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Two-parameter Perturbation Theory for Cosmologies with Non-linear Structure

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Declaration

I, Sophia Rachel Goldberg, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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'Cosmology on all scales: a two-parameter perturbation expansion'
Sophia R. Goldberg, Timothy Clifton and Karim A. Malik
Physical Review D **95**, 043503 (2017)

'Perturbation theory for cosmologies with nonlinear structure'
Sophia R. Goldberg, Christopher S. Gallagher and Timothy Clifton
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Abstract

We propose and construct a two-parameter expansion around a Friedmann-Lemaître-Robertson-Walker geometry which uses both large-scale and small-scale perturbations analogous to cosmological perturbation theory and post-Newtonian gravity. We justify this observationally, derive a set of field equations valid on a fraction of the horizon size and perform a detailed investigation of the associated gauge problem. We find only the Newtonian gauge, out of the standard gauges used in cosmological perturbation theory, is applicable to post-Newtonian perturbations; we can identify a consistent set of perturbed quantities in the matter and gravity sectors and construct corresponding gauge-invariant quantities. The field equations, written in terms of these quantities, takes on a simpler form, and allows the effects of small-scale structure on the large-scale properties of the Universe to be clearly identified and discussed for different physical scenarios. With a definition of statistical homogeneity, we find that the cosmological constant and the average energy density, of radiation and dust, source the Friedmann equation, whereas only the inhomogeneous part of the Newtonian energy density sources the Newton-Poisson equation – even though both originate from the same equation. There exists field equations at new orders in our formalism, such as a frame-dragging field equation a hundred times larger than expected from using cosmological perturbation theory alone. Moreover, we find non-linear gravity, mode-mixing and a mixing-of-scales at orders one would not expect from intuition based on cosmological perturbation theory. By recasting the field equations as an effective fluid we observe that these non-linearities lead to, for example, a large-scale effective pressure and anisotropic stress. We expect our formalism to be useful for accurately modelling our Universe, and for investigating the effects of non-linear gravity in the era of ultra-large-scale surveys.

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1. Introduction

"How is the cosmos, darling?"

— Annette Goldberg

Modern cosmology has the challenge of modelling a vast array of historical epochs of the Universe whilst including the range of structure we observe on many different scales. From the epochs of radiation domination, structure formation, to the late-time accelerated expansion, and the gravitationally-bound structures of planets, stars, galaxies, clusters and superclusters that make up the cosmic web we see today: modelling these crucial features of the observable Universe is a huge task for theoretical cosmologists. If we wish to consider the Universe on distance scales as large as the Hubble radius and structure with large density contrasts, we must understand how gravitation works on the largest scales during the evolution of the Universe. Indeed, answering this is exactly what is addressed in this thesis.

Furthermore, there are motivations for this observationally, as the next generation of astronomical surveys [1–3], which will collect data on scales comparable to the cosmological horizon, will have sufficient precision to provide a new testing ground for non-linear relativistic gravity [11, 23, 111, 121]. This is a particularly exciting prospect as, to date, non-linear gravitational effects have only been observed in the solar system [44, 46, 89, 90], binary pulsar systems [158, 170], and the newly discovered binary black hole mergers observed using LIGO [7–9]. The observation of similar effects in cosmology would allow general relativity to be investigated on unprecedented length and time scales, as well as in an entirely different physical environment. This would give us a new insight into Einstein's theory.

Gravitational physicists use Einstein's general theory of relativity to model both isolated astrophysical systems and cosmology, on many different scales in our Universe. Presented just over a hundred years ago at the Royal Prussian Academy of Sciences [86], this revolutionary theory transformed our understanding of gravitation. Gravity is not merely a force given by Newton's law but is a manifestation of space-time. Moreover, space-time itself is dynamical and dependent on its content. Structure in the Universe is normally modelled with gravitation taken in two limits of Einstein's field equations: on small scales Newtonian or post-Newtonian gravity is used, whereas on horizon-sized scales cosmological perturbation theory is used. We will now focus on these limits in turn.

The application of general relativity to isolated astrophysical objects, on scales much less than the horizon, from high-precision observations of our solar system to extraordinary exotic astrophysical systems such as binary pulsars, has furthered our fundamental understanding of our Universe. Solving the full Einstein equations is necessary when studying strong gravitational systems such as near binary black holes, where non-perturbative methods break down. However, in the case of isolated systems, which are weak-field and slowly varying, Newtonian gravity is a good approximation to the dynamics of the system. Indeed, it is one of the great successes of general relativity that in the appropriate small-scale limit Newtonian gravity can be derived from it. It can be used to model structures in our Solar System to a precision of one part in 10^5. In fact, non-linear structure on scales up to of about 100Mpc, in the late Universe, are modelled using Newtonian gravity. With the inclusion of dark matter, baryons and dark energy realistic large-scale simulations of the Universe can be used to model the dynamics of our Universe accurately [55]. These simulations exclude the existence of highly relativistic objects such as neutron stars and black holes in their model of the Universe. Nevertheless, these large-scale structure simulations have shown the formation of filaments, walls and voids, which are very close to what we observe in the late Universe.

Relativistic corrections to such Newtonian systems are small, nevertheless, they have been quantified accurately using the aptly named 'post-Newtonian' formalism. The post-Newtonian book-keeping is a methodology for counting magnitudes of perturbations to the geometry and strictly keeping track of the time derivatives on such metric potentials in the dynamics of the field equations. Given a metric theory of gravity the post-Newtonian formalism can be used to derive the slow-motion and weak-field limit of it. Furthermore, one can parametrize the post-Newtonian limit of a large class of metric theories of gravity, known as parametrized-post-Newtonian gravity. This formalism has enabled gravitational physicists to test gravitation for weak-field systems on the smallest scales, like in our Solar System, to high accuracy [44, 46, 89]. These tests range from the classical tests of light-bending around the Sun and Mercury's perihilion precession [171], to the modern tests of frame-dragging due to the gravito-magnetic potential of the Earth [90]. These small-scale tests of gravity have confirmed the parameterized post-Newtonian parameters are consistent with those derived from Einstein's theory of general relativity. In other words, it is the small-scale tests of gravity, along with parametrized post-Newtonian gravity, which has given us faith in general relativity. It has enabled gravitational physicists to test gravity to successively higher and higher order in accuracy and indeed shown us that gravity is consistent with general relativity. Nevertheless, arguably the most fascinating application of general relativity has been to cosmology, in studying the evolution of our entire Universe.

The concordance model of cosmology, ΛCDM, is a simple theory that has been very successful in furthering our understanding the large-scale properties of the Universe. It assumes the existence of the dark sector with dark energy, which is taken to be a cosmological constant, Λ, and cold dark matter. Dark matter is necessary for structure formation and the cosmological constant accounts for the accelerated expansion in the late Universe. Moreover, it assumes that the large-scale evolution of the Universe can be determined from a direct application of Einstein's theory of general relativity and that the cosmological principle holds. In this case space-time is globally homogeneous and isotropic and therefore described by a Friedmann-Lemaître-Robertson-Walker (FLRW) metric. This simplistic picture is arguably one of the greatest successes of Einstein's theory because, with the inclusion of baryonic matter and radiation, it is consistent with a variety of high-precision observations of our Universe. Such evidence is derived from observations that span vast epochs and scales, and includes the Cosmic Microwave Background (CMB) [12], type Ia supernovae [45, 118] and large-scale structure [10, 21, 25]. Furthermore, the Universe is taken to be very close to spatially flat on large scales, justified from Planck's CMB anisotropy measurements [12, 78], and baryon acoustic oscillation (BAO) data [28].

Presently we believe that large-scale structure initially formed from the gravitational instability of a Gaussian random field of super-horizon-size primordial density perturbations generated from quantum fluctuations, which were expanded to the largest scales during inflation. These fluctuations were so small that the density contrast was of order 10^{-5} and is justified from precision observations of the temperature of the CMB. It is these fluctuations that are believed to source the formation of the large-scale structure we see today. Moreover, it is observations of the CMB that tell us that at these very early times the Universe was homogeneous and isotropic on horizon-sized scales and therefore we can model it by a uniformly expanding FLRW metric. To model the evolution of small fluctuations in the early Universe cosmological perturbation theory is used, this is a weak-field expansion where perturbations vary on large scales, of order the horizon-size. This formalism allows the dynamics of the background, and higher-order perturbed quantities, to be tractable and solved for too, order-by-order in perturbation theory. Furthermore, high-precision cosmology in the early Universe has rigorously allowed us to precisely constrain the cosmological parameters [20].

The standard model of cosmology has gone a long way in enabling us to understand our Universe. However there are several open questions, probably the most well known of which are related to the dark sector. Efforts from theoretical cosmologists have been undertaken to further understand these components of the Universe [43, 76]. Experimentally, direct and indirect searches of dark matter have been undertaken, which are searches for dark matter candidates through non-gravitational interactions [35]. Additionally, observational cosmologists will use large-scale structure in upcoming high-precision observations to find a more precise measurement of dark energy and matter through their gravitational interactions [11, 63, 121].

There are also fundamental questions about how we apply general relativity to structure on the largest scales. As discussed, the standard approach is to consider two limits of the field equations, assume a global FLRW background, and seperately consider cosmological perturbation theory above the homogeneity scale or Newtonian gravity below 100Mpc in order to model the effects of weak gravitational fields, see Refs. [55, 83, 128]. This looks very natural at linear order in the gravitational fields, partly because the linear equations of Newtonian gravity can be recovered from the quasi-static limit of cosmological perturbation theory, when the gravitational fields slowly vary in time. Moreover, as we have discussed above, this approach works extremely well for a wide variety of situations. However, it starts to become problematic when one tries to correctly consider non-linear relativistic gravity on scales of order 100Mpc or in the late Universe, where we expect contributions to the field equations from perturbations on small and large scales. The reason that standard cosmological perturbation theory is not ideal for modelling structure on these scales is that below this scale both density contrasts and velocities become large, in comparison to the background energy density and gravitational potentials, respectively. Also, spatial gradients of gravitational potentials become large with respect to time derivatives of gravitational potentials.

This implies perturbations to the metric can appear at the same order in the field equations as the dynamical background [143]. Perturbing around a background geometry which is much smaller than higher-order perturbations to it corresponds to a breakdown of the perturbation theory itself. This has led to much study of the idea that the formation of clumpy structure in the Universe could have a strong "back-reaction" effect on the large-scale expansion, as the perturbative expansion may start to breakdown [60–62, 71, 74, 88, 165]. Back-reaction is generally calculated from the difference between the large-scale expansion of an inhomogeneous cosmology and a homogeneous and isotropic cosmology, both of which are solutions to Einstein's field equations. The homogeneous and isotropic cosmology is formed from a best-fit to either the average of observables or dynamics in the inhomogeneous solution. Although many authors believe the back-reaction on the FLRW background to be small, this does not necessarily mean that the effect of small-scale structure on large-scale perturbations must also be small. Therefore we require an approach that can systematically and consistently track the effects of a realistic Universe with both non-linear small-scale and linear large-scale structures order-by-order in perturbation theory. This will enable us to write a hierarchical set of field equations which can be solved systematically, which is exactly what our two-parameter expansion does.

In some respects our treatment of gravity on small scales (and post-Newtonian gravity), can be viewed as a formalised version of the quasi-static (or slow-motion) limit of cosmological perturbation theory. This approach has often been used in the literature to describe small-scale structure [107], and, at lowest order, gives a set of equations that look a lot like those of Newtonian gravity. The basic idea in this approach is to neglect terms with time derivatives in the field equations, as these are generally expected to be small in comparison to spatial derivatives. Studies with this goal have already been performed using second-order cosmological perturbation theory [38, 39, 48, 50, 51, 58, 161], and we expect it to be a matter of significant interest to determine whether a framework that formalizes the quasi-static limit can be used to simplify or extend them. Hints that this should be possible come from studies of second-order gravitational fields that average to the size of first order fields [14, 56, 72, 115, 143], and calculations that suggest the second-order vector potential to be a hundred times larger than naively expected from second order cosmological perturbation theory [26]. In fact, both of these turn out to be natural results of the two-parameter formalism, which may therefore prove useful for gaining a full understanding of the results from upcoming high-precision surveys [1–3].

In Refs. [36, 138] a quasi-static limit of second-order perturbation theory is used to account for non-linear gravity on large scales in an effective field theory approach, in spite of the relegation of terms with time derivatives not being systematic. Specifically, this is because, when considering the quasi-static limit, the terms that have been relegated to higher-order can no longer be entirely neglected; they can and should be expected to appear in the next-to-leading-order gravitational field equations. This could be at second-order on small scales, but could in principle be at what is usually thought of as first-order on large scales. What is unclear in the usual application of the quasi-static limit is how this approach can be extended to non-linear gravity. Additionally, it may or may not be necessary to adjust the order-of-smallness of velocities or vector potentials in order to make the entire system of equations consistent with the results expected observationally. The question of how to construct a perturbative expansion that can systematically perform the required re-ordering, and produce a self-consistent and well-motivated set of field equations on all scales, is the purpose of our two-parameter expansion.

We should note that Newtonian perturbation theory applied to cosmology, is Newtonian theory linearly perturbed in an expanding space [137, 169], includes some, but by no means all, of the relativistic corrections expected from a metric theory of gravity – like general relativity – and derived from, for example, the quasi-static limit of cosmological perturbation theory or post-Newtonian gravity. Note that whereas Newtonian perturbation theory and the quasi-static limit require that perturbations to a homogeneous background energy density remain small, this is not required from post-Newtonian gravity. In spite of its successes, the application of post-Newtonian gravity to our entire Universe, in cosmology, is somewhat limited as it only describes isolated systems. This is due to the fact the expansion is an expansion around a Minkowski space-time where velocities and time derivatives are small and there exists asymptotic flatness. However, on large-scales, for example when we approach the cosmological horizon, this is not the case for gravitational fields as the time scale of cosmic evolution is not negligible and there are no asymptotically flat regions in cosmology.

Furthermore, applying the standard post-Newtonian formalism to the entire evolution of the Universe is not realistic. We need to add extra matter fields, which go beyond the use of non-relativistic baryonic matter alone (which is all that is normally considered in post-Newtonian gravity). To apply post-Newtonian gravity to cosmology we would require the inclusion of dark matter. We also require both radiation and a cosmological constant to describe epochs of the Universe which are not matter dominated. Therefore, such matter components would need to be included formally. There has been research into addressing this problem [149] by adding radiation and a cosmological constant to the post-Newtonian expansion book-keeping. This is a practical way of allowing for these extra matter sources. However, as both radiation and the cosmological constant are large-scale quantities associated with the horizon-size they behave like perturbations that are 'cosmological' in our two-parameter expansion. Hence, from our expansion we can derive that for non-linear structure on very small scales the cosmological constant only affects the dynamics at an order well beyond that of the leading-order relativistic effects. We find the two-parameter formalism developed in this thesis naturally incorporates extra matter sources (with barotropic equations of state and a cosmological constant) as well as non-relativistic matter.

To develop a mathematical formalism for investigating the non-linear properties of gravity in cosmology is a non-trivial task, but there is now a substantial literature dedicated to developing different approaches to this. In summary, the most common approach is a direct implementation of second-order cosmological perturbation theory [106, 127, 134], which allows relativistic gravitational perturbations around a homogeneous and isotropic background to be modelled in the presence of linear density contrasts. Other approaches, however, have started to import techniques from post-Newtonian gravity [15, 16, 130], where gravitational fields are assumed to be slowly varying and where non-linear density contrasts can be consistently modelled.

However, if one wants to consider non-linear gravity in a universe that *simultaneously* contains linear structures on large scales *and* non-linear structures on small scales, then one must adopt a more sophisticated approach. This is exactly what our two-parameter expansion does. We note that two-parameter and N-parameter expansions of tensorial quantities have previously been studied in Refs. [57, 154], but not in the context of different types of perturbations which vary differently in space-time, and which vary on different length scales, as we do here. From our two-parameter formalism we derive that although non-linear structure does not affect the scale factor, it does affect higher-order large-scale corrections to it. Moreover, by writing our equations in terms of an effective fluid, they are much more tractible and the relativistic effects of non-linear gravity on large-scales can be identified more easily. For example we can clearly see mode-mixing[1], a mixing-of-scales, and small-scale non-linear gravity sourcing an effective large scale pressure and anisotropic stress at linear order in cosmological pertrubations (this is something normally observed at second order in cosmological perturbations). These types of terms offer exciting possibilities for testing non-linear gravity with upcoming surveys. This approach can be compared to the effective fluid approach studied previously in Refs. [36, 64], as well as the large and small wavelength split used in Refs. [99, 100]. Our approach simultaneously expands the metric and energy-momentum tensor using both cosmological and post-Newtonian perturbation theories [141, 171]. The result of this can formally be described as a perturbative expansion in two parameters, which is a consistent and valid description of both non-linear structure on small scales and

[1]Note that we use "mode-mixing" to describe the coupling of scalar, vector and tensor perturbations. We will use "mixing-of-scales" to refer to the coupling of large-scale and short-scale perturbations due to quadratic terms.

linear fluctuations on horizon-sized scales. Such a formalism therefore enables one to model the effects of non-linear structure on the dynamics of large-scale cosmological perturbations, as well as determine if non-linear structure affects the dynamics of the cosmological background. It provides a more representative picture of the real Universe than either cosmological perturbation theory or post-Newtonian theory could by themselves, and may be of use for consistently modelling the relativistic effects that future surveys will seek to detect.

Furthermore, the two-parameter formalism, in the case where long-wavelength cosmological perturbations are neglected, successfully reduces to single parameter post-Newtonian gravity on an expanding background. This is similar to the expansions in Refs. [15, 16, 130] which relegate terms in the field equations with time derivatives, some more systematically than others. If the scale of the post-Newtonian system is small enough, then the background expansion only influences the local physics of that system at high orders in perturbation theory – our formalism shows this assertion holds. This means we end up with a set of equations that are consistent with post-Newtonian gravity up to the accuracy of current observations but which differ from post-Newtonian gravity at higher-order. Our framework could therefore be used to quantify the effects of cosmological expansion and cosmological potentials on local weak-field systems, if it were required.

Additionally, our two-parameter expansion has the potential to allow us to test gravity on the largest scales. This is a new paradigm for testing Einstein's theory. Whereas parametrized-post-Newtonian gravity allows gravitational physicists to test non-linear gravity on the smallest scales, there has been much recent development by theoretical physicists to derive a parametrization which allows non-linear gravity to be tested on cosmological scales [110]. Metric theories of gravity [75] have been parametrized for cosmology in parametrized post-Friedmann approaches [24, 30, 31, 105, 119, 153], parametrized post-Newtonian cosmology [150], and effective field theories [29, 33, 34, 37, 47, 102, 119, 138]. These parametrizations vary from the very general, which allow for exotic modified gravity theories to be tested, to the more conservative, characterising small deviations from general relativity. Nevertheless, to correctly test non-linear structure on the largest scales we must initially provide a framework which allows us to consistently describe non-linear structures on such scales. While employing general relativity, this is what our framework does.

Of course, the main application of constructing a two-parameter perturbation expansion is to determine the signatures of concordance cosmology, and even Einstein's theory, in cosmological data. Galaxy surveys are now aiming for 1% precision, the same as CMB measurements. For example, the detection of BAO by SDSS has allowed the first 1% level cosmological constraint by a galaxy survey [25]. Moreover, future surveys such as Euclid [2, 22, 121] and SKA [1, 108, 111, 152] and LSST [3] will reach scales of order the horizon-size. It is crucial that our theories catch up with the precision of our observations, this means developing a framework which accounts for relativistic effects in observations. There has been much work on this within the literature [40, 42, 52, 53, 65, 162]; this is also what our two-parameter expansion achieves, with the additional beneficial features discussed above.

The effects of small-scale nonlinearities on cosmological observables may need to be accounted for, such as in galaxy number counts [48, 85, 135]. These late-time observations for lensing and redshift space distortions will allow the precise measurement of the growth of clustering [136, 147]. Studies suggest that inhonmogeneities may also bias the dark-energy equation of state [93]. Additionally, it has been found that the impact of small perturbations on distance measurements is not negligible and can be a probe for cosmology [6, 54] or an additional effective noise in measurements [38, 164] that may be relevent to observations from Planck [94, 129]. Furthermore, relativistic effects such as Doppler magnification, which causes the observed size of galaxies to change, should also be detectable in current and upcoming optical and radio surveys [49]. Other relativistic effects, such as lightcone projection effects [109, 163], need to be accurately accounted for. Additionally, through simple parameterizations of the Newton-Poisson equation, small deviations from general relativity have been accounted for as a slip between the scalar gravitational potentials via $E_g(z)$. The impact of the lensing contribution to galaxy number counts on the $E_g(z)$ statistics has been studied in Ref. [81]. Such a parameterization could be achieved at beyond leading-order using our two-parameter formalism.

On large scales the galaxy power spectrum contains signatures of local primordial non-Gaussianity and horizon-scale general relativistic effects, calculating these effects has been of much interest in the literature. For example, the authors in Ref. [95] show a multi-tracer method (which benefits from large bias differences between two tracers of the underlying dark matter distribution) and the combination of two surveys (a large neutral hydrogen intensity mapping survey in SKA Phase 1 and a Euclid-like photometric survey) would provide unprecedented constraints on primordial non-Gaussianity and general relativistic effects. The authors forecast that the error on local primordial non-Gaussianity will break the cosmic variance limit on CMB surveys. In Ref. [96] they also calculate that the SKA precursor (MeerKAT) and DES, can be combined using a multitracer technique to deliver an accuracy on measurement of the non-Gaussianity parameter $f_{\rm NL}$ up to three times better than Planck. It has been found that lightcone and mode-coupling contributions at second-order in cosmological perturbations mean that relativistic corrections are non-negligible at smaller scales for the bispectrum than in the case of the power spectrum [163]. There are also other works which calculated observed galaxy number counts using standard perturbation theory and estimate the corresponding non-Gaussianity parameter $f_{\rm NL}$ [79, 123]. These relativistic effects, if ignored in the analysis of observations, could be mistaken for primordial non-Gaussianity.

For predictions of future observations to be accurate to of order 1% it may be necessary to go beyond the Newtonian approximation in N-body simulations. Newtonian N-body computer simulations [55] have been crucial in modelling structure formation, and have kept pace with the increasing data quality of cosmological surveys due to technological advancements. The use of Newton's law of gravitation, and ignoring any general relativistic corrections, has spurred recent debate on whether such an approximation is justified given the current era of high-precision cosmology [17, 67, 91, 92, 101, 145]. It is therefore important to formalise and account for relativistic effects on these scales. N-body simulations beyond ΛCDM, which include modified gravity theories in quasi-static limits, have already been used in modified Newtonian codes [122, 124, 142, 151] and relativistic corrections have been accounted for in Refs. [16–19, 67, 69, 91, 92, 101, 145, 159]. These perturbative approaches have the benefit of accounting for relativistic effects without employing full numerical relativity simulations, which solve the full Einstein equations. However, the open question remains how exactly we account for relativistic effects in nonlinear gravity correctly. This may also prove useful for gaining a full understanding of the results from upcoming high-precision surveys.

This is an exciting time for cosmology because we are living at an intersection where there exists significant motivations from both theoretical and observational cosmology to study gravitation on the largest scales. Not only are we answering fundamental questions about how gravity works, but the improved quality of observations mean it is necessary we do so.

1.1. Notation

At this stage it is convenient to define some notation we use throughout this thesis. We use Latin indices and Greek indices to denote space (*e.g.* $x^i, i \in \{1,2,3\}$) and space-time (*e.g.* $x^\mu, \mu \in \{0,1,2,3\}$) indices, respectively. Commas and dots denote partial derivatives and derivatives with respect to coordinate time t, respectively, such that

$$f_{,\mu} \equiv \frac{\partial f}{\partial x^\mu}\,, \qquad \dot{f} \equiv \frac{\partial f}{\partial t}\,,$$

where x^μ are space-time coordinates and f is any function of space-time. Dashes denote derivatives with respect to conformal time τ such that

$$f' \equiv \frac{\partial f}{\partial \tau}\,,$$

single spatial derivatives are given by $f_{,i} \equiv \nabla f$, and ∇^2 refers to the Laplacian associated with spatial partial derivatives with respect to comoving coordinates. Note that we use Einstein's summation convention throughout: we sum over all repeated indices. Additionally, we choose units such that $c = G = 1$, so that Einstein's field equations have dimensions length^{-2}.

1.2. Overview

We first introduce general relativity in Chapter 2 before discussing the relevant perturbative expansions for our formalism: cosmological perturbation theory and post-Newtonian gravity, in Chapters 3 and 4, respectively. The first four chapters contain introductory material. We construct our two-parameter formalism and justify it observationally in Chapter 5. In Chapter 6 we derive the two-parameter perturbed Ricci and energy-momentum tensors generally and then the field equations for large-scale structure, on the order of a fraction of the horizon-size. In Chapter 7 we define a two-parameter coordinate transformation that can be applied to the metric and energy-momentum tensor. This enables us to construct gauge-invariant quantities which simplifies the field equations and allows us to determine at which orders we expect perturbations to appear in our expansion. We also find that most gauges studied in standard cosmological perturbation theory are not applicable to post-Newtonian potentials. In Chapter 8 we write gauge-invariant versions of the field equations, discuss the relativistic features of them, and consider how to derive standard perturbation theory and post-Newtonian gravity from our approach. We also discuss how our formalism can be applied to the smallest and largest gravitationally-bound structures that exist in the Universe. We recast terms that arise in our equations as an effective fluid in Chapter 9. We conclude in Chapter 10. Finally, the Appendices contain calculations using our two-parameter formalism for the case of dust only (as opposed to the case of dust, radiation and a cosmological constant – which is presented throughout this thesis), for simplicity and use in applications to the late Universe.

2. General Relativity

2.1. Introduction

Our standard model of gravitation is general relativity, created by Einstein more than a hundred years ago. It holds up to high-precision observational scrutiny by accurately describing gravitational phenomena on a range of scales, and in tests devised long after its formulation. Moreover, general relativity has been a huge success in the study of fundamental physics, through its elegant simplicity and by resolving flaws in Newtonian gravity.

General relativity is the simplest 'reasonable' metric theory of gravity. Such 'reasonable' metric theories are characterised by serveral key assumptions, listed by Dicke [80] and with the additions of Will [171], which restrict the number of theories gravitational physicists wish to consider and we now state:

1. The theory has the correct Newtonian limit.
2. Space-time is a four-dimensional differentiable Lorentzian manifold.
3. General coordinate covariance manifests.
4. The theory is relativistic.
5. The field equations can be derived from an invariant action principle.
6. There does not exist a priori geometric structure in the theory.
7. The theory is simple.

