# Hyperkähler Geometry and Teichmüller Space

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A thesis presented for the
degree of Doctor of Philosophy
of the University of London.

Declaration

Except where references are given or it is explicitly stated otherwise, this dissertation is entirely my own work.

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## Abstract

We consider the hyperkähler extension of Teichmüller space with the Weil-Petersson metric. We describe its recent construction as a hyperkähler quotient and examine the defining equations for the resulting moduli space. We examine relations between this moduli space and the quasi-Fuchsian deformation space of the surface, with particular attention to the connection with the canonical holomorphic symplectic structure. We also consider the connection with Taubes' moduli space of hyperbolic germs and whether it is possible to extend the hyperkähler structure in any fashion.

# Contents

# Chapter 1

## 1.1 Introduction

The Teichmüller space of complex structures on a surface is a natural object of study. It is known to be manifold of dimension $-3\chi(\Sigma)$ where $\chi(\Sigma)$ is the Euler characteristic of the surface under consideration. Further, Teichmüller space itself has a complex structure and a natural metric, the Weil-Petersson metric, that together make it into a Kähler manifold. Results of Feix and Kaledin imply that any real analytic Kähler manifold possesses a hyperkähler extension realised as some neighbourhood of the zero section inside its cotangent bundle. In the case of Teichmüller space Donaldson [5] explicitly constructed this extension as a hyperkähler quotient. In this thesis we outline the necessary geometry to define and understand the hyperkähler extension of Teichmüller space. Consideration of the defining equations for the extension lead us to analyse a particular family of elliptic partial differential equations; we present results on the existence and uniqueness of solutions to these equations. We examine the relations between the hyperkähler extension and the so-called quasi-Fuchsian space, relating the various symplectic structures with those of Goldman [10], [11], and Platis [21]. We also consider the recent work of Taubes on minimal hyperbolic germs [26].

The thesis is organised as follows:

In chapter 2 we describe the necessary background material in symplectic and hyperkähler geometry, including the relevant finite dimensional quotient constructions, before turning to a discussion of complex structures on surfaces and finally a description of Teichmüller space and the Weil-Petersson metric.

In chapter 3 we present the results of Feix and Kaledin on the existence of hyperkähler extensions before looking in some detail at a specific example; the hyperkähler extension of the hyperbolic plane. We explicitly construct a map from the hyperkähler extension of the hyperbolic plane to the product $H^2 \times \overline{H^2}$ that is used in the sequel. Next we discuss the construction by Donaldson of the hyperkähler extension of Teichmüller space as an infinite dimensional hyperkähler qotient.

In chapter 4 we analyse the non-linear partial differential equation arising in the definition of Donaldson's moduli space. We obtain results about its existence and uniqueness which allow us to construct an embedding of the moduli space $\mathcal{M}$ into the cotangent bundle of Teichmüller space. We describe a certain explict subset of the image of $\mathcal{M}$ inside the cotangent bundle of Teichmüller space, as well as proving that this space is the Feix-Kaledin hyperkähler extension of Teichmüller space with the Weil-Petersson metric.

In chapter 5 we introduce the quasi-Fuchsian deformation space $\mathcal{QF}(\Sigma)$ before explicitly constructing a map from the moduli space $\mathcal{M}$ into the quasi-Fuchsian deformation space. We show that the restriction of Goldman's holomorphic symplectic form on the representation variety of $\Sigma$ to the image of $\mathcal{M}$ coincides with a natural holomorphic symplectic form defined by the hyperkähler structure on $\mathcal{M}$.

In chapter 6 we discuss the question of what subset of the quasi-Fuchsian deformation space lies in the image of $\mathcal{M}$ and whether, if it is not a surjection we may extend the hyperkähler structure off $\mathcal{M}$ to some larger open set. Then we describe the construction of Taubes' moduli space of hyperbolic germs, a natural extension of the moduli space $\mathcal{M}$. We present results about when we can extend the hyperkähler structure from $\mathcal{M}$ to some larger set in Taubes' space.

## 1.2 Acknowledgements

It would be remiss of me not to recall the assistance I have received in the course of writing this PhD. from my joint supervisors Prof. S. Donaldson and Dr. R. Thomas. I would also like to thank the teachers who have encouraged and stimulated my pursuit of academic knowledge: J. Genton, R. Jowett, N. Allott, A. Etheridge, L. Mason, M. Dunajski. In addition I observe that none of this would be possible without the constant support of my family, to whom I am eternally grateful.

# Chapter 2

Here we summarise results that are basic to our exposition and recall constructions to be used in the sequel. We assume throughout out this section that all quotients of manifolds by group actions are themselves smooth manifolds.

## 2.1 Symplectic and hyperkähler geometry

Let $M$ be a $2n$ dimensional manifold. A symplectic form $\omega$ on $M$ is a closed non-degenerate 2-form. Most of the material in this section may be found in [20].

### 2.1.1 The symplectic structure on a cotangent bundle

Let now $N$ be any manifold, there is a canonical symplectic structure on the cotangent bundle $T^*N$. To see this we define a canonical one form, $\lambda$, on the total space $T^*N$. Let $q = (p, \sigma) \in T^*N$, $X \in T_qT^*N$, and $\pi : T^*N \to N$ be the canonical projection map, then:

$$\lambda_q(X) := \sigma(\pi_* X).$$

We define the canonical symplectic structure $\omega := d\lambda$.

If $N$ is a complex manifold then this construction applied to the holomorphic cotangent space $T^{*1,0}N$ gives us a holomorphic symplectic structure $\omega_{\mathbb{C}}$ on the total space of $T^{*1,0}N$. If we take $z_i = x_i + iy_i$ as local complex coordinates on $N$ and $dz_i = dx_i + idy_i$ a basis for $T^{*1,0}N$. Writing, for $w \in T^{*1,0}N$, $w = w_i dz_i$ gives local complex coordinates on the fibres of

$T^{*1,0}N$. We may check that:

$$\omega_{\mathbb{C}} = dz_i \wedge dw_i.$$

Given the suggestive nomenclature the following is not a surprise:

**Lemma 2.1.1.** *The canonical symplectic structure is natural. That is, if $M$, $N$ are manifolds and there exists some diffeomorphism $\phi : M \to N$, then the symplectic manifolds $(T^*M, \omega_M)$ and $(T^*N, \omega_N)$ are symplectomorphic. Here $\omega_M$ and $\omega_N$ are the canonical symplectic structures on the cotangent bundles.*

*Proof.* Since $\phi : M \to N$ we can immediately construct a map $\tilde{\phi} : T^*M \to T^*N$,

$$\tilde{\phi}(p, \sigma) = (\phi(p), \phi_*(\sigma)),$$

where here $\phi_* := (\phi^{-1})^*$ on the cotangent vectors, i.e. we are pulling back by the inverse map. We have the following commutative diagram:

$$\begin{array}{ccc} T^*M & \xrightarrow{\tilde{\phi}} & T^*N \\ {\scriptstyle \pi_1}\big\downarrow & & {\scriptstyle \pi_2}\big\downarrow \\ M & \xrightarrow{\phi} & N \end{array}$$

where the $\pi_i$ are the obvious projections. Now let $\lambda_N$ be the canonical one-form on $N$, and consider $\tilde{\phi}^*\lambda_N$. Let $q = (p, \sigma) \in T^*M$ and suppose $X \in T_qT^*M$. We have that

$$\begin{aligned} \tilde{\phi}^*\lambda_N(X) &= \lambda_N(\tilde{\phi}_*X) \\ &= \phi_*(\sigma)(\pi_{2*}\tilde{\phi}_*X) \\ &= \sigma((\phi^{-1})_*\pi_{2*}\tilde{\phi}_*X) \\ &= \sigma(\pi_{1*}X) \\ &= \lambda_M(X). \end{aligned}$$

where we use the relations between the differentials implied by the commutative diagram.

Since the canonical one forms are identified by $\tilde{\phi}$ we must have that the symplectic structures are identified, as pullback commutes with exterior derivative. $\square$

### 2.1.2 Symplectic reduction

Under certain circumstances the technique of *symplectic reduction* allows us to construct new symplectic manifolds by considering the action of some group on a given symplectic manifold. Let $(M, \omega)$ be a symplectic manifold and let $G$ be a compact Lie group. Suppose $G$ acts on $M$ via Hamiltonian symplectomorphisms. That is, for each $g \in G$ the action of $g$ on $M$ preserves the symplectic form, and in addition for any $\xi \in \mathfrak{g}$ there exists a Hamiltonian function $H_\xi$ such that

$$\iota(X_\xi)\omega = dH_\xi,$$

where $X_\xi$ is the vector field generated by the infinitesimal action and the map $\xi \mapsto H_\xi$ defines a Lie algebra homomorphism $\mathfrak{g} \to C^\infty(M)$

**Definition 2.1.2.** A *moment map* for the action is a map $\mu : M \to \mathfrak{g}^*$ such that

$$H_\xi(p) = \langle \mu(p), \xi \rangle,$$

where the map $\xi \mapsto H_\xi$ defines a Lie algebra homomorphism $\mathfrak{g} \to C^\infty(M)$.

Here $\langle \, , \rangle$ is the natural pairing of $\mathfrak{g}$ with its dual. The moment map is G-equivariant. The main theorem in finite dimensions is:

**Theorem 2.1.3** (Marsden-Weinstein)**.** *Take $(M, \omega)$ and $G$ as above, and let the moment map for the action be given by $\mu : M \to \mathfrak{g}^*$. Since the moment map is G-equivariant and $0 \in \mathfrak{g}^*$ is a fixed point for the coadjoint action, the preimage of $\mu^{-1}(0)$ is invariant under $G$. Assume that*

1. *$0$ is a regular value of $\mu$ so that $\mu^{-1}(0)$ is a submanifold of $M$.*

2. *$G$ is acting freely and properly on $\mu^{-1}(0)$ so that the quotient $\mu^{-1}(0)/G$ is a manifold.*

*Then there is a natural symplectic structure on the quotient $\mu^{-1}(0)/G$.*

*If, in addition, $M$ is a Kähler manifold with Kähler form $\omega$ and $G$ preserves the complex structure $I$, then the quotient $\mu^{-1}(0)/G$ is also Kähler manifold.*

In order to prove this theorem we require the following result about linear symplectic reduction.

**Lemma 2.1.4.** *Let $V$ be a symplectic vector space, that is, a $2n$ dimensional vector space equipped with a non-degenerate skew bilinear form $\omega$. Let $W$ be a subspace of $V$ and denote by $W^\omega$ the* symplectic complement *of $W$:*

$$\{v \in V \,|\, \omega(v, w) = 0 \ \forall \ w \in W\}.$$

*Suppose $W^\omega \subseteq W$, then the symplectic structure on $W$ descends to a symplectic structure on the quotient $W/W^\omega$.*

*Proof.* Let $\tilde{\omega}([u], [v]) = \omega(u, v)$. Since $\omega(u, v) = \omega(u + u^\omega, v + v^\omega)$ for any $u^\omega, v^\omega \in W^\omega$ we have that $\tilde{\omega}$ is well defined. Suppose for some $[v] \in W/W^\omega$ we have $\tilde{\omega}([v], [w]) = 0$ for all $[w] \in W/W^\omega$. Then picking a representative $v$ for $[v]$ we see this would contradict the fact that $\omega$ is non degenerate unless $[v] = 0$. □

Now we return to the proof of theorem 2.1.3

*Proof.* Let $N := \mu^{-1}(0)/G$ be the quotient manifold. We need to define a symplectic form on $T_nN$ for each $n \in N$. Let $\pi : \mu^{-1}(0) \to N$ be the natural projection map and $\iota : \mu^{-1}(0) \to M$ be inclusion, we will show that $\tilde{\omega}$ given by

$$\pi^*\tilde{\omega} = \iota^*\omega.$$

is a symplectic form on $N$.

Define $\mathcal{O}(p) := \{g(p) \,|\, g \in G\}$, the orbit of $G$ through $p$. Firstly we show that, if $\mu(p) = 0$, $T_p\mathcal{O}(p) = (T_p(\mu^{-1}(0)))^\omega$ and hence by lemma 2.1.4 we have a well defined non-degenerate skew bilinear from on each quotient $T_p(\mu^{-1}(0))/T_p\mathcal{O}(p)$.

Now, $\mu(p) = 0$ implies $H_\xi(p) = \langle \mu(p), \xi \rangle = 0$ for all $\xi \in \mathfrak{g}$. It follows that $\omega(X_\xi, Y) = dH_\xi(Y) = 0$ for all $Y \in T_p(\mu^{-1}(0))$, since $H_\xi$ is constant on $\mu^{-1}(0)$. But $X_\xi(p) \in T_p\mathcal{O}(p)$ and conversely every element of $T_p\mathcal{O}(p)$ is of the form $X_\xi(p)$ for some $\xi \in \mathfrak{g}$ therefore $T_p\mathcal{O}(p) \subseteq (T_p(\mu^{-1}(0)))^\omega$. Then we have

$$\dim(T_p(\mu^{-1}(0)))^\omega = \dim(M) - \dim(T_p(\mu^{-1}(0)) = \dim(G)$$

which is the dimension of $T_p\mathcal{O}(p)$ so $T_p\mathcal{O}(p) = (T_p(\mu^{-1}(0)))^\omega$.

So on each $T_p(\mu^{-1}(0))/T_p\mathcal{O}(p)$ we have that $\tilde{\omega}$ is well defined; we now need to check it is uniquely defined at each point of the quotient $N$. But

$$\begin{aligned}\pi^*\tilde{\omega}_{\pi(p)}(u,v) &= \omega_p(u,v)\\ &= (\psi_g^*\omega)_{\psi_g(p)}(u,v)\\ &= \omega_{\psi_g(p)}(\psi_{g*}u,\psi_{g*}v)\\ &= \pi^*\tilde{\omega}_{\pi(\psi_g(p))}(\psi_{g*}u,\psi_{g*}v).\end{aligned}$$

Therefore

$$\tilde{\omega}_{\pi(p)}(\pi_*u,\pi_*v) = \tilde{\omega}_{\pi(\psi_g(p))}(\pi_*u,\pi_*v),$$

as desired, and all tangent vectors on $N$ are of the form $\pi_*u$ for some tangent vector $u$ on $M$ since $\pi$ is surjective.

We know $\tilde{\omega}$ is non degenerate since each $\tilde{\omega}_p$ is, so it remains to check that it is closed. But this follows since

$$0 = \iota^*d\omega = d\iota^*\omega = d\pi^*\tilde{\omega} = \pi^*d\tilde{\omega}$$

and $\pi$ is a surjection.

We now need to prove the last assertion. Suppose $M$ has a complex structure $I$ compatible with $\omega$, so that we obtain a real metric $g$ on $M$ defined by

$$g(X,Y) = \omega(X,IY),$$

for any $X,Y \in T_pM$. Now this metric restricts to give us a metric on the subset $\mu^{-1}(0)$, and the decomposition of the tangent space of $\mu^{-1}(0)$ as

$$T_p\mu^{-1}(0) = T_p\mathcal{O}(p) \oplus T_p\mathcal{O}(p)^{\perp},$$

defines a connection in the prinicipal $G$ bundle $\mu^{-1}(0)$ over $\mu^{-1}(0)/G$.

Now we can define a complex structure on the quotient $\mu^{-1}(0)/G$ as follows. Let $X \in T_{[p]}\left(\mu^{-1}(0)/G\right)$, take its unique horizontal lift $\tilde{X} \in T_p\mu^{-1}(0)$. Since the horizontal subspace of the connection is defined to be $T\mathcal{O}(p)^{\perp}$ we must have that for all $\xi \in \mathfrak{g}$,

$$\begin{aligned}0 &= g(X_\xi,\tilde{X})\\ &= \omega(X_\xi,I\tilde{X})\\ &= dH_\xi(I\tilde{X})\\ &= d(\langle\mu,\xi\rangle)(I\tilde{X}).\end{aligned}$$

Since this holds for all $\xi$ we must have that $\mu$ is not changing in the direction of $I\tilde{X}$ and hence that $I\tilde{X} \in T_p\mu^{-1}(0)$. Define the almost complex structure $\tilde{I}$ on the quotient by

$$\tilde{I}X = \pi_*(I\tilde{X}),$$

for $\tilde{X}$ any horizontal lift of $X$. We need to show that this is well defined, the proceeding discussion shows $\pi_*(I\tilde{X}) \in T_{[p]}((\mu^{-1}(0)/G)$ so it certainly lies in the right space, we need it to be independent of the point $q \in [p]$ we lifted to. This follows since $G$ preserves both the complex structure $I$ and the horizontal subspaces of the connection. To see that $\tilde{I}$ is an almost complex structure observe that for $\tilde{X} \in T_p\mu^{-1}(0)$ and $\xi \in \mathfrak{g}$:

$$\begin{aligned} g(X_\xi, I\tilde{X}) &= -\omega(X_\xi, \tilde{X}) \\ &= -d(\langle \mu, \xi \rangle)(\tilde{X}) \\ &= 0. \end{aligned}$$

Thus $I\tilde{X}$ is orthogonal to the orbit, and if $\tilde{X}$ is horizontal we therefore have that $I\tilde{X}$ is horizontal.

We need to show that $I$ is integrable. This follows immediately from the fact that for any horizontal vector fields $\tilde{X}, \tilde{Y}$ on $\mu^{-1}(0)$:

$$\pi_*[\tilde{X}, \tilde{Y}] = [\pi_*\tilde{X}, \pi_*\tilde{Y}].$$

$$\pi_* I\tilde{X} = \tilde{I}\pi_*\tilde{X},$$

so that if the Nijenhuis tensor vanishes on $M$ it must also on $\mu^{-1}(0)/G$.

This complex structure is clearly compatible with the symplectic form on the quotient and the quotient is therefore Kähler. □

### 2.1.3 Symplectic reduction of cotangent bundles

Suppose $M$ is a manifold and let $\omega$ be the canonical symplectic structure on $T^*M$. Suppose that we have some group $G$ acting by diffeomorphisms on $M$. It follows from the fact that the canonical symplectic structure on the cotangent bundle is natural that $G$ acts by symplectomorphisms on the symplectic manifold $(T^*M, \omega)$. There exists a moment map for the action of $G$ on $T^*M$ and we may form the Marsden-Weinstein quotient $T^*M//G$. Let

$\tilde{G}$ be another group of diffeomorphisms of $M$ such that $G \subseteq \tilde{G}$. If we have an identification of the symplectic quotient $T^*M//G$ with the cotangent bundle of the quotient of $M$ by the action of $\tilde{G}$ then we may hope that the induced symplectic structure on the symplectic quotient coincides with the canonical symplectic structure on the cotangent bundle $T^*(M/\tilde{G})$. We consider such a result now.

Let $M$, $G$, $\tilde{G}$ be as above, with $\omega$ the canonical symplectic structure on the cotangent bundle $T^*M$ and $\hat{\omega}$ the canonical symplectic structure on the cotangent bundle $T^*(M/\tilde{G})$. Let $\mu$ be the moment map for the action of $G$ on $T^*M$ and suppose that if $q = (p, \sigma) \in \mu^{-1}(0)$ then $\sigma$ is in the annihilator of the kernel of the map $\pi_{2*} : T_pM \to T_{[p]}(M/\tilde{G})$. Since the tangent space at a point $[p] \in M/G$ is given by

$$T_{[p]}(M/G) \cong T_pM/\ker(\pi_{2*}),$$

we see that if $q = (p, \sigma) \in \mu^{-1}(0)$ then $\sigma$ defines a point in $T^*_{[p]}(M/G)$. Suppose that the resulting map

$$\tilde{\iota} : \mu^{-1}(0)/G \to T^*(M/\tilde{G}),$$

is injective and the following diagram commutes:

$$\begin{array}{ccccc} \mu^{-1}(0) & \xrightarrow{\iota} & T^*M & \xrightarrow{\pi} & M \\ {\scriptstyle \pi_1}\downarrow & & \downarrow & & {\scriptstyle \pi_2}\downarrow \\ \mu^{-1}(0)/G & \xrightarrow{\tilde{\iota}} & T^*(M/\tilde{G}) & \xrightarrow{\tilde{\pi}} & M/\tilde{G} \end{array}$$

In this situation we have the following proposition:

**Proposition 2.1.5.** *The canonical injection $\tilde{\iota} : T^*M//G \hookrightarrow T^*(M/\tilde{G})$ satisfies*

$$\tilde{\iota}^*\hat{\omega} = \tilde{\omega},$$

*where $\tilde{\omega}$ is the symplectic structure on $\mu^{-1}(0)/G$ and $\hat{\omega}$ is the canonical sympectic structure on $T^*(M/\tilde{G})$.*

*Proof.* We prove this by considering the behavior of the canonical one forms on $\mu^{-1}(0)$ and $T^*(M/\tilde{G})$. Take $q = (p, \sigma) \in \mu^{-1}(0)$ and let $X \in T_q\mu^{-1}(0)$, since the diagram commutes we have immediately that

$$\pi_{2*} \circ \pi_* \circ \iota_*(X) = \tilde{\pi}_* \circ \tilde{\iota}_* \circ \pi_{1*}(X).$$

Writing $\hat{\lambda}$ for the canonical one form on $T^*(M/\tilde{G})$ and $\lambda$ for the canonical one form on $T^*M$,

$$\begin{aligned}(\pi_1^* \circ \tilde{\iota}^* \hat{\lambda})(X) &= \hat{\lambda}(\tilde{\iota}_* \circ \pi_{1*}(X)) \\ &= \sigma(\tilde{\pi}_* \circ \tilde{\iota}_* \circ \pi_{1*}(X)) \\ &= \sigma(\pi_{2*} \circ \pi_* \circ \iota_*(X)) \\ &= \sigma(\pi_* \circ \iota_*(X)) \\ &= (\iota^* \lambda)(X).\end{aligned}$$

Here we have used the fact that $\sigma$ is in the annihilator of $\ker \pi_{2*}$.

Recall that the symplectic structure $\tilde{\omega}$ on the Marsden-Weinstein quotient $T^*M//G$ is defined by

$$\pi_1{}^* \tilde{\omega} = \iota^* \omega.$$

Since we have seen that

$$\pi_1^* \circ \tilde{\iota}^* \hat{\lambda} = \iota^* \lambda,$$

we must have that

$$\pi_1^* \circ \tilde{\iota}^* \hat{\omega} = \iota^* \omega,$$

which implies that

$$\tilde{\iota}^* \hat{\omega} = \tilde{\omega},$$

which is what we were trying to show. □

### 2.1.4 Hyperkähler geometry

The material in this section may be found in the paper [14]. Let $M$ be a $4n$ dimensional smooth manifold.

**Definition 2.1.6.** A hyperkähler structure on $M$ is a quadruple $(g, I, J, K)$, where $g$ is a metric on $M$ and each of $I, J, K$ is an integrable orthogonal complex structure on $M$ parallel with respect to the Levi-Civita connection. Further the complex structures satisfy the quaternion algebra identities:

$$I^2 = J^2 = K^2 = -1,$$

$$IJ = -JI = K,$$

$$KI = -IK = J,$$

$$JK = -KJ = I.$$

Since each complex structure is parallel, each induces in combination with $g$ a Kähler metric on $M$. We have corresponding Kähler forms:

$$\omega_1(X,Y) = g(IX,Y), \quad \omega_2(X,Y) = g(JX,Y), \quad \omega_3(X,Y) = g(KX,Y).$$

The following is clearly a generalisation of the Marsden-Weinstein quotient from the previous section. We suppose that the group action in what follows is sufficiently nice for the quotient to be a manifold.

