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INTERNAL RATIONALITY, LEARNING AND IMPERFECT INFORMATION

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Internal Rationality, Learning and Imperfect Information*

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Abstract

We construct, estimate and explore the monetary policy consequences of a New Keynesian (NK) behavioural model with bounded-rationality and heterogeneous agents. We radically depart from most existing models of this genre in our treatment of bounded rationality and learning. Instead of the usual Euler learning approach, we assume that agents are internally rational (IR) given their beliefs of aggregate states and prices. The model is inhabited by fully rational (RE) and IR agents where the latter use simple heuristic rules to forecast aggregate variables exogenous to their micro-environment. We find that IR results in an NK model with more persistence and a smaller policy space for rule parameters that induce stability and determinacy. In the most general form of the model, agents learn from their forecasting errors by observing and comparing them with those under RE making the composition of the two types endogenous. In a Bayesian estimation with fixed proportions of RE and IR agents and a general heuristic forecasting rule we find that a pure IR model fits the data better than the pure RE case. However, the latter with imperfect rather than the standard perfect information assumption outperforms IR (easily) and RE-IR composites (slightly), but second moment comparisons suggest that the RE-IR composite can match data better. Our findings suggest that Kalman-filtering learning with RE can match bounded-rationality in matching persistence seen in the data.

JEL Classification: E03, E12, E32.
Keywords: New Keynesian Behavioural Model, Internal Rationality, Heterogeneous Expectations, Reinforcement Learning, Imperfect Information

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Contents

1 Introduction

This paper constructs, estimates and explores the monetary policy consequences of a New Keynesian (NK) behavioural model with bounded-rationality and heterogeneous agents. It departs from existing models of this genre in its approach to bounded rationality and learning. There are broadly two choices made by the learning literature at this point: Euler learning or internal rationality. The first, *Euler learning* (EL), follows the pioneering work of Evans and Honkapohja (2001) and assumes, in the case of households, that agents forecast their own consumption decision next period. Furthermore they know the minimum state variable (MSV) form of the equilibrium (equivalent to the saddle-path under rational expectations) and use direct observations or VAR estimates of these states to update their estimates each period using a discounted least-squares estimator. Then a statistical learning equilibrium is one where this perceived law of motion and the actual one coincide. For firms the same applies except the decision is on prices made by firms who are no longer locked into a contract.

Although this form of bounded rationality responds to what many regard as an extreme assumption of model-consistent expectations, the departure is only a modest one in that agents still need to know the MSV form of the equilibrium. The defining characteristic of behavioural macro-models is to limit the cognitive skills of at least a group of agents in the model and this is achieved by introducing simple 'heuristic' learning rules. However this raises the opposite concern regarding the bounds on bounded rationality: with heuristic rules agents may fall considerably short of building rational expectations and such models are particularly vulnerable to the Lucas critique when policy scenarios are studied. The problem is that agents can depart from rationality in an infinite number of ways leading into the 'wilderness' of Sims (1980).

In response to the wilderness concern, the literature on behavioural models adopts a basic general framework pioneered by Brock and Hommes (1997). To limit the departure from rationality and rule out stupid behaviour the approach of *reinforcement learning* proposes that, although adaptation can be slow and there can be a random component of choice, the higher the 'payoff' (defined appropriately) from taking an action in the past, the more likely it will be taken in the future.[1]

The alternative approach to learning adopted in this paper assumes that agents are *internally rational* (IR) given their beliefs of aggregate states and prices which are exogenous to their decisions.[2] As with the Euler equation approach, agents cannot form model-consistent expectations and instead learn about these variables using their knowledge of the MSV form of the equilibrium. The two approaches then differ with respect to what agents learn about - their own decisions in the first approach, and variables exogenous to the agents in the second

[1] See Young (2004) for a general treatment of this approach.

[2] See Adam and Marcet (2011) who apply the concept to asset-pricing, Eusepi and Preston (2011) for an RBC model with IR, Woodford (2013) who adopts a similar NK framework as in this paper and Branch and McGough (2016) for a recent discussion of of IR, also referred to as the 'infinite time-horizon approach to learning'. We adopt the general definition of internal rationality used in the first paper: namely that "agents maximize utility under uncertainty, given their constraints and given a consistent set of probability beliefs about payoff-relevant variables that are *beyond their control or external*". Then beliefs can take the form of a well-defined probability measure over a stochastic process (the 'fully Bayesian' plan), or they can adopt an 'anticipated utility' framework of Kreps (1998). Adam and Marcet (2011) adopt the former approach whereas this paper and the other applications mentioned adopt the latter. Cogley and Sargent (2008) compares the two and encouragingly find that anticipated utility can be seen as a good approximation to fully Bayesain optimization.

approach.[3]

Proceeding to the linearization about a deterministic steady state, as is usual in the literature, we show that if we require non-rational agents in the model to forecast only macro-variables exogenous to their decision rules, EL of Evans and Honkapohja (2001) then makes two implicit assumptions: agents (1) know they are all identical and (2) observe the state vector including the shock processes. Our formulation, by contrast, makes neither of these assumptions. We adopt heuristic rules for IR agents which can be thought of as parsimonious forms of forecasting rules (as in Branch and Evans (2011)) which, for sophisticated agents, would take the form of high-order VARs. This, we argue, fits well the behavioural approach of assuming agents in the model with limited cognitive skills.

The main contributions of this paper are as follows: first, we start with the full non-linear formulation to provide rigorous foundations for NK behavioural models based on rational expectations (RE) or internal rationality (IR) without assumptions (1) and (2) above; second, we examine empirically the support for a composite RE-IR model of the Brock-Hommes variety by Bayesian estimation; third, in our comparisons of different composites including the pure RE and IR cases, we impose what we term *informational consistency* where RE and IR agents in the model share the same information as the econometrician estimating the model.

The nearest paper to ours is Massaro (2013) which presents a calibrated composite heterogeneous expectations model of RE and IR-anticipated-utility agents. As in our paper he emphasizes the need for policymakers to design robust rules that stabilize the economy across different composite models; but here we focus on the informational assumptions made by the two sets of agents and we seek empirical support for the modelling choices. We also relax an implied assumption in his and other models of this genre, that the two groups of agents do not lend to each other thus leading to a wealth distribution.[4]

The rest of the paper is structured as follows. Section 2 sets out the standard linear NK RE model used in the literature and then proceeds to the Brock-Hommes composite model of rational and boundedly rational agents. Section 3 goes back to the non-linear foundations of the model and demonstrates why assumption (1) above is required in the Euler learning set-up. Section 4 examines the information assumptions that are made explicitly or implicitly in the RE and boundedly rational forms of the NK model. Section 5 sets out our IR model with heuristic adaptive expectations forecasting rules. Then Section 6 provides numerical results on the dynamic properties of three possible models of expectations, rational (RE), boundedly rational with Euler learning (EL) and boundedly but internally rational (IR).[5] This section assumes homogeneous expectations for which all agents (households and firms) form either RE or IR or EL or expectations. Then in Section 7 we introduce heterogeneity in a full Brock-Hommes NK model with a composite model of IR and RE agents allowing for a wealth distribution between the two groups. Section 8 estimates the latter, alongside the pure IR and RE models by Bayesian methods, and conducts a likelihood race. This section estimates the behavioural model

[3]See Graham (2011) for a discussion of this distinction.

[4]IR also fits into the Agent-Based Modelling (ABM) framework: Sinitskaya and Tesfatsion (2014) introduce forward-looking optimizing agents into an ABM model. They use essentially the IR concept which they refer to as constructive rational decision-making. This results in a novel AB macro-model in having internally rational optimizers: households maximize expected intertemporal utility over an infinite time-horizon and firms do the same with their utility being taken as profit.

[5]Jump and Levine (2017) provides analytical results for stability.

in which the adaptive expectations assumption used by IR agents is generalized to a heuristic forecasting rule. The section provides alternative estimation results imposing different fixed proportions of rational agents. It first assumes RE agents have perfect information regarding current state variables. Then it adds an additional learning mechanism assuming that RE agents do not observe all current state variables and only have an imperfect information set. Section 9 examines the ability of these estimated variants of the NK model to match the second moments in the data. Section 10 examines the impulse response functions of the estimated model and discusses endogenous persistence. Section 11 concludes the paper. A summary of the full non-linear model is set out in a separate on-line Appendix which also contains details of the estimation results and the imperfect information solution procedure.

2 The Standard NK Behavioural Model

This section discusses the standard New Keynesian behavioural model framework used by Jang and Sacht (2012), Jang and Sacht (2014), De Grauwe (2012a), De Grauwe (2012b), Branch and McGough (2010), Massaro (2013), Cornea *et al.* (2014), Di Bartolomeo *et al.* (2016) and others.

2.1 The Workhorse NK Model

We first set out the most basic three-equation linearized workhorse NK model with RE

$$
\begin{aligned}
y_t &= \mathbb{E}_t y_{t+1} - (r_{n,t} - \mathbb{E}_t \pi_{t+1}) + u_{1,t} && (1)\\
\pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \lambda y_t + u_{2,t} && (2)\\
r_{n,t} &= \rho_r r_{n,t-1} + (1-\rho_r)(\theta_\pi \pi_t + \theta_y y_t) + u_{3,t} && (3)
\end{aligned}
$$

where y_t, π_t and $r_{n,t}$ are the output gap, the inflation rate and the nominal interest rate respectively. All variables are expressed in log-deviation form about the steady state. The shock processes $u_{i,t}$, $i = 1, 2, 3$ should be interpreted as exogenous shocks to demand (or preferences), the supply side and monetary policy respectively and are usually AR(1) processes.[6] Expectations up to now are formed assuming RE and perfect information of the state vector (which includes the shock processes). Equation (1) is the linearized Euler equation for consumption which is equated with output in equilibrium (there is no government expenditure). (2) is the NK Phillips curve and (3) is the nominal interest rate rule in 'implementable form' in that it responds to output relative to the steady state rather than the output gap.

Before relaxing the RE assumption two points about this formulation need to be made. First, there are no the lagged term in y_t in the demand curve (1) nor a lagged term in π_t in the Phillips curve (2). These can enter through the introduction of external habit in the consumers' utility function and price indexing respectively. But we choose to focus on learning as a persistence mechanism, so both these features are omitted. Second, the linearization even without these persistence terms is only correct about a zero-inflation steady state.

[6]In fact, the supply side shock is a composite of technology and marginal cost processes in the model developed in this paper. The AR(1) feature of shock processes is criticized by De Grauwe (2012b) as it implies persistence is exogenously generated. This paper addresses this critique in developing strong endogenous mechanisms through learning.

2.2 The Brock-Hommes Behavioural NK Model

In the Brock-Hommes framework, which we later follow, the model becomes behavioural by a departure from the RE assumption and the introduction of two groups of agents. One group is rational and the other forms expectations through simple 'heuristic' learning rules. RE agents form model-consistent expectations fully aware of the existence of IR agents in the composite model. General adaptive learning rules[7] that encompass those adopted by Brock and Hommes (1997), Hommes (2013), Branch and McGough (2010), De Grauwe (2012b), and De Grauwe (2012a) are

$$\mathbb{E}_t^* y_{t+1} = \mathbb{E}_{t-1}^* y_t + \lambda_y (y_{t-j} - \mathbb{E}_{t-1}^* y_t)\,; \quad \lambda_y \in [0,1],\, j = 0,1 \tag{4}$$

$$\mathbb{E}_t^* \pi_{t+1} = \mathbb{E}_{t-1}^* \pi_t + \lambda_\pi (\pi_{t-j} - \mathbb{E}_{t-1}^* \pi_t)\,; \quad \lambda_\pi \in [0,1],\, j = 0,1 \tag{5}$$

where we can in principle allow for both current and lagged observations of output and inflation, $j = 0, 1$, respectively. Throughout the rest of the paper we make the following information assumptions: for observations of *aggregate* output and inflation, $j = 1$, which is assumed in the EL approach. Later in the IR approach we need to model observations of *market-specific* variables consisting of factor prices, profits and marginal costs. These we assume can be observed without a lag and therefore $j = 0$.

Let $n_{y,t}$, $n_{\pi,t}$ be the proportions of rational agents forecasting output and inflation respectively. The IS and NK equations then become

$$y_t = n_{y,t}\mathbb{E}_t y_{t+1} + (1 - n_{y,t})\mathbb{E}_t^* y_{t+1} - [r_{n,t} - (n_{\pi,t}\mathbb{E}_t \pi_{t+1} + (1 - n_{\pi,t})\mathbb{E}_t^* \pi_{t+1})] + u_{1,t} \tag{6}$$

$$\pi_t = \beta[n_{\pi,t}\mathbb{E}_t \pi_{t+1} + (1 - n_{\pi,t})\mathbb{E}_t^* \pi_{t+1}] + \lambda(y_t - y_t^F) + u_{2,t} \tag{7}$$

To complete the model we need expressions for the weights $n_{y,t}$ and $n_{\pi,t}$. These follow the reinforcement learning literature by choosing probabilities

$$n_{x,t} = \frac{\exp(-\gamma \Phi_{x,t}^{RE}(\{x_t\}))}{\exp(-\gamma \Phi_{x,t}^{RE}(\{x_t\})) + \exp(-\gamma \Phi_{x,t}^{AE}(\{x_t\}))} \tag{8}$$

where $\Phi_{x,t}^{RE}(\{x_t\})\})$ and $\Phi_{x,t}^{AE}(\{x_t\})\})$ are 'fitness' measures respectively of the forecast performance of the rational and non-rational predictor of outcome $\{x_t\} = \{y_t\}, \{\pi_t\}$ given by a discounted least squares error predictor

$$\Phi_{x,t}^{RE}(\{x_t\}) = \mu_{RE}\Phi_{x,t-1}^{RE}(\{x_t\}) + (1 - \mu_{RE})([x_t - \mathbb{E}_{t-1}\, x_t]^2 + C_x) \tag{9}$$

$$\Phi_{x,t}^{AE}(\{x_t\}) = \mu_{AE}\Phi_{x,t-1}^{AE}(\{x_t\}) + (1 - \mu_{AE})[x_{t-j} - \mathbb{E}_{t-1-j}^*\, x_{t-1}]^2\,; j = 0,1 \tag{10}$$

where C_x represents the relative costs of being rational in learning about variable x_t. Thus the proportion of rational agents in the steady state is given by

$$n_x = \frac{\exp(-\gamma C_x)}{\exp(-\gamma C_x) + 1}$$

[7] Anufriev *et al.* (2015) provide lab-based support for such rules.

which is pinned down by the γC_x. Equations (3) and (4) – (10) constitute the linearized NK behavioural model.[8]

3 The Non-Linear New Keynesian Model

So far in the linearized model the justification for the form of adaptive forecasts needs to be established. In order to address this we step back to the underlying non-linear model and introduce the distinction between internal decisions and aggregate macro-variables. We start with the non-linear RE model and proceed from full to bounded rationality in stages.

3.1 Households

Household j chooses between work and leisure and therefore how much labour it supplies. Let $C_t(j)$ be consumption and $H_t(j)$ be the proportion of this available for work or leisure spent at the former. The single-period utility we choose, compatible with a balanced growth steady state, is

$$U_t(j) = U(C_t(j), H_t(j)) = \log(C_t(j)) - \frac{H_t(j)^{1+\phi}}{1+\phi}$$

and the value function of the representative household at time t dependent on its assets B is given by

$$V_t(j) = V_t(B_{t-1}(j)) = \mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s U(C_{t+s}(j), H_{t+s}(j))\right] \tag{11}$$

The household's problem at time t is to choose paths for consumption $\{C_t(j)\}$, labour supply $\{H_t(j)\}$ and holdings of financial savings to maximize $V_t(j)$ given by (11) given its budget constraint in period t

$$B_t(j) = R_t B_{t-1}(j) + W_t H_t(j) + \Gamma_t - C_t(j) - T_t \tag{12}$$

where $B_t(j)$ is the given net stock of financial assets at the end of period t, W_t is the wage rate, T_t are lump-sum taxes, Γ_t are profits from wholesale and retail firms owned by households and R_t is the real interest rate paid on assets held at the beginning of period t given by

$$R_t = \frac{R_{n,t-1}}{\Pi_t}$$

where $R_{n,t}$ and Π_t are the nominal interest and inflation rates respectively. W_t, $R_{n,t}$, Π_t and Γ_t are all exogenous to household j. As usual all real variables are expressed relative to the price of final output. The standard first order conditions are

$$\mathbb{E}_t\left[\Lambda_{t,t+1}(j)R_{t+1}\right] = 1$$

[8] De Grauwe (2012b), and De Grauwe (2012a) construct a rather different composite EL-type model consisting of 'fundamentalist' rather than rational agents alongside adaptive learners. For the former RE $\mathbb{E}(\cdot)$ are replaced with $\mathbb{E}^f y_{t+1} = y_t^F$ and $\mathbb{E}^f \pi_{t+1} = 0$. Thus fundamentalists always believe next period's output gap is zero and the net inflation rate will return to its steady-state value of zero. The same author also assumes $C_x = 0$ in (9). Aurissergues (2017) studies a composite model closer to that in our paper, but again in an EL framework, where non-RE agents learn the autocorrelation of endogenous variables. He shows that these agents can actually form better forecasts and dominate in the long run through switching.

$$\frac{U_{H,t}(j)}{U_{C,t}(j)} = -W_t$$

where $\Lambda_{t,t+1}(j) \equiv \beta \frac{U_{C,t+1}(j)}{U_{C,t}(j)}$ is the stochastic discount factor for household j, over the interval $[t, t+1]$. For our choice of utility function $U_{C,t} = \frac{1}{C_t}$ and $U_{H,t} = -H_t^{\phi}$ so these become

$$\frac{1}{C_t(j)} = \beta \mathbb{E}_t \left[\frac{R_{t+1}}{C_{t+1}(j)} \right] \tag{13}$$

$$C_t(j) H_t(j)^{\phi} = W_t \Rightarrow H_t(j) = \left(\frac{W_t}{C_t(j)} \right)^{\frac{1}{\phi}} \tag{14}$$

We now express the solution in a form suitable for moving from a RE to a learning equilibrium. Solving (12) forward in time and imposing the transversality condition on debt we can write

$$B_{t-1}(j) = \mathrm{PV}_t(C_t(j)) - \mathrm{PV}_t(W_t H_t(j)) - \mathrm{PV}_t(\Gamma_t) + \mathrm{PV}_t(T_t) \tag{15}$$

where the present (expected) value of a series $\{X_{t+i}\}_{i=0}^{\infty}$ at time t is defined by

$$\mathrm{PV}_t(X_t) \equiv \mathbb{E}_t \sum_{i=0}^{\infty} \frac{X_{t+i}}{R_{t,t+i}} = \frac{X_t}{R_t} + \frac{1}{R_t} \mathrm{PV}_{t+1}(X_{t+1}) \tag{16}$$

writing $R_{t,t+i} \equiv R_t R_{t+1} R_{t+2} \cdots R_{t+i}$ as the real interest rate over the interval $[t-1, t+i]$.