The first condition states that, in the limit of weak fields and slow motions, the given theory of gravity must reproduce Newton's laws. This should be true for any theory of gravity, no matter how exotic, with or without a metric. Clearly this is supported observationally as there exists a huge amount of data supporting the idea that Newtonian gravity dominates, at leading-order, for weak-field slow-motion systems such as our Solar System.

The second condition means we are only considering *metric* theories of gravity where the metric and affine connection are not predefined but solved for through the dynamics of the field equations. Moreover, each point in the space-time manifold corresponds to a physical event and the manifold itself is four-dimensional and Lorentzian, which means it possesses a metric and corresponding Levi-Civita connection. The Lorentzian character of the space-time manifold is crucial as it implies there exists causal structure, an important feature of reality.

The condition of general coordinate covariance means the equations of gravity and the mathematical entities in them are to be expressed in a form that is independent of the particular coordinates used. This means it is the position of the events in space-time and its geometry that are of significance, not how we label them. This implies gravitation should be formulated in terms of tensors.

The fourth condition is that we require that in the limit of zero gravity the non-gravitational laws of physics must reduce to the laws of special relativity. The evidence for this comes largely from high-energy physics and is true for most smooth Lorentzian manifolds, where for all points on the manifold it is possible to make a local coordinate transformation to normal coordinates that recover special relativity at leading order. This is a manifestation of the weak equivalence principal, which states that a freely falling frame in a gravitational field is the same as an inertial frame in the absence of gravity (up to tidal forces).

The condition that the field equations can be derived from an invariant action principle is justified by two ideas. Firstly, given the requirement of varying an action in quantum mechanics it may be necessary to include this method in a theory of gravity if we are to one day unify both quantum mechanics and gravitation. Furthermore, it is through variational principles that allows us to find covariantly conserved quantities and enables us to form conservation laws, such as the conservation of energy and momentum, via Noether's theorem.

The condition that there does not exist a priori geometric structure in the theory means all gravitational fields can be solved for through the dynamics of the field equations, *i.e.* there is no prior geometry. Finally, requiring a simple theory is a condition justified through what might be attributed to Occam's razor: that a theory with minimal functions and greater simplicity is a better model. Nevertheless, this has often been relaxed to study exotic alternative theories of gravity.

General relativity satisfies all the above conditions and several others [126]:

1. The strong equivalence principle.

2. Newton's 'constant' is truly a constant in space and time.

3. The action is linear in second derivatives of the metric.

4. The metric is the only dynamical gravitational degree of freedom.

The strong equivalence principle is the idea that massive, astronomical, bodies with gravitational binding energy will follow geodesics of space-time. This is an extension of the weak equivalence principle, which only holds for freely falling (test) particles. The second condition means that not only should this theory give the correct Newtonian limit in our Solar System, but the same over time and space. This condition is violated in more complex theories of gravity with a varying Newton's constant, *e.g.* Brans-Dicke theory. The third condition ensures that the field equations in general relativity are of no higher than second order in derivatives of the metric. If this does not hold then new degrees of freedom exist in the theory of gravity. Often these new degrees of freedom have negative energy and the theory is plagued by ghost instabilities. The final condition means we do not consider other fields beyond (or rather than) the metric tensor. It is important to note that, given these conditions, general relativity is the simplest theory of gravity that can be conceived of, and any modification to it increases the complexity of the dynamics in the field equations.

2.2. Differential geometry

The mathematics of general relativity is differential geometry. In this section we outline the crucial features of differential geometry relevant for general relativity. For further introduction see Refs. [131, 133, 168].

2.2.1. Metrics

Given a four-dimensional differentiable manifold, $\mathcal{M}$, we can define space-time points $p \in \mathcal{M}$ with coordinates $x^\mu(p)$. Furthermore, at all points we can associate vectors, covectors and tensors with tangent and cotangent spaces. We can also define a metric tensor, $g_{\mu\nu}$, such that it is a symmetric (0,2) tensor field, $g_{\mu\nu} = g_{\nu\mu}$, and non-degenerate, $g_{\mu\nu} u^\mu v^\nu = 0$ for all vectors u^μ if and only if $v^\nu = 0$. Moreover, in general relativity we require $\mathcal{M}$ to be a Lorentzian manifold; so throughout this thesis we choose $\mathcal{M}$ to have signature (-,+,+,+).

Given that the metric is non-degenerate this implies it is invertible

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu \,, \tag{2.1}$$

where δ^μ_ν is a Kronecker delta: $\delta^\mu_\nu = 1$ if $\mu = \nu$ and $\delta^\mu_\nu = 0$ otherwise. The metric provides an isomorphism between vectors and covectors, such that given a vector v^μ we can define a covector $v_\nu = v^\mu g_{\mu\nu}$, and given a covector u_μ we can define a vector $u^\nu = u_\mu g^{\mu\nu}$. The metric allows us to raise and lower space-time indices.

We note that the metric allows us to encode geometric notions of "orthogonality" and "norms" of vectors. The norm of a vector v^μ is given by $|v|^2 = g_{\mu\nu}v^\mu v^\nu$, and two vectors, u^μ and v^ν, are orthogonal if $g_{\mu\nu}u^\mu v^\nu = 0$. Note that this can be thought of as the generalisation of the dot product of two vectors in Euclidean space. This property makes the metric tensor essential in physics for the measuring of invariant distances in manifolds.

2.2.2. Geodesics and Christoffel symbols

The geodesic equation can be derived from a variation of the action principle and provides a definition of the Christoffel symbol. Given a metric we can define the action for two timelike separated points λ_0 and λ_1 by

$$S = \int_{\lambda_0}^{\lambda_1} ds\,, \tag{2.2}$$

where

$$ds = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} \tag{2.3}$$

is the line element. We can minimise this action, reparameterise it in terms of proper time $d\tau^2 = -ds^2$, and use the Euler-Lagrange equation to find the geodesic equation

$$\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\alpha\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} = 0\,, \tag{2.4}$$

where we have defined the Christoffel symbol as

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}\left(g_{\rho\lambda,\nu} + g_{\nu\rho,\lambda} - g_{\nu\lambda,\rho}\right)\,. \tag{2.5}$$

Note that $\Gamma^\mu_{\nu\lambda}$ is symmetric in lower indices and does not define a tensor.

Finally, a vector u^μ is said to be timelike, null, or spacelike depending on whether $g_{\mu\nu}u^\mu u^\nu$ is negative, zero, or positive, respectively, given the metric $g_{\mu\nu}$ has signature (-,+,+,+).

2.2.3. Covariant derivatives and curvature

The covariant derivative provides the notion of a partial derivative with tensorial properties. A metric $g_{\mu\nu}$ allows us to define the covariant derivative ∇_μ of tensorial quantities in $\mathcal{M}$, in association with the Levi-Civita connection. Specifically, the Levi-Civita connection is the unique connection that is defined as being torsion-free and gives $\nabla^\mu g_{\mu\nu} = 0$. In terms of the Christoffel symbol we define the covariant derivative of a tensor T^ν_λ as

$$\nabla_\mu T^\nu_\lambda = T^\nu_{\lambda,\mu} + \Gamma^\nu_{\sigma\mu} T^\sigma_\lambda - \Gamma^\sigma_{\lambda\mu} T^\nu_\sigma \,. \tag{2.6}$$

In the neighbourhood of a point $p \in \mathcal{M}$ there is a coordinate system, normal coordinates, in which the components of the Christoffel symbols vanish and so the covariant derivative is simply given by a partial derivative.

The notion of curvature arises from the commutator of covariant derivatives acting on a vector v^μ. More concretely, one has that

$$\nabla_\alpha \nabla_\beta v^\mu - \nabla_\beta \nabla_\alpha v^\mu = R^\mu_{\delta\alpha\beta} v^\delta \,, \tag{2.7}$$

where $R^\mu_{\delta\alpha\beta}$ is the Riemann curvature tensor. The Riemann tensor can be written as

$$R^\mu_{\nu\lambda\rho} = \Gamma^\mu_{\nu\rho,\lambda} - \Gamma^\mu_{\nu\lambda,\rho} + \Gamma^\mu_{\lambda\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\mu_{\rho\sigma}\Gamma^\sigma_{\nu\lambda} \,. \tag{2.8}$$

Contracting the Riemann tensor gives the Ricci tensor, $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$, and Ricci scalar, $R = g^{\mu\nu} R_{\mu\nu}$. At this stage we will also define the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \,. \tag{2.9}$$

We observe that the Riemann tensor satisfies the following relations:

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta} \,, \quad \text{and} \quad R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0 \,, \tag{2.10}$$

the former relations are known as skew and interchange symmetries; the latter relation is known as the first Bianchi identity. The Riemann tensor also follows a differential identity, known as the second Bianchi identity, given by

$$\nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\beta R_{\gamma\alpha\delta\epsilon} + \nabla_\gamma R_{\alpha\beta\delta\epsilon} = 0 \,. \tag{2.11}$$

Contracting the second Bianchi identity twice gives the well-known conservation of

$G_{\mu\nu}$:

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = 0 \,. \tag{2.12}$$

2.2.4. Lie derivatives

Another type of invariant derivative on a manifold is the Lie derivative, the infinitesimal diffeomorphism. This measures the change of a tensor T^μ_ν as it is transported along the direction given by a vector field ξ^α, and is denoted by $\mathcal{L}_\xi$ such that

$$\mathcal{L}_\xi T^\mu_\nu = \xi^\alpha T^\mu_{\nu,\alpha} - \xi^\mu_{,\alpha} T^\alpha_\nu + \xi^\alpha_{,\nu} T^\mu_\alpha \,, \tag{2.13}$$

and can be given on any differential manifold, with or without a metric. Moreover Eq. (2.13) is itself a tensor.

If $\mathcal{L}_\xi T = 0$ for some tensor T then the diffeomorphism due to ξ is said to be a symmetry transformation of the tensor T. For the metric tensor g, the diffeomorphisms that are symmetries of g are isometries and the vector fields satisfying $\mathcal{L}_\xi g = 0$ (which generate isometries) are defined as Killing vectors.

2.3. Einstein's field equations

Space-time corresponds to the pair $(\mathcal{M}, g_{\mu\nu})$, where the metric satisfies Einstein's field equations [86]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \,. \tag{2.14}$$

However, throughout this work we will consider the field equations in the equivalent form:

$$R_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \tfrac{1}{2} T g_{\mu\nu} \right) + \Lambda g_{\mu\nu} \,, \tag{2.15}$$

where $T \equiv T^\mu_\mu$, and we have introduced the energy-momentum tensor, $T_{\mu\nu}$, and the cosmological constant Λ. The energy-momentum tensor encodes information about the matter in the space-time. The field equations can be derived by varying the Einstein-Hilbert action together with the action for the matter fields.

Additionally, from the Eq. (2.12), and the field equations, Eq. (2.14), the conservation of energy-momentum is implied

$$\nabla^\mu T_{\mu\nu} = 0 \,, \tag{2.16}$$

because by definition $\nabla^\mu g_{\mu\nu} = 0$. The energy-momentum conservation equation, above, implies that special relativity can be recovered in the neighbourhood of every point in space-time, up to tidal forces.

Finally, we note that in general relativity test particles move along timelike geodesics while rays of light move along null geodesics.

2.4. Exact solutions

There are many exact solutions of the field equations that are of physical interest in our Universe, see Ref. [155]. These range from the Schwarzschild and Kerr solutions for black holes to the Friedmann solutions for cosmology, and the Minkowski space-time, useful for studies of weak gravitational systems. It is the latter two space-times and their applications that we are concerned with here.

2.4.1. Minkowski solution

The simplest example of a solution to Eq. (2.14) is given by a line element

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \,, \tag{2.17}$$

with metric

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \,, \tag{2.18}$$

which implies $R_{\mu\nu\alpha\beta} = 0$. Given the vacuum field equations, where $T_{\mu\nu} = 0$ and $\Lambda = 0$, this implies $R_{\mu\nu} = 0$. Therefore, the Minkowski metric is a solution to the vacuum field equations with no cosmological constant [155]. Moreover, any metric related to the Minkowski metric, Eq. (2.18), by a coordinate transformation is also a solution to the vacuum field equation with no cosmological constant.

The fact that the Minkowski space-time is a solution to the vacuum field equations, with no cosmological constant, has crucial physical significance in fundamental physics. It means that, in the limit where there is no matter in the Universe there is a space-time solution, the Minkowski solution, where the non-gravitational laws of physics reduce exactly to the laws of special relativity.

Moreover, for an isolated system which is both weak-field and slowly-varying, modelled by small metric perturbations of the Minkowski space-time and large density contrasts, the Newton-Poisson equation and Newton's acceleration equation can be derived from the field equations and geodesic equations, respectively. This is an essential feature of a realistic gravitational theory and a key justification for general relativity. Additionally, the relativistic corrections to Newtonian gravity are accounted for in the post-Newtonian limit of general relativity, see Chapter 4.

2.4.2. FLRW solution

The cosmological principle states that: *there is no preferred point or direction in space*. In cosmology this is applied to the Universe after some coarse-graining, and is justified by observations from the CMB. Geometrically, we take this to require spatial homogeneity and isotropy, so no point or direction in space is special. For an expanding space-time the unique homogeneous and isotropic space-time geometry is the FLRW space-time given by the line-element [82, 169]

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right) , \tag{2.19}$$

where $a(t)$ is the scale factor in terms of cosmic time, k is the spatial curvature and we have written the above line-element in spherical polar coordinates. The terms in the large parentheses, above, correspond to the metric on a homogeneous and isotropic three-dimensional spatial geometry, for which there are three possibilities: a flat, closed or open three-space corresponding k equal to $0\,, +1$ and -1, respectively. Note that k has dimensions length^{-2}. For this work we will consider geometries which are flat $k = 0$. We can write the above metric in terms of conformal time τ via a conformal transformation $a^2(\tau)d\tau^2 = dt^2$. Minkowski space-time is equivalent to the FLRW space-time in the case where $a = 1$, and $k = 0$.

The precise functional form of $a(t)$ depends on the matter content of the space-time and the value of the curvature constant k. The energy-momentum tensor for standard cosmology is often taken to be a perfect fluid, given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \,, \tag{2.20}$$

where ρ and p are the energy density and pressure, respectively, measured by observers with four-velocity

$$u^\mu \equiv \frac{\partial x^\mu}{\partial t_p} \,, \tag{2.21}$$

where t_p is the proper time comoving with the fluid. This implies the constraint

$$u^\mu u_\mu = -1 \,. \tag{2.22}$$

For comoving observers cosmic time is equal to proper time, such that $t = t_p$.

For a homogeneous and isotropic perfect fluid and FLRW metric the field equations imply that $\rho(t)$ and $p(t)$. Moreover, given the conservation equations, and for an equation of state $p = w\rho$, where the equation of state parameter is given by w, and $k = 0$ we can write the well-known solutions for different cosmological epochs in our Universe:

$$\text{Radiation domination:} \quad w = \frac{1}{3} \Rightarrow a(t) \propto t^{1/2} \tag{2.23}$$

$$\text{Matter domination:} \quad w = 0 \Rightarrow a(t) \propto t^{2/3} \tag{2.24}$$

$$\Lambda \text{ domination:} \quad w = -1 \Rightarrow a(t) \propto \exp\left(\sqrt{\frac{\Lambda}{3}}t\right) . \tag{2.25}$$

At this stage we define the Hubble parameter $H = \frac{\dot{a}}{a}$, and the conformal Hubble parameter $\mathcal{H} = \frac{a'}{a}$.

2.5. Introduction to perturbative solutions

Perturbative expansions are used extensively in gravitational physics, as the full Einstein equations, Eq. (2.15), can be very difficult to solve exactly for many important physcial scenarios. These expansions come in a variety of different forms, and are usually constructed or adapted to be used in particular situations of physical interest [27, 141]. The two perturbation expansions we discuss in detail in the following chapters are the post-Newtonian expansion and cosmological perturbation theory. These are by no means the only perturbative constructions that can be applied to understand relativistic gravity, but they are well suited to understanding it in cosmology.

The post-Newtonian expansion is valid for systems of with large density contrasts, on scales of order 100Mpc or less. Formally, it is a weak-field and slow-motion expansion. On the other hand, cosmological perturbation theory is a weak-field expansion that describes an entire universe. It is normally applied to the largest scales of order the horizon-size down to the homogeneity scale, of order of 100Mpc, where density contrasts are small. Moreover, spatial derivatives do not add largeness, nor do time derivatives add smallness, to gravitational potentials or matter sources, as is the case for post-Newtonian gravity.

It is important to note that for the rest of this chapter post-Newtonian gravity and cosmological perturbation theory are perturbative expansions which in fact correspond to two limiting regimes of the same equations, Einstein's field equations, Eq. (2.15), and are associated with the so-called near and wave zones, respectively (see Sections 3.2.1 and 4.3.1). Furthermore, we describe how to formally define perturbations in the field equations, which can allow us to systematically write the field equations in a hierarchical manner.

2.5.1. Limits of the field equations

The starting point for both the post-Newtonian and cosmological perturbation expansions is the realisation that the Einstein equations can be written non-perturbatively as a set of wave equations, which take the form [120]

$$\Box\psi = -4\pi\mu\,, \tag{2.26}$$

where $\Box$ is the D'Alembertian operator associated with the metric of space-time, ψ represents the various gravitational potentials associated with the metric, and μ is a source term (derived from the components of the energy-momentum tensor, and the components of the metric with up to one derivative).

Equation (2.26) is a wave equation with null characteristics, so its retarded solution, assuming certain boundary conditions, is given by a time dependent Green's function solution, of the form [141]

$$\psi(t,\mathbf{x}) = \int_{\mathcal{C}_-} \frac{\mu(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x'\,, \tag{2.27}$$

where $\mathcal{C}_-$ the past light cone of the point $x = (t,\mathbf{x})$. The retarded time $t - |\mathbf{x}-\mathbf{x}'|$ is the latest time at which a light signal emitted from position $\mathbf{x}'$ would be received at position $\mathbf{x}$ before time t. These solutions, in general, represent a set of waves, with a characteristic wavelength and frequency that are determined by the source, μ. We will refer to these as λ_c and ω_c, respectively. Because Eq. (2.27) represents a set of null waves, these quantities are related by $\lambda_c = 2\pi/\omega_c = t_c$, where t_c is the characteristic time-scale of the source.

So far, we have not used perturbation theory at all. If we wish to use perturbation theory to solve the field equations in Eq. (2.26) we need to understand how the integral in Eq. (2.27) behaves under the relevant approximations. Specifically, we need to know if the length scale under investigation is smaller or greater than λ_c. These regimes are often referred to in the relativistic astrophysics literature as the "near zone" and the "wave zone", respectively [141]. We will use the same ideas, but apply them to cosmology instead. We will then refer to these two regimes as the "Newtonian", or "post-Newtonian", and the "cosmological", regimes respectively.

The relevant expansion for these regimes are, unsurprisingly, the post-Newtonian expansion and cosmological perturbation theory, respectively.

More formally, it is now from the retarded solution to Eq. (2.27), in the limiting regimes of the near-zone on length scales $L_N \ll \lambda_c$, and wave-zone on length scales of order $L_C \sim \lambda_c$, that we find the field equations are dominated by a Laplacian equation and wave equation at leading-order, respectively. This is discussed in more detail in Sections 3.2.1 and 4.3.1, respectively.

2.5.2. Defining perturbations

Perturbation theory applied to cosmology is often such that the Universe is modelled by a manifold, $\mathcal{M}$. In the perturbative formalisms of both cosmological perturbation theory and post-Newtonian gravity the perturbed manifold includes inhomogeneous structure whereas the background manifold is homogenous and isotropic[1]. We label this fiducial background manifold $\bar{\mathcal{M}}$. A diffeomorphism exists between these two manifolds. This means that there exists a function, with an inverse, both of which are continuously differentiable, between them.

Diffeomorphisms, however, are relatively weak conditions, they do not mean the physical properties are the same for these two manifolds (in spite of them having the same physical laws for all points). After all, we do not want these manifolds to be the same physically, we want one to include inhomogeneous structure and the other not to in both cosmological perturbation theory and post-Newtonian gravity. We illustrate this by writing the field equations in the background manifold, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, assuming $\Lambda = 0$ for simplicity. We then write the field equations in the perturbed manifold as $G_{\mu\nu} + \delta G_{\mu\nu} = 8\pi(T_{\mu\nu} + \delta T_{\mu\nu})$, due to differences in the metric and sources of energy-momentum. These two sets of equations do not have the same physical properties, and they do not describe the same physical situations, due to non-zero $\delta G_{\mu\nu}$ and $\delta T_{\mu\nu}$.

Both the background and perturbed manifold are related by a corresponding vector field, or gauge generator. Formally, we note that all tensorial quantities Q can be split up into a background $Q^{(0)}$ and perturbed δQ part such that

$$Q \equiv Q^{(0)} + \delta Q\,, \tag{2.28}$$

where $Q^{(0)}$ and δQ are tensorial quantities in the tangent spaces of a point in the

[1] A covariant approach to cosmological perturbation theory has been developed by Ellis and Bruni [87], and earlier work by [103].

background and perturbed manifolds, respectively. Moreover, δQ itself can further be split up as a power series such that

$$\delta Q = \sum_{n=1}^{\infty} \frac{1}{n!} Q^{(n)} \,, \tag{2.29}$$

where n denotes the order in perturbations, which we write in terms of a smallness parameter $\mathcal{E}$ such that $\mathcal{O}\left(Q^{(n)}\right) \sim \mathcal{O}\left(\mathcal{E}^n\right)$. This is a single parameter expansion. Each of these perturbations $Q^{(n)}$ can be pulled-back to all manifolds $\{\mathcal{M}_{\mathcal{E}^i}\}$, where $n \leq i$. These manifolds are taken to approximate the physical manifold $\mathcal{M}$ with increasing precision with higher i. All quantities Q in the field equations are perturbed, this includes the matter sources and the metric and allows us to derive a hierarchical set of field equations, where solutions to the lower-order field equations are sources to higher-order ones. Hence perturbation theory allows quantities in the real Universe, Q, to be modelled to high precision.

Note that we relate points on these manifolds by writing them in terms of coordinates x^μ, originally defined on the background manifold $\bar{\mathcal{M}}$. Furthermore, by studying small changes in these coordinates and ensuring background quantities remain unchanged, we are able to see how perturbations change under an infinitesimal coordinate, or gauge, transformation. In doing this we are able to write the field equations in terms of perturbed quantities which do not change under infinitesimal gauge transformations and therefore represent physical degrees of freedom.

In the following two chapters we consider the book-keeping for both cosmological perturbation theory and post-Newtonian gravity, before considering them simultaneously in our two-parameter expansion.

3. Cosmological Perturbation Theory

3.1. Introduction

The standard model of cosmology assumes a spatially homogeneous and isotropic FLRW background metric to describe the expansion of the Universe on large scales. This successfully describes the large-scale expansion of the Universe from a hot and dense initial state, dominated by radiation, to the cool and diffuse state dominated by non-relativistic matter, and the cosmological constant dominated epoch of the present day. This model only requires a handful of parameters such as the present temperature of the CMB radiation and the density parameters.

However, this homogeneous model cannot describe the complexity of inhomogeneous structure we see in the late Universe where galaxies consist of stars and galaxies make up clusters, groups and superclusters, over a huge range of scales. For the study of structure we require inhomogeneity and anisotropy. For modelling this cosmological perturbation theory is often used. This is a simple, systematic approach which starts from the exact spatially homogeneous and isotropic FLRW model as a background solution and adds inhomogeneous perturbations to it, and allows for the introduction of increasing complexity order-by-order in perturbation theory.

In this chapter we review the key features of cosmological perturbation theory.

3.2. The formalism

3.2.1. An expansion in the wave-zone

Cosmological perturbation theory is a weak-field, but not a slow-motion expansion. Cosmological perturbation theory is valid in a limiting regime of Einstein's field equations, sometimes known as the wave-zone, on scales up to and beyond the particle horizon of the observable Universe. Such length scales are, by definition, comparable to the characteristic wavelength, λ_c, defined in Section 2.5.1, such that [120]

$$L_C \sim \lambda_C = \frac{2\pi}{\omega_c} = t_c\,, \tag{3.1}$$

where L_C is the typical length scale associated with the regime of cosmological perturbation theory. This means that characteristic velocities go like $V \sim L_C/t_c \sim 1$ (*i.e.* they are not small). Moreover, the variation in time of gravitational potentials and matter fields cannot be considered small when compared to their variation in space, which gives

$$\dot{\psi} \sim |\nabla\psi| \sim \frac{\psi}{L_C}\,, \quad \text{and} \quad \dot{\mu} \sim |\nabla\mu| \sim \frac{\mu}{L_C}\,, \tag{3.2}$$

respectively. These facts mean that, unlike the case of post-Newtonian gravity, we cannot use V to track the smallness of gravitational potentials or matter fields. Instead we have to hypothesize, or construct [73, 148], a background solution to Einstein's equations that can be used as a background to perturb around. In cosmological perturbation theory this is taken to be the Friedmann solutions, see Section 2.4.2.

3.2.2. Defining perturbations

Perturbations are defined such that tensorial quantities Q are perturbed around $Q^{(0)}$, which exists in the background manifold, as seen in Eqs. (2.28) and (2.29) and discussed in Section 2.5.2. Moreover, perturbations are associated with perturbed manifolds as is Q. For cosmological perturbation theory the expansion parameter is given by $\mathcal{E} \equiv \epsilon$. Note that throughout this chapter we will consider perturbations up to second order in cosmological perturbation theory.

It is often convenient to slice the space-time manifolds, $\bar{\mathcal{M}}\,, \mathcal{M}_{\epsilon^n}$, into foliations of spatial hypersurfaces of constant time, which is the standard $3+1$ decomposition of space-time [83, 128, 131]. In this chapter the foliation of space-time by spatial hypersurfaces is given in terms of conformal time.