**Theorem 2.1.7.** *Let $(M, g, I, J, K)$ be a hyperkähler manifold and suppose there is an action of a group $G$ of isometries on $M$ preserving the three symplectic forms $\omega_i$. Suppose in addition that we have three moment maps $\mu_i$, one for each symplectic form. Then the quotient*

$$M///G = \frac{\bigcap_i \mu_i^{-1}(0)}{G}$$

*has a natural hyperkähler structure.*

*Proof.* We shall reduce the proof of this to three applications of theorem (2.1.3). Define the complex valued map $\mu_I : M \to \mathfrak{g}^* \otimes \mathbb{C}$,

$$\mu_I := \mu_2 + i\mu_3.$$

Further, for $\xi \in \mathfrak{g}$ define the complex valued function $\mu_I^\xi : M \to \mathbb{C}$,

$$\mu_I^\xi := \langle \mu_I, \xi \rangle = \langle \mu_2, \xi \rangle + i\langle \mu_3, \xi \rangle.$$

Here the angle brackets denote the natural pairing between $\mathfrak{g}$ and its dual. From these definitions we see immediately that for $Y$ a vector field on $M$,

$$\begin{aligned} d\mu_I^\xi(Y) &= dH_\xi^2(Y) + idH_\xi^3 \\ &= \omega_2(X_\xi, Y) + i\omega_3(X_\xi, Y) \\ &= g(JX_\xi, Y) + ig(KX_\xi, Y), \end{aligned}$$

and

$$\begin{aligned} d\mu_I^\xi(IY) &= g(JX_\xi, IY) + ig(KX_\xi, IY) \\ &= -g(KX_\xi, Y) + ig(JX_\xi, Y) \\ &= id\mu_I^\xi(Y). \end{aligned}$$

Since $d = \partial + \overline{\partial}$ we see by comparing types that the above relationship must hold for $\overline{\partial}\mu_I^\xi(Y)$ as well. Now taking $Y \in T_I^{1,0}M$ we see that

$$i\overline{\partial}\mu_I^\xi(Y) = \overline{\partial}\mu_I^\xi(IY) = -i\overline{\partial}\mu_I^\xi(Y),$$

so that

$$\overline{\partial}\mu_I^\xi = 0.$$

Therefore $\mu_I^\xi$ is a holomorphic function on $M$ for every $\xi \in \mathfrak{g}$. This tells us precisely that the map $\mu_I : M \to \mathfrak{g}^* \otimes \mathbb{C}$ is holomorphic since we may think of the $\xi$ as giving complex coordinate functions on $\mathfrak{g}^* \otimes \mathbb{C}$. Now consider the submaniold of $M$ defined by $\mu_I^{-1}(0) = \mu_2^{-1}(0) \cap \mu_3^{-1}(0)$. Since we know from the above that $\mu_I$ is a holomorphic map this is a complex submanifold of $M$ with respect to the complex structure $I$. Therefore the metric induced by the restriction of $g$ is Kähler. By hypothesis, the group $G$ acts on $\mu_I^{-1}(0)$ preserving the symplectic form $\omega_1$ and complex structure $I$, with moment map given by the rstriction of $\mu_1$ to $\mu_I^{-1}(0)$. Applying theorem (2.1.3) we see that the quotient $(\mu_I^{-1}(0) \cap \mu_1^{-1}(0))/G$ is a Kähler manifold with the complex structure induced by $I$ and the symplectic structure induced by $\omega_1$. Repeating the argument with the complex structures $J$ and $K$ we arrive at the desired conclusion. □

## 2.2 Complex structures

In this section we examine the theory of complex structures on surfaces. We start by discussing the case of linear complex structures on $\mathbb{R}^2$ before generalising this construction to surfaces. Finally we describe the construction and properties of the Teichmüller space of a surface.

### 2.2.1 Linear complex structures on $\mathbb{R}^2$

We fix conventions. Take the natural coordinates $(x, y)$ on the vector space $\mathbb{R}^2$, and fix the orientation $dx \wedge dy$. A complex structure on $\mathbb{R}^2$ is an endomorphism of $\mathbb{R}^2$ that squares to minus the identity. We have for example the standard complex structure $J_0$ on $\mathbb{R}^2$,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The set of all complex structures on $\mathbb{R}^2$ is written $\mathcal{J}(\mathbb{R}^2)$. By definition,

$$\mathcal{J}(\mathbb{R}^2) = \{J \in \mathrm{End}(\mathbb{R}^2) | J^2 = -\mathbf{1}\}.$$

A complex structure on $\mathbb{R}^2$ induces a natural orientation by taking $\{\partial_x, J\partial_x\}$ as an ordered basis for the tangent space. The space of complex structures on $\mathbb{R}^2$ compatible with the fixed orientation is written $\mathcal{J}^+(\mathbb{R}^2)$.

There is a natural left action of $\mathrm{SL}_2\mathbb{R}$ on $\mathcal{J}^+(\mathbb{R}^2)$ by conjugation:

$$A : J \mapsto AJA^{-1}.$$

This action is transitive and the stabiliser of the standard complex structure $J_0$ is $\mathrm{SO}_2\mathbb{R}$, hence we may identify $\mathcal{J}^+(\mathbb{R}^2)$ with the coset space $\mathrm{SL}_2\mathbb{R}/\mathrm{SO}_2\mathbb{R}$.

The tangent space at $J$ in $\mathcal{J}^+(\mathbb{R}^2)$ is given by the set of endomorphisms of $\mathbb{R}^2$ that anticommute with $J$,

$$T_J\mathcal{J}^+(\mathbb{R}^2) = \{H \in \mathrm{End}(\mathbb{R}^2) \mid HJ = -JH\}.$$

An immediate consequence of this is that the tangent vectors at $J_0$ are symmetric and trace free matrices. That is, given $H \in T_{J_0}\mathcal{J}^+(\mathbb{R}^2)$ we may write it

$$H = \begin{pmatrix} u & v \\ v & -u \end{pmatrix}.$$

There is a natural inner product on $\mathcal{J}^+(\mathbb{R}^2)$ given by the standard trace form. That is, given $H_1, H_2 \in T_J\mathcal{J}^+(\mathbb{R}^2)$, define

$$\langle H_1, H_2 \rangle = \frac{1}{2}\mathrm{Tr}(H_1 H_2).$$

This metric is trivially $\mathrm{SL}_2\mathbb{R}$ equivariant because the trace form is.

There is a natural complex structure $\Phi$ on $\mathcal{J}^+(\mathbb{R}^2)$. This is given at a point $J \in \mathcal{J}^+(\mathbb{R}^2)$ by multiplication by $J$. That is:

$$\Phi : T_J\mathcal{J}^+(\mathbb{R}^2) \to T_J\mathcal{J}^+(\mathbb{R}^2)$$
$$H \mapsto JH.$$

Given a $J \in \mathcal{J}^+(\mathbb{R}^2)$ we may consider the natural splitting:

$$\mathbb{R}^2 \otimes \mathbb{C} = V_J^{1,0} \oplus V_J^{0,1}.$$

Here $V_J^{1,0}$ is the $+i$ eigenspace, and $V_J^{0,1}$ the $-i$ eigenspace for the complexified action of $J$.

**Lemma 2.2.1.** *There are natural identifications*

$$T_J\mathcal{J}^+(\mathbb{R}^2) \cong \mathrm{Hom}(V_J^{1,0}, V_J^{0,1}) \cong V_J^{*1,0} \otimes V_J^{0,1}.$$

*Proof.* Firstly we construct a map $T_J\mathcal{J}^+(\mathbb{R}^2) \to \mathrm{Hom}(V_J^{1,0}, V_J^{0,1})$. Let $H \in T_J\mathcal{J}^+(\mathbb{R}^2)$ and $v \in V_J^{1,0}$, define a map

$$\lambda_H : V_J^{1,0} \to V_J^{0,1},$$

by $\lambda_H(v) := Hv$ where we extend the action of $H$ to $\mathbb{R}^2 \otimes \mathbb{C}$ linearly. Then

$$J\lambda_H(v) = JHv = -HJv = -iHv = -i\lambda_H(v),$$

so that $\lambda_H(v) \in V_J^{0,1}$ as claimed.

Now let $\lambda \in \mathrm{Hom}(V_J^{1,0}, V_J^{0,1})$. We define a map $H_\lambda \in \mathrm{End}(\mathbb{R}^2)$ by

$$Hw = \lambda(\tilde{w}) + \overline{\lambda(\tilde{w})},$$

where $2\tilde{w} = (w - iJw) \in V_J^{1,0}$. Since for all $w \in \mathbb{R}^2$ we have:

$$HJw = \lambda(J\tilde{w}) + \overline{\lambda(J\tilde{w})} = i\lambda(\tilde{w}) - i\overline{\lambda(\tilde{w})} = -J\lambda(\tilde{w}) + J\overline{\lambda(\tilde{w})} = -JHw,$$

we see that $H$ anticommutes with $J$ as desired. □

Suppose now we have fixed a non-degenerate skew bilinear form $\rho$ on $\mathbb{R}^2$, compatible with the usual orientation. Given a complex structure $J \in \mathcal{J}^+(\mathbb{R}^2)$ we may construct a complex pairing on $\mathbb{R}^2$:

$$X, Y \mapsto \rho(X, JY) - i\rho(X, Y).$$

This extends complex linearly to $\mathbb{R}^2 \otimes \mathbb{C}$ and induces a hermitian inner product on $V_J^{1,0}$, for $u, v \in V_J^{1,0}$,

$$h(u, v) = -2i\rho(u, \overline{v}).$$

This extends to give us a hermitian inner product which we shall also denote by $h$ on $V_J^{*1,0} \otimes V_J^{*1,0}$. In addition, the hermitian inner product allows us to identify $V_J^{0,1}$ with $V_J^{*1,0}$ so we may map a tangent vector $H \in T_J\mathcal{J}^+(\mathbb{R}^2)$ to a quadratic form $\xi_H$, that is an element of $V_J^{*1,0} \otimes V_J^{*1,0}$. Let $\sigma \in V_J^{*1,0} \otimes V_J^{*1,0}$, and $H \in T_J\mathcal{J}^+(\mathbb{R}^2)$, then we define a pairing $\langle \sigma, H\rangle := h(\sigma, \xi_H)$. This then identifies the cotangent bundle $T_J^*\mathcal{J}(\mathbb{R}^2)$ with the space $V_J^{*1,0} \otimes V_J^{*1,0}$.

**Lemma 2.2.2.** *The hermitian inner product $h$ defined above induces an $\mathrm{SL}_2\mathbb{R}$ invariant metric on $\mathcal{J}^+(\mathbb{R}^2)$.*

*Proof.* It is clear that $h$ induces a metric, because we have defined an inner product on the cotangent vectors, we only need to check the asserted equivariance. Fix $X \in \mathbb{R}^2$, let $v = \frac{1}{2}(X - iJX)$ be a generator for $V_J^{1,0}$, and $\nu \in V_J^{*1,0}$ dual to $v$. Suppose that the tangent vectors $H_\eta, H_\xi \in T_J\mathcal{J}^+(\mathbb{R}^2)$ correspond to the quadratic forms $\eta, \xi \in V_J^{*1,0} \otimes V_J^{*1,0}$. Chasing the chain of identifications through we see that:

$$\eta = -2i\rho(v, H_\eta v)\nu \otimes \nu,$$

$$\xi = -2i\rho(v, H_\xi v)\nu \otimes \nu.$$

This means that with respect to this basis,

$$\langle \eta, \xi\rangle = -\frac{\rho(v, H_\eta v)\overline{\rho(v, H_\xi v)}}{\rho^2(v, \overline{v})}.$$

Now let $g \in \mathrm{SL}_2\mathbb{R}$, so that $g(J) = gJg^{-1}$ and

$$g_*(H_\eta) = gH_\eta g^{-1}, \ g_*(H_\xi) = gH_\xi g^{-1}.$$

Consider the vector $g^{-1}X \in \mathbb{R}^2$, this leads to the vector $\tilde{v} = \frac{1}{2}(gX - igJX)$ generating the vector space $V_{g(J)}^{1,0}$. With respect to this basis we have from above that

$$\langle g_*\eta, g_*\xi\rangle = -\frac{\rho(\tilde{v}, gH_\eta g^{-1}\tilde{v})\overline{\rho(\tilde{v}, gH_\xi g^{-1}\tilde{v})}}{\rho^2(\tilde{v}, \overline{\tilde{v}})}.$$

Observe that $\tilde{v} = gv$ so that this reduces to,

$$\langle g_*\eta, g_*\xi \rangle = -\frac{\rho(gv, gH_\eta v)\overline{\rho(gv, gH_\xi v)}}{\rho^2(gv, g\overline{v})}.$$

Since $\rho$ is invariant under the $\mathrm{SL}_2\mathbb{R}$ action we have that the group acts by isometries as desired. □

**Lemma 2.2.3.** *The real part of the hermitian metric defined above is the natural metric from the trace form defined earlier.*

*Proof.* We only need to verify this at $J_0$ because both metrics are $\mathrm{SL}_2\mathbb{R}$ equivariant. Let $H_1, H_2 \in T_{J_0}\mathcal{J}^+(\mathbb{R}^2)$,

$$H_1 := \begin{pmatrix} u_1 & v_1 \\ v_1 & -u_1 \end{pmatrix}, \; H_2 = \begin{pmatrix} u_2 & v_2 \\ v_2 & -u_2 \end{pmatrix}.$$

Then

$$\mathrm{Tr}(H_1, H_2) = 2(u_1u_2 + v_1v_2).$$

On the other hand as in the proof of the previous proposition,

$$\begin{aligned} h(\xi_{H_1}, \xi_{H_2}) &= -\frac{\rho(\partial_z, H_1\partial_z)\overline{\rho(\partial_z, H_2\partial_z)}}{\rho^2(\partial_z, \overline{\partial_z})} \\ &= (iu_1 + v_1)(-iu_2 + v_2). \end{aligned}$$

The real part of this is

$$u_1u_2 + v_1v_2,$$

which completes the proof. □

### 2.2.2 The hyperbolic plane

In this section we introduce the hyperbolic plane and its relationship with the linear complex structures on $\mathbb{R}^2$. We take as a model of the hyperbolic plane the upper half plane:

$$H^2 = \{x + iy \in \mathbb{C} \mid y > 0\},$$

equipped with the hyperbolic metric:

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2),$$

and the obvious complex structure inherited from $\mathbb{C}$: multiplication by $i$. The group of conformal isometries of $H^2$ is the set $\text{Möb}_{\mathbb{R}}$ of real Möbius transformations. This is the set of maps

$$\text{Möb}_{\mathbb{R}} := \left\{ z \mapsto \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{R},\ ad-bc>0 \right\}.$$

There is a natural map

$$\text{SL}_2\mathbb{R} \to \text{Möb}_{\mathbb{R}},$$

defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az-b}{-cz+d}.$$

This map is a group homomorphism and allows us to define an action of $\text{SL}_2\mathbb{R}$ on $H^2$. This action is transitive and the stabiliser of $i \in H^2$ is $\text{SO}_2\mathbb{R}$. Therefore we identify $H^2$ with the coset space $\text{SL}_2\mathbb{R}/\text{SO}_2\mathbb{R}$. This allows us to identify $H^2$ with the space $\mathcal{J}^+(\mathbb{R}^2)$ of the previous section. Under the action of $\text{SL}_2\mathbb{R}$ on $H^2$, the element

$$\begin{pmatrix} \sqrt{y} & \frac{-x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \in \text{SL}_2\mathbb{R},$$

maps $i \in H^2$ to $x+iy \in H^2$. On the other hand the action of $\text{SL}_2\mathbb{R}$ on $\mathcal{J}^+(\mathbb{R}^2)$ takes the complex structure $J_0$ to the complex structure

$$J = \begin{pmatrix} \sqrt{y} & \frac{-x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{y}} & \frac{x}{\sqrt{y}} \\ 0 & \sqrt{y} \end{pmatrix} = \begin{pmatrix} \frac{-x}{y} & -y\left(1-\frac{x^2}{y^2}\right) \\ \frac{1}{y} & \frac{x}{y} \end{pmatrix}.$$

We therefore identify a point $x+iy \in H^2$ with the complex structure $J$, this map is equivariant by construction.

**Lemma 2.2.4.** *The natural equivariant map $\iota : H^2 \to \mathcal{J}^+(\mathbb{R}^2)$ is a biholomorphic isometry with respect to the natural metrics and complex structures.*

*Proof.* Firstly observe that the equivariance and transitivity of the action mean that we only have to verify these properties hold at $i \in H^2 \mapsto J_0 \in \mathcal{J}^+(\mathbb{R}^2)$. Now the tangent map, $\iota_*$, at $i \in H^2$ takes the tangent vector $(u,v)$ to the matrix

$$\begin{pmatrix} -u & -v \\ -v & u \end{pmatrix} \in T_{J_0}\mathcal{J}^+(\mathbb{R}^2).$$

Therefore

$$\iota_* \circ i(u, v) = \iota_*(-v, u) = \begin{pmatrix} v & -u \\ -u & -v \end{pmatrix},$$

and

$$\Phi \circ \iota_* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -u & -v \\ -v & u \end{pmatrix} = \begin{pmatrix} v & -u \\ -u & v \end{pmatrix}.$$

So that the map is a biholomorphism. To see it is an isometry suppose we have the two tangent vectors $(u_1, v_1)$ and $(u_2, v_2)$ at $i \in H^2$. With respect to the standard basis for $V_{J_0}^{1,0}$ given by

$$v = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

we can calculate as in lemma (2.2.2) $\langle \iota_*(u_1, v_1), \iota_*(u_2, v_2)\rangle$ to be

$$-\frac{\rho(v, H_1 v)\overline{\rho(v, H_2 v)}}{\rho^2(v, \overline{v})}.$$

Since

$$H_1 v = \begin{pmatrix} -u_1 & -v_1 \\ -v_1 & u_1 \end{pmatrix} \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) = -(u_1 - iv_1)\overline{v},$$

we see that $\rho(v, H_1 v) = -(u_1 - iv_1)\rho(v, \overline{v})$ and similarly for the term in $H_2$, hence

$$\langle \iota_*(u_1, v_1), \iota_*(u_2, v_2)\rangle = (u_1 - iv_1)(u_2 + iv_2) = u_1u_2 + v_1v_2 - i(v_1u_2 - u_1v_2).$$

But the real part of this is exactly the evaluation of the hyperbolic metric on the two tangent vectors, as desired. □

Observe that it follows from this lemma that the complex structure $\Phi$ on $\mathcal{J}^+(\mathbb{R}^2)$ is the one induced by the $\mathrm{SL}_2\mathbb{R}$ equivariant metric, a fact we had not elicited in the previous section.

### 2.2.3 Complex structures on surfaces

Let $\Sigma$ be a fixed oriented smooth surface of genus 2 or more. We shall always take $\Sigma$ to be closed, that is compact and without boundary though this is not essential in this current section. Since the genus is greater than one we have that $\chi(\Sigma)$, the Euler characteristic of $\Sigma$, is strictly negative.

**Definition 2.2.5.** An *almost complex structure* on $\Sigma$ is a smooth assignment for each $p \in \Sigma$ of an automorphism $J : T_p\Sigma \to T_p\Sigma$ such that $J^2 = -\mathbf{1}$.

Given an almost complex structure $J$, the existence of isothermal coordinates guarantees the existence of an atlas for $\Sigma$ consisting of complex valued charts with biholomorphic transition maps, so that $(\Sigma, J)$ may be thought of as a complex curve. For this reason we shall not draw the usual distinction between almost complex structures on the tangent bundle of $\Sigma$ and complex structures on $\Sigma$ in the sense of giving $\Sigma$ the structure of a complex curve, referring to both notions as complex structures.

A complex structure on $\Sigma$ induces an orientation by taking $\{\partial_x, J\partial_x\}$ as an ordered basis on each tangent plane. We shall only be concerned with those complex structures $J$ that induce the orientation on $\Sigma$ that we fixed earlier. We denote the set of all complex structures on $\Sigma$ compatible with the fixed orientation $\mathcal{J}(\Sigma)$. It is immediate from the definitions that $\mathcal{J}(\Sigma)$ is a subset of the space of smooth sections of $T^*\Sigma \otimes T\Sigma$. We shall write $(\Sigma, J)$ to denote the pair consisting of the surface $\Sigma$ together with the complex structure $J$. When there is no danger of confusion we shall leave the surface implicit.

The results of the previous section imply that the formal tangent space at $J$ to the space of complex structures may be identified as:

$$T_J\mathcal{J}(\Sigma) = \{H \in \mathrm{End}(T\Sigma) : HJ = -JH\},$$

and that there is a canonical metric on $\mathcal{J}(\Sigma)$ induced by the trace form and a canonical complex structure $\Phi$ on $\mathcal{J}(\Sigma)$ given on the tangent space at $J$ by multiplication by $J$.

Now fix a volume form $\rho$ on $\Sigma$ and let $P$ be the principal $\mathrm{SL}_2\mathbb{R}$ bundle of volume one frames for $T\Sigma$. Since we have an $\mathrm{SL}_2\mathbb{R}$ action on $\mathcal{J}^+(\mathbb{R}^2)$ we can construct the associated fibre bundle over $\Sigma$ with fibre $\mathcal{J}^+(\mathbb{R}^2)$. It is immediate from the work of the previous section that we have the following identification:

$$P \times_{\mathrm{SL}_2\mathbb{R}} \mathcal{J}^+(\mathbb{R}^2) \cong P \times_{\mathrm{SL}_2\mathbb{R}} H^2.$$

We denote the latter bundle by $\underline{H}^2$. Letting $\Gamma(\underline{H}^2)$ denote the vector space of smooth sections of the fibre bundle $\underline{H}^2$, it follows that $\mathcal{J}(\Sigma) = \Gamma(\underline{H}^2)$.

Further, using $\rho$, we may construct as in the previous sections an identification

$$T_J^*\mathcal{J}(\Sigma) \cong \Gamma(T_J^{*1,0}\Sigma \otimes T_J^{*1,0}\Sigma),$$

where $\Gamma(T_J^{*1,0}\Sigma \otimes T_J^{*1,0}\Sigma)$ is the vector space of *smooth quadratic differentials*, smooth sections of the bundle $T_J^{*1,0}\Sigma \otimes T_J^{*1,0}\Sigma$. This allows us to define a hermitian metric on $\mathcal{J}(\Sigma)$, given $\sigma, \eta$ two smooth quadratic differentials in $T_J^*\mathcal{J}(\Sigma)$ we define

$$h_\rho(\sigma, \eta) = \int_\Sigma h(\sigma, \eta)\rho.$$

where $h$ is the metric on $\Sigma$ induced by $\rho$ and $J$.

### 2.2.4 Splitting the tangent space of $\mathcal{J}(\Sigma)$

In this section we show that there is a canonical decomposition of the tangent space of $\mathcal{J}(\Sigma)$; a result we shall require in the section on Teichmüller space. This material is adapted from the presentation in Tromba [27].

Fix a complex structure $J \in \mathcal{J}(\Sigma)$. We know that the volume form $\rho$ and complex structure $J$ induce a Riemannian metric $g$ on $\Sigma$ defined by:

$$g(X, Y) = \rho(X, JY).$$

In turn this induces a natural $L^2$ metric on vector fields on $\Sigma$:

$$\langle X, Y \rangle = \int_\Sigma g(X, Y)\rho.$$

We can extend this metric in a natural way to $1, 1$ tensors, we find that for $H_1, H_2 \in \Gamma(T^*\Sigma \otimes T\Sigma)$,

$$\langle H_1, H_2 \rangle = \frac{1}{2}\int_\Sigma \mathrm{Tr}(H_1 H_2)\rho.$$

When $H_1$ and $H_2$ are in $T_J\mathcal{J}(\Sigma)$ this is just the real part of the hermitian metric $h_\rho$ defined in the previous paragraph .

Now we define an operator:

$$\nabla : \Gamma(T\Sigma) \to \Gamma(T^*\Sigma \otimes T\Sigma),$$
$$X \mapsto \mathcal{L}_X J.$$

It is immediate that in fact $\nabla : \Gamma(T\Sigma) \to T_J\mathcal{J}(\Sigma)$.