The forward-looking budget constraint (15) holds for the representative household. In aggregate because agents only borrow from or lend to one another there is no net debt so $B_{t-1} = 0$. Then in a symmetric equilibrium with $C_t(j) = C_t$ and $H_t(j) = H_t$, (15) and (14) become

$$\mathrm{PV}_t(C_t) = \mathrm{PV}_t \left(\frac{W_t^{1+\frac{1}{\phi}}}{C_t^{\frac{1}{\phi}}} \right) + \mathrm{PV}_t(\Gamma_t) - \mathrm{PV}_t(T_t)$$

$$H_t = \left(\frac{W_t}{C_t} \right)^{\frac{1}{\phi}}$$

Solving (13) forward in time and using the law of iterated expectation we have for $i \geq 1$

$$\frac{1}{C_t} = \beta^i \mathbb{E}_t \left[\frac{R_{t+1,t+i}}{C_{t+i}} \right] \; ; \; i \geq 1 \tag{17}$$

We now express the solution to the household optimization problem for C_t and H_t that are functions of *point expectations* $\{\mathbb{E}_t^* W_{t+i}\}_{i=1}^{\infty}$, $\{\mathbb{E}_t^* R_{t+1,t+i}\}_{i=1}^{\infty}$ and $\{\mathbb{E}_t^* \Gamma_{t+i}\}_{i=0}^{\infty}$ treated as exogenous processes given at time t. With point expectations we use (17) to obtain the following optimal decision for C_{t+i} given point expectations $\mathbb{E}_t^* R_{t+1,t+i}$

$$C_{t+i} = C_t \beta^i \mathbb{E}_t^* R_{t+1,t+i} \; ; \; i \geq 1 \tag{18}$$

$$\mathbb{E}_t^*(W_{t+i} H_{t+i}) = \frac{(\mathbb{E}_t^* W_{t+i})^{1+\frac{1}{\phi}}}{C_{t+i}^{\frac{1}{\phi}}} \tag{19}$$

Substituting (18) and (19) into the forward-looking household budget constraint, using $\sum_{i=0}^{\infty} \beta^i =$

$\frac{1}{1-\beta}$ and $\mathbb{E}_t^* R_{t,t+i} = R_t \mathbb{E}_t^* R_{t+1,t+i}$ for $i \geq 1$, we arrive at

$$
\begin{aligned}
\frac{C_t}{(1-\beta)} &= \frac{1}{C_t^{\frac{1}{\phi}}} \left(W_t^{1+\frac{1}{\phi}} + \sum_{i=1}^{\infty} (\beta^{\frac{1}{\phi}})^{-i} \left(\frac{\mathbb{E}_t^* W_{t+i}}{\mathbb{E}_t^* R_{t+1,t+i}} \right)^{1+\frac{1}{\phi}} \right) \\
&+ \Gamma_t - T_t + \sum_{i=1}^{\infty} \frac{\mathbb{E}_t^* (\Gamma_{t+i} - T_{t+i}))}{\mathbb{E}_t^* R_{t+1,t+i}}
\end{aligned}
$$

which can be written in recursive form as

$$
\begin{aligned}
\frac{C_t}{(1-\beta)} &= \frac{1}{C_t^{\frac{1}{\phi}}} \left(W_t^{1+\frac{1}{\phi}} + \Omega_{1,t} \right) + \Gamma_t - T_t + \Omega_{2,t} \qquad (20) \\
\Omega_{1,t} &\equiv \sum_{i=1}^{\infty} (\beta^{\frac{1}{\phi}})^{-i} \left(\frac{\mathbb{E}_t^* W_{t+i}}{\mathbb{E}_t^* R_{t+1,t+i}} \right)^{1+\frac{1}{\phi}} \\
&= (\beta^{\frac{1}{\phi}})^{-1} \left(\frac{\mathbb{E}_t^* W_{t+1}}{\mathbb{E}_t^* R_{t+1,t+1}} \right)^{1+\frac{1}{\phi}} + \frac{\Omega_{1,t+1}}{\beta^{\frac{1}{\phi}} \mathbb{E}_t^* R_{t+1}} \\
\Omega_{2,t} &\equiv \sum_{i=1}^{\infty} \frac{\mathbb{E}_t^* (\Gamma_{t+i} - T_{t+i})}{\mathbb{E}_t^* R_{t+1,t+i}} = \frac{\mathbb{E}_t^* (\Gamma_{t+1} - T_{t+1})}{\mathbb{E}_t^* R_{t+1,t+1}} + \frac{\Omega_{2,t+1}}{\mathbb{E}_t^* R_{t+1}}
\end{aligned}
$$

Consumption is then given by (20) assuming point expectations or by the symmetric form of the Euler equation (13) under full rationality (i.e. households know symmetric nature of equilibrium with $C_t(j) = C_t$). C_t is a function of *non-rational point expectations* $\{\mathbb{E}_t^* W_{t+i}\}_{i=1}^{\infty}$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=i}^{\infty}$ and $\{\mathbb{E}_t^* \Gamma_{t+i}\}_{i=1}^{\infty}$ treated as exogenous processes given at time t as opposed to rational model-consistent expectations $\{\mathbb{E}_t W_{t+i}\}_{i=0}^{\infty}$ etc. Since $E_t f(X_t) \approx f(E_t(X_t))$; $E_t f(X_t Y_t)) \approx f(E_t(X_t) E_t(Y_t))$ up to a first-order Taylor-series expansion, assuming point expectations is equivalent to using a linear approximation (given below) as is usually done in the literature.

3.2 Firms

Wholesale firms employ a Cobb-Douglas production function to produce a homogeneous output

$$
Y_t^W = F(A_t, H_t) = A_t H_t^{\alpha}
$$

where A_t is total factor productivity. Profit-maximizing demand for labour results in the first order condition

$$
W_t = \frac{P_t^W}{P_t} F_{H,t} = \alpha \frac{P_t^W}{P_t} \frac{Y_t^W}{H_t} \qquad (21)
$$

The retail sector costlessly converts a homogeneous wholesale good into a basket of differentiated goods for aggregate consumption

$$
C_t = \left(\int_0^1 C_t(m)^{(\zeta-1)/\zeta} dm \right)^{\zeta/(\zeta-1)} \qquad (22)
$$

where ζ is the elasticity of substitution. For each m, the consumer chooses $C_t(m)$ at a price $P_t(m)$ to maximize (22) given total expenditure $\int_0^1 P_t(m) C_t(m) dm$. Assuming government services are similarly differentiated, this results in a set of demand equations for each differentiated good m with price $P_t(m)$ of the form

$$Y_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} Y_t \tag{23}$$

where $P_t = \left[\int_0^1 P_t(m)^{1-\zeta} dm\right]^{\frac{1}{1-\zeta}}$. P_t is the aggregate price index. C_t and P_t are Dixit-Stigliz aggregates – see Dixit and Stiglitz (1977).

Following Calvo (1983), we assume that there is a probability of $1-\xi$ at each period that the price of each retail good m is set optimally to $P_t^O(m)$. If the price is not re-optimized, then it is held fixed. For each retail producer m, given its real marginal cost $MC_t = \frac{P_t^W}{P_t}$, the objective is at time t to choose $\{P_t^O(m)\}$ to maximize discounted real profits

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \frac{\Lambda_{t,t+k}}{P_{t+k}} Y_{t+k}(m) \left[P_t^O(m) - P_{t+k} MC_{t+k}\right]$$

subject to (23), where $\Lambda_{t,t+k} \equiv \beta^k \frac{U_{C,t+k}}{U_{C,t}}$ is the stochastic discount factor over the interval $[t, t+k]$. The solution to this is standard and give by

$$\frac{P_t^O(m)}{P_t} = \frac{\zeta}{\zeta-1} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \Lambda_{t,t+k} \left(\Pi_{t,t+k}\right)^{\zeta} Y_{t+k} MC_{t+k}}{\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \Lambda_{t,t+k} \left(\Pi_{t,t+k}\right)^{\zeta} \left(\Pi_{t,t+k}\right)^{-1} Y_{t+k}}$$

Denoting the numerator and denominator by J_t and JJ_t respectively, and introducing a mark-up shock MS_t to MC_t, we write in recursive form

$$\frac{P_t^O(m)}{P_t} = \frac{J_t}{JJ_t} \tag{24}$$

$$J_t - \xi \mathbb{E}_t[\Lambda_{t,t+1} \Pi_{t+1}^{\zeta} J_{t+1}] = \frac{1}{1-\frac{1}{\zeta}} Y_t MC_t MS_t \tag{25}$$

$$JJ_t - \xi \mathbb{E}_t[\Lambda_{t,t+1} \Pi_{t+1}^{\zeta-1} JJ_{t+1}] = Y_t \tag{26}$$

(see the lemma in Appendix C). Using the fact that all resetting firms will choose the same price, by the Law of Large Numbers we can find the evolution of inflation given by

$$1 = \xi \left(\Pi_{t-1,t}\right)^{\zeta-1} + (1-\xi)\left(\frac{P_t^O}{P_t}\right)^{1-\zeta} \tag{27}$$

Price dispersion lowers aggregate output as follows. Market clearing in the labour market gives

$$H_t = \sum_{m=1}^{n} H_t(m) = \sum_{m=1}^{n} \left(\frac{Y_t(m)}{A_t}\right)^{\frac{1}{\alpha}} = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{\alpha}} \sum_{m=1}^{n} \left(\frac{P_t(m)}{P_t}\right)^{-\frac{\zeta}{\alpha}}$$

using (23). Hence equilibrium for good m gives

$$Y_t = \frac{Y_t^W}{\Delta_t^{\alpha}} \tag{28}$$

where price dispersion is defined by

$$\Delta_t \equiv \left(\sum_{m=1}^{n} \left(\frac{P_t(m)}{P_t} \right)^{-\frac{\zeta}{\alpha}} \right)$$

Assuming as before that the number of firms is large we obtain the following dynamic relationship:

$$\Delta_t = \xi \Pi_t^{\frac{\zeta}{\alpha}} \Delta_{t-1} + (1-\xi) \left(\frac{J_t}{JJ_t} \right)^{-\frac{\zeta}{\alpha}} \tag{29}$$

3.3 Closing the Model

To close the model we first require total profits from retail and wholesale firms, Γ_t, is remitted to households. This is given in real terms by

$$\Gamma_t = \underbrace{Y_t - \frac{P_t^W}{P_t} Y_t^W}_{\text{retail}} + \underbrace{\frac{P_t^W}{P_t} Y_t^W - W_t H_t}_{\text{Wholesale}} = Y_t - \alpha \frac{P_t^W}{P_t} Y_t^W$$

using the first-order condition (21). Then to complete closure we have resource and balanced government budget constraints:

$$\begin{aligned} Y_t &= C_t + G_t \\ G_t &= T_t \end{aligned}$$

where G_t is an exogenous demand process, and a monetary policy rule for the nominal interest rate given by the following implementable Taylor-type rule:

$$\begin{aligned} \log \left(\frac{R_{n,t}}{R_n} \right) &= \rho_r \log \left(\frac{R_{n,t-1}}{R_n} \right) + (1-\rho_r) \Big(\theta_\pi \log \left(\frac{\Pi_t}{\Pi_{targ,t}} \right) \\ &+ \theta_y \log \left(\frac{Y_t}{Y} \right) + \theta_{dy} \log \left(\frac{Y_t}{Y_{t-1}} \right) \Big) + \epsilon_{MP,t} \qquad (30) \\ \log A_t - \log A &= \rho_A (\log A_{t-1} - \log A) + \epsilon_{A,t} \\ \log G_t - \log G &= \rho_G (\log G_{t-1} - \log G) + \epsilon_{G,t} \\ \log MS_t - \log MS &= \rho_{MS} (\log MS_{t-1} - \log MS) + \epsilon_{MS,t} \\ \log \Pi_{targ,t} - \log \Pi &= \rho_\pi (\log \Pi_{targ,t-1} - \log \Pi) + \epsilon_{\pi_{targ},t} \end{aligned}$$

and $\epsilon_{MP,t}$ is an i.i.d. shock to monetary policy. $\Pi_{targ,t}$ is a time-varying inflation target following an AR(1) process. This completes the model.

3.4 Recovering the NK Workhorse Model

We now pose the question: can the linearized form of the non-linear model about the steady state reduce to the standard workhorse model in Section 2.1 where rational expectations $\mathbb{E}_t y_{t+1}$ and $\mathbb{E}_t \pi_{t+1}$ or non-RE $\mathbb{E}_t^* y_{t+1}$ and $\mathbb{E}_t^* \pi_{t+1}$ can be treated as expectations by individual households and firms respectively of *aggregate* future output and inflation respectively? To answer this consider the linearized form of the above set-up about a zero inflation and growth deterministic steady state. With RE the household j's first order conditions take one of two forms. Either:

$$\begin{aligned} \alpha_1 c_t(j) &= \alpha_2 w_t + \alpha_3(\omega_{2,t} + r_t) + \alpha_4 \omega_{1,t} \\ \omega_{1,t} &= \alpha_5 \mathbb{E}_t w_{t+1} - \alpha_6 \mathbb{E}_t r_{t+1} + \beta \mathbb{E}_t \omega_{1,t+1} \\ \omega_{2,t} &= (1-\beta)(\gamma_t - g_t) - r_t + \beta \mathbb{E}_t \omega_{2,t+1} \\ \gamma_t &= \frac{1}{\gamma_y} y_t - \frac{\alpha}{\gamma_y}(w_t + h_t) \end{aligned} \tag{31}$$

from (20) where lower case variables $x_t \equiv \log(X_t/X)$ where X is the steady state of X_t; $c_y \equiv \frac{C}{Y}$, $\gamma_y \equiv \frac{\Gamma}{Y}$, $g_y \equiv \frac{G}{Y}$ and γ_t is *exogenous* profit per household (a function of aggregate consumption and hours). Positive coefficients are given by $\alpha_1 \equiv 1 + \frac{\alpha}{\phi c_y}$, $\alpha_2 \equiv (1-\beta)(1+\frac{1}{\phi})\frac{\alpha}{c_y}$, $\alpha_3 \equiv \frac{\gamma_y}{c_y}$, $\alpha_4 \equiv \frac{\beta\alpha}{c_y}$, $\alpha_5 \equiv (1-\beta)(1+\frac{1}{\phi})$ and $\alpha_6 \equiv (1+\frac{1}{\phi})$.

Alternatively from the Euler equation (13):

$$c_t(j) = \mathbb{E}_t c_{t+1}(j) - \mathbb{E}_t r_{t+1} \tag{32}$$

If now we make the assumption that households are identical and know this symmetric nature of the equilibrium then we have that $\mathbb{E}_t c_{t+1}(j) = \mathbb{E}_t c_{t+1}$ which is now an expectation of a variable exogenous to household j. Then in a symmetric equilibrium.

$$c_t = \mathbb{E}_t c_{t+1} - \mathbb{E}_t r_{t+1} \tag{33}$$

Linearizing the household supply of hours decision, the resource constraint and the Fisher equation we have,

$$y_t = (1-g_y)c_t + g_y g_t \tag{34}$$

$$r_t = r_{n,t-1} - \pi_t + rs_{t-1} \tag{35}$$

$$h_t = \frac{1}{\phi}(w_t - c_t)$$

Then in a special case where $G_t = 0$ and there is no distinction between public and private consumption, $g_y = 0$ and $y_t = c_t$. Equations (33)–(35) with $rs_t = u_{1,t}$ reduces to (1) where $\mathbb{E}_t y_{t+1}$ is the forecast of *aggregate* output. With RE using (31) or (32) results in the same equilibrium, but under bounded rationality with the same beliefs considered below this is no longer the case.

Turning to the supply side, for the wholesale sector:

$$\begin{aligned} y_t &= a_t + \alpha h_t \\ mc_t &= w_t - y_t + h_t \end{aligned}$$

For retail firm m, linearizing (24)–(26) and (27) about a zero net equation steady state we have:

$$p_t^o(m) - p_t = \beta\xi \mathbb{E}_t[\pi_{t+1} + p_{t+1}^o(m) - p_{t+1}] + (1-\beta\xi)(mc_t + ms_t) \tag{36}$$

$$\xi \pi_t = (1-\xi)(p_t^o - p_t) \tag{37}$$

Solving forward

$$p_t^o(m) - p_t = \mathbb{E}_t \sum_{i=0}^{\infty} (\beta\xi)^i [\beta\xi \pi_{t+i+1} + (1-\beta\xi)(mc_{t+i} + ms_{t+i})]$$

Then in a symmetric equilibrium we have

$$\pi_t = \frac{(1-\xi)}{\xi} \left(\mathbb{E}_t \sum_{i=0}^{\infty} (\beta\xi)^i [\beta\xi \pi_{t+i+1} + (1-\beta\xi)(mc_{t+i} + ms_{t+i})] \right) \tag{38}$$

where $\mathbb{E}_t[\pi_{t+i+1}]$ and $\mathbb{E}_t[mc_{t+i} + ms_{t+i}]$ are expectations of aggregate inflation and real marginal costs, both variables exogenous to individual price-setters. However, if we assume price-setters know they are identical then we can use (37) to obtain

$$p_t^o(m) - p_t = p_t^o - p_t = \frac{\xi}{(1-\xi)} \pi_t$$

Then substituting back into (36) we arrive at

$$\pi_t = \frac{(1-\xi)(1-\beta\xi)}{\xi} \mathbb{E}_t^* \sum_{i=0}^{\infty} \beta^i (mc_{t+i} + ms_{t+i}) \tag{39}$$

which omits learning about aggregate inflation. (39) is the familiar linearized Phillips curve. Under RE, (38) and (39) are equivalent. Putting $mc_t = w_t - a_t + h_t = (1+\phi)h_t = (1+\phi)(y_t - a_t)/\alpha$, (39) in recursive form gives (2) with $\lambda = \frac{(1-\xi)(1-\beta\xi)(1+\phi)}{\alpha\xi}$ and $u_{2,t} = \lambda ms_t$.

To summarize, the 'Euler Learning' form of the workhorse linearized model expressed in terms of expectations of *aggregate* output (or the output gap) and inflation is valid under bounded rationality provided that individual households and price-setting firms *know the symmetric nature of the equilibrium.* Then Euler learning is equivalent to internal rationality. If we drop this assumption, then (31) and (38) must be used given non-RE beliefs of these same aggregates and in addition expectations of the wage rate, interest rate, profits and government spending. This will be the form of the model we use under internal rationality.

4 Perfect versus Imperfect Information

We now examine the information assumptions that are made explicitly or implicitly in the RE and boundedly rational forms of the NK model. In linearized form of the NK model this can has a state-space form:

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ \mathbb{E}_t \mathsf{x}_{t+1} \end{bmatrix} = E \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} + \begin{bmatrix} F \\ 0 \end{bmatrix} \epsilon_{t+1}\,; \quad \mathsf{w}_t = G \begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} \tag{40}$$

where z_t is a $(n-m) \times 1$ vector of predetermined variables at time t with z_0 given, x_t, is a $m \times 1$ vector of non-predetermined variables and w_t is a vector observable macro-economic variables which when we come to estimation will be the data used by the econometrician. All variables are expressed as proportional deviations about a steady state. E, F and G are fixed matrices,

ϵ_t as a vector of random zero-mean shocks. RE under perfect information are formed assuming a full information set $\{z_s, x_s, \epsilon_s\}$, $s \leq t, E, F, G$.

We now proceed to the assumption that there are non-rational agents who are unable to form model-consistent expectations. For such agents, in the learning literature pioneered by Evans and Honkapohja (2001) learning rules are specified in terms of the minimum state variable representation of the perfect information model-consistent solution to (40). If the number of eigenvalues outside the unit circle is equal to the number of non-predetermined variables, the system has a unique equilibrium which is also stable with saddle-path $x_t = -Nz_t$ where $N = N(D)$ and depends on the rule (see Blanchard and Kahn (1980); Currie and Levine (1993)). Instability (indeterminacy) occurs when the number of eigenvalues of E outside the unit circle is larger (smaller) than the number of non-predetermined variables.

Partitioning E conformably with z_t and x_t, the RE perfect information solution takes the form of a first-order VAR

$$\mathsf{z}_t = [E_{11} - E_{12}N]\mathsf{z}_{t-1} + F\epsilon_t \tag{41}$$

$$\mathsf{x}_t = -N\mathsf{z}_t \tag{42}$$

$$\mathbb{E}_t\mathsf{x}_{t+1} = -N\mathbb{E}_t\mathsf{z}_{t+1} = -N[E_{11} - E_{12}N]\mathsf{z}_t \tag{43}$$

In the learning literature with 'Euler-learning' (also termed by Ellison and Pearlman (2011) as 'saddle-path learning') agents are the assumed to make their forecast (43) by using (41) to estimate a first order VAR in z_t. As we have seen this implies that agents know they are all identical. But perfect information makes a further assumption that agents *observe the state vector including the shock processes.*

We now express learning rules in terms of a subset of $\mathsf{w}_t = [y_t, \pi_t, r_{n,t}]'$. Observing these three time-series under RE enables agents (and the econometrician) to back out the shocks and to express w_t as an infinite VAR (Fernandez-Villaverde *et al.* (2007) and Levine *et al.* (2012)). To show this write the RE solution as the following ARMA process

$$\mathsf{z}_t = A\mathsf{z}_{t-1} + B\epsilon_t \tag{44}$$

$$\mathsf{w}_t = C\mathsf{z}_{t-1} + D\epsilon_t \tag{45}$$

Because we have three shocks and three observables, the matrix D is square. Assume now it is also non-singular which is only possible if w_t are *observations without lags.* Then $\epsilon_t = D^{-1}(\mathsf{w}_t - C\mathsf{z}_{t-1})$ and substituting into (44) and denoting the lag operator by L, we have

$$[(I - (A - BD^{-1}C)L]\mathsf{z}_t = BD^{-1}\mathsf{w}_t \tag{46}$$

Hence combining (41) – (46) we have

$$\mathsf{z}_t = \sum_{i=0}^{\infty}(A - BD^{-1}C)^i BD^{-1}\mathsf{w}_{t-i} \tag{47}$$

$$\mathsf{w}_t = C\sum_{i=1}^{\infty}(A - BD^{-1}C)^i BD^{-1}\mathsf{w}_{t-i} + D\epsilon_t \tag{48}$$

Convergence of the summations in (47) and (48) requires that the matrix $(A - BD^{-1}C)$ has all eigenvalues within the unit circle. Then equation (48) is an infinite VAR for the three observables $\mathsf{w}_t = [y_t, \pi_t, r_{n,t}]'$ which is estimatable from output, inflation and interest rate data.[9] It follows that the RE forecast is:

$$\mathbb{E}_t \mathsf{w}_{t+1} = C \sum_{i=0}^{\infty} (A - BD^{-1}C)^i BD^{-1} \mathsf{w}_{t-i} \tag{49}$$

whereas the adaptive heuristic rules (4) and (5) are parsimonious representations of (49):

$$\begin{aligned}
\mathbb{E}_t^* y_{t+1} &= \sum_{i=0}^{\infty} \lambda_y^i y_{t-i}\,; \quad \text{or} \quad \mathbb{E}_t^* y_{t+1} = \sum_{i=1}^{\infty} \lambda_y^i y_{t-i} \\
\mathbb{E}_t^* \pi_{t+1} &= \sum_{i=0}^{\infty} \lambda_\pi^i \pi_{t-i}\,; \quad \text{or} \quad \mathbb{E}_t^* \pi_{t+1} = \sum_{i=1}^{\infty} \lambda_\pi^i \pi_{t-i}
\end{aligned}$$

Thus we can interpret the heuristic rules as parsimonious forecasting models in which non-rational agents choose under-parameterized predictors (see Branch and Evans (2011)).