3.2.3. Perturbed metric

In cosmological perturbation theory the background space-time is often described by a spatially flat (justified observationally [12, 78], and assumed throughout this thesis) FLRW metric, $g^{(0)}_{\mu\nu}(t)$. The FLRW metric is given by Eq. (2.19) in terms of cosmic time with curvature, and given here in terms of conformal time τ [83]

$$ds^2 = g^{(0)}_{\mu\nu} dx^\mu dx^\nu = a^2(\tau)\left(-d\tau^2 + \delta_{ij} dx^i dx^j\right) , \tag{3.3}$$

where $k = 0$ and δ_{ij} is defined as the comoving background spatial metric. Given Eqs. (2.28) and (2.29) we can write perturbations to the metric to second order in the following way [128]

$$g_{\mu\nu} = g^{(0)}_{\mu\nu}(\tau) + g^{(1)}_{\mu\nu}(x^\alpha) + \frac{1}{2} g^{(2)}_{\mu\nu}(x^\alpha) + \dots , \tag{3.4}$$

where $g^{(n)}_{\mu\nu}(x^\alpha)$ such that $n > 0$, corresponds to perturbations. Nevertheless, contributions to the metric at linear order have, to date, been the only ones required to calculate cosmological gravitational phenomena. The ellipsis in this equation denote terms that are smaller than $g^{(2)}_{\mu\nu} \sim \mathcal{O}(\epsilon^2)$ (they should not be confused with quantities perturbed in the post-Newtonian expansion, as outlined in the next chapter).

We can write the perturbed components of the metric in the following way

$$g_{00} = -a^2\left(1 + 2\phi^{(1)} + \phi^{(2)}\right) + \dots \tag{3.5}$$

$$g_{0i} = a^2\left(h^{(1)}_{0i} + \frac{1}{2} h^{(2)}_{0i}\right) + \dots \tag{3.6}$$

$$g_{ij} = a^2\left(\delta_{ij} + 2C^{(1)}_{ij} + C^{(2)}_{ij}\right) + \dots , \tag{3.7}$$

where the perturbations to the space-time and space-space parts of the metric, h_{0i} and C_{ij}, respectively, can be split into divergenceless vectors and transverse and traceless tensors. This is given by

$$h_{0i} = B_{,i} - \mathrm{S}_i \tag{3.8}$$

$$C_{ij} = -\psi\delta_{ij} + \mathrm{E}_{,ij} + \mathrm{F}_{(i,j)} + \mathrm{h}_{ij} , \tag{3.9}$$

the former is known as the shift and the latter are perturbations to the spatial three-metric[1]. Also, we have omitted superscripts for simplicity. For the above perturbations ϕ, B, ψ and E are scalar perturbations, S_i and F_i are vector perturbations and h_{ij} is a tensor perturbation which we will now define.

Scalar perturbations are constructed such that they are necessarily curl-free, *e.g.* $B_{,[ij]} = 0$. Vector perturbations are divergenceless, *e.g.* $\mathrm{F}_{i,}{}^{i} = 0$. Note that here

[1]The g_{00} component of the metric is directly related to the lapse [131]

divergence-free is defined with respect to the flat-space metric, rather than using covariant derivatives, since perturbations are defined with respect to a spatially flat background[2]. Raising and lowering spatial indices of vector and tensor perturbations uses the comoving background spatial metric, δ_{ij}, not the full metric, so for example $\mathrm{h}_i{}^j \equiv \delta^{jk}\mathrm{h}_{ik}$. Tensor perturbations are transverse, $\mathrm{h}_{ij,}{}^j = 0$, and trace-free $\mathrm{h}_i{}^i = 0$, using this we can also define the inverse perturbed metric in terms of these scalars, vectors and tensors.

3.2.4. Perturbed matter sources

Given the four-velocity, defined in Eq. (2.21), the constraint equation (2.22), and the above perturbed metric we can calculate the perturbed four-velocity up to second order in ϵ

$$u^0 = a^{-1}\left(1 - \phi^{(1)} - \frac{1}{2}\phi^{(2)} + \frac{3}{2}\phi^{(1)2} + \frac{1}{2}v_k^{(1)}v^{(1)k} + v^{(1)k}h_{k0}^{(1)}\right) + \dots \quad (3.10)$$

$$u^i = a^{-1}\left(v^{(1)i} + \frac{1}{2}v^{(2)i}\right) + \dots , \quad (3.11)$$

where we have defined the spatial part of the four-velocity such that $u^i = a^{-1}v^i$, in terms of the three-velocity v^i. This is due to the fact that proper time t_p is cosmic time t for comoving observers, and cosmic time and conformal time are related by $dt = ad\tau$, therefore we can define the three-velocity

$$u^i \equiv \frac{\partial x^i}{\partial t_p} = \frac{\partial x^i}{\partial t} = \frac{1}{a}\frac{\partial x^i}{\partial \tau} \equiv \frac{1}{a}v^i . \quad (3.12)$$

Moreover, the three-velocity, at all orders in perturbation theory, can be decomposed into scalar and divergenceless vector parts such that

$$v^i \equiv \delta^{ij}v_{,j} + \hat{v}^i . \quad (3.13)$$

The remaining quantities to perturb are the energy density and pressure to second order, given by

$$\rho = \rho^{(0)} + \rho^{(1)} + \frac{1}{2}\rho^{(2)} + \dots \quad (3.14)$$

$$p = p^{(0)} + p^{(1)} + \frac{1}{2}p^{(2)} + \dots , \quad (3.15)$$

[2] The decomposition of a four-vector into a curl-free and divergence-free part in Euclidean space is known as Helmholtz's theorem [113].

where the quantities $\rho^{(0)}$ and $p^{(0)}$ are the energy density and pressure in the background FLRW geometry, respectively, and both have dimensions L_C^{-2}. Allowing for not only a perturbed energy density but also a perturbed pressure implies that the field equations describe matter and radiation dominated epochs of the Universe.

In cosmological perturbation theory the energy-momentum tensor can be taken to be a perfect fluid, with no anisotropic stress, this is given in Eq. (2.20). As in Ref. [112], the proper energy density is the eigenvalue of the energy-momentum tensor with eigenvector equal to the four-velocity

$$T^{\mu}_{\nu} u^{\nu} = -\rho u^{\mu} \, . \tag{3.16}$$

Given the perturbed energy density, pressure, four-velocity and metric we can write the fully perturbed energy-momentum tensor, T^{μ}_{ν}, to second order

$$T^{\mu}_{\nu} \;=\; T^{(0)\mu}_{\nu} + T^{(1)\mu}_{\nu} + \frac{1}{2} T^{(2)\mu}_{\nu} + \dots \, , \tag{3.17}$$

which includes the background, linear and second order parts. Explicitly to linear order we have:

$$T_{00} \;=\; \rho^{(0)} + \rho^{(1)} - g^{(1)}_{00}\rho^{(0)} + \dots \tag{3.18}$$

$$T_{0i} \;=\; -\rho^{(0)}(v^{(1)}_i + g^{(1)}_{0i}) - p^{(0)} v^{(1)}_i + \dots \tag{3.19}$$

$$T_{ij} \;=\; p^{(0)}(g^{(0)}_{ij} + g^{(1)}_{ij}) + p^{(1)} g^{(0)}_{ij} + \dots \, . \tag{3.20}$$

Note that from the field equations, we require that the energy density and pressure are homogeneous at lowest order – time dependent perturbations are allowed beyond leading-order. The cosmological constant is unperturbed and background-order, $\Lambda \equiv \Lambda^{(0)}$, but it has dimensions, $[\Lambda] \sim L_C^{-2}$. Additionally, note that we are able to model a universe of multiple fluids by using the total energy-momentum tensor, which is simply the sum of the energy-momentum tensors for each fluid.

3.2.5. Summary of book-keeping

We summarise the book-keeping of cosmological perturbation theory by noting that all perturbations to the metric and matter fields are taken to have the same order-of-smallness, ϵ, such that

$$\epsilon \sim |v^{(1)i}| \sim g^{(1)}_{\mu\nu} \sim L_C^2 \rho^{(1)} \sim L_C^2 p^{(1)} \tag{3.21}$$

(this is in contrast with post-Newtonian gravity as we will show in Section 4.3.4). Cosmological perturbation theory is a simple perturbative expansion in a single parameter ϵ around an exact solution to the field equations. Furthermore, the book-keeping in cosmological perturbation theory is such that time derivatives are not small with respect to spatial derivatives when acting on gravitational fields, see Eq. (3.2), this is in contrast to what occurs in post-Newtonian gravity.

Additionally, the reader may notice we have included factors of L_C^2 above, the characteristic length scale cosmological gravitational fields vary on. This allows us to compare the dimensionless expansion parameter, peculiar velocity and gravitational potentials to dimensional quantities like the perturbed energy density and pressure, as above. This is necessary, strictly speaking, in order to establish that quantities are of the same order of smallness. These additional factors are usually excluded in the literature [128], but are crucial in understanding much of the work we present in this thesis.

Substituting both the perturbed metric and matter sources into Einstein's equation allows us to solve for each of the components of the metric. However, it is first simpler to write the field equations in terms of gauge invariant quantities, this is what is discussed in the following section. For further explanation of cosmological perturbation theory the reader is referred to the review by Malik & Wands [128].

3.3. Gauges

General relativity is a diffeomorphism invariant, or covariant, theory meaning that the form of the tensor equations that we use to describe it must be valid for any set of coordinates. Diffeomorphisms obey a strict group structure, which guarantees that we can transform any given solution into a new set of coordinates, and that the result will still obey Einstein's equations. However, splitting the field equations into a background part and perturbations is not a covariant procedure, and therefore introduces a coordinate or gauge dependence. By construction, under infinitesimal gauge transformations the background remains the same and only perturbations are affected. Given general perturbations about a fixed background, there is then a freedom in coordinate re-parametrization of perturbations, and this is referred to as a "gauge freedom". This allows for the construction of gauge invariant variables, and the field equations written in terms of these quantities appear greatly simplified.

3.3.1. Gauge transformations

The gauge group of general relativity is the group of diffeomorphisms. The general form of an infinitesimal diffeomorphism or gauge transformation can be written as an infinitesimal change of coordinates [59, 132]

$$x^\mu \mapsto \tilde{x}^\mu = e^{\xi^\alpha \partial_\alpha} x^\mu \,, \tag{3.22}$$

where ξ^μ is known as the "gauge generator" and, for cosmological perturbation theory, is a small quantity in the perturbation expansion, which to second order is given by

$$\xi^\mu \equiv \xi^{(1)\mu} + \frac{1}{2}\xi^{(2)\mu} + \dots \,. \tag{3.23}$$

The expansion of the gauge generator, above, is necessary to ensure a closed system of perturbations, so that no 'new' perturbations are generated in the field equations under this transformation, for example at order $\mathcal{O}\left(\epsilon^{3/2}\right)$. Note that the gauge generator has dimensions of length L_C because Eq. (3.22) refers to a change in coordinates (which have dimensions of length). Moreover, these gauge generators, at any order $\mathcal{O}\left(\epsilon^n\right)$ can be decomposed into scalar and divergenceless vector parts such that

$$\xi^{(n)0} \equiv \delta t^{(n)} \tag{3.24}$$

$$\xi^{(n)i} \equiv \delta x^{(n),i} + \delta x^{(n)i} \,, \tag{3.25}$$

where $\delta x^i_{,i} = 0$. The Stewart-Walker lemma states that linear perturbations of a tensor field are gauge-invariant if the background of the tensor field is zero, constant in time, or a linear combination of products of delta functions [156]. From this lemma, a transformation of the above type leaves all background quantities and constants invariant, but changes the form of the perturbations. In this expression we have used the exponential map between coordinate systems, which guarantees that the group structure of the manifold is preserved. We also note that ξ^μ is a four-vector with indices which should be raised and lowered with the metric. However, we use the convention set out in [128]; that ξ^i is lowered using the flat spatial metric δ_{ij} such that[3] $\xi_j \equiv \delta_{ij}\xi^i$.

Given a tensor field $\mathcal{T}$, with coordinates x^μ defined on a background manifold

[3]For completeness, with this notation, we write the inverse of ξ^j as $\bar{\xi}_j = g_{j\nu}\xi^\nu$, rather than ξ_j.

$\bar{\mathcal{M}}$, the Lie derivative tells us how the components of this tensor transform under an infinitesimal transformation to coordinates $\tilde{x}^\mu$ (see Eq. (3.22)). To arbitrarily high-order the explicit form of the transformation applied to the tensor $\mathcal{T}$, under the coordinate transformation presented in Eq. (3.22), is given by the exponential map [59, 128, 132]

$$\tilde{\mathcal{T}} = e^{\mathcal{L}_\xi}\mathcal{T} = \mathcal{T} + \mathcal{L}_\xi\mathcal{T} + \tfrac{1}{2}\mathcal{L}_\xi^2\mathcal{T} + \dots\,, \tag{3.26}$$

where $\tilde{\mathcal{T}}$ is the transformed tensor, $\mathcal{L}_\xi$ is the Lie derivative along ξ^μ (see Eq. (2.13)). So the Lie derivative can be used to compute the change in the tensors $\mathcal{T}$ and $\tilde{\mathcal{T}}$ at a point.

Given the above transformation rule we can now transform the entire metric, and matter sources. All background quantities, like $a(t)$, $\rho^{(0)}$, $p^{(0)}$ and Λ remain invariant, and perturbations all transform. For the metric tensor, from Eq. (3.26), at linear order we have

$$\tilde{g}^{(1)}_{\mu\nu} = g^{(1)}_{\mu\nu} + \mathcal{L}_\xi g^{(0)}_{\mu\nu}\,. \tag{3.27}$$

Furthermore, at first order, we find all scalar, vector and tensor sources in the field equations transform linearly and can all be split by taking derivatives, divergences and traces of these transformations such that [128]

$$\tilde{\phi}^{(1)} = \phi^{(1)} + \mathcal{H}\delta t^{(1)} + \delta t^{(1)\prime} \tag{3.28}$$

$$\tilde{\psi}^{(1)} = \psi^{(1)} - \mathcal{H}\delta t^{(1)} \tag{3.29}$$

$$\tilde{B}^{(1)} = B^{(1)} - \delta t^{(1)} + \delta x^{(1)\prime} \tag{3.30}$$

$$\tilde{\mathrm{E}}^{(1)} = \mathrm{E}^{(1)} + \delta x^{(1)} \tag{3.31}$$

$$\tilde{\mathrm{S}}^{(1)i} = \mathrm{S}^{(1)i} - \delta x^{(1)i\,\prime} \tag{3.32}$$

$$\tilde{\mathrm{F}}^{(1)i} = \mathrm{F}^{(1)i} + \delta x^{(1)i} \tag{3.33}$$

$$\tilde{\mathrm{h}}^{(1)}_{ij} = \mathrm{h}^{(1)}_{ij}\,, \tag{3.34}$$

where we can observe that, along with the background quantities, $\mathrm{h}^{(1)}_{ij}$ is gauge invariant. For the matter perturbations we find

$$\tilde{\rho}^{(1)} = \rho^{(1)} + \rho^{(0)\prime}\delta t^{(1)} \tag{3.35}$$

$$\tilde{p}^{(1)} = p^{(1)} + p^{(0)\prime}\delta t^{(1)} \tag{3.36}$$

$$\tilde{v}^{(1)} = v^{(1)} - \delta x^{(1)\prime} \tag{3.37}$$

$$\tilde{\hat{v}}^{(1)i} = \hat{v}^{(1)} - \delta x^{(1)i\,\prime}. \tag{3.38}$$

Note that p and ρ transform as four-scalars, whereas ψ, v, *etc.* transform as three-scalars.

We could also consider the transformations of the metric and matter sources at second order, which can be split into scalars, vectors and tensors by taking derivatives and traces of these transformations, given explicitly in Ref. [128]. These second-order transformations are highly non-linear and complex, for example the second-order metric transforms according to:

$$\begin{aligned}
\tilde{g}^{(2)}_{\mu\nu} = \; & g^{(2)}_{\mu\nu} + g^{(0)}_{\mu\nu,\lambda}\xi^{(2)\lambda} + g^{(0)}_{\mu\lambda}\xi^{(2)\lambda}_{,\nu} + g^{(0)}_{\lambda\nu}\xi^{(2)\lambda}_{,\mu} + 2\Big(g^{(1)}_{\mu\nu,\lambda}\xi^{(1)\lambda} + g^{(1)}_{\mu\lambda}\xi^{(1)\lambda}_{,\nu} + g^{(1)}_{\lambda\nu}\xi^{(1)\lambda}_{,\mu}\Big) \\
& + g^{(0)}_{\mu\nu,\lambda\alpha}\xi^{(1)\lambda}\xi^{(1)\alpha} + g^{(0)}_{\mu\nu,\lambda}\xi^{(1)\lambda}_{,\alpha}\xi^{(1)\alpha} + g^{(0)}_{\mu\lambda}\Big(\xi^{(1)\lambda}_{,\nu\alpha}\xi^{(1)\alpha} + \xi^{(1)\lambda}_{,\alpha}\xi^{(1)\alpha}_{\ \ \nu}\Big) \\
& + 2\Big(g^{(0)}_{\mu\lambda,\alpha}\xi^{(1)\alpha}\xi^{(1)\lambda}_{,\nu} + g^{(0)}_{\lambda\nu,\alpha}\xi^{(1)\alpha}\xi^{(1)\lambda}_{,\mu} + g^{(0)}_{\lambda\alpha}\xi^{(1)\lambda}_{,\mu}\xi^{(1)\alpha}_{,\nu}\Big) \\
& + g^{(0)}_{\lambda\nu}\Big(\xi^{(1)\lambda}_{,\mu\alpha}\xi^{(1)\alpha} + \xi^{(1)\lambda}_{,\alpha}\xi^{(1)\alpha}_{,\mu}\Big).
\end{aligned} \tag{3.39}$$

3.3.2. Gauge invariant quantities

Having performed infinitesimal coordinate transformations of the metric and sources of energy-momentum, we are now in a position to isolate and remove the superfluous degrees of freedom associated with infinitesimal diffeomorphisms. This is normally undertaken in cosmological perturbation theory to represent the physical degrees of freedom in the problem only, and will remove the possibility of any interference from spurious gauge modes. These problems were circumvented by Bardeen, who was the first to construct combinations of perturbations that remained invariant under general gauge transformations [32]. There are, in fact, an infinite number of such quantities. This removes all ambiguity, and allowed perturbed field equations to be written down that were guaranteed to be free from all gauge freedoms.

The methodology for calculating such quantities was pioneered by Bardeen and developed for use in second-order cosmological perturbation theory [128]. We will use the example of calculating gauge invariant quantities that correspond to the longitudinal gauge. These gauge invariant quantities reduce to the metric perturbations in longitudinal gauge when $\mathrm{E} = B = 0$ and $\mathrm{F}_i = 0$ (we omit superscript labels here for simplicity, but the methodology is true for all orders in perturbation).

We start by choosing gauge generators, $\delta x, \delta x^i$ and δt, such that $\tilde{\mathrm{E}} = \tilde{B} = \tilde{\mathrm{F}}_i = 0$, this fixes four degrees of freedom. We will then substitute these quantities back into the expressions for all of the transformed perturbations in the metric and matter fields. The results will be gauge invariant, as the original gauge transformations were written down in a completely arbitrary coordinate system. This means that newly constructed quantities cannot depend on any choice of gauge, and hence must be gauge invariant.

Bardeen's potentials are gauge-invariant quantities calculated from $\tilde{\mathrm{E}} = \tilde{B} = 0$, and are given by

$$\boldsymbol{\phi}^{(1)} \equiv \phi^{(1)} + \mathcal{H}\left(B^{(1)} - \mathrm{E}^{(1)\prime}\right) + \left(B^{(1)} - \mathrm{E}^{(1)\prime}\right)' \tag{3.40}$$

$$\boldsymbol{\psi}^{(1)} \equiv \psi^{(1)} - \mathcal{H}\left(B^{(1)} - \mathrm{E}^{(1)\prime}\right) . \tag{3.41}$$

When the field equations are written in terms of these gauge invariant quantities they correspond to the field equations in the longitudinal gauge. Moreover, gauge invariant quantities can be calculated for metric potentials S_i or F_i and matter sources $\rho^{(1)}, p^{(1)}, v^{(1)}$ and $\hat{v}^{(1)i}$ in the longitudinal gauge. For the matter sources we find:

$$\mathsf{\rho}^{(1)} \equiv \rho^{(1)} + \rho^{(0)\prime}\left(B^{(1)} - \mathrm{E}^{(1)\prime}\right) \tag{3.42}$$

$$\mathrm{p}^{(1)} \equiv p^{(1)} + p^{(0)\prime}\left(B^{(1)} - \mathrm{E}^{(1)\prime}\right) \tag{3.43}$$

$$\mathrm{v}^{(1)} \equiv v^{(1)} + E^{(1)\prime} \tag{3.44}$$

$$\mathrm{v}^{(1)i} \equiv \hat{v}^{(1)} + F^{(1)i\,\prime} \tag{3.45}$$

$\mathrm{h}_{\mathrm{ij}}^{(1)}$ is trivially gauge-invariant.

3.3.3. Choice of gauge

As discussed in Section 3.3.2 the field equations contain not only the essential degrees of freedom required to describe the physical situation at hand, but also four superfluous degrees of freedom from the gauge generator, and so can simplify greatly by choosing a gauge. There are, in fact, an infinite number of gauges we can take, but there are several which are normally chosen, one of which is the longitudinal gauge, discussed in Section 3.3.2. The field equations in terms of a particular gauge are of the same form as the field equations written in terms of gauge invariant quantities (which in fact contain *all* metric potentials). There are several that are traditionally chosen in cosmological perturbation theory, we will list them and their conditions

[128]:

Longitudinal or Newtonian gauge: $B = \mathrm{E} = 0$ and usually $\mathrm{F}_i = 0$, or $\mathrm{S}_i = 0$.

Spatially flat gauge: $\psi = \mathrm{E} = 0$ and $\mathrm{F}_i = 0$.

Synchronous gauge: $\phi = B = 0$ and $\mathrm{S}_i = 0$.

Comoving orthogonal gauge: $v = B = 0$ and $\hat{v}^i = 0$.

Total matter gauge: $v + B = 0, \mathrm{E} = 0$ and $\mathrm{F}_i = 0$.

Uniform density gauge: $\rho^{(1)} = 0$, and sometimes $\mathrm{E} = 0$ and $\mathrm{F}_i = 0$.

3.4. Dynamics

3.4.1. The field equations

By substituting the perturbed metric and energy-momentum tensor into the field equations (2.14), we can write the field equations to lowest order. This gives the Friedmann equations

$$\mathcal{H}^2 = \frac{8\pi}{3}a^2\rho^{(0)} + \frac{1}{3}a^2\Lambda \tag{3.46}$$

$$\mathcal{H}' = -\frac{4\pi}{3}a^2\left(\rho^{(0)} + 3p^{(0)}\right) + \frac{1}{3}a^2\Lambda\,, \tag{3.47}$$

and solutions discussed in Section 2.4.2.

At first order, in terms of Bardeen's variables and the gauge invariant quantities corresponding to the matter sector, see Section 3.3.2, the scalar field equations give

$$3\mathcal{H}(\boldsymbol{\psi}^{(1)\prime} + \mathcal{H}\boldsymbol{\phi}^{(1)}) - \nabla^2\boldsymbol{\psi}^{(1)} = -4\pi a^2 \mathrm{\rho}^{(1)} \tag{3.48}$$

$$\boldsymbol{\psi}^{(1)\prime} + \mathcal{H}\boldsymbol{\phi}^{(1)} = -4\pi a^2\left(\rho^{(0)} + p^{(0)}\right)\mathrm{v}^{(1)}\,. \tag{3.49}$$

Furthermore, we can write the vector and tensor field equations

$$\nabla^2\mathbf{S}_i^{(1)} = -16\pi a^2\left(\rho^{(0)} + p^{(0)}\right)\left(\mathrm{v}_i^{(1)} - \mathbf{S}_i^{(1)}\right) \tag{3.50}$$

$$\mathrm{h}_{ij}^{(1)\prime\prime} + 2\mathcal{H}\mathrm{h}_{ij}^{(1)\prime}\nabla^2\mathrm{h}_{ij}^{(1)} = 0\,, \tag{3.51}$$

where $\mathbf{S}_i = \mathrm{S_i} + \mathrm{F}^{(1)\mathrm{i}\prime}$ is a gauge invariant quantity in terms of $\mathrm{F}^{(1)i}$. All the above field equations are now in terms of gauge invariant quantities (all background quantities are gauge invariant). These equations are therefore valid in any gauge but when the longitudinal gauge is fixed then these equations look the same as field equations in the longitudinal gauge (*e.g.* $\mathbf{S}_i$ and $\boldsymbol{\phi}$ are simply replaced by $\mathrm{S_i}$ and ϕ, respectively). Moreover, we have split the equations into scalar, vector and tensor parts via derivatives and traces of the field equations. In cosmological perturbation theory scalar, vector and tensor perturbations decouple with each other at first order in perturbations, and so the equations that govern each of them can be solved independently of the other two sectors. We refer the reader to Ref. [128] for the field equations at second order.

3.4.2. Conservation equations

At lowest order, the temporal part of the conservation equation (2.16), gives us the continuity equation

$$\rho^{(0)\prime} = -3\mathcal{H}(\rho^{(0)} + p^{(0)})\,, \tag{3.52}$$

and at linear order

$$\rho^{(1)\prime} \;=\; -3\mathcal{H}\left(\rho^{(1)} + p^{(1)}\right) + 3\psi'\left(\rho^{(0)} + p^{(0)}\right) - \left(\left(\rho^{(0)} + p^{(0)}\right)v^i\right)_{,i}\,, \tag{3.53}$$

which is not gauge-fixed. The spatial part of the conservation equation at leading-order is linear and gives us the Euler equation

$$v^{(1)i\prime} + \mathcal{H}v^{(1)i} - 3\mathcal{H}\frac{p^{(0)\prime}}{\rho^{(0)\prime}}v^{(1)i} \;=\; -\frac{p^{(1)}_{,i}}{\rho^{(0)} + p^{(0)}} - \phi^{(0)}_{,i}\,, \tag{3.54}$$

which again is not gauge fixed. The conservation equations can also be derived beyond linear order [128].

These equations can be used to help solve Einstein's field equations for the gravitational potentials and matter sources. It can also be noted that we require each of the components of the metric only up to first order in perturbations, in order to consistently write the equations of motion of a time-like particle to first order. This is a departure from the more complicated situation that arises in post-Newtonian gravity.

Cosmological perturbation theory can be applied, given a fluid with known equation of state, and one can derive both the background and perturbed equations. This allows the theory to be applied to both the radiation-dominated and matter-dominated stages of the Universe's evolution, as well as to the current cosmological-constant-dominated epoch. This is a versatility that is absent from the standard approach in post-Newtonian gravity, as radiation and Λ are almost completely negligible for the study of gravity in the Solar System, binary pulsars, and other such small-scale astrophysical environments. If one wants to apply such expansions to super-clusters in a cosmological context, however, then more care may be required. For further explanation of cosmological perturbation theory, the reader is referred to the review by Malik and Wands [128].