**Lemma 2.2.6.** *The $L^2$ adjoint of $\nabla$ is the vector field given in standard index notation by:*

$$(\nabla^* H)^n = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g} H^i_j J^i_n\right) g^{mn} + \frac{1}{2} H^j_i \left(\frac{\partial}{\partial x^n} J^i_j\right) g^{mn}.$$

*Proof.* Let $X$ be any vector field on $\Sigma$,

$$\begin{aligned}
\langle H, \nabla X \rangle &= \frac{1}{2} \int_\Sigma \mathrm{Tr}(H \mathcal{L}_X J) \rho \\
&= \frac{1}{2} \int_\Sigma \left( H^j_i \left(\frac{\partial}{\partial x^k} J^i_j\right) X^k + H^j_i J^i_k \frac{\partial}{\partial x^j} X^k - H^j_i J^k_j \frac{\partial}{\partial x^k} X^i \right) \rho \\
&= \frac{1}{2} \int_\Sigma \left( H^j_i \left(\frac{\partial}{\partial x^k} J^i_j\right) X^k + 2 H^j_i J^i_k \frac{\partial}{\partial x^j} X^k \right) \rho \\
&= \frac{1}{2} \int_\Sigma \left( H^j_i \left(\frac{\partial}{\partial x^n} J^i_j\right) g^{mn} - \frac{2}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g} H^j_i J^i_n\right) g^{mn} \right) g_{mk} X^k \rho \\
&=: \langle (\nabla^* H), X \rangle.
\end{aligned}$$

□

We would like to use standard material about elliptic operators, applied to the operator

$$\nabla^* \nabla : \Gamma(T\Sigma) \to \Gamma(T\Sigma),$$

to conclude that we can decompose $T_J \mathcal{J}(\Sigma)$ as

$$\begin{aligned}
T_J \mathcal{J}(\Sigma) &= \mathrm{range}(\nabla) \oplus \mathrm{range}(\nabla)^\perp \\
&= \mathrm{range}(\nabla) \oplus \ker(\nabla^*).
\end{aligned}$$

We sketch this material now. Firstly we require that $\nabla^* \nabla$ is elliptic, that is it has invertible symbol, but this is clear from the expression in coordinates (see [27]). Secondly we have that $T_J \mathcal{J}(\Sigma)$ is not a Hilbert space, however we can work with the space of sections with any amount of weak differentiability and use elliptic regularity to conclude the result in the limit of smooth sections. In conclusion we obtain the following theorem.

**Theorem 2.2.7.** *We may decompose any $H \in T_J \mathcal{J}(\Sigma)$ as*

$$H = \mathcal{L}_X J + H^0,$$

*where $X$ is a vector field on $\Sigma$ and $H^0$ satisfies $\nabla^* H^0 = 0$.*

We call a smooth quadratic differential on $(\Sigma, J)$ holomorphic if it is a holomorphic section of $T^{*1,0}\Sigma \otimes T^{*1,0}\Sigma$. In local holomorphic coordinates, a quadratic differential may be written

$$\sigma = \sigma' dz \otimes dz,$$

for some function $\sigma'$. Then $\sigma$ is holomorphic if $\overline{\partial}\sigma' = 0$. We denote the vector space of all holomorphic differentials for a given complex structure $J$ by $Q(J)$.

**Lemma 2.2.8.** *Let $H \in T_J\mathcal{J}(\Sigma)$ and suppose*

$$H \in \ker(\nabla^*),$$

*then the corresponding quadratic differential $\sigma_H$ is holomorphic.*

*Proof.* Let $H \in \ker(\nabla^*)$ we have immediately that

$$-\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}H^i_j J^i_n\right) g^{mn} + \frac{1}{2}H^j_i \left(\frac{\partial}{\partial x^n} J^i_j\right) g^{mn} = 0.$$

Choosing local conformal coordinates in which $g = \mathrm{e}^{2\phi}\delta_{ij}$ and using the fact that $H$ is trace free, this simplifies to

$$\frac{\partial}{\partial x^j}\left(\mathrm{e}^{2\phi}H^j_i\right) = 0.$$

However, in these coordinates, the tangent vector $H \in T_J\mathcal{J}(\Sigma)$ given by

$$H = \begin{pmatrix} u & v \\ v & -u \end{pmatrix},$$

corresponds to the quadratic differential

$$\begin{aligned} \sigma_H &= -2i\rho\left(\frac{\partial}{\partial z}, H\frac{\partial}{\partial z}\right) dz \otimes dz \\ &= \mathrm{e}^{2\phi}(u - iv) dz \otimes dz. \end{aligned}$$

So the above equation reduces to

$$\begin{aligned} \frac{\partial}{\partial x}(\mathrm{e}^{2\phi}u) + \frac{\partial}{\partial y}(\mathrm{e}^{2\phi}v) =& 0 \\ \frac{\partial}{\partial x}(\mathrm{e}^{2\phi}v) - \frac{\partial}{\partial y}(\mathrm{e}^{2\phi}v) =& 0. \end{aligned}$$

But these are just the cauchy Riemann equations for the local function

$$\sigma' = \mathrm{e}^{2\phi}(u - iv),$$

so that $\sigma_H$ is a holomorphic quadratic differential as required. □

Since we may pair quadratic differentials with tangent vectors using the hermitian metric, we identify the dual of the space $\ker(\nabla^*)$ as the space $Q(J)$ of holomorphic quadratic differentials.

### 2.2.5 Teichmüller space

Fix a closed oriented surface $\Sigma$ and as in the previous section let $\mathcal{J}(\Sigma)$ be the set of complex structures compatible with this orientation. Let $\mathrm{Diff}_0(\Sigma)$ be the identity component of the diffeomorphism group of $\Sigma$. This acts on $\mathcal{J}(\Sigma)$ by pullback:

$$J \mapsto \phi^* J = \phi_*^{-1} \circ J \circ \phi_*.$$

The resulting quotient $\mathcal{J}(\Sigma)/\mathrm{Diff}_0(\Sigma)$ is called the *Teichmüller space of* $\Sigma$ and we denote it $\mathcal{T}(\Sigma)$. The following is well known, see for example [27]:

**Theorem 2.2.9.** *The Teichmüller space $\mathcal{T}(\Sigma)$ is a smooth manifold of dimension $-3\chi(\Sigma)$.*

We would like to do differential geometry on the Teichmüller space. If we let $\pi : \mathcal{J}(\Sigma) \to \mathcal{T}(\Sigma)$ denote the canonical projection map, then we can identify the tangent space at an equivalence class $[J] \in \mathcal{T}(\Sigma)$,

$$T_{[J]}\mathcal{T}(\Sigma) \cong \frac{T_J\mathcal{J}(\Sigma)}{\ker(\pi_*)}.$$

**Proposition 2.2.10.** *The cotangent space $[J]$ to $\mathcal{T}(\Sigma)$ may be identified with the space of holomorphic quadratic differentials $Q(J)$.*

*Proof.* From the previous section we know we can write any $H \in T_J\mathcal{J}(\Sigma)$ as

$$H = \mathcal{L}_X J + H^0,$$

where $\nabla^* H^0 = 0$. We also know that we can identify the dual of the subspace $\ker(\nabla^*)$ with the space of holomorphic quadratic differentials. Since it is obvious that the elements of $\ker(\pi_*)$ are precisely those that can be written as $\mathcal{L}_X J$ for some vector field $X$ it follows that

$$T^*_{[J]}\mathcal{T}(\Sigma) \cong Q(J).$$

□

The canonical complex structure on $\mathcal{J}(\Sigma)$ induces one on Teichmüller space.

**Theorem 2.2.11** ([27])**.** *The almost complex structure $\Phi$ on $\mathcal{J}(\Sigma)$ descends to Teichmüller space $\mathcal{T}(\Sigma)$. The resulting almost complex structure is integrable, thus Teichmüller space is a complex manifold.*

There is a canonical hermitian metric, the *Weil-Petersson* metric, on $\mathcal{T}(\Sigma)$. We shall detail the definition of the Weil-Petersson metric. We need the following result.

**Theorem 2.2.12** (Poincaré)**.** *Given a complex structure $J$ there exists a unique metric $g$ of constant Gauss curvature $-1$ in the conformal class of $J$.*

Let $[J]$ be a point of $\mathcal{T}(\Sigma)$, we know that $T^*_{[J]}\mathcal{T}(\Sigma)$ can be identified with the vector space of holomorphic quadratic differentials $Q(J)$. Let $g$ be the unique metric of Gauss curvature $-1$ in the conformal class of $J$ provided by theorem 2.2.12. This metric induces a volume form $\rho$, and in conjunction with $J$ we may then construct the hermitian inner product $h_\rho$ as in section 2.2.3.

Suppose that we had chosen a different representative $J'$ for $[J]$, then $J' = \phi^* J$ for some $\phi \in \mathrm{Diff}_0(\Sigma)$. The unique metric $g'$ of Gauss curvature $-1$ in $J'$ is just $\phi^* g$ so that

$$\int_\Sigma h'(\phi^*\sigma, \phi^*\tau)\rho' = \int_\Sigma h(\sigma, \tau)\rho,$$

and hence the pairing descends to Teichmüller space to give us a hermitian metric $h_{\mathrm{WP}}$.

The Weil-Petersson metric $h_{\mathrm{WP}}$ is the natural one on Teichmüller space with its canonical complex structure.

**Theorem 2.2.13** ([27])**.** *The Weil-Petersson metric is Kähler with respect to the canonical complex structure on Teichmüller space.*

# Chapter 3

In this chapter we discuss the existence of hyperkähler extensions of Kähler manifolds following Feix-Kaledin. We examine in detail the hyperkähler extension of the hyperbolic plane as well as presenting an identification of this with the product of two copies of the hyperbolic plane. We then move on to consider the hyperkähler extension of Teichmüller space and present its construction by Donaldson.

## 3.1 Hyperkähler extensions

Let $N$ be a Kähler manifold. In this section we will present results that show there exists a canonical hyperkähler extension of $N$. In addition we exhibit a particular example, the extension of the hyperbolic plane. A construction of the author then identifies this hyperkähler extension of the hyperbolic plane with the hyperbolic plane crossed with the hyperbolic plane with reversed orientation.

### 3.1.1 Results of Feix and Kaledin

In the papers [16], [8] of Feix and Kaledin hyperkähler metrics are constructed on neighbourhoods of the zero section in the cotangent bundles of real analytic Kähler manifolds. We have the following theorem:

**Theorem 3.1.1** (Feix)**.** *Let $N$ be a real analytic Kähler manifold. Then there exists a neighbourhood, $N^c$, of the zero section of the cotangent bundle on which there is a hyperkähler metric $(g, I, J, K)$. This metric is compatible with the canonical holomorphic-symplectic structure $\omega_{\mathbb{C}}$ on $T^{*1,0}N$ in the* *sense that*

$$\omega_{\mathbb{C}} = \omega_2 + i\omega_3.$$

*Furthermore, the* $\mathrm{S}^1$*-action given by scalar multiplication in the fibres is isometric and the restriction of the hyperkähler metric to the zero section induces the original Kähler metric. This hyperkähler metric is unique.*

### 3.1.2 The circle action

Let $N$ be a Kähler manifold, and $N^c$ the hyperkähler thickening inside its cotangent bundle provided by the Feix theorem. We know we have a circle action on $N^c$ induced by scalar multiplication in the fibres of the cotangent bundle. This action is isometric on the fibres and $N^c$ contains $N$ as the fixed submanifold of this action. Let $X$ be a vector field generated by this action.

**Lemma 3.1.2** ([13])**.** *The circle action fixes* $\omega_1$ *and rotates the* $\omega_2, \omega_3$ *plane. Therefore*

$$\mathcal{L}_X\omega_1 = 0,\ \mathcal{L}_X\omega_2 = \omega_3,\ \mathcal{L}_X\omega_3 = -\omega_2.$$

Suppose $H^1(N;\mathbb{R}) = 0$, since $\mathcal{L}_X\omega_1 = 0$ we have a Hamiltonian function $H : N^c \to \mathbb{R}$ vanishing along the fixed set $N \subset N^c$.

**Proposition 3.1.3** ([13])**.** *The Hamiltonian function for the circle action with respect to the symplectic form* $\omega_1$ *provides a Kähler potential for* $\omega_2$*,*

$$\omega_2 = -2i\partial_J\overline{\partial}_J H.$$

*Proof.* Let $Y$ be a tangent vector to $N^c$ then,

$$(\iota(X)\omega_1)(JY) = dH(JY) = (\partial_J H + \overline{\partial}_J H)(JY) = i(\partial_J H - \overline{\partial}_J H)(Y).$$

On the other hand

$$\begin{aligned}
(\iota(X)\omega_1)(JY) &= \omega_1(X, JY) \\
&= g(IX, JY) \\
&= g(KX, Y) \\
&= \omega_3(X, Y) = (\iota(X)\omega_3)(Y).
\end{aligned}$$

Therefore

$$i(\partial_J H - \overline{\partial}_J H) = \iota(X)\omega_3,$$

and so by taking the exterior derivative of both sides:

$$\mathcal{L}_X\omega_3 = d(\iota(X)\omega_3) = -2i\partial_J\overline{\partial}_J H.$$

But by the previous lemma this is just $-\omega_2$ implying the desired result. $\square$

### 3.1.3 Hyperkähler extension of Hyperbolic space

A basic example of this idea is provided by the Hyperkähler extension of $H^2$, with its metric of constant curvature $-1$, inside $T^*H^2$. Explicitly, following Feix [8], we take coordinates $z = x + iy$ on the upper half plane model of $H^2$. We then have natural coordinates $(u, v)$ on $T^*_{(x,y)}H^2$ for the 1-form $udx + vdy$. We consider the unit disc sub-bundle (with respect to the hyperbolic metric) $D$ inside $T^*H^2$ given in these coordinates by

$$\{(x, y, u, v)|z = x + iy \in H^2,\ y(u^2 + v^2)^{\frac{1}{2}} < 1\}.$$

We set $\gamma := (1 - y^2(u^2 + v^2))^{\frac{1}{2}}$, then the Hyperkähler metric is given by:

$$g = \frac{\gamma}{y^2}(dx^2 + dy^2) + \frac{1}{\gamma y^2}((yvdx - uydy - y^2du)^2 + (yudx + yvdy + y^2dv)^2).$$

We have one obvious complex structure, $I$, induced by the standard complex structure on $H^2$, that acts on $D$:

$$I := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The other two, which we shall denote by $J$ and $K$, come from taking as symplectic forms the real and imaginary parts of the canonical holomorphic symplectic structure on $T^*H^2$. In these coordinates this symplectic form is

$$\omega_{\mathbb{C}} = \omega_2 + i\omega_3 = (dx + idy) \wedge (du - idv).$$

Which leads to

$$J := \begin{pmatrix} \frac{yv}{\gamma} & -\frac{yu}{\gamma} & -\frac{y^2}{\gamma} & 0 \\ -\frac{yu}{\gamma} & -\frac{yv}{\gamma} & 0 & -\frac{y^2}{\gamma} \\ \frac{1}{y^2\gamma} & 0 & -\frac{yv}{\gamma} & \frac{yu}{\gamma} \\ 0 & \frac{1}{y^2\gamma} & \frac{yu}{\gamma} & \frac{yv}{\gamma} \end{pmatrix}.$$

One can verify by direct calculation that the $K = IJ$ and the quaternionic identities are satisfied. The complex structures are $\mathrm{SL}_2\mathbb{R}$ equivariant by construction.

We now construct the Kähler potential for the symplectic form $\omega_2$ by finding the hamiltonian with respect to $\omega_1$ for the circle action given by multiplication by $i$ on $T^*H^2$. Consider the action of $\mathrm{e}^{i\theta}$ on $D$.

$$\mathrm{e}^{i\theta} : (x, y, u, v) \mapsto (x, y, u\cos\theta + v\sin\theta, -u\sin\theta + v\cos\theta).$$

This therefore generates the vector field:

$$\begin{aligned} X_\theta &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{e}^{it\theta}(x, y, u, v) \\ &= \theta(0, 0, v, -u) \\ &= \theta\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\right). \end{aligned}$$

Then the exterior derivative of the Hamiltonian function $H_\theta$ is by definition the contraction of $X_\theta$ into the symplectic form $\omega_1$ so,

$$\begin{aligned} dH_\theta &= \iota(X_\theta)\omega_1 \\ &= \iota(IX_\theta)g \\ &= -\theta\iota\left(v\frac{\partial}{\partial v} - u\frac{\partial}{\partial u}\right) \\ &= -\frac{\theta}{\gamma}(yu^2dy + yv^2dy + y^2udu + y^2vdv) \\ &= \theta d\gamma, \end{aligned}$$

using the above expressions for the metric $g$ and the definition of $\gamma$. There, and with the constraint that $H_\theta$ vanishes on the fixed set of the circle action, we have that

$$H_\theta = \theta(\gamma - 1).$$

Observe from the definition of $\gamma$ that this is just a function of the norm in the $T^*H^2$ fibres.

### 3.1.4 A holomorphic map from $D$ to $H^2 \times \overline{H^2}$

We construct a map from $D$ to the product $H^2 \times \overline{H^2}$, where $\overline{H^2}$ is the hyperbolic plane with the complex structure $-J$. It turns out that this map is holomorphic with respect to the complex structure $J$ on $D$ and the natural complex structure $\tilde{I} := J \oplus (-J)$ on $H^2 \times \overline{H^2}$.

**Lemma 3.1.4.** *There exists a map $\alpha : D \to H^2 \times \overline{H^2}$ which is an* $\mathrm{SL}_2\mathbb{R}$ *equivariant holomorphic bijection with respect to the complex structure $J$ on $D$ and the natural complex structure $\tilde{I}$ on $H^2 \times \overline{H^2}$.*

*Proof.* Consider the map $\alpha : D \to H^2 \times \overline{H^2}$ defined by

$$\alpha : (z, w) \mapsto (\exp_z(Ifw)_\flat, \exp_z(-Ifw)_\flat),$$

here $\exp_z : T_pH^2 \to H^2$ is the exponential map at $z$, $I$ is the natural complex structure on the cotangent space,

$$f = f(\|w\|) = \frac{1}{\|w\|}\operatorname{arc\,tanh}(-\|w\|),$$

is a real function of $\|w\|$, the length of $w$ in the hyperbolic metric, and we represent by $\flat$ the natural identification $T^*H^2 \to TH^2$ induced by the metric. The $\mathrm{SL}_2\mathbb{R}$ equivariance then follows immediately since the geodesic through $z \in H^2$ with initial tangent vector $(Ifw)_\flat$ is clearly mapped by any $g \in \mathrm{SL}_2\mathbb{R}$ to the geodesic through $g(z) \in H^2$ with initial tangent vector $g_*((Ifw)_\flat)$ since $g$ is an isometry for the hyperbolic metric.

With respect to the coordinates $(x, y, u, v)$ on $D$ considered above the map $\alpha$ is given by

$$\alpha(x, y, u, v) = \left(x - \frac{y^2v}{1 - yu}, \frac{\gamma y}{1 - yu}, x + \frac{y^2v}{1 + yu}, \frac{\gamma y}{1 + yu}\right),$$

where as before $\gamma^2 = 1 - y^2u^2 - y^2v^2 = 1 - \|w\|^2$.

For $\alpha$ to be $J$-holomorphic we must have

$$d\alpha \circ J = \tilde{I} \circ d\alpha.$$

We verify this by direct calculation, which we simplify using the equivariance of $\alpha$ and the fact that $\mathrm{SL}_2\mathbb{R}$ acts transitively on cotangent vectors of fixed length to reduce to checking it is true at $(0,1,u,0)$. Here we see that $d\alpha$, $J$ and $\tilde{I}$ are given as follows:

$$J := \begin{pmatrix} 0 & -\frac{u}{\gamma} & -\frac{1}{\gamma} & 0 \\ -\frac{u}{\gamma} & 0 & 0 & -\frac{1}{\gamma} \\ \frac{1}{\gamma} & 0 & 0 & \frac{u}{\gamma} \\ 0 & \frac{1}{\gamma} & \frac{u}{\gamma} & 0 \end{pmatrix},$$

$$d\alpha := \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \frac{1+u-u^2}{\gamma(1-u)} & \frac{1}{\gamma(1-u)} & 0 \\ 1 & 0 & 0 & \frac{1}{1+u} \\ 0 & \frac{1-u-u^2}{\gamma(1+u)} & \frac{1}{\gamma(1+u)} & 0 \end{pmatrix},$$

$$\tilde{I} := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

The result follows.

We would like to show that $\alpha$ is injective. Since the map is $\mathrm{SL}_2\mathbb{R}$ equivariant we may assume $\alpha(0,1,u_1,0) = \alpha(x,y,u_2,v_2)$ for some points $(0,1,u_1,0),(x,y,u_2,v_2) \in D$. Let $\gamma_2^2 = 1 - y^2u_2^2 - y^2v_2^2$, then we must have that:

$$\left(0, \sqrt{\frac{1+u_1}{1-u_1}}, 0, \sqrt{\frac{1-u_1}{1+u_1}}\right) = \left(x - \frac{y^2v_2}{1-yu_2}, \frac{\gamma_2 y}{1-yu_2}, x + \frac{y^2v_2}{1+yu_2}, \frac{\gamma_2 y}{1+yu_2}\right).$$

It follows that $(x,y,u_2,v_2) = (0,1,u_1,0)$ and the map is injective as desired.

To prove the surjectivity it is enough, due to the $\mathrm{SL}_2\mathbb{R}$ equivariance, to show that the map surjects onto $\{i\} \times \overline{H^2} \subset H^2 \times \overline{H^2}$. One may check that the point $(i, \tilde{x} + i\tilde{y}) \in \{i\} \times \overline{H^2}$ is the image under $\alpha$ of $(x,y,u,v)$ for:

$$\begin{aligned} x &= \tilde{x}(\tilde{y}+1)^{-1}, \\ y &= (1+\tilde{y})^{-1}\tilde{y}^{\frac{1}{2}}(\tilde{x}^2 + (1+\tilde{y})^2)^{\frac{1}{2}}, \\ u &= (1-\tilde{y})\tilde{y}^{-\frac{1}{2}}(\tilde{x}^2 + (1+\tilde{y})^2)^{-\frac{1}{2}}, \\ v &= 2\tilde{x}(\tilde{x}^2 + (1+\tilde{y})^2)^{-1}. \end{aligned}$$

□

## 3.2 Hyperkähler extension of Teichmüller space

In this section we present Donaldson's construction of the Feix-Kaledin hyperkähler extension of Teichmüller space. This material appears in the paper [5].

We fix a closed surface $\Sigma$ with a volume form $\rho$. Let $P$ be the principal $\mathrm{SL}_2\mathbb{R}$ bundle of volume one frames over $\Sigma$. Since $\mathrm{SL}_2\mathbb{R}$ acts naturally on $H^2$ it acts naturally on $T^*H^2$ and hence on $D$. Therefore we may form the associated bundle $\underline{D}$ over $\Sigma$ with fibres $D$. Let $\Gamma(\underline{D})$ denote the set of smooth sections of $\underline{D}$. Let $s \in \Gamma(\underline{D})$. At each point $p \in \Sigma$ we get a point $s(p)$ in the fibre $\underline{D}_p \cong D \subset T^*H^2$. By the results of chapter two we may therefore identify a section $s$ with the pair $(J, \sigma)$ where $J \in \mathcal{J}(\Sigma)$ and $\sigma$ is a smooth quadratic differential for the complex structure $J$.

Let $T_V\underline{D}$ be the vertical subbundle of the tangent bundle $T\underline{D}$, that is, the kernel of the canonical projection $\pi : T\underline{D} \to \Sigma$.

**Definition 3.2.1.** We define the tangent space at a section $s$ to $\Gamma(\underline{D})$ to be the space of sections of $s^*T_V\underline{D}$.

To see that this is reasonable let $s_t : \Sigma \to \underline{D}$ for $t \in (a, b)$ some interval in $\mathbb{R}$, be a family of sections of $\underline{D}$. For each point $p \in \Sigma$ we obtain a map $s_t(p) : (a, b) \to \underline{D}$ and moreover the image of this map lies entirely in the fibre over the point $p$. Therefore $s_t(p)$ defines a tangent vector $v(p)$ at $s_0(p)$ to $\underline{D}$ that lies in $T_V\underline{D}$ at $s_0(p)$, i.e. is tangent to the fibre of the bundle over $p$. In addition we have that the projection of $v(p)$ to $\underline{D}$ is $s_0(p)$ and so $(p, v(p))$ lies in the pullback bundle $s^*T_V\underline{D}$. Since we obtain such a tangent vector for each $p \in \Sigma$ we see that tangent vectors at $s \in \Gamma(\underline{D})$ may certainly be identified as sections of this bundle.

