\Gamma$ -extensions of some algebraic structure. After the construction of this algebraic structure, the notions of substructures are defined and their different properties are investigated. Zadeh [2] introduced the concept of fuzzy set in 1965, after that many extensions of fuzzy set such as rough set, fuzzy rough set, bipolar fuzzy set and intuitionistic fuzzy set etc. have been introduced so far. In all fuzzy set and its generalizations only positive meanings of information have been used. Jun et al.[12] used the negative meanings of information for fuzzy set and introduced the notion of N-fuzzy set. Still there is a need of some generalization of intuitionistic fuzzy set by considering the negative meanings of information to handle the real life problems using uncertainty and negativity. This gives the idea for the constructions of intuitionistic N-fuzzy set. In the second part of this research, the concept of fuzzy, N-fuzzy (negative fuzzy) and intuitionistic N-fuzzy set is applied to the algebraic structure which we have defined in first part of this research to get the corresponding fuzzy, N-fuzzy and intuitionistic N-fuzzy algebraic structures. At the end, some works on the fuzzy soft and bipolar fuzzy soft ideals of Γ -semigroup $\text{bi}\Gamma$ -ternary semigroup are done and some characterizations of these ideals in terms of some special operations defined between them are given.

1.4 Aim of the Research

The aim of this research is to construct an algebraic structure $\text{bi}\Gamma$ -ternary semigroup, its fuzzifications and the characterizations of this algebraic structure as well as the characterization of its fuzzifications. To achieve this aim, first of all an algebraic structure $\text{bi}\Gamma$ -ternary semigroup and its substructures (ideals) are defined and characterized. After the construction of ideal theory of $\text{bi}\Gamma$ -ternary semigroup, the concepts of fuzzy set and N-fuzzy set are applied to form the corresponding fuzzy and N-fuzzy structures. In the second step, intuitionistic N-fuzzy set is defined by considering the negative meanings of information and the basic operations between the intuitionistic N-fuzzy sets are discussed. Then the concept of intuitionistic N-fuzzy set is applied to some already existing algebraic structures like, semigroup, Γ -semigroup, ternary semigroup to check the validity of intuitionistic N-fuzzy set and finally it is applied to algebraic structure $\text{bi}\Gamma$ -ternary semigroup to obtain the final

structure intuitionistic N-fuzzy $\text{bi}\Gamma$ -ternary semigroup. Moreover the concept of fuzzy soft set and bipolar fuzzy soft set is applied to characterize Γ -semigroups and $\text{bi}\Gamma$ -ternary semigroup in terms of its fuzzy soft and bipolar fuzzy soft ideals.

1.5 Research Objective

This research has some specific objectives which are outlined below:

- 1 To develop a new algebraic structure $\text{bi}\Gamma$ -ternary semigroup and its examples .
- 2 To define the notions of ideals for this structure and characterized it by these ideals.
- 3 To develop the fuzzifications and N-fuzzifications of the algebraic structure.
- 4 To develop the theory of intuitionistic N-fuzzy set and intuitionistic N-fuzzifications of this structure.
- 5 To develop the theory of fuzzy soft and bipolar fuzzy soft ideals of $\text{bi}\Gamma$ -ternary semigroup.

1.6 Scope and Limitation

The scope and limitations of this work are given as under.

- 1 A new algebraic structure $\text{bi}\Gamma$ -ternary semigroup is defined and explained by constructing some examples and counter examples.
- 2 Some substructures and different ideals of $\text{bi}\Gamma$ -ternary semigroup are defined and characterized in this study.
- 3 The fuzzifications and N-fuzzifications of $\text{bi}\Gamma$ -ternary semigroup are defined and characterized by their ideals.
- 4 The intuitionistic N-fuzzy set and its basic operations along with examples are defined.
- 5 The applications of intuitionistic N-fuzzy set on some algebraic structures, particularly on $\text{bi}\Gamma$ -ternary semigroup are studied.
- 6 The scope of this study also covers some characterizations of fuzzy soft and bipolar fuzzy soft ideals of $\text{bi}\Gamma$ -ternary semigroup.

1.7 Research Contributions

This research provides the following contributions to the knowledge in the fields of semigroup theory and its generalizations, fuzzy set and its generalizations, fuzzy algebraic structures, intuitionistic and bipolar fuzzy algebraic structures and their generalizations:

- 1 A new algebraic structure $\text{bi}\Gamma$ -ternary semigroup and its ideals are defined and characterized.
- 2 A new generalization "intuitionistic N-fuzzy set" of fuzzy set and intuitionistic fuzzy set is given. The basic operations between these intuitionistic N-fuzzy sets along with suitable examples are discussed.
- 3 The fuzzifications and N-fuzzifications of ideals of $\text{bi}\Gamma$ -ternary semigroup are defined and characterized.
- 4 Intuitionistic N-fuzzy $\text{bi}\Gamma$ -ternary semigroup and its intuitionistic N-fuzzy $\text{bi}\Gamma$ -ideals are defined. Furthermore, the relationships between these intuitionistic N-fuzzy $\text{bi}\Gamma$ -ideals of $\text{bi}\Gamma$ -ternary semigroup are discussed and characterized using their properties.
- 5 Some special operations between the fuzzy soft and bipolar fuzzy soft ideals of $\text{bi}\Gamma$ -ternary semigroup are discussed.

1.8 Thesis Organization

This thesis is organized and divided into seven chapters including the introduction and conclusion.

In Chapter 1, an overview on the background of the research as well as the problem statement, research objectives, scope of the research and the research contributions are outlined.

In Chapter 2, a detailed survey of the works done previously in the related fields with references as literature review is given. The preliminary concepts required to read the subsequent chapters are also given.

In Chapter 3, an algebraic structure "Bi Γ -Ternary Semigroup" as a new generalization of ternary semigroup and Γ -semigroup is proposed. Some ideals such as $\text{bi}\Gamma$ -ideals (left, right, lateral, quasi and bi) for this structure are defined and different

properties of these ideals are investigated. The regular $\text{bi}\Gamma$ -ternary semigroup and its ideal are also defined and characterized.

Chapter 4 is divided into two sections. In Section 4.1, the concept of fuzzy set is applied to the propose structure and its ideals to get their fuzzifications and characterize them by using different properties of these fuzzy ideals. In Section 4.2, the concept of N-fuzzy set is applied to the algebraic structure "Bi Γ -Ternary Semigroups" and its ideals to get the corresponding N-fuzzifications and characterize them by using these ideals.

There are four sections in Chapter 5. In Section 5.1, intuitionistic N-fuzzy set is defined as a new generalization of fuzzy set and intuitionistic fuzzy set. Some basic operations between these intuitionistic N-fuzzy sets along with some examples are also defined. In Section 5.2 and Section 5.3, the concept of intuitionistic N-fuzzy set is applied to the ideal theory of semigroups and Γ -semigroups to produce the corresponding intuitionistic N-fuzzy ideals of these structures and characterize them by these intuitionistic N-fuzzy ideals. In Section 5.4, both of the proposed concepts i.e., "Intuitionistic N-Fuzzy Set" and "Bi Γ -Ternary semigroup" are combined to proposed "Intuitionistic N-fuzzy bi Γ -ternary semigroup". We define intuitionistic N-fuzzy bi Γ -ideals (left, right, lateral, quasi and bi) of intuitionistic N-fuzzy bi Γ -ternary semigroup and characterize it by these ideals.

In Chapter 6, there are two sections. In Section 6.1, fuzzy soft sets over Γ -semigroup and bi Γ -ternary semigroup are discussed. We define different fuzzy soft Γ -ideals (left, right, quasi and bi) of Γ -semigroup and bi Γ -ternary semigroup. We also define and characterize the special operations "AND","OR", extended union, extended intersection, restricted union and restricted intersection between these ideals. In Section 6.2, the bipolar fuzzy soft set over Γ -semigroup and bi Γ -ternary semigroup are discussed. The bipolar fuzzy soft ideals of Γ -semigroup and bi Γ -ternary semigroup are defined and the special operations, which we discussed in Section 6.1 for fuzzy soft Γ -ideals are also discussed for bipolar fuzzy soft ideals.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This chapter is a description of literature survey of the previously done research work in the directions of our research. In this research, a latest generalization of some algebraic structure and its fuzzy extensions has been proposed. For our proposed research, it need to discuss the ideal theory of "semigroup", "ternary semigroup", Γ - semigroup and the corresponding fuzzy and intuitionistic fuzzy algebraic structures.

The algebraic theory of semigroups was widely studied by Clifford and Preston [13, 14], Petrich [15, 16, 17] and Ljapin [18]. They all studied the notion of the ideal in semigroups. Good and Hughes [19] and Lajos [20] introduced the notion of bi-ideals in semigroup. Lajos [21] and Szasz [22, 23] gave the notion of interior ideals in semigroup. Steinfeld [24] introduced the notion of quasi-ideals in semigroups. Prime ideals in semigroup were presented by Grimble [25]. The ideal theory in general semigroups was developed by Hoehnke [26], Schwarz [27], Anjaneyulu [28, 29], Giri and Wazalwar [30].

In 1932 Lehmer [31] gave the formal definition of a ternary semigroup but earlier such structures were studied by Kasner [32] and Prufer [33]. Sioson [34] worked on the ideal theory in ternary semigroups. He introduced the notion of prime ideal, semiprime ideal, quasi-ideal and study regular ternary semigroup by using these ideals. Dixit and Dewan [35] studied the notions of quasi-ideal and bi-ideal in ternary semigroups. Santiago [36] developed the theory of ternary semigroups and semiheaps. Dutta et al. [37] characterized regular ternary semigroups by their ideals.