4. Post-Newtonian gravity

4.1. Introduction

The standard model of gravitation assumes general relativity, which fulfils the assumptions stated in Section 2.1 and is justified observationally by local tests of gravity. From general relativity in the appropriate weak-field slow-motion limit we can derive Newtonian gravity. Moreover, we can derive small relativistic corrections to Newtonian gravity using post-Newtonian gravity. It is the post-Newtonian formalism, applied to general relativity, that we discuss in this section [66, 141, 171]. This approximation is sufficiently accurate to encompass all solar system tests that have been performed, tests which occur on scales greater than the Schwarzschild radius of the constituent objects but much less than the Hubble radius.

Post-Newtonian gravity is designed to describe large density contrasts and isolated systems, such that peculiar velocities remain small, this is known as the near-zone. For peculiar velocities to remain small we require that the velocity due to the Hubble flow, HL_N, must be smaller than or equal to the peculiar velocities of the constituent objects, this is true for lengths scales less than $\sim$ 100Mpc. Traditionally, for post-Newtonian gravity to hold, we require that far from the source of gravitational potentials we expect the metric to be described by the flat Minkowski metric [66, 171]. Note, however, that the post-Newtonian expansion has also been constructed around a time dependent background metric, $g^{(0)}_{\mu\nu}(t)$, the FLRW metric. Indeed, a small enough region of perturbed FLRW can be shown to be entirely equivalent to perturbed Minkowski space at both Newtonian [73] and post-Newtonian orders [148]. We consider the post-Newtonian expansion on an FLRW background throughout this chapter.

The post-Newtonian formalism is not adequate, however, for studies of systems which include compact objects where variations in time are comparable to variations in space for constituent objects, such as near binary pulsars, or for gravitational radiation and Horizon-sized scales, such as in cosmology, where the slow motion assumption no longer holds. In these systems the gravitational potentials vary on similar length and time scales, a regime that is sometimes known as the wave-zone. In some respects, post-Newtonian gravity resembles the quasi-static (or slow-motion) limit of cosmological perturbation theory. We will comment on this link, and its limitations, in later sections.

In this chapter we review the key features of Newtonian gravity and the post-Newtonian formalism. For further details on the post-Newtonian expansion we refer the reader to [141, 171].

4.2. Newtonian gravity

4.2.1. Newton's laws

In the Solar System gravitation is weak enough that Newtonian theory is adequate to describe all but small relativistic effects. To an accuracy of 10^{-5} light rays travel in straight lines and test particles move according to Newton's acceleration equation [131]

$$\dot{v}_i \;=\; U_{,i}\,, \tag{4.1}$$

where v_i is the peculiar velocity and U is the Newtonian gravitational potential produced by a rest-mass energy density according to

$$\nabla^2 U \;=\; -4\pi\rho\,. \tag{4.2}$$

This is known as Newton's law of gravitation. For reasonable relativistic theories of gravity, in the slow-motion and weak-field limit, we expect to derive the acceleration equation, (4.1), from the geodesic equation, (2.4), and Newtonian gravitation equation, (4.2), from the field equations for the theory of gravity in question.

A perfect fluid obeys the Eulerian equations of hydrodynamics given by [131]

$$\partial_t\rho + (\rho v^i)_{,i} \;=\; 0 \tag{4.3}$$

$$(\partial_t + v_j\partial^j)v_i \;=\; \rho U_{,i} - p_{,i}\,, \tag{4.4}$$

which are known as the continuity (or mass conservation) equation and momentum conservation (or Euler) equation, respectively. Here p is pressure. From a metric theory of gravity we expect this equation to be derived from the Bianchi identity or the derived conservation of stress-energy, see Eq. (2.16).

Newtonian perturbation theory is formulated from perturbing each term in the dynamical equations for Newtonian gravity, above. This produces dynamics that are a subset of those derived from the post-Newtonian limit of Einstein's field equations, (2.15), the geodesic equation, (2.4), and the conservation equations, (2.16). However, the dynamics from the post-Newtonian limit are more complex because Einstein's field equations, (2.15), and the geodesic equation, (2.4), are non-linear, whereas Newton's law of gravity, Eq. (4.2), and Newton's acceleration equation, (4.1), are linear. Moreover, the conservation of energy-momentum, field equations and geodesic equations derived using post-Newtonian gravity allow for all perturbations from the space-time metric to be solved for, this includes non-scalar gravitational potentials and scalar gravitational fields other than U. As the concept of a space-time metric does not exist in Newtonian gravity such gravitational potentials do not appear in Newtonian perturbation theory.

For the post-Newtonian book-keeping, outlined in the next section, Eqs. (4.1)-(4.4) are all derived at leading-order from the Newtonian limit of general relativity via the geodesic equation, field equations and conservation of energy-momentum. Beyond-leading-order post-Newtonian corrections can also be derived.

4.2.2. Newtonian N-body simulations

N-body simulations have been crucial in our understanding of physical cosmology, in the evolution of large-scale structure and the justification of ΛCDM cosmology. The Millenium simulation, which uses a Newtonian approximation in an expanding background, enables cosmologists to study the processes which lead to the formation of structure with large density contrasts on the largest scales, such as galaxy halos and filaments, due to dark matter [55]. The latest simulation, the Millennium XXL simulation, used a cube of length $3000\text{Mpc}\,\text{h}^{-1}$ with periodic boundary conditions, and 6720^3 particles, each of mass $\sim 10^9 M_\odot$. The motion of each particle is determined from Newton's law of gravitation and the acceleration equation for each particle. It also requires and solves the Eulerian equations of hydrodynamics. Additionally, the Friedmann equations are solved for given specified cosmological parameters. The initial conditions obey constraints from the CMB and the evolution is taken to the present day. Cosmologists use this simulation to study the distribution of dark matter halos and galaxies, as the resolution is taken to be valid from $\sim$ 10Gpc down to $\sim$ 10kpc. Indeed, the Millenium simulation has shown that our models can produce the voids and filaments that form the cosmic web we observe today, and which remains crucial for our understanding of cosmology [45]. Finally, note that N-body simulations, beyond Newtonian ΛCDM, which include relativistic effects, have been studied in Refs. [16–19, 67, 69, 91, 92, 101, 145, 159].

4.3. Post-Newtonian formalism

4.3.1. A slow-motion expansion

There are two key features of the post-Newtonian expansion: it is both a slow-motion expansion, valid on small-scales, and a weak-field expansion.

Firstly, the post-Newtonian formalism is valid in a regime where distance scales are small compared to the characteristic wavelength, λ_c, such that [120, 141]

$$L_N \ll \lambda_c = \frac{2\pi}{\omega_c} = t_c \,, \tag{4.5}$$

where we have introduced the typical length scale associated with the Newtonian regime, L_N. This is analogous to how the near-zone is treated for isolated systems. Another way of stating this condition would be to say that the velocities of the sources are, in some sense, slow. This follows from the fact that characteristic dimensionless velocities are of the order $v \sim L_N/t_c \ll 1$. In this sense, small scales tend to correspond to slow motions. Post-Newtonian gravity is therefore appropriate for modelling isolated astrophysical systems, but (by itself) is not appropriate for modelling an entire universe. This is in contrast to cosmological perturbation theory, valid in the opposite extreme (a regime analogous to the wave-zone).

Now consider the consequences of the assumption of small scales for derivatives of the source term, μ, derived from Eqs. (2.26) and (2.27). Spatial derivatives are of the order $|\nabla\mu| \sim \mu/L_N$, while time derivatives are of order $\dot{\mu} \sim \mu/t_c$. We therefore have

$$\dot{\mu} \ll |\nabla\mu| \,. \tag{4.6}$$

In words, the typical variation of the sources in time is small compared to their variation in space. It is also apparent that the order-of-smallness should be expected to be of the same size as the dimensionless velocity, v.

Let us now consider the size of the gravitational potentials that are represented by ψ, and how they vary in space and time. It is apparent from Eq. (2.27) that if $L_N \sim |\mathbf{x} - \mathbf{x}'| \ll t \sim \lambda_C$, and we Taylor expand the time-dependent part of the integrand, then the leading-order part of ψ is given by

$$\psi = \int_{\mathcal{V}} \frac{\mu(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \,, \tag{4.7}$$

where $\mathcal{V}$ denotes a space-like volume of constant time. Note that this can also be derived straight from the wave equation (2.26) given the assumption that $L_N \sim |\mathbf{x} - \mathbf{x}'| \ll t \sim \lambda_C$. This wave equation then becomes a Laplacian equation $\nabla^2\psi = -4\pi\mu$, because our assumption implies quantities in Eq. (2.26) are approximately constant in time, and so variations in time are approximately zero. To solve this Laplacian equation, one requires the time-independent (rather than the time-dependent) Green's function solution, which is given by Eq. (4.7).

We can see from Eqs. (4.6) and (4.7) that when $|\mathbf{x} - \mathbf{x}'| \ll t_c$ the derivatives of ψ satisfy [141]

$$\dot{\psi} \ll |\nabla\psi| \,. \tag{4.8}$$

Again, the order of smallness of the time derivative, compared to the space derivatives, is found to be of the order of v. It can also be seen that $\psi \sim \mu L_N^2$.

4.3.2. Defining perturbations

The second requirement of the post-Newtonian expansion is that the magnitudes of the gravitational potentials are small. Defining this smallness is complicated by the fact that there exists a number of gravitational potentials, the time-time, time-space and space-space gravitational potentials, in Einstein's theory (represented schematically by one potential given in Eq. (2.26)). These potentials may all be small, but they may also be different in magnitude. The magnitude of these potentials is determined through the geodesic equations for freely falling time-like particles and the field equations, via the sources of energy-momentum. This is quite different to cosmological perturbation theory where metric potentials and sources of energy-momentum are perturbed from the onset in terms of $\epsilon\,, \epsilon^2$, and so on[1], as discussed in detail in Chapter 3.

The magnitude of any given potential can also be linked to the velocity of the matter fields in the space-time through the equations of motion of those fields. In

[1]Another comparison is that cosmological perturbation theory is an expansion around an exact solution that is a good approximation to the perturbed solution. Whereas the post-Newtonian expansion is closer to an asymptotic expansion, it allows for small perturbations in the geometry and large perturbations in the energy-density. So when these perturbations are set to zero we derive the vacuum solution, which is not close in magnitude to the perturbed solution, when considering, for example, curvature scalars.

order to do this, it is convenient to define the smallness-parameter

$$\eta \sim v \sim \frac{|\partial/\partial t|}{|\partial/\partial x|}, \tag{4.9}$$

where the spatial part of the four-velocity is of order the three-velocity such that $v \equiv |v^{(1)i}| \sim |u^{(1)i}| \sim \eta$, see the relation in Eq. (3.12). This book-keeping is used to keep track of the order-of-smallness of a quantity within this expansion and implies the smallness parameter is η for post-Newtonian gravity. Post-Newtonian gravity, unlike cosmological perturbation theory, is not only a weak-field, but also a slow motion expansion, (it has been observed that this corresponds to a two or potentially three parameter expansion, in the context of the quasi-static limit of cosmological perturbation theory in [143]). Nevertheless, post-Newtonian gravity relates the smallness of time derivatives with the small magnitude of weak gravitational fields, so both are related by a single perturbation parameter η.

In post-Newtonian gravity, perturbations are also defined such that tensorial quantities are perturbed around a background value, which exists in the background manifold. Perturbations, $Q^{(n)} \sim \mathcal{O}(\eta^n)$ where $n > 0$, are associated with perturbed manifolds, and space-time is foliated into spatial hypersurfaces of constant time. The superscript in parentheses now denote the order of smallness of a term in η, and should not be confused with the perturbed quantities in the previous chapter (which were perturbed in ϵ).

4.3.3. Perturbed metric and matter sources

Let us now consider how the post-Newtonian book-keeping works for the leading-order parts of each of the components of the metric. At leading order, the space-components of the geodesic equation, (2.4), the equations of motion for freely falling time-like particles, tells us that $\dot{v} \sim |\nabla g_{00}|$. Furthermore as velocities and time derivatives each have smallness η ($v \equiv |v^{(1)i}|$ and $\partial/\partial t \sim \eta/L_N$) this implies the metric is strictly perturbed in the following way [141, 171]

$$g_{00} = g_{00}^{(0)}(t) + g_{00}^{(2)}(t, \mathbf{x}) + \dots, \tag{4.10}$$

where ellipsis denote terms that are smaller than η^2. There can be no such metric potentials that depend on spatial position which are larger than order η^2, as this would be incompatible with the leading-order part of the geodesic equation, Eq. (4.1) (and in post-Newtonian book-keeping $\dot{v}_i \sim \eta^2$).

Meanwhile, the leading-order part of the time-time component of the field equations, (2.15), gives

$$\nabla^2 g_{00} \sim \rho \,, \tag{4.11}$$

where the energy-momentum tensor is taken to be a perfect fluid, Eq. (2.20), and ρ is the leading-order part of the energy density of the matter fields. The relation in Eq. (4.11) tells us that ρ, which actually corresponds to the mass density in Newtonian and post-Newtonian gravity, can be no larger than $\eta^2 L_N^{-2}$, so

$$\rho \equiv \rho^{(2)} + \dots \,. \tag{4.12}$$

The similarity between Eq. (4.11) and the Newton-Poisson equation, (4.2), justifies associating $g_{00}^{(2)}(t, \mathbf{x})$ with the Newtonian gravitational potential, U. Furthermore, for freely falling time-like particles the we find

$$U \sim v^2 \,, \tag{4.13}$$

from the virial theorem. This was traditionally justified observationally because nowhere in the Solar System is the gravitational potential larger than $U \sim v^2 \sim 10^{-5}$.

To go to higher-order in g_{00}, and to find the other leading-order components of the metric, we need to consider the higher-order components of the energy-momentum tensor. To do this we first expand the energy density and pressure as $\rho = \rho^{(2)} + \rho^{(4)} + \dots$ and $p = p^{(4)} + \dots$, respectively. These high-order perturbations are traditionally justified observationally [171] as other forms of energy density (other than the rest-mass energy density at leading-order), such as compressional energy, radiation, thermal energy *etc.*, which are small (no larger than $\eta^4 L_N^{-2}$ in the Solar System). Also, astrophysical bodies such as the Sun are gravitationally stable, so we expect $p \sim \rho U$ and this implies $p \sim \eta^4 L_N^{-2}$ at leading order. Note that p and ρ have the same dimensions because we set $c = G = 1$, so that the field equations have dimensions length^{-2}, see Section 1.1.

We can now derive the components of the energy-momentum tensor given in Eq. (2.20), up to $\mathcal{O}(\eta^5 L_N^{-2})$, which are then

$$T_{00}^{(2)} = -g_{00}^{(0)} \rho^{(2)} \tag{4.14}$$

$$T_{00}^{(4)} = -g_{00}^{(0)} \rho^{(4)} - \rho^{(2)} \left(g_{00}^{(0)} u^{(1)i} u_i^{(1)} + g_{00}^{(2)} \right) \tag{4.15}$$

$$T_{0i}^{(3)} = -\sqrt{-g_{00}^{(0)}} \rho^{(2)} u_i^{(1)} \tag{4.16}$$

$$T^{(4)}_{ij} = \rho^{(2)} u^{(1)}_i u^{(1)}_j + p^{(4)} g^{(0)}_{ij}, \tag{4.17}$$

where we assume $g^{(0)}_{0i} = 0$, this is the case for the Minkowski and FLRW space-times. In each of these expressions we have continued the practice of using superscripts in brackets to denote the order-of-smallness of a quantity. However, when a quantity is dimensional, such as $p^{(4)}$, then the reader should take this to mean, for example, $p^{(4)} \sim \eta^4 L_N^{-2}$.

Through the geodesic equations, (2.4), for freely falling time-like particles and the field equations, (2.15), the gravitational fields that result from Eqs. (4.14)-(4.17) are given by

$$g_{00} = g^{(0)}_{00}(t) + g^{(2)}_{00}(t, \mathbf{x}) + \tfrac{1}{2} g^{(4)}_{00}(t, \mathbf{x}) \dots \tag{4.18}$$

$$g_{ij} = g^{(0)}_{ij}(t) + g^{(2)}_{ij}(t, \mathbf{x}) + \dots \tag{4.19}$$

$$g_{0i} = g^{(3)}_{0i}(t, \mathbf{x}) + \dots , \tag{4.20}$$

where we have assumed a background time-dependent metric $g^{(0)}_{\mu\nu}(t)$. This background is useful for the studies of post-Newtonian perturbed Minkowski and FLRW space-times. The metric components $g^{(4)}_{00}$, $g^{(2)}_{ij}$, and $g^{(3)}_{0i}$ are usually referred to as "post-Newtonian potentials".

One may note that the first spatially dependent term in g_{0i} occurs at $O(v^3)$. This is because the first non-zero source term for this potential is of order $\mathcal{O}\left(\rho^{(2)} v^{(1)}_i\right)$, from Einstein's field equations. It can also be noted that the orders of the gravitational potentials required for them to be labelled "post-Newtonian" are different in different parts of the metric [171]. This is because time derivatives add an order-of-smallness, compared to space derivatives, and because these two types of derivatives on the different components of the metric in the equations of motion of time-like particles and field equations.

One may also note that there are a number of missing terms in both the energy-momentum tensor and the metric, see Eqs (4.14)-(4.17) and (4.18)-(4.20), respectively. For example, there are no terms in T_{00} of $\mathcal{O}\left(\eta^3 L_N^{-2}\right)$, and no terms in g_{00} of $\mathcal{O}(\eta^3)$. As far as the energy-momentum tensor is concerned, this can be considered a choice of the type of matter that one wishes to model. For example, matter with a pressure term at $\mathcal{O}(\eta^2 L_N^{-2})$ is traditionally excluded from the expansion. This is no accident, however, as if such a term were to be included then energy-momentum conservation equations would imply that it would need to be spatially homogeneous (as found in [149], and in our two-parameter expansion in Section 5.2.4). This means that barotropic fluids with $p = w\rho$ and $w \neq 0$ do not fit into post-Newtonian gravity in a natural way at leading-order, unless they are diffuse enough to be considered post-Newtonian in order (*i.e.* occur at only $\mathcal{O}(\eta^4)$ or above). This is because post-Newtonian gravity was formulated to include dust [66, 171], but not radiation or a cosmological constant. This is very different to cosmological perturbation theory, reviewed in the previous chapter, which was formulated to describe all epochs of the Universe (radiation, dust, and cosmological dominated epochs) on horizon-sized scales. Such versatility is absent from the standard approach in post-Newtonian gravity, as radiation and Λ are almost completely negligible for the study of gravity in the Solar System, binary pulsars, and other such very-small-scale astrophysical environments. If one wants to apply the post-Newtonian expansion to super-clusters in a cosmological context, therefore, then more care may be required[2]. The situation with the metric, however, is quite different.

The required order-of-smallness of the different components of the metric is not specified from the outset. It is determined by solving the field equations, and by using the equations of motion of the matter fields [171]. We end up with a metric and an energy-momentum tensor that are expanded at even orders in η in their time-time and space-space components, and at odd orders in η in their time-space components (a trend that continues until gravitational waves are generated), and the lowest-order gravitational potentials are at either $\mathcal{O}(\eta^2)$ or $\mathcal{O}(\eta^3)$. One could, for example, have tried to include a $g_{00}^{(3)}$ term in the time-time component of the metric. However, there would be no matter fields to source such a term, and so it would end up satisfying a homogeneous version of the equation satisfied by $g_{00}^{(2)}$. This means that the hypothesized $g_{00}^{(3)}$ term describes no new physics, and can be absorbed into $g_{00}^{(2)}$ without loss of generality, and it is not necessary or helpful to consider such a term independently. We will return to this point later on.

4.3.4. Summary of book-keeping

We summarise the book-keeping of post-Newtonian gravity by noting that all perturbations to the metric and matter fields are given in terms of the order-of-smallness parameter η, such that

$$\eta^2 \sim |v^{(1)i}|^2 \sim g_{00}^{(2)} \sim L_N^2 \rho^{(2)} \sim L_N^2 \frac{p^{(4)}}{|v^{(1)i}|^2} \sim \left(\frac{\partial_t}{\partial_i}\right) . \tag{4.21}$$

[2] Additionally, one could consider the inclusion of heat flow or anisotropic pressure, traditionally excluded from the post-Newtonian expansion.

Time derivatives are small (by a factor of η) with respect to spatial derivatives, which is in stark contrast with cosmological perturbation theory, Eq. (3.21). Cosmological perturbation theory is a simple perturbative expansion in a single parameter ϵ around a known solution to the field equations.

The reader may notice we have included factors of L_N^2 above, the characteristic length scale over which gravitational fields vary in for post-Newtonian systems, as this is necessary to compare the dimensionless expansion parameter, peculiar velocity and gravitational potentials to dimensional quantities like the perturbed energy density and pressure. This is necessary, strictly speaking, in order to establish that quantities are of the same order of smallness. These additional factors are usually excluded in the literature [141, 171], but are crucial in understanding much of the work we present in this thesis.

Substituting both the perturbed metric and matter sources into Einstein's equation and the geodesic equation allows us to solve for each of the components of the metric. However, it is first simpler to write the field equations in terms of a gauge, this is what is discussed in the following section. For further details about post-Newtonian expansions the reader is referred to the textbooks by Will [171] and Poisson & Will [141].

4.4. Post-Newtonian equations

4.4.1. Field equations and standard post-Newtonian gauge

The post-Newtonian equations for general relativity are normally derived for given a Minkowski background metric [141, 171], therefore this is how we preceded. The field equations are usually written in terms of the Ricci tensor, not the Einstein tensor, on the left-hand side, (with no cosmological constant) in the form of Eq. (2.15). Therefore, the leading-order components of the Ricci tensor are given by [141, 171]

$$R_{00}^{(2)} = -\frac{1}{2}\nabla^2 g_{00}^{(2)} \tag{4.22}$$

$$R_{0i}^{(3)} = -\frac{1}{2}\left(-\dot{g}_{ik,k}^{(2)} + \dot{g}_{kk,i}^{(2)} + \nabla^2 g_{0i}^{(3)} - g_{0k,ki}^{(3)}\right) \tag{4.23}$$

$$R_{ij}^{(2)} = \frac{1}{2}\left(g_{ik,jk}^{(2)} + g_{jk,ik}^{(2)} - \nabla^2 g_{ij}^{(2)} - g_{kk,ij}^{(2)} + g_{00,ij}^{(2)}\right) . \tag{4.24}$$

At post-Newtonian order, the only other component of the Ricci tensor required is the 00-component to order η^4

$$R^{(4)}_{00} = 2g^{3}_{0k,0k} - \frac{1}{2}\left(\nabla g^{(2)}_{00}\right)^2 - \frac{1}{2}g^{(4)}_{00} - \ddot{g}^{(2)}_{kk} + g^{(2)}_{00,i}\left(g^{(2)}_{ik,k} - \frac{1}{2}g^{(2)}_{kk,i}\right) + \frac{1}{2}g^{(2)}_{ik}g^{(2)}_{00,ik}\,. \tag{4.25}$$

The above perturbed Ricci tensor can be compared directly to the Ricci tensor derived in our two-parameter expansion, see Section 6.1.1.

To write the field equations it is often convenient to make a gauge choice in order to eliminate superfluous degrees of freedom. Unlike in cosmological perturbation theory, the choice of gauge traditionally taken in post-Newtonian gravity, the standard post-Newtonian gauge, was done by setting the solutions to part of the field equations to zero, rather than setting metric potentials to zero (as is done in cosmological perturbation theory). Then once solving the field equations, if the gauge conditions hold, then they are valid gauge conditions. The standard post-Newtonian gauge was popularised by Chandrasekhar [66], and has been widely used by researchers in the post-Newtonian community[3]. The standard post-Newtonian gauge assumes [171]

$$\frac{1}{2}g^{(2)}_{00,i} + g^{(2)}_{ik,k} - \frac{1}{2}g^{(2)}_{kk,i} = 0 \tag{4.26}$$

$$g^{(3)}_{0i,i} - \frac{1}{2}\dot{g}^{(2)}_{kk} = 0\,. \tag{4.27}$$

One should note that the transformation into the gauge where the above equations hold, is such that the Newtonian rest-mass energy density, $\rho^{(2)}$, is invariant because it is the rest-mass energy density measured in the local Lorentz frame. Through the field equations at lowest order, that is Newtonian gravity, we can see this directly implies $g^{(2)}_{00}$ is invariant too.

If one were to calculate the field equations beyond post-Newtonian order, analogous constraints to those above would need to be defined for higher-order potentials. These gauge conditions can then be substituted into the field equations for simplification. Similarly, this gauge condition can be used to simplify the geodesic equation and conservation equations. Moreover, traditionally the introduction of

[3]Current researchers in the field, however, often prefer to use the harmonic gauge, this is where for a potential at order $\mathcal{O}(\eta^n)$ we have $g^{(n)\mu\nu}_{,\mu} = 0$ [141, 171]. This differs from traditional gauge choices in cosmological perturbation theory, see Section 3.3.3, where perturbations to the metric (rather than derivatives of perturbations to the metric) are chosen to be zero.

certain "post-Newtonian potentials" is used to simplify the field equations further and is discussed in the next section.

4.4.2. Parameterised post-Newtonian gravity

The success of post-Newtonian gravity has been in establishing parametrized-post-Newtonian gravity (PPN gravity) [170]. This enables us to parametrize the post-Newtonian limits of many alternative theories of gravity with the use of "post-Newtonian potentials" and ten parameters with fixed values which can be determined for a given metric theory of gravity. PPN gravity allows experimental gravitational physicists, from local tests of gravitation, to determine these parameters. Tests have shown these parameters are consistent with those derived from general relativity and rule out large classes of exotic theories of gravity. In the standard post-Newtonian gauge the metric in terms of the PPN potentials is given by [141, 171]

$$\begin{aligned} g_{00} &= -1 + 2U - 2\beta U^2 - 2\xi\Phi_W + (2\gamma + 2 + \alpha_3 + \zeta_1 - 2\xi)\Phi_1 \\ &\quad +2(3\gamma - 2\beta + 1 + \zeta_2 + \xi)\Phi_2 + 2(1 + \zeta_3)\Phi_3 \\ &\quad +2(3\gamma + 3\zeta_4 - 2\xi)\Phi_4 - (\zeta_1 - 2\xi)A \end{aligned} \tag{4.28}$$

$$g_{0i} = \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi)V_i - \frac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi)W_i \tag{4.29}$$

$$g_{ij} = (1 + 2\gamma U)\delta_{ij}\,, \tag{4.30}$$

where all gravitational potentials above are defined to obey specific differential equations [171], similar to the Poisson equation used in Newtonian gravity. Here, the ten PPN parameters are given by $\gamma, \beta, \xi, \alpha_1, \alpha_2, \alpha_3, \zeta_1, \zeta_2, \zeta_3$ and ζ_4. The Newtonian potential is given by $U \sim \eta^2$, from the 00-component of the metric. From the 0i-component of the field equations we have post-Newtonian potentials V_i and W_i at order η^3. Finally, at order η^4 we have post-Newtonian potentials $\Phi_W, \Phi_1, \Phi_2, \Phi_3, \Phi_4$ and A.