### 3.2.1 The hyperkähler structure on $\Gamma(\underline{D})$

We claim there is a natural hyperkähler structure on $\Gamma(\underline{D})$ induced by that on $D$. For each of the symplectic forms $\omega_i$ of the hyperkähler structure on $D$ we define a two-form $\Omega_i$ on $\Gamma(\underline{D})$ as follows. Let $\xi, \eta \in T_s\Gamma(\underline{D})$ be two tangent vectors at $s$, the symplectic form $\omega_i$ on $D$ induces a pairing between these vectors so we may define

$$\Omega_i(\xi, \eta) = \int_\Sigma \omega_i(\xi, \eta)\rho.$$

**Lemma 3.2.2.** *For each* $i \in \{1, 2, 3\}$*,* $\Omega_i$ *defines a (formal) symplectic structure on* $\Gamma(\underline{D})$*.*

*Proof.* We need to show that each $\Omega_i$ is closed and non-degenerate. We deal first with closure. Let $\xi, \eta, \tau \in T_s\Gamma(\underline{D})$, then we have that

$$\begin{aligned} d\Omega_i(\xi, \eta, \tau) = \xi\Omega(\eta, \tau) - \eta\Omega(\xi, \tau) &+ \tau\Omega(\xi, \eta) \\ &- \Omega([\xi, \eta], \tau) + \Omega([\xi, \tau], \eta) - \Omega([\eta, \tau], \xi). \end{aligned}$$

Therefore

$$\begin{aligned} d\Omega_i(\xi, \eta, \tau) =& \xi \int_\Sigma \omega_i(\eta, \tau)\rho + \eta \int_\Sigma \omega_i(\xi, \tau)\rho + \tau \int_\Sigma \omega_i(\xi, \eta)\rho \\ &- \int_\Sigma \omega_i([\xi, \eta], \tau)\rho + \int_\Sigma \omega_i([\xi, \tau], \eta)\rho - \int_\Sigma \omega_i([\eta, \tau], \xi)\rho. \end{aligned}$$

Since

$$\tau \int_\Sigma \omega_i(\xi, \eta)\rho = \int_\Sigma \tau\omega_i(\xi, \eta)\rho,$$

we have

$$\begin{aligned} d\Omega_i(\xi, \eta, \tau) = \int_\Sigma &\{\xi\omega_i(\eta, \tau) + \eta\omega_i(\xi, \tau) + \tau\omega_i(\xi, \eta) \\ &- \omega_i([\xi, \eta], \tau) + \omega_i([\xi, \tau], \eta) - \omega_i([\eta, \tau], \xi)\}\rho. \end{aligned}$$

Therefore

$$d\Omega_i(\xi, \eta, \tau) = \int_\Sigma d\omega_i(\xi, \eta, \tau)\rho = 0,$$

since the closure of each $\omega_i$ implies the integrand is zero.

Now let us suppose that for $\xi \in T_s\Gamma(\underline{D})$ we have $\Omega_i(\xi, \eta) = 0$ for every $\eta$. Recalling that we have a complex structure compatible with the symplectic structure $\omega_i$ on each fibre of the bundle $\underline{D}$ we can see this would imply that

$$\int_\Sigma g(\xi, \xi)\rho = 0,$$

and hence that $\xi$ is zero. We conclude that each $\Omega_i$ is non-degenerate and that each defines a symplectic structure on $\Gamma(\underline{D})$. □

The three complex structures $I, J, K$ on $D$ induce three complex structures on $\Gamma(\underline{D})$, we shall also denote these $I, J, K$. In conjunction with the three symplectic forms $\Omega_i$ we then obtain a formal hyperkähler structure on $\Gamma(\underline{D})$, the metric being given by

$$\langle \xi, \eta \rangle = \int_\Sigma g(\xi, \eta)\rho.$$

### 3.2.2 The hyperkähler quotient of $\Gamma(\underline{D})$

We now proceed to describe a certain hyperkähler quotient of $\Gamma(\underline{D})$ as constructed by Donaldson in [5].

**Lemma 3.2.3.** *The group* $\mathrm{Ham}(\Sigma, \rho)$ *of Hamiltonian symplectomorphisms of* $(\Sigma, \rho)$ *acts on* $\Gamma(\underline{D})$ *preserving the hyperkähler structure.*

*Proof.* Firstly we require that the action of $\mathrm{Ham}(\Sigma, \rho)$ preserves the three $\Omega_i$. Let $\phi \in \mathrm{Ham}(\Sigma, \rho)$ and $\xi, \eta \in T_s\Gamma(\underline{D})$. We have that

$$\begin{aligned}
\left.(\phi^*\Omega_i)\right|_s(\xi, \eta) &= \left.\Omega_i\right|_{\phi(s)}(\phi_*\xi, \phi_*\eta) \\
&= \int_\Sigma \omega_i(\phi_*\xi, \phi_*\eta)\rho \\
&= \int_\Sigma \phi^*(\omega_i(\xi, \eta)\rho) \\
&= \int_\Sigma \omega_i(\xi, \eta)\rho = \left.\Omega_i\right|_s(\xi, \eta).
\end{aligned}$$

Here we have used the fact that $\phi^*\rho = \rho$ to go from line two to line three. Therefore the symplectic forms are preserved. Similarly the metric is preserved, implying the result. □

We might now like to find moment maps for the action of $\mathrm{Ham}(\Sigma, \rho)$ and hope that the natural quotient inherits a hyperkähler structure from that on $\Gamma(\underline{D})$. However, the spaces under consideration are not finite dimensional so the results on hyperkähler reduction from the previous chapter cannot be applied directly. None the less we have the following result of Donaldson.

**Theorem 3.2.4** (Donaldson [5])**.** *There exist three moment maps for the action of* $\mathrm{Ham}(\Sigma, \rho)$ *on* $\Gamma(\underline{D})$. *The quotient*

$$\Gamma(\underline{D})//\mathrm{Ham}(\Sigma, \rho)$$

*exists and has the induced hyperkähler structure.*

We write $(g, \omega_1, \omega_2, \omega_3)$ for the hyperkähler structure induced on the quotient. Formally it is constructed in exactly the same way as in the finite dimensional case, we now illustrate this. Assume that the quotient exists as a manifold. Denote the zero set of the three moment maps by $\underline{\mu}^{-1}(0)$, we assume that this is a submanifold of $\Gamma(\underline{D})$. The quotient is

$$\underline{\mu}^{-1}(0)/\mathrm{Ham}(\Sigma, \rho).$$

To define, for example, any of the symplectic structures $\omega_i$ let

$$X, Y \in T_{[p]}\left(\underline{\mu}^{-1}(0)/\mathrm{Ham}(\Sigma, \rho)\right),$$

we lift these to any tangent vectors to $\underline{\mu}^{-1}(0)$ at $p$. In view of our definition of the tangent space to $\Gamma(\underline{D})$ as the sections of the pullback of the vertical bundle we see that tangent vectors to $\underline{\mu}^{-1}(0)$ must be some subset of this vector space of sections. We have a symplectic structure $\Omega_i$ on $T_p\Gamma(\underline{D})$ and we use this to evaluate the symplectic product of the lifted tangent vectors in $T_p\underline{\mu}^{-1}(0)$. By the definition of moment map, the tangent vectors to the orbit of $p$ evaluate to zero against those tangent to $\underline{\mu}^{-1}(0)$, this allows us to define a closed skew form $\omega_i$ on the quotient. This form will be non-degenerate provided the symplectic complement of $T_p\underline{\mu}^{-1}(0)$ and the tangent space to the orbit coincide, assuming this is the case we have the desired symplectic structure. The existence of the Kähler structure for each $\omega_i$ also proceeds formally as in the finite dimensional case.

In fact Donaldson shows that we may define a hyperkähler structure on the quotient of a subset $\mathcal{A}$ of $\Gamma(\underline{D})$ by the group $\mathrm{Symp}_0(\Sigma, \rho)$ which is the identity component of the group of symplectomorphisms of $(\Sigma, \rho)$. The set $\mathcal{A}$ is essentially the zero set of the moment map of theorem 3.2.4. This is the quotient by a larger set of diffeomorphisms of $\Sigma$, the hyperkähler structure is induced by that on the hyperkähler quotient $\Gamma(\underline{D})///\mathrm{Ham}(\Sigma, \rho)$ above.

In order to describe this set we fix conventions. Let us assume that we have scaled $\rho$ so that

$$2\int_\Sigma \rho = 2\pi\chi(\Sigma).$$

**Definition 3.2.5.** Let $\mathcal{A}$ be the set of pairs $(J, \sigma)$ such that $\sigma$ is a holomorphic quadratic differential, $|\sigma|_g < 1$ and

$$K_g + \frac{1}{2}\Delta \log(1 + \sqrt{1 - |\sigma|_g^2}) = -2.$$

Here $K_g$ is the Gaussian curvature of the metric induced by $J$ and $\rho$, and $|\sigma|_g$ denotes the norm of $\sigma$ in this metric.

**Theorem 3.2.6** (Donaldson [5])**.** *The quotient $\mathcal{N} := \mathcal{A}/\mathrm{Symp}_0(\Sigma)$ has a hyperkähler structure.*

### 3.2.3 The moduli space $\mathcal{M}$

In order to interpret the moduli space $\mathcal{N}$ described in the previous section, we follow Donaldson in [5] and recast it in terms more amenable to geometric analysis.

**Definition 3.2.7.** Let $\mathcal{B}$ be the set of pairs $(g, \sigma)$ such that $g$ is a metric on $\Sigma$, $\sigma$ is a holomorphic quadratic differential with $|\sigma|_g < 1$ and

$$K_g + |\sigma|_g^2 = -1.$$

Define $\mathcal{M} := \mathcal{B}/\mathrm{Diff}_0(\Sigma)$.

**Theorem 3.2.8.** *There is a bijection $\mathcal{N} \to \mathcal{M}$. This bijection is induced by the map $\mathcal{A} \to \mathcal{B}$ defined by*

$$(J, \sigma) \mapsto \left((1 + \sqrt{1 - |\sigma|_g^2})g, \sigma\right).$$

The proof of this falls into two parts, one algebraic in nature and the other a piece of differential geometry. Firstly, let $\mathcal{A}'$ denote the set of pairs $(g, \sigma)$ where $g$ is a metric on $\Sigma$ and $\sigma$ is a holomorphic quadratic differential with respect to the complex structure defined by the conformal class of $g$ and $(g, \sigma)$ satisfy the equation

$$K_g + \frac{1}{2}\Delta \log(1 + \sqrt{(1 - |\sigma|_g^2)}) = -2.$$

Define the space

$$\mathcal{N}' := \mathcal{A}'/\mathrm{Diff}_0(\Sigma).$$

We prove that the obvious injection induces a formal diffeomorphism $\mathcal{N} \to \mathcal{N}'$. Secondly we define a diffeomorphism from $\mathcal{N}'$ to the space $\mathcal{M}$, this is induced by the map $\mathcal{A}' \to \mathcal{B}$ defined by

$$(J, \sigma) \mapsto \left((1 + \sqrt{1 - |\sigma|_g^2})g, \sigma\right).$$

We address the first point. We require some preliminary results from symplectic geometry.

**Theorem 3.2.9** (Moser's stability theorem [20])**.** *Let $M$ be a closed manifold and suppose $\omega_t$ is a smooth family of cohomologous symplectic forms on $M$. Then there is a family of diffeomorphisms $\psi_t$ of $M$ such that*

$$\psi_0 = \mathrm{id}, \qquad \psi_t^* \omega_t = \omega_0.$$

This is the principal tool in the proof of the first part of theorem 3.2.8, and also in the proof of the following technical result.

**Proposition 3.2.10.** *Suppose we have a symplectomorphism, $\phi$, of $(\Sigma, \omega)$. Then if $\phi$ is isotopic to the identity through diffeomorphisms it is isotopic to the identity through symplectomorphisms. That is,*

$$\mathrm{Symp}_0(\Sigma, \omega) = \mathrm{Diff}_0(\Sigma) \cap \mathrm{Symp}(\Sigma, \omega).$$

*Proof.* Let $\phi$ be as in the statement of the theorem. Hence there exists a family, $\phi_t$, of diffeomorphisms connecting $\phi$ to the identity, say $\phi_0 = \mathrm{id}$, $\phi_1 = \phi$. This generates a loop in the space of symplectic forms on $\Sigma$ by the prescription $\omega_t := \phi_t^* \omega$. On a surface the space of symplectic forms compatible with a fixed orientation is convex and we can construct an explicit homopy between the loop $\omega_t$ and the form $\omega$ by

$$H(s,t) = (1-s)\omega + s\omega_t. \qquad s,t \in [0,1].$$

Then for each fixed $t$, $H_s := H(s,t)$ is a cohomologous family of symplectic forms, and hence by Moser's theorem there exists some family $\psi_s^t$ of diffeomorphisms such that $\psi_0^t = \mathrm{id}$ and $(\psi_s^t)^* H_s = H_0$. In particular

$$(\psi_1^t)^* H_1 = (\psi_1^t)^* \omega_t = \omega.$$

Now consider the family of diffeomorphisms $\theta_t$ defined by

$$\theta_t = \phi_t \circ \psi_1^t.$$

We see that $\theta_0 = \mathrm{id}$, $\theta_1 = \phi$ and that

$$\theta_t^* \omega = (\phi_t \circ \psi_1^t)^* \omega = (\psi_1^t)^* \omega_t = \omega.$$

Therefore the family $\theta_t$ provides a family of symplectomorphisms connecting $\phi$ to the identity and the result is proved. □

**Proposition 3.2.11.** *Let $(\hat{\rho}, J, \sigma)$ be a triple where $\hat{\rho}$ is a volume form on $\Sigma$, $J$ is a complex structure on $\Sigma$ and $\sigma$ is a holomorphic quadratic differential with respect to $J$. Suppose $(\hat{\rho}, J, \sigma)$ satisfy:*

$$K_{\hat{g}} + \frac{1}{2}\Delta \log(1 + \sqrt{1 - |\sigma|^2_{\hat{g}}}) = -2,$$

*where the metric $\hat{g}$ is induced by $\hat{\rho}$ and $J$. Then there exists $\phi \in \mathrm{Diff}_0(\Sigma)$ such that the pair $(\phi^* J, \phi^* \sigma)$ satisfies*

$$K_g + \frac{1}{2}\Delta \log(1 + \sqrt{1 - |\phi^*\sigma|^2_g}) = -2,$$

*where the metric $g$ is now that induced by $\phi^* J$ and our previously fixed volume form $\rho$. Further this diffeomorphism is unique up to an element of* $\mathrm{Symp}_0(\Sigma, \rho)$.

*Proof.* Since we are on a surface we know that $\mathrm{H}^2_{dR}(\Sigma, \mathbb{R}) = \mathbb{R}$, hence for some $a \in \mathbb{R}$ and $\tau \in \Omega^1(\Sigma)$,

$$\rho - \hat{\rho} = a\rho + d\tau.$$

Integrating this expression over $\Sigma$ we see that $a = 0$. Define:

$$\rho_t := \hat{\rho} + (1 - t)d\tau$$

For each $t$ these forms are pointwise non-degenerate because the second exterior power of the cotangent bundle has rank one. Therefore we have a smooth family of cohomologous symplectic forms. Using Moser's theorem we can conclude that there exists a family of diffeomorphisms $\phi_t : \Sigma \to \Sigma$ such that $\phi_0 = id$ and

$$\phi_t^* \rho_t = \hat{\rho} \quad \forall t \in [0, 1].$$

Moreover it is clear that the pair $(\phi_1^* J, \sigma)$ satisfies the desired equation.

For the partial uniqueness suppose there were two diffeomorphisms $\phi$, $\psi$ such that

$$(\phi^{-1})^* \rho = \hat{\rho} = (\psi^{-1})^* \rho,$$

then

$$\psi^* (\phi^{-1})^* \rho = \rho,$$

which implies the two diffeomorphisms differ by a symplectomorphism of $(\Sigma, \rho)$. We see that this is an element of $\mathrm{Symp}_0(\Sigma, \rho)$ by appealing to proposition 3.2.10. $\square$

**Corollary 3.2.12.** *The canonical injection $\mathcal{A} \to \mathcal{A}'$ induces a diffeomorphism*

$$\mathcal{N} \cong \mathcal{N}'.$$

*Proof.* The map $(J, \sigma) \mapsto (g_J, \sigma)$, where we write $g_J$ for the metric induced by $\rho$ and $J$, descends to a well defined map

$$\mathcal{A}/\mathrm{Symp}_0(\Sigma, \rho) \to \mathcal{A}'/\mathrm{Diff}_0(\Sigma).$$

Now let $(g, \sigma) \in \mathcal{A}'$, proposition 3.2.11 tells us precisely that there exists a $\phi \in \mathrm{Diff}_0(\Sigma)$ such that the pair $(\phi^* J, \phi^* \sigma) \in \mathcal{A}$. Hence the canonical map is a surjection. Since we know that the diffeomorphism $\phi$ is unique upto an element of $\mathrm{Symp}_0(\Sigma, \rho)$ we obtain that the map is also injective and the result is proved. □

This completes the necessary work for the first part of the proof of theorem 3.2.8. We turn our attention to the second part which is a calculation contained in [5].

**Lemma 3.2.13** ([5])**.** *The map*

$$(g, \sigma) \mapsto \left((1 + \sqrt{1 - |\sigma|_g^2})g, \sigma\right),$$

*is a diffeomorphism $\mathcal{A}' \to \mathcal{B}$.*

*Proof.* To see this, let $(g, \sigma) \in \mathcal{A}'$ and define

$$F := 1 + \sqrt{1 - |\sigma|_g^2}.$$

Write $\hat{g} = Fg$ so the map is given by $(g, \sigma) \mapsto (\hat{g}, \sigma)$. We can calculate the Gauss curvature of the metric $\hat{g}$ to be

$$K_{\hat{g}} = \frac{1}{2F}\Delta_g \log F + \frac{1}{F} K_g.$$

But since $(g, \sigma) \in \mathcal{A}'$ this reduces to

$$K_{\hat{g}} = -\frac{2}{F}.$$

We know from its definition that

$$(F - 1)^2 = 1 - |\sigma|_g^2,$$

so that

$$F^2 = 2F - |\sigma|_g^2.$$

We then have the following

$$\begin{aligned} K_{\hat{g}} + |\sigma|_{\hat{g}}^2 &= -\frac{2}{F} + |\sigma|_{\hat{g}}^2 \\ &= -\frac{2}{F} + \frac{|\sigma|_g^2}{F^2} \\ &= -1, \end{aligned}$$

so that the pair $(\hat{g}, \sigma) \in \mathcal{B}$. The map is a bijection because $|\sigma|_g < 1$ and the map

$$x \mapsto \frac{x}{1 + \sqrt{1 - x^2}},$$

is a bijection from $[0, 1)$ to itself. □

We return to the proof of theorem 3.2.8.

*proof of theorem 3.2.8.* This is now just a question of fitting the above pieces together. We know from corollary 3.2.12 that $\mathcal{N} \cong \mathcal{N}'$, then lemma 3.2.13 induces a diffeomorphism $\mathcal{N}' \to \mathcal{M}$ proving the result. The above proofs exhibit that the map $\mathcal{A} \to \mathcal{B}$ is of the stated form. □

We may summarise the various relationships between the spaces in the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\iota} & \mathcal{A}' & \xrightarrow{\cong} & \mathcal{B} \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_2}\downarrow & & {\scriptstyle \pi_3}\downarrow \\ \mathcal{N} & \xrightarrow{\cong} & \mathcal{N}' & \xrightarrow{\cong} & \mathcal{M} \end{array}$$

We therefore have a hyperkähler structure on the space $\mathcal{M}$ induced by that on $\mathcal{N}$ and the identifications above. We write $(h_{\mathrm{HK}}, I, J, K)$ for the hyperkähler quadruple on $\mathcal{M}$.

### 3.2.4 Teichmüller space and $\mathcal{M}$

Consider the space $T^*\mathcal{J}(\Sigma)$, it follows from the work of the previous chapter that there is a canonical map

$$\mathcal{B} \to T^*\mathcal{J}(\Sigma),$$

given by taking the pair $(g, \sigma)$ to the pair $(J, \sigma)$ where $J$ is the conformal class generated by $J$. It will turn out from work of the next chapter that this map is in fact an injection. However for now we can only say the following:

**Proposition 3.2.14.** *We can identify $\mathcal{J}(\Sigma)$ with the subset of $\mathcal{B}$ defined by $\{(g, 0) \in \mathcal{B}\}$. This descends to the quotient to define a canonical injection:*

$$\iota : \mathcal{T}(\Sigma) \hookrightarrow \mathcal{M}.$$

*Proof.* Let $J \in \mathcal{J}(\Sigma)$, by Poincarés theorem, there exists a unique metric $g$ in the conformal class of $J$ with Gauss curvature $-1$. This allows us to define a map $\mathcal{J}(\Sigma) \to \mathcal{B}$ by $J \mapsto (g, 0)$. This map is surjective since we may take $(g, 0)$ to the conformal class $J$ defined by $g$. The descent to the quotient is obvious. □

Next we obtain a result about the restriction of the hyperkähler metric to the subset defined by the Teichmüller space as above.

**Proposition 3.2.15.** *The restriction of the hyperkähler metric on $\mathcal{M}$ to Teichmüller space is the Weil-Petersson metric $h_{\mathrm{WP}}$:*

$$\iota^* h_{\mathrm{HK}} = h_{\mathrm{WP}}.$$

*Proof.* Given a class in $\mathcal{M}$ there exists a unique class $[(J, 0)] \in \mathcal{N}$ such that the image of $[(J, 0)]$ is our chosen point in $\mathcal{M}$. We know from the above that we can choose to represent the class as $[(g, 0)] \in \mathcal{M}$ where the metric $g$ induces our fixed volume form $\rho$. Given a pair of (co)tangent vectors $\xi, \eta \in T_{[J,0]}\mathcal{N}$ we know that the metric is given by:

$$\langle \xi, \eta \rangle = \int_\Sigma \tilde{h}(\xi, \eta)\rho,$$

where now $\tilde{h}$ is the metric on the vertical tangent bundle of the fibre bundle $\underline{D}$ induced by the hyperkähler metric $h$ on $D \subset T^*H^2$. But we are restricting to tangent vectors to $\mathcal{T}(\Sigma) \subset \mathcal{M}$ and since we are in $\mathcal{N}$ we must have $g$ has constant curvature $-2$ so the metric is just (up to scale) the Weil-Petersson metric as required. □

# Chapter 4

In this chapter we consider the defining equations for the moduli space $\mathcal{M}$ constructed in the previous chapter. This requires analysis of a certain non-linear partial differential equation. We obtain results about the existence and uniqueness of solutions to this partial differential equation. This enables us to conclude that the moduli space $\mathcal{M}$ is embedded in the cotangent bundle of Teichmüller space and to determine certain explicit subsets of $\mathcal{M}$ in terms of the cotangent bundle $T^*\mathcal{T}(\Sigma)$. In addition we can deduce that $\mathcal{M}$ is the Feix-Kaledin extension of Teichmüller space.

## 4.1 The Gauss equation

Let $\Sigma$ be our fixed closed surface of genus greater than one. Suppose we have a complex structure $J$ on $\Sigma$. We can think of the complex structure $J$ on $\Sigma$ as a conformal equivalence class of metrics. Given a holomorphic quadratic differential $\sigma$ on $\Sigma$ we would like to know when there exists a metric $g \in J$ such that:

$$K_g + |\sigma|_g^2 = -1, \tag{4.1}$$

where $K_g$ is the Gaussian curvature of the metric $g$, and $|\sigma|_g$ is the norm of $\sigma$ in this metric. Furthermore, if such a metric exists under what conditions we might find it unique. We call solutions $g$ of equation (4.1) solutions of the *Gauss equation*.

Analysis of the solutions to the Gauss equation turns out to be related to previous work by Kazdan and Warner [17]. We shall use standard methods in the analysis of elliptic partial differential equations as expounded in eg [9].