As a generalization of semigroup and ternary semigroup, Sen [1] introduced the notion of Γ -semigroup in 1981 and developed some theory on Γ -semigroups [38].

Many classical notions of semigroups have been extended to Γ -semigroups by Saha [39, 40]. The notion of quasi ideal and bi-ideals in Γ -semigroups were introduced by Chinram [41], Chinram and Jirojukul [42], Moin and Khan [43].

In 1965 Zadeh [2] gave the concept of fuzzy set. The fuzzy set developed by Zadeh and its extensions by others have many applications in the domain of mathematics. The study of fuzzy algebraic structures started with the introduction of the concepts of fuzzy subgroup (subgroupoid) and fuzzy (left, right) ideal in the pioneering paper by Rosenfeld [3] in 1971. The detailed study of fuzzy semigroup was done by Kuroki [4, 5, 6, 7, 8]. Many authors have worked on fuzzy semigroups, some detailed work can be found in [44] and [45, 46, 47, 48, 49, 50].

Petchkhaew and Chinram [51] proposed fuzzy rough and rough fuzzy ideals of ternary semigroups. Kar and Sarkar [52, 53] worked on fuzzy ideals, fuzzy quasi and fuzzy bi-ideals of ternary semigroups. Rehman and Shabir [54, 55, 56] worked on some special type of fuzzy ideals of ternary semigroups. Williams et al. [57] introduced the concept of fuzzy bi- Γ -ideals of Γ -semigroups. Dutta et. al. [58, 59] and Sardar et al. [60, 61] introduced several notions of fuzzy ideals of Γ -semigroups. In literature, we can find a lot of work done by many authors on fuzzy ideals of ternary semigroups and Γ -semigroups [see, [62], [63], [64], [65], [66], [67], [68], [69]].

The concept of intuitionistic fuzzy set was introduced by Atanassov [9] as a generalization of fuzzy set in 1983 (see also [70, 71]). Biswas [10] introduced the concept of intuitionistic fuzzy subgroupoids. Kim and Jun [11, 72] applied the concept of intuitionistic fuzzy sets to the ideal theory of semigroup and defined several ideals of semigroup. Hur et al. [73], Hong and Jiang [74], Kim and Lee [75], Ozturk et al. [76], Shabir et al. [77] and many other authors have applied the concept of intuitionistic fuzzy set to the ideal theory of semigroups. Lekkoksung [78] worked on intuitionistic fuzzy bi-ideals of ternary semigroups. Akram [79] discussed ternary semigroups in terms of intuitionistic fuzzy points and intuitionistic fuzzy ideals. Yaqoob et al. [80], Akram and Tyagi [81] discussed different intuitionistic fuzzy ideals of ternary semigroups.

Mustafa et al. [82] applied the concept of intuitionistic fuzzy set to Γ -semigroup and characterized several ideals of Γ -semigroup in terms of intuitionistic fuzzy sets. Sardar et al. [83], Davvaz and Majumder [84], Palaniappan and Lalithamani

[85], Akram [86] and many other authors have applied intuitionistic fuzzy sets to the ideal theory of Γ -semigroups [see, [87], [88], [89]]. The concept of N-fuzzy set was given by Jun et al. [12]. They applied this concept to BCK/BCI-algebras which was extended to other structures by Khan et al. [90, 91], Jun et al. [92], and Akram et al. [93], Williams and Saeid [94].

The concept of soft set was initiated by Molodtsov [95] in 1999. He used this concept for the modeling of uncertainty. Maji et al. [96], defined some binary operations on soft sets, which were later corrected by Ali et al. [97]. Shabir and Ali [98], introduced the notion of soft semigroups. The soft ternary semigroups were studied by Shabir and Ahmad [99]. Changphas and Thongkam [100], gave the notion of soft Γ -semigroups. In 2001 Maji et al. [101], introduced the notion of fuzzy soft set as a combination of fuzzy set and soft set. They studied the union, intersection, compliment and De Morgan Law etc. for fuzzy soft sets. Ahmad and Kharal [102], improved the results of [101]. Aygunoglu and Aygun [103] extended Aktas and Cagman [104], soft groups concept for fuzzy soft groups. Irfan and Shabir [105] redefined the basic operations between the fuzzy soft sets defined by Maji et al. in [101]. Yang [106], introduced the notion of fuzzy soft semigroups and fuzzy soft ideals. Recently, Bora et al. [107], defined some operations of fuzzy soft sets and explained them with examples.

Zhang [108, 109], introduced the notion of bipolar fuzzy sets and used it for modeling and decision analysis. Lee [110, 111], used the term bipolar valued fuzzy sets and applied to algebraic structures. The notion of fuzzy bipolar soft sets and bipolar fuzzy soft was introduced by Naz and Shabir [112]. They defined their special union and special intersection and also showed that the both notions are equivalent. Earlier Yang and Li [106], used the term bipolar-value fuzzy soft set for this notion and defined their operations. Abdullah et al. [113], also worked on bipolar fuzzy soft sets and their special union and intersection.

2.2 Preliminaries

In this section some preliminary concepts related to this work are outlined.

2.2.1 Semigroup

The formal study of semigroup was started during the second quarter of 20th century, which became an important subject for the researcher in very short time. Most of the mathematicians worked on semigroups and proposed a lot of notions in this structure. The following definitions about semigroups can be found in [114].

A nonempty set S together with a closed and associative binary operation is called a semigroup and a nonempty subset A of a semigroup S is called a subsemigroup of S , if it is itself a semigroup under the same binary operation defined in S . Ordinarily, we take the binary operation in S as the usual multiplication of elements. The product AB of two nonempty subsets of a semigroup S is defined as the set of all elements ab , for $a \in A$ and $b \in B$. Since the associativity hold in S so it holds in A , then the condition that a subset A of S is a subsemigroup of S becomes that when, $AA \subseteq A$. A nonempty subset A of S is called a left ideal of S if $SA \subseteq A$ and a right ideal of S if $AS \subseteq A$. An ideal or a two sided ideal of S is a nonempty subset of S , which is both a left and a right ideal of S . A nonempty subset Q of S is called quasi-ideal of S if $QS \cap SQ \subseteq Q$. A subsemigroup B of S is called a bi-ideal of S if $BSB \subseteq B$.

2.2.1.1. Proposition. [114] Let S be a semigroup and X be a nonempty subset of S then

- (i) SX is a left ideal of S .
- (ii) XS is a right ideal of S .
- (iii) SXS is an ideal of S .
- (iv) $SX \cap XS$ is a quasi ideal of S .

2.2.2 Ternary semigroup

Although the study of ternary structures started earlier in the 20th century but the formal definition of ternary semigroup was given by Lehmer [31] in 1932 as:

2.2.2.1. Definition. [31] A ternary semigroup T is a nonempty set whose elements are closed under the ternary operation (usually multiplication) and satisfies the associative law defined as $[[abc]de] = [a[bcd]e] = [ab[cde]]$, for all $a, b, c, d, e \in T$. For simplicity we shall write $[abc]$ as abc .

2.2.2.2. Definition. [34] A nonempty subset A of a ternary semigroup T is called a ternary subsemigroup of T if $AAA \subseteq A$ and is called an idempotent if $AAA = A^3 = A$. A left ideal of a ternary semigroup T is a nonempty subset A of T such that $TTA \subseteq A$ and a right ideal of T if $ATT \subseteq A$ and a lateral ideal of T if $TAT \subseteq A$.

A nonempty subset A of T is called an ideal of T if it is a left, a right and a lateral ideal of T .

2.2.2.3. Definition. [34] A nonempty subset Q of T is called a quasi ideal of T if it satisfies $QTT \cap TQT \cap TTQ \subseteq Q$ and $QTT \cap TTQTT \cap TTQ \subseteq Q$.

2.2.2.4. Definition. [35] A subsemigroup B of a ternary semigroup T is called a bi-ideal of T if $BTBTB \subseteq B$.

2.2.3 Γ -semigroup

Compared to semigroup and ternary semigroup the concept of Γ -semigroup is very new. It was studied by Sen [1], in 1981 but later in 1986, Sen and Saha [38] redefined Γ -semigroup. Sen and Saha [38], Saha [39, 40], developed the ideal theory of Γ -semigroups, which became a popular area of research. They defined Γ -semigroup and its substructures as:

2.2.3.1. Definition. [38] Let $S = \{x, y, z, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be two nonempty sets. Then S is called a Γ -semigroup if it satisfies,

$$(i) \quad x\alpha y \in S$$

$$(ii) \quad (x\alpha y)\beta z = x\alpha(y\beta z), \text{ for all } x, y, z \in S \text{ and } \alpha, \beta \in \Gamma.$$

2.2.3.2. Definition. [38] A nonempty subset A of a Γ -semigroup S is called Γ -subsemigroup of S if $A\Gamma A \subseteq A$. By a left (right) Γ -ideal of a Γ -semigroup S we mean a nonempty subset A of S such that $ST A \subseteq A$ ($A\Gamma S \subseteq A$). A Γ -ideal or a two sided Γ -ideal of S is that which is both a left and a right Γ -ideal of S .

2.2.3.3. Definition. [41] A nonempty subset Q of T is called a quasi ideal of T if $Q\Gamma S \cap STQ \subseteq Q$.

2.2.3.4. Definition. [42] A Γ -subsemigroup B of a Γ -semigroup S is called a Γ -bi-ideal of S if $B\Gamma S\Gamma B \subseteq B$.

2.2.3.5. Proposition. [115] Let S be a Γ -semigroup and X be a nonempty subset of S then

- (i) $S\Gamma X$ is a Γ -left ideal of S .
- (ii) $X\Gamma S$ is a Γ -right ideal of S .
- (iii) $S\Gamma X\Gamma S$ is a Γ -ideal of S .
- (iv) $S\Gamma X \cap X\Gamma S$ is a Γ -quasi ideal of S .