The PPN parameters are defined for their physical significance, such that these parameters can be determined directly from observations. For example, γ corresponds to the space-time curvature per unit rest-mass (and is measured from time delay and light deflection) and β is related to the degree of nonlinearity in the gravitational laws (and is measured from Mercury's perihelion shift and the Nordtvedt effect[4]),

[4] The Nordtvedt effect is determined from the difference in acceleration between the Earth and the Moon falling towards the Sun because of the small difference in their internal gravitational binding energy per unit mass [170].

both are used in the classical tests of gravity. The parameter ξ is non-zero for a theory of gravity which predicts preferred-location effects, such as a galaxy-induced anisotropy in the local gravitational constant.

The values of the PPN parameters allow theorists to easily identify some of the key features of a given metric theory of gravity. For general relativity these parameters are given by $\gamma = \beta = 1$ and $\xi = \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$. This means that general relativity is a fully conservative theory of gravity ($\alpha_i = \zeta_i = 0$) with no preferred-frame effects ($\alpha_i = 0$). Scalar-tensor theories also have only γ and β non-zero. Fully conservative theories can have, at most, three non-zero PPN parameters, γ, β and ξ [171].

5. Two-parameter formalism

5.1. Introduction

In the previous chapters we considered the post-Newtonian and cosmological perturbation expansions separately. These expansions give different field equations at leading-order and so describe different physics. We therefore cannot use post-Newtonian gravity to describe the CMB, nor cosmological perturbation theory to describe Newtonian systems (such as the Solar System). These expansions are normally considered separately but in reality, both types of perturbations are expected to be present in any realistic model of the Universe [140], which should contain structure from galaxies all the way through to super-horizon fluctuations. So it is necessary to understand the relativistic contributions from both small-scale non-linearities and large-scale linear structure as they are both important for future high precision observations. We therefore want to construct a two-parameter framework, which we expand around an FLRW geometry and that incorporates them both. By considering these two expansions simultaneously we aim to shed light on the link between the gravitational fields of highly non-linear virialized objects, and the large-scale properties of the Universe. We expect this interplay will become increasingly important as we move to higher orders in perturbation theory.

In this chapter we begin by defining two-parameter perturbations formally. We then summarise the two-parameter book-keeping before detailing its derivation carefully, discussing how the perturbations of the matter sector, metric and derivatives were constructed. We then justify our expansion observationally.

5.2. Formalism

5.2.1. Defining perturbations

To introduce the idea of a two-parameter expansion, let us start by considering a dimensionless function, or tensorial quantity, $\mathbf{F}(x^\mu)$, that exists in a manifold, $\mathcal{M}$. By expanding in both ϵ and η, the smallness parameters associated with our two expansions, we can write this function as

$$\mathbf{F}(x^\mu) = \sum_{n,m} \frac{1}{n'!m'!} \mathbf{F}^{(n,m)}(x^\mu) \,, \tag{5.1}$$

where $\mathbf{F}^{(n,m)}(x^\mu)$ is an order $\mathcal{O}(\epsilon^n \eta^m)$ quantity: all such perturbations exist on perturbed manifolds diffeomorphic to $\mathcal{M}$ and the background manifold, $\bar{\mathcal{M}}$. The superscripts n and m on these quantities label their order-of-smallness in ϵ and η, respectively. The quantities n' and m', on the other hand, are set by whether the term in question is leading-order in ϵ or η, or next-to-leading-order, *etc*[1]. Of course, such an expansion only converges if both ϵ and $\eta \ll 1$. In the limit where one of the parameters vanishes, ϵ or $\eta \to 0$, the expansion in Eq. (5.1) reduces to the expansion of a tensorial quantity for a single parameter, set out earlier in the introductory chapters in Eqs. (2.28) and (2.29).

Expansions of this kind have already been considered in the literature [57, 154], but have been studied in the context of studies of time-dependent perturbations of isolated stationary axisymmetric rotating stars [114, 125, 146] using a spherical background, rather than in the context of cosmology. Reference [154] explains that the advantage of a multiparameter expansion is that "it allows us to make distinctions between different types of perturbations corresponding to different parameters, so that we can study their coupling and some non-linear effects without having to compute the whole set of higher-order perturbations". We consider two parameters where η corresponds to small-scale perturbations and ϵ corresponds to horizon-size perturbations analogous to the near-zone and wave-zone, respectively, for isolated systems. Space-time derivatives act differently, *i.e.* add smallness or do not add smallness, when operated on different types of perturbations[2].

The geometry of our set-up is illustrated in Fig. 5.1. The reader should note that perturbed tensors, such as $\mathbf{F}^{(n,m)}$, are pulled-back to the background manifold, $\bar{\mathcal{M}}$, and can therefore be written in terms of the background coordinates, x^μ. This then enables us to compare perturbed tensors with unperturbed tensors, just as in single-parameter perturbation theories. Physically, $\mathbf{F}(x^\mu)$ corresponds to a quantity that is close to $\mathbf{F}^{(0,0)}(x^\mu)$ in magnitude, but is perturbed in *two* different ways. This

[1] We turn to the case where $\epsilon \to 0$, to observe that m does not necessarily equal m'. In this case we expect Eq. (5.1) to recover an expansion in η alone. Given that η corresponds to post-Newtonian perturbations, we now consider when $\mathbf{F}$ is the metric. The leading-order component of the $0i$-metric perturbation has $m = 3$, see the previous chapter. In this case $m' = 1$, so m does not necessarily equal m'.

[2] Whereas the work in [57, 154] assumes smallness is not associated derivatives acting on different types of perturbations and perturbations vary on the same characteristic length scales.

is the picture we have in mind when we perturb both the FLRW metric, and the matter fields.

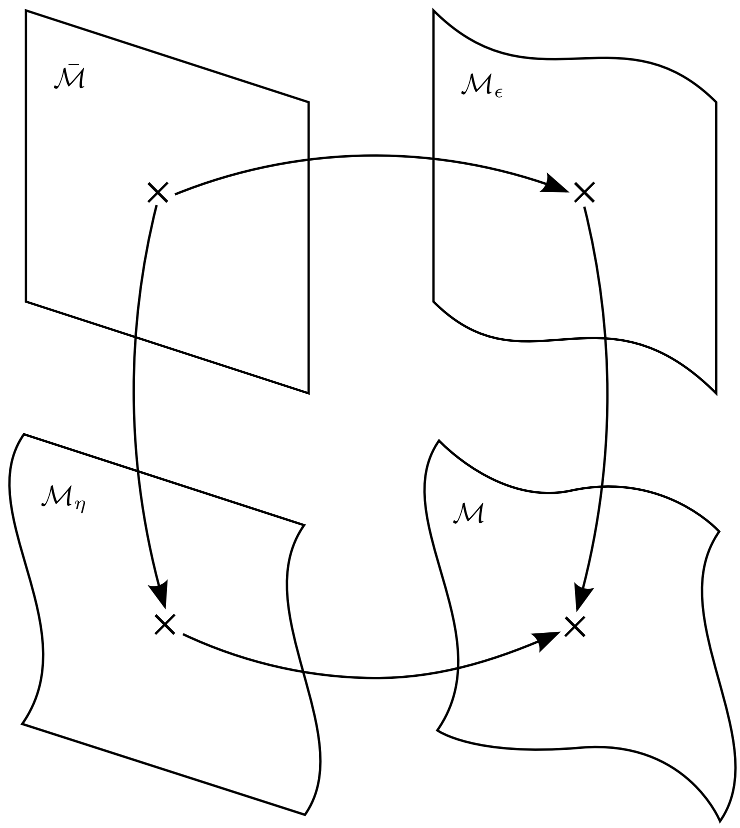

Figure 5.1.: An illustration of the maps between the background manifold $\bar{\mathcal{M}}$, and the manifold of the perturbed space-time, $\mathcal{M}$. The manifolds $\mathcal{M}_\epsilon$ and $\mathcal{M}_\eta$ correspond to perturbations in ϵ and η only. The two different routes between points on $\bar{\mathcal{M}}$ and $\mathcal{M}$ must be identical if the overall map is invertible.

As a simple illustrative example of the scenario we envisage, we could consider a one-dimensional function $\mathbf{F}(x)$ that satisfies a given differential equation. If we imagine that $\mathbf{F}(x)$ is close to being a sinusoidal wave, then we could write $\mathbf{F}^{(0,0)}(x) = \sin(2\pi x/\lambda)$. However, if $\mathbf{F}(x)$ is not exactly sinusoidal then we may want to calculate the corrections that are required in order to accurately model this function. One way of doing this would be to transform these corrections into a Fourier series, and to split the Fourier modes into those that have a wavelength shorter than λ, and those that have a wavelength greater than λ. We can then associate the smallness of the former of these fluctuations with η, and the latter with ϵ. Specifically, these perturbations vary on characteristic length scales L_N and L_C, respectively. As long as both η and ϵ are small, we can then use perturbation theory in order to determine the coefficients $\mathbf{F}^{(n,m)}$, order by order in smallness. The benefit of using two parameters in this situation is that we are able to consider scenarios in which the small-scale corrections behave differently to those that occur on large-scales, as happens in cosmology. It also allows us to investigate the way in which small-scale perturbations affect their large-scale counterparts, and *vice versa*.

Let us now return to considering cosmology, and continue by expanding both the metric and the matter fields in terms of both ϵ and η. These two parameters need not necessarily be of the same size, and, for now, we will keep our expansion general by not assuming anything about the relationship between them. This means, specifically, that we will not assume a relationship of the form $\epsilon = \epsilon(\eta)$, and we will not assume anything about the relationship between the scales L_N and L_C (later on we will restrict ourselves to particular situations of more direct physical interest, in order to write down the field equations, and perform calculations, in a sensible way).

5.2.2. Summary of book-keeping

In this section we summarise the two-parameter frameworks developed in Refs. [97, 98] that simultaneously performs a perturbation expansion in two-parameters and includes dust, radiation and a cosmological constant, in subsequent sections we justify this expansion carefully.

The first step is to expand the total energy density and pressure in both ϵ and η:

$$\rho = \rho^{(0,0)} + \rho^{(0,2)} + \rho^{(1,0)} + \rho^{(1,1)} + \rho^{(1,2)} + \tfrac{1}{2}\rho^{(0,4)} + \dots \qquad (5.2)$$

$$p = p^{(0,0)} + p^{(1,0)} + p^{(1,2)} + \tfrac{1}{2}p^{(0,4)} + \dots . \qquad (5.3)$$

The terms $\rho^{(0,0)}$ and $p^{(0,0)}$ correspond to what would normally be considered as the background energy density and pressure in cosmological perturbation theory, as they are not perturbed in either ϵ or η. All other terms correspond to perturbations at the order indicated by the superscript. To be even more precise, the orders-of-magnitude of these perturbed quantities are given by

$$\rho^{(0,0)} \sim \frac{1}{L_C^2}\,, \quad \rho^{(n,0)} \sim \frac{\epsilon^n}{L_C^2}\,, \quad \rho^{(0,m)} \sim \frac{\eta^m}{L_N^2} \quad \text{and} \quad \rho^{(n,m)} \sim \frac{\epsilon^n \eta^m}{L_N^2}\,, \qquad (5.4)$$

where $\{m, n\} \in \mathbb{N}^+$, and again L_C and L_N are the characteristic length scales of the cosmological and post-Newtonian systems, respectively. A similar expression holds for the expansion of p. The length scales are necessary in the denominators of these expressions, as ρ is a quantity with dimensions inverse length squared, and because it only makes sense to compare the magnitude of quantities with the same dimensions. Note that we have included a background energy density and pressure, this will be useful for the inclusion of radiation and the justification for such an expansion is provided in subsequent sections.

We expand the metric in both ϵ and η, which we do as follows:

$$\begin{aligned} g_{00} &= g_{00}^{(0,0)} + g_{00}^{(0,2)} + g_{00}^{(1,0)} + g_{00}^{(1,1)} + g_{00}^{(1,2)} + \tfrac{1}{2} g_{00}^{(0,4)} + \dots \\ &= -1 + h_{00}^{(0,2)} + h_{00}^{(1,0)} + h_{00}^{(1,1)} + h_{00}^{(1,2)} + \tfrac{1}{2} h_{00}^{(0,4)} + \dots \end{aligned} \tag{5.5}$$

$$\begin{aligned} g_{ij} &= g_{ij}^{(0,0)} + g_{ij}^{(0,2)} + g_{ij}^{(1,0)} + g_{ij}^{(1,1)} + g_{ij}^{(1,2)} + \tfrac{1}{2} g_{ij}^{(0,4)} + \dots \\ &= a^2 \left(\delta_{ij} + h_{ij}^{(0,2)} + h_{ij}^{(1,0)} + h_{ij}^{(1,1)} + h_{ij}^{(1,2)} + \tfrac{1}{2} h_{ij}^{(0,4)} \right) + \dots \end{aligned} \tag{5.6}$$

$$\begin{aligned} g_{0i} &= g_{0i}^{(1,0)} + g_{0i}^{(0,3)} + g_{0i}^{(1,2)} + \dots \\ &= a \left(h_{0i}^{(1,0)} + h_{0i}^{(0,3)} + h_{0i}^{(1,2)} \right) + \dots \,, \end{aligned} \tag{5.7}$$

where in the second line of each of these equations we have chosen our background metric $g_{\mu\nu}^{(0,0)}$ to be the flat[3] FLRW metric from Eq. (2.19), and simultaneously defined the perturbations $h_{\mu\nu}$. The orders of magnitude of each of the perturbations to each of the components of this metric are the minimal set required to self-consistently account for the gravitational fields of the two-parameter perturbed perfect fluid discussed above, in any arbitrary coordinate system. We can compare the above expansion of the metric to the expansion of the metric in cosmological perturbation theory, Eq. (3.4), and post-Newtonian gravity, Eqs. (4.18)-(4.20): we find the two-parameter perturbed metric includes perturbations new at mixed-orders.

The final ingredient of the field equations that must be perturbed is the peculiar velocity, v^i. This is split into post-Newtonian and cosmological parts such that

$$v^i = v^{(0,1)i} + v^{(1,0)i} + \dots \,, \tag{5.8}$$

which leads to the following components of the reference four-velocity u^μ:

$$u^0 = 1 + \frac{1}{2}\left(h_{00}^{(0,2)} + h_{00}^{(1,0)} \right) + \frac{1}{2} v^{(0,1)i} v_i^{(0,1)} + \dots \tag{5.9}$$

[3]Note it would be of interest to extend this two-parameter framework to include some positive or negative curvature. This has been undertaken in a perturbative manner in the context of post-Newtonian gravity in Ref. [149].

$$u^i = \frac{1}{a}\left(v^{(0,1)i} + v^{(1,0)i}\right) + \dots \tag{5.10}$$

$$u_0 = -1 + \frac{1}{2}\left(h_{00}^{(0,2)} + h_{00}^{(1,0)}\right) - \frac{1}{2}v^{(0,1)i}v_i^{(0,1)} + \dots \tag{5.11}$$

$$u_i = a\left(v_i^{(0,1)} + v_i^{(1,0)} + h_{0i}^{(1,0)}\right) + \dots\,, \tag{5.12}$$

derived from the constraint Eq. (2.22), the first perturbations to the metric, Eqs. (5.5)-(5.7), and the expansion of the three-velocity, Eq. (5.8). Note that we have defined the spatial components of u^μ to leading order such that $u^{(1,0)i} = a^{-1}v^{(1,0)i}$ and $u^{(0,1)i} = a^{-1}v^{(0,1)i}$, as is done in cosmological perturbation theory, see Eqs. (3.11) and (3.12). The ellipses correspond to terms $\mathcal{O}(\eta^3)$, $\mathcal{O}(\epsilon\eta)$, $\mathcal{O}(\epsilon^2)$ and smaller – none of these perturbations are necessary up to the order we wish to consider the field equations to. The components of the total two-parameter perturbed energy-momentum tensor that arises from these equations is given in Section 6.1.2 and the components of the Ricci tensor are given in Section 6.1.1.

Within the context of the two-parameter formalism, time derivatives are taken to add an extra order-of-smallness, η, compared to spatial derivatives whenever they act on an object that contains any non-zero perturbation in its post-Newtonian sector. So, for example, we take

$$\dot{\rho}^{(0,2)} \sim \eta\,|\nabla\rho^{(0,2)}| \sim \frac{\eta^3}{L_N^3} \qquad \text{and} \qquad \dot{\rho}^{(1,1)} \sim \eta\,|\nabla\rho^{(1,1)}| \sim \frac{\epsilon\eta^2}{L_N^3}\,, \tag{5.13}$$

whilst

$$\dot{\rho}^{(1,0)} \sim |\nabla\rho^{(1,0)}| \sim \frac{\epsilon}{L_C^3}\,. \tag{5.14}$$

As in Eq. (5.4), the purpose of this is to reflect the expectation that quantities perturbed in the post-Newtonian sector should be slowly varying in time and change over spatial length scales L_N, while quantities that are perturbed only in the cosmological sector should vary equally over both time and length scales L_C. Note that no higher-order perturbations are necessary to the order we wish to consider the field equations to. In this section we summarised the two-parameter framework, we now justify this book-keeping carefully in the following sections.

5.2.3. Matter perturbations

Let us start by expanding the energy-momentum tensor for non-relativistic matter, given in Eq. (2.20), in both ϵ and η using Eq. (5.1). This gives

$$\rho_M = \rho_M^{(0,2)} + \rho_M^{(1,0)} + \rho_M^{(1,1)} + \rho_M^{(1,2)} + \tfrac{1}{2}\rho_M^{(0,4)} + \dots, \tag{5.15}$$

in the single stream case (the multi-stream generalization should follow straightforwardly), where $\rho_M^{(n,0)}$, $\rho_M^{(0,m)}$ and $\rho_M^{(n,m)}$ are the cosmological, post-Newtonian and mixed perturbations of the matter energy density, respectively, and $n \geq 0$ and $m > 0$. Subscripts M differentiates the matter energy-density from the radiation energy-density (given by subscripts R). Equation (5.15) is the non-relativistic matter component of the total energy density, Eq. (5.2). The quantities $\rho_M^{(0,2)}$ and $\rho_M^{(0,4)}$ are the post-Newtonian contributions to the energy density which correspond to the energy density in the rest mass of the matter fields (the Newtonian energy density) and their internal energy density, respectively [171]. Meanwhile, $\rho_M^{(1,0)}$ is a large-scale cosmological fluctuation in the energy density, and both $\rho_M^{(1,1)}$ and $\rho_M^{(1,2)}$ are small-scale perturbations on top of large-scale fluctuations (or *vice versa*). In Fig. 5.2 some of these different contributions to the perturbed energy density are represented visually.

The reader may note that we have omitted a time-dependent background-level contribution to the matter energy density, which would otherwise have occurred as $\rho_M^{(0,0)}(t) \sim L_C^{-2}$. This is intentional, and indeed necessary, if we are to construct a sensible two-parameter expansion in both ϵ and η. The reason for this is that such a term, while being usual in single-parameter cosmological perturbation theory, would be highly unusual in post-Newtonian gravity. It would correspond to a contribution to the energy density that is much larger than the rest mass of the matter fields within the space-time, there is no discernible homogeneous fluid of non-relativistic matter with this magnitude in the real Universe and the existence of such a component corresponds to a breakdown of standard perturbation theory [143]. This is because the leading-order contribution to ρ_M is in fact dominated by the (inhomogeneous) rest mass of galaxies, dust *etc.*, which is exactly what $\rho_M^{(0,2)}(x^\mu)$ corresponds to. We therefore set $\rho_M^{(0,0)} = 0$, and find out that it is instead the spatial average of $\rho_M^{(0,2)}$ that plays the role of (what would otherwise be) the background energy density in the Friedmann equations, (3.46) and (3.47). This will be explained in more detail in Chapter 8.2.

We derived the expansion of the matter energy density, given in Eq. (5.15),

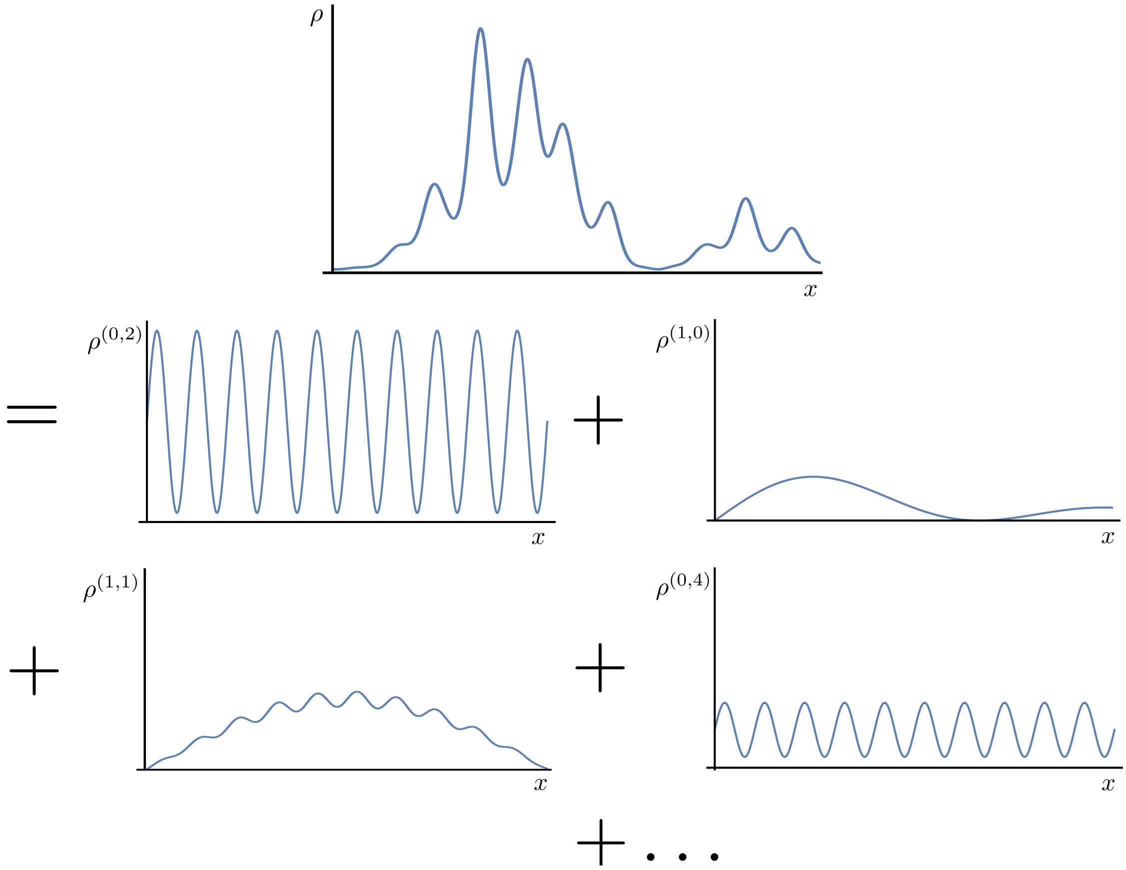

Figure 5.2.: A sketch of the different contributions to the total energy density of matter (top), where $\rho \equiv \rho_M$. These contributions include the Newtonian (middle left), first cosmological perturbation (middle right), first mixed-order perturbation (bottom left), and higher-order contributions to internal energy density (bottom right). Other, high-order contributions to the energy density are denoted by the ellipsis.

so that it contains the minimum number of perturbations necessary to describe a two-parameter system. To do this we wrote an initial ansatz for the perturbed energy density that was given by the sum of the post-Newtonian perturbed energy density, the cosmological inhomogeneous perturbed energy density and mixed-order perturbations which are products of the leading-order Newtonian and cosmological perturbations. However, after performing a gauge transformation our initial ansatz energy-momentum tensor, via the transformations given in Chapter 7, we produced a term of the form $\rho^{(0,2)}_{M,i}\xi^{(1,0)i}$ (where $\xi^{(1,0)i}$ is a part of the gauge generator – see Chapter 7), a matter source of energy density of $\mathcal{O}(\epsilon\eta L_N^{-2})$, see Eq. (7.62)[4]. This

[4]This source is of this order because we chose $L_N \sim \eta L_C$. Note that for other relationships between the two length scales there should not be a term $\rho^{(1,1)}_M$ of $\mathcal{O}(\epsilon\eta L_N^{-2})$ in the expansion of the matter energy density.

implies there must in general exist a term $\rho_M^{(1,1)}$ in the expansion of ρ_M, because even if we artificially exclude it in one coordinate system, it will be generated in another[5]. This means that mixed-order terms do not always appear at the same order as the product of post-Newtonian and cosmological terms (i.e. we have included $\rho^{(1,1)}$, even though there is no $\mathcal{O}(\eta)$ term in the post-Newtonian expansion). Our procedure gives the perturbed energy density in Eq. (5.15). This perturbed energy density after gauge transformation is consistent with the original energy density, *i.e.* there are no new potentials generated at new orders, and therefore have the minimal number of perturbations necessary to describe a two-parameter system.

The remaining contributions to the energy-momentum tensor of dust come from the isotropic pressure, p_M, and the peculiar velocity, v_M^i. These are expanded in ϵ and η such that they are the sum of the peculiar velocities and pressures used in post-Newtonian gravity and cosmological perturbation theory. No other perturbations are necessary up to the order we wish to consider. Therefore we write

$$v_M^i = v_M^{(0,1)i} + v_M^{(1,0)i} + \dots\,, \qquad (5.16)$$

and

$$p_M = p_M^{(1,0)} + p_M^{(1,2)} + \frac{1}{2} p_M^{(0,4)} + \dots\,, \qquad (5.17)$$

where the peculiar velocity, defined as the spatial part of of the four-velocity u_M^μ, corresponds to the deviation of the paths of matter fields from the background Hubble flow. If it is zero, then the matter moves only with the expansion of the Universe. If $\eta > \epsilon$ the post-Newtonian velocity $v^{(0,1)i}$ is greater than the velocity allowed by cosmological perturbation theory alone, $v^{(1,0)i}$ (this is the case for the field equations we derive in the following sections). Both Eqs. (5.16) and (5.17) are the non-relativistic matter components of the total peculiar velocity and pressure, see Eqs. (5.8) and (5.3).