We now recall standard facts from the study of analysis on surfaces. Given a surface $\Sigma$ of genus greater than one, with a metric $g$; we can represent $g$ in local conformal coordinates:

$$g = \mathrm{e}^{2\phi}(dx^2 + dy^2).$$

In these coordinates the Laplacian on functions induced by $g$ is,

$$\Delta_g = \mathrm{e}^{-2\phi}\Delta_{\mathrm{euc}},$$

where $\Delta_{\mathrm{euc}}$ is the normal Euclidean Laplacian:

$$\Delta_{\mathrm{euc}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

It should be noted that we are therefore using an "analysts' Laplacian" which differs from the "geometers" model by a factor of $-1$. Given a metric $g$ represented in local conformal coordinates by $g = \mathrm{e}^{2\phi}(dx^2 + dy^2)$ its Gauss curvature is given by:

$$K_g = -\Delta_g\phi.$$

Recall from chapter one the theorem of Poincaré that given a conformal class $J$ of metrics on $\Sigma$ (equivalently a complex structure) there exists a unique metric in this conformal class whose Gauss curvature is $-1$. In what follows we shall always denote by $g$ the metric on $\Sigma$ of constant Gauss curvature $-1$. We let $\rho$ be the volume form of this metric, $\Delta$ the Laplacian with respect to this metric, and $|\sigma|$ the norm of $\sigma$ with respect to this metric. Writing in local conformal coordinates $g := \mathrm{e}^{2\phi}(dx^2 + dy^2)$ we therefore have:

$$K_g = -\Delta\phi = -1.$$

### 4.1.1 The local equation

We shall attempt to find a solution to equation (4.1) by deforming the hyperbolic metric on $\Sigma$ inside its conformal class. Explicitly, we would like to find a $u \in C^\infty(\Sigma)$ such that the metric $g' := e^{2u}g$ satisfies (4.1). In local conformal coordinates we have

$$\begin{aligned} K_{g'} &= -\Delta_{g'}(u + \phi) \\ &= -\frac{1}{e^{2u}}(\Delta u + 1). \end{aligned}$$

Therefore in order to satisfy equation (4.1) we must have

$$-\frac{1}{e^{2u}}(\Delta u + 1) + \frac{|\sigma|^2}{e^{4u}} = -1,$$

which leads us to the non linear partial differential equation for $u$:

$$e^{2u}(\Delta u + 1) - e^{4u} - |\sigma|^2 = 0. \tag{4.2}$$

**Lemma 4.1.1.** *Given a solution $u$ of (4.2) we have:*

- $\Delta u + 1 \geq 2|\sigma|$.
- $u \leq 0, \;\; e^{2u} \leq 1$.
- $\int_\Sigma |\sigma|\rho \leq -\pi\chi(\Sigma)$,

*where we write $\chi(\Sigma)$ for the Euler characteristic of $\Sigma$.*

*Proof.* The first assertion follows immediately from re-writing (4.2) as:

$$\Delta u + 1 = (e^u - e^{-u}|\sigma|)^2 + 2|\sigma|.$$

To prove the second observe that

$$\Delta u + 1 - e^{2u} = e^{-2u}|\sigma| \geq 0.$$

At a maximum of $u$ we have $\Delta u \leq 0$, which implies $e^{2u} \leq 1$ at a maximum of $u$ and hence everywhere.

The third follows from integrating the first over $\Sigma$ and the fact that since $\rho$ is the volume form for the metric of constant Gauss curvature $-1$ we have by Gauss-Bonnet:

$$\int_\Sigma \rho = -2\pi\chi(\Sigma).$$

□

### 4.1.2 Solutions with $|\sigma|_g < 1$

This is the class of solutions we are most interested in. In view of the fact that $g' = e^{2u}g$ we see that $|\sigma| < e^{2u}$ is equivalent to the condition that the size of the holomorphic quadratic differential $\sigma$ in the rescaled metric is less than one pointwise.

**Definition 4.1.2.** We define $\mathcal{C}$ to be the set of holomorphic quadratic differentials $\sigma$ on $\Sigma$ that satisfy $|\sigma| < 1$.

We define $\mathcal{D}$ to be the set of holomorphic quadratic differentials $\sigma$ on $\Sigma$ such that there exists a solution $u$ of

$$e^{2u}(\Delta u + 1) - e^{4u} - |\sigma|^2 = 0$$

and moreover the solution satisfies the pointwise estimate $|\sigma| < e^{2u}$.

Observe that if $\sigma \in \mathcal{D}$ then the pair $(\mathrm{e}^{2u} g, \sigma) \in \mathcal{B}$ where $\mathcal{B}$ is as defined in the last chapter.

Suppose we have a $\sigma \in \mathcal{D}$, in view of lemma (4.1.1) we must have that $|\sigma| < 1$ and hence $\sigma \in \mathcal{C}$. This gives us immediately

$$\mathcal{D} \subseteq \mathcal{C}.$$

Determining exactly what defines the subset $\mathcal{D}$ is a question of considerable interest. The answer would lead to an explicit description of the moduli space $\mathcal{M}$ as a subset of the cotangent bundle of Teichmüller space, which we do not currently possess.

**Uniqueness of solutions**

**Proposition 4.1.3.** *Suppose $\sigma \in \mathcal{D}$, and let $u$ be a solution of*

$$e^{2u}(\Delta u + 1) - e^{4u} - |\sigma|^2 = 0$$

*satisfying $|\sigma| \leq e^{2u}$ on $\Sigma$ then this is the unique solution with this property.*

*Proof.* Suppose we have two solutions $u_1$ and $u_2$ satisfying the hypothesis of the proposition. Then their difference $w = u_1 - u_2$ satisfies:

$$\begin{aligned} &\Delta w - e^{2u_2}(e^{2w} - 1) - e^{-2u_2}(e^{-2w} - 1)|\sigma|^2 = 0 \\ &\Rightarrow \Delta w - e^{2u_2}(e^{2w} - 1)(1 - e^{-2w} e^{-4u_2}|\sigma|^2) = 0. \end{aligned}$$

Since $\Sigma$ is compact and $w$ is continuous, $w$ attains its bounds on $\Sigma$. At the maximum of $w$ we have that $\Delta w \leq 0$ so that,

$$e^{2u_2}(e^{2w} - 1)(1 - e^{-2w} e^{-4u_2}|\sigma|^2) \leq 0.$$

If $(e^{2w} - 1) > 0$ we must have

$$(1 - e^{-2w} e^{-4u_2} |\sigma|^2) \leq 0$$

$$\Rightarrow e^{2w} \leq e^{-4u_2} |\sigma|^2.$$

But this is not possible since by hypothesis $|\sigma| < e^{2u_2}$. Hence $e^{2w} \leq 1$ at the maximum of $w$ and hence everywhere on $\Sigma$. This implies $u_1 \leq u_2$ everywhere on $\Sigma$. By symmetry we must have the same argument for $\hat{w} = u_2 - u_1$, concluding that $u_2 \leq u_1$ everywhere on $\Sigma$. Hence we have the desired uniqueness. □

**Existence of solutions**

We shall show that there exist solutions to the equation (4.2) under the additional hypothesis that

$$\sup_\Sigma |\sigma| < \frac{1}{2}.$$

We do this by solving a slightly more general problem and applying the results to the present case. Firstly we need to establish some a priori estimates:

**Lemma 4.1.4.** *Let $\psi \in L^2_2$ be such that $\psi \geq 0$. Suppose $u \in L^2_4$ solves*

$$e^{2u}(\Delta u + 1) - e^{4u} = \psi,$$

*with $\|e^{-4u}\psi\|_{C^0} < 1$ then*

$$\frac{1}{2} < e^{2u} < 1.$$

*Proof.* The first inequality comes from observing that if $\psi \geq 0$ then

$$\Delta u + 1 - e^{2u} \geq 0.$$

At a maximum of $u$ we have $\Delta u \leq 0$, which implies $e^{2u} \leq 1$ at a maximum of $u$ and hence everywhere. We obtain the second by using the fact that $\|e^{-4u}\psi\|_{C^0} < 1$ so that,

$$\Delta u + 1 < 2e^{2u}.$$

At a minimum of $u$ we have $\Delta u \geq 0$, which implies

$$e^{2u} \geq \frac{1}{2}$$

at a minimum of $u$ and hence everywhere. □

**Lemma 4.1.5.** *Let $\psi \in L^2_k$ with $k \geq 2$ and let $u \in L^2_2$ solve*

$$e^{2u}(\Delta u + 1) - e^{4u} = \psi.$$

*Then $u \in L^2_{k+2}$.*

*Proof.* For $l \geq 2$ analytic operations map $L^2_l \to L^2_l$, hence

$$\Delta u = e^{2u} + e^{-2u}\psi - 1$$

is in $L^2_2$. Note that this is true because we are in two dimensions. The Sobolev inequality

$$\|u\|_{L^2_4} \leq c\left(\|\Delta u\|_{L^2_2} + \|u\|_{L^2}\right)$$

then ensures $u \in L^2_4$. Iterating this argument gives the result. □

**Theorem 4.1.6.** *Let $\Sigma$ be a closed oriented surface of genus greater than one. If $\psi \in L^2_2$ satisfies*

$$\psi \geq 0,$$
$$\|\psi\|_{C^0} < \frac{1}{4},$$

*then there exists a solution $u \in L^2_4$ to the equation*

$$e^{2u}(\Delta u + 1) - e^{4u} = \psi \tag{4.3}$$

*satisfying $\|e^{-4u}\psi\|_{C^0} < 1$.*

*Proof.* We follow a continuity method. Let $S$ denote the functions $\psi \in L^2_2$ satisfying $\psi \geq 0$, $\|\psi\|_{C^0} < 0.25$. It is immediate that $S$ is convex and hence connected. Let $S' \subset S$ be the subset of $S$ for which (4.3) has a solution $u \in L^2_4$. We shall show that $S'$ is both open and closed in $S$ with respect to the induced $L^2_2$ topology.

*$S'$ is open:*

We consider the map $F : L^2_4 \to L^2_2$ defined by

$$F(u) = e^{2u}(\Delta u + 1) - e^{4u}.$$

The derivative of $F$ at $u$ is given by

$$DF_u(\phi) = e^{2u}(\Delta\phi - 2e^{2u}(1 - e^{-4u}\psi)\phi).$$

This linear operator is elliptic and self-adjoint. If $\psi \in S'$ and $u$ is the associated solution, then hypothesis the operator

$$\phi \mapsto \Delta\phi - 2e^{2u}(1 - e^{-4u}\psi)\phi,$$

is strictly negative, so we conclude that $DF_u$ has no kernel. Hence by the Fredholm alternative $DF_u$ is an isomorphism $L^2_4 \to L^2_2$. The inverse function theorem in Banach spaces then ensures that each point $\psi \in S'$ has a neighbourhood on which $F$ is invertible, thus $S'$ is open.

*$S'$ is closed:*

Take a sequence $\psi_n \in S'$ converging in $L^2_2$ to some $\psi \in S$. We need to show that $\psi \in S'$. Let $u_n \in L^2_4$ solve,

$$e^{2u_n}(\Delta u_n + 1) - e^{4u_n} = \psi_n.$$

Since $\psi_n$ converges in $L^2_2$ we must have that $\|\psi_n\|_{L^2}$ is bounded. The Sobolev embedding theorem gives $L^2_k \hookrightarrow C^0$ for $k \geq 2$. Hence $u_n \in C^0$. From (4.1.4) we have a priori bounds:

$$\frac{1}{2} < e^{2u_n} < 1,$$

thus $\|e^{2u_n}\|_{C^0}$ and hence $\|u_n\|_{C^0}$ are bounded. This implies $\|e^{2u_n}\|_{L^2}$ and $\|u_n\|_{L^2}$ are bounded. Since

$$\Delta u_n = e^{-2u_n}\psi_n + e^{2u_n} - 1,$$

we establish that $\Delta u_n$ is bounded in $L^2$. The Sobolev inequality

$$\|u_n\|_{L^2_2} \leq c\left(\|\Delta u_n\|_{L^2} + \|u_n\|_{L^2}\right)$$

for some constant $c$, then ensures that $u_n \in L^2_2$. Since we know $\psi_n \in L^2_2$ we may essentially repeat the above argument to obtain that $\Delta u_n \in L^2_2$ and hence using the Sobolev inequality

$$\|u_n\|_{L^2_4} \leq c'\left(\|\Delta u_n\|_{L^2_2} + \|u_n\|_{L^2}\right)$$

we conclude that $u_n \in L^2_4$. Using the Rellich lemma we have that $L^2_4 \hookrightarrow C^2$ compactly. Hence $u_n$ has a $C^2$ convergent subsequence, $u_n \to u \in C^2$. From the Sobolev embedding theorem we know that $\psi_n \to \psi$ in $C^0$. Thus $u$ is a $C^2$ solution of

$$e^{2u}(\Delta u + 1) - e^{4u} = \psi.$$

By the regularity lemma (4.1.5) we have that $u \in L^2_4$.

We know that

$$e^{4u} \geq \frac{1}{4},$$

and by hypothesis,

$$\|\psi\|_{C^0} < \frac{1}{4},$$

hence we have $\|e^{-4u}\psi\|_{C^0} < 1$ as desired. □

It should be observed that the condition

$$\|\psi\|_{C^0} < \frac{1}{4},$$

is key to this proof to work with the estimate $\|e^{-4u}\psi\|_{C^0} < 1$ holding, otherwise it is certainly possible that at some point $e^{-4u}\psi = 1$.

We now apply this to the situation we have been considering.

**Theorem 4.1.7.** *Let $\Sigma$ be a closed oriented surface of genus greater than one. Let $\sigma$ be a holomorphic quadratic differential satisfying*

$$\sup_\Sigma |\sigma| < \frac{1}{2}.$$

*Then there exists a unique solution $u \in C^\infty$ to the equation*

$$e^{2u}(\Delta u + 1) - e^{4u} = |\sigma|^2,$$

*satisfying $|\sigma| < e^{-2u}$ on $\Sigma$.*

*Proof.* Since $\sigma$ is holomorphic we have that $|\sigma| \in C^\infty$. Setting $\psi = |\sigma|^2$ and using theorem 4.1.6 we obtain a solution $u \in L^2_4$. Since $|\sigma| \in C^\infty$ we see $|\sigma|^2 \in L^2_k$ for every $k$, applying the regularity lemma 4.1.5 gives $u \in C^\infty$. Uniqueness follows from 4.1.3. □

## 4.2 The map $\mathcal{M} \to T^*\mathcal{T}(\Sigma)$

Recall from the last chapter that we have a map

$$\mathcal{B} \to T^*\mathcal{J}(\Sigma),$$

given by mapping the pair $(g, \sigma)$ to the pair $(J, \sigma)$ where $J$ is the conformal class of $g$. In view of the uniqueness theorem 4.1.3 we see that this map is in fact injective as suggested earlier. Since for a point $(g, \sigma) \in \mathcal{B}$ the quadratic differential $\sigma$ is holomorphic we may construct a natural map

$$\mathcal{B}/\mathrm{Diff}_0(\Sigma) = \mathcal{M} \to T^*\mathcal{T}(\Sigma),$$

so we have the following theorem:

**Theorem 4.2.1.** *The moduli space $\mathcal{M}$ is embedded in the cotangent bundle of Teichmüller space.*

The existence statement of theorem 4.1.7 allows us to identify a certain subset of the cotangent bundle of Teichmüller space that must be contained in the image of the moduli space $\mathcal{M}$.

**Proposition 4.2.2.** *The subset of $T^*\mathcal{T}(\Sigma)$ defined by those holomorphic quadratic differentials satisfying the pointwise size estimate*

$$|\sigma| < \frac{1}{2},$$

*is contained in the moduli space $\mathcal{M}$. Recall that here $|\sigma|$ is the size of $\sigma$ in the metric of constant scalar curvature $-1$.*

We now consider some properties of the embedding of $\mathcal{M}$ in the cotangent bundle of Teichmüller space.

**Proposition 4.2.3.** *The injection $\mathcal{M} \hookrightarrow T^*\mathcal{T}(\Sigma)$ is a holomorphic map from $(M, I)$ to the cotangent bundle with the complex structure coming from the canonical complex structure on Teichmüller space.*

*Proof.* The complex structure $I$ on $\mathcal{M}$ is that induced by the complex structure $I$ on the hyperkähler extension on the hyperbolic plane. Since we have the holomorphic identification

$$\Gamma(\underline{D}) \cong T^*\mathcal{J}(\Sigma),$$

and the complex structure on the cotangent bundle of Teichmüller space is that induced by the structure $I$ on $T^*\mathcal{J}(\Sigma)$ the result follows. $\square$

**Proposition 4.2.4.** *The restriction of the canonical holomorphic symplectic structure on $T^*\mathcal{T}(\Sigma)$ to the subset defined by $\mathcal{M}$ coincides with the holomorphic symplectic structure on $\mathcal{M}$ defined since $\mathcal{M}$ is derived from a symplectic reduction of a cotangent bundle.*

*Proof.* We need to use the infinite dimensional analogue of the theorem about the reduction of cotangent bundles from chapter two, proceeding formally at the level of tangent spaces we then have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{A} & \longrightarrow & T^*\mathcal{J}(\Sigma) & \longrightarrow & \mathcal{J}(\Sigma) \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow & & {\scriptstyle \pi_2}\big\downarrow \\ \mathcal{A}/\mathrm{Symp}_0(\Sigma,\rho) & \xrightarrow{\tilde{\iota}} & T^*(\mathcal{T}(\Sigma)) & \longrightarrow & \mathcal{J}(\Sigma)/\mathrm{Diff}_0(\Sigma) = \mathcal{T}(\Sigma). \end{array},$$

where the map $\tilde{\iota}$ is injective. Applying the earlier theorem in this situation gives the result. $\square$

We can use these propositions together with results of the last chapter to conclude that the moduli space $\mathcal{M}$ with the hyperkähler structure constructed in the previous chapter is in fact the Feix-Kaledin extension of the Weil-Petersson metric on Teichmüller space.

**Theorem 4.2.5.** *The moduli space $\mathcal{M}$ is the Feix-Kaledin hyperkähler extension of the Weil-Petersson metric.*

*Proof.* We need to show that:

- the metric on $\mathcal{M}$ restricts to the Weil-Petersson metric on Teichmüller space $\mathcal{T}(\Sigma) \subset \mathcal{M}$,
- the canonical holomorphic symplectic structure on $T^*\mathcal{T}(\Sigma)$ is compatible with the form $\Omega_2 + i\Omega_3$ on $\mathcal{M}$,
- the circle action on $T^*\mathcal{T}(\Sigma)$ given by multiplication by $i$ preserves the metric.

The first point was dealt with in the last chapter, whilst the second is the result of the previous proposition. The circle action induced from $I$ on $\mathcal{M}$ coincides with that from the complex structure on $\mathcal{T}(\Sigma)$. Since multiplication by $i$ on $D \subset T^*H^2$ preserves the metric on $D$, and the complex structure $I$ on $\mathcal{M}$ is the one induced by multiplication by $i$ on $D$, the circle action on $\mathcal{M}$ from $I$ preserves the metric on $\mathcal{M}$. $\square$

# Chapter 5

In this chapter we define the representation variety associated to the surface $\Sigma$ and the Lie group $\mathrm{PSL}_2\mathbb{C}$. This has a canonical holomorphic-symplectic structure and we present a result about the restriction of this holomorphic symplectic structure to the real subspace of $\mathrm{PSL}_2\mathbb{R}$ representations; Teichmüller space. Next we identify a certain subset of the representation variety, the quasi-Fuchsian space. We proceed to define a map from the moduli space $\mathcal{M}$ into the quasi-Fuchsian space and discuss its properties.

## 5.1 Deformation spaces of representations

Let $\pi_1(\Sigma)$ be the fundamental group of our fixed surface $\Sigma$. Denote the set of all representations $\theta : \pi_1(\Sigma) \to \mathrm{PSL}_2\mathbb{C}$ of the fundamental group of $\Sigma$ into the Lie group $\mathrm{PSL}_2\mathbb{C}$, $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2\mathbb{C})$. There is a natural action of $\mathrm{PSL}_2\mathbb{C}$ on $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2\mathbb{C})$ given by conjugating representations, $g : \theta \mapsto g\theta g^{-1}$.

**Definition 5.1.1.** The representation variety is defined to be the quotient $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2\mathbb{C})/\mathrm{PSL}_2\mathbb{C}$.

We shall be concerned only with that subset of the representation variety that is a smooth manifold, we shall denote this subset $\mathcal{V}(\Sigma)$, it is a complex manifold of dimension $-3\chi(\Sigma)$[10]. It is a result of Goldman [10] that $\mathcal{V}(\Sigma)$ has a canonical holomorphic-symplectic structure induced by the standard trace form Tr on $\mathrm{PSL}_2\mathbb{C}$.

### 5.1.1 Fuchsian and quasi-Fuchsian groups and their deformation spaces

**Definition 5.1.2.** We now recall some standard facts about discrete groups of Möbius transformations, see for example [4], [18], [12]. Let $G$ be a discrete group of Möbius transformations acting properly discontinuously on a non-trivial subset of $\mathbb{C}P^1$. We define the *region of discontinuity* of $G$ to be the maximal subset of $\mathbb{C}P^1$ on which the action is properly discontinuous. We define the *limit set* $C$ of $G$ to be the complement of the region of discontinuity.

We call a subset of $\mathbb{C}P^1$ a *circle* if it is a great circle of $\mathbb{C}P^1$ thought of as a sphere.

**Definition 5.1.3.** A *Fuchsian* group is a discrete group of Möbius transformations whose limit set is a circle. A *quasi-Fuchsian* group is a discrete group of Möbius transformations whose limit set is a Jordan curve.

If $G$ is Fuchsian, then we may find a Möbius transformation mapping its limit set to the real line $\mathbb{R}P^1 \subset \mathbb{C}P^1$, showing that $G$ is conjugate in $\mathrm{PSL}_2\mathbb{C}$ to a discrete group of Möbius transformations that fix the upper and lower half planes that is it is conjugate in $\mathrm{PSL}_2\mathbb{C}$ to a subgroup of $\mathrm{PSL}_2\mathbb{R}$.

Suppose that $G$ is a quasi-Fuchsian group and let $C$ be its limit set. Now $G$ acts properly discontinuously on the complement of $C$ in $\mathbb{C}P^1$, indeed the complement of $C$ in $\mathbb{C}P^1$ falls into two $G$ invariant connected components $\Gamma^+$ and $\Gamma^-$ and the quotient

$$(\mathbb{C}P^1 \backslash C)/G \cong \Gamma^+/G \sqcup \Gamma^-/G,$$

is the disjoint union of two homeomorphic Riemann surfaces,

$$\Sigma_1 := \Gamma^+/G, \quad \Sigma_2 := \Gamma^-/G.$$

We say that the group $G$ *represents* the pair $\Sigma_1, \Sigma_2$. If $G$ were Fuchsian then in fact the quotient

$$(\mathbb{C}P^1 \backslash C)/G = \Sigma_J \sqcup \overline{\Sigma_J},$$

the disjoint union of the Riemann surface $\Sigma_J$ and that obtained by taking the conjugate complex structure $\overline{\Sigma_J}$.

The quasi-Fuchsian group $G$ acts properly discontinuously on the hyperbolic three space $H^3$ and the quotient $H^3/G$ is a hyperbolic 3-manifold $M$. We call the hyperbolic three manifolds obtained by quotienting $H^3$ by a quasi-Fuchsian group *quasi-Fuchsian manifolds*. The two Riemann surfaces obtained as the quotient of $\mathbb{C}P^1 \backslash C$ then arise from the action of $G$ on the sphere at infinity in $H^3$ where we have a conformal but not a metric structure. It is a result of Marden [19] that the manifold $M$ is topologically $\Sigma \times \mathbb{R}$ so we may think of $M$ as a hyperbolic cobordism between the two Riemann surfaces.

These two classes of subgroup of $\mathrm{PSL}_2\mathbb{C}$ allows us to consider two subsets of the representation variety. Let $\mathcal{R}(\Sigma)$ be the subset of $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2\mathbb{C})$ consisting of those representations $\theta$ whose image $\theta(\pi_1(\Sigma))$ is Fuchsian, and $\mathcal{Q}(\Sigma)$ those whose image is quasi-Fuchsian. It is immediate that $\mathcal{R}(\Sigma) \subset \mathcal{Q}(\Sigma)$.

**Definition 5.1.4.** The *Fuchsian deformation space* $\mathcal{F}(\Sigma)$ is the quotient of $\mathcal{R}(\Sigma)$ by the conjugation action of $\mathrm{PSL}_2\mathbb{C}$. The *quasi-Fuchsian deformation space* $\mathcal{QF}(\Sigma)$ is the quotient of $\mathcal{Q}(\Sigma)$ by the conjugation action of $\mathrm{PSL}_2\mathbb{C}$.

It is immediate from these definitions that both the Fuchsian and quasi-Fuchsian deformation spaces are subsets of the representation variety, and in addition that $\mathcal{F}(\Sigma) \subset \mathcal{QF}(\Sigma)$. In addition we have that the quasi-Fuchsian deformation space is a $-3\chi(\Sigma)$ complex submanifold of the representation variety [11].