2.2.4 Fuzzy set and intuitionistic fuzzy set

2.2.4.1. Definition. [2] A fuzzy set in a nonempty set X is a function $\mu : X \rightarrow [0, 1]$, where $\mu(x)$ denotes the degree of membership of $x \in X$ in $[0, 1]$.

2.2.4.2. Definition. [2] Let μ, ν be two fuzzy sets in X then μ is called a fuzzy subset of ν , written as $\mu \leq \nu$ if, $\mu(x) \leq \nu(x)$, for all $x \in X$.

2.2.4.3. Definition. [2] Let μ and ν be two fuzzy sets in X then their union and intersection is also a fuzzy set defined as

$$\begin{aligned} (\mu \cup \nu)(x) &= \max\{\mu(x), \nu(x)\}, \text{ for all } x \in X \\ \text{and } (\mu \cap \nu)(x) &= \min\{\mu(x), \nu(x)\}, \text{ for all } x \in X. \end{aligned}$$

Denote the maximum by \vee and the minimum by \wedge . Then $(\mu \cup \nu)(x) = \mu(x) \vee \nu(x)$ and $(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x)$.

2.2.4.4. Definition. [70] An intuitionistic fuzzy set (briefly, IFS), A in a nonempty set X is an object having the form $A = \{\langle x, \mu_A(x), \gamma_A(x) \rangle \mid x \in X\}$, where the functions, $\mu_A : X \rightarrow [0, 1]$ and $\gamma_A : X \rightarrow [0, 1]$ denotes the degree of membership (namely, $\mu_A(x)$) and the degree of non-membership (namely, $\gamma_A(x)$), for each $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$.

For simplicity, we shall use the notation $A = (\mu_A, \gamma_A)$ for an intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x) \rangle \mid x \in X\}$ in X .

2.2.4.5. Definition. [70] Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy sets in X then A is called an intuitionistic fuzzy subset of B , written as $A \subseteq B$ if, $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$, for all $x \in X$.

2.2.4.6. Definition. [70] Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy sets in X then their union and intersections is defined as

$$A \cup B = \{ \langle x, \max(\mu_A(x), \mu_B(x)), \min(\gamma_A(x), \gamma_B(x)) \rangle \mid x \in X \}$$

$$A \cap B = \{ \langle x, \min(\mu_A(x), \mu_B(x)), \max(\gamma_A(x), \gamma_B(x)) \rangle \mid x \in X \}.$$

CHAPTER 3

BI Γ -TERNARY SEMIGROUP AND ITS BI Γ -IDEALS

In this chapter the concept of bi Γ -ternary semigroup is given. The notions of bi Γ -ternary subsemigroup, bi Γ -left (right, lateral) ideals, bi Γ -quasi and bi Γ -bi-ideals of this structure are defined and characterized. Moreover, the regular bi Γ -ternary semigroups and their properties are studied in terms of bi Γ -ideals.

3.1 Bi Γ -ternary semigroup

As a generalization of semigroup and ternary semigroup, Sen [1], introduced the notion of Γ -semigroup and developed a theory on Γ -semigroups. Many classical notions of semigroups and ternary semigroups have been extended to Γ -semigroups.

We have inspired from the concept of ternary semigroup and Γ -semigroup and obtained a new algebraic structure called bi Γ -ternary semigroup. The word bi Γ is used due to the double appearance of the nonempty set Γ in the structure. Here, the notions of bi Γ -ternary subsemigroup, bi Γ -left (right, lateral) ideal, bi Γ -quasi ideal and bi Γ -bi-ideal are presented with the characterization of regular bi Γ -ternary semigroup by these ideals.

3.1.1 Basic definitions and examples

Some basic definitions of bi Γ -ternary semigroup and its substructures are defined in this section.

3.1.1.1. Definition. Let $T = \{x, y, z, u, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \delta, \dots\}$ be two nonempty sets and $() : T \times \Gamma \times T \times \Gamma \times T \rightarrow T$ be a mapping. Then we call T as a bi Γ -ternary semigroup if it satisfies,

$$(i) (x\alpha y\beta z) \in T$$

$$(ii) (x\alpha y\beta z)\gamma u\delta v = x\alpha(y\beta z\gamma u)\delta v = x\alpha y\beta(z\gamma u\delta v),$$

for all $x, y, z, u, v \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$.

The following examples shows the existence of above defined structures.

3.1.1.2. Example. Let $T = \{4n + 3, n \in \mathbb{N}\}$ and $\Gamma = \{4n + 1, n \in \mathbb{N}\}$. Define the mapping $() : T \times \Gamma \times T \times \Gamma \times T \rightarrow T$ as $(x\gamma y\delta z) = x + \gamma + y + \delta + z$, for $x, y, z \in T$ and $\gamma, \delta \in \Gamma$. Then

$$\begin{aligned} (x\alpha y\beta z) &= x + \alpha + y + \beta + z \\ &= 4n_1 + 3 + 4n' + 1 + 4n_2 + 3 + 4n'' + 1 + 4n_3 + 3 \\ &= 4(n_1 + n' + n_2 + n'' + n_3 + 2) + 3 \\ &= 4n + 3, \end{aligned}$$

(where, $n = n_1 + n' + n_2 + n'' + n_3 + 2 \in \mathbb{N}$, for $n_1, n', n_2, n'', n_3 \in \mathbb{N}$)

Also it is clear that $(x\alpha y\beta z)\gamma u\delta v = x\alpha(y\beta z\gamma u)\delta v = x\alpha y\beta(z\gamma u\delta v)$, for all $x, y, z, u, v \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Hence T is a bi Γ -ternary semigroup.

3.1.1.3. Example. Let $T = \{2n, n \in \mathbb{N}\}$, $\Gamma = \{\alpha, \beta, \gamma, \dots\}$. Define $(x\alpha y\beta z) = x + y + z$ for all $x, y, z \in T$ and $\alpha, \beta \in \Gamma$. Then T is a bi Γ -ternary semigroup.

3.1.1.4. Remark. Every Γ -semigroup is a bi Γ -ternary semigroup.

The following example shows that the converse of above Remark is not true.

3.1.1.5. Example. Let $T = \mathbb{Z}^-$ and $\Gamma \subseteq \mathbb{Z}^+$. Define $(x\alpha y\beta z)$, for $x, y, z \in T$ and $\alpha, \beta \in \Gamma$ as the usual multiplication of integers. Then for $x, y, z \in T$ and $\alpha, \beta \in \Gamma$, $(x\alpha y\beta z) \in T$ and $(x\alpha y\beta z)\gamma u\delta v = x\alpha(y\beta z\gamma u)\delta v = x\alpha y\beta(z\gamma u\delta v)$, for all $x, y, z, u, v \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Hence T is a bi Γ -ternary semigroup. Now for $x, y \in T$ and $\alpha \in \Gamma$, $x\alpha y \notin T$. Which shows that T is not a Γ -semigroup.

3.1.1.6. Example. Let $T = i\mathbb{R}$ where, $i = \sqrt{-1}$ and \mathbb{R} is the set of real numbers. If $\Gamma \subseteq \mathbb{R}$ and $(x\alpha y\beta z)$ is defined as the usual multiplication of complex numbers. Then,

for $x, y, z \in T$ there exist $a, b, c \in \mathbb{R}$ so that $x = ai, y = bi$ and $z = ci$. For, $\alpha, \beta \in \Gamma$,

$$(x\alpha y\beta z) = ai\alpha bi\beta ci = abc\alpha\beta i^3 = -abc\alpha\beta i = ri, \text{ where } r = -abc\alpha\beta \in R.$$

$$\text{Also, } (x\alpha y\beta z)\gamma u\delta v = x\alpha(y\beta z\gamma u)\delta v = x\alpha y\beta(z\gamma u\delta v),$$

for all $x, y, z, v, w \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Hence T is a bi Γ -ternary semigroup. But, for $x = ai, y = bi \in T = i\mathbb{R}$ and $\alpha \in \Gamma, x\alpha y = ai\alpha bi = ab\alpha i^2 = -ab\alpha \notin T = i\mathbb{R}$. This shows that T is not a Γ -semigroup.

3.1.1.7. Definition. Let T be a bi Γ -ternary semigroup and A be a nonempty subset of T . Then A is called a bi Γ -ternary subsemigroup of T if $A\Gamma A\Gamma A \subseteq A$.

3.1.1.8. Example. let $T = \mathbb{N} = \{1, 2, 3, \dots\}$ and $\Gamma = \{4n + 2, n \in \mathbb{N}\}$.

Define $(x\alpha y\beta z) = x + \alpha + y + \beta + z$. Under this operation T is a bi Γ -ternary semigroup.

Let $A = \{4n, n \in \mathbb{N}\}$ be a nonempty subset of T . For $a, b, c \in A$ and $\alpha, \beta \in \Gamma$,

$$\begin{aligned} (a\alpha b\beta c) &= a + \alpha + b + \beta + c \\ &= 4n_1 + 4n' + 2 + 4n_2 + 4n'' + 2 + 4n_3 \\ &= 4(n_1 + n' + n_2 + n'' + n_3 + 1) = 4n \in A \end{aligned}$$

Where, $n = n_1 + n' + n_2 + n'' + n_3 + 1 \in \mathbb{N}$, for $n_1, n', n_2, n'', n_3 \in \mathbb{N}$.

This implies that $A\Gamma A\Gamma A \subseteq A$. Hence A is a bi Γ -ternary subsemigroup.