There are a couple of points that the reader may want to note about these expansions. Firstly, the usual velocity in post-Newtonian gravity does not exactly correspond to the small-scale peculiar velocity $v_M^{(0,1)i}$. In fact, it is the sum of the small-scale peculiar velocity $v_M^{(0,1)i}$ *and* the Hubble flow. This is because velocities in normal post-Newtonian gravity are relative to a Minkowski background, whereas in our formalism velocities are peculiar velocities relative to an expanding FLRW

[5]This is analogous to what would happen in cosmological perturbation theory: if one were to start naively perturbing the metric in terms of ϵ, ϵ^3 and so on, and then infinitesimally gauge transform the perturbed metric, one would find that terms in the metric of order ϵ^2 are generated in general. Therefore, it is necessary to have a metric perturbation of order ϵ^2 to begin with – otherwise, this term will always be generated under a gauge transformation.

space-time. This is an important difference.

Secondly, we have not included a contribution to the pressure of the form $p_M^{(0,2)} \sim \eta^2 L_N^{-2}$ because for non-relativistic matter we require $p_M \ll \rho_M$. The term $p_M^{(0,2)}$ corresponds to a barotropic fluid with an energy density comparable to that of dark matter and baryonic matter. Such a fluid could be used to model the effects of radiation in the early Universe, because for radiation the pressure is proportional to the energy density: $\rho_R \propto p_R$ for all orders. This was done in Ref. [149], and is discussed in the following section, where we add radiation. We instead allow for some small cosmological, post-Newtonian and mixed-order pressure $p_M^{(1,0)}, p_M^{(0,4)}$ and $p_M^{(1,2)}$, respectively. This completes our discussion of the perturbations necessary for the dust-dominated stage of the Universe's evolution.

5.2.4. Including radiation and a cosmological constant

Non-relativistic matter is all that is normally considered in post-Newtonian gravity and cosmological perturbation theory applied to the dust-dominated epoch. We now wish to include radiation and a cosmological constant Λ, to our two-parameter expansion, as these are important in cosmological modelling. For radiation this can be achieved by writing the total energy-momentum tensor[6]

$$T_{\mu\nu} = T_{M\mu\nu} + T_{R\mu\nu}\,, \tag{5.18}$$

where $T_{M\mu\nu}$ and $T_{R\mu\nu}$ are the energy-momentum tensors for the non-relativistic matter and radiation, respectively, and are taken to be perfect fluids[7]. This provides a definition for the total energy density and pressure

$$\rho \;=\; \rho_M + \rho_R\,, \qquad p = p_M + p_R\,, \tag{5.19}$$

where ρ_R and p_R are the energy density and pressure of radiation. Given the peculiar velocity for radiation, v_R^i, we define the total peculiar velocity as $v^i \equiv v_I^i$, where $I \in \{M, R\}$. The total peculiar velocity can be expanded for, see Eq. (5.8), and

[6]Note that the definition of the total energy-momentum tensor does not include the cosmological constant because Λ appears in addition to $T_{\mu\nu}$ on the right hand side of the field equations (2.15).

[7]A theoretical extension of this two-parameter framework could include applications to non-perfect fluids, for example viscus fluids, and fluids with non-barotropic equations of state. However, the book-keeping of these new terms would require care, as the book-keeping of stress-energy terms are physically motivated, as seen in this chapter.

allows us to use a shorthand notation[8]:

$$X(v^i)^n \equiv \sum_{\forall I} X_I(v_I^i)^n = X_M(v_M^i)^n + X_R(v_R^i)^n\,, \tag{5.20}$$

throughout this thesis, where $X \in \{p, \rho\}$ and $n \in \mathbb{N}^+$. This shorthand notation is useful for writing the total energy-momentum tensor and field equations.

We now need to explicitly expand the energy density, pressure and peculiar velocities, in ϵ and η, which we do according to

$$\rho_R = \rho_R^{(0,0)} + \rho_R^{(1,0)} + \rho_R^{(1,2)} + \tfrac{1}{2}\rho_R^{(0,4)} + \dots \tag{5.21}$$

$$p_R = p_R^{(0,0)} + p_R^{(1,0)} + p_R^{(1,2)} + \tfrac{1}{2}p_R^{(0,4)} + \dots\,. \tag{5.22}$$

For the peculiar velocity for radiation we have

$$v_R^i = v_R^{(0,1)i} + v_R^{(1,0)i} + \dots\,. \tag{5.23}$$

These equations, (5.19)-(5.23), and the perturbations for non-relativistic matter, Eqs. (5.15)-(5.17), can be compared to Eqs. (5.2), (5.3) and (5.8) to read off values for the perturbations to the total energy density, pressure and peculiar velocity. Note that we include the mixed-order term $p_M^{(1,2)}$ so the expansion of p_M is comparible to the radiation fluid's pressure[9]. We have expanded the radiation contribution to the peculiar velocity in the same way for radiation as matter. We will discuss, in detail, the above expansions of the energy density and pressure for radiation in what follows.

The expansion of the matter and radiation contributions to the energy density and pressure have been performed in the same way. So all energy-momentum quantities relating to radiation have the minimum number of perturbations necessary to describe a two-parameter system such that under an infinitesimal gauge transformation, no terms of new orders are generated. Nevertheless, the expansions of the matter and radiation contributions to the energy density are not equivalent: we have omitted (i) a time-dependent background-level contribution to the matter energy density and pressure, and (ii) a Newtonian-level contribution to the radiation energy density and pressure, the former is justified in Section 5.2.3. The latter is

[8] So for $n \in \mathbb{N}^+$, $X(v^i)^n \neq (X_M + X_R)(v^i)^n$, and $n = 0$ recovers $X = X_M + X_R$, Eq. (5.19).

[9] We also include a factor of 1/2 in front of $p_M^{(0,4)}$, this is a notational change from [97], but here is necessary to make p_M directly comparable to the expansion of the radiation pressure.

such that

$$\rho_R^{(0,2)} = p_R^{(0,2)} = 0\,.$$

Radiation pressure and energy-density perturbations all occur at the same order because, for radiation, the pressure is proportional to the energy density: $\rho_R \propto p_R$ for all orders, see Section 2.4.2. So given that there is a low-order contribution to the energy density $\rho_R^{(0,0)}$ this implies there also exists $p_R^{(0,0)}$ in the expansion of pressure; this is different to the expansion of p_M where $p_M \ll p_R$, see Section 2.4.2.

Let us now consider the expansion of ρ_R and p_R given in Eqs. (5.21) and (5.22). For this purpose we consider the conservation of the total energy-momentum tensor $T_{\mu\nu}$, defined in Eq. (5.18), such that:

$$\nabla^\mu T_{\mu\nu} = \nabla^\mu (T_{M\mu\nu} + T_{R\mu\nu}) = 0\,, \tag{5.24}$$

where $T_{M\mu\nu}$ and $T_{R\mu\nu}$ are the matter and radiation contributions to the total energy-momentum tensor, respectively. This implies $\nabla^\mu T_{M\mu\nu} = Q_\nu$ and $\nabla^\mu T_{R\mu\nu} = -Q_\nu$, where $Q_\nu \neq 0$ for interacting fluids and $Q_\nu = 0$ for non-interacting fluids. In either case, the lowest-order part of Eq. (5.24) is given by

$$\nabla p_R^{(0,0)} \;\;=\;\; 0\,, \tag{5.25}$$

from Eqs. (5.2), (5.3) and (5.8). Equation (5.25) corresponds to the lowest-order part of the Euler equation for a barotropic fluid and implies that the lowest-order pressure is time dependent only, $p_R^{(0,0)}(t)$. If we now take $p_R = \frac{1}{3}\rho_R$, then this result implies that the leading-order part of the energy density in radiation must also be spatially homogeneous, such that $\rho_R^{(0,0)} = \rho_R^{(0,0)}(t)$. This is, in fact, exactly what is required for a background-level contribution to the energy density in an FLRW model[10]. We therefore find that the lowest order at which inhomogeneous perturbations to radiation exist is at order $\mathcal{O}(p_R^{(1,0)}) \sim \mathcal{O}(\epsilon L_C^{-2})$ in our two-parameter expansion, which corresponds to a cosmological-scale perturbation.

A similar argument can now be used to understand why it would be inappropriate to include a term $\rho_R^{(0,2)}$ in Eq. (5.21). Such a term would imply the existence of $p_R^{(0,2)}$ which, again through the conservation equations, can be shown to be necessarily spatially homogeneous. Such a term would therefore be functionally degenerate

[10]A similar result is found in Ref. [149]. They found this results from the lowest-order post-Newtonian radiation contribution to the pressure (analogous to a term $p_R^{(0,2)}(t)$ in our two-parameter expansion), whereas our result corresponds to the large-scale cosmological radiation pressure – something that only exists when considering cosmological perturbation theory or a two-parameter perturbed universe.

with $\rho_R^{(0,0)}$. As they are both functions of time only they would show up in every conceivable set of equations in exactly the same way. We can therefore neglect both $\rho_R^{(0,2)}$ and $p_R^{(0,2)}$ without any loss of generality. Moreover, the term $\rho_R^{(0,2)}(t)$ would be Newtonian in size, and such a term would be highly unusual in normal post-Newtonian gravity which is normally associated with small-scale fluctuations whereas radiation is associated with horizon-size fluctuations in the early Universe. Radiation therefore fits naturally into our two-parameter expansion at lowest-order as a cosmologically perturbed quantity.

The reader may also note that there is no term $\rho_R^{(1,1)}$ in Eq. (5.21), whereas there is a term $\rho_M^{(1,1)}$ in Eq. (5.15). This is because if we omit $\rho_M^{(1,1)}$ then a term of that order is always generated under an infinitesimal coordinate transformation, therefore we include it for generality. However, a similar argument does not apply to $\rho_R^{(1,1)}$, because the gauge transformation of $\rho_R^{(0,0)}$ does not generate any terms of the same order as $\rho_R^{(1,1)}$. This can be seen to be true because $\rho_R^{(0,0)}$ is a function of time only, such that $\rho_{R,i}^{(0,0)}\xi^{(1,0)i} = 0$. The same argument would apply to the term $\rho_R^{(0,2)}$, if it had been included, as this term would also be time dependent. This means that we can set $\rho_R^{(1,1)} = p_R^{(1,1)} = 0$ in any coordinate system, and the same result will hold in any other coordinate system related by an infinitesimal gauge transformation.

Finally, let us consider the cosmological constant Λ. We assign an order of magnitude and dimensions to the cosmological constant in the following way:

$$\Lambda = \Lambda^{(0,0)} \sim \frac{1}{L_C^2} \,. \tag{5.26}$$

This choice is motivated by the fact that the cosmological constant in the standard model of cosmology must be of background order in order for it to be influential in the Friedmann equations at late times. There is also no point in perturbing it in either ϵ or η, as it is a constant, the Taylor expansion is trivial. The cosmological constant therefore fits naturally into our two-parameter expansion at lowest-order as a cosmological background quantity of magnitude L_C^{-2}, because it is a horizon-scale phenomenon affecting scales which correspond to lengths of order L_C. A very nice feature of our book-keeping is that the contribution of the cosmological constant in the field equations is something implied directly from our book-keeping (specifically, the relationship between the two length scales L_N and L_C), it is not something put in by hand. Normally, for non-linear structure on large-scales L_N we have to set $\Lambda \sim \eta^2 L_N^{-2}$ [149] so Λ appears in the leading-order field equations, and on small scales L_N we have to set $\Lambda = 0$ so Λ does not contribute to the leading-order field equations. With our two-parameter expansion, on the other hand, the cosmological constant has dimensions $\Lambda \sim L_C^{-2}$ whereas for non-linear structure on the smallest scales L_N, like in the Solar System, we automatically find $\eta^2 L_N^{-2} \gg L_C^{-2}$, so Λ does not appear in the lowest-order field equations. Likewise, for non-linear structure on the largest scales L_N our two parameter expansion automatically implies Λ appears in the lowest order field equations because $\eta^2 L_N^{-2} \sim L_C^{-2}$. This is a product of our two-parameter expansion which allows for small-scale and large-scale phenomena on lengths L_N and L_C, respectively.

5.2.5. Derivatives

Let us now consider what happens when derivatives act on the perturbed quantities defined above. We start with the assumption that the rate at which an object changes in space and time can be determined from its order of smallness in ϵ and η. If an object is perturbed in η only, we will say that it is post-Newtonian. We denote all such objects by N, so that $N \sim \eta^m$. Similarly, all objects perturbed in ϵ only will be called cosmological, and are denoted by $C \sim \epsilon^n$. The remaining objects, perturbed in both ϵ and η, will be called mixed, and are denoted by $M \sim \epsilon^n \eta^m$.

Following the discussion in Chapter 4, we will assume that derivatives act on all Newtonian quantities such that

$$N_{,i} \sim \frac{N}{L_N} \quad \text{and} \quad \dot{N} \sim \frac{\eta N}{L_N}\,. \tag{5.27}$$

Similarly, following the discussion in Chapter 3, we take derivatives to act on all cosmological quantities (and background quantities, such as the scale factor $a(t)$) such that

$$C_{,i} \sim \frac{C}{L_C} \quad \text{and} \quad \dot{C} \sim \frac{C}{L_C}\,. \tag{5.28}$$

It now remains to decide the order of smallness of the derivatives of mixed terms. This is more complicated.

We start our consideration of the derivatives of mixed terms by noting that they vary in space and time on both Newtonian and cosmological length scales, as illustrated in Fig. 5.2. In order to determine which of these contributions dominate the derivative on a mixed-order quantity we need to relate L_N and L_C. In order to do this it is useful to define a new quantity, l, such that

$$l \equiv \frac{L_N}{L_C}\,, \tag{5.29}$$

this enables us to compare the sizes of derivatives on different types of potentials.

Also, we observe that we want to consider post-Newtonian perturbed structure, on scales L_N, such that the post-Newtonian expansion (around Minkowski space) still holds. For this to be true we need the velocity due to the Hubble flow, HL_N, to be smaller than or equal to the peculiar velocities of the constituent objects, η, hence $HL_N \leqslant \eta$. Otherwise, such systems would have velocities larger than η with respect to a Minkowski background, and so post-Newtonian gravity would break down. Given that $H \sim L_C^{-1}$, and using the definition from Eq. (5.29), we then have the requirement that

$$l \leqslant \eta\,, \tag{5.30}$$

which implies that two parameter perturbed systems which saturate this limit are such that $l \sim \eta$, and implies $L_N \sim \eta L_C$, which corresponds to non-linear structure up to the homogeneity scale of order 100Mpc and linear structure beyond that. Furthermore, Eq. (5.30) implies two things: (i) spatial derivatives acting on cosmological terms are strictly smaller than spatial derivatives acting on Newtonian terms, and (ii) time derivatives acting on cosmological terms are strictly less than or equal to time derivative acting on Newtonian terms. Therefore, post-Newtonian spatial and temporal derivatives dominate over or are equal to cosmological ones. Hence we can write

$$M_{,i} \sim \frac{M}{L_N} \quad \text{and} \quad \dot{M} \sim \frac{\eta M}{L_N}\,, \tag{5.31}$$

because, at most, derivatives of mix-ordered terms go like derivatives of post-Newtonian perturbed quantities.

At this point we can make two more comments related to Eqs. (5.29) and (5.30). The first arises because we can write

$$\rho^{(1,0)} \sim \frac{\epsilon}{L_C^2} \sim \frac{\epsilon l^2}{L_N^2}\,, \tag{5.32}$$

from Eqs. (5.4) and (5.29). Equation (5.32), together with Eq. (5.30), means that $\rho^{(1,0)} \ll \rho^{(0,2)}$. In other words, the total energy density is *always* such that the small-scale rest mass dominates over the large-scale "cosmological" fluctuations to the energy density, independent of the relative magnitude of the gravitational potentials on small and large scales (*i.e.* independent of the relationship between ϵ and η). This will be important when it comes to writing the field equations order by order in Chapter 6.

The second point is that the above book-keeping of derivatives on Newtonian, cosmological and mixed-order terms can be considered in units of either L_N or L_C. If we consider the field equations in units of L_N then we relegate certain terms to higher orders, by adding orders of smallness in η and l. If we consider the field equations in units of L_C we move terms to lower orders, by adding largeness via η^{-1} and l^{-1}. The former book-keeping is often referred to as a slow-motion expansion, whereas the latter is known as a gradient (or large gradient) expansion. Either is perfectly acceptable, as they provide the same resulting field equations, but we choose to employ the former. This is because it is easier to omit terms which become higher order under a derivative, rather than to go through all possible higher-order terms in order to see which terms might be larger under a derivative.

5.2.6. Metric

To complete the description, let us now expand the metric in both ϵ and η. As stated previously, we assume a background geometry, $g^{(0,0)}_{\mu\nu}$, which is taken to be the flat FLRW space-time, see Eq. (2.19), with $k = 0$. Such a background is quite standard for cosmological perturbation theory, but little used for post-Newtonian gravity (see however Refs. [15, 130]). Nevertheless, it is entirely compatible with the discussion in Section 4 [73, 148], which we kept general, *i.e.* time dependent in order to allow for this possibility. Our two parameter perturbed metric is given in Section 5.2.2, see Eqs. (5.5)-(5.7). The orders of magnitude of each of the components of this metric are derived using the method outlined in Chapter 4 for post-Newtonian gravity. That is, they are derived from the orders-of-smallness of each of the components of the total energy-momentum tensor $T_{\mu\nu}$, cosmological constant, and the orders-of-smallness and dimensions of space-time derivatives, along with the field equations. We also require the relationship between the two length scales L_N and L_C (we consider the case where $l \sim \eta$, discussed Section 5.3). Importantly, the perturbed metric, Eqs. (5.5)-(5.7), is the same for radiation, dust and a cosmological constant, as for dust alone [97][11]. In other words, the perturbed metric does not require the introduction of any new metric potentials at any new orders with the inclusion of radiation or a cosmological constant, compared to dust alone [98].

Alternatively, we also derived the two-parameter expansion of the metric in the same way as the energy density, discussed earlier in Section 5.2.3. *i.e.* the metric contains the minimum number of perturbations necessary to describe a two-parameter system. We write an initial ansatz for the perturbed metric given by the sum of the FLRW metric, the usual post-Newtonian metric, the cosmologically perturbed

[11] This also implies that the perturbed Ricci tensor, presented fully in Section 6.1.1, is the same with the inclusion of not only matter but also radiation and a cosmological constant [98] as the perturbed Ricci tensor for matter only [97].

metric and mixed-order perturbations which are products of the leading order Newtonian and cosmological perturbations. However, after a gauge transformation (see Chapter 7) we produced metric potentials in the 00, $0i$ and ij parts of the metric at $\mathcal{O}(\epsilon\eta)$, $\mathcal{O}(\epsilon\eta^2)$ and $\mathcal{O}(\epsilon\eta)$, from Eqs. (7.6), (7.16) and (7.13), respectively (this was under the choice $l \sim \eta$)[12]. Therefore, we include metric potentials of order $g_{00}^{(1,1)}$, $g_{ij}^{(1,1)}$ and $g_{0i}^{(1,2)}$ in our new ansatz, giving the perturbed metric, Eqs. (5.5)-(5.7). Now, the new perturbed metric after gauge transformation is consistent with the original metric, and therefore has the minimal number of perturbations necessary to describe a two-parameter system.

Note that the full expressions for the perturbed total energy-momentum and Ricci tensors, along with the two-parameter perturbed field equations, are given in the next chapter. Next, we justify this two-parameter expansion observationally.

5.3. Observational justifications

Here we address what is the observational justification of using a two-parameter expansion in cosmology. After all, this is the key motivation for constructing such an expansion. In the previous chapters we considered the different ways that perturbation theory can be applied to gravitational fields on both horizon-sized and sub-horizon-sized regions of space-time. This resulted in a derivation of both the post-Newtonian and cosmological perturbation theories, using little more than the fact that Einstein's equations can be written as null wave equations. We then considered how these two different expansions could be formally combined into a two-parameter expansion that could be used to describe the Universe on both large and small scales. Throughout all of this we tried to keep the discussion as general as possible, without specifying any specific relationship between either the expansion parameters ϵ and η, or the length scales L_C and L_N.

We now consider observations of the specific astrophysical systems that exist on different scales in the Universe. The aim of this is to see which types of systems are best described by post-Newtonian expansions, and which are best described using cosmological perturbation theory. This allows us to consider the physical scenarios that could potentially be described using our two-parameter expansion, as well as the particular values of ϵ and η that are appropriate in each case. Of course, each pair of systems also comes with its own values of L_C and L_N, which can also be

[12] Note that for other relationships between the two length scales L_N and L_C there should not be terms $g_{00}^{(1,1)}$ and $g_{ij}^{(1,1)}$ at order $\mathcal{O}(\epsilon\eta)$. However, for all relationships between L_N and L_C there would exist a metric potential at order $g_{0i}^{(1,2)}$, after gauge transformation.

related to the expansion parameters. This is necessary in order to write down the field equations of our two-parameter expansion perturbatively, order-by-order in perturbations. On the scales relevant to large-scale structure in cosmology (of order a fraction of the horizon-size) we find the conditions that $\epsilon \sim \eta^2$ and $L_N \sim \eta L_C$. This is then used in Chapter 6 to write a hierarchical set of field equations.

5.3.1. Post-Newtonian gravity

The post-Newtonian expansion is usually applied to describe the gravitational physics of astrophysical bodies that range in size: from binary pulsar systems (about a million kilometres), to the size of the orbits of the planets in our solar system (a few hundred million kilometres). Let us begin by considering these systems, before moving on to the larger astrophysical systems that are of more interest for cosmology. To do this, we will quote estimates for the largest velocities that occur within them, and compare these to estimates of the largest gravitational potentials that we can find. We do this using the order-of-magnitude estimator

$$U = \frac{GM_N}{c^2 L_N}\,, \tag{5.33}$$

where M_N and L_N are observational estimates of the mass and length scale of the system, and are in units of kilograms and meters, respectively.This implies here, in this section, we require dimensional constants $G, c \neq 1$. Equation (5.33) will allow us to estimate η, as well as establish whether or not a given system is indeed suitably described using a post-Newtonian perturbative expansion. The results are summarized in Table 5.1.

The largest velocities in the Solar System correspond to coronal mass ejections, which can erupt at up to $450\text{km}\,\text{s}^{-1}$ (see p. 375 of Ref. [77]). This corresponds to $v \sim 10^{-3}$, in units where $c = 1$. As well as this, the mass of the Sun is about $M_\odot \sim 2\times10^{30}$kg, and its radius is approximately $L_N \sim L_\odot \sim 7\times10^8$m. This means that Eq. (5.33) implies $U \sim 10^{-6}$. This means that the post-Newtonian expansion is indeed applicable, because $v^2 \sim U$, as expected from Eq. (4.13). It also means that the value of the expansion parameter in this system is given by $\eta \sim 10^{-3}$, this can be seen from Eq. (4.21) and is how η is estimated in what follows.

There are a number of systems that one could consider above the scale of the Sun, but to speed the discussion let us move directly up to the scale of spiral galaxies. These systems are typically made up of billions of stars, and typically have a bulge, a disk, and a dark matter halo. The observed velocities of stars can be as high

System	v	L_N/Mpc	$M_N/M_\odot$	U
Sun	10^{-3}	2×10^{-14}	1	10^{-6}
Galaxy	10^{-3}	10^{-2}	10^{12}	10^{-6}
Group	10^{-3}	0.8	10^{13}	10^{-6}
Cluster	$10^{-2.5}$	2	10^{15}	10^{-5}
Supercluster	$10^{-2.5}$	100	10^{16}	10^{-5}

Table 5.1.: Summary of the magnitude of v and U in a variety of gravitationally bound systems, covering a wide range of scales.

as 300km s^{-1} (see p. 571, 578 & 580 of Ref. [77]). This again corresponds to $v \sim 10^{-3}$. If we consider a bulge of radius $L_N \sim 10$kpc, and mass $M_N \sim 10^{11} M_\odot$, then this gives $U \sim 10^{-6}$. We again have $v^2 \sim U$, meaning that a post-Newtonian perturbative expansion seems appropriate to describe the gravitational field, and we again have $\eta \sim 10^{-3}$.

Typical galaxy groups contain 3-30 galaxies that are gravitationally bound, and it is estimated that $\sim$ 55% of galaxies exist within groups. The maximum radial dispersion in groups of galaxies is observed to be about 500km s^{-1} (see p. 614 of Ref. [77]), again implying $v \sim 10^{-3}$. We estimate the mass of a typical group, including dark matter, is $M_N \sim 10^{13} M_\odot$, and that the radius of a typical group is $L_N \sim 0.8$Mpc (this is an average of the range given in p. 614 [77]). This implies that $U \sim 10^{-6}$ in galaxy groups, and that the post-Newtonian perturbative expansion seems to apply here as well. We even have $\eta \sim 10^{-3}$, as above.

Moving up in scale still further, we have clusters of galaxies. Typical galaxy clusters contain 30-300 gravitationally bound galaxies. The dispersion velocities of galaxies within clusters can be as large as 1400km s^{-1}, or $v \sim 10^{-2.5}$ in units where $c = 1$. We take the mass of a typical cluster to be about $M_N \sim 10^{15} M_\odot$, and the average radius to be around $L_N \sim 2$Mpc (averages of quantities given on p. 614 of Ref. [77]). Similarly we average to find the typical radius of a cluster which is around $L_N \sim L_{\text{cluster}} \sim 2$Mpc. The maximum gravitational potentials expected in clusters are therefore $U \sim 10^{-5}$. We again have $v^2 \sim U$, but now with $\eta \sim 10^{-2.5}$.

Super-clusters are the largest virialized objects we currently observe in the Universe. They make up the filaments and walls that form the cosmic web, and are made from clusters, groups and other smaller gravitationally bound systems. Observations show that peculiar velocities within of our own local supercluster are around 1000km s^{-1} [68, 160], which corresponds to $v \sim 10^{-2.5}$. There are typically 2-15 clusters per supercluster, which implies the mass of a supercluster is at least

$10^{16}M_{\odot}$ (see p. 635 of Ref. [77]). They have typical scales of $L_N \sim 100$Mpc. This gives $U \sim 10^{-5}$. Even on these extraordinarily large scales, we have $v^2 \sim U$ and $\eta \sim 10^{-2.5}$.