### 5.1.2 Uniformisation; the Fuchsian deformation space

The uniformisation theorem for Riemann surfaces tells us that any such can be expressed as the quotient of the hyperbolic plane by the action of a group of real Möbius transformations.

**Theorem 5.1.5** (Uniformisation)**.** *Given a Riemann surface $\Sigma_J$ there exists a discrete subgroup $G$ of $\mathrm{PSL}_2\mathbb{R}$ such that:*

$$\Sigma_J = H^2/G.$$

Using this theorem we shall identify the Teichmüller space of the surface $\Sigma$ in our current context of group representations. We fix a realisation of our surface $\Sigma$ as the quotient $H^2/\pi_1(\Sigma)$ where we are not concerned with the complex structure this induces on $\Sigma$ in this instance. Given any (discrete, faithful) representation

$$\theta_J : \pi_1(\Sigma) \to \mathrm{PSL}_2\mathbb{R},$$

write $\hat{\theta}_J$ for the image of $\pi_1(\Sigma)$. We have the following commutative diagram

$$\begin{array}{ccc} H^2 & \xrightarrow{\tilde{f}} & H^2 \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{f} & H^2/\hat{\theta}_J, \end{array}$$

where $f$ is a homeomorphism from $\Sigma$ onto the Riemann surface $H^2/\hat{\theta}_J$ given by the map

$$[z]_{\pi_1(\Sigma)} \mapsto [z]_{\hat{\theta}_J},$$

and $\tilde{f}$ is the lift of $f$ to the covering space. In view of this we define a complex structure on $\Sigma$ by pulling back the complex structure on $H^2/\hat{\theta}_J$, we write $\Sigma_J$ to denote the resulting Riemann surface.

**Proposition 5.1.6.** *The Fuchsian deformation space of the surface $\Sigma$ is homeomorphic to its Teichmüller space,*

$$\mathcal{F}(\Sigma) \cong \mathcal{T}(\Sigma).$$

*Proof.* Let $[\theta] \in \mathcal{F}(\Sigma)$ and choose any $\theta_{\mathbb{R}} \in [\theta]$ consisting of real Möbius transformations. We define the map $\mathcal{F}(\Sigma) \to \mathcal{T}(\Sigma)$ as that induced by taking the equivalence class $[\theta]$ to the equivalence class of the complex structure on $\Sigma$ induced by $H^2/\theta_{\mathbb{R}}(\pi_1(\Sigma))$. We need to see that this map is well defined and bijective.

To see it is well-defined suppose we have two real representations $\theta_1, \theta_2 \in [\theta]$. So there exists an $h \in \mathrm{PSL}_2\mathbb{C}$ such that $\theta_2 = h\theta_1 h^{-1}$. Since the map $\mathbb{C}P^1 \to \mathbb{C}P^1$ given by $z \to hz$ must map the limit set of $\hat{\theta}_1$ onto that of $\hat{\theta}_2$ we see that $h \in \mathrm{PSL}_2\mathbb{R}$. Hence $h$ induces a conformal map $H^2 \to H^2$ by mapping $z \mapsto hz$. Since

$$h\hat{\theta}_1 z = h\hat{\theta}_1 h^{-1} hz = \hat{\theta}_2 hz,$$

this therefore maps orbits of $\hat{\theta}_1$ onto those of $\hat{\theta}_2$ and hence descends to a conformal map

$$h : H^2/\hat{\theta}_1 \to H^2/\hat{\theta}_2.$$

In addition we have the isomorphism

$$\begin{aligned} \mu : \hat{\theta}_1 &\to \hat{\theta}_2, \\ g &\mapsto hgh^{-1} \end{aligned}$$

so $\mu(\theta_1(\gamma)) = \theta_2(\gamma)$ for any $\gamma \in \pi_1(\Sigma)$.

We have, for $i \in \{1, 2\}$, homeomorphisms

$$f_i : \Sigma \to H^2/\hat{\theta}_i,$$

given by the obvious projections. These define a necessarily conformal map $\phi : \Sigma_1 \to \Sigma_2$ by

$$\phi := f_2 \circ h \circ f_1^{-1},$$

as in the following commutative diagram:

$$\begin{array}{ccc} H^2/\hat{\theta}_1 & \xrightarrow{h} & H^2/\hat{\theta}_2 \\ {\scriptstyle f_1^{-1}}\downarrow & & {\scriptstyle f_2^{-1}}\downarrow \\ \Sigma & \xrightarrow{\phi} & \Sigma. \end{array}$$

Now $\phi$ induces a map $\phi_*$ on the fundamental group of $\Sigma$ and

$$\begin{aligned} \phi_*(\gamma) &= f_{2*} \circ h_* \circ {f_1^{-1}}_* \\ &= \theta_2 \circ \mu \circ \theta_1^{-1}(\gamma) \\ &= \gamma. \end{aligned}$$

But then $\phi$ induces the same isomorphism on $\pi_1(\Sigma)$ as the identity map on $\Sigma$. Since we have $\Sigma = H^2/\pi_1(\Sigma)$ we can construct a homotopy equivalence between the two diffeomorphisms of $\Sigma$ (see for example [18]). Hence $\phi \in \mathrm{Diff}_0(\Sigma)$ and therefore the two complex structures from $\theta_1$ and $\theta_2$ define the same point of Teichmüller space.

Let $\Sigma_1$ and $\Sigma_2$ be the Riemann surfaces $(\Sigma, J_1)$ and $(\Sigma, J_2)$ for $J_1, J_2 \in \mathcal{J}(\Sigma)$. Suppose $\Sigma_1$ and $\Sigma_2$ are uniformised by the Fuchsian groups $G_1$ and $G_2$ respectively. Further suppose they are conformally equivalent by a map

$\phi$ homotopic to the identity, thus $[J_1] = [J_2] \in \mathcal{T}(\Sigma)$. Since the surfaces are conformally equivalent we can lift this map to a conformal map on the covering space $H^2$. This must then be a Möbius transformation represented by an element $h \in \mathrm{PSL}_2\mathbb{R}$. This induces a map:

$$\begin{aligned} \mu : G_1 &\to G_2 \\ g &\mapsto hgh^{-1}. \end{aligned}$$

Now $\phi_*$, the induced map on the fundamental group, must be the identity. Hence using the commutative diagram:

$$\begin{array}{ccc} \pi_1(\Sigma) & \xrightarrow{\phi_*} & \pi_1(\Sigma) \\ {\scriptstyle\theta_1}\big\downarrow & & {\scriptstyle\theta_2}\big\downarrow \\ G_1 & \xrightarrow{\mu} & G_2 \end{array}$$

we have,

$$h^{-1}\theta_2 h = \theta_1.$$

So the representations are conjugate and we have injectvity.

That the map is surjective follows immediately from the uniformisation theorem. □

### 5.1.3 Simultaneous uniformisation; the quasi-Fuchsian deformation space

We have identifed the Fuchsian deformation space of representations as an object we are already familiar with. Results of Bers' ([1], [2], [3]) enable us to do the same with the quasi-Fuchsian deformation space using the so called simultaneous uniformisation theorem.

**Theorem 5.1.7** (Bers)**.** *Given a pair of Riemann surfaces $\Sigma_1, \Sigma_2$ and a homotopy class $[f]$ of orientation-reversing homeomorphisms between them, there exists a quasi-Fuchsian group representing these surfaces and this class. The group is determined uniquely up to conjugation in* $\mathrm{PSL}_2\mathbb{C}$.

As a corollary of this theorem we identify the quasi-Fuchsian deformation space as a smooth manifold with the product of two copies of Teichmüller space:

$$\mathcal{T}(\Sigma) \times \overline{\mathcal{T}(\Sigma)} \cong \mathcal{QF}(\Sigma).$$

Secondly we may identify the complex structure on $\mathcal{QF}(\Sigma)$ with the natural one on the product of the Teichmüller spaces:

**Theorem 5.1.8** (Bers)**.** *The natural map induced by simultaneous uniformisation is a biholomorphism*

$$\mathcal{T}(\Sigma) \times \overline{\mathcal{T}(\Sigma)} \to \mathcal{QF}(\Sigma).$$

Now it is clear that there is a canonical embedding of the Teichmüller space into the quasi-Fuchsian deformation space given by the diagonal map,

$$\mathcal{T}(\Sigma) \hookrightarrow \mathcal{T}(\Sigma) \times \overline{\mathcal{T}(\Sigma)}.$$

This embeds Teichmüller space as a $-3\chi(\Sigma)$ dimensional real submanifold of the complex manifold $\mathcal{QF}(\Sigma)$. Consideration of the quotient $(\mathbb{C}P^1 \backslash C)/G$ for $G$ a Fuchsian group implies that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(\Sigma) & \xrightarrow{\iota} & \mathcal{QF}(\Sigma) \\ \cong\downarrow & & \cong\downarrow \\ \mathcal{T}(\Sigma) & \xrightarrow{\iota} & \mathcal{T}(\Sigma) \times \overline{\mathcal{T}(\Sigma)}, \end{array}$$

where we have written $\iota$ for the obvious injections.

The canonical holomorphic symplectic structure [10] on the representation variety restricts to give one on the submanifold $\mathcal{QF}(\Sigma)$, remarkably it also restricts to give a symplectic structure on the real submanifold given by the Fuchsian deformation space.

**Theorem 5.1.9** (Goldman [11])**.** *The holomorphic symplectic structure on the* $\mathrm{PSL}_2\mathbb{C}$ *representation variety restricts to a symplectic structure on the Fuchsian deformation space. This resulting symplectic form is the Kähler form for the Weil-Petersson metric under the identification with Teichmüller space.*

Note that the holomorphic symplectic structure on the $\mathrm{PSL}_2\mathbb{C}$ representation variety is a complex form, its imaginary part restricts to zero on the Fuchsian deformation space.

## 5.2 Embedding $\mathcal{M}$ in the quasi-Fuchsian deformation space

In this section we construct a map from the moduli space constructed in chapter two to the quasi-Fuchsian space introduced above. We show that this map embeds $\mathcal{M}$ as an open set in $\mathcal{QF}(\Sigma)$. Thus we obtain a hyperkähler structure on an open subset of the quasi-Fuchsian deformation space. In addition we show that the map $(\mathcal{M}, J) \to (\mathcal{QF}(\Sigma), I)$ is holomorphic where $I$ is the natural complex structure on the quasi-Fuchsian deformation space.

### 5.2.1 Quasi-Fuchsian three manifolds

Let $\Sigma$ be a fixed closed surface. In this section we construct a Riemannian 3-manifold $M_{(g,\sigma)}$ associated to any pair $(g, \sigma)$ where $g$ is a metric on $\Sigma$ and $\sigma$ is a smooth quadratic differential with $|\sigma|_g < 1$. We show that there is a natural embedding $\Sigma \hookrightarrow M_{(g,\sigma)}$ that is a minimal isometric immersion. Under the additional constraint that $\sigma$ is holomorphic and

$$K_g + |\sigma|^2 = -1,$$

we find that the $M_{(g,\sigma)}$ is a quasi-Fuchsian hyperbolic 3-manifold. This is essentially an elucidation of the work of Uhlenbeck in [28].

Let $(g, \sigma)$ be a pair consisting of a metric $g$ and a smooth quadratic differential $\sigma$ satisfying $|\sigma| < 1$.

**Lemma 5.2.1.** *The quadratic differential $\sigma$ induces a smooth, symmetric, trace-free bilinear form $h$ on the real tangent bundle to $\Sigma$.*

*Proof.* Let $X, Y \in T_{\mathbb{R}}\Sigma$, then since $T_{\mathbb{R}}\Sigma \cong T^{1,0}\Sigma$ we may consider their images $\tilde{X}, \tilde{Y}$ in $T^{1,0}\Sigma$. Clearly $\sigma(\tilde{X}, \tilde{Y}) \in \mathbb{C}$, and we define

$$h(X, Y) := \mathrm{Re}(\sigma(\tilde{X}, \tilde{Y})).$$

The symmetry and smoothness properties follow immediately from the fact that $\sigma$ is a smooth quadratic differential on $\Sigma$. To see that it is trace free let $X \in T_{\mathbb{R}}\Sigma$, then $\{X, JX\}$ span $T_{\mathbb{R}}\Sigma$, where of course $J$ is the complex structure induced by $g$. Now:

$$\begin{aligned} h(JX, JX) &= \mathrm{Re}(\sigma(\widetilde{JX}, \widetilde{JX})) \\ &= \mathrm{Re}(\sigma(i\widetilde{X}, i\widetilde{X})) \\ &= -\mathrm{Re}(\sigma(\tilde{X}, \tilde{X})) \\ &= -h(X, X). \end{aligned}$$

But then we must have that the symmetric form $h$ is trace free. □

Now we define $M_{(g,\sigma)}$ topologically as the product $\Sigma \times \mathbb{R}$, and equip it with a symmetric bilinear form

$$\Lambda = dt^2 + (\cosh^2 t + |\sigma|_g^2 \sinh^2 t)g - (2\cosh t \sinh t)h,$$

where we take $t$ as a global coordinate in the $\mathbb{R}$ direction.

**Lemma 5.2.2.** *The form $\Lambda$ gives $M_{(g,\sigma)}$ a Riemannian structure.*

*Proof.* We need to show that $\Lambda$ is non-degenerate on $M_{(g,\sigma)}$. Take local conformal coordinates on $\Sigma$ so that $g = \mathrm{e}^{2\phi}(dx^2 + dy^2)$. In these coordinates we write $\sigma = (\alpha + i\beta)dz \otimes dz$, so that $h = \alpha dx^2 - 2\beta dxdy - \alpha dy^2$. We find that in the coordinates $(x, y, t)$ the metric $\Lambda$ is represented by the matrix

$$\Lambda_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{2\phi} A^2 \end{pmatrix},$$

where the $2 \times 2$ matrix $A$ is defined as

$$A := \begin{pmatrix} \cosh t + \sinh t \mathrm{e}^{-2\phi}\alpha & -\sinh t \mathrm{e}^{-2\phi}\beta \\ -\sinh t \mathrm{e}^{-2\phi}\beta & \cosh t - \sinh t \mathrm{e}^{-2\phi}\alpha \end{pmatrix}.$$

We see that $\Lambda$ is non-degenerate if and only if $\det(A) > 0$ on $M_{(g,\sigma)}$, since $\det(A) > 0$ at $t = 0$. Now

$$\det(A) = \cosh^2 t - \mathrm{e}^{-4\phi}(\alpha^2 + \beta^2)\sinh^2 t,$$

so we see that $\det(A) > 0$ on $M_{(g,\sigma)}$ if and only if

$$\mathrm{e}^{-4\phi}(\alpha^2 + \beta^2)\tanh^2 t < 1, \ \forall t \in \mathbb{R}.$$

Since $0 \leq \tanh^2 t < 1$ we certainly have the desired relation if

$$\mathrm{e}^{-4\phi}(\alpha^2 + \beta^2) = |\sigma|_g^2 < 1,$$

on $\Sigma$ as we supposed. □

From now on when we refer to the manifold $M_{(g,\sigma)}$ it will be implicit that it has the above defined metric.

**Lemma 5.2.3.** *The slice $\Sigma \times \{0\}$ is a minimal isometric embedding $\Sigma \hookrightarrow M_{(g,\sigma)}$. The second fundamental form for this embedding is given by $h$.*

*Proof.* It is immediate from the definition of $\Lambda$ that this defines an isometric embedding. We need to show that it is in fact minimal, that is the mean curvature is zero. First we calculate the second fundamental form of the embedding. Let $X, Y \in T_p\Sigma$ and write $\partial_t$ for the unit normal vector in the $\mathbb{R}$ direction, by definition of the second fundamental form $S$,

$$S(X,Y) := \Lambda(\nabla_X \partial_t, Y),$$

where $\nabla$ is the Levi-Civita connection on $M_{(g,\sigma)}$. We have that this is given as follows:

$$\begin{aligned}\Lambda(\nabla_X \partial_t, Y) &= \frac{1}{2}(X\Lambda(Y,\partial_t) - \partial_t\Lambda(X,Y) + Y\Lambda(\partial_t, X) \\ &\quad - \Lambda(X,[Y,Z]) + \Lambda(\partial_t,[X,Y]) - \Lambda(Y,[\partial_t,X])) \\ &= -\frac{1}{2}\partial_t\Lambda(X,Y).\end{aligned}$$

Here we have used the fact that $\partial_t$ is perpendicular to the plane spanned by $X$ and $Y$ and we may choose $X, Y$ so that all the commutators evaluate to zero. Then

$$\begin{aligned}2S(X,Y) &= -\left.\frac{\partial}{\partial t}\right|_{t=0}((\cosh^2 t + |\sigma|_g^2 \sinh^2 t)g(X,Y) - (2\cosh t \sinh t)h(X,Y)) \\ &= 2h(X,Y).\end{aligned}$$

We have from an earlier lemma that $h$ is trace free, so by definition the mean curvature of $\Sigma \times \{0\}$ vanishes and the embedding is minimal. $\square$

Under additional conditions on $\sigma$ and $g$ we can say more about the manifold $M_{(g,\sigma)}$. We shall make essential use of the following result of Uhlenbeck from [28].

**Proposition 5.2.4** (Uhlenbeck [28])**.** *Suppose that $\sigma$ is holomorphic with $|\sigma|_g < 1$, and*

$$K_g + |\sigma|_g^2 = -1,$$

*then $M_{(g,\sigma)}$ is a quasi-Fuchsian 3-manifold and the embedding $\Sigma \hookrightarrow M_{(g,\sigma)}$ as the slice $\Sigma \times \{0\}$ is the only minimal surface in $M_{(g,\sigma)}$.*

The conditions on the curvature and the holomorphicity of the quadratic differential turn out to be equivalent to the Gauss-Codazzi equations for the surface $\Sigma$ embedded in $M_{(g,\sigma)}$ in this way.

### 5.2.2 The map $\mathcal{M} \to \mathcal{QF}(\Sigma)$

Using the theorem of Uhlenbeck from the previous section we can construct a map $\mathcal{M} \hookrightarrow \mathcal{QF}(\Sigma)$ from the one moduli space into the other.

Recall the set $\mathcal{B}$ consisting of the set of pairs $(g, \sigma)$ where $g$ is a metric on $\Sigma$, $\sigma$ is a holomorphic quadratic differential and the pair staisfies the Gauss equation:

$$K_g + |\sigma|_g^2 = -1.$$

Clearly we have a map

$$\mathcal{B} \hookrightarrow \mathcal{Q}(\Sigma),$$

given by taking the pair $(g, \sigma) \in \mathcal{B}$ to the quasi-Fuchsian three manifold $M_{(g,\sigma)}$. In fact this map descends to give us a bona fide map at the quotient level.

**Proposition 5.2.5.** *There is an injective map $\iota$ from the moduli space $\mathcal{M}$ into the quasi-Fuchsian space $\mathcal{QF}(\Sigma)$.*

*Proof.* We need to show that the map defined above from $\mathcal{B} \to \mathcal{Q}(\Sigma)$ descends to give us a map between the quotients:

$$\mathcal{M} = \mathcal{B}/\mathrm{Diff}_0(\Sigma) \to \mathcal{Q}(\Sigma)/\mathrm{PSL}_2\mathbb{C} = \mathcal{QF}(\Sigma).$$

The map is well defined since if we have two equivalent pairs $(g_1, \sigma_1)$ and $(g_2, \sigma_2)$ then the corresponding three manifolds $M_{(g_1,\sigma_1)}$, $M_{(g_2,\sigma_2)}$ are obviously isometric. Now suppose that we have $(g_1, \sigma_1), (g_2, \sigma_2) \in \mathcal{B}$ and we know that $M_{(g_1,\sigma_1)}$ is equivalent to $M_{(g_2.\sigma_2)}$ in $\mathcal{QF}(\Sigma)$. Let $G_1$ and $G_2$ be the quasi-Fuchsian groups corresponding to the three manifolds, these are therefore conjugate in $\mathrm{PSL}_2\mathbb{C}$. Suppose $G_1 = hG_2h^{-1}$, then we get an isometry of $H^3$ by mapping $w \mapsto hw$ which descends to an isometry $\tilde{h}$ between the three manifolds. Since this map $\tilde{h}$ is an isometry, it must map the unique minimal surface inside $M_{(g_1,\sigma_1)}$ to that inside $M_{(g_2,\sigma_2)}$. But then the pairs $(g_1, \sigma_1), (g_2, \sigma_2)$ are related by $\tilde{h}$ so that our map is injective as desired. $\square$

We should observe that the methods of chapter three imply immediately that the image of $\mathcal{M}$ is an open set in quasi-Fuchsian space, therefore we have a hyperkähler structure on an open subset of $\mathcal{QF}(\Sigma)$.

### 5.2.3 Holomorphicity of the embedding

In this section we show that the embedding of $\mathcal{M}$ as an open set in the quasi-Fuchsian deformation space is holomorphic with respect to the complex structure $J$ on $\mathcal{M}$ and the natural complex structure on $\mathcal{QF}(\Sigma)$.

Recall from chapter three the set $\mathcal{A}$ is defined to consist of pairs$(J, \sigma)$ where $J$ is a complex structure on $\Sigma$, $\sigma$ is a holomorphic quadratic differential such that $|\sigma|_g < 1$ and the following equation is satisfied:

$$K_g + \frac{1}{2}\Delta \log(1 + \sqrt{1 - |\sigma|_g^2}) = -4.$$

Here $g$ is the metric induced by $\rho$ and $J$. Suppose we have a point $(J, \sigma) \in \mathcal{A}$, this corresponds to some section $s$ of the fibre bundle $\underline{D}$ over $\Sigma$. Recalling from chapter three that we have a map $\alpha : D \to H^2 \times \overline{H^2}$ we see that we have an induced map, which we shall also denote $\alpha$, from $\Gamma(\underline{D}) \to \Gamma(\underline{H^2 \times \overline{H^2}})$. In other words we have a map

$$\alpha : \Gamma(\underline{D}) \to \mathcal{J}(\Sigma) \times \overline{\mathcal{J}(\Sigma)}.$$

From the result in lemma 3.1.4 concerning the map $\alpha : D \to H^2 \times \overline{H^2}$ it follows immediately that $\alpha$ intertwines the complex structure (induced by) $J$ on $\Gamma(\underline{D})$ and the canonical complex structure on $\mathcal{J}(\Sigma) \times \overline{\mathcal{J}(\Sigma)}$. Therefore we have a holomorphic map

$$\alpha : \mathcal{A} \hookrightarrow \mathcal{J}(\Sigma) \times \overline{\mathcal{J}(\Sigma)}.$$

Recall also that we have a map $\tilde{\pi} : \mathcal{A} \to \mathcal{M}$ given by mapping the pair $(J, \sigma)$ to the $\mathrm{Diff}_0(\Sigma)$ equivalence class of the pair $(\tilde{g}, \sigma)$, where $\tilde{g} = (1 +$

$\sqrt{1-|\sigma|^2})g$. Consider the following diagram:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\alpha} & \mathcal{J}(\Sigma) \times \overline{\mathcal{J}(\Sigma)} \\ \tilde{\pi}\big\downarrow & & \pi\big\downarrow \\ \mathcal{M} & \xrightarrow{\iota} & \mathcal{T}(\Sigma) \times \overline{\mathcal{T}(\Sigma)} \end{array} . \tag{5.1}$$

From the discussion above we see that the maps $\alpha, \tilde{\pi}$, and $\pi$ are holomorphic. Since the results of chapter three tell us $\tilde{\pi}$ is a surjection we will have shown that the injection $\iota : \mathcal{M} \hookrightarrow \mathcal{T}(\Sigma) \times \overline{\mathcal{T}(\Sigma)}$ is holomorphic if we can show that the diagram commutes.