3.1.1.9. Definition. Let T be a bi Γ -ternary semigroup and A be a nonempty subset of T . Then A is called a bi Γ -left ideal of T if $T\Gamma T\Gamma A \subseteq A$.

3.1.1.10. Definition. Let T be a bi Γ -ternary semigroup and A be a nonempty subset of T . Then A is called a bi Γ -right ideal of T if $A\Gamma T\Gamma T \subseteq A$.

3.1.1.11. Definition. Let T be a bi Γ -ternary semigroup and A be a nonempty subset of T . Then A is called a bi Γ -lateral ideal of T if $T\Gamma A\Gamma T \subseteq A$.

3.1.1.12. Definition. Let T be a bi Γ -ternary semigroup and A be a nonempty subset of T . Then A is called a bi Γ -ideal of T if it is a bi Γ -left, a bi Γ -right and a bi Γ -lateral ideal of T .

3.1.1.13. Example. Let $T = \{2n, n \in \mathbb{N}\}, \Gamma = \{\alpha, \beta, \gamma, \dots\}$ and $A = \{4n, n \in \mathbb{N}\}$.

Define, $(x\alpha y\beta z) = 2x + 2y + z$, for, $x, y, z \in T$ and $\alpha, \beta \in \Gamma$. Then T is a $\text{bi}\Gamma$ -ternary semigroup. Let $x, y \in T$, $a \in A$ and $\alpha, \beta \in \Gamma$, then $x = 2n_1, y = 2n_2$ and $a = 4n'$ for $n_1, n_2, n' \in \mathbb{N}$, we have

$$(x\alpha y\beta a) = 2x + 2y + a = 2(2n_1 + 2n_2) + 4n' = 4(n_1 + n_2 + n') = 4n \in A,$$

where, $n = n_1 + n_2 + n' \in \mathbb{N}$. This implies that $T\Gamma T\Gamma A \subseteq A$. Hence A is a $\text{bi}\Gamma$ -left ideal of T . Now, consider

$$(a\alpha x\beta y) = 2a + 2x + y = 8n' + 4n_1 + 2n_2 = 4(2n' + n_1) + 2n_2.$$

$$\text{Taking } n' = n_1 = n_2 = 1, \Rightarrow (a\alpha x\beta y) = 4(2.1 + 1) + 2.1 = 14 \notin A.$$

This implies that $A\Gamma T\Gamma T \notin A$. Similarly we can show that $T\Gamma A\Gamma T \notin A$. Hence A is neither a $\text{bi}\Gamma$ -right nor a $\text{bi}\Gamma$ -lateral ideal of T .

3.1.1.14. Remark. If we define, $(x\alpha y\beta z) = x + 2y + 2z$ and $(x\alpha y\beta z) = 2x + y + 2z$ respectively, then A is a $\text{bi}\Gamma$ -right and a $\text{bi}\Gamma$ -lateral ideal of T .

3.1.1.15. Example. In the above example if we define, $(x\alpha y\beta z) = 2x + 2y + 2z$, then A is a $\text{bi}\Gamma$ -left, a $\text{bi}\Gamma$ -right and a $\text{bi}\Gamma$ -lateral ideal of T . Hence A is a $\text{bi}\Gamma$ -ideal of T .

3.1.2 Bi Γ -ideals of $\text{bi}\Gamma$ -ternary semigroup

Let T denotes a $\text{bi}\Gamma$ -ternary semigroup then we shall use $B\Gamma TS(s)$, $B\Gamma TSS(s)$, $B\Gamma LI(s)$, $B\Gamma RI(s)$, $B\Gamma MI(s)$ and $B\Gamma I(s)$ for $\text{bi}\Gamma$ -ternary semigroup(s), $\text{bi}\Gamma$ -ternary subsemigroup(s), $\text{bi}\Gamma$ -left ideal(s), $\text{bi}\Gamma$ -right ideal(s), $\text{bi}\Gamma$ -lateral ideal(s) and $\text{bi}\Gamma$ -ideal(s) of a $\text{bi}\Gamma$ -ternary semigroup respectively.

3.1.2.1. Proposition. Let T be a $B\Gamma TS$ and X be a nonempty subset of T . Then

- (i) $T\Gamma T\Gamma X$ is a $B\Gamma LI$ of T .
- (ii) $X\Gamma T\Gamma T$ is a $B\Gamma RI$ of T .
- (iii) $T\Gamma X\Gamma T \cup T\Gamma T\Gamma X\Gamma T \Gamma T$ is a $B\Gamma MI$ of T .

Proof. Let T be a $B\Gamma TS$ and X be a nonempty subset of T .

- (i) Let $L = T\Gamma T\Gamma X$ and $z \in T\Gamma T\Gamma L$ then $z = t_1\alpha_1 t_2\alpha_2 a$ for some $t_1, t_2 \in T, \alpha_1, \alpha_2 \in \Gamma$ and $a \in L$. Since, $a \in L = T\Gamma T\Gamma X$ implies that $a = t_3\alpha_3 t_4\alpha_4 x_1$

for $t_3, t_4 \in T, \alpha_3, \alpha_4 \in \Gamma$ and $x_1 \in X$. Then

$$\begin{aligned}
 z &= t_1\alpha_1t_2\alpha_2a = t_1\alpha_1t_2\alpha_2(t_3\alpha_3t_4\alpha_4x_1) \\
 &= t_1\alpha_1(t_2\alpha_2t_3\alpha_3t_4)\alpha_4x_1 \\
 &= t_1\alpha_1t_5\alpha_4x_1, \text{ where } t_5 = t_2\alpha_2t_3\alpha_3t_4 \in T. \\
 &\in T\Gamma T\Gamma X = L \\
 &\Rightarrow T\Gamma T\Gamma L \subseteq L.
 \end{aligned}$$

Hence, $L = T\Gamma T\Gamma X$ is a $B\Gamma LI$ of T .

(ii) Let $R = X\Gamma T\Gamma T$ and $z \in R\Gamma T\Gamma T$ then $z = r\alpha_1t_1\alpha_2t_2$ for some $t_1, t_2 \in T, \alpha_1, \alpha_2 \in \Gamma$ and $r \in R$. Since, $r \in R = X\Gamma T\Gamma T$ implies that $r = x_1\alpha_3t_3\alpha_4t_4$ for $t_3, t_4 \in T, \alpha_3, \alpha_4 \in \Gamma$ and $x_1 \in X$. Then

$$\begin{aligned}
 z &= r\alpha_1t_1\alpha_2t_2 = (x_1\alpha_3t_3\alpha_4t_4)\alpha_1t_1\alpha_2t_2 \\
 &= x_1\alpha_3(t_3\alpha_4t_4\alpha_1t_1)\alpha_2t_2 \\
 &= x_1\alpha_3t_5\alpha_2t_2, \text{ where } t_5 = t_3\alpha_4t_4\alpha_1t_1 \in T. \\
 &\in X\Gamma T\Gamma T = R \\
 &\Rightarrow R\Gamma T\Gamma T \subseteq R.
 \end{aligned}$$

Hence, $R = X\Gamma T\Gamma T$ is a $B\Gamma RI$ of T .

(iii) Let $M = T\Gamma X\Gamma T \cup T\Gamma T\Gamma X\Gamma T \Gamma T$ and $z \in M$. Then $z \in T\Gamma X\Gamma T$ or $z \in T\Gamma T\Gamma X\Gamma T \Gamma T$.

If $z \in T\Gamma X\Gamma T$ then $z = t_1\alpha_1x_1\alpha_2t_2$ for some $t_1, t_2 \in T, \alpha_1, \alpha_2 \in \Gamma$ and $x_1 \in X$. Now, for any $t_3, t_4 \in T, \alpha_3, \alpha_4 \in \Gamma$

$$\begin{aligned}
 t_3\alpha_3z\alpha_4t_4 &= t_3\alpha_3(t_1\alpha_1x_1\alpha_2t_2)\alpha_4t_4 \\
 &= t_3\alpha_3t_1\alpha_1(x_1\alpha_2t_2\alpha_4t_4) \\
 &\in T\Gamma T\Gamma X\Gamma T\Gamma T.
 \end{aligned}$$

$$\begin{aligned}
 &\Rightarrow t_3\alpha_3z\alpha_4t_4 \in T\Gamma X\Gamma T \cup T\Gamma T\Gamma X\Gamma T\Gamma T = M \\
 &\Rightarrow T\Gamma M\Gamma T \subseteq M.
 \end{aligned}$$

If $z \in T\Gamma T\Gamma X\Gamma T \Gamma T$ then $z = t_1\alpha_1t_2\alpha_2x_1\alpha_3t_3\alpha_4t_4$ for some $t_1, t_2, t_3, t_4 \in T$, $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Gamma$ and $x_1 \in X$. Now, for any $t_5, t_6 \in T$, $\alpha_5, \alpha_6 \in \Gamma$

$$\begin{aligned} t_5\alpha_5z\alpha_6t_6 &= t_5\alpha_5(t_1\alpha_1t_2\alpha_2x_1\alpha_3t_3\alpha_4t_4)\alpha_6t_6 \\ &= (t_5\alpha_5t_1\alpha_1t_2)\alpha_2x_1\alpha_3(t_3\alpha_4t_4\alpha_6t_6) \\ &\in T\Gamma X\Gamma T \end{aligned}$$

$$\Rightarrow t_5\alpha_5z\alpha_6t_6 \in T\Gamma X\Gamma T \cup T\Gamma T\Gamma X\Gamma T\Gamma T = M$$

$$\Rightarrow T\Gamma M\Gamma T \subseteq M.$$

Hence, $M = T\Gamma X\Gamma T \cup T\Gamma T\Gamma X\Gamma T\Gamma T$ is $B\Gamma MI$ of T

Alternative proof.