We conclude that unless shock processes are either known or observed then at best with the number of shocks equal to the number of observables and no lags in the latter, a well-specified forecasting rule in the form of an infinite VAR is available and may be e-stable converging to the RE equilibrium.[10] Otherwise the ARMA solution (44)–(45) is not invertible. In fact none of these conditions are satisfied in the set-up we consider when we come to estimation: we have *more* shocks than observables and our heuristic rules assume aggregate variables are observed with a lag. Thus if we are to compare like with like, rational agents also observe with a lag and we must therefore solve under *imperfect information*. We return to this issue in Section 8.

5 Internal Rationality

With internal rationality and anticipated utility (also known as the 'infinite horizon approach'), our model of learning is one in which agents are rational regarding their internal decisions, but have no macroeconomic model to form expectations of aggregate variables. We draw a clear distinction between aggregate and internal quantities so that identical agents in our model are not aware of this equilibrium property (nor any others). We now drop the key assumption for Euler learning that agents know they are all identical.

We utilize the internal household and retail firm decision rules set out in Section 3.4. To close the model, we need to specify the manner in which internally rational households and firms form their expectations. To do so, we assume that variables which are local to the agents, in a geographical sense, are observable within the period, whereas variables that are strictly macroeconomic are only observable with a lag. This categorization regarding information about the current state of the economy follows Nimark (2014). He distinguishes between the local information that agents acquire directly through their interactions in markets and statistics

[9] If this matrix is not stable, then the Spectral Factorization Theorem states that provided A is a stable matrix, then there exists an infinite VAR representation; but in this case the estimated shocks are not the fundamental ones ϵ_t. See Fernandez-Villaverde *et al.* (2007) for an example.

[10] Approximating the infinite lag with a finite one introduces a further degree of missecification.

that are collected and summarised, usually by governments, and made available to the wider public.[11] The only exception to this is the nominal interest rate, which we assume is observable within the period given the timing structure of NK models. Given this, we assume a strict form of naive expectations. Thus internally rational household expectations are given by

$$\begin{aligned}
\mathbb{E}_t^* r_{t+1} &= r_{n,t} - \mathbb{E}_t^* \pi_{t+1} & (50)\\
\mathbb{E}_t^* r_{t+i} &= r_{n,t+i-1} - \mathbb{E}_t^* \pi_{t+i}\,; \quad i \geq 2 & (51)\\
\mathbb{E}_t^* r_{n,t+i} &= \mathbb{E}_t^* r_{n,t+1}\,; \quad i \geq 1 & (52)\\
\mathbb{E}_{h,t}^* \pi_{t+i} &= \mathbb{E}_{h,t}^* \pi_{t+1}\,; \quad i \geq 1 & (53)\\
\mathbb{E}_t^* w_{t+i} &= \mathbb{E}_t^* w_{t+1}\,; \quad i \geq 1 & (54)\\
\mathbb{E}_t^* \gamma_{t+i} &= \mathbb{E}_t^* \gamma_{t+1}\,; \quad i \geq 1 & (55)
\end{aligned}$$

Then expressing $\mathbb{E}_t \omega_{1,t+1}$ and $\mathbb{E}_t \omega_{2,t+1}$ in (31) as forward-looking summations and using (50)–(55), we arrive at the *IR consumption equation*

$$\begin{aligned}
\alpha_1 c_t &= \alpha_2 w_t + \alpha_3(\omega_{2,t} + r_t) + \alpha_4 \omega_{1,t}\\
\omega_{1,t} &= \frac{1}{1-\beta}\left[\alpha_5 \mathbb{E}_t^* w_{t+1} - \alpha_6(\beta \mathbb{E}_t^* r_{n,t+1} - \mathbb{E}_t^* \pi_{t+1})\right] - \alpha_6 r_{n,t}\\
\omega_{2,t} &= (1-\beta)(\gamma_t - g_t) - r_t + \frac{\beta}{1-\beta}((1-\beta)(\mathbb{E}_t^* \gamma_{t+1} - \mathbb{E}_t^* g_{t+1}) - \mathbb{E}_t^* r_{t+1})
\end{aligned}$$

which is now expressed in terms of one-step ahead forecasts by

$$\mathbb{E}_t^* x_{t+1} = \mathbb{E}_t^* x_t + \lambda_x(x_{t-j} - \mathbb{E}_t^* x_t)\,; \quad x = w,\, r_n,\, \pi,\, \gamma\,; \quad j = 0, 1$$

Internally rational households make rational inter-temporal decisions for their consumption and hours supplied given adaptive expectations of the wage rate, the nominal interest rate, inflation and profits. These macro-variables may in principle be observed with or without a one-period lag ($j = 1, 0$), but as stated earlier we assume $j = 0$ for market-specific variables w_t, γ_t, and $j = 1$ for aggregate inflation π_t. However we assume the current nominal interest rate, $r_{n,t}$ is announced and therefore also observed without a lag.

For retail firm m with adaptive expectations

$$\begin{aligned}
\mathbb{E}_t^* \pi_{t+i+1} &= \mathbb{E}_t^* \pi_{t+1}\,; \quad i \geq 0\\
\mathbb{E}_t^*(mc_{t+i} + ms_{t+i}) &= \mathbb{E}_t^*(mc_{t+1} + ms_{t+1})\,; \quad i \geq 1
\end{aligned}$$

so that

$$p_t^o(m) - p_t = \frac{\beta\xi}{1-\beta}\mathbb{E}_{f,t}^* \pi_{t+1} + (1-\beta\xi)(mc_t + ms_t) + \frac{\beta}{1-\beta}\mathbb{E}_t^*(mc_{t+1} + ms_{t+1})$$

One-step ahead forecasts are given by

$$\mathbb{E}_t^* x_{t+1} = \mathbb{E}_t^* x_t + \lambda_x(x_{t-j} - \mathbb{E}_t^* x_t)\,; \quad x = \pi_f,\, (mc + ms); \quad j = 0, 1$$

[11] His paper actually focuses on a third category, information provided by the news media, and allows for imperfect information in the form of noisy signals, issues which go beyond the scope of our paper.

Internally rational retail firms make rational inter-temporal decisions for their price and output given adaptive expectations of the aggregate inflation rate and their post-shock real marginal shock wage rate. As before these variables may be observed with or without a one-period lag ($j = 1, 0$), but for aggregate inflation we assume $j = 1$ as for households, but $j = 0$ for the market-specific variable mc_t. Note that we can in principle distinguish between households' and firms' expectations of inflation.

6 Stability Analysis

We now have three possible models of expectations, rational (i.e. model consistent), boundedly rational with Euler learning and boundedly but internally rational. We denote these three cases by RE, EL and IR respectively. In this section we consider homogeneous expectations for which all agents (households and firms) form either RE or IR or EL expectations. In the next section we then allow for the possibility that households and firms are heterogenous across these groups (but retain intra-group homogeneity).

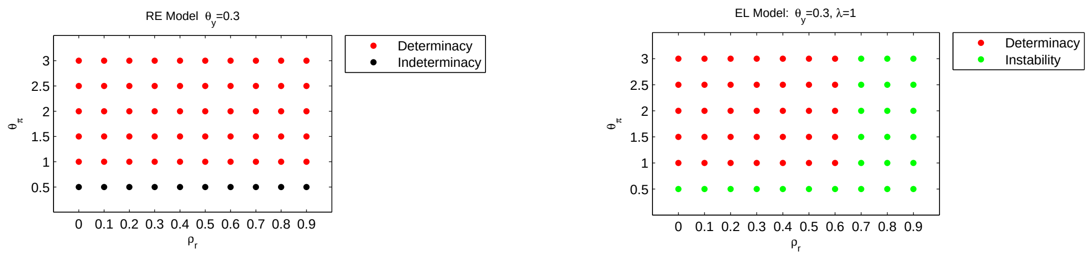
In the numerical results below we fix parameters at their priors used later in the Bayesian estimation apart from the adaptive learning parameter λ_x which we set at unity. As stated above we make the following information assumptions: for observations of *aggregate* output and inflation $j = 1$ which is assumed in the EL approach. Later in the IR approach we need to model observations of *market-specific* variables consisting of factor prices, profits and marginal costs. These we assume can be observed without a lag and therefore $j = 0$. Note this only applies to the EL and IR agents but the RE equilibrium for now assumes perfect information where agents observe all current values of state variables. Later in Section 8 we address this inconsistency and assume all agents have the *same* imperfect information (II) set as for IR agents. However for rational agents the stability conditions considered now can be derived from a perfect foresight equilibrium and are independent of the information assumption.

Figures 1 and 2 compare the models in (ρ_r, θ_π) space with $\theta_y = 0.3$ and $\theta_{dy} = 0$. Finally Figure 3 sets $\rho_r = 1$ and compares EL and IR models in (α_y, α_π) space having re-parameterized the rule as $r_{n,t} = \rho_r r_{n,t-1} + \alpha_\pi \pi_t + \alpha_y y_t$. Note that this rule reduces to a *price-level rule* when $\alpha_y = 0$. The differences in the sizes of the policy spaces that result in a saddle-path stable equilibrium are significant. Furthermore a clear ranking of the sizes of these spaces emerges with $RE \supset EL \supset IR$. This means that unless the policy rule is designed for the IR model, uncertainty as to which model of expectations is correct can lead to a rule that is unstable or has infinite multiple equilibria (i.e., is indeterminate).

7 Heterogeneous Expectations across Households and Firms

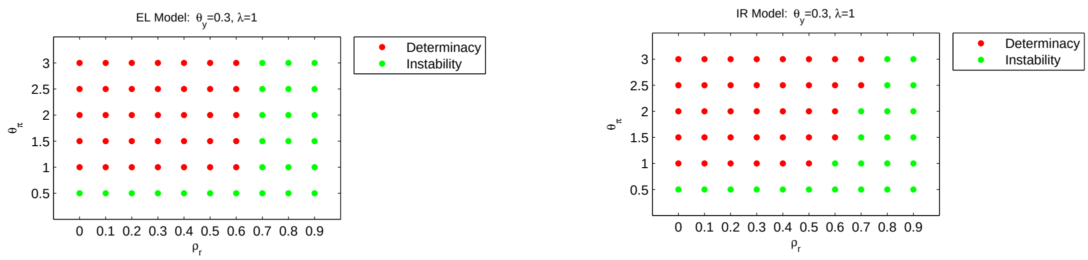
Now we come to the full Brock-Hommes NK model but with IR rather than EL boundedly rational agents. The composite RE-IR model then has an equilibrium (in the original non-linear form)

$$
\begin{aligned}
H_t^d &= n_{h,t} \left(H_t^s\right)^{RE} + (1 - n_{h,t}) \left(H_t^s\right)^{IR} \\
C_t &= n_{h,t} \left(C_t\right)^{RE} + (1 - n_{h,t}) \left(C_t\right)^{IR} = Y_t - G_t
\end{aligned}
$$



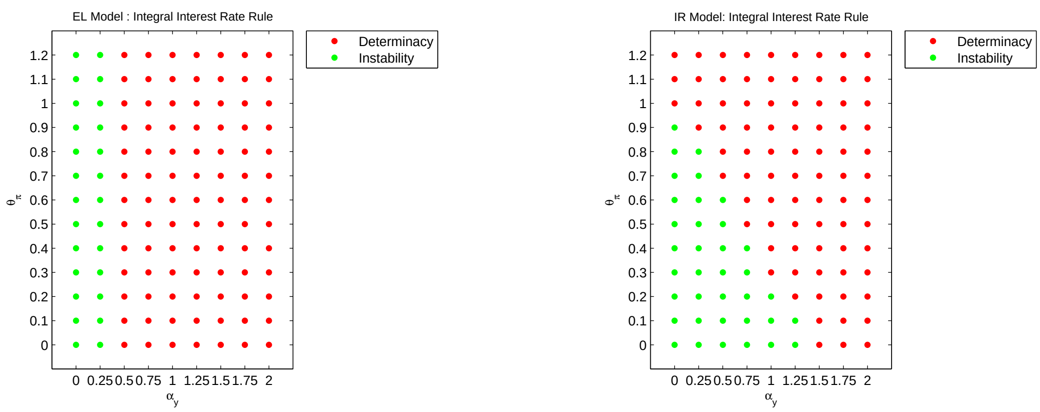
(a) RE (b) EL

Figure 1: **Comparison of Stability Properties of RE and EL Models.** $\rho_r > 0$, $\lambda_x = 1$.



(a) EL (b) IR

Figure 2: **Comparison of Stability Properties of EL and IR Models.** $\rho_r > 0$, $\lambda_x = 1$.



(a) EL (b) IR

Figure 3: **Comparison of Stability Properties of EL and IR Models.** $\rho_r = 1$, $\lambda_x = 1$.

$$\frac{P_t^o}{P_t} = n_{f,t}\left(\frac{P_t^o}{P_t}\right)^{RE} + (1-n_{f,t})\left(\frac{P_t^o}{P_t}\right)^{IR}$$

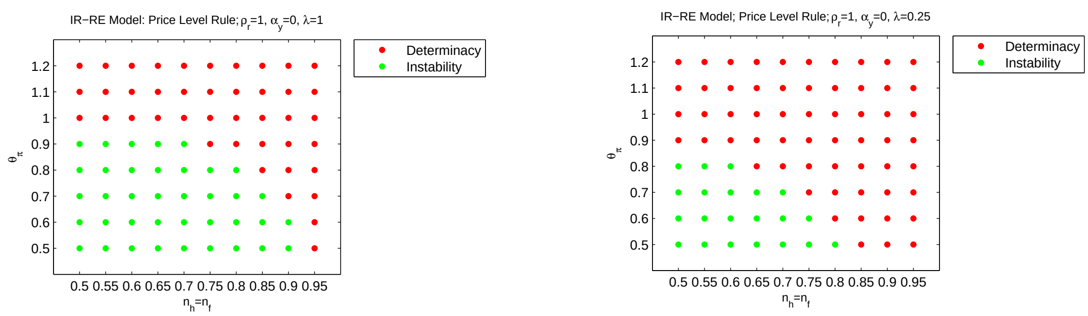
Note that rational agents in this model form model-consistent expectations taking into account the presence of internally rational agents.

We first consider the properties of the model with fixed exogenous proportions of RE and IR agents. Then we allow these proportions to be determined endogenously. Finally we model the wealth distribution between RE and IR agents.

7.1 Exogenous Proportions of RE and IR Agents

Figure 4 provides a stability analysis with a price level rule ($\rho_r = 1$, $\alpha_y = 0$ in the re-parameterized rule $r_{n,t} = \rho_r r_{n,t-1} + \alpha_\pi \pi_t + \alpha_y y_t$), and $n_h = n_f = n$ in the steady state. We can see that fast learning ($\lambda_x = 1$) results in a larger regions of instability (a smaller policy space) than the case of slower learning ($\lambda_x = 0.25$).

So far we have confined the simulations to parameter regions of the model and policy rule that result in saddle-path stability. If we enter a region of local instability, but global boundedness, we see chaotic dynamics as highlighted generally in Hommes (2013) and for an NK model with Euler learning in Branch and McGough (2010). Two points should be made concerning this possible outcome. First, there is then enormous inflation volatility under chaos so the model is one of hyper-inflation. Second, we have seen that this clearly undesirable outcome can be avoided by an appropriate choice of monetary policy rule.



(a) fast learning.

(b) slow learning

Figure 4: **Stability of RE-IR heterogeneous-agent model with price-level rule under fast and slow learning.**

Figure 5 plots the impulse response functions (IRFs) with standard parameters for the rule for a shock to monetary policy under fast and slow learning. Figures 10 to 11 in the Online Appendix show IRFs for shocks to technology and the mark-up shock. Not surprisingly fast learning sees an IRF converge faster to the RE case, but in either case IR introduces *more persistence* compared with RE. This suggests this feature should lead to a better fit of the data without relying on other persistence mechanisms (shocks, habit or price indexing). This we examine in the estimation of our model.

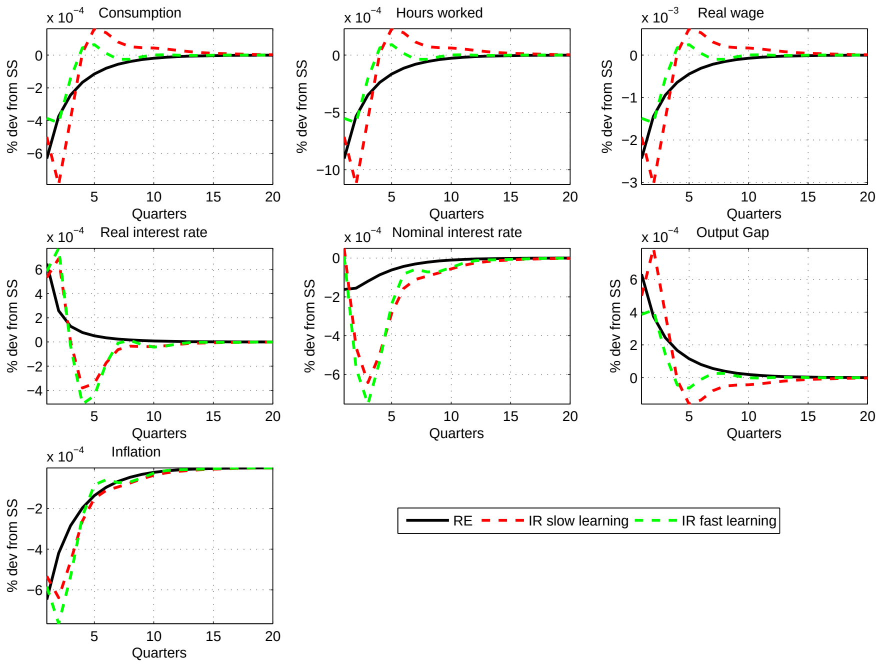


Figure 5: **RE versus RE-IR Composite Expectations with $n_h = n_f = 0.5$, $\lambda_x = 0.25, 1.0$; Taylor rule with $\rho_r = 0.7$, $\theta_\pi = 1.5$ and $\theta_y = 0.3$, $\theta_{dy} = 0$, Monetary Policy Shock**

7.2 Endogenous Proportions of RE and IR Agents

Proportions of rational households and firms are given by

$$
\begin{aligned}
n_{h,t} &= \frac{\exp(-\gamma\Phi_{h,t}^{RE})}{\exp(-\gamma\Phi_{h,t})^{RE} + \exp(\gamma\Phi_{h,t}^{IR})} \\
n_{f,t} &= \frac{\exp(-\gamma\Phi_{f,t}^{RE})}{\exp(-\gamma\Phi_{f,t}^{RE}) + \exp(\gamma\Phi_{f,t}^{IR})}
\end{aligned}
$$

where fitness for households given by

$$
\begin{aligned}
\Phi_{h,t}^{RE} &= \mu_h^{RE}\Phi_{h,t-1}^{RE} + \Big(\text{weighted sum of forecast errors} + C_h\Big) \\
\Phi_{h,t}^{IR} &= \mu_h^{IR}\Phi_{h,t-1}^{IR} + \Big(\text{weighted sum of forecast errors}\Big)
\end{aligned}
$$

with similar expressions for firms with a subscript f replacing h. Using the estimated model of the next section, Table 1 provides a third order perturbation solution of non-linear NK RE-IR Model. In the estimation the model is linearized and the proportions $n_{h,t}$ and $n_{f,t}$ are fixed. Non-linear estimation is required to pin down the parameters n_h, n_f in the steady state, and $\mu_h^{RE,IR}$, $\mu_f^{RE,IR}$ and γ in the reinforcement learning process. So here we impose them as reported in the table. We also scale the estimated standard deviations of the shocks using a parameter $\sigma = 1, 2$. The results are that our heterogenous agent model with IR alongside RE agents

Variable	Stochastic Mean	Standard Deviation (%)	Skewness	Kurtosis
$\frac{C_t}{C}$	0.9993	2.47	0.2792	0.0371
$\frac{H_t}{H}$	1.0002	0.19	0.0192	0.0327
$\frac{W_t}{W}$	0.9996	2.15	0.2771	0.0215
$\frac{\Pi_t}{\Pi}$	0.9999	0.46	0.0159	0.0645
$\frac{R_{n,t}}{R_n}$	0.9999	0.46	0.0070	0.0651
$\Phi^{RE}_{h,t} - C_h$	-0.000065	0.000020	-0.7589	0.9487
$\Phi^{AE}_{h,t}$	-0.000084	0.000054	-1.8238	5.7852
$\Phi^{RE}_{f,t} - C_f$	-0.000011	0.000009	-0.7203	0.7834
$\Phi^{AE}_{f,t}$	-0.000069	0.000053	-2.2156	8.8686
$n_{h,t}(\gamma = 1; \sigma = 1)$	0.093301	0.000004	1.8039	6.0897
$n_{f,t}(\gamma = 1; \sigma = 1)$	0.098603	0.000004	2.2688	9.2725
$n_{h,t}(\gamma = 100; \sigma = 1)$	0.094221	0.003634	1.8039	6.0897
$n_{f,t}(\gamma = 100; \sigma = 1)$	0.101751	0.004303	2.2688	9.2725
$n_{h,t}(\gamma = 1000; \sigma = 1)$	0.102506	0.036343	1.8039	6.0897
$n_{f,t}(\gamma = 1000; \sigma = 1)$	0.130105	0.043030	2.2688	9.2725
$n_{h,t}(\gamma = 1000; \sigma = 2)$	0.129993	0.146939	1.8403	6.6096
$n_{f,t}(\gamma = 1000; \sigma = 2)$	0.224367	0.174046	2.3668	10.5098

Table 1: **Third Order Solution of the Estimated NK RE-IR Model**; $\mu_h^{RE} = \mu_h^{IR} = \mu_f^{RE} = \mu_f^{IR} = 0.0$; $\gamma = 1, 100, 1000$

introduces *high kurtosis and skewness*[12] in macro variables and learning results in the numbers of rational agents increasing from the estimated deterministic steady state value of 0.093 and 0.099 to 0.13 and 0.22 for households and firms respectively in the stochastic steady state.