**Proposition 5.2.6.** *The diagram 5.1 is commutative, implying that the map*

$$\iota : (\mathcal{M}, J) \hookrightarrow \mathcal{T}(\Sigma) \times \overline{\mathcal{T}(\Sigma)},$$

*is holomorphic.*

*Proof.* Let $(J, \sigma) \in \mathcal{A}$ and choose local conformal coordinates on $\Sigma$. In these coordinates the fixed volume form $\rho$ is represented as $\rho = \mathrm{e}^{2\phi}(dx \wedge dy)$ and the induced metric $g = \mathrm{e}^{2\phi}(dx^2+dy^2)$. The section $s$ of $\underline{D}$ giving rise to $(J, \sigma)$ is locally a map to $T_i^* H^2$, $(x, y) \mapsto (u(x, y), v(x, y))$, and $|\sigma|^2 = y^2(u^2 + v^2)$. Then under the map $\alpha : \mathcal{A} \to \underline{H^2 \times \overline{H^2}}$ we obtain the section:

$$\alpha(s) = \left( -\frac{v}{1-u}, \frac{\sqrt{1-|\sigma|^2}}{1-u}, \frac{v}{1+u}, \frac{\sqrt{1-|\sigma|^2}}{1+u} \right).$$

This corresponds to the pair of complex structures:

$$J_+ = \begin{pmatrix} \frac{v}{\sqrt{1-|\sigma|^2}} & -\frac{1+u}{\sqrt{1-|\sigma|^2}} \\ \frac{1-u}{\sqrt{1-|\sigma|^2}} & -\frac{v}{\sqrt{1-|\sigma|^2}} \end{pmatrix}, \quad J_- = \begin{pmatrix} -\frac{v}{\sqrt{1-|\sigma|^2}} & -\frac{1-u}{\sqrt{1-|\sigma|^2}} \\ \frac{1+u}{\sqrt{1-|\sigma|^2}} & \frac{v}{\sqrt{1-|\sigma|^2}} \end{pmatrix}.$$

These complex structures are induced by the metrics

$$g_\pm = \frac{\mathrm{e}^{2\phi}}{\sqrt{1-|\sigma|^2}}((1 \mp u)dx^2 \mp 2v dx dy + (1 \pm u)dy^2).$$

Now, as a quadratic differential, we have that $\sigma = -\mathrm{e}^{2\phi}(u - iv)dz \otimes dz$, and writing $h$ for the real part of $\sigma$ as discussed earlier in this chapter we find $h = -\mathrm{e}^{2\phi}(udx^2 + 2vdxdy - udy^2)$, and in conclusion:

$$g_\pm = \frac{1}{\sqrt{1-|\sigma|^2}}(g \pm h).$$

Now consider the image of $(J, \sigma)$ in $\mathcal{B}$, it consists of the pair $(\tilde{g}, \sigma)$ where $\tilde{g} = (1 + \sqrt{1 - |\sigma|_g^2})g$. Recall from lemma 3.2.13 that we have

$$|\sigma|_{\tilde{g}} = \frac{|\sigma|_g}{1 + \sqrt{1 - |\sigma|_g^2}}.$$

Using this point in $\mathcal{B}$ we know from the previous section that we may construct the quasi-Fuchsian three-manifold $M_{(\tilde{g},\sigma)}$ whose metric is given by:

$$\Lambda = dt^2 + (\cosh^2 t + |\sigma|_g^2 \sinh^2 t)g - (2\cosh t \sinh t)h,$$

where the form $h$ is the real part of $\sigma$ as described earlier. We know that this three manifold corresponds to the point in the product of Teichmüller spaces $\mathcal{T}(\Sigma) \times \overline{\mathcal{T}(\Sigma)}$ defined by the conformal structures induced at infinity by this metric. The conformal structure induced on the slice $\Sigma \times \{t\}$ of $M_{(\tilde{g},\sigma)}$ by $\Lambda$ is induced by the metric

$$(1 + |\sigma|_{\tilde{g}}^2 \tanh^2 t)\tilde{g} - 2\tanh t\, h.$$

Therefore the conformal structures at infinity are induced by the metrics

$$\begin{aligned} g_{\pm\infty} &= (1 + |\sigma|_{\tilde{g}}^2)\tilde{g} \mp 2h \\ &= \left(1 + \frac{|\sigma|_g^2}{(1 + \sqrt{1 - |\sigma|_g^2})^2}\right)\left(1 + \sqrt{1 - |\sigma|_g^2}\right) g \pm 2h \\ &= 2(g \pm h). \end{aligned}$$

These maps therefore give rise to the same conformal structures and we have that diagram (5.1) commutes. □

## 5.3 Holomorphic symplectic structures on $\mathcal{QF}(\Sigma)$

In this section we pick a holomorphic symplectic form on the moduli space $\mathcal{M}$ with the complex structure $J$. Identifying $\mathcal{M}$ as an open subset of $\mathcal{QF}(\Sigma)$ we then compare this form with the restriction of Goldman's natural holomorphic symplectic form on the $\mathrm{PSL}_2\mathbb{C}$ representation variety.

### 5.3.1 The holomorphic symplectic structure on $\mathcal{M}$

Recall that Donaldson's moduli space $\mathcal{M}$ is equipped with a hyperkähler structure $(g, I, J, K)$ and three symplectic structures $\Omega_i,\ i \in \{1, 2, 3\}$.

**Lemma 5.3.1.** *The complex two form*

$$\omega_{\mathbb{C}} := \Omega_1 - i\Omega_3$$

*is a holomorphic-symplectic structure on the complex manifold* $(\mathcal{M}, J)$.

*Proof.* It is immediate that $\omega_{\mathbb{C}}$ defines a non-degenerate skew complex two form on $\mathcal{M}$. We need to show $\omega_{\mathbb{C}} \in \Omega^{2,0}_J(\mathcal{M})$. Since both $\omega_1$ and $\omega_3$ are closed we see that $d\omega_{\mathbb{C}} = 0$ but then by comparing types we see that the image of $\overline{\partial} : \Omega^{2,0}_J(\mathcal{M}) \to \Omega^{2,1}_J(\mathcal{M})$ is zero, so that $\omega_{\mathbb{C}}$ is holomorphic. $\square$

Since $\iota : (\mathcal{M}, J) \to \mathcal{QF}(\Sigma)$ is a holomorphic embedding we see that we can push the form forward to get a form $\omega_{\mathbb{C}}$ on the subset of $\mathcal{QF}(\Sigma)$ defined by the image of $\mathcal{M}$, and this form will be a holomorphic symplectic form with respect to the restriction of the natural complex structure on the quasi-Fuchsian deformation space.

### 5.3.2 Comparing holomorphic-symplectic structures

In this section we show that the two holomorphic-symplectic forms coincide on the image of Donaldson's moduli space inside the quasi-Fuchsian moduli space. We use the fact that both symplectic structures restrict to the Weil-Petersson form on Teichmüller space to conclude they must be equal as holomorphic 2-forms on the diagonal. We then use an analytic continuation argument to show they are equal on the whole of $\mathcal{M}$. Throughout this section we shall denote by $\hat{\mathcal{T}}$ the natural diagonal embedding $\mathcal{T}(\Sigma) \hookrightarrow \mathcal{QF}(\Sigma)$, and will use the fact that $\mathcal{M}$ may be thought of as an open subset of $\mathcal{QF}(\Sigma)$ without further comment.

First we establish the following piece of linear algebra:

**Lemma 5.3.2.** *Let $W$ be a $n$ dimensional complex vector space, $V$ an $n$ dimensional real vector space. Suppose that we have an inclusion $\iota_* : V \to W$ and under this identification*

$$W = V \oplus iV,$$

*then the natural map $\pi_* : W \to V$ given by projection induces an isomorphism*

$$\pi^* : \wedge^2 V^* \otimes_{\mathbb{R}} \mathbb{C} \to \wedge^2 W^*.$$

*Proof.* Let $\{\nu_1, \nu_2, ..., \nu_n\}$ be a basis for $V^*$ and $\{v_1, v_2, ..., v_n\}$ be a dual basis for $V$. For each $i$ choose $w_i$ such that $\pi_* w_i = v_i$. Let

$$\sum_{i<j} \alpha_{ij} \otimes \nu_i \wedge \nu_j \in \wedge^2 V^* \otimes_{\mathbb{R}} \mathbb{C},$$

for $\alpha_{ij}$ not all zero, be in the kernel of $\pi^*$. But then we have for all $l < m$

$$\begin{aligned} 0 &= \left( \pi^* \sum_{i<j} \alpha_{ij} \otimes \nu_i \wedge \nu_j \right) (w_l \wedge w_m) \\ &= \left( \sum_{i<j} \alpha_{ij} \otimes \nu_i \wedge \nu_j \right) (v_l \wedge v_m) \\ &= \alpha_{lm}. \end{aligned}$$

Therefore $\pi^*$ has no kernel. Since the complex dimensions of the spaces under consideration are equal we have the result. □

This now allows us to identify the two holomorphic-symplectic forms on the subset $\hat{\mathcal{T}}$ of quasi-Fuchsian deformation space.

**Proposition 5.3.3.** *The holomorphic symplectic forms $\omega_G$ and $\omega_{\mathbb{C}}$ are equal on $\hat{\mathcal{T}} \subset \mathcal{QF}(\Sigma)$.*

*Proof.* Let $\iota : \mathcal{T}(\Sigma) \to \hat{\mathcal{T}} \subset \mathcal{QF}(\Sigma)$ be the diagonal embedding. Since by theorem 5.1.9 and proposition 3.2.15 the restriction of both the forms $\omega_G$ and $\omega_{\mathbb{C}}$ to $\hat{\mathcal{T}}$ is the Weil-Petersson form we have by definition that:

$$\iota^* \omega_G = \omega_{\mathrm{WP}} = \iota^* \omega_{\mathbb{C}}.$$

Let $V$ be the tangent space to $\mathcal{T}(\Sigma)$ at $[J]$, and $W$ the tangent space to $\mathcal{QF}(\Sigma)$ at $\iota([J])$. Considering $V$ as a subset of $W$ and recalling that the complex structure on $\mathcal{QF}(\Sigma)$ is the natural one on $\mathcal{T}(\Sigma) \times \overline{\mathcal{T}(\Sigma)}$ we see that $V \oplus iV = W$ so that we may apply lemma 5.3.2. We conclude that holomorphic two-forms $\omega_G = \omega_{\mathbb{C}}$ are equal on the subset $\hat{\mathcal{T}}$. □

We now seek to use analytic continuation to conclude the forms are equal on the whole of $\mathcal{M}$. We remark that this is very similar to the method employed by Platis in [21]. To begin we prove a lemma about analytic continuation in $\mathbb{C}^n$. A small piece of notation is required here, if $U \subset \mathbb{C}^n$ we denote by $\mathbb{R}(U)$ the set of *real points* of $U$, that is $\mathbb{R}^n \cap U$ where the embedding of $\mathbb{R}^n$ in $\mathbb{C}^n$ is the standard one.

**Lemma 5.3.4.** *Let $U \subset \mathbb{C}^n$ be an open connected region with a non-empty set of real points $\mathbb{R}(U)$. Suppose we have a holomorphic function $\psi : U \to \mathbb{C}$ such that $\psi(z_1, z_2, ..., z_n) = 0$ if $z_i \in \mathbb{R}\ \forall i \in \{1...n\}$. Then $\psi$ is identically zero in $U$.*

*Proof.* Let $(\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}(U)$, since $U$ is open we can certainly find $r \in \mathbb{R}$ such that $r > 0$ and the open polydisc $P_r(\alpha_1, \alpha_2, ...\alpha_n) = \bigotimes_{i=1}^{i=n} D_r(\alpha_i)$ is a subset of $U$. Here of course $D_r(\alpha_i)$ is the disc of radius $r$ in $\mathbb{C}$. Now consider the function $\tilde{\psi}_1 : D(\alpha_1, r_1) \to \mathbb{C}$ defined by

$$\tilde{\psi}_1(z) = \psi(z, \alpha_2, ..., \alpha_n).$$

Now $\tilde{\psi}_1 = 0$ on $(\mathbb{R} \cap D(\alpha_1)$ which is a non-empty subset of $\mathbb{C}$ and by the identity theorem for one complex variable we conclude that $\tilde{\psi}_1$ is identically zero on $D(\alpha_1)$. Now let $\beta_1 \in D_r(\alpha_1)$ and consider the map $\tilde{\psi}_2 : \mathbb{C} \to \mathbb{C}$ defined by

$$\tilde{\psi}_2(z) = \psi(\beta_1, z, ..., \alpha_n).$$

As above we can conclude $\tilde{\psi}_2$ is identically zero on $D_r(\alpha_2)$, and since $\beta_1$ was arbitrary we find we have that the map $\psi$ is identically zero on the set

$$(z_1, z_2, \alpha_3, ..., \alpha_n),$$

for $(z_1, z_2) \in D_r(\alpha_1) \times D_r(\alpha_2)$. It is clear that we can continue in this fashion to obtain that $\psi$ is zero on the polydisc $P_r(\alpha_1, \alpha_2, ...\alpha_n)$. Repeating for the other points in $\mathbb{R}(U)$ we see that $\psi$ is zero on an open neighbourhood of $\mathbb{R}(U)$ in $U$. The identity theorem for several complex variables allows us to conclude that $\psi$ is zero in $U$. □

This is the final ingredient we need to show that the two holomorphic-symplectic forms on $\mathcal{M}$ are the same.

**Proposition 5.3.5.** *The restriction of Goldman's holomorphic-symplectic form $\omega_G$ to Donaldson's moduli space $\mathcal{M}$ coincides with the holomorphic-symplectic form $\omega_{\mathbb{C}}$.*

*Proof.* Let $U$ be an open neighbourhood of a point in $\hat{\mathcal{T}}$, and take local complex coordinates for $\mathcal{QF}(\Sigma)$, $(z_1, ...z_n)$ on $U$. In these coordinates we have that the difference $\eta := \omega_G - \omega_{\mathbb{C}}$ can be written,

$$\eta = \Sigma_{i,j}\psi_{ij}dz_i \wedge dz_j,$$

for some holomorphic functions $\psi_{ij}$. Since by proposition 5.3.3 we must have that $\psi_{ij} = 0$ on $U \cap \hat{\mathcal{T}}$ we may apply 5.3.4 to conclude that $\psi_{ij} = 0$ on $U$. It follows that $\eta$ is identically zero on an open neighbourhood of $\hat{\mathcal{T}}$, and from the identity theorem for complex manifolds hence identically zero on the connected set $\mathcal{M}$. □

We note that Platis in [21] constructs a holomorphic symplectic form on $\mathcal{QF}(\Sigma)$ coinciding with Goldman's form and hence by the preceding theorem, ours.

# Chapter 6

In this chapter we consider the image of the hyperkähler moduli space $\mathcal{M}$ inside the quasi-Fuchsian deformation space. We start by discussing the existence of a family of minimal surfaces inside a family of quasi-Fuchsian manifolds. This being established we discuss sufficient conditions for the map $\mathcal{M} \to \mathcal{QF}(\Sigma)$ to be a surjection. We then show how, under certain assumptions, we may extend the hyperkähler structure on $\mathcal{M}$ to a strictly larger open neighbourhood $\tilde{\mathcal{M}}$ of $\mathcal{M}$. Lastly we introduce Taubes' moduli space of minimal hyperbolic germs. Our moduli space sits inside his in a canonical fashion, we examine whether we might induce a hyperkähler structure off the image of our moduli space inside Taubes'.

## 6.1 Families of minimal surfaces

Fix $M = \Sigma \times \mathbb{R}$ as a smooth manifold, and write $M_\lambda$ to denote $M$ with the Riemannian metric $\lambda$. The purpose of this section is to deduce results about the existence of families of minimal surfaces associated to families of quasi-Fuchsian manifolds. We follow the program for finding harmonic maps laid out in [7] and [24]. It is immediate from results of Sacks and Uhlenbeck [28], that given any hyperbolic three manifold we can find a minimal immersion of $\Sigma$ in it, therefore given any open set in the space of quasi-Fuchsian metrics on $M$ we can define a map to the space of maps from $\Sigma$ to $M$. However, this is not enough for our eventual purposes; we would like to deduce, at the least, continuity of the resulting family of minimal surfaces. We proceed to discuss these matters.

Let $\Lambda_{\mathrm{QF}}$ be the set of all quasi-Fuchsian metrics on $M$, and let $\Phi$ denote the set of all smooth embeddings $\phi : \Sigma \hookrightarrow M$. Let $\lambda \in \Lambda_{\mathrm{QF}}$, by definition, a minimal embedding $\phi : \Sigma \to M_\lambda$ is a critical point of the area functional:

$$A(\phi) := \int_\Sigma \rho_\phi,$$

where $\rho_\phi$ is the area form on $\Sigma$ induced by the embedding $\phi$. It is well known, see for example [6], that critical points of this functional are those embeddings whose mean curvature vanishes, the mean curvature being the trace of the second fundamental form of the embedding. We shall denote the second fundamental form of the embedding $\phi : \Sigma \to M_\lambda$ by $S_\lambda(\phi)$ and the mean curvature of the embedding by $\overline{S}_\lambda(\phi)$.

Fix now a metric $\lambda$ on $M$ and suppose we have a smooth family $\phi_s$ of embeddings of $\Sigma$ in $M_\lambda$. We have the following formula for the derivative of mean curvature.

**Lemma 6.1.1** ([25])**.** *Let $\overline{S}_\lambda : \Phi \to C^\infty(\Sigma)$ denote the mean curvature operator, and $\phi_s$ a smooth family of embeddings $\phi_s : \Sigma \to M$, then*

$$\left.\frac{d}{ds}\right|_{s=0} \overline{S}_\lambda(\phi_s) = \Delta\psi - (1 - |S_\lambda(\phi_0)|^2)\psi.$$

*Where,*

$$\psi := \left( \left.\frac{d}{ds}\right|_{t=0} \phi_s \right)^N,$$

*is the normal component of the derivative of $\phi_s$, and $|S_\lambda(\phi_0)|$ is the pointwise norm of the second fundamental form of the embedding in the induced metric on $\Sigma$.*

We call the resulting operator on derivatives to the space of smooth embeddings of $\Sigma$ the *Jacobi operator* and denote it $\mathbf{J}_\phi$.

**Corollary 6.1.2.** *Suppose that $\phi_s$ is a family of embeddings such that the norm of the second fundamental form is not constant and satisfies*

$$|S_\lambda(\phi_0)|^2 \leq 1,$$

*pointwise on $\Sigma$, then the Jacobi operator is invertible.*

*Proof.* Suppose $\psi \in \ker(\mathbf{J}_\phi)$, then

$$\begin{aligned} 0 &= \int_\Sigma \mathbf{J}_\phi(\psi)\psi\rho \\ &= \int_\Sigma \left(\Delta\psi - (1 - |S_\lambda(\phi_0)|^2)\psi\right)\psi\rho \\ &= \int_\Sigma -|\nabla\psi|^2\rho - \int_\Sigma (1 - |S_\lambda(\phi_0)|^2)\psi^2\rho. \end{aligned}$$

But, for $\psi$ not identically zero, the hypothesis ensure this last is strictly negative. Hence $\mathbf{J}_\phi$ has trivial kernel. □

The remainder of this section is a discussion of the ideas needed to establish the following result.

**Theorem 6.1.3.** *Let $\lambda_0 \in \Lambda$, and suppose $\phi_0 : \Sigma \to M_{\lambda_0}$ is a minimal embedding, with the Jacobi operator invertible. Then there exists an open set $U \subset \Lambda$ containing $\lambda_0$ and such that given any $\lambda \in U$ there exists a minimal embedding $\phi_\lambda : \Sigma \to M_\lambda$. The association $\lambda \mapsto \phi_\lambda$ is continuous.*

Denote by $\Lambda$ the space of metrics on $M$. Let $(\lambda_0, \phi_0) \in \Lambda \times \Phi$ be such that $\overline{S}_{\lambda_0}(\phi_0) = 0$ and $\mathbf{J}_{\phi_0}$ is invertible. We aim to use the implicit function theorem to find a neighbourhood $U \subset \Lambda$ of any $\lambda_0$ for which there exists a $\phi_0$ satisfying $\overline{S}_{\lambda_0}(\phi_0) = 0$, such that for all $\lambda \in U$ we may find a $\phi_\lambda$, varying continuously with $\lambda$, such that $\overline{S}_\lambda(\phi_\lambda) = 0$.

*proof of theorem 6.1.3.* Let $\Lambda^k$ be the space of metrics on $M$ that are $k$ times weakly differentiable, that is sections of $T^*M \otimes T^*M$ that are $k$ times weakly differentiable and are metrics. It is immediate that $\Lambda^k$ is an open set in the Sobolev space $\Gamma^k(T^*M \otimes T^*M)$ consisting of $k$ times weakly differentiable square integrable sections of $T^*M \otimes T^*M$. Here we use the metric $\lambda_0$ to define the notion of square integrable.

Consider a tubular neighbourhood of $\phi_0(\Sigma)$ in $M_{\lambda_0}$, since $M \cong \Sigma \times \mathbb{R}$ we know the tubular neighbourhood is of the form $\Sigma \times \mathrm{I}$ for some interval $\mathrm{I} \subset \mathbb{R}$. Let $\Phi^k$ be the set maps $\phi : \Sigma \to \Sigma \times \mathrm{I}$ given by

$$p \mapsto (p, f(p)),$$

where $f \in L^2_k(\Sigma)$ is $k$ times weakly differentiable.

Since calculating the mean curvature involves taking derivatives of both the function and the components of the metric tensor metric, the mean curvature map $\overline{S}$ maps $\Lambda^k \times \Phi^k$ to $L^2_{k-2}(\Sigma)$, for $k > 2$ as we can see from working in coordinates, see [6].

The hypothesis that the Jacobi operator associated to $(\lambda_0, \phi_0)$ is invertible, implies precisely that the derivative of the mean curvature map in the direction of the embeddings is invertible at $(\lambda_0, \phi_0)$. Therefore, by the implicit function theorem in Banach spaces we obtain a neighbourhood $U^k \subset \Lambda^k$ of $\lambda_0$ and a continuous map $U^k \to \Phi^k$, $\lambda \mapsto \phi_\lambda$ such that

$$\overline{S}(\lambda, \phi_\lambda) = 0.$$

Now by restriction we obtain a neighbourhood $U' \subset \Lambda$ of the smooth metrics and an associated map $\phi_\lambda : \Sigma \to M_\lambda$ that is a minimal embedding. Results on Harmonic maps [6] ensure that $\phi_\lambda$ is itself in fact smooth allowing the conclusion of the theorem. □

Given the results on families of harmonic maps associated to varying metrics in [24] and [7], we expect that we can actually take the map $\lambda \mapsto \phi_\lambda$ to be smooth. We have not examined this.

## 6.2 The image of $\mathcal{M}$ in quasi-Fuchsian space

In the previous chapter we defined an embedding of the hyperkähler extension of Teichmüller space $\mathcal{M}$ into the quasi-Fuchsian defomation space $\mathcal{QF}(\Sigma)$. A natural question is whether this map is in fact a surjection or not. This appears to be a delicate matter to which we do not know the answer at present. In this section we discuss the work of Uhlenbeck [28] which enables a partial answer to be given.

Suppose we have a point $[\theta]$ in the quasi-Fuchsian deformation space. Here $\theta$ is a representation of the fundamental group of $\Sigma$ into $\mathrm{PSL}_2\mathbb{C}$ whose image is quasi-Fuchsian. We recall that given such a representation, then the quotient of the hyperbolic three space $H^3$ by $\hat{\theta} := \theta(\pi_1(\Sigma))$ is a hyperbolic three manifold $M_\theta$; by definition this is what we mean by a quasi-Fuchsian three manifold. Equivalently we get a pair $(J_1, \overline{J_2}) \in \mathcal{J}(\Sigma) \times \overline{\mathcal{J}(\Sigma)}$, which we may think of as the conformal structures induced by the action of $\hat{\theta}$ on the $H^2$ at infinity in $H^3$. Further recall that the latter identification extends to give us an identification

$$\mathcal{QF}(\Sigma) \cong \mathcal{T}(\Sigma) \times \overline{\mathcal{T}(\Sigma)},$$

associating to $[\theta] \in \mathcal{QF}(\Sigma)$ the point $([J_1], [\overline{J_2}])$ in the product of the Teichmüller spaces.

We have the following proposition due to Uhlenbeck,

**Theorem 6.2.1** (Uhlenbeck [28])**.** *Suppose for some metric $g$ on $\Sigma$ we have a minimal isometric embedding of $\Sigma$ into a quasi-Fuchsian three manifold $M \cong \Sigma \times \mathbb{R}$, then the second fundamental form of the embedding defines a holomorphic quadratic differential $\sigma$ on $\Sigma$ and the pair $(g, \sigma)$ satisfy the Gauss equation*

$$K_g + |\sigma|_g^2 = -1.$$

*If in addition the holomorphic quadratic differential satisfies $|\sigma|_g < 1$ then the minimal surface is unique.*

We arrive at the following characterisation of the moduli space $\mathcal{M}$ in terms of its image in the quasi-Fuchsian moduli space.