(i) $T\Gamma T\Gamma X$ is a $B\Gamma LI$ of T .

Let $L = T\Gamma T\Gamma X$ then

$$\begin{aligned} T\Gamma T\Gamma L &= T\Gamma T\Gamma(T\Gamma T\Gamma X) \\ &= T\Gamma T\Gamma T\Gamma T\Gamma X \\ &\subseteq T\Gamma T\Gamma X = L \end{aligned}$$

$$\Rightarrow T\Gamma T\Gamma L \subseteq L.$$

Hence, $L = T\Gamma T\Gamma X$ is $B\Gamma LI$ of T .

(ii) $X\Gamma T\Gamma T$ is a $B\Gamma RI$ of T .

Let $R = X\Gamma T\Gamma T$ then

$$\begin{aligned} R\Gamma T\Gamma T &= (X\Gamma T\Gamma T)\Gamma T\Gamma T \\ &= X\Gamma T\Gamma T\Gamma T\Gamma T \\ &\subseteq X\Gamma T\Gamma T = R \end{aligned}$$

$$\Rightarrow X\Gamma T\Gamma T \subseteq R.$$

Hence, $R = X\Gamma T\Gamma T$ is a $B\Gamma RI$ of T .

(iii) $T\Gamma X\Gamma T \cup T\Gamma T\Gamma X\Gamma T \Gamma T$ is a $B\Gamma MI$ of T .

Let $M = T\Gamma X\Gamma T \cup T\Gamma T\Gamma X\Gamma T \Gamma T$ then

$$\begin{aligned}
 T\Gamma M\Gamma T &= T\Gamma(T\Gamma X\Gamma T \cup T\Gamma T\Gamma X\Gamma T\Gamma T)\Gamma T \\
 &= (T\Gamma T\Gamma X\Gamma T \cup T\Gamma T\Gamma T\Gamma X\Gamma T\Gamma T)\Gamma T \\
 &= T\Gamma T\Gamma X\Gamma T\Gamma T \cup T\Gamma T\Gamma T\Gamma X\Gamma T\Gamma T\Gamma T \\
 &\subseteq T\Gamma T\Gamma X\Gamma T\Gamma T \cup T\Gamma X\Gamma T, \text{ since } T\Gamma T\Gamma T \subseteq T. \\
 &= T\Gamma X\Gamma T \cup T\Gamma T\Gamma X\Gamma T\Gamma T = M
 \end{aligned}$$

$$\Rightarrow T\Gamma M\Gamma T \subseteq M.$$

Hence, $M = T\Gamma X\Gamma T \cup T\Gamma T\Gamma X\Gamma T \Gamma T$ is $B\Gamma MI$ of T . \square

3.1.2.2. Lemma. Let T be a $B\Gamma TS$, for any $t \in T$, define,

$$(i) (t)_l = \{t\} \cup T\Gamma T\Gamma t$$

$$(ii) (t)_r = \{t\} \cup t\Gamma T\Gamma T$$

$$(iii) (t)_m = \{t\} \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T$$

$$(iv) (t) = \{t\} \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T.$$

Then $(t)_l, (t)_r, (t)_m$ and (t) are $B\Gamma LI, B\Gamma RI, B\Gamma MI$ and $B\Gamma I$ of T respectively.

Proof. Let T be a $B\Gamma TS$, for any, $t \in T$,

(i) Since, $(t)_l = \{t\} \cup T\Gamma T\Gamma t$ then

$$\begin{aligned}
 T\Gamma T\Gamma (t)_l &= T\Gamma T\Gamma (\{t\} \cup T\Gamma T\Gamma t) = T\Gamma T\Gamma t \cup T\Gamma T\Gamma T\Gamma T\Gamma t \\
 &\subseteq T\Gamma T\Gamma t \cup T\Gamma T\Gamma t = T\Gamma T\Gamma t \subseteq \{t\} \cup T\Gamma T\Gamma t = (t)_l \\
 \Rightarrow T\Gamma T\Gamma (t)_l &\subseteq (t)_l.
 \end{aligned}$$

Hence, $(t)_l$ is $B\Gamma LI$ of T .

(ii) Since, $(t)_r = \{t\} \cup t\Gamma T\Gamma T$ then

$$\begin{aligned}
 (t)_r\Gamma T\Gamma T &= (\{t\} \cup t\Gamma T\Gamma T)\Gamma T\Gamma T = t\Gamma T\Gamma T \cup t\Gamma T\Gamma T\Gamma T\Gamma T \\
 &\subseteq t\Gamma T\Gamma T \cup t\Gamma T\Gamma T \quad (\text{since, } T\Gamma T\Gamma T \subseteq T) \\
 &= t\Gamma T\Gamma T \subseteq \{t\} \cup t\Gamma T\Gamma T = (t)_r \\
 \Rightarrow (t)_r\Gamma T\Gamma T &\subseteq (t)_r.
 \end{aligned}$$

Hence, $(t)_r$ is $B\Gamma RI$ of T .

(iii). Since, $(t)_m = \{t\} \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T$. Then

$$\begin{aligned}
 T\Gamma(t)_m\Gamma T &= T\Gamma(\{t\} \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T)\Gamma T \\
 &= T\Gamma t\Gamma T \cup T\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T \cup T\Gamma T\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T\Gamma T \\
 &\subseteq T\Gamma t\Gamma T \cup T\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \quad (\text{since, } T\Gamma T\Gamma T \subseteq T) \\
 &= T\Gamma t\Gamma T \cup T\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T \\
 &\subseteq \{t\} \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T = (t)_m \\
 T\Gamma(t)_m\Gamma T &\subseteq (t)_m.
 \end{aligned}$$

Hence, $(t)_m$ is $B\Gamma MI$ of T .

(iv) $(t) = \{t\} \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T$. As,

$$\begin{aligned}
 T\Gamma T\Gamma(t) &= T\Gamma T\Gamma(t \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T) \\
 &= T\Gamma T\Gamma t \cup T\Gamma T\Gamma T\Gamma T\Gamma t \cup T\Gamma T\Gamma t\Gamma T\Gamma T \cup T\Gamma T\Gamma T\Gamma t\Gamma T \\
 &\quad \cup T\Gamma T\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T \\
 &\subseteq T\Gamma T\Gamma t \cup T\Gamma T\Gamma t \cup T\Gamma T\Gamma t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T \\
 &= T\Gamma T\Gamma t \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T \\
 &\subseteq \{t\} \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T = (t) \\
 \Rightarrow T\Gamma T\Gamma(t) &\subseteq (t).
 \end{aligned}$$

This implies that (t) is $B\Gamma LI$. Similarly, we can show that it is $B\Gamma RI$. Now consider,

$$\begin{aligned}
 T\Gamma(t)\Gamma T &= T\Gamma(t \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T)\Gamma T \\
 &= T\Gamma t\Gamma T \cup T\Gamma T\Gamma T\Gamma T\Gamma t\Gamma T \cup T\Gamma t\Gamma T\Gamma T\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T \\
 &\quad \cup T\Gamma T\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T\Gamma T \\
 &\subseteq T\Gamma t\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \\
 &= T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T \\
 &\subseteq t \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T = (t) \\
 \Rightarrow T\Gamma(t)\Gamma T &\subseteq (t).
 \end{aligned}$$

This implies that (t) is $B\Gamma MI$ of T . Hence (t) is $B\Gamma I$ of T . □

3.1.2.3. Remark. The ideals $(t)_l, (t)_m, (t)_r, (t)$ are called principal bi Γ -left, bi Γ -right, bi Γ -lateral and bi Γ -ideal of T generated by t . Note that for any $a \in A \subseteq T$, $\bigcup_{a \in A} (a)_l = (A)_l$, $\bigcup_{a \in A} (a)_m = (A)_m$, $\bigcup_{a \in A} (a)_r = (A)_r$ and $\bigcup_{a \in A} (a) = (A)$ are bi Γ -left ideal, bi Γ -right ideal, bi Γ -lateral ideal and bi Γ -ideal of T generated by A .

3.1.2.4. Lemma. Let T be a $B\Gamma TS$ then,

- (i) The arbitrary intersection of $B\Gamma TSS(s)$ of T is again a $B\Gamma TSS$ of T .
- (ii) The arbitrary intersection of $B\Gamma LI(s)$ ($B\Gamma RI(s)$, $B\Gamma MI(s)$, $B\Gamma I(s)$) of T is a $B\Gamma LI$ ($B\Gamma RI$, $B\Gamma MI$, $B\Gamma I$) of T .

Proof. Let T be a $B\Gamma TS$.

- (i) Let $\{A_i, i \in I\}$ be a collection of bi Γ -ternary subsemigroups of T , then $A_i \Gamma A_i \Gamma A_i \subseteq A_i$, for all $i \in I$. Also $\bigcap_{i \in I} A_i \subseteq A_i$ for all $i \in I$ then,

$$\begin{aligned} \left(\bigcap_{i \in I} A_i \right) \Gamma \left(\bigcap_{i \in I} A_i \right) \Gamma \left(\bigcap_{i \in I} A_i \right) &\subseteq A_i \Gamma A_i \Gamma A_i \subseteq A_i, \text{ for all } i \in I. \\ \Rightarrow \left(\bigcap_{i \in I} A_i \right) \Gamma \left(\bigcap_{i \in I} A_i \right) \Gamma \left(\bigcap_{i \in I} A_i \right) &\subseteq \bigcap_{i \in I} A_i. \end{aligned}$$

Hence $\bigcap_{i \in I} A_i$ is a bi Γ -ternary subsemigroup of T .