7.3 Wealth Distribution

Up to now we have assumed that there is no net lending of borrowing *between* each of the RE and IR households. We now relax this assumption and allow for a wealth distribution between these groups. To achieve a stationary path for bond holdings we need to introduce a portfolio adjustment cost. Consider the jth RE household with a budget constraint:

$$B_t^{RE}(j) = R_t B_{t-1}^{RE}(j) + W_t H_t(j)^{RE} + \Gamma_t - C_t(j)^{RE} - T_t - \frac{\varpi}{2}(B_{t-1}^{RE}(j) - B)^2$$

Then zero net wealth *in aggregate* implies that $n_{h,t} B_t^{RE} = -(1 - n_{h,t}) B_t^{IR}$.

Define the Lagrangian at time $t = 0$ as

$$\begin{aligned} & \mathbb{E}_t \Big[\sum_{t=0}^{\infty} \beta^t [U(C_t^{RE}(j), H_t^{RE}(j)) \\ + \quad & \lambda_t (R_t B_{t-1}^{RE}(j) + W_t H_t^{RE}(j) - \Gamma_t - C_t^{RE}(j) - T_t - \frac{\varpi}{2}(B_{t-1}^{RE}(j) - B)^2) - B_t^{RE}(j))\Big] \end{aligned}$$

[12]The absence of kurtosis in the standard NK model, often highlighted in the literature (see, for example, De Grauwe (2012a) is in part simply the consequence of linearization and non-normality is a feature of higher order approximations.

Then given $B_0^{RE}(j)$ the first order conditions are

$$
\begin{aligned}
C_t^{RE} &: \; U_C^{RE}(j) - \lambda_t = 0 \\
B_t^{RE} &: \; \mathbb{E}_t \left[\beta \lambda_{t+1}(R_{t+1} - \varpi(B_t^{RE}(j) - B)) - \lambda_t \right] = 0
\end{aligned}
$$

Hence the consumption Euler equation becomes

$$
\mathbb{E}_t \left[\beta \frac{U_{C,t+1}^{RE}(j)(R_{t+1} - \varpi(B_t(j) - B))}{U_{C,t}(j)} \right] = \mathbb{E}_t \left[\Lambda_{t,t+1}^{RE}(j)(R_{t+1} - \varpi(B_t^{RE}(j) - B)) \right] = 1
$$

The remaining change to the model is to replace C_t^{IR} with $C_t^{IR} - B_t^{IR}$.

With the same choice of parameter values as before and ϖ chosen to be very small, Figure 6 compares the impulse responses of the RE model with the heterogeneous agent RE-IR model with exogenous and equal proportions of RE and IR households and firms. Figures 12–13 in the Online Appendix provide impulse response functions for technology and government spending shocks. The case where the wealth distribution between RE and IR households is included is compared with that where (as in all the heterogeneous NK model literature) it is suppressed. The figures suggest that with our calibration the wealth distribution effect does not significantly change the equilibrium, at least up to first order for which the impulse responses are computed.

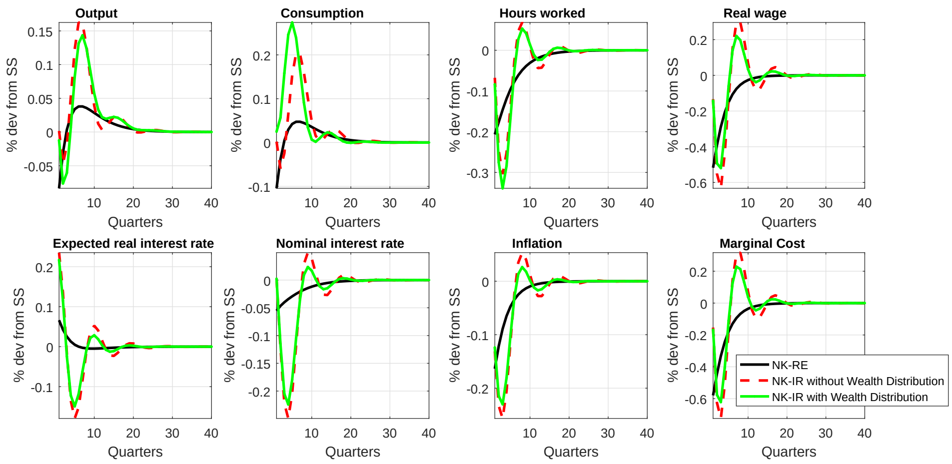


Figure 6: **Wealth Distribution and Impulse Responses – Monetary Policy Shock**

8 Bayesian Estimation

We now turn to the estimation of an empirical NK behavioural model which differs from the linearized form used up to now in two respects: first, we assume that the a steady state about which the perturbation solution is computed has a non-zero net growth and inflation. The former is stochastic and given by $g_t = (1+g)\exp(\epsilon_{Atrend}) - 1$ where ϵ_{Atrend} is a shock to technology trend. The estimation then is conducted to be consistent with the long-term trend of output and inflation in the data used in the estimation. Second, we generalize the adaptive expectations assumption used by IR agents in the previous section drawing upon Hommes *et al.* (2015), Anufriev *et al.* (2015) and Hommes (2011). For any variable with outcome X_t we study heuristic forecasting rules of the form:

$$X_t^e = X_{t-1}^{\lambda_1}(X_{t-1}^e)^{1-\lambda_1}\left(\frac{X_{t-1}}{X_{t-2}}\right)^{\lambda_2} \quad ; \quad \lambda_1 \in [0,1],\, \lambda_2 \in [-1,1]$$

where $X_t^e \equiv \mathbb{E}_{t-1}^* X_t$. If we put $\lambda_2 = 0$, this reduces to the adaptive expectations case of the previous sections.

We estimate three models with wealth distribution: the NK RE model, the NK model with individual rationality (IR Model) and the behavioural composite model with heterogeneous expectations (RE-IR Model). For the RE agents in either the 'pure' or composite RE model we assume and compare perfect or imperfect information sets as discussed in Section 4. Bayesian methods are employed using Dynare adapted to handle imperfect information.[13] We use a subset of the observable set used in Smets and Wouters (2007) in first difference at quarterly frequency but extend the sample length to the second quarter of 2008, before the outbreak of the 2008-09 crisis. Thus the sample period is 1984:1-2008:2. These observable variables are the log differences of real GDP and the GDP deflator, and the federal funds rate. All series are seasonally adjusted and taken from the FRED Database available through the Federal Reserve Bank of St.Louis and the US Bureau of Labour Statistics.

8.1 The Measurement Equations and Priors

The corresponding measurement equations for the 3 observables are:[14]

$$\begin{bmatrix} D(\log GDP_t) * 100 \\ \log(GDPDEF_t/GDPDEF_{t-1}) * 100 \\ FEDFUNDS_t/4 * 100 \end{bmatrix} = \begin{bmatrix} \log\left(\frac{Y_t}{\overline{Y}_t}\right) - \log\left(\frac{Y_{t-1}}{\overline{Y}_{t-1}}\right) + \text{trend} + \epsilon_{y,t} - \epsilon_{y,t-1} + \epsilon_{A,t} \\ \log\left(\frac{\Pi_t}{\Pi}\right) + \text{cons}_\pi + \epsilon_{\pi,t} \\ \log\left(\frac{R_{n,t}}{R_n}\right) + \text{cons}_r \end{bmatrix}$$

where constants *trend*, $cons_\pi$ and $cons_r$ are related to the steady state of our model by

$$\begin{aligned} \Pi &= \text{cons}_\pi/100 + 1 \\ \log(1+g) &= \text{trend}/100 \\ R_n &= \frac{\Pi}{\beta_g} = \frac{\Pi(1+g)}{\beta} = \text{cons}_r/100 + 1 \end{aligned}$$

This implies that β is determined empirically as

$$\beta = \left(\frac{\text{cons}_\pi + 100}{\text{cons}_r + 100}\right)(1+g)$$

We introduce measurement errors on two observables, output and inflation ($\epsilon_{y,t}$ and $\epsilon_{\pi,t}$) so in total there are 3 variables in the observations, 4 exogenous AR(1) processes (A_t, G_t, MS_t, $\Pi_{targ,t}$) and 4 further i.i.d shocks including measurement errors, ($\epsilon_{MP,t}$, $\epsilon_{Atrend,t}$ and $\epsilon_{y,t}$, $\epsilon_{\pi,t}$).

[13] Levine *et al.* (2017a) provides full details of this addition to Dynare.

[14] $Y_t = GDP_t$, $\overline{Y}_t$ =trend and trend growth $=\log \overline{Y}_t - \log \overline{Y}_{t-1} = \log(1+g) + \epsilon_{A,t}$. $\epsilon_{y,t}$ and $\epsilon_{\pi,t}$ are measurement equations for output and inflation respectively.

Thus there are 8 shocks and 3 observables meaning that the invertibilty condition discussed in Section 4 is not satisfied. A number of the structural parameters are fixed, so as to match their sample means or in accordance with previous studies and are collected into Θ_f:

$$\Theta_f \equiv [\zeta,\, \alpha,\, \mu_h^{RE},\, \mu_h^{IR},\, \mu_f^{RE},\, \mu_f^{IR},\, \gamma] = [7.0,\, 0.7,\, 0.5,\, 0.5,\, 0.5,\, 0.5,\, 1.0]$$

These parameters are necessary to solve and linearize the models but are problematic for estimation (e.g. identification). From Section 7 the parameters in the RE-IR model, $[\mu_h^{RE}\, \mu_h^{IR}\, \mu_f^{RE}\, \mu_f^{IR}\, \gamma]$, do not enter into the first-order solution for the linearized model but only affect the second-order or higher solutions. They cannot be identified in the first-order solution that is used for estimation so are imposed at their mid-point values as above. As in De Grauwe (2011) we fix γ to unity so that allow for a moderate degree in the intensity of individual choice. The remaining calibration values for $[\zeta, \alpha]$ are standard choices in the DSGE literature.

For the remainder of parameters gamma and inverse gamma distributions are used as priors when non-negativity constraints are necessary, and beta distributions for fractions or probabilities. Normal distributions are used when more informative priors seem to be necessary. The prior means and distributions of these parameters can be found in Table 2. The values of priors are in line with those in Smets and Wouters (2007). The Calvo coefficient ξ is assumed to be beta distributed with prior mean of 0.5 and prior standard deviation of 0.2, implying that prices are sticky for two quarters. We draw all the AR(1) parameters ρ_A, ρ_{MS}, ρ_π and ρ_G, and the lagged interest rate ρ_r from the beta distribution in order to restrict them to the open unit interval. Similarly, the beta distribution we use on the adaptive expectations learning parameter λ_1 also restricts it to the open unit interval, but we set a generalized beta prior for λ_2 with support $[-1, 1]$ and 0 mean. For all these beta distribution parameters we centre the prior density in the middle of the unit interval.

A common theme in papers that study empirical RBC/DSGE models is the difficulty in pinning down the parameter of labour supply elasticity ϕ. Inference on the inverse Frisch elasticity of labour supply has been found susceptible to model specifications, and exhibiting wide posterior probability intervals. So we assume a normal distribution with mean 2.0 and standard deviation of 0.5 for the parameter which is well within the range of point estimates reported in the RBC and labour literature. For the Taylor rule parameter on inflation the prior is set to obey the Taylor principle is centred at the value suggested by Taylor. With regard to output level and growth the response of interest rate is smaller but we do not rule out negative responses for both parameters. Finally the priors on the standard deviations of the exogenous shocks and measurement errors are assumed to have inverse gamma distributions. The uncertainty held about these elements motivates an open interval for their priors that excludes zero and is unbounded.

8.2 Identification Checks and Estimation of the Posterior Distribution

Based on the prior information, we first conduct some pre-estimation identification diagnostics and report them in detail in Appendix D. We find that the sensitivity effect of n_f at its posterior mean point is relatively weak and also at their estimated values, the pair-wise collinearity between n_f and ξ is very close to an exact linear dependence between the pair (0.9927). From

Parameter	Notation	Prior distribution		
		Density	Mean	S.D/df
Calvo prices	ξ	$\mathcal{B}$	0.50	0.10
Labour supply elasticity	ϕ	$\mathcal{N}$	2.00	0.50
Adaptive learning $\in [0,1]$	λ_1	$\mathcal{B}$	0.50	0.20
Adaptive learning $\in [-1,1]$	λ_2	$\mathcal{B}$	0.00	0.30
Interest rate rule				
Inflation	θ_π	$\mathcal{N}$	1.50	0.25
Output	θ_y	$\mathcal{N}$	0.12	0.05
Output growth	θ_{dy}	$\mathcal{N}$	0.12	0.05
Interest rate smoothing	ρ_r	$\mathcal{B}$	0.75	0.10
AR(1) coefficient				
Technology	ρ_A	$\mathcal{B}$	0.50	0.20
Government spending	ρ_G	$\mathcal{B}$	0.50	0.20
Price mark-up	ρ_{MS}	$\mathcal{B}$	0.50	0.20
Inflation target	ρ_π	$\mathcal{B}$	0.50	0.20
Standard deviation of shocks				
Technology trend	$sd(\epsilon_{Atrend})$	$\mathcal{IG}$	0.10	2.00
Technology	$sd(\epsilon_A)$	$\mathcal{IG}$	0.10	2.00
Government spending	$sd(\epsilon_G)$	$\mathcal{IG}$	0.10	2.00
Price mark-up	$sd(\epsilon_{MS})$	$\mathcal{IG}$	0.10	2.00
Inflation target	$sd(\epsilon_{\pi targ})$	$\mathcal{IG}$	0.10	2.00
Monetary policy	$sd(\epsilon_{MP})$	$\mathcal{IG}$	0.10	2.00
Standard deviation of measurement errors				
Observation error (output)	$sd(\epsilon_y)$	$\mathcal{IG}$	0.10	2.00
Observation error (inflation)	$sd(\epsilon_\pi)$	$\mathcal{IG}$	0.10	2.00

Table 2: **Prior Distributions**

high correlations to near-exact collinearity one may suspect some weak identification. Figure 15 in the Online Appendix shows the identification strength and sensitivity component in the moments using the composite RE-IR estimation results and shows again the sensitive strength in the moments of n_h is very weak. Therefore in this section we compare the cases without estimating n_f and n_h so the proportions $n_h = n_f = n = 0.5, 0.1$ are fixed to the values we used in the stability section earlier in the paper.

Turning to the estimation, the joint posterior distribution of the estimated parameters is obtained in two steps. First, the posterior mode and the Hessian matrix are obtained via standard numerical optimization routines. The Hessian matrix is then used in the Metropolis-Hastings (MH) algorithm to generate a sample from the posterior distribution. Two parallel chains are used in the Monte-Carlo Markov Chain Metropolis-Hastings (MCMC-MH) algorithm. Thus, 100,000 random draws (though the first 25% 'burn-in' observations are discarded to remove any dependance from the initial conditions) from the posterior density are obtained via the MCMC-MH algorithm, with the variance-covariance matrix of the perturbation term in the algorithm being adjusted in order to obtain reasonable acceptance rates (between 20%-40%). We run an iterative process of MCMC simulations in order to calibrate the scaling factor to achieve the desired rate of acceptance which is key for the speed of convergence of the MCMC-MH chains, which are also sensitive to the number of MCMC iterations. The former ensures that more of the parameter region is searched more regularly, but at the expense of reducing the acceptance ratio. In this estimation the number of draws we choose is sufficient to allow for convergence. To formally test and to check the convergence, besides calibrating the acceptance rate, we use the convergence indicators recommended by Brooks and Gelman (1998) and Gelman *et al.* (2003).

8.3 Bayes Factor Comparison

We first focus on the pure RE, pure IR and the composite RE-IR models when RE agents have a perfect information set. We employ the Bayes Factor (BF) from the model marginal likelihoods to gauge the relative merits across the four models in Table 3.

Model	Pure RE (PI)	Pure IR	RE(PI)-IR (n=0.5)	RE(PI)-IR (n=0.1)
LL	-143.05	-138.90	-139.38	-138.15
Prob	0.0042	0.2666	0.1649	0.5643

Table 3: **Marginal Log-likelihood Values and Posterior Model Odds: RE Agents with Perfect Information (PI)**

Models IR (Pure IR) and RE-IR ($n = 0.1, 0.5$) all substantially outperform their RE counterpart which is firmly rejected by the data. Formally, using the Bayesian statistical language of Kass and Raftery (1995), a BF, the quotient of the probabilities reported, greater than 100 (marginal log-likelihood difference over 4.61) offers "decisive evidence". Thus we have decisive support for the pure IR and some composite behaviour from the US data we observe. However the BF differences between the non-RE models are not strong.

Next we assume an imperfect information set for the RE agents of the form:

$$I_t = [Y_{s-1}, \Pi_{s-1}, R_{n,s}\,;\, s \leq t]$$

The policy maker is assumed observe current output, inflation and to know to know her own current inflation target. The implemented rule therefore is still (30), but the *perceived* rule for RE agents with II, again imposing point expectations, is now given by the rule:

$$\begin{aligned}\log\left(\frac{R_{n,t}}{R_n}\right) &= \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + (1-\rho_r)\Big(\theta_\pi \log\left(\frac{\mathbb{E}_t\Pi_t}{\mathbb{E}_t\Pi_{targ,t}}\right) \\ &+ \theta_y \log\left(\frac{\mathbb{E}_t Y_t}{Y}\right) + \theta_{dy}\log\left(\frac{\mathbb{E}_t Y_t}{Y_{t-1}}\right)\Big) + \epsilon_{MP,t} \qquad (56)\end{aligned}$$

where rational expectations under II of current inflation, output and the inflation target are now required to implement the rule. An important point to stress is that this is the *same information set* we assume for IR agents when they come to update their heuristic rule. In this sense we now have *informational consistency* across IR and RE agents, and also with the econometrician estimating the model. This feature we believe is new for the heterogeneous behavioural NK model literature.[15] The results for the likelihood race are reported in Table 4.