It is interesting to note the maximum amplitude of the gravitational potential is roughly $\sim 10^{-5}$ for all of the systems considered above. This ranges over just about all astrophysical objects, from the Sun to our local supercluster. We therefore have an expansion parameter $\eta \sim 10^{-3}$ for all of these systems. The similarity in the size of the gravitational potential, no matter what system is being considered, indicates that the mass of the system under consideration increases approximately in proportion to its length scale. This type of self-similarity will break down whenever a system's mass is much larger than about 10^{-5} of its length scale, at which point we expect the post-Newtonian expansion should start to break down. This happens, for example, in the case of neutron stars.

Although post-Newtonian perturbation theory appears to be applicable to superclusters, we do not expect it to be valid on scales that are much larger. This is because the square of the velocity due to the Hubble flow starts to become comparable to the order of the Newtonian potentials, *i.e.* $H^2 L_N^2 \sim 10^{-5}$. Going to even larger scales would therefore mean that the square of the Hubble flow velocity would start to exceed the magnitude of the gravitational potentials. If this is the case then post-Newtonian expansions are no longer applicable, refer to the discussion leading to the limit in Eq. (5.30), and cosmological perturbation theory must be used. It is expected that the next generation of surveys, such as Euclid, LSST and SKA, will start to probe this new regime [1–3].

5.3.2. Cosmological perturbation theory

Let us now consider the largest of all scales in the observable Universe; those comparable to the size of the horizon. In terms of the CMB, this corresponds to about one degree at decoupling [83]. In the late Universe this distance translates to scales of around 30Gpc. In this case we expect the cosmological perturbation theory expansion outlined in Chapter 3 to be applicable. The principle distinction between the size of the perturbed quantities in this expansion, when compared to the post-Newtonian expansion, is that time derivatives do not add any extra orders of smallness. This means that velocity cannot be used as an expansion parameter. The separation of objects is instead dominated by the Hubble flow, with only small peculiar velocities (of the order of gravitational potentials) being allowed in addition.

The discussion of superclusters, in the previous section, should already have made it clear that cosmological perturbation theory is not the appropriate framework for discussing the dynamics of astrophysical systems that exist below $\sim$ 100Mpc. This is essentially because the time variation of both gravitational and matter fields are slow compared to their variation in space, meaning that $U \sim v^2$. On larger scales, however, we expect to find $U \sim v$. There do not currently exist any galaxy surveys that probe these scales directly, but we can use the CMB to justify the application of cosmological perturbation theory on horizon-sized length scales and above.

The temperature fluctuations in the CMB, after the dipole has been subtracted, are all at the level of about 10^{-5} [41]. The main contribution to these fluctuations, on large scales, is expected to come from the Sachs-Wolfe effect [83]. This is essentially a redshifting of the CMB radiation as it escapes the gravitational potentials that existed at the surface of last scattering, and the redshift is related to the temperature in a well-known way [131]. We therefore expect

$$\frac{\delta T}{T} \sim U\,, \tag{5.34}$$

where U should be understood as a typical gravitational potential at last scattering. The observations of the temperature fluctuations at the level of one part in 10^5 therefore very directly imply that gravitational potentials at last scattering were of the size $U \sim 10^{-5}$.

If we now consider the polarization of the CMB, then we can gain information about the magnitude of peculiar velocities at last scattering. This is because polarization of the CMB radiation, $\mathcal{E}$, is primarily due to quadrupole anisotropy in the velocity field of the plasma at last scattering [144]. We expect the mean-free path of photons at last scattering to be of the order of the inverse Hubble rate (so that $1/n_e\sigma_t \sim L_C$, where n_e is the number density of electrons, and σ_t is the Thomson cross section). The polarisation is therefore given by

$$\mathcal{E} \sim \Delta v\,, \tag{5.35}$$

where Δv is the difference in peculiar velocity of matter, in orthogonal directions on the sky (for details see Ref. [144]). Observations of CMB polarization now measure $\mathcal{E} \sim 10^{-6}$ [116], which means that peculiar velocities at last scattering are order $v \sim 10^{-6}$.

Taken together, these observations therefore suggest that $v \sim U$ on horizon-sized scales, as expected. These results clearly indicate that a post-Newtonian expansion is *not* the appropriate framework to be describing gravity on these scales, and that cosmological perturbation theory should be used instead. What is more, it can be seen that the expansion parameter for cosmological perturbation theory should be of magnitude $\epsilon \sim 10^{-5}$. Although it has not yet been directly observed, we very strongly expect similar results to hold at and above $\sim$ 1Gpc in the late Universe.

5.3.3. A realistic universe

In previous chapters we have outlined the key features of both cosmological perturbation theory and post-Newtonian gravity, they provide formalisms with different equations at leading-order (and subsequent order) and so describe different physics. Exactly which physical systems are best described by which formalism is the subject of the preceding sections. We found that planetary systems, galaxies, groups, clusters and superclusters are all well described by post-Newtonian gravity. That is, their observed velocities and inferred gravitational potentials satisfy $v^2 \sim U \sim 10^{-5}$. Additionally, we find that observed fluctuations on the scale of the horizon are well described by cosmological perturbation theory, as $v \sim U \sim 10^{-5}$. Moreover, time derivatives are order v smaller than spatial derivatives for post-Newtonian gravity, this is not the case for cosmological perturbation theory. This very strongly indicates that post-Newtonian gravity cannot be used to describe structure on the scale of the horizon, and that cosmological perturbation theory cannot be used to describe non-linear structure on the scale of 100Mpc or less.

In order to model a realistic Universe, that has non-linear structure on small scales, as well as linear structure on large scales, we therefore need to expand in both ϵ and η. This is exactly the type of two-parameter expansion that we formulate in this thesis. In what follows, we will take $\epsilon \sim \eta^2 \sim 10^{-5}$, this excludes very dense compact objects, but fits almost all large astrophysical structures that exist in the Universe (see Table 5.1), and that we wish to describe with our formalism.

We will also consider cosmologically perturbed structure on scales of order the horizon $L_C \sim$ 30Gpc (the horizon size at present time) down to the homogeneity scale 100Mpc. These are the scales on which cosmological perturbation theory can be applied. We also consider post-Newtonian structure on scales of order[13] $L_N \sim$ 100Mpc down to 100kpc, the scales on which structures such as superclusters down to clusters and groups exist. These length scales imply $l \sim \eta$, this corresponds to the saturation of the bound in Eq. (5.30), and describes a two-parameter system for large-scale structure. In fact the restriction $l \sim \eta$ implies that the field equations

[13]This length scale roughly corresponds to that of the largest gravitationally bound objects that have so far been observed to exist in the Universe [68].

derived are valid for all systems where the post-Newtonian structure varies on length scales 100-1000 times smaller than the cosmologically perturbed structure.

For the system where $\epsilon \sim \eta^2 \sim 10^{-5}$ and $L_N \sim \eta L_C$, in what follows, we will write the field equations order by order in a two-parameter expansion.

6. Einstein's field equations with a two-parameter expansion

Here we provide the energy-momentum and Ricci tensors derived using our two-parameter expansion. They are derived generally, for no specified physical system. So the relationships between ϵ and η, and length scales L_N and L_C, have not been specified. We provide these quantities for the derivation of the field equations, but also for use in future applications. The field equations provided are for a specific cosmological scenario, valid on a fraction of the horizon size, where $\epsilon \sim \eta^2$ and $L_N \sim \eta L_C$. Throughout this chapter we do not fix a gauge, to allow for the most general expressions. We hope this will be useful, in particular, for future studies requiring different gauges other than the Newtonian gauge, which we discuss in the following chapter. Finally, the tensor algebra packages xAct and xPand [4, 5, 139], were used to derive some of the equations presented in this work.

6.1. Ricci and total energy-momentum tensors

6.1.1. Ricci tensor

We now provide detailed expressions for the perturbed Ricci tensor and the perturbed energy-momentum tensor. We make no assumptions about the relative magnitude of ϵ and η in this section, nor do we assume anything about the relationships between length scales L_C and L_N. We begin by expanding the components of the Ricci tensor in our two parameters. We find that the non-vanishing contributions to each component are given by the following equations:

$$R_{00} = R_{00}^{(0,0)} + R_{00}^{(0,2)} + R_{00}^{(0,3)} + \frac{1}{2}R_{00}^{(0,4)} + R_{00}^{(1,0)} + R_{00}^{(1,1)} + R_{00}^{(1,2)} + \dots \quad (6.1)$$

$$R_{0i} = R_{0i}^{(0,2)} + R_{0i}^{(0,3)} + R_{0i}^{(1,0)} + R_{0i}^{(1,2)} + \dots \quad (6.2)$$

$$R_{ij} = R_{ij}^{(0,0)} + R_{ij}^{(0,2)} + R_{ij}^{(0,3)} + \frac{1}{2}R_{ij}^{(0,4)} + R_{ij}^{(1,0)} + R_{ij}^{(1,1)} + R_{ij}^{(1,2)} + \dots, \quad (6.3)$$

where ellipses denote higher-order terms which we will not require in this thesis.

Any term in each of these equations has an order of smallness in ϵ and η, as indicated by the superscript in brackets. They also have a length scale associated with them, given by L_N^{-2}, L_C^{-2} or $L_C^{-1}L_N^{-1}$ as the Ricci tensor contains two derivatives of the dimensionless metric. We have not indicated this directly on each of the terms in the expansion above, but it is important when using the perturbed Ricci tensor to determine the field equations presented later in this chapter. We will therefore be careful to keep track of these length scales in the expressions that follow.

The terms on the right-hand side of Eq. (6.1) are given explicitly by

$$R_{00}^{(0,0)} = -3\frac{\ddot{a}}{a} \sim \frac{1}{L_C^2} \tag{6.4}$$

$$R_{00}^{(0,2)} = -\frac{1}{2a^2}h_{00,ii}^{(0,2)} \sim \frac{\eta^2}{L_N^2} \tag{6.5}$$

$$R_{00}^{(0,3)} = \frac{\dot{a}}{a^2}h_{0i,i}^{(0,3)} - \frac{\dot{a}}{a}h_{ii,0}^{(0,2)} - \frac{3\dot{a}}{2a}h_{00,0}^{(0,2)} \sim \frac{\eta^3}{L_C L_N} \tag{6.6}$$

$$\begin{aligned} R_{00}^{(0,4)} &= -\frac{1}{2a^2}\left(h_{00,i}^{(0,2)}\right)^2 - \frac{1}{2a^2}h_{00,ii}^{(0,4)} - h_{ii,00}^{(0,2)} + \frac{2}{a}h_{0i,0i}^{(0,3)} + \frac{1}{a^2}h_{00,ij}^{(0,2)}h_{ij}^{(0,2)} \\ &\quad + \frac{1}{2a^2}h_{00,i}^{(0,2)}\left(2h_{ij,j}^{(0,2)} - h_{jj,i}^{(0,2)}\right) \\ &\sim \frac{\eta^4}{L_N^2} \end{aligned} \tag{6.7}$$

$$R_{00}^{(1,0)} = -\frac{1}{2a^2}h_{00,ii}^{(1,0)} - \frac{1}{2}h_{ii,00}^{(1,0)} + \frac{\dot{a}}{a^2}h_{0i,i}^{(1,0)} - \frac{\dot{a}}{a}h_{ii,0}^{(1,0)} + \frac{1}{a}h_{0i,0i}^{(1,0)} - \frac{3\dot{a}}{2a}h_{00,0}^{(1,0)} \sim \frac{\epsilon}{L_C^2} \tag{6.8}$$

$$R_{00}^{(1,1)} = -\frac{1}{2a^2}h_{00,ii}^{(1,1)} \sim \frac{\epsilon\eta}{L_N^2} \tag{6.9}$$

$$\begin{aligned} R_{00}^{(1,2)} &= -\frac{1}{2a^2}h_{00,ii}^{(1,2)} + \frac{1}{2a^2}h_{00,ij}^{(0,2)}h_{ij}^{(1,0)} + \text{terms of size}\left[\frac{\epsilon\eta^2}{L_N L_C}\right] \\ &\sim \frac{\epsilon\eta^2}{L_N^2} + \frac{\epsilon\eta^2}{L_N L_C}. \end{aligned} \tag{6.10}$$

The terms in Eq. (6.2) are given by

$$R_{0i}^{(0,2)} = -\frac{\dot{a}}{a}h_{00,i}^{(0,2)} \sim \frac{\eta^2}{L_C L_N} \tag{6.11}$$

$$R_{0i}^{(0,3)} = \frac{1}{2a}\left(h_{0j,ij}^{(0,3)} - h_{0i,jj}^{(0,3)} + ah_{ij,0j}^{(0,2)} - ah_{jj,0i}^{(0,2)}\right) + \text{terms of size}\left[\frac{\epsilon\eta^3}{L_C^2}\right] \tag{6.12}$$

$$\sim \frac{\eta^3}{L_N^2} + \frac{\eta^3}{L_C^2}$$

$$R_{0i}^{(1,0)} = \frac{1}{2a}\Big(h_{0j,ij}^{(1,0)} - h_{0i,jj}^{(1,0)} + ah_{ij,0j}^{(1,0)} - ah_{jj,0i}^{(1,0)} - 2\dot{a}h_{00,i}^{(1,0)} + 4\dot{a}^2 h_{0i}^{(1,0)} + 2a\ddot{a}h_{0i}^{(1,0)}\Big)$$

$$\sim \frac{\epsilon}{L_C^2} \tag{6.13}$$

$$R_{0i}^{(1,1)} = -2\dot{a}h_{00,i}^{(1,1)} \sim \frac{\epsilon\eta}{L_N L_C} \tag{6.14}$$

$$R_{0i}^{(1,2)} = \frac{1}{2a}\left(h_{0j,ij}^{(1,2)} - h_{0i,jj}^{(1,2)} + ah_{ij,0j}^{(1,1)} - ah_{jj,0i}^{(1,1)}\right) + \frac{1}{2a}h_{0j}^{(1,0)}h_{00,ij}^{(0,2)} \tag{6.15}$$

$$+\,\text{terms of size}\left[\frac{\epsilon\eta^2}{L_C^2} + \frac{\epsilon\eta^2}{L_N L_C}\right]$$

$$\sim \frac{\epsilon\eta^2}{L_N^2} + \frac{\epsilon\eta^2}{L_C^2} + \frac{\epsilon\eta^2}{L_N L_C}\,.$$

Finally, the terms in Eq. (6.3) are given by

$$R_{ij}^{(0,0)} = \left(2\dot{a}^2 + a\ddot{a}\right)\delta_{ij} \sim \frac{1}{L_C^2} \tag{6.16}$$

$$R_{ij}^{(0,2)} = \frac{1}{2}\left(h_{00,ij}^{(0,2)} + 2h_{k(i,j)k}^{(0,2)} - h_{kk,ij}^{(0,2)} - h_{ij,kk}^{(0,2)}\right) + \left(2\dot{a}^2 + a\ddot{a}\right)\left(h_{ij}^{(0,2)} + h_{00}^{(0,2)}\delta_{ij}\right)$$

$$\sim \frac{\eta^2}{L_N^2} + \frac{\eta^2}{L_C^2} \tag{6.17}$$

$$R_{ij}^{(0,3)} = \frac{1}{2}a\dot{a}h_{00,0}^{(0,2)}\delta_{ij} - 2\dot{a}h_{0(i,j)}^{(0,3)} - \dot{a}h_{0k,k}^{(0,3)}\delta_{ij} + \frac{3}{2}a\dot{a}h_{ij,0}^{(0,2)} + \frac{1}{2}a\dot{a}h_{kk,0}^{(0,2)}\delta_{ij} \tag{6.18}$$

$$\sim \frac{\eta^3}{L_C L_N}$$

$$R_{ij}^{(0,4)} = \frac{1}{2}\left(h_{00,ij}^{(0,4)} - h_{ij,kk}^{(0,4)} - h_{kk,ij}^{(0,4)}\right) + a^2 h_{ij,00}^{(0,2)} + \frac{1}{2}h_{00,k}^{(0,2)}\left(h_{ij,k}^{(0,2)} - 2h_{k(i,j)}^{(0,2)}\right)$$

$$+h_{kl,ij}^{(0,2)}h_{kl}^{(0,2)} + h_{ij,kl}^{(0,2)}h_{kl}^{(0,2)} - 2h_{k(i,j)l}^{(0,2)}h_{kl}^{(0,2)} + h_{kl,l}^{(0,2)}\left(h_{ij,k}^{(0,2)} - 2h_{k(i,j)}^{(0,2)}\right)$$

$$+h_{ik,l}^{(0,2)}\left(h_{jk,l}^{(0,2)} - h_{jl,k}^{(0,2)}\right) + h_{k(i,j)k}^{(0,4)} + h_{00,ij}^{(0,2)}h_{00}^{(0,2)} + h_{kk,l}^{(0,2)}\left(2h_{l(i,j)}^{(0,2)} - h_{ij,l}^{(0,2)}\right)$$

$$+\frac{1}{2}h_{00,i}^{(0,2)}h_{00,j}^{(0,2)} + \frac{1}{2}h_{kl,i}^{(0,2)}h_{kl,j}^{(0,2)} - 2ah_{0(i,j)0}^{(0,3)} + \text{ terms of size}\left[\frac{\eta^4}{L_C^2}\right] \tag{6.19}$$

$$\sim \frac{\eta^4}{L_N^2} + \frac{\eta^4}{L_C^2}$$

$$R_{ij}^{(1,0)} = \frac{1}{2}\left(h_{00,ij}^{(1,0)} - h_{ij,kk}^{(1,0)} - h_{kk,ij}^{(1,0)}\right) + h_{k(i,j)k}^{(1,0)} + a\ddot{a}h_{ij}^{(1,0)} + a\ddot{a}h_{00}^{(1,0)}\delta_{ij}$$

$$
\begin{aligned}
&+\frac{1}{2}a\dot{a}h^{(1,0)}_{00,0}\delta_{ij} - 2\dot{a}h^{(1,0)}_{0(i,j)} - \dot{a}h^{(1,0)}_{0k,k}\delta_{ij} + \frac{3}{2}a\dot{a}h^{(1,0)}_{ij,0} + \frac{1}{2}a\dot{a}h^{(1,0)}_{kk,0}\delta_{ij} \\
&+\frac{1}{2}a^2h^{(1,0)}_{ij,00} + 2\dot{a}^2h^{(1,0)}_{00}\delta_{ij} + 2\dot{a}^2h^{(1,0)}_{ij} - ah^{(1,0)}_{0(i,j)0} \qquad (6.20) \\
\sim\ & \frac{\epsilon}{L_C^2}
\end{aligned}
$$

$$
\begin{aligned}
R^{(1,1)}_{ij} &= \frac{1}{2}(h^{(1,1)}_{00,ij} - h^{(1,1)}_{ij,kk} - h^{(1,1)}_{kk,ij}) + h^{(1,1)}_{k(i,j)k} + \text{terms of size } \left[\frac{\epsilon\eta}{L_C^2}\right] \qquad (6.21) \\
&\sim \frac{\epsilon\eta}{L_N^2} + \frac{\epsilon\eta}{L_C^2}
\end{aligned}
$$

$$
\begin{aligned}
R^{(1,2)}_{ij} &= \frac{1}{2}\left(h^{(1,2)}_{00,ij} - h^{(1,2)}_{ij,kk} - h^{(1,2)}_{kk,ij}\right) + h^{(1,2)}_{k(i,j)k} + \frac{1}{2}h^{(0,2)}_{00,ij}h^{(1,0)}_{00} + \frac{1}{2}h^{(0,2)}_{kl,ij}h^{(1,0)}_{kl} \\
&+\frac{1}{2}h^{(0,2)}_{ij,kl}h^{(1,0)}_{kl} - h^{(0,2)}_{k(i,j)l}h^{(1,0)}_{kl} + \text{terms of size } \left[\frac{\epsilon\eta^2}{L_C^2} + \frac{\epsilon\eta^2}{L_N L_C}\right] \qquad (6.22) \\
&\sim \frac{\epsilon\eta^2}{L_N^2} + \frac{\epsilon\eta^2}{L_C^2} + \frac{\epsilon\eta^2}{L_N L_C}\,.
\end{aligned}
$$

We note that in Eq. (6.17) the two orders or magnitude after the $\sim$ indicate the term in the first parentheses and the subsequent terms, respectively.

6.1.2. Total energy-momentum tensor

We provide detailed expressions for the total perturbed energy-momentum tensor, which will be used to derive the field equations presented later. We make no assumptions about the relative magnitude of ϵ and η here, nor do we assume anything about the relationships between length scales L_C and L_N. We substitute the total perturbed energy-density, (5.2), pressure, (5.3), and four-velocity, (5.9)-(5.12), into the perturbed total energy-momentum tensor, (2.20). This total energy-momentum tensor includes two fluids, radiation and matter, see Eq. (5.18).

Expanding the total energy-momentum tensor in both ϵ and η the non-vanishing components of the energy-momentum tensor are given by

$$
T_{00} = T^{(0,0)}_{00} + T^{(0,2)}_{00} + T^{(1,0)}_{00} + T^{(1,1)}_{00} + T^{(1,2)}_{00} + \frac{1}{2}T^{(0,4)}_{00} + \dots \qquad (6.23)
$$

$$
T_{0i} = T^{(0,1)}_{0i} + T^{(0,3)}_{0i} + T^{(1,0)}_{0i} + T^{(1,2)}_{0i} + \dots \qquad (6.24)
$$

$$
T_{ij} = T^{(0,0)}_{ij} + T^{(0,2)}_{ij} + T^{(1,0)}_{ij} + T^{(1,1)}_{ij} + T^{(1,2)}_{ij} + \frac{1}{2}T^{(0,4)}_{ij} + \dots\,, \qquad (6.25)
$$

where ellipses again indicate higher-order terms that we will not consider in this thesis. There are significant differences between the above expansion of the total-stress energy tensor for matter and radiation compared to matter only (for the latter, refer to Appendix A) which are all due to the inclusion of the terms $\rho^{(0,0)}, p^{(0,0)}$ and $p^{(1,2)}$.

The terms on the right-hand side of Eq. (6.23) are given by

$$T_{00}^{(0,0)} = \rho^{(0,0)} \sim \frac{1}{L_C^2} \tag{6.26}$$

$$T_{00}^{(0,2)} = \rho^{(0,2)} - \rho^{(0,0)} h_{00}^{(0,2)} + (\rho^{(0,0)} + p^{(0,0)}) v^{(0,1)i} v_i^{(0,1)} \sim \frac{\eta^2}{L_N^2} + \frac{\eta^2}{L_C^2} \tag{6.27}$$

$$T_{00}^{(0,4)} = \rho^{(0,4)} - 2h_{00}^{(0,2)} \rho^{(0,2)} + 2\rho^{(0,2)} v^{(0,1)i} v_i^{(0,1)} + \text{ terms of size } \left[\frac{\eta^4}{L_C^2}\right] \tag{6.28}$$

$$\sim \frac{\eta^4}{L_N^2} + \frac{\eta^4}{L_C^2}$$

$$T_{00}^{(1,0)} = \rho^{(1,0)} - \rho^{(0,0)} h_{00}^{(1,0)} \sim \frac{\epsilon}{L_C^2} \tag{6.29}$$

$$T_{00}^{(1,1)} = \rho^{(1,1)} + \text{ terms of size } \left[\frac{\epsilon\eta}{L_C^2}\right] \sim \frac{\epsilon\eta}{L_N^2} + \frac{\epsilon\eta}{L_C^2}$$

$$T_{00}^{(1,2)} = \rho^{(1,2)} - h_{00}^{(1,0)} \rho^{(0,2)} + \text{ terms of size } \left[\frac{\epsilon\eta^2}{L_C^2}\right] \sim \frac{\epsilon\eta^2}{L_N^2} + \frac{\epsilon\eta^2}{L_C^2},$$

while the terms in Eq. (6.24) are given by

$$T_{0i}^{(0,1)} = -a(\rho^{(0,0)} + p^{(0,0)}) v_i^{(0,1)} \sim \frac{\eta}{L_C^2} \tag{6.30}$$

$$T_{0i}^{(0,3)} = -a\rho^{(0,2)} v_i^{(0,1)} + \text{ terms of size } \left[\frac{\eta^3}{L_C^2}\right] \sim \frac{\eta^3}{L_N^2} + \frac{\eta^3}{L_C^2}$$

$$T_{0i}^{(1,0)} = -a\rho^{(0,0)} (v_i^{(1,0)} + h_{0i}^{(1,0)}) - ap^{(0,0)} v_i^{(1,0)} \sim \frac{\epsilon}{L_C^2} \tag{6.31}$$

$$T_{0i}^{(1,2)} = -a\rho^{(0,2)} \left(v_i^{(1,0)} + h_{0i}^{(1,0)}\right) - a\rho^{(1,1)} v_i^{(0,1)} + \text{ terms of size } \left[\frac{\epsilon\eta^2}{L_C^2}\right] \tag{6.32}$$

$$\sim \frac{\epsilon\eta^2}{L_N^2} + \frac{\epsilon\eta^2}{L_C^2},$$

and the terms in Eq. (6.25) are given by

$$T_{ij}^{(0,0)} = a^2 p^{(0,0)} \delta_{ij} \sim \frac{1}{L_C^2} \tag{6.33}$$

$$T_{ij}^{(0,2)} = a^2\left(\rho^{(0,0)}+p^{(0,0)}\right)v_i^{(0,1)}v_j^{(0,1)} + a^2 p^{(0,0)} h_{ij}^{(0,2)} \sim \frac{\eta^2}{L_C^2} \tag{6.34}$$

$$T_{ij}^{(1,0)} = a^2 p^{(1,0)}\delta_{ij} + a^2 h_{ij}^{(1,0)} p^{(0,0)} \sim \frac{\epsilon}{L_C^2} \tag{6.35}$$

$$T_{ij}^{(1,1)} = \text{terms of size}\left[\frac{\epsilon\eta}{L_C^2}\right] \sim \frac{\epsilon\eta}{L_C^2} \tag{6.36}$$

$$T_{ij}^{(1,2)} = a^2 p^{(1,2)}\delta_{ij} + \text{terms of size}\left[\frac{\epsilon\eta^2}{L_C^2}\right] \sim \frac{\epsilon\eta^2}{L_N^2} + \frac{\epsilon\eta^2}{L_C^2} \tag{6.37}$$

$$T_{ij}^{(0,4)} = 2a^2\rho^{(0,2)}v_i^{(0,1)}v_j^{(0,1)} + a^2 p^{(0,4)}\delta_{ij} + \text{ terms of size}\left[\frac{\eta^4}{L_C^2}\right] \tag{6.38}$$

$$\sim \frac{\eta^4}{L_N^2} + \frac{\eta^4}{L_C^2}\,.$$

This completes the list of expanded tensor components that are required to derive the field equations in the next section. We also provide the energy-momentum tensor for dust only in Appendix A.