- (ii) Let $\{L_i, i \in I\}$ be a collection of bi Γ -left ideals of T then $T\Gamma T\Gamma L_i \subseteq L_i$, for all $i \in I$. Also $\bigcap_{i \in I} L_i \subseteq L_i$ for all $i \in I$ then,

$$\begin{aligned} T\Gamma T\Gamma \left(\bigcap_{i \in I} L_i \right) &\subseteq T\Gamma T\Gamma L_i \subseteq L_i, \text{ for all } i \in I. \\ T\Gamma T\Gamma \left(\bigcap_{i \in I} L_i \right) &\subseteq L_i, \text{ for all } i \in I. \\ \Rightarrow T\Gamma T\Gamma \left(\bigcap_{i \in I} L_i \right) &\subseteq \bigcap_{i \in I} L_i. \end{aligned}$$

Hence $\bigcap_{i \in I} L_i$ is a bi Γ -left ideal of T . Similarly, we can prove for bi Γ -right and bi Γ -lateral ideal and bi Γ -ideal of T .

□

3.1.3 Bi Γ -quasi and bi Γ -bi-ideals of bi Γ -ternary semigroup

3.1.3.1. Definition. A nonempty subset Q of a bi Γ -ternary semigroup T is called a bi Γ -quasi-ideal of T if

$$Q\Gamma T\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma T\Gamma Q \subseteq Q \text{ and}$$

$$Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q \subseteq Q.$$

3.1.3.2. Definition. A nonempty subset B of a bi Γ -ternary semigroup T is called a bi Γ -bi-ideal of T if

- (i) B is a bi Γ -ternary subsemigroup of T .
- (ii) $B\Gamma T\Gamma B\Gamma T\Gamma B \subseteq B$.

We will write $B\Gamma QI(s)$ and $B\Gamma BI(s)$ for bi Γ -quasi-ideal(s) and bi Γ -bi-ideal(s), respectively.

3.1.3.3. Proposition. Let T be a $B\Gamma TSS$. Then every $B\Gamma QI$ of T is a $B\Gamma TSS$ of T .

Proof. We suppose that Q is a bi Γ -quasi-ideal of T . Since

$$Q\Gamma Q\Gamma Q \subseteq Q\Gamma T\Gamma T, \quad Q\Gamma Q\Gamma Q \subseteq T\Gamma Q\Gamma T \quad \text{and} \quad Q\Gamma Q\Gamma Q \subseteq T\Gamma T\Gamma Q.$$

$$\Rightarrow Q\Gamma Q\Gamma Q \subseteq Q\Gamma T\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma T\Gamma Q \subseteq Q.$$

$$\Rightarrow Q\Gamma Q\Gamma Q \subseteq Q, \quad \text{since } Q \text{ is bi}\Gamma\text{-quasi-ideal.}$$

Hence, Q is bi Γ -ternary subsemigroup of T . □

3.1.3.4. Proposition. The arbitrary intersection of $B\Gamma QI(s)$ of T is a $B\Gamma QI$ of T .

Proof. The proof is simple and can be done by routine calculations so omitted. □

3.1.3.5. Remark. Note that a $B\Gamma LI$ ($B\Gamma RI$, $B\Gamma MI$) of T is also $B\Gamma QI$ of T but any $B\Gamma QI$ of T may not be a $B\Gamma LI$ ($B\Gamma RI$, $B\Gamma MI$) of T , so we have following lemma.

3.1.3.6. Lemma. Let T be a $B\Gamma TSS$. Then every $B\Gamma LI$ ($B\Gamma RI$, $B\Gamma MI$) of T is a $B\Gamma QI$ of T .

Proof. Let L be a bi Γ -left ideal of T , then $T\Gamma T\Gamma L \subseteq L$, implies that

$$L\Gamma T\Gamma T \cap T\Gamma L\Gamma T \cap T\Gamma T\Gamma L \subseteq L, \quad \text{and}$$

$$L\Gamma T\Gamma T \cap T\Gamma T\Gamma L\Gamma T\Gamma T \cap T\Gamma T\Gamma L \subseteq L.$$

Hence L is bi Γ -quasi-ideal of T . Other cases are similar. □

3.1.3.7. Lemma. A nonempty subset Q of T is a $B\Gamma QI$ of T if and only if it is an intersection of a $B\Gamma LI$, a $B\Gamma MI$ and a $B\Gamma RI$ of T .

Proof. Let L, M and R be the bi Γ -left, bi Γ -lateral and bi Γ -right ideals of T . Let $Q = R \cap M \cap L$, then

$$\begin{aligned} Q\Gamma\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma\Gamma T Q \\ &= (R \cap M \cap L)\Gamma\Gamma\Gamma T \cap T\Gamma(R \cap M \cap L)\Gamma T \cap T\Gamma\Gamma T(R \cap M \cap L) \\ &\subseteq R\Gamma\Gamma\Gamma T \cap T\Gamma M\Gamma T \cap T\Gamma\Gamma T L \\ &\subseteq R \cap M \cap L, \text{ since, } L, M \text{ and } R, \text{ are bi}\Gamma\text{-left, lateral and right ideals.} \\ &= Q. \end{aligned}$$

Similarly, $Q\Gamma\Gamma\Gamma T \cap T\Gamma\Gamma T Q\Gamma\Gamma\Gamma T \cap T\Gamma\Gamma T Q \subseteq Q$. Hence Q is a bi Γ -quasi-ideal of T .

Conversely, let Q be a bi Γ -quasi-ideal of T . For any $q \in Q$, $(q)_l, (q)_m, (q)_r$, be the bi Γ -left, bi Γ -lateral and bi Γ -right ideals of T generated by q , then

$$\begin{aligned} q &\in (q)_r \cap (q)_m \cap (q)_l \\ \bigcup_{q \in Q} \{q\} &\subseteq \bigcup_{a \in Q} (a)_r \cap \bigcup_{q \in Q} (q)_m \cap \bigcup_{q \in Q} (q)_l \\ Q &\subseteq (Q)_r \cap (Q)_m \cap (Q)_l. \end{aligned}$$

Since, $(Q)_l = Q \cup Q\Gamma\Gamma\Gamma T$, $(Q)_m = Q \cup T\Gamma Q\Gamma T \cup T\Gamma\Gamma T Q\Gamma\Gamma\Gamma T$ and $(Q)_r = Q \cup T\Gamma\Gamma T \Gamma Q$, then

$$\begin{aligned} &(Q)_r \cap (Q)_m \cap (Q)_l \\ &= (Q \cup Q\Gamma\Gamma\Gamma T) \cap (Q \cup T\Gamma Q\Gamma T \cup T\Gamma\Gamma T Q\Gamma\Gamma\Gamma T) \cap Q \cup T\Gamma\Gamma T \Gamma Q \\ &= Q \cup (Q\Gamma\Gamma\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma\Gamma T Q) \cup (Q\Gamma\Gamma\Gamma T \cap T\Gamma\Gamma T Q\Gamma\Gamma\Gamma T \cap T\Gamma\Gamma T \Gamma Q) \\ &\subseteq Q, \text{ since } Q \text{ is bi}\Gamma\text{-quasi-ideal of } T. \end{aligned}$$

This implies that $Q = (Q)_r \cap (Q)_m \cap (Q)_l$, where, $(Q)_r, (Q)_m$, and $(Q)_l$ are bi Γ -left ideal, bi Γ -lateral ideal and a bi Γ -right ideal of T . Hence the proof. \square

3.1.3.8. Lemma. Let T be a B Γ T S and R_s, M_s, L_s be the smallest B Γ R I , B Γ M I , B Γ L I of T . Then $R_s \cap M_s \cap L_s$ is the smallest B Γ Q I of T .

Proof. Let R_s, M_s, L_s be the smallest B Γ R I , B Γ M I , B Γ L I of T . Then by Lemma 3.1.3.7, $Q = R_s \cap M_s \cap L_s$ is B Γ Q I of T . Now, we have to show that $Q = R_s \cap M_s \cap L_s$ is smallest B Γ Q I of T . Suppose that P be any other B Γ Q I of T . Then $P\Gamma\Gamma\Gamma T$ is a B Γ R I of T and $P\Gamma\Gamma\Gamma T \subseteq Q\Gamma\Gamma\Gamma T \subseteq R_s$ but R_s is smallest B Γ R I of T so

REFERENCES

1. Sen, M. K. On Γ -semigroups. *Proc. Int. Conf. Algebra Appl.* New York: Decker Publication, 1981. pp. 301.
2. Zadeh, L. A. Fuzzy sets. *Inform. and Control.*, 1965. 8: 338–353.
3. Rosenfeld, A. Fuzzy groups. *J. Math. Anal. Appl.*, 1971. 35: 512–517.
4. Kuroki, N. On fuzzy ideals and fuzzy bi-ideals in semigroups. *Fuzzy Sets and Systems*, 1981. 5: 203-215.
5. Kuroki, N. Fuzzy bi-ideals in semigroups. *Comput. Math. Univ. St. Paul.*, 1979. 28: 17-21.
6. Kuroki, N. On fuzzy semigroups. *Information Sciences*, 1991. 53(3): 203-236.
7. Kuroki, N. Fuzzy generalized bi-ideal in semigroups. *Information Sciences*, 1992. 66(3): 235-243.
8. Kuroki, N. Fuzzy semiprime quasi ideals in semigroups. *Information Sciences*, 1993. 75(3): 201-211.
9. Atanassov, K. T. Intuitionistic fuzzy sets. In : *V. Sgurev, Ed., VII ITKR's Sessions*, (Central Sci. and Tech. Library, Bulg. Academy of Science, Sofia, Bulgaria, Rep. No. 1697/84) Sofia, Jun 1983.
10. Biswas, R. Intuitionistic fuzzy subgroupoids. *Mathematical Forum*, 1989. 10: 37-46.
11. Kim, K. H. and Jun, Y. B. Intuitionistic fuzzy ideals of semigroups. *Indian J. Pure and Applied Math.*, 2002. 33(4): 443-449.
12. Jun, Y. B., Lee, K. J. and Song, S. Z. N-ideals of BCK/BCI algebras. *Journal of Chungcheong Mathematical Society*, 2009. 22(3): 417-437.
13. Clifford, A. H. and Preston, G. B. *The Algebraic Theory of Semigroups Vol. I*, Mathematical Surveys No.7. American Mathematical Society, Providence,