[15] If we assume informational consistency for the policymaker as well then her information set would be $I_t = [Y_{s-1}, \Pi_{s-1}, R_{n,s}, \Pi_{targ,t}]$. Then the implemented rule becomes (56), rather than (30), but with $\mathbb{E}_t\Pi_{targ,t}$ replaced with $\Pi_{targ,t}$ (since the policymaker knows her own target). But then the set-up involves *two* imperfect information sets and goes beyond our II framework with only one. This more general case is studied in Lubik *et al.* (2017) who show that this generalization results in different Blanchard-Kahn stability conditions for perfect and imperfect information.

Model	Pure RE (II)	Pure IR	RE(II)-IR (n=0.5)	RE(II)-IR (n=0.1)
LL	-135.60	-138.90	-136.83	-137.88
Prob	0.6986	0.0258	0.2042	0.0715

Table 4: **Marginal Log-likelihood Values and Posterior Model Odds: RE Agents with Imperfect Information (II)**

Now a very different picture emerges when comparing the RE model with the behavioural alternatives. RE with imperfect information (RE(II)) actually wins the likelihood race. In formal Bayesian language, a BF of 10-100 or a marginal likelihood range of [2.30, 4.61] is "strong to very strong evidence" so the RE(II) strongly dominates the pure II and RE(II)-IR with n=0.1. But the likelihood race cannot separate the RE(II) and RE(II)-IR composite model with n=0.5. In Section 9 we examine whether the ability to match second moments of the data is able to separate these two models. But first we turn to the parameter estimation results.

8.4 Parameter Estimation Results

Table 5 contains summary statistics of the posterior distributions of the NK models. We report posterior means of the parameters of interest and 95% probability intervals alongside the posterior model odds for all 7 models so far: RE(PI), RE(II), IR and RE(PI)-IR or RE(II)-IR with $n = 0.5, 0.1$.

The price stickiness parameter, ξ, is estimated to be larger than assumed in the prior distribution (0.59). This implies that there is some degree of price stickiness and the implied average contract duration is about 2.44 quarters from this model. The posteriors of this model also indicate a Frisch labour supply elasticity, $\phi^{-1} = 0.61$ and a strong response to inflation that satisfies the Taylor principle, $\theta_\pi = 1.77$. In terms of the persistence of the exogenous shocks, the estimates of the AR(1) coefficients show that the technology and inflation shocks are inertial. According to the estimated standard deviation, the technology shock stands out as being the most volatile structural shock in this economy. The interest rate policy shock is less volatile and is less important in driving inflation, consumption and output. The variations in measurement error of output is relatively moderate in this model but there is a sizeable estimate of the inflation measurement error. Overall these estimates are in the range often found in the existing literature.

The IR solution equilibrium we propose departs from the standard RE solutions and allows a process of adaptive learning driven by the speed of learning parameter $\lambda_1 \in [0, 1]$ and $\lambda_2 \in [-1, 1]$ for the household and firms respectively. The closer λ is to zero slower the learning process is, which is the key mechanism of this setup because this introduces more dynamics into the model.

Focusing on the parameter characterising the degree of price stickiness, ξ, again, the mean estimates report an average price contract duration of around 1.96 and 1.92 quarters for IR and RE-IR. Their estimated 95% intervals imply that price contacts change in the ranges of $\in (1.54, 2.50)$ suggesting that the firms of IR and RE-IR economies change prices as frequently as once every 1.5 quarters. The estimated contract length is shorter in the non-pure-RE models. The estimates of the AR(1) coefficients show that the technology shock is significantly inertial.

With IR in the model, the exogenous technology shock volatility contributes the most to the variation in the data and the monetary/fiscal policy volatility mattes much less for this aspect of the fit. The price mark-up shock (the uncertainty interval) is sightly larger than that of the RE model because the expectation heterogeneity in the model increases inflation volatility (uncertainty) and acts as a persistent force in this behavioural economy in the inflation fluctuations (this is also evident in the following section when we examine the implied model moments). The measurement error on inflation also has some sizeable contribution.

For the policy rule, we find that for the behavioural models the parameter estimate for the degree of interest rate smoothing indicates that there is a low degree of persistence in the nominal interest rate which is much lower than observed in the literature. The responses to output (θ_y, θ_{dy}) are very low, nearly non-existent, while the feedback to inflation (θ_π) is strong, implying a stronger concern from the monetary authorities about inflation variability, relative to the moments in output, which is caused by the varying forecast behaviours from agents' heterogenous expectations.

Overall, the parameter estimates are reasonably robust across information specifications, despite the fact that the II alternative leads to a much better model fit based on the corresponding posterior marginal likelihood. It is interesting to note that the point estimates of almost every single parameter under II are tighter and more strongly determined compared with the case under the standard PI assumption, i.e., the confidence intervals are more tightly estimated with II, so this helps to explain its superior performance in the likelihood race.

9 Matching Second Moments

In this section we examine the model second moments, which has been a standard practice for researchers in the RBC tradition. We consider second moments and autocorrelations in turn. In this section, we mainly focus our analysis on the baseline RE model with its II variant, the behavioural IR and the outperforming composite, including conditional second moments implied by the estimated models such as impulse responses.

In terms of matching volatility the behavioural composite RE(II)-IR is able to match precisely the rate of change of output (henceforth referred to as 'output') standard deviation in the data and performs very well at getting much closer to the interest rate data, whereas the pure RE model (including II) performs very poorly at capturing inflation and interest rate volatility, lying well-outside the 95% confidence bands. In the pure RE(PI) economy, the central bank can reduce output variability by applying a policy regime with strong output responses, but this comes at a cost of much higher inflation volatility. However, for the behavioural composite, there is room for improvement in matching inflation volatility. The model's ability of matching inflation moments is distorted, generating much volatility in inflation than the data and as noted this can be explained by the role played by the more volatile pricing shock (ϵ_{MS}) found in the estimated models which gives rise to the amplification effects on inflation dynamics caused by the expectation heterogeneity in the behavioural economy. The pure IR model is able to reduce this volatility while still matching output well.

Table 6 also reports the cross-correlations of the 3 observable variables vis-a-vis output. All the estimated models do well and predict the correct sign for the output-inflation cross-

	Pure RE(PI)	RE-IR(n=0.5)	RE-IR(n=0.1)	Pure IR	Pure RE(II)	RE-IR(n=0.5,II)	RE-IR(n=0.1,II)
ξ	0.59 [0.50:0.68]	0.49 [0.35:0.62]	0.48 [0.35:0.61]	0.49 [0.35:0.64]	0.60 [0.54:0.67]	0.43 [0.30:0.58]	0.42 [0.27:0.56]
ϕ	1.64 [0.71:2.66]	1.73 [0.87:2.58]	1.74 [0.88:2.65]	1.63 [0.72:2.57]	1.44 [0.64:2.22]	2.00 [1.75:2.32]	1.81 [1.59:2.05]
Adaptive learning							
λ_1	-	0.17 [0.04:0.38]	0.29 [0.04:0.65]	0.29 [0.04:0.66]	-	0.66 [0.53:0.86]	0.51 [0.30:0.73]
λ_2	-	0.58 [0.37:0.82]	0.47 [0.16:0.82]	0.40 [0.04:0.85]	-	0.14 [-0.03:0.34]	0.20 [0.01:0.51]
Proportion of rationality (imposed)							
n_h	1.00	0.50	0.10	0.00	1.00	0.50	0.10
n_f	1.00	0.50	0.10	0.00	1.00	0.50	0.10
Interest rate rule							
θ_π	1.77 [1.46:2.11]	1.28 [1.12:1.43]	1.22 [1.05:1.46]	1.17 [1.06:1.26]	1.55 [1.22:1.85]	1.10 [1.04:1.17]	1.09 [1.04:1.13]
θ_y	0.09 [0.01:0.17]	-0.03 [-0.06:0.00]	-0.02 [-0.05:0.00]	-0.01 [-0.04:0.01]	0.14 [0.06:0.22]	0.01 [-0.00:0.03]	0.00 [-0.01:0.01]
θ_{dy}	0.11 [0.03:0.20]	0.02 [-0.03:0.07]	0.02 [-0.03:0.07]	0.00 [-0.01:0.01]	0.10 [0.01:0.18]	0.11 [0.05:0.18]	-0.00 [-0.04:0.03]
ρ_r	0.55 [0.40:0.68]	0.40 [0.33:0.47]	0.39 [0.33:0.45]	0.41 [0.36:0.46]	0.49 [0.37:0.61]	0.36 [0.30:0.44]	0.38 [0.32:0.44]
AR(1) coefficients							
ρ_A	0.86 [0.78:0.94]	0.96 [0.93:0.99]	0.96 [0.92:0.99]	0.96 [0.92:0.99]	0.36 [0.08:0.65]	0.96 [0.94:0.99]	0.96 [0.93:0.99]
ρ_G	0.49 [0.23:0.71]	0.51 [0.19:0.84]	0.54 [0.24:0.86]	0.51 [0.14:0.80]	0.41 [0.25:0.58]	0.51 [0.21:0.88]	0.51 [0.29:0.68]
ρ_{MS}	0.50 [0.16:0.84]	0.50 [0.18:0.79]	0.50 [0.18:0.81]	0.53 [0.21:0.84]	0.47 [0.18:0.80]	0.58 [0.41:0.76]	0.56 [0.24:0.83]
ρ_π	0.97 [0.94:0.99]	0.73 [0.40:0.96]	0.57 [0.25:0.95]	0.54 [0.21:0.89]	0.98 [0.96:0.99]	0.66 [0.49:0.86]	0.48 [0.34:0.69]
Standard deviation of shocks							
ϵ_{Atrend}	0.55 [0.48:0.62]	0.09 [0.06:0.11]	0.09 [0.06:0.12]	0.09 [0.07:0.12]	0.59 [0.53:0.66]	0.08 [0.05:0.11]	0.09 [0.06:0.12]
ϵ_A	0.06 [0.03:0.10]	0.56 [0.49:0.62]	0.55 [0.47:0.62]	0.54 [0.48:0.61]	0.07 [0.03:0.12]	0.60 [0.54:0.66]	0.60 [0.53:0.65]
ϵ_G	0.14 [0.02:0.30]	0.06 [0.03:0.10]	0.06 [0.02:0.09]	0.14 [0.02:0.29]	0.08 [0.02:0.19]	0.07 [0.02:0.11]	0.10 [0.02:0.24]
ϵ_{MS}	0.07 [0.03:0.11]	0.07 [0.02:0.12]	0.10 [0.03:0.22]	0.10 [0.02:0.22]	0.08 [0.03:0.14]	0.10 [0.02:0.23]	0.10 [0.02:0.24]
ϵ_π	0.08 [0.05:0.10]	0.05 [0.03:0.06]	0.04 [0.03:0.06]	0.04 [0.02:0.06]	0.07 [0.04:0.09]	0.04 [0.02:0.06]	0.04 [0.02:0.06]
ϵ_{MP}	0.07 [0.02:0.12]	0.04 [0.02:0.05]	0.04 [0.03:0.06]	0.04 [0.02:0.05]	0.06 [0.02:0.09]	0.04 [0.02:0.05]	0.04 [0.02:0.05]
Standard deviation of measurement errors							
ϵ_y	0.06 [0.03:0.10]	0.06 [0.03:0.11]	0.07 [0.03:0.13]	0.07 [0.02:0.12]	0.06 [0.03:0.09]	0.06 [0.03:0.09]	0.06 [0.02:0.10]
ϵ_π	0.52 [0.45:0.65]	0.47 [0.40:0.54]	0.48 [0.42:0.55]	0.48 [0.42:0.57]	0.60 [0.53:0.67]	0.53 [0.47:0.59]	0.53 [0.47:0.61]
Price contract length							
$\frac{1}{1-\xi}$	2.44	1.96	1.92	1.96	2.50	1.75	1.72
Marginal likelihood and posterior model odd							
LL	-143.05	-139.38	-138.15	-138.90	-135.60	-136.83	-137.88
Prob.	0.0004	0.0149	0.0509	0.0241	0.6523	0.1907	0.0667

Table 5: **Bayesian Posterior Distributions for RE, IR and Composite RE-IR Models**: Perfect Information (PI) and Imperfect Information (II) Assumptions for RE Agents. For all estimated models we use observations with a lag and the information set for lag 1 case at time t is $I_t = \{Y_{t-1}, \Pi_{t-1}, R_{n,t}\}$. $n = n_h = n_f = 0.5, 0.1$ are imposed in this estimation. The trend or mean of the data variables are calculated directly from the data and not estimated with the rest of the model. The steady state is consistent with these values.

correlation and the best performing behavioural composite is also highly successful in reproducing the co-movement in the data. However, in terms of the output-interest rate correlation, all models perform poorly and have the wrong sign although the RE-II assumption improves in this dimension, getting slightly closer to the data. Overall, the strength of the composite-II behaviour reproducing business cycles lies in the output and interest rate moments as the estimated model matches most of the US data and the empirical moments are captured well-within the 95% uncertainty bands in the data.

		Standard Deviation	
	Output	Inflation	Interest rate
US Data	0.58	0.24	0.61
	(0.50, 0.69)	(0.21, 0.27)	(0.55, 0.70)
Pure RE(PI)	0.54	1.02	0.84
Pure IR	0.53	**0.68**	0.45
Pure RE(II)	0.60	1.17	0.97
Heuristic-RE(II)-IR ($n = 0.5$)	**0.58**	0.78	**0.58**
		Cross-correlation with Output	
US Data	1.00	-0.12	0.22
	(-)	(-0.31, 0.10)	(0.02, 0.39)
Pure RE(PI)	1.00	-0.02	-0.02
Pure IR	1.00	-0.04	-0.05
Pure RE(II)	1.00	-0.01	**-0.01**
Heuristic-RE(II)-IR ($n = 0.5$)	1.00	**-0.07**	-0.08

Table 6: **Selected Second Moments (At the Posterior Means)**: For the empirical moments computed from the dataset the bootstrapped 95% confidence bounds based on the sample estimates are presented in parentheses.

If we look at the autocorrelations up to 10 lags in Figure 7, the picture is also somewhat mixed. Overall it shows very good goodness-of-fit of RE-IR Composite under II to data in terms of successfully capturing the autocorrelations up to many lags – in any case, almost all of the moments are inside the 95% confidence intervals of the empirical moments of autocorrelations, which leads to some confidence in the estimated models. Model RE with and without II is problematic in reproducing the output autocorrelations at the first two and three orders, ACF lying outside of the lower interval and having the wrong sign. The behavioural composite with the generalized forecasting rule is capable of generating more persistence in inflation and interest rate than the IR special case and the reason for this lies in the estimated learning mechanism of the adaptive expectations scheme in, for example, forecasting inflation movements from their RE counterparts. These autocorrelations are able to reproduce an important stylized fact, namely the persistence of aggregate inflation usually observed in empirical data, generating much inertia in the time path to match the actual inflation (also shown in the IRF predictions below). This is more effective than the pure RE case with II learning and/or exogenous shock dynamics which generates too much inertia. Finally switching the information set from PI to II for the RE model produces a little more persistence, captured by the implied correlograms of inflation.

The findings in this section are generally in line with those in Jang and Sacht (2014), who conduct an empirical investigation on moment matching using a bounded rationality behavioural model à la De Grauwe (2011) estimated by the Simulated Method of Moments for the Euro

Area. They find that their results can mimic the real data well, slightly outperforming the linear RE counterpart in some of the moments, or are at least as good as the RE model in terms of providing fits for auto- and cross-covariances of the data. Perhaps the main message to emerge from this RBC type of model validity exercise is that it can be misleading to assess model fit using a selective choice of second moment comparisons as there are trade-offs in terms of fitting some second moments well, at the expense of others. As pointed out the most comprehensive form of assessment of competing models is via likelihood comparisons. In this moments analysis there is some evidence that shows a good fit of both RE(II) and Composite RE-IR, in particular how they capture the autocorrelation dynamics and output volatility, to some dimensions of the data but this needs to be analysed with some caution and the probabilistic assessment using the marginal likelihoods provides the most decisive support. Our estimated models so far replicate the stylized facts, yielding persistence in aggregate data, obtaining reasonable inertia to get close to the data with the endogenized learning mechanisms – this shows an improved ability of the DSGE model with II and IR behaviour to generate endogenous propagation mechanisms. This explains the improved overall model fit in the comparison section.

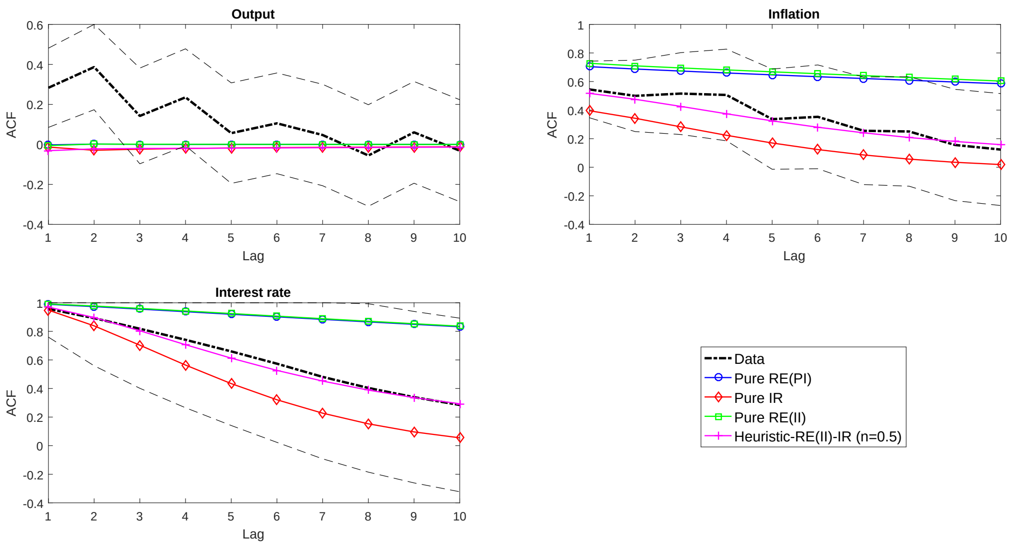


Figure 7: **Autocorrelations of Observables in the Actual Data and in the Estimated Models**: The approximate 95% confidence bands are constructed using the large-lag standard errors (see Anderson (1976)).

10 Posterior Impulse Responses and Endogenous Persistence

As shown above from the estimated models and the moment analysis, both the heuristic rules and RE-II learning mechanisms introduces more dynamics (persistence) into the model solutions. As a result, the empirical models incorporating either form of endogenous learning can significantly outperform the standard RE-PI model in the likelihood comparison. The empirical impulse response functions from the estimated models in this section support these conclusions. In Figures 8 - 9, relaxing PI in particular introduces more persistence compared with RE-PI, generating more hump-shaped trajectories after the system is shocked suggesting this feature

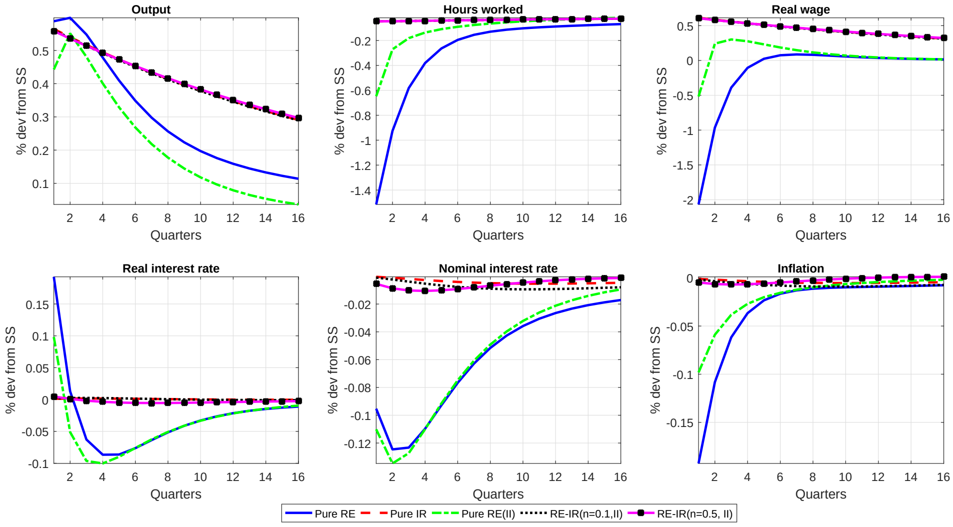


Figure 8: **Estimated Impulse Responses – Technology Shock**

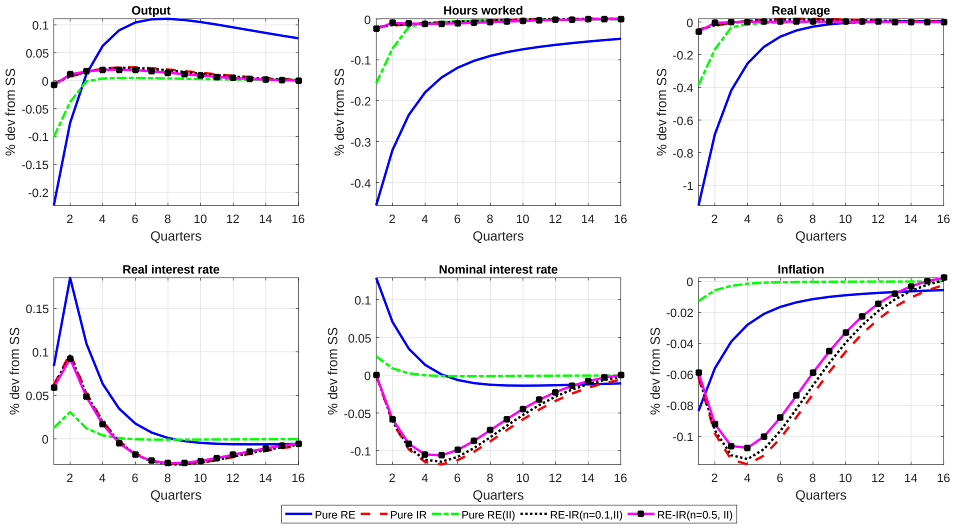


Figure 9: **Estimated Impulse Responses – Monetary Policy Shock**

should lead to a better fit of the data without relying on other model internal inertia mechanisms.