**Proposition 6.2.2.** *The hyperkähler moduli space $\mathcal{M}$ is identified with that subset of $\mathcal{QF}(\Sigma)$ defined by the set of points $q \in \mathcal{QF}(\Sigma)$ such that for any representation $\theta$ with $[\theta] = q$ the hyperbolic manifold $M_\theta$ contains a minimal $\Sigma$ whose second fundamental form induces a holomorphic quadratic differential $\sigma$ with $|\sigma|_g < 1$.*

*Proof.* Denote by $\mathcal{M}'$ the set of points $q \in \mathcal{QF}(\Sigma)$ satisfying the hypothesis of the statement of the theorem. Suppose we have a pair $(g, \sigma) \in \mathcal{B}$, the work of the previous chapter tells us precisely that the map

$$\Sigma \hookrightarrow M_{(g,\sigma)},$$

is a minimal isometric immersion of $(\Sigma, g)$ into the quasi-Fuchsian three manifold $M_{(g,\sigma)}$. The second fundamental form is given by the real part of $\sigma$ and by hypothesis $|\sigma|_g < 1$. The deck transformations on the universal cover $H^3$ associated to this manifold give a representation of $\pi_1(\Sigma)$ into $\mathrm{PSL}_2\mathbb{C}$ that is quasi-Fuchsian by construction. It follows immediately that $\mathcal{M}$ is identified with some subset of $\mathcal{M}'$.

Now suppose we have a $q \in \mathcal{M}'$, pick a representative representation

$$\theta : \pi_1(\Sigma) \to \mathrm{PSL}_2\mathbb{C},$$

so that $q = [\theta]$. Consider the quasi-Fuchsian manifold $H^3/\hat{\theta}$. By hypothesis there exists a metric $g$ on $\Sigma$ and a minimal isometric embedding

$$\Sigma \hookrightarrow M_\theta,$$

so that the associated holomorphic quadratic differential satisfies $|\sigma|_g < 1$. In view of the proposition above we see that the pair $(g, \sigma)$ satisfies the equation

$$K_g + |\sigma|_g^2 = -1,$$

and hence lies in $\mathcal{B}$. The uniqueness part of theorem 6.2.1 ensures that this is a well defined assignment of a point in $\mathcal{B}$ to a quasi-Fuchsian representation $\theta$. It follows that this is the inverse of the map that constructs a quasi-Fuchsian manifold from a point in $\mathcal{B}$. Since the map from $\mathcal{B}$ descends to the quotient as a map $\mathcal{M} \to \mathcal{QF}(\Sigma)$ we must have that the map from $\mathcal{M}'$ to $\mathcal{M}$ is well defined, which completes the proof. □

In summary we see that we have a hyperkähler structure on some open subset $\mathcal{M} \subset \mathcal{QF}(\Sigma)$.

### 6.2.1 Extending the hyperkähler metric off $\mathcal{M}$

In this section we suppose that in fact the map from the moduli space $\mathcal{M}$ to the quasi-Fuchsian deformation space is not onto and attempt to extend the hyperkähler structure off $\mathcal{M}$ to some strictly larger open set in $\mathcal{QF}(\Sigma)$. We know the moduli space $\mathcal{M}$ is hyperkähler with complex structures $I, J, K$, and corresponding symplectic structures $\omega_1, \omega_2, \omega_3$. Recall from chapter three that the action of multiplication by $I$ has a Hamiltonian $A$ which provides a Kähler potential for the symplectic form $\omega_2$, that is:

$$\omega_2 = 2i\partial_J\overline{\partial}_J A.$$

On $D$, the hyperkähler extension of $H^2$, this potential function was just given by

$$\sqrt{1-|\sigma|^2}-1,$$

therefore on the moduli space $\mathcal{N}$ it is just given by the function

$$A=\int_\Sigma\left(\sqrt{1-|\sigma|_g^2}-1\right)\rho,$$

where $g$ is the metric induced by $\rho$ and $J$. But the point $(J,\sigma)\in\mathcal{A}$ corresponds to the point

$$\left((1+\sqrt{1-|\sigma|_g^2})g,\sigma\right)\in\mathcal{B},$$

thus up to a constant $A$ is just the area of the metric in $\mathcal{B}$, which corresponds to the area of the minimal surface in the quasi-Fuchsian manifold $M_{(g,\sigma)}$.

Consider the structures we have on the complex manifold $\mathcal{QF}(\Sigma)$. We have the complex structure, which we know restricts to the complex structure $J$ on $\mathcal{M}$, and we have the canonical holomorphic symplectic form of Goldman, which we know from the last chapter restricts to the holomorphic symplectic form $\omega_1-i\omega_3$ on $\mathcal{M}$. We are therefore missing one symplectic form on $\mathcal{QF}(\Sigma)$ that would turn it into a hyperkähler manifold. However, if we have a well defined area functional $A$, the preceding paragraph suggests we might attempt to define the third symplectic structure as

$$\omega_2:=2i\partial_J\overline{\partial}_JA.$$

We pursue this idea now.

Recall from chapter three the definition of the set $\mathcal{B}$ as the set of pairs $(g,\sigma)$ with $\sigma$ holomorphic, $|\sigma|_g<1$ and

$$K_g+|\sigma|_g^2=-1.$$

Define the area function $A$ on $\mathcal{B}$ that associates to a pair $(g,\sigma)$ the area of $\Sigma$ with the volume form $\rho_g$ induced by $g$:

$$A[(g,\sigma)]=\int_\Sigma\rho_g.$$

It is clear that this is well defined on $\mathcal{B}$ and also clear, since diffeomorphisms preserve area, that it is well defined on the quotient $\mathcal{M}$.

**Proposition 6.2.3.** *Let $q \in \mathcal{QF}(\Sigma)$ and suppose that there exists quasi-Fuchsian manifold representing $q$ containing a minimal surface with invertible Jacobi operator. Then there exists a neighbourhood $U$ of $q$ and a function $\tilde{A} : U \to \mathbb{R}$ induced by the area of a family of minimal surfaces in the neighbourhood of any quasi-Fuchsian manifold representing $q$ . We call $\tilde{A}$ a local area function at $q$.*

*Proof.* Suppose $\lambda_0$ is the quasi-Fuchsian metric on $M = \Sigma \times \mathbb{R}$ representing the point $q \in \mathcal{QF}(\Sigma)$ guaranteed by the hypotheses. Then $M_{\lambda_0}$ must contain a minimal surface $\phi_0 : \Sigma \to M_{\lambda_0}$ with invertible Jacobi operator. Therefore we may apply theorem 6.1.3 to obtain a neighbourhood of $\lambda_0$ in the space of quasi-Fuchsian metrics and an associated family of minimal embeddings of $\Sigma$. We define the area function to be the area of these minimal embeddings. □

Note that a priori this construction depends on the lift of $q$ to a quasi-Fuchsian manifold and on the choice of stable minimal surface in this manifold, that is there may be several different local area functions at $q$. However, if we chose $q \in \mathcal{M}$ then it follows from the uniqueness of minimal surfaces in quasi-Fuchsian manifolds representing classes in $\mathcal{M}$ that there is a unique local area function at $q$ and indeed it is the area function $A$ defined earlier.

**Theorem 6.2.4.** *Let $q \in \mathcal{QF}(\Sigma)$ be in $\overline{\mathcal{M}}$, the closure of $\mathcal{M}$. Suppose that there exists quasi-Fuchsian manifold representing $q$ containing a minimal surface with invertible Jacobi operator and that the resulting local area functional $\tilde{A}$ is analytic. Then we may extend the hyperkähler metric on $\mathcal{M}$ to an open neighbourhood containing $\mathcal{M} \bigcup \{q\}$.*

*Proof.* Let the analytic local area functional $\tilde{A}$ be defined on the neighbourhood $U$ of $q \in \mathcal{QF}(\Sigma)$. Since by hypothesis $q \in \overline{\mathcal{M}}$ we have that $U \bigcap \mathcal{M}$ is open in $\mathcal{QF}(\Sigma)$ and non-empty. Since the area functional $A$ is uniquely defined on $\mathcal{M}$ we must have that on $U \bigcap \mathcal{M}$, $\tilde{A} = A$. This allows us to take $\tilde{A}$ as an analytic extension of $A$ to $U \bigcap \mathcal{M}$. We write $A$ for the extended function and define a holomorphic two form on $U \bigcap \mathcal{M}$ as

$$\tilde{\omega}_2 = 2i\partial_J \overline{\partial}_J A.$$

Clearly this agrees with $\omega_2$ on $\mathcal{M}$ and is analytic on $U \bigcap \mathcal{M}$. We are therefore free to write the resulting two form as $\omega_2$.

Now write $\omega_1$ and $\omega_3$ for the symplectic structures on $U \bigcap \mathcal{M}$ coming from setting $\omega_1 - i\omega_3$ to be Goldman's canonical holomorphic symplectic form on $\mathcal{QF}(\Sigma)$. We know these restrict to the suggested symplectic forms on $\mathcal{M}$. Let $n$ be the dimension of $\mathcal{QF}(\Sigma)$, on $\mathcal{M}$ we have

$$\omega_1^n = \omega_2^n = \omega_3^n.$$

Now $\omega_1$ and $\omega_3$ are analytic and non-degenerate on the whole of $\mathcal{QF}(\Sigma)$, since we have that $\omega_2$ is analytic on $U \bigcap \mathcal{M}$ this identity must hold on $U \bigcap \mathcal{M}$ so that $\omega_2$ is non-degenerate on $U \bigcap \mathcal{M}$. Thus $\omega_2$ is a symplectic form on $U \bigcap \mathcal{M}$. We require the three forms $\omega_1, \omega_2, \omega_3$ satisfy the correct algebraic identities for a hyperkähler structure. But since the algebraic identities hold on the open set $\mathcal{M}$ they must hold on the extension $U \bigcap \mathcal{M}$. A theorem of Hitchin [13] then ensures that since the symplectic forms are closed the complex structures they induce are integrable so that we indeed have a hyperkähler structure extending that on $\mathcal{M}$. □

Now define $\tilde{\mathcal{M}}$ to be the set of points in the closure of $\mathcal{M}$ for which we can find an analytic local area function. Using the above theorem we can obtain the following:

**Corollary 6.2.5.** *There is a hyperkähler structure on $\tilde{\mathcal{M}}$ extending that on $\mathcal{M}$.*

*Proof.* This follows immediately from the methods of the previous theorem once we observe that given any two $q_1, q_2 \in \tilde{\mathcal{M}}$, the local area functionals $\tilde{A}_1, \tilde{A}_2$ defined on the respective open neighbourhoods $U_1$ and $U_2$ of $\mathcal{QF}(\Sigma)$ are analytic. Their respective analytic extensions to $\mathcal{M} \bigcup U_i$ agree on the open set $\mathcal{M}$ and so we have a well defined analytic function on $\mathcal{M} \bigcup U_1 \bigcup U_2$. □

To conclude this section we note that in his thesis [22] Platis constructs a hyperkähler structure on the quasi-Fuchsian space. His structure has the complex structure $J$ and the holomorphic symplectic form $\omega_1 - i\omega_3$ and therefore must agree with ours on its domain of definition.

## 6.3 Taubes' moduli space of hyperbolic germs

In this section we shall describe a recent construction of Taubes [26]. There is an obvious injection of the moduli space $\mathcal{M}$ into Taubes' space, we discuss whether or not the resulting hyperkähler structure can be extended from this subset. Note, that in what follows we use a slightly different normalisation to that employed in Taubes' paper, this is to facilitate comparison with the rest of the work presented in this thesis.

As usual $\Sigma$ denotes a closed compact smooth surface of genus at least 2. Let $\chi(\Sigma)$ be the Euler characteristic of $\Sigma$, so that $\chi(\Sigma) < 0$.

**Definition 6.3.1.** A pair $(g, \sigma)$ consisting of a metric $g$ and a holomorphic quadratic differential $\sigma$ is called a *minimal hyperbolic germ* on $\Sigma$ if

$$K_g + |\sigma|_g^2 = -1.$$

This clearly contains the space $\mathcal{B}$ described earlier, indeed the only difference is that we have removed the restriction on the pairs $(g, \sigma)$ that $|\sigma|_g < 1$. In view of this, we denote the space of minimal hyperbolic germs $\tilde{\mathcal{B}}$.

The identity component of the diffeomorphism group of $\Sigma$, $\mathrm{Diff}_0(\Sigma)$, acts on the set of minimal hyperbolic germs on $\Sigma$. Taubes defines his *moduli space of minimal hyperbolic germs* $\mathcal{H}$ to be the resulting quotient . That is,

$$\mathcal{H} = \tilde{\mathcal{B}}/\mathrm{Diff}_0(\Sigma).$$

We have the following result about the analytic structure of this space.

**Theorem 6.3.2** (Taubes)**.** *The moduli space $\mathcal{H}$ has the structure of a smooth, orientable manifold of dimension $-6\chi(\Sigma)$.*

It is immediate from the definition of the space $\mathcal{H}$ that we have an embedding

$$\mathcal{M} \hookrightarrow \mathcal{H},$$

induced by the embedding of $\mathcal{B}$ into $\tilde{\mathcal{B}}$. Thus the hyperkähler moduli space $\mathcal{M}$ sits inside Taubes' moduli space of minimal hyperbolic germs $\mathcal{H}$ in a natural way.

Given a minimal hyperbolic germ $(g, \sigma)$ we now construct in exactly the same fashion as in the last chapter a three manifold $M_{(g,\sigma)}$.

**Proposition 6.3.3.** *Let* $(g, \sigma) \in \tilde{\mathcal{B}}$ *be a minimal hyperbolic germ. Then there exists a hyperbolic three manifold* $M_{(g,\sigma)}$*, that is topologically* $\Sigma \times \mathrm{I}$ *for some subinterval* $\mathrm{I}$ *of* $\mathbb{R}$ *and contains the manifold* $\Sigma$ *with the metric* $g$ *as a minimal embedding.*

*Proof.* The idea here is to define as in the last chapter a symmetric bilinear form

$$\Lambda = dt^2 + (\cosh^2 t + |\sigma|_g^2 \sinh^2 t)g - (2\cosh t \sinh t)h,$$

with $h$ the real part of $\sigma$. This metric is non degenerate on $\Sigma \times \mathbb{R}$ provided

$$\cosh^2 t - |\sigma|_g^2 \sinh^2 t > 0.$$

So that we get a metric $\Lambda$ on the subset of $\Sigma \times \mathbb{R}$ where

$$|\sigma|_g < \frac{\cosh t}{|\sinh t|}.$$

We define the interval I to be the maximal one for which this inequality holds $\forall t \in \mathrm{I}$, and define $M_{(g,\sigma)}$ to be the manifold $\Sigma \times \mathrm{I}$ with the metric $\Lambda$. The results of the last chapter tell us that $\Sigma$ is minimally isometrically embedded as the zero slice, and the result of Uhlenbeck tells us that the metric is hyperbolic. □

So we have constructed a hyperbolic thickening of the surface $\Sigma$ to a three manifold, this explains the nomenclature minimal hyperbolic germ for the pair $(g, \sigma) \in \tilde{\mathcal{B}}$.

The definition of $\mathcal{H}$ allows us to define a canonical map into the cotangent bundle of Teichmüller space,

$$\mathcal{H} \to T^*\mathcal{T}(\Sigma),$$

extending the map from the hyperkähler moduli space $\mathcal{M}$. We have no uniqueness result for solutions of the equation

$$K_g + |\sigma|_g^2 = -1,$$

if $|\sigma|_g > 1$ at any point of $\Sigma$, therefore in contrast to the situation with $\mathcal{M}$ we do not have an embedding of $\mathcal{H}$ in $T^*\mathcal{T}(\Sigma)$. However, recall from section

6.1 that given an embedding $\phi$ of $\Sigma$ into $\Sigma \times \mathbb{R}$ with metric $\lambda$ we obtain the Jacobi operator

$$\mathbf{J}_\phi(\psi) = \Delta\psi - (1 - |S_\lambda(\phi_0)|^2)\psi,$$

for $\psi$ a normal deformation of $\phi$. This carries over to our current situation where we thing of $\Sigma$ as embedded as the zero slice in the hyperbolic thickening associated to $(g, \sigma)$. We can therefore associate a Jacobi operator to a minimal hyperbolic germ $(g, \sigma)$. If this is invertible then the operator associated to any pair $(g', \sigma')$ in the orbit of $(g, \sigma)$ under the diffeomorphism group of $\Sigma$ is also invertible, so that invertibility of the Jacobi operator is a well defined concept on $\mathcal{H}$. We denote by $\tilde{\mathcal{H}}$ the subset of $\mathcal{H}$ on which the Jacobi operator is invertible. We have the following:

**Proposition 6.3.4** (Taubes [26])**.** *The subset of $\mathcal{H}$ on which the canonical map to the coatngent bundle of Teichmüller space is an immersion is precisely $\tilde{\mathcal{H}}$.*

We note that the above proposition also follows from our analysis of the Gauss equation in chapter three.

**Proposition 6.3.5.** *The subset $\tilde{\mathcal{H}}$ is a complex manifold with a holomorphic symplectic structure.*

*Proof.* We just pull back the complex structure $I'$ and the canonical holomorphic symplectic form $\omega_2' + i\omega_3'$ from $T^*\mathcal{T}(\Sigma)$ using the map from Taubes' moduli space to the cotangent bundle of Teichmüller space. On $\tilde{\mathcal{H}}$ this map is locally injective so that the pulled back structures define what we require. We note that on the subset $\mathcal{M}$ the complex structure and holomorphic symplectic form coincide with the relevant ones from the hyperkähler structure. □

### 6.3.1 The map from $\mathcal{H}$ to the $\mathrm{PSL}_2\mathbb{C}$ representation variety

Let $(g, \sigma)$ be a minimal hyperbolic germ. Denote by $h$ the hermitian metric on $\Sigma$ induced by $g$ and the complex structure $J_g$ induced by $g$. In this section we follow Donaldson [5] and show how to associate to a such an $h$ a representation of the fundamental group of $\Sigma$ into $\mathrm{PSL}_2\mathbb{C}$, this allows us to define a map from Taubes' moduli space to the $\mathrm{PSL}_2\mathbb{C}$ representation variety. This is all motivated by the work of Hitchin on self-duality equations [13] though here we keep this rather in the background.

The Chern connection induced by $h$ is a $\mathrm{U}(1)$ connection on the holomorphic tangent, and hence cotangent, bundles of $\Sigma$ with the complex structure $J_g$. Let $K^{\frac{1}{2}}$ be a square root of the holomorphic cotangent bundle of $\Sigma$, so that

$$T^{*1,0}\Sigma = K^{\frac{1}{2}} \otimes K^{\frac{1}{2}}.$$

Consider the vector bundle

$$E := K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}}.$$

The Chern connection induces a $\mathrm{U}(1)$ connection which we denote $a$ on $K^{-\frac{1}{2}}$ and a connection $-a$ on $K^{\frac{1}{2}}$. Further we have,

$$\sigma \in \Omega^{1,0}(T^{*1,0}\Sigma) \cong \Omega^{1,0}(K^{\frac{1}{2}} \otimes K^{\frac{1}{2}}) \cong \Omega^{1,0}(\mathrm{Hom}(K^{-\frac{1}{2}}, K^{\frac{1}{2}}),$$

and, using the hermitian metric $h$ and conjugation:

$$\overline{\sigma} \in \Omega^{0,1}(\mathrm{Hom}(K^{\frac{1}{2}}, K^{-\frac{1}{2}})).$$

These identifications allow us to write down the following connection matrix on $E = K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}}$:

$$A := \begin{pmatrix} a & \overline{\sigma} \\ \sigma & -a \end{pmatrix}.$$

Now let $P$ be the principal $\mathrm{SU}(2)$ bundle associated to $E$ and define a so called *Higgs field* $\Phi \in \Omega^{1,0}(\mathrm{ad}P \otimes \mathbb{C})$ by

$$\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We are working with respect to the decomposition $E = K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}}$ and we think of 1 as identified with an element of $\Omega^{1,0}(\mathrm{End}E)$ as the canonical section of

$$T^{*1,0}\Sigma \otimes \mathrm{Hom}(K^{\frac{1}{2}}, K^{-\frac{1}{2}}) \cong T^{*1,0}\Sigma \otimes T^{1,0}\Sigma \cong \mathbb{C}.$$

Observe that $\Phi$ is thus holomorphic.

Consider the $\mathrm{PSL}_2\mathbb{C}$ connection defined by

$$B := A + \Phi + \Phi^*,$$

this is a connection on the principal bundle $P \otimes \mathbb{C}$. The curvature of this connection is given by:

$$F_B = \begin{pmatrix} K_g + |\sigma|_g^2 + 1 & 0 \\ 0 & -(K_g + |\sigma|_g^2 + 1) \end{pmatrix} \frac{i\rho_g}{2}.$$

Thus the connection is flat since by hypothesis the pair $(g, \sigma)$ is a minimal hyperbolic germ. Since $B$ is flat it corresponds to a representation of the fundamental group of $\Sigma$ in $\mathrm{PSL}_2\mathbb{C}$. Thus we obtain a map associating to a minimal hyperbolic germ a representation of $\pi_1(\Sigma)$ in $\mathrm{PSL}_2\mathbb{C}$. It is a result of Donaldson [5] that this map agrees on $\mathcal{M}$ with the explicit construction of the quasi-Fuchsian three manifold.

**Theorem 6.3.6** (Taubes [26])**.** *The subset of $\mathcal{H}$ on which the map into the* $\mathrm{PSL}_2\mathbb{C}$ *representation variety is an immersion is precisely the set $\tilde{\mathcal{H}}$ defined earlier.*

This theorem allows us to put further structures on $\tilde{\mathcal{H}}$.

**Proposition 6.3.7.** *$\tilde{\mathcal{H}}$ is a complex manifold with a complex structure $J''$ and a holomorphic symplectic structure $\omega_1'' - i\omega_3''$, this complex structure extends the complex structure $J$ and the holomorphic symplectic structure $\omega_1 - i\omega_3$ on $\mathcal{M} \subseteq \tilde{\mathcal{H}}$.*

*Proof.* We define of $J''$ and $\omega_1'' - i\omega_3''$ by pulling back the complex structure and Goldmans holomorphic symplectic structure from the representation variety. It is immediate, since we have an immersion into the representation variety by hypothesis, that this defines a complex structure and a holomorphic symplectic structure on $\tilde{\mathcal{H}}$. That the structures extend those on $\mathcal{M}$ follows from properties of the map from $\mathcal{M}$ into the quasi-Fuchsian deformation space which is a submanifold of the representation variety. $\square$

### 6.3.2 The hyperkähler structure on $\tilde{\mathcal{H}}$

It is immediate that the moduli space $\mathcal{M}$ is a subset of $\tilde{\mathcal{H}}$. In addition we have two pairs

$$
\begin{aligned}
&I',\ \omega_2' + i\omega_3' \\
&J',\ \omega_1'' - i\omega_3'',
\end{aligned}
$$

on $\tilde{\mathcal{H}}$ consisting of a complex structure and a holomorphic symplectic form. Restricted to the moduli space $\mathcal{M}$ these structures agree with the equivalent ones of the hyperkähler structure on $\mathcal{M}$. Explicitly writing $I, J, K$ for the complex structures on $\mathcal{M}$ and $\omega_1, \omega_2, \omega_3$ for the symplectic structures coming from the hyperkähler metric, we have the following identities:

$$
\begin{aligned}
&I = I',\ J = J'', \\
&\omega_1 = \omega_1', \\
&\omega_2 = \omega_2', \\
&\omega_3 = \omega_3' = \omega_3''.
\end{aligned}
$$

**Proposition 6.3.8.** *Suppose the real analytic structures induced on $\tilde{\mathcal{H}}$ by $I'$ and $J''$ coincide, then there is a hyperkähler structure on the space $\tilde{\mathcal{H}}$ extending the structure on the moduli space $\mathcal{M}$.*

*Proof.* Since the real analytic structures coincide, and the identities above hold on $\mathcal{M}$, they must hold on the whole of $\tilde{\mathcal{H}}$. In addition algebraic relations between the identities must also hold, so that the quaternion identities are satisfied and the structure is hyperkähler. □

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