- R.I.1961.
14. Clifford, A. H. and Preston, G. B. *The Algebraic Theory of Semigroups. Vol. II*, Mathematical Surveys No.7. American Mathematical Society, Providence, R.I.1967.
 15. Petrich, M. *Introduction to semigroups*. Merrill, Columbus, Ohio, 1973.
 16. Petrich, M. *Lectures in semigroups*. Wiley, 1977.
 17. Petrich, M. *Inverse semigroups*. Wiley, 1984.
 18. Ljapin, E. S. Semigroups. In : *Translations of Mathematical Monographs. Vol.3*, American Mathematical Society, Providence, R. I., 1963.
 19. Good, R. A. and Hughes D. R. Associated groups for a semigroup. *Bull. Amer. Math. Soc.*, 1952. 58: 624–625.
 20. Lajos, S. On the bi-ideals in semigroups. *Proc. Japan Acad.*, 45, 1969.
 21. Lajos, S. (m; k; n)-ideals in semigroups. In : *Notes on Semigroups II*, Karl Marx Univ. Econ.,Dept. Math. Budapest, No.1, 1976. 12-19.
 22. Szasz, G. Interior ideals in semigroups. In : *Notes on Semigroups IV*, Karl Marx Univ. Econ.,Dept. Math. Budapest, No.5, 1977. 1-7.
 23. Szasz, G. Remark on interior ideals of semigroups. *Studia Scient. Math. Hung.*, 1981. 16: 61–63.
 24. Steinfeld, O. Uber die quasi-ideals von halbgruppen. *Publ. Math. (Debrecen)*, 1956. 4: 262 – 275.
 25. Grimble, H. B. *Prime Ideals in Semigroups*. Master thesis, University of Tennessee - Knoxville, 1950.
 26. Hoehnke, H. J. Structure of semigroups. *Canad. J. Math.*, 1966. 18: 449–491.
 27. Schwarz, S. Prime ideals and maximal ideals in semigroups. *Czechoslovak Mathematical Journal*, 1969. 19(1): 72 – 79.
 28. Anjaneyulu, A. Primary ideals in semigroups. *Semigroup Forum*, 1980. 20: 129–144.
 29. Anjaneyulu, A. Semigroups in which prime ideals are maximal. *Semigroup Forum*, 1981. 22(1): 151–158.
 30. Giri, R. D. and Wazalwar, A. K. Prime ideals and prime radicals in non-commutative semigroups. *Kyungpook Mathematical Journal*, 1993. 33(1):

- 37–48.
31. Lehmer, D. H. A ternary analogue of abelian groups. *Amer. J. Math.*, 1932. 59: 329–338.
 32. Kasner, E. An extension of the group concept. *Bull. Amer. Math. Soc.*, 1904. 10: 290–291.
 33. Prufer, H. Theorie der abel schen gruppen. *Mathematische Zeitschrift*, 1924. 20(1): 166–187.
 34. Sioson, F. M. Ideal theory in ternary semigroups. *Math. Japonica*, 1965. 10: 63–84.
 35. Dixit, V. N. and Dewan, S. A note on quasi and bi-ideals in ternary semigroups. *Int. J. Math. Math. Sci.*, 1995. 18: 501–508.
 36. Santiago, M. L. *Some contribution to the study of ternary semigroups and semi-heaps*. Ph.D. Thesis. University of Madras, India, 1983.
 37. Dutta, T. K. and Kar, S. and Maity, B. K. On ideals in regular ternary semigroups. *Math. Gen. Algebra Appl.*, 2008. 28: 147–159.
 38. Sen, M. K. and Saha, N. K. On Γ -semigroup-1. *Bull. Cal. Math. Soc.*, 1986. 78: 180–186.
 39. Saha, N. K. On Γ -semigroup-11. *Bull. Cal. Math. Soc.*, 1987. 79: 331–335.
 40. Saha, N. K. On Γ -semigroup-111. *Bull. Cal. Math. Soc.*, 1988. 80: 1–12.
 41. Chinram, R. On quasi-gamma-ideals in gamma-semigroups. *Science Asia*, 2006. 32: 351–353.
 42. Chinram, R. and Jirojkul, C. On bi- Γ -ideals in Γ -semigroups. *Songklanakarin Journal of Science and Technology*, 2007. 29: 231–234.
 43. Moin, A. and Khan, M. R. Notes on (m,n) bi- Γ -ideals in Γ -semigroups. *Rendiconti del Circolo Matematico di Palermo*, 2011. 60((1-2)): 31–42.
 44. Mordeson, J. N., Malik, D. S. and Kuroki, N. Fuzzy Semigroups. *Studies in fuzziness and Soft Computing*, 131. Springer Verlag, 2003.
 45. Ahsan, J., Latif, R. M. and Shabir, M. Fuzzy quasi-ideals in semigroups. *Jour. of Fuzzy Mathematics*, 2001. 9: 259-270.
 46. Ahsan, J., Li, K. Y. and Shabir M. Semigroups characterized by their fuzzy bi-ideals. *Journal of Fuzzy Mathematics*, 2002. 10: 441-449.
 47. Ahsan, J., Saifullah, K. and Khan, M. F. Semigroups characterized by their

- fuzzy ideals. *Fuzzy Systems and Mathematics*, 1995. 9: 29-32.
48. Shabir, M. Fully fuzzy prime semigroups. *International Journal of Mathematics and Mathematical Sciences*, 2005. 1: 163-168.
49. Shabir, M. and Ali, M. Characterizations of semigroups by their $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy ideals. *Iranian journal of science and technology-Trans A*, 2013. 37A2: 117-131.
50. Shabir, M., Jun, Y. B. and Bano, M. On prime fuzzy bi-ideals of semigroups. *Iranian journal of fuzzy systems*, 2010. 7(3): 115-128.
51. Petchkhaew, P. and Chinram, R. Fuzzy, rough and rough fuzzy ideals and of ternary semigroup. *International Journal of pure and applied Mathematics*, 2009. 56(1): 21-36.
52. Kar, S and Sarkar, P. Fuzzy ideals of ternary semigroup. *Fuzzy Information and Engineering*, 2012. (4)2: 181-193.
53. Kar, S and Sarkar, P. Fuzzy quasi-ideals and fuzzy bi-ideals of ternary semigroups. *Annals of Fuzzy Mathematics and Informatics*, 2012. 4(2): 407- 423.
54. Rehman, N. and Shabir, M. Characterizations of ternary semigroups by (α, β) -fuzzy ideals. *World Applied Sciences Journal*, 2012. 18(11): 1556-1570.
55. Rehman, N. and Shabir, M. $(\overline{\alpha}, \overline{\beta})$ -fuzzy ideals of ternary semigroups. *World Applied Sciences Journal*, 2012. 17(12): 1736-1758.
56. Rehman, N. and Shabir, M. Some kinds of $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ -fuzzy ideals of ternary semigroups. *Iranian Journal of Science and Technology*, 2013. 37(3)(Special issue-Mathematics): 365-378.
57. Williams, D. R. P., and Latha, K. B. and Chandrasekeran, E. Fuzzy bi- Γ -ideals in Γ -semigroups. *Hacettepe Journal of Mathematics and Statistics*, 2009. 38(1): 1-15.
58. Dutta, T. K., Sardar, S. K. and Majumder, S. K. Fuzzy ideal extensions of Γ -semigroups. *International Mathematical Forum*, 2009. 4(42): 2093-2100.
59. Dutta, T. K., Sardar, S. K. and Majumder, S. K. Fuzzy ideal extensions of Γ -semigroups via its operator semigroups. *Int. J. Contemp. Math. Sciences*, 2009. 4(30): 1455-1463.

60. Sardar, S. K., Majumder, S. K. and Mandal, D. . A note on characterization of prime ideals of Γ -semigroups in terms of fuzzy subsets. *Int. J. Contemp. Math. Sciences*, 2009. 4(30): 1465-1472.
61. Sardar, S. K. and Majumder, S. K . A note on characterization of semiprime ideals of Γ -semigroups in terms of fuzzy subsets. *International Journal of Pure and Applied Mathematics*, 2009. 56(3): 451-457.
62. Sardar, S. K. and Kayal, S. Prime and semiprime fuzzy bi-ideals in Γ -semigroups-revisited. *Annals of Fuzzy Mathematics and Informatics*, 2014. 7(1): 31-44.
63. Khan, F. M., Sarmin, N. H. and Khan, A. A novel approach towards fuzzy Γ -ideals in ordered Γ -semigroups. *Indian Journal of Pure and Applied Mathematics*, 2014. 45(3): 343-362.
64. Kanlaya, A. and Iampan, A. Coincidences of different types of fuzzy ideals in ordered Γ -semigroups. *Korean Journal of Mathematics*, 2014. 22(2): 367-381.
65. Pal, P., Majumder, S. K. and Davvaz, B. Regularity of po- Γ -semigroups in terms of fuzzy subsemigroups and fuzzy bi-ideals. *Fuzzy Information and Engineering*, 2015. 7(2): 165-182.
66. Chinnadurai, V. and Bharathivelan, K. Cubic ideals of Γ -semigroups. *International Journal of Current Research and Modern Education*, 2016. 1(2): 138-150.
67. Rehman, N and Shabir, M. Some characterizations of ternary semigroups by the properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy ideals. *Journal of Intelligent and Fuzzy Systems*, 2014. 26(5): 2107-2117.
68. Rehman, N and Shabir, M. A new generalization of fuzzy ideals of ternary semigroups. *Applied Mathematics and Information Sciences*, 2015. 9(3): 1623-1637.
69. Saha, B. C., Majumder, S. K., Pal, P. and Sardar, S. K. Fuzzy, rough, rough fuzzy interior ideals of ternary semigroups. *Annals of Fuzzy Mathematics and Informatics*, 2015. 10(1): 181-192.
70. Atanassov, K. T. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 1986. 20: 87-96.