The IRFs also attempt to address the difficulty of generating reasonable endogenous persistence in DSGE frameworks and replicating the observed business cycle stylized facts. As already seen in Table 5, our baseline RE model with II learning statistically dominates all other modelling assumptions. Relaxing 50% pure rationality in the baseline model with the general heuristic learning rule also performs well. Model fit can be much improved without resorting to building a large number of frictions and shocks, offering a parsimonious approach while relaxing the extreme RE and PI. Of particular interest for the evaluation of using internal propagation mechanisms, relaxing full rationality leads to a reduction in the estimated degree of price stickiness ξ. In addition, relaxing the RE and perfect information restrictions generally leads to a reduction in the estimated persistence of the shock processes (e.g. ρ_A or ρ_π in particular).

We also find that the lagged interest rate is highly significant in the estimated policy rule, but the estimated inertia is much reduced when IR and II are introduced, suggesting a reduction in the persistence needed in the rule. The monetary policy volatility matters much less for explaining the data variation aspect of the fit when the model is no longer pure RE. Overall taking the results reported in Sections 8, 9 and 10, we can capture business cycle movements without having to assume either highly autocorrelated shocks, high policy rule persistence and/or the presence of endogenous inertia in the model due to, for example, habit formation in consumption and lengthy price-setting contracts. This contrasts with standard DSGE models in a RE-PI environment.

11 Conclusions

This paper studies an NK behavioural model for which boundedly rational beliefs of internally rational (IR) economic agents are about payoff-relevant macroeconomic variables that are exogenous to their decision rules. IR agents do not know they are identical, as opposed to Euler-learning (EL) where agents are (implicitly) assumed to know the symmetric nature of the equilibrium. We compare pure forms of RE, IR and EL models before proceeding to construct a Brock-Hommes composite model with reinforcement learning.

We examine the policy space of feedback parameters in a Taylor-type rule with interest rate persistence. We find that the pure IR model has a smaller policy space than pure EL which in turn is smaller than pure RE, making it more prone to local instability and the possibility of chaos. The differences in the sizes of the policy spaces that result in a saddle-path stable equilibrium are significant. Furthermore a clear ranking of the sizes of these spaces emerges with $RE \supset EL \supset IR$. This means that unless the policy rule is designed for the IR model, uncertainty as to which model of expectations is correct can lead to a rule that is unstable or indeterminate.

In a Bayesian estimation of the RE-IR composite model with exogenous proportions of RE and IR agents, informational assumptions are central to the paper. In comparisons of different composites including the pure RE and IR cases, we impose what we term *informational consistency* where RE and IR agents in the model share the same imperfect information as the econometrician estimating the model. We contrast this with the standard assumption that RE agents have perfect information of the current state variables. We find in a likelihood race that the RE model with imperfect information (II) outperforms the IR model which in turn outperforms RE with perfect information (PI). When we examine the behavioural composite model with a general heuristic forecasting rule and the RE agents having only II, the behavioural model cannot be statistically distinguished from RE with II. Second moment comparisons with the data are mixed, but the RE-IR composite with II captures more dimensions than RE with II with the latter projecting more drawn-out impulse response trajectories. These results suggest that persistence can be injected into the NK model to improve data fit in two contrasting ways: bounded-rationality with learning, and retaining RE, but with imperfect information Kalman-filtering learning.

Our results for a very simple NK model suggest a new agenda for constructing empirical medium-sized NK models. Future work will embed the RE-IR composite model into a richer

NK model along the lines of Smets and Wouters (2007), use non-linear estimation methods to identify a number of parameters involving reinforcement learning that are not identified using linear Bayesian estimation and examine optimal monetary policy. Future work on the policy aspect will follow Hall and Mitchell (2007), Geweke and Amisano (2012) and Deak *et al.* (2017) and estimate an optimal pool of RE and RE-IR composites to design a robust rule across such model variants.

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ONLINE APPENDICES (Not for Publication)

A Summary of Composite RE-IR Model

In stationarized form the model for exogenous proportions $n_{h,t}$ and $n_{f,t}$ we have:

RE Households:

$$\begin{aligned}
U_t^{RE} &= U(C_t^{RE}, H_t^{RE}) = \log C_t^{RE} - \frac{(H_t^{RE})^{1+\phi}}{1+\phi} \\
U_{C,t}^{RE} &= \mathbb{E}_t[\beta_{g,t+1} U_{C,t+1}^{RE} R_{t+1}] \\
\beta_{g,t} &= \beta/(1+g_t) \\
g_t &= (1+g)\exp(\epsilon_{Atrend}) - 1 \\
R_t &= \frac{R_{n,t-1}}{\Pi_t} \\
U_{C,t}^{RE} &= \frac{1}{C_t^{RE}} \\
U_{H,t}^{RE} &= -(H_t^{RE})^{\phi} \\
-\frac{U_{H,t}^{RE}}{U_{C,t}^{RE}} &= W_t
\end{aligned}$$

IR Households:

$$\begin{aligned}
U_t^{IR} &= U(C_t^{IR}, H_t^{IR}) = \log C_t^{IR} - \frac{(H_t^{IR})^{1+\phi}}{1+\phi} \\
\frac{C_t^{IR}}{(1-\mathbb{E}_t\beta_{g,t+1})} &= \frac{1}{(C_t^{IR})^{\frac{1}{\phi}}}\left(W_t^{1+\frac{1}{\phi}} + \frac{\left(\left(\frac{\mathbb{E}_t^* R_{n,t+1}}{R_{n,t}}\right)\mathbb{E}_t^* W_{t+1}\right)^{1+\frac{1}{\phi}}}{(\mathbb{E}_t\beta_{g,t+1})^{\frac{1}{\phi}}(\mathbb{E}_t^* R_{t+1}^{ex})^{1+\frac{1}{\phi}} - 1}\right) \\
&+ \Gamma_t - G_t + \frac{\left(\frac{\mathbb{E}_t^* R_{n,t+1}}{R_{n,t}}\right)\mathbb{E}_t^*(\Gamma_{t+1} - G_{t+1})}{\mathbb{E}_t^* R_{t+1}^{ex} - 1} \\
&\equiv \frac{1}{(C_t^{IR})^{\frac{1}{\phi}}}\left(W_t^{1+\frac{1}{\phi}} + \left(\frac{\mathbb{E}_t^* R_{n,t+1}}{R_{n,t}}\right)^{1+\frac{1}{\phi}}\Omega_{1,t}\right) \\
&+ \Gamma_t - G_t + \left(\frac{\mathbb{E}_t^* R_{n,t+1}}{R_{n,t}}\right)\Omega_{2,t}
\end{aligned}$$

where

$$\begin{aligned}
\Omega_{1,t} &= \frac{(\mathbb{E}_t^* W_{t+1})^{1+\frac{1}{\phi}}}{(\mathbb{E}_t\beta_{g,t+1})^{\frac{1}{\phi}}(\mathbb{E}_t^* R_{t+1}^{ex})^{1+\frac{1}{\phi}} - 1} \\
\Omega_{2,t} &= \frac{\mathbb{E}_t^*(\Gamma_{t+1} - G_{t+1})}{\mathbb{E}_t^* R_{t+1}^{ex} - 1} \\
\mathbb{E}_t^* R_{t+1}^{ex} &= \frac{\mathbb{E}_t^* R_{n,t+1}}{\mathbb{E}_{h,t}^* \Pi_{t+1}} \\
U_{C,t}^{IR} &= \frac{1}{C_t^{IR}} \\
U_{H,t}^{IR} &= -(H_t^{IR})^{\phi} \\
-\frac{U_{H,t}^{IR}}{U_{C,t}^{IR}} &= W_t
\end{aligned}$$

Wholesale Firms:

$$\begin{aligned}
Y_t^W &= F(A_t, H_t) = A_t H_t^{\alpha} = A_t(n_{h,t}H_t^{RE} + (1-n_{h,t})H_t^{IR})^{\alpha} \\
Y_t &= \frac{Y_t^W}{\Delta_t^{\alpha}} \\
\frac{P_t^W}{P_t}F_{H,t} &= \frac{P_t^W}{P_t}\frac{\alpha Y_t^W}{H_t} = W_t \\
1 &= \xi\Pi_t^{\zeta-1} + (1-\xi)\left(n_{f,t}\left(\frac{J_t^{RE}}{JJ_t^{RE}}\right)^{1-\zeta} + (1-n_{f,t})\left(\frac{J_t^{IR}}{JJ_t^{IR}}\right)^{1-\zeta}\right) \\
\Delta_t &= \xi\Pi_t^{\frac{\zeta}{\alpha}}\Delta_{t-1} + (1-\xi)\left(n_{f,t}\left(\frac{J_t^{RE}}{JJ_t^{RE}}\right)^{-\frac{\zeta}{\alpha}} + (1-n_{f,t})\left(\frac{J_t^{IR}}{JJ_t^{IR}}\right)^{-\frac{\zeta}{\alpha}}\right) \\
MC_t &= \frac{P_t^W}{P_t} = \frac{W_t}{F_{H,t}} \\
\Gamma_t &= Y_t - \alpha MC_t Y_t^W
\end{aligned}$$

RE Retail Firms:

$$\begin{aligned}
JJ_t^{RE} - \xi E_t[\Pi_{t+1}^{\zeta-1}JJ_{t+1}^{RE}\beta_{g,t+1}] &= Y_t\left(n_{h,t}U_{C,t}^{RE} + (1-n_{h,t})U_{C,t}^{IR}\right) \\
J_t^{RE} - \xi E_t[\Pi_{t+1}^{\zeta}J_{t+1}^{RE}\beta_{g,t+1}] &= \left(\frac{1}{1-\frac{1}{\zeta}}\right)Y_t MC_t MS_t\left(n_{h,t}U_{C,t}^{RE} + (1-n_{h,t})U_{C,t}^{IR}\right) \\
\left(\frac{P_t^0}{P_t}\right)^{RE} &= \frac{J_t^{RE}}{JJ_t^{RE}}
\end{aligned}$$

IR Retail Firms:

$$\begin{aligned}
J_t^{IR} &= \left(\frac{1}{1-\frac{1}{\zeta}}\right)(Y_t MC_t MS_t + \Omega_{3,t}) \\
JJ_t^{IR} &= Y_t + \Omega_{4,t} \\
\left(\frac{P_t^0}{P_t}\right)^{IR} &= \frac{J_t^{IR}}{JJ_t^{IR}}
\end{aligned}$$

where

$$\begin{aligned}
\Omega_{3,t} &= \frac{\xi(\mathbb{E}_{f,t}^*\Pi_{t+1})^{\zeta}\mathbb{E}_t^*Y_{t+1}\mathbb{E}_t^*MC_{t+1}\mathbb{E}_t^*MS_{t+1}}{\mathbb{E}_{f,t}^*R_{t+1} - \xi(\Pi_{t+1})^{\zeta}} \\
\Omega_{4,t} &= \frac{\xi(\mathbb{E}_{f,t}^*\Pi_{t+1})^{\zeta-1}\mathbb{E}_t^*Y_{t+1}}{\mathbb{E}_{f,t}^*R_{t+1} - \xi(\mathbb{E}_{f,t}^*\Pi_{t+1})^{\zeta-1}}
\end{aligned}$$

where

$$\mathbb{E}_{f,t}^*R_{t+1} = \mathbb{E}_{f,t}^*\left[\frac{R_{n,t}}{\Pi_{t+1}}\right] = \frac{R_{n,t}}{\mathbb{E}_{f,t}^*\Pi_{t+1}}$$

One-Period Ahead Adaptive Expectations:

$$\mathbb{E}_t^*[\beta_{g,t+1}] = \mathbb{E}_{t-1}^*[\beta_g, t] + \lambda_{1,\beta_g}\left(\beta_{g,t-1} - \mathbb{E}_{t-1}^*[\beta_{g,t}]\right) + \lambda_{2,\beta_g}\left(\beta_{g,t-1} - \beta_{g,t-2}\right)\ ;\ \lambda_{i,\beta_g} \in [0,1]$$

$$
\begin{aligned}
\mathbb{E}_t^*[G_{t+1}] &= \mathbb{E}_{t-1}^*[G_t] + \lambda_{1,G}\left(G_t - \mathbb{E}_{t-1}^*[G_t]\right) + \lambda_{2,G}\left(G_t - G_{t-1}\right) \; ; \; \lambda_{i,G} \in [0,1] \\
\mathbb{E}_t^*[W_{t+1}] &= \mathbb{E}_{t-1}^*[W_t] + \lambda_W\left(W_t - \mathbb{E}_{t-1}^*[W_t]\right) + \lambda_{2,W}\left(W_t - W_{t-1}\right) \; ; \; \lambda_{i,W} \in [0,1] \\
\mathbb{E}_t^*[\Gamma_{t+1}] &= \mathbb{E}_{t-1}^*[\Gamma_t] + \lambda_{1,\Gamma}\left(\Gamma_t - \mathbb{E}_{t-1}^*[\Gamma_t]\right) + \lambda_{2,\Gamma}\left(\Gamma_t - \Gamma_{t-1}\right) \; ; \; \lambda_{i,\Gamma} \in [0,1] \\
\mathbb{E}_t^*[R_{n,t+1}] &= \mathbb{E}_{t-1}^*[R_{n,t}] + \lambda_{1,R_n}\left(R_{n,t} - \mathbb{E}_{t-1}^*[R_{n,t}]\right) + \lambda_{2,R_n}\left(R_{n,t} - R_{n,t-1}\right) \; ; \; \lambda_{i,R_n} \in [0,1] \text{ (households)} \\
\mathbb{E}_{h,t}^*[\Pi_{t+1}] &= \mathbb{E}_{t-1}^*[\Pi_t] + \lambda_{1h,\Pi}\left(\Pi_{t-1} - \mathbb{E}_{t-1}^*[\Pi_t]\right) + \lambda_{2h,\Pi}\left(\Pi_{t-1} - \Pi_{t-2}\right) \; ; \; \lambda_{ih,\Pi} \in [0,1] \text{ (households)} \\
\mathbb{E}_{f,t}^*[\Pi_{t+1}] &= \mathbb{E}_{t-1}^*[\Pi_t] + \lambda_{1f,\Pi}\left(\Pi_{t-1} - \mathbb{E}_{t-1}^*[\Pi_t]\right) + \lambda_{2h,\Pi}\left(\Pi_{t-1} - \Pi_{t-2}\right) \; ; \; \lambda_{if,\Pi} \in [0,1] \text{ (firms)} \\
\mathbb{E}_t^*[Y_{t+1}] &= \mathbb{E}_{t-1}^*[Y_t] + \lambda_{1,Y}\left(Y_{t-1} - \mathbb{E}_{t-1}^*[Y_t]\right) + \lambda_{2,Y}\left(Y_{t-1} - Y_{t-2}\right) \; ; \; \lambda_{i,Y} \in [0,1] \\
\mathbb{E}_t^*[\tilde{MC}_{t+1}] &= \mathbb{E}_{t-1}^*[\tilde{MC}_t] + \lambda_{1,MC}\left(\tilde{MC}_t - \mathbb{E}_{t-1}^*[\tilde{MC}_t]\right) + \lambda_{2,MC}\left(\tilde{MC}_t - \tilde{MC}_{t-1}\right) \; ; \; \lambda_{i,MC} \in [0,1]
\end{aligned}
$$

where $\tilde{MC}_t \equiv MC_t MS_t$. Note that we have used the first order approximation $\log \frac{X_t}{X} \approx \frac{X_t - X}{X}$.

Wealth Distribution:
First define bond holdings of IR households by

$$
B_t^{IR} = R_t B_{t-1}^{IR} + W_t H_t^{IR} + \Gamma_t - C_t^{IR} - T_t - \frac{\varpi}{2}(B_{t-1}^{IR} - B)^2
$$

having introduced a portfolio cost adjustment with a small ϖ. Then replace C_t^{IR} and Euler equation above with

$$
\begin{aligned}
\frac{C_t^{IR} - B_t^{IR}}{(1 - \mathbb{E}_t^*\beta_{g,t+1})} &= \frac{1}{(C_t^{IR})^{\frac{1}{\phi}}}\left(W_t^{1+\frac{1}{\phi}} + \frac{\left(\left(\frac{\mathbb{E}_t^* R_{n,t+1}}{R_{n,t}}\right)\mathbb{E}_t^* W_{t+1}\right)^{1+\frac{1}{\phi}}}{(\mathbb{E}_t^*\beta_{g,t+1})^{\frac{1}{\phi}}(\mathbb{E}_t^* R_{t+1}^{ex})^{1+\frac{1}{\phi}} - 1}\right) + \Gamma_t - G_t \\
&+ \frac{\left(\frac{\mathbb{E}_t^* R_{n,t+1}}{R_{n,t}}\right)\mathbb{E}_t^*(\Gamma_{t+1} - G_{t+1})}{\mathbb{E}_t^* R_{t+1}^{ex} - 1} \\
&\equiv \frac{1}{(C_t^{IR})^{\frac{1}{\phi}}}\left(W_t^{1+\frac{1}{\phi}} + \left(\frac{\mathbb{E}_t^* R_{n,t+1}}{R_{n,t}}\right)^{1+\frac{1}{\phi}}\Omega_{1,t}\right) \\
&+ \Gamma_t - G_t + \left(\frac{\mathbb{E}_t^* R_{n,t+1}}{R_{n,t}}\right)\Omega_{2,t} \\
U_{C,t}^{RE} &= \mathbb{E}_t\left[\beta_{g,t+1} U_{C,t+1}^{RE}(R_{t+1} - \varpi(B_t^{RE} - B))\right]
\end{aligned}
$$

where zero net wealth implies $n_{h,t} B_t^{RE} = -(1 - n_{h,t}) B_t^{IR}$.

Closure of Model:

$$
\begin{aligned}
Y_t &= n_{h,t} C_t^{RE} + (1 - n_{h,t}) C_t^{IR} + G_t \\
G_t &= T_t \\
\log\left(\frac{R_{n,t}}{R_n}\right) &= \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + (1 - \rho_r)\Big(\theta_\pi \log\left(\frac{\Pi_t}{\Pi_{targ,t}}\right) \\
&+ \theta_y \log\left(\frac{Y_t}{Y}\right) + \theta_{dy} \log\left(\frac{Y_t}{Y_{t-1}}\right)\Big) + \epsilon_{MP,t} \textbf{ (Perfect Information)} \\
\log\left(\frac{R_{n,t}}{R_n}\right) &= \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + (1 - \rho_r)\Big(\theta_\pi \log\left(\frac{\mathbb{E}_t[\Pi_t]}{\Pi_{targ,t}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \theta_y \log\left(\frac{\mathbb{E}_t[Y_t]}{Y}\right) + \theta_{dy} \log\left(\frac{\mathbb{E}_t[Y_t]}{Y_{t-1}}\right)\Bigg) + \epsilon_{MP,t} \ \textbf{(Imperfect Information)}
\end{aligned}
$$

$$
\begin{aligned}
\log A_t - \log A &= \rho_A(\log A_{t-1} - \log A) + \epsilon_{A,t} \\
\log G_t - \log G &= \rho_G(\log G_{t-1} - \log G) + \epsilon_{G,t} \\
\log MS_t - \log MS &= \rho_{MS}(\log MS_{t-1} - \log MS) + \epsilon_{MS,t} \\
\log \Pi_{targ,t} - \log \Pi &= \rho_\pi(\log \Pi_{targ,t-1} - \log \Pi) + \epsilon_{\pi,t}
\end{aligned}
$$

Endogenous Proportions of RE and IR Agents:
The payoff for households and firms is expressed on terms of a discounted sum of past weighted forecast errors, $\Phi_{h,t}$ say, starting at $t = 0$ for with rational and non-rational households respectively:

$$
\begin{aligned}
\Phi_{h,t}^{RE} &= \mu_h^{RE} \Phi_{h,t-1}^{RE} - (1 - \mu_h^{RE})\Big(w_{\beta_g}(\beta_{g,t} - E_{h,t-1}\beta_{g,t})/\beta_g)^2 + w_G((G_t - E_{h,t-1}G_t)/G)^2 \\
&+ w_W((W_t - E_{h,t-1}W_t)/W)^2 + w_{h,\Pi}((\Pi_t - E_{h,t-1}\Pi)/\Pi)^2 \\
&+ w_\Gamma((\Gamma_t - E_{h,t-1}\Gamma_t)/\Gamma)^2 + w_R((R_{n,t} - E_{t-1}R_{n,t})/R_n)^2 + C_h\Big) \\
\Phi_{h,t}^{IR} &= \mu_h^{IR} \Phi_{h,t-1}^{IR} - (1 - \mu_h^{IR})\Big(w_{\beta_g}(\beta_{g,t} - E_{h,t-1}^{*}\beta_{g,t})/\beta_g)^2 + w_G((G_t - E_{h,t-1}^{*}G_t)/G)^2 \\
&+ w_W((W_t - E_{h,t-1}^{*}W_t)/W)^2 + w_{h,\Pi}((\Pi_t - E_{h,t-1}^{*}\Pi)/\Pi)^2 + w_\Gamma((\Gamma_t - E_{h,t-1}^{*}\Gamma_t)/\Gamma)^2 \\
&+ w_R((R_{n,t} - E_{t-1}R_{n,t})/R_n)^2)\Big)
\end{aligned}
$$

The parameter C_h is a fixed cost of being rational for households. For firms this becomes

$$
\begin{aligned}
\Phi_{f,t}^{RE} &= \mu_f^{RE} \Phi_{f,t-1}^{RE} - (1 - \mu_f^{RE})\Big(w_Y((Y_t - E_{f,t-1}Y_t)/Y)^2 + w_{f,\Pi}((\Pi_t - E_{f,t-1}\Pi)/\Pi)^2 \\
&+ w_{MC}((\tilde{MC}_t - E_{f,t-1}\tilde{MC}_t)/MC)^2 + C_f\Big) \\
\Phi_{f,t}^{IR} &= \mu_f^{IR} \Phi_{f,t-1}^{IR} - (1 - \mu_f^{IR})\Big(w_Y((Y_t - E_{f,t-1}^{*}Y_t)/Y)^2 + w_{f,\Pi}((\Pi_t - E_{f,t-1}^{*}\Pi)/\Pi)^2 \\
&+ w_{MC}((\tilde{MC}_t - E_{f,t-1}^{*}\tilde{MC}_t)/MC)^2\Big)
\end{aligned}
$$

where the parameter C_f is a fixed cost of being rational for firms and we allow for the possibility that $C_h \neq C_f$.

Note that for variable X_t, $E_{t-1}X_t$ above denotes *rational* expectations so that putting $(EX)_{t-1} \equiv E_{t-1}X_t$ we have the Dynare set-up

$$(EX)_t = E_t X_{t+1} \text{ coded as } EiX = X(+1) \text{ for } i = h, f \text{ where appropriate}$$

Then the proportions of rational households and firms is given by

$$
\begin{aligned}
n_{h,t} &= \frac{\exp(\gamma\Phi_{h,t}^{RE})}{\exp(\gamma\Phi_{h,t})^{RE} + \exp(\gamma\Phi_{h,t}^{IR})} = \frac{\exp(\gamma(\Phi_{h,t}^{RE} - \Phi_{h,t}^{IR}))}{\exp(\gamma(\Phi_{h,t}^{RE} - \Phi_{h,t}^{IR})) + 1} \\
n_{f,t} &= \frac{\exp(\gamma\Phi_{f,t}^{RE})}{\exp(\gamma\Phi_{f,t})^{RE} + \exp(\gamma\Phi_{f,t}^{IR})} = \frac{\exp(\gamma(\Phi_{f,t}^{RE} - \Phi_{f,t}^{IR}))}{\exp(\gamma(\Phi_{f,t}^{RE} - \Phi_{f,t}^{IR})) + 1}
\end{aligned}
$$

Thus the proportion of rational agents in the steady state is given by

$$
\begin{aligned}
n_h &= \frac{\exp(-\gamma C_h)}{\exp(-\gamma C_h)+1} \\
n_f &= \frac{\exp(-\gamma C_f)}{\exp(-\gamma C_f)+1}
\end{aligned}
$$

which is pinned down by the cost parameters (C_h, C_f) (which can be positive or negative).

Welfare and Consumption Equivalence:

$$
\begin{aligned}
U_t &= \log((n_{h,t}C_t^{RE} + (1-n_{h,t})C_t^{IR}) - \frac{(n_{h,t}H_t^{RE} + (1-n_{h,t}H_t)^{IR})^{1+\phi}}{1+\phi} \\
wel_t &= (1-\beta_{g,t})U_t + \mathbb{E}_t[\beta_{g,t+1}wel_{t+1}] \\
wel_t^{RE} &= (1-\beta_{g,t})U_t^{RE} + \mathbb{E}_t[\beta_{g,t+1}wel_{t+1}^{RE}] \\
wel_t^{IR} &= (1-\beta_{g,t})U_t^{IR} + \mathbb{E}_t[\beta_{g,t+1}wel_{t+1}^{IR}] \\
CE_t &= \log(1.01C_t) - \log(C_t)
\end{aligned}
$$

B Balanced Growth Steady State

In recursive form the zero-growth zero-inflation ($\Pi = 1$) steady state of can be written

$$
\begin{aligned}
R &= \frac{1}{\beta} \\
\Lambda &= \beta \\
MC = \frac{P^W}{P} &= 1 - \frac{1}{\zeta} \\
\frac{C}{Y} &= 1 - g_y \\
H &= \frac{\alpha\Delta^\alpha MC}{\kappa(1-g_y)} \\
Y^W &= (AH)^\alpha \\
Y &= \frac{Y^W}{\Delta^\alpha} \\
W &= \alpha\frac{P^W}{P}\frac{Y^W}{H} \\
J &= \frac{YMCU_C}{(1-\frac{1}{\zeta})(1-\xi\beta\Pi^\zeta)} \\
JJ &= \frac{YU_C}{(1-\xi\beta\Pi^{\zeta-1})} \\
\text{Hence with } \Pi = 1, J &= JJ \\
\Delta &= 1 \\
\Gamma &= Y - \alpha MCY^W
\end{aligned}
$$

For a particular steady state inflation rate $\Pi > 1$ the NK features of the steady state become

$$\begin{aligned}
\frac{J}{JJ} &= \left(\frac{1-\xi\Pi^{\zeta-1}}{1-\xi}\right)^{\frac{1}{1-\zeta}} \\
MC = \frac{P^W}{P} &= \left(1-\frac{1}{\zeta}\right)\frac{J(1-\beta\xi\Pi^{\zeta})}{JJ(1-\beta\xi\Pi^{\zeta-1})} \\
\Delta &= \frac{(1-\xi)^{\alpha}\left(\frac{J}{JJ}\right)^{-\zeta}}{1-\xi\Pi^{\zeta}}
\end{aligned}$$

Then $P^W Y^W/PY = MC\Delta$.

We can now easily set up the model with a balanced-exogenous-growth steady state. Now the process for A_t is replaced with

$$\begin{aligned}
A_t &= \bar{A}_t A_t^c \\
\bar{A}_t &= (1+g)\bar{A}_{t-1}\exp(\epsilon_{A,t}) \\
\log A_t^c - \log A^c &= \rho_A(\log A_{t-1}^c - \log A^c) + \epsilon_{A,t}
\end{aligned}$$

where A_t is a labour-augmenting technical progress parameter which we decompose into a cyclical component, A_t^c, modelled as a temporary AR(1) process and a stochastic trend, whose log is a random walk with drift, $\bar{A}_t$. Thus the balanced growth deterministic steady state path (bgp) is driven by labour-augmenting technical change growing at a net rate g. If we put $g = \epsilon_{trend,t} = 0$ and $\bar{A}_t = 1$, we arrive at our previous formulation with $A_t^c = A_t$.

Now stationarize variables by defining cyclical and stationary components:

$$\begin{aligned}
(Y_t^W)^c &\equiv \frac{Y_t^W}{\bar{A}_t} = A_t^c H_t^{\alpha} \\
C_t^c &\equiv \frac{C_t}{\bar{A}_t} \\
W_t^c &\equiv \frac{W_t}{\bar{A}_t} \\
U_t^c &\equiv \log C_t^c - \kappa\frac{H_t^{1+\phi}}{1+\phi} \\
U_{C,t}^c &\equiv \frac{1}{C_t^c} \\
\Lambda_{t,t+1} &= \beta\frac{U_{C,t+1}}{U_{C,t}} = \beta_{g,t+1}\frac{U_{C,t+1}^c}{U_{C,t}^c}
\end{aligned}$$

for all non-stationary variables where

$$\begin{aligned}
g_t &\equiv \frac{(\bar{A}_t - \bar{A}_{t-1})}{\bar{A}_t} = (1+g)\exp(\epsilon_{A,t}) - 1 \\
\beta_{g,t} &\equiv \beta(1+g_t)
\end{aligned}$$

is the stochastic steady state growth rate and the stationarized Euler equation and the Calvo pricing become

$$E_t\left[\Lambda_{t,t+1}R_{t+1}\right]=E_t\left[\beta_{g,t+1}\frac{U^c_{C,t+1}}{U^c_{C,t}}R_{t+1}\right]=1$$

and

$$\begin{aligned}\widehat{JJ}^c_t-\xi E_t[\Pi^{\zeta-1}_{t+1}\widehat{JJ}^c_{t+1}\Lambda_{t,t+1}] &= Y^c_t\\ \widehat{J}^c_t-\xi E_t[\Pi^{\zeta}_{t+1}\widehat{J}^c_{t+1}\Lambda_{t,t+1}] &= Y^c_t MC_t MS_t\end{aligned}$$

or equivalently

$$\begin{aligned}\widehat{JJ}^c_t-\xi E_t[\Pi^{\zeta-1}_{t+1}\widehat{JJ}^c_{t+1}\beta_{g,t+1}] &= Y^c_t U^c_t\\ \widehat{J}^c_t-\xi E_t[\Pi^{\zeta}_{t+1}\widehat{J}^c_{t+1}\beta_{g,t+1}] &= Y^c_t U^c_t MC_t MS_t\end{aligned}$$

The steady state for the rest of the system is the same as the zero-growth one except for the following relationships:

$$R = \frac{1}{\beta_g}=\frac{R_n}{\Pi}$$

where R and R_n are the real and nominal steady state interest rates and Π is inflation.

C Proof of Lemma

In the first order conditions for Calvo contracts and expressions for value functions we are confronted with expected discounted sums of the general form

$$\Omega_t=\mathbb{E}_t\left[\sum_{k=0}^{\infty}\beta^k X_{t,t+k}Y_{t+k}\right]$$

where $X_{t,t+k}$ has the property $X_{t,t+k}=X_{t,t+1}X_{t+1,t+k}$ and $X_{t,t}=1$ (for example an inflation, interest or discount rate over the interval $[t,t+k]$).

Lemma

Ω_t can be expressed as

$$\Omega_t=Y_t+\beta\mathbb{E}_t\left[X_{t,t+1}\Omega_{t+1}\right]$$

Proof

$$\begin{aligned}\Omega_t &= X_{t,t}Y_t+\mathbb{E}_t\left[\sum_{k=1}^{\infty}\beta^k X_{t,t+k}Y_{t+k}\right]\\ &= Y_t+\mathbb{E}_t\left[\sum_{k'=0}^{\infty}\beta^{k'+1}X_{t,t+k'+1}Y_{t+k'+1}\right]\\ &= Y_t+\beta\mathbb{E}_t\left[\sum_{k'=0}^{\infty}\beta^{k'}X_{t,t+1}X_{t+1,t+k'+1}Y_{t+k'+1}\right]\\ &= Y_t+\beta\mathbb{E}_t\left[X_{t,t+1}\Omega_{t+1}\right] \qquad \square\end{aligned}$$

C.1 Proof of Equation 29

In the next period, ξ of these firms will keep their old prices, and $(1-\xi)$ will change their prices to P^O_{t+1}. By the law of large numbers, we assume that the distribution of prices among those firms that do not change their prices is the same as the overall distribution in period t. It follows that we may write

$$\begin{aligned}
\Delta_{t+1} &= \xi \sum_{j_{no\ change}} \left(\frac{P_t(j)}{P_{t+1}}\right)^{-\zeta} + (1-\xi)\left(\frac{J_{t+1}}{JJ_{t+1}}\right)^{-\zeta} \\
&= \xi \left(\frac{P_t}{P_{t+1}}\right)^{-\zeta} \sum_{j_{no\ change}} \left(\frac{P_t(j)}{P_t}\right)^{-\zeta} + (1-\xi)\left(\frac{J_{t+1}}{JJ_{t+1}}\right)^{-\zeta} \\
&= \xi \left(\frac{P_t}{P_{t+1}}\right)^{-\zeta} \sum_{j} \left(\frac{P_t(j)}{P_t}\right)^{-\zeta} + (1-\xi)\left(\frac{J_{t+1}}{JJ_{t+1}}\right)^{-\zeta} \\
&= \xi \Pi^{\zeta}_{t+1}\Delta_t + (1-\xi)\left(\frac{J_{t+1}}{JJ_{t+1}}\right)^{-\zeta} \qquad \square
\end{aligned}$$

D Addition to Section 7

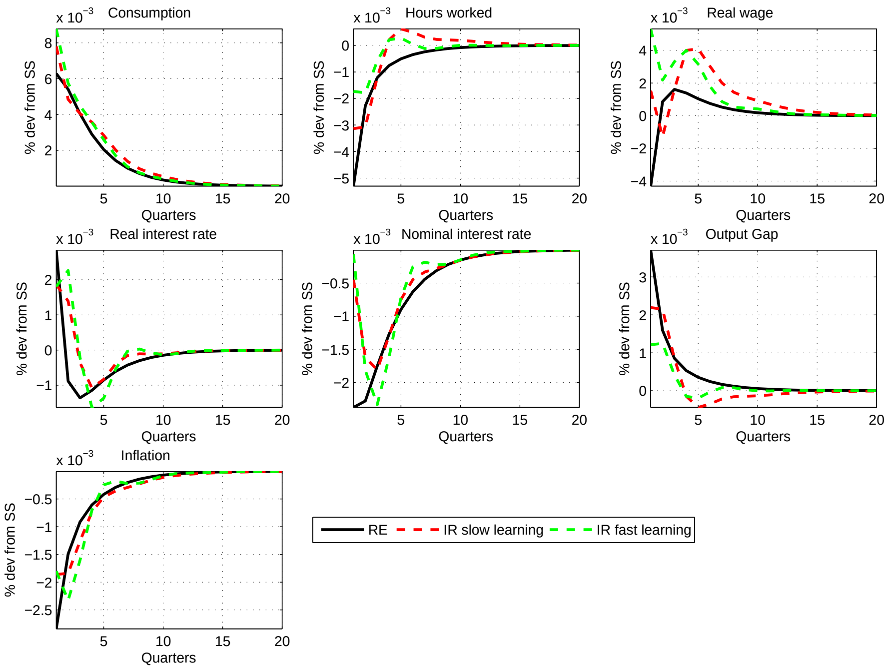

Figure 10: **RE versus RE-IR Composite Expectations with $n_h = n_f = 0.5$, $\lambda_x = 0.25, 1.0$; Taylor rule with $\rho_r = 0.7$, $\theta_\pi = 1.5$ and $\theta_y = 0.3$. Technology Shock**

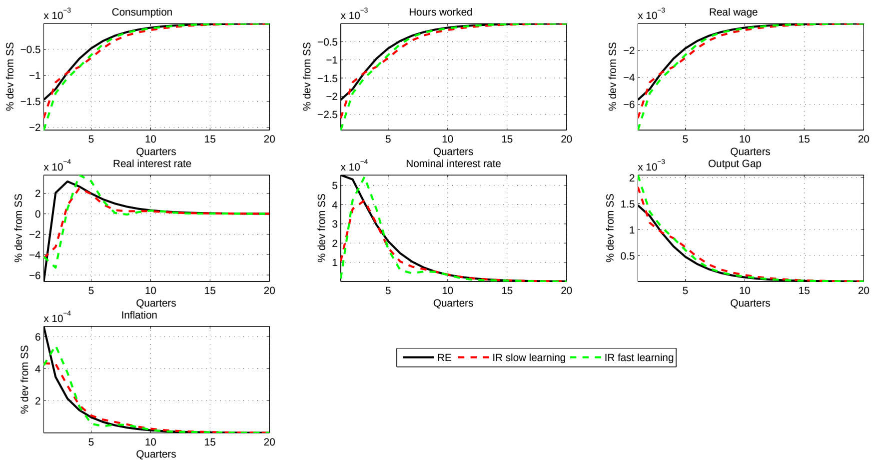

Figure 11: **RE versus RE-IR Composite Expectations with $n_h = n_f = 0.5$, $\lambda_x = 0.25, 1.0$; Taylor rule with $\rho_r = 0.7$, $\theta_\pi = 1.5$ and $\theta_y = 0.3$. Mark-up Shock**

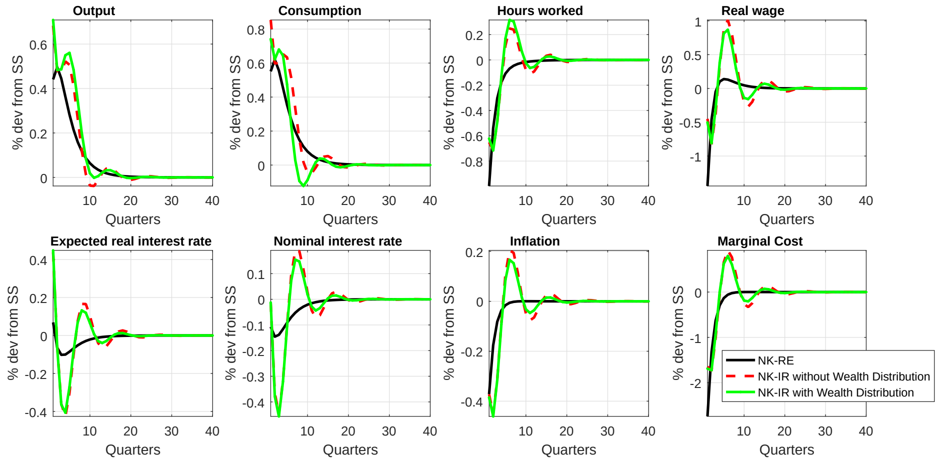

Figure 12: **Wealth Distribution and Impulse Responses – Technology Shock**

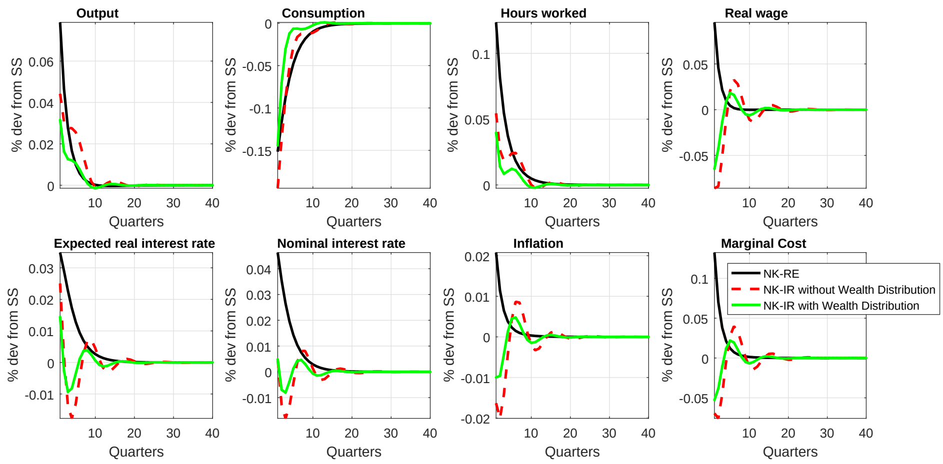

Figure 13: **Wealth Distribution and Impulse Responses – Government Spending Shock**

E Identification Strength at Priors and Posteriors

It is necessary to confront the question of parameter identifiability in our DSGE models before taking them to the data, as model or parameter identification is a prerequisite for the informativeness of different estimators, and their effectiveness when one uses the models to address policy questions. In this section we focus on detecting parameter identification difficulties that are inherent in the structure of the models. As mentioned we fix some parameters before estimation because of their non-identification in the model solution (at first order). The aim of this section is to scan the parameters we choose to estimate in terms of their identification in our models. Among the authors who have made the most recent contributions to addressing the identification issues in DSGE models are Iskrev (2008) and Iskrev (2010b), Canova and Sala (2009), and Komunjer and Ng (2011). We use Iskrev (2010b)'s computational toolbox to perform formal identification checks on the reduced form parameters and structural parameters. This approach is based on evaluating analytically the information matrix of the reduced-form model and checking for rank deficiency of gradient matrix (the Jacobian). Our checks are performed in terms of a local analysis that is based on the identification evaluation at the point values of the prior means in Table 2 and a 'global' prior exploration of point identification properties by taking a Monte Carlo samples from the prior space. The identifiability of each draw including the mean prior is established by studying the ranks of Jacobian of the model and given the set of observable variables and the sample size (the sample moments).