71. Atanassov, K. T. New operations defined over the intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 1994. 61(2): 137-142.
72. Kim, K. H. and Jun, Y. B. Intuitionistic fuzzy interior ideals of semigroups. *Inter. J. Math. and Mathematical Sci.*, 2001. 27(5): 261-267.
73. Hur, K., Kim, K. H. and Song, H. K. Intuitionistic fuzzy ideals and bi-ideals. *Honam Mathematical Journal*, 2004. 26(3): 309-330.
74. Hong, Y. and Jiang, C. Characterizing regular semigroups using intuitionistic fuzzy sets. *Notes on Intuitionistic Fuzzy Sets*, 2004. 10(2): 20-31.
75. Kim, K. H. and Lee, J. G. On intuitionistic fuzzy bi-ideals of semigroups. *Turk J. Math.*, 2005. 29: 201-210.
76. Ozturk, M. A., Ceven, Y. and Jun, Y. B. Intuitionistic fuzzy sets in semigroups. *Scientiae Mathematicae Japonicae Online*, 2007. e-2007: 37-42.
77. Shabir, M., Arif, M. S., Khan, A. and Aslam, M. On intuitionistic fuzzy prime bi-ideals of semigroups. *Bulletin of the Malaysian Mathematical Sciences Society*, 2012. 35(4): 983-996.
78. Lekkoksung, S. Intuitionistic fuzzy bi-ideals of ternary semigroups. *International Mathematical Forum*, 2012. 7(8): 385-389.
79. Akram, M. Intuitionistic fuzzy points and ideals of ternary semigroups. *International Journal of Algebra and Statistics*, 2012. 1(1): 74-82.
80. Yaqoob, N., Khan, M., Akram, M. and Khan, A. Interval valued intuitionistic (s,t)- fuzzy ideals of ternary semigroups. *Indian Journal of Science and Technology*, 2013. 6(11): 5418-5428.
81. Akram, M. and Tyagi S. K. Intuitionistic fuzzy prime bi-ideals of ternary semigroups. *International Journal of Fuzzy Mathematics and Systems*, 2012. 2(2): 179-190.
82. Mustafa, U., Ozturk, M. A. and Jun, Y. B. Intuitionistic fuzzy sets in Γ -semigroups. *Bull. Korean Math. Soc.*, 2007. 44(2): 359-367.
83. Sardar, S. K., Majumder, S. K. and Mandal, M. Atanassov's intuitionistic fuzzy ideals of Γ -semigroups. *International Journal of Algebra*, 2011. 5(7): 335-353.
84. Davvaz, B. and Majumder, S. K. Atanassov's intuitionistic fuzzy interior

- ideals of Γ -semigroups. *U.P.B. Sci. Bull. Series A*, 2011. 73(3): 45-60.
85. Palaniappan, N. and Lalithamani, N. Characterization of intuitionistic fuzzy ideals in Γ -semigroups. *Int. J. of Mathematical Sciences and Applications*, 2011. 1(3): 1239-1246.
 86. Akram, M. Characterizing Γ -semigroups by intuitionistic fuzzy points. *ARPJN Journal of Systems and Software*, 2012. 2(12): 359-365.
 87. Akram, M. Intuitionistic fuzzy prime bi- Γ -ideals of Γ -semigroups. *Annals of Fuzzy Mathematics and Informatics*, 2013. 5(2): 309-320.
 88. Li, B. and Feng, Y. Intuitionistic (λ, μ) -fuzzy sets in Γ -semigroups. *Journal of Inequalities and Applications*, 2013. Vol.2013: 1-9.
 89. Iampan, A. Characterizing intuitionistic fuzzy Γ -ideals of ordered Γ -semigroups by means of intuitionistic fuzzy points. *Notes on Intuitionistic Fuzzy Sets*, 2015. 21(3): 24-39.
 90. Khan, A., Jun, Y. B. and Shabir, M. N-fuzzy ideals in ordered semigroups. *International Journal of Mathematics and Mathematical Sciences*, 2009. 2009: 1-14.
 91. Khan, A., Jun, Y. B. and Shabir, M. N-fuzzy quasi-ideals in ordered semigroups. *Quasigroups and Related Systems*, 2009. 17: 237-252.
 92. Jun, Y. B., Kavikumar, J. and So, K. S. N-ideals of subtraction algebras. *Commun. Korean Math. Soc.*, 2010. 25(2): 173-184.
 93. Akram, M., Kavikumar, J. and Khamis, A. On n- Γ -ideals in Γ -AG-groupoids. *Analele Universitatii Oradea Fasc. Matematica*, 2014. XXI(1): 137-144.
 94. Williams, D. R. P., and Saeid, A. B. Generalized N-ideals of subtraction algebras. *Journal of Uncertain Systems*, 2015. 9(1): 31-48.
 95. Molodtsov, D. Soft set theory-first results. *Comput. Math. Appl.*, 1999. 37: 19-31.
 96. Maji, P. K., Roy, A. R. and Biswas, R. Soft set theory. *Comput. Math. Appl.*, 2003. 45: 555-562.
 97. Ali, M. I., Feng, F., Liu, X. Y. and Shabir, M. On some new operations in soft set theory. *Comput. Math. Appl.*, 2009. 57: 1547-1553.
 98. Shabir, M. and Ali, M. I. Soft ideals and generalized fuzzy ideals in semigroups. *New Math. and Nat. Comput.*, 2009. 5(3): 599-615.

99. Shabir, M. and Ahmad, A. On soft ternary semigroups. *Annals of Fuzzy Math. Inform.*, 2012. 3(1): 39-59.
100. Changphas, T. and Thongkam, B. On Γ -soft semigroups. *Annals of Fuzzy Mathematics and Informatics*, 2012. 4(2): 217-223.
101. Maji, P. K., Roy, A. R. and Biswas, R. Fuzzy soft sets. *The journal of fuzzy mathematics*, 2001. 9(3): 589-602.
102. Ahmad, B. and Kharal, A. On fuzzy soft sets. *Advances in fuzzy systems*, 2009. 2009: 1-6.
103. Aygunoglu, A. and Aygun, H. Introduction to fuzzy soft groups. *Comput. Math. appl.*, 2009. 58: 1279-1286.
104. Aktas, H. and Cagman, N. Soft sets and soft groups. *Information Sciences*, 2007. 177(13): 2726-2735.
105. Ali, M. I. and Shabir, M. Comments on de Morgan's law in fuzzy soft set. *The Journal of Fuzzy Mathematics*, 2010. 18(3): 1547-1553.
106. Yang, W. and Li, S. Bipolar value fuzzy soft sets. *Computers Engineering and Applications*, 2012. 48(35): 15-18.
107. Bora, M., Neog, T. J. and Sut, D. K. A study on some operations of fuzzy soft sets. *International Journal of Modern Engineering Research*, 2012. 2(2): 219-225.
108. Zhang, W. R. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. *Proceedings of IEEE Conf.*, 1994. 305-309.
109. Zhang, W. R. Bipolar fuzzy sets. *Proceedings of World Congress on Computational Intelligence Fuzzy-IEEE*, 1998. 835-840.
110. Lee, K. M. Bi-polar-valued fuzzy sets and their operations. *Proc. Int. Conf. Intelligent Technologies, Bangkok, Thailand*, 2000. 307-312.
111. Lee, K. M. Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bi-polar-valued fuzzy sets. *J. Fuzzy Logic Intel. Syst.*, 2004. 14: 125-129.
112. Naz, M. and Shabir, M. On fuzzy bipolar soft sets, their algebraic structures and applications. *Journal of Intelligent and Fuzzy Systems*, 2014. 26(4): 1645-1656.

113. Abdullah, S., Aslam, M. and Ullah, K. Bipolar fuzzy soft sets and its application in decision making problem. *Journal of Intelligent and Fuzzy Systems*, 2013. 27(2): 729-742.
114. Howie, J. M. *An Introduction to Semigroup Theory*. New York. Academic Press, 1976.
115. Hila, K. On regular, semiprime and quasi-reflexive Γ -semigroup and minimal quasi-ideals. *Lobachevskii Journal of Mathematics*, 2008. 29(3): 141-152.
116. Ahn, S. S. and Ko, J. M. Coupled n-structures applied to ideals in d-algebras. *Commun. Korean Math. Soc.*, 2013. 28(4): 709-721.
117. Seo, M. J. and Ahn, S. S. Applications of coupled N-structures in BH-algebras. *Honam Mathematical Journal*, 2012. 34(4): 585-596.
118. Wang, G. J. and He, Y. Y. Intuitionistic fuzzy sets and I-fuzzy sets. *Fuzzy sets and systems*, 2000. 110: 271-274.