We take our models to the identification toolbox that computes the Jacobian numerically of the model (the solution) and the moments for rank evaluations prior to estimating them. To completely rule out a flat likelihood at the local point we also check collinearity between the effects of different parameters on the likelihood. If there exists an exact linear dependence between a pair and among all possible combinations their effects on the moments are not distinct which must indicate a flat likelihood and lack of identification. We find that the Jacobian matrix has full rank and that the models can be identified locally within the prior space. This includes our key parameters for behaviourial heterogeneity λ_1, n_h and n_f.

A further output of Iskrev (2010b)'s identification routine is the analysis of identification strength, i.e., focusing on weak identification, summarized in Figures 13 - 16 in Appendix B. The procedures are based on either the asymptotic or a moment information matrix. The first can be obtained given a sample of size T, whereas the second can be computed based on Monte Carlo simulations for samples of size T, from which sample moments of the observed variables are computed, forming a sample of N replicas of simulated moments. The corresponding information matrix is then obtained as $I_T(\theta|\mathbf{m_T}) = H_T \Sigma_{\mathbf{m_T}} H_T$, where $\Sigma_{\mathbf{m_T}}$ is the covariance matrix of simulated moments and H_T the derivatives of the vector collecting all the reduced-form coefficients. We now examine more carefully all our parameters at the means of the prior and posterior distributions and using the prior uncertainty. We focus this analysis on the two behaviourial models and report the sensitivity measure and collinearity results for all parameters evaluated at the prior mean, relative to the prior standard deviation and at the posterior mean obtained using the estimated models in the next section. Note that, in Appendix B, all their parameters are sensitive in affecting the likelihood through their effects on the moments of the observed variables. Table 7 highlights the effects of λ_1, n_h and n_f on the likelihood which are very strong (at least at the prior means). Although some similarities in terms of pair-wise collinearity are detected it is important to confirm that no linear dependance (non-identification) is found across the estimated parameters in these models.

	Sensitivity			Collinearity			
	Prior Mean		Posterior Mean	Prior Mean		Posterior Mean	
θ_i	$\Delta_i\theta_i$	$\Delta_i^{prior}\theta_i$	$\Delta_i\theta_i$	ϱ_i	θ_j	ϱ_i	θ_j
λ_1	25.0845	15.2038	19.5426	0.6532	θ_π	0.5649	ϵ_π
n_h	47.6276	27.7389	0.5148	0.9253	ϵ_{Atrend}	0.8685	θ_π
n_f	186.7911	37.1511	5.3974	0.8390	θ_y	0.9927	ξ

Table 7: **Identification at Priors and Posteriors (Parameters λ_1, n_h and n_f)**

We follow Iskrev (2010a)'s procedures and measure the identification strength, based on the information matrix $I_T(\theta)$, as sensitivity of the information derived from the likelihood to the parameters and collinearity between the effects of different parameters on the likelihood. The 'strength' of identification can be decomposed into a 'sensitivity' and 'correlation' component. The first referring to the case when weak identification arises when the moments do not change with θ_i and the second when collinearity dampens the effect of θ_i. The former is defined as

$$\Delta_i = \sqrt{\theta_i^2 \cdot I_T(\theta)_{(i,i)}}$$

which can also be normalised relative to the prior standard deviation for θ_i: $\sigma(\theta_i)$, weighting the information matrix using the prior uncertainty:

$$\Delta_i^{prior} = \sigma(\theta_i) \cdot \sqrt{I_T(\theta)_{(i,i)}}$$

It is possible to show the standard error of a parameter:

$$s.e.(\theta_i) = \frac{1}{\Delta_i}\frac{1}{\sqrt{1-\varrho_i^2}}$$

where ϱ_i denotes collinearity between the effects of different parameters so that lack of identification and a flat likelihood may be due to either $\Delta_i = 0$ or $\varrho_i = 1$.

F Solution of Linearized Models under Imperfect Information

We write a RE model in the general non-linear form:

$$\mathbb{E}_t[f(\mathsf{y}_t, \mathsf{y}_{t+1}, \mathsf{y}_{t-1}, \epsilon_t)] = 0 \tag{F.1}$$

with $\mathbb{E}_t$ now referring to expectations subject to an information set that may be imperfect. Either analytically, or numerically using the methods of Levine and Pearlman (2011), a log-linearized form state-space representation can be obtained as

$$\begin{bmatrix} z_{t+1} \\ \mathbb{E}_t x_{t+1} \end{bmatrix} = G \begin{bmatrix} z_t \\ x_t \end{bmatrix} + H \begin{bmatrix} \mathbb{E}_t z_t \\ \mathbb{E}_t x_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \epsilon_{t+1} \tag{F.2}$$

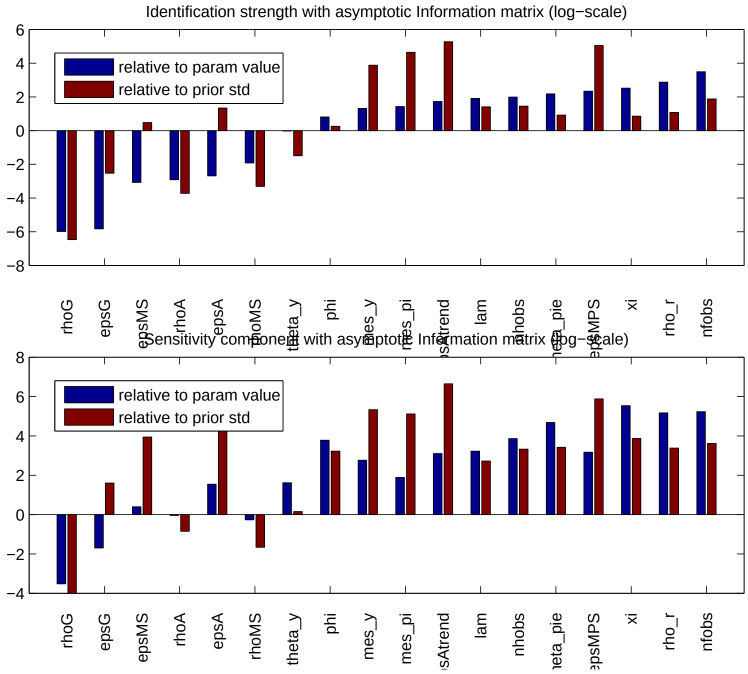

Figure 14: **Identification Strength at Prior Means in Model RE-IR with Estimated n_h, n_f**

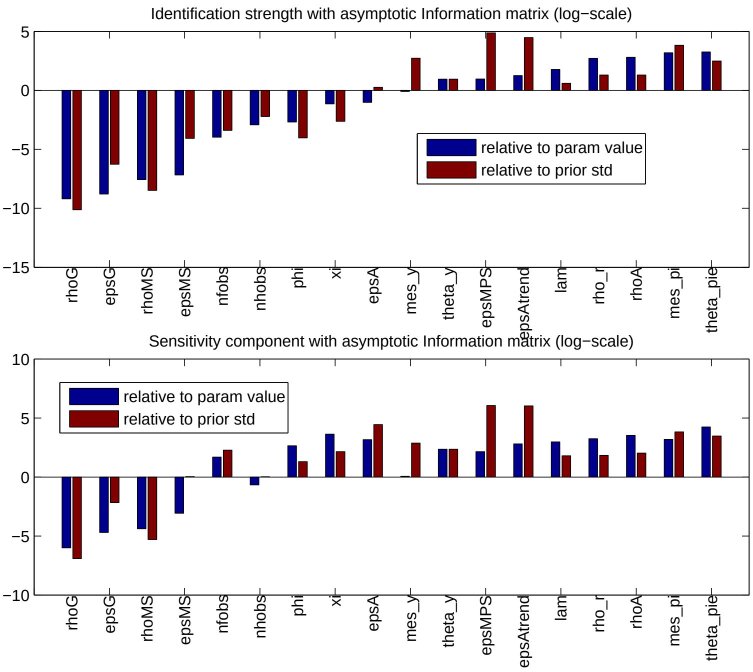

Figure 15: **Identification Strength at Posterior Means in Model RE-IR with Estimated n_h, n_f**

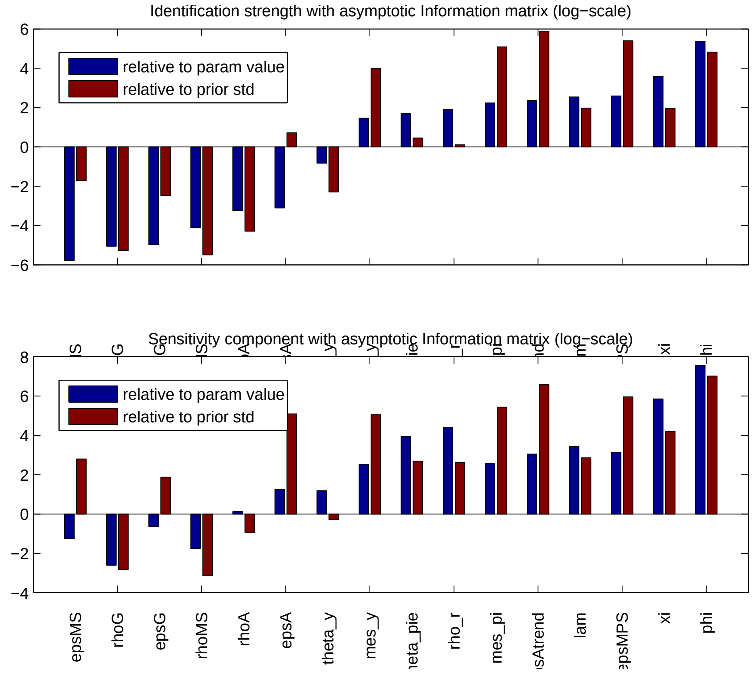

Figure 16: **Identification Strength at Prior Means in Model IR**

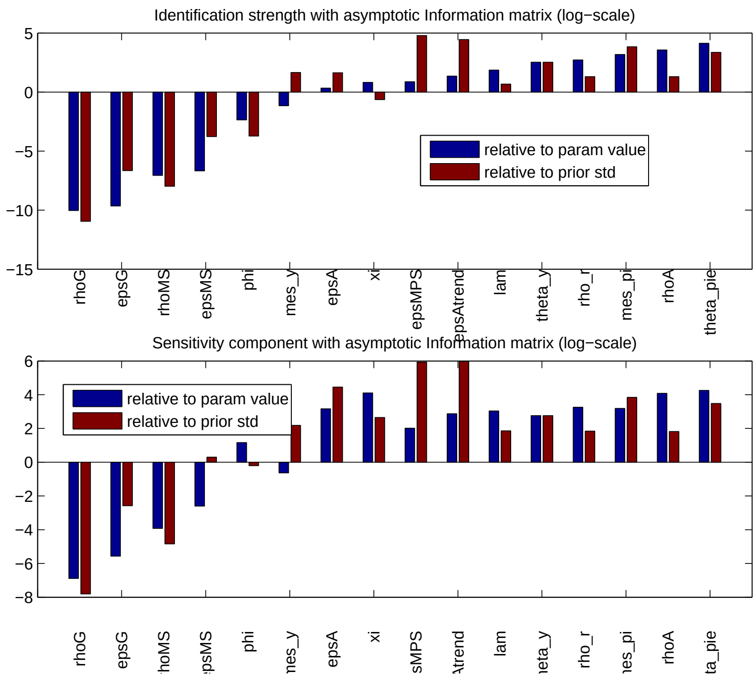

Figure 17: **Identification Strength at Posterior Means in Model IR**

where z_t, x_t are vectors of backward and forward-looking variables, respectively, and ϵ_t is a vector of shock variables. We define $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$, with H similarly defined. The reason for transforming the equations of the model from the linearized version of (F.1) is that the corresponding solution method of Sims (2002) does not extend easily to imperfect information. In addition we assume that agents all make the same observations at time t, which are given, in non-linear and subsequently linearized forms respectively, by

$$\begin{aligned} M_t^{obs} &= m(y_t) \\ m_t &= \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} L_1 & L_2 \end{bmatrix} \begin{bmatrix} \mathbb{E}_t z_t \\ \mathbb{E}_t x_t \end{bmatrix} \end{aligned} \tag{F.3}$$

Note that the expressions involving $\mathbb{E}_t z_t, \mathbb{E}_t x_t$ arise from rewriting the model in Blanchard-Kahn form (F.2). The presence of these terms is what distinguishes our results on invertibility from those of Baxter *et al.* (2011), and in addition we do not make the assumption that agents have full current information on all variables for which forward expectations are present in the model.

Thus the information set at time t for all agents is $\{m_s : \ s \leq t\}$. For ease of notation we assume that if any variables are observed with measurement error, then these variables are included in the state space, and the measurement errors are then part of the vector ϵ_t. Given the fact that expectations of forward-looking variables depend on the information set, it is hardly surprising that the absence of perfect information will impact on the path of the system. A full derivation of the solution for the general linear setup above is provided in Pearlman *et al.* (1986), but is outlined below.

F.1 Perfect Information Case

We first consider the solution for (F.2) and (F.3) under *perfect information*; in this case we assume that all stocks dated $t-1$ and other variables dated t in (F.2) are fully observed during the course of period t. These would include beginning-of-period capital stock k_{t-1}, beginning-of-period net worth n_{t-1}, all flows such as output, consumption, investment, all output and factor prices, inflation over the period and all end-of-period realizations of exogenous stochastic processes such as a_t, g_t etc.

For this perfect information case (where $\mathbb{E}_t z_t = z_t, \mathbb{E}_t x_t = x_t$) there is a saddle path satisfying:

$$x_t + N z_t = 0 \quad \text{where} \quad \begin{bmatrix} N & I \end{bmatrix} (G + H) = \Lambda^U \begin{bmatrix} N & I \end{bmatrix} \tag{F.4}$$

where Λ^U is a matrix with unstable eigenvalues. If the number of unstable eigenvalues of $(G+H)$ is the same as the dimension of x_t, then the system will be determinate.[16] We then ask whether observations by the econometrician of the form (F.3) will lead to invertibility.

From the saddle path relationship (F.4), it is clear that the reduced-form representation of

[16]Note that in general the dimension of x_t will not match the number of expectational variables in (F.1). The algorithm in Levine and Pearlman (2011) will eliminate linear dependency among expectational variables and will also convert the system $a_t = \rho a_{t-1} + \varepsilon_t, \ \ b_t = \mathbb{E}_t a_{t+1}$ into $a_t = \rho a_{t-1} + \varepsilon_t, \ \ b_t = \rho a_t$.

the model is now

$$z_t = (G_{11} + H_{11} - (G_{12} + H_{12})N)z_{t-1} + B\epsilon_t \qquad m_t = (M_1 + L_1 - (M_2 + L_2)N)z_t$$

Expressing m_t in terms of z_{t-1} and ϵ_t, from Fernandez-Villaverde *et al.* (2007) we deduce that a necessary and sufficient condition for invertibility (see (45)) is that $\tilde{D} = EB$ is invertible, where $E = M_1 + L_1 - (M_2 + L_2)N$.

F.2 Imperfect Information Case

We first briefly outline how the imperfect information setup is solved, and the provide the conditions for invertibility. Following Pearlman *et al.* (1986), we use the Kalman filter updating given by

$$\begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} = \begin{bmatrix} z_{t,t-1} \\ x_{t,t-1} \end{bmatrix} + J\left[m_t - \begin{bmatrix} M_1 & M_2 \end{bmatrix}\begin{bmatrix} z_{t,t-1} \\ x_{t,t-1} \end{bmatrix} - \begin{bmatrix} L_1 & L_2 \end{bmatrix}\begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix}\right]$$

where we denote $z_{t,t} \equiv \mathbb{E}_t[z_t]$ etc. The Kalman filter was developed in the context of backward-looking models, but extends as we see here to forward-looking models. The basic idea behind it is that the best estimate of the states $\{z_t, x_t\}$ based on current information is a weighted average of the best estimate using last period's information and the new information m_t. Thus the best estimator of the state vector at time $t-1$ is updated by multiple J of the error in the predicted value of the measurement as above, where J is given by

$$J = \begin{bmatrix} PD' \\ -NPD' \end{bmatrix}\Gamma^{-1}$$

and $D \equiv M_1 - M_2G_{22}^{-1}G_{21}$, $M \equiv [M_1\ M_2]$ is partitioned conformably with $\begin{bmatrix} z_t \\ x_t \end{bmatrix}$, $\Gamma \equiv EPD'$ where $E \equiv M_1 + L_1 - (M_2 + L_2)N$ and P satisfies the Riccati equation (F.2) below.

With only one imperfect information set, the same saddle path relationship (F.4) as for perfect information holds.[17] Then using the Kalman filter, the solution as derived by Pearlman *et al.* (1986) [18] is given by the following processes describing the pre-determined and non-predetermined variables $z_t = \tilde{z}_t + z_{t,t-1}$ and x_t, and a process describing the innovations $\tilde{z}_t \equiv z_t - z_{t,t-1}$:

$$\begin{aligned} \text{Predetermined :} \quad z_{t+1,t} &= Cz_{t,t-1} + CPD'(DPD')^{-1}D\tilde{z}_t \\ \text{Non-predetermined :} \quad x_t &= -Nz_{t,t-1} - G_{22}^{-1}G_{21}\tilde{z}_t - (N - G_{22}^{-1}G_{21})PD'(DPD')^{-1}D\tilde{z}_t \\ \text{Innovations :} \quad \tilde{z}_{t+1} &= A\tilde{z}_t - APD'(DPD')^{-1}D\tilde{z}_t + B\epsilon_{t+1} \end{aligned}$$

where

$$C \equiv G_{11} + H_{11} - (G_{12} + H_{12})N, \quad A \equiv G_{11} - G_{12}G_{22}^{-1}G_{21}, \quad D \equiv M_1 - M_2G_{22}^{-1}G_{21}$$

[17] But see footnote 15 and Lubik *et al.* (2017) for the case of more than one imperfect information set.

[18] A less general solution procedure for linear models with imperfect information is in Lungu *et al.* (2008) with an application to a small open economy model, which they also extend to a non-linear version.

and P is the solution of the Riccati equation given by

$$P = APA' - APD'(DPD')^{-1}DPA' + BB'$$

where we assume that the shocks are normalized such that their covariance matrix is given by the identity matrix.

The measurement m_t, as shown by Pearlman *et al.* (1986), can now be expressed as

$$m_t = Ez_{t,t-1} + EPD'(DPD')^{-1}D\tilde{z}_t$$

We can see that the solution procedure above is a generalization of the Blanchard-Kahn solution for perfect information and that the determinacy of the system is independent of the choice of a perfect or a single imperfect information information set.

F.3 When is the System Invertible?

We now pose the question: given the econometrician's information set, under what conditions do the RE solutions under agents' different information sets actually differ? When can the econometrician infer the full state vector, including shocks?

Fernandez-Villaverde *et al.* (2007) do not attempt to answer this question. Their focus is on the complete reduced form of the solution from the perspective of the econometrician; the source of this reduced form i.e its dependence on the information set of the agents is not discussed at all. As we have seen above, the reduced form under any information set is of the standard state space type investigated by Fernandez-Villaverde *et al.* (2007), but the invertibility properties depend on the information set.

Levine *et al.* (2017b) show the following: If EB is of full rank (i.e. number of observables = number of shocks) but D is *not* of full row rank, then imperfect information is not equivalent to full information, and the system is then not invertible. This is a new result in the literature, which says that if a limited information set under perfect information is invertible, it does not follow that the same limited information set under imperfect information is also invertible.