6.2. The field equations

It is straightforward to expand the field equations (2.15) in both ϵ and η, but the results are somewhat lengthy. This is partly due to the fact that we are using two parameters in our perturbative expansion, but is also a result of the freedom in choosing coordinates that exists within general relativity. Nevertheless, we want to present our results in the most general form possible at this stage. We therefore wrote the full versions of the two-parameter perturbed Ricci tensor and energy-momentum tensor above. The form of these equations is particularly complicated not only because of gauge freedoms and that each component of every tensor contains a large number of terms, but because each term is itself associated with a different length scale (or set of scales) and two parameters.

In practise, we want to apply our formalism to specific examples of physical interest. That is we need a relationship l between L_N and L_C and a given relationship between η and ϵ. Once such an example scenario has been chosen, then the expansion parameters and length scales can be written in terms of one another. This reduces the complexity, and allows the field equations to be written out explicitly, order-by-order, and without ambiguity.

In Section 5.3 we carefully analysed different astrophysical systems that exist on different scales in the Universe to see which are best described by the post-Newtonian expansion or cosmological perturbation theory. In this section we will present results for the choice

$$\epsilon \sim \eta^2 \sim \frac{L_N^2}{L_C^2} \sim 10^{-5}\,, \tag{6.39}$$

because, as described at the end of Section 5.3, $\epsilon \sim \eta^2 \sim 10^{-5}$ fits almost all non-linear astrophysical structures that exist in the Universe and the length scales $L_N \sim \eta L_C$ imply models where the non-linear structure exists on scales η smaller than the linear cosmological perturbations. Our field equations therefore correspond to dynamics on a fraction of the horizon-size. These results will be presented without fixing coordinates to any particular gauge, and are therefore still quite lengthy. In the following chapters we will exploit the gauge freedom associated with coordinate re-parametrization, and use this to present the same field equations in a much more compact form in Chapter 8.

At this stage it is useful to define some new notation, so that we can present the trace-free part of various quantities in the most efficient way possible. We define angular brackets on a pair of indices to mean that they are symmetric and trace-free, such that

$$\mathcal{T}_{\langle ij \rangle} \equiv \mathcal{T}_{(ij)} - \frac{1}{3}\delta_{ij}\mathcal{T}_{kk}\,, \tag{6.40}$$

where $\mathcal{T}$ is a rank-2 tensor, and where indices are now being raised and lowered with the Kronecker delta, δ_{ij}. The round brackets in this expression denote symmetrization, and repeated indices are summed over, as usual. We will also use vertical lines around indices if they are to be excluded from a symmetrization or trace-free operation.

Additionally, we define a symmetric and trace-free second derivative operator by the following equation:

$$D_{ij}\varphi \equiv \varphi_{,(ij)} - \frac{1}{3}\delta_{ij}\nabla^2\varphi\,, \tag{6.41}$$

where φ is any tensorial quantity (not necessarily a scalar), and where, here, ∇ represents the Laplacian on Euclidean space. For a tensor $\mathcal{T}$, of any rank, we observe the equivalence $\mathcal{T}_{,\langle ij \rangle} = D_{ij}\mathcal{T}$. We will use this notation to write out the trace and trace-free parts of the field equations, order by order in perturbations.

6.2.1. Background-order potentials

The two-parameter book-keeping implies the leading-order field equations, in our formalism, are not just at zeroth order in perturbations, but also include leading-order Newtonian perturbations. They come in at order $\mathcal{O}(L_C^{-2}) \sim \mathcal{O}(\eta^2 L_N^{-2})$ given the conditions in Eq. (6.39). The leading-order part of the 00-field equation is therefore given by

$$3\frac{\ddot{a}}{a} + \frac{1}{2a^2}\nabla^2 h_{00}^{(0,2)} = -4\pi\left(\rho^{(0,0)} + \rho^{(0,2)} + 3p^{(0,0)}\right) + \Lambda^{(0,0)} . \tag{6.42}$$

This equation results from Eqs. (6.4), (6.5), (6.26) and (6.27), and is a combination of the Raychaudhuri equation from Friedmann cosmology, and the Newton-Poisson equation from post-Newtonian gravity. It is interesting to see that the rest mass density, $\rho^{(0,2)}$, is the source of both the Newtonian gravitational field and the large-scale acceleration equation[1]. This is compatible with the usual understanding of how these phenomena are generated, but usually we do not see these terms in the same equation, at the same order in perturbations, they are normally derived initially as separate equations. Here we see that $a \sim 1$ and $\ddot{a} \sim 1/L_C^2$, as the time variation of $a(t)$ is over cosmological scales.

At the same order of accuracy, we find that the leading-order contribution to the trace of the ij-field equations is at $\mathcal{O}(\eta^2 L_N^{-2})$ and given by

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{6a^2}\left(\nabla^2 h_{ii}^{(0,2)} - h_{ij,ij}^{(0,2)}\right) = \frac{8\pi}{3}\left(\rho^{(0,2)} + \rho^{(0,0)}\right) + \frac{1}{3}\Lambda^{(0,0)} . \tag{6.43}$$

This equation is derived from Eqs. (6.16), (6.17), (6.33), (6.27) and simplified with the field equation (6.42). This derived equation is a combination of the Friedmann equation and the Newton-Poisson equation for the trace of the post-Newtonian potential $h_{ii}^{(0,2)}$. Again, we expect to see the Friedmann equation sourced by radiation $\rho^{(0,0)}$ and a cosmological constant at lowest order, as they are considered background-order quantities. However, it is unusual to see them with a mixture of first-order Newtonian perturbations to the metric and energy density, if one was using single-parameter cosmological perturbation theory. Finally, the trace-free part of the ij-field equations is also at $\mathcal{O}(\eta^2 L_N^{-2})$, and is given by

$$D_{ij}\left(h_{00}^{(0,2)} - h_{kk}^{(0,2)}\right) + 2h_{k\langle i,j\rangle k}^{(0,2)} - \nabla^2 h_{\langle ij\rangle}^{(0,2)} = 0 , \tag{6.44}$$

where we have made use of the notation introduced in Eqs. (6.40) and (6.41). This equation is the same for dust only and radiation, dust and a cosmological

[1]We find later, with a definition of the homogeneity scale, that the inhomogeneous part of the Newtonian rest mass does not affect the expansion rate, only the average of it does, see Section 8.2.1.

constant because neither the cosmological constant nor radiation contribute trace-free components. This equation looks like the quasi-static limit of a first-order equation from cosmological perturbation theory, see for example [36, 138].

6.2.2. Vector potentials

Now let us consider the $0i$-field equations, which usually result in the governing equations for the vector gravitational potentials. The leading-order contribution to these equations are $\mathcal{O}(\eta^3 L_N^{-2})$, and is given by

$$\begin{aligned} & \nabla^2 h_{0i}^{(0,3)} - h_{0j,ij}^{(0,3)} - a\dot{h}_{ij,j}^{(0,2)} + a\dot{h}_{jj,i}^{(0,2)} + 2\dot{a}h_{00,i}^{(0,2)} \\ = \;& 16\pi a^2 \left(\rho^{(0,0)} + \rho^{(0,2)} + p^{(0,0)}\right) v_i^{(0,1)} \,. \end{aligned} \tag{6.45}$$

This equation is the result of using Eqs. (6.11), (6.12), (6.30) and (6.31). This is the equation for the small-scale post-Newtonian vector potential, responsible for phenomena such as the Lense-Thirring effect, and is the one studied in Ref. [130]. However, it is unusual in post-Newtonian gravity to see contributions from the radiation energy density and pressure in this equation too because post-Newtonian gravity normally considers dust only. Interestingly, this field equation implies the gravitomagnetic ($0i$-metric) potential is ~ 100 times larger than second-order perturbation theory predicts. This is because the first non-decaying $0i$-metric potential is of order ϵ^2 in second-order cosmological perturbation theory. This metric potential is 100 times smaller than the post-Newtonian metric potential in Eq. (6.45), of size η^3, because $\epsilon \sim \eta^2 \sim 10^{-5}$. A similar result was found in the calculation of the vector potential in Ref. [26].

At next-to-leading-order in the $0i$-field equation, at $\mathcal{O}(\eta^4 L_N^{-2})$, from Eqs. (6.13)-(6.15), (6.31) and (6.32), we find that

$$\begin{aligned} & \nabla^2 \left(h_{0i}^{(1,0)} + h_{0i}^{(1,2)}\right) - \left(h_{0j}^{(1,0)} + h_{0j}^{(1,2)}\right)_{,ij} - h_{0j}^{(1,0)} h_{00,ij}^{(0,2)} - a\left(h_{ij}^{(1,0)} + h_{ij}^{(1,1)}\right)^{\cdot}_{,j} \\ & + a\left(h_{jj}^{(1,0)} + h_{jj}^{(1,1)}\right)^{\cdot}_{,i} + 2\dot{a}\left(h_{00}^{(1,0)} + h_{00}^{(1,1)}\right)_{,i} - 2h_{0i}^{(1,0)}\left(2\dot{a}^2 + a\ddot{a}\right) \\ = \;& 8\pi a^2 \Big(2\left(\rho^{(0,0)} + \rho^{(0,2)} + p^{(0,0)}\right) v_i^{(1,0)} + \left(\rho^{(0,0)} + \rho^{(0,2)} + 3p^{(0,0)}\right) h_{0i}^{(1,0)} \\ & + 2\rho^{(1,1)} v_i^{(0,1)}\Big) - 2a^2 \Lambda^{(0,0)} h_{0i}^{(1,0)} \,. \end{aligned} \tag{6.46}$$

This equation is the governing equation for the large-scale vector potentials. It is more complicated than Eq. (6.45), and shows that non-linear gravitational effects could potentially source the growth of large-scale vector potentials at late times. This equation can also be seen to have contributions from the cosmological constant, unlike Eq. (6.45).

6.2.3. Higher-order scalar potentials

The next-to-leading-order 00-field equation occurs at $\mathcal{O}(\eta^3 L_N^{-2})$, and is given by

$$\nabla^2 h_{00}^{(1,1)} = -8\pi a^2 \rho^{(1,1)} \,. \tag{6.47}$$

This is a Newton-Poisson equation, derived from Eqs. (6.9) and (6.30). It is sourced only by a mixed order matter energy density $\rho^{(1,1)} = \rho_M^{(1,1)}$. This is not usual in post-Newtonian gravity because the Newton-Poisson equation is normally only at leading order and, of course, is not normally associated with a mixed-order perturbed quantity. Also, for radiation or cosmological constant domination, using our two-parameter expansion, we find $\nabla^2 h_{00}^{(1,1)} = 0$ which implies $h_{00}^{(1,1)} = 0$ for these epochs, given boundary conditions.

The governing equations for the cosmological potentials $h_{00}^{(1,0)}$ and $h_{ii}^{(1,0)}$ occur along with post-Newtonian and mixed order potentials at $\mathcal{O}(\eta^4 L_N^{-2})$ – this was also the case for the vector potentials considered above. From the 00-field equation at this order (once multiplied by a factor of $-2a^2$), we find that

$$\begin{aligned}
&\nabla^2 \left(h_{00}^{(1,0)} + h_{00}^{(1,2)} + \frac{1}{2} h_{00}^{(0,4)} \right) + \frac{1}{2} \left(\nabla h_{00}^{(0,2)} \right)^2 + a^2 \left(h_{ii}^{(0,2)} + h_{ii}^{(1,0)} \right)^{\cdot\cdot} \\
&-2 \left[a \left(h_{0i}^{(0,3)} + h_{0i}^{(1,0)} \right)_{,i} \right]^{\cdot} + 2a\dot{a} \left(h_{ii}^{(0,2)} + h_{ii}^{(1,0)} \right)^{\cdot} - \frac{1}{2} h_{00,i}^{(0,2)} \left(2h_{ij,j}^{(0,2)} - h_{jj,i}^{(0,2)} \right) \\
&-h_{00,ij}^{(0,2)} \left(h_{ij}^{(1,0)} + h_{ij}^{(0,2)} \right) + 3a\dot{a} \left(h_{00}^{(0,2)} + h_{00}^{(1,0)} \right)^{\cdot} \\
= \; & -8\pi a^2 \left[\rho^{(1,0)} + \rho^{(1,2)} + \frac{1}{2} \rho^{(0,4)} - \left(\rho^{(0,0)} + \rho^{(0,2)} + 3p^{(0,0)} \right) \left(h_{00}^{(1,0)} + h_{00}^{(0,2)} \right) \right. \\
&\left. +3 \left(p^{(1,0)} + p^{(1,2)} + \frac{1}{2} p^{(0,4)} \right) \right] - 16\pi a^2 \left(v_i^{(0,1)} \right)^2 \left(\rho^{(0,0)} + \rho^{(0,2)} + p^{(0,0)} \right) \\
&-2a^2 \Lambda^{(0,0)} \left(h_{00}^{(0,2)} + h_{00}^{(1,0)} \right) \,.
\end{aligned} \tag{6.48}$$

There are a number of interesting things to note about this equation. These include the fact that the cosmological scalar $h_{00}^{(1,0)}$ is sourced by terms that are quadratic in the small-scale Newtonian potential, $h_{00}^{(0,2)}$, as well as terms that are linear in the vector potential, $h_{0i}^{(0,3)}$, and post-Newtonian potential $h_{00}^{(0,4)}$. This kind of mixing in scales and modes is a product of the approach we have used in our two-parameter perturbative expansion and could explain why studies of second-order gravitational fields using cosmological perturbation theory average to the size of first order gravitational fields [14, 56, 72, 115, 143]. It suggests that interesting relativistic phenomenology in the late Universe could result at linear order in large-scale potentials. This equation can be seen to have additional sources due to the presence of radiation and a cosmological constant, compared to the corresponding equation in the presence of dust only (see Appendix B).

The ij-field equation, at $\mathcal{O}(\eta^3 L_N^{-2})$, can be split into its trace and trace-free parts. The trace-free part will be presented in the next section. The trace gives

$$\nabla^2 h_{ii}^{(1,1)} - h_{ij,ij}^{(1,1)} = -16\pi a^2 \rho^{(1,1)} \,. \tag{6.49}$$

This equation is derived from Eqs. (6.21) and (6.30), and is a Poisson equation for the trace of the mixed order potential $h_{ii}^{(1,1)}$, and has only a dust source as $\rho^{(1,1)} = \rho_M^{(1,1)}$. Again, this is not usual because such an equation is normally at post-Newtonian order and is normally not associated with a mixed-order quantity.

Similarly the ij-field equation at $\mathcal{O}(\eta^4 L_N^{-2})$ can also be split into its trace and trace-free parts. The trace of this equation gives

$$\begin{aligned}
& \left(\delta_{ij}\nabla^2 - \partial_i\partial_j\right)\left(h_{ij}^{(1,0)} + h_{ij}^{(1,2)} + \frac{1}{2}h_{ij}^{(0,4)}\right) + 4\dot{a}\left(h_{0i}^{(1,0)} + h_{0i}^{(0,3)}\right)_{,i} \\
& -2a\dot{a}\left(h_{ii}^{(1,0)} + h_{ii}^{(0,2)}\right)^{\cdot} - \left(2\dot{a}^2 + a\ddot{a}\right)\left(h_{ii}^{(1,0)} + h_{ii}^{(0,2)} + 3h_{00}^{(1,0)} + 3h_{00}^{(0,2)}\right) \\
= \; & -16\pi a^2\left[\rho^{(1,0)} + \frac{1}{2}\rho^{(0,4)} + \rho^{(1,2)} + \left(\rho^{(0,0)} + \rho^{(0,2)} + p^{(0,0)}\right)\left(v_i^{(0,1)}\right)^2\right] \\
& -4\pi a^2\left[\left(\rho^{(0,0)} + \rho^{(0,2)} - p^{(0,0)}\right)\left(h_{ii}^{(0,2)} + h_{ii}^{(1,0)}\right)\right. \\
& \left. - \left(\rho^{(0,0)} + \rho^{(0,2)} + 3p^{(0,0)}\right)\left(h_{00}^{(0,2)} + h_{00}^{(1,0)}\right)\right] \\
& -a^2\Lambda^{(0,0)}\left[h_{00}^{(0,2)} + h_{00}^{(1,0)} + h_{ii}^{(0,2)} + h_{ii}^{(1,0)}\right] + \mathcal{A}\,,
\end{aligned} \tag{6.50}$$

where we have simplified this expression using Eq. (6.48) multiplied by a factor of a^2. The $\mathcal{A}$ in Eq. (6.50) represents the sum of all terms that are quadratic in lower-order potentials, and is given by

$$\mathcal{A} \equiv \frac{3}{4}\left(h_{ij,k}^{(0,2)}\right)^2 + h_{ij,j}^{(0,2)}\left(h_{kk,i}^{(0,2)} - h_{ik,k}^{(0,2)}\right) - \frac{1}{2}h_{ij,k}^{(0,2)}h_{ik,j}^{(0,2)} - \frac{1}{4}h_{ii,j}^{(0,2)}h_{kk,j}^{(0,2)} \tag{6.51}$$

$$+\frac{1}{2}\nabla^2 h_{00}^{(0,2)}\left(h_{00}^{(1,0)}+h_{00}^{(0,2)}\right)+\frac{1}{2}\left(h_{00,ij}^{(0,2)}+\nabla^2 h_{ij}^{(0,2)}\right)\left(h_{ij}^{(1,0)}+h_{ij}^{(0,2)}\right)$$
$$+\left(\frac{1}{2}h_{ii,jk}^{(0,2)}-h_{ij,ik}^{(0,2)}\right)\left(h_{jk}^{(0,2)}+h_{jk}^{(1,0)}\right).$$

If $\mathcal{A}$ is non-zero, then this indicates that non-linear relativistic effects could be important in the determination of scalar gravitational fields on large scales – this is what we expect generally. One may also note that small-scale peculiar velocities are now a source for linear cosmological scalar gravitational fields – these terms are highly non-linear. Furthermore, as $\rho^{(0,2)}$ is inhomogeneous, the term $\rho^{(0,2)}\left(v_i^{(0,1)}\right)^2$ would normally appear at third order in cosmological perturbation theory. This equation includes additional source terms due to the inclusion of matter, radiation and cosmological constant when compared to matter alone. The trace-free part of this equation is presented below.

6.2.4. Tensor potentials

The next-to-leading-order trace-free ij-field equation occurs at $\mathcal{O}(\eta^3 L_N^{-2})$, and is given by

$$D_{ij}\left(h_{00}^{(1,1)}-h_{kk}^{(1,1)}\right)+2h_{k\langle i,j\rangle k}^{(1,1)}-\nabla^2 h_{\langle ij\rangle}^{(1,1)}=0\,. \tag{6.52}$$

where we have used Eqs. (6.21) and (6.30). We note that this equation has the same form as the lowest order trace-free ij-field equation, given in Eq. (6.44), and for dust only (as the cosmological constant and radiation are isotropic).

The remaining part of the field equations that we wish to consider is the trace-free part of the ij-component. At $\mathcal{O}(\eta^4 L_N^{-2})$ we find that this equation is given by

$$\begin{aligned}
&\nabla^2\left(h_{\langle ij\rangle}^{(1,0)}+h_{\langle ij\rangle}^{(1,2)}+\frac{1}{2}h_{\langle ij\rangle}^{(0,4)}\right)-2\left(h_{k\langle i}^{(1,0)}+h_{k\langle i}^{(1,2)}+\frac{1}{2}h_{k\langle i}^{(0,4)}\right)_{,j\rangle k} \qquad (6.53)\\
&-D_{ij}\left(h_{00}^{(1,0)}+h_{00}^{(1,2)}+\frac{1}{2}h_{00}^{(0,4)}-h_{kk}^{(1,0)}-h_{kk}^{(1,2)}-\frac{1}{2}h_{kk}^{(0,4)}\right)\\
&-2\left(2\dot{a}^2+a\ddot{a}\right)\left(h_{\langle ij\rangle}^{(1,0)}+h_{\langle ij\rangle}^{(0,2)}\right)+\frac{2}{a}\left[a^2\left(h_{0\langle i}^{(1,0)}+h_{0\langle i}^{(0,3)}\right)\right]^{\cdot}_{,j\rangle}\\
&-a^2\left(h_{\langle ij\rangle}^{(1,0)}+h_{\langle ij\rangle}^{(0,2)}\right)^{\cdot\cdot}-3a\dot{a}\left(h_{\langle ij\rangle}^{(1,0)}+h_{\langle ij\rangle}^{(0,2)}\right)^{\cdot}\\
=\;&-2a^2\Lambda^{(0,0)}\left(h_{\langle ij\rangle}^{(0,2)}+h_{\langle ij\rangle}^{(1,0)}\right)-8\pi a^2\left[\left(\rho^{(0,0)}+\rho^{(0,2)}-p^{(0,0)}\right)\left(h_{\langle ij\rangle}^{(0,2)}+h_{\langle ij\rangle}^{(1,0)}\right)\right.
\end{aligned}$$

$$+2\left(\rho^{(0,0)}+\rho^{(0,2)}+p^{(0,0)}\right)v^{(0,1)}_{\langle i}v^{(0,1)}_{j\rangle}\Big]+\mathcal{B}_{ij}\,,$$

where we used $\mathcal{B}_{ij}$ to denote the summation of all terms that are quadratic in lower-order potentials

$$\begin{aligned}\mathcal{B}_{ij} \;\equiv\;& \frac{1}{2}h^{(0,2)}_{00,\langle i|}h^{(0,2)}_{00,|j\rangle}+\frac{1}{2}h^{(0,2)}_{kl,\langle i|}h^{(0,2)}_{kl,|j\rangle}+D_{ij}h^{(0,2)}_{00}\left(h^{(1,0)}_{00}+h^{(0,2)}_{00}\right) \\ &+\frac{1}{2}\left(h^{(0,2)}_{00,k}+2h^{(0,2)}_{kl,l}-h^{(0,2)}_{ll,k}\right)\left(h^{(0,2)}_{\langle ij\rangle,k}-2h^{(0,2)}_{k\langle i,j\rangle}\right) \\ &+\left(D_{ij}h^{(0,2)}_{kl}+h^{(0,2)}_{\langle ij\rangle,kl}-2h^{(0,2)}_{k\langle i,j\rangle l}\right)\left(h^{(1,0)}_{kl}+h^{(0,2)}_{kl}\right)+h^{(0,2)}_{\langle i|k,l}\left(h^{(0,2)}_{|j\rangle k,l}-h^{(0,2)}_{|j\rangle l,k}\right)\,.\end{aligned} \tag{6.54}$$

These expressions show that trace-free large-scale tensor potentials are, in this formalism, sourced by small-scale peculiar velocities, as well as by terms that are quadratic in lower-order potentials; effects only found at second order or third order in standard perturbation theory. This again indicates the possibility of mode-mixing between scales, a mixing of different fluids, and the sourcing of gravitational phenomena in ways that are impossible at first order in standard cosmological perturbation theory. This completes the full set of field equations, to the order at which we require them.

The two-parameter perturbed field equations for a Universe with non-relativistic matter only are given in Appendix B. In the next chapter we will consider how gauge transformations affect the perturbations that we have been considering. This information will then be used to simplify the field equations that are given above, as well as to present them in a gauge-invariant form.

7. Two-parameter gauge transformations

As discussed in Section 3.3, general relativity is a covariant theory. This means that the form of the tensor equations that we use to describe general relativity must be valid for any set of coordinates. Diffeomorphisms obey a strict group structure, which guarantee that we can transform any given solution into a new set of coordinates, and that the result will still obey Einstein's equations. When considering general perturbations about a fixed background, this freedom in coordinate re-parametrization is referred to as infinitesimal diffeomorphisms or "gauge freedoms", and are given by the Lie derivative at leading-order or the exponential map at beyond-leading-order, see Eq. (3.26). When it comes to solving Einstein's equations, coordinate re-parametrization invariance and gauge freedom are both a blessing and a curse. In general, they mean that perturbations, such as perturbations to the metric, contain not only the essential degrees of freedom required to describe the physical situation at hand, but also a number of superfluous degrees of freedom that relate only to the arbitrary coordinates used to describe the problem. However, while it takes some care to remove these extra degrees of freedom, the process of doing so, where we calculate gauge invariant variables, can be used to simplify the equations that result, which are ultimately the same form as the field equations in terms of the longitudinal gauge. This is especially welcome in our case, as the equations presented in Chapter 6 are particularly unwieldy.

In this chapter we will outline how gauge transformations are performed in a two-parameter perturbation expansion – this is non-trivial because our expansion requires perturbations which vary on two different length scales. These transformations differ significantly from gauge transformations in single parameter cosmological perturbation theory[1], discussed in Section 3.3. The form of these transformations will then be used to construct a set of variables that have the superfluous gauge

[1]Gauge transformations are also necessary to form a complete set of two-parameter perturbations of metric and matter sources, for example see the transformations of $h_{00}^{(1,1)}$ and $\rho^{(1,1)}$, Eqs. (7.6) and (7.73), respectively.

freedoms removed. We also assert which gauge choices, out of those traditionally used in cosmological perturbation theory, are allowed in post-Newtonian gravity and therefore in our two-parameter expansion. The gauge invariant quantities we construct will allow us not only to write the field equations in a more compact form, but also to present a set of equations that represents only the degrees of freedom required to characterise the physical problem itself. Additionally, a full understanding of the gauge transformations of the matter fluids and metric fields also allows us to identify certain terms in our two-parameter expansion summarised in Section 5.2.2.

7.1. Two-parameter gauge transformations

Two-parameter gauge transformations have much the same form as those constructed in cosmological perturbation theory, Section (3.3.1). To be precise, the general form of the infinitesimal gauge transformation still holds, see Eq. (3.22), where ξ^μ is gauge generator (small in the perturbative expansion). For our two-parameter expansion this implies the gauge generator is expanded in two-parameters simultaneously, ϵ and η. A transformation of this type leaves all background quantities invariant, but changes the form of the perturbations, it uses the exponential map between coordinates systems, given in Eq. (3.26), which guarantees that the group structure of the manifold is preserved, and the Lie derivative is defined in Eq. (2.13). Now with Eqs. (3.22) and (3.26), and a two-parameter perturbed tensor $\mathcal{T}$ in hand, we can specify how the gauge generator ξ^μ should be expanded in two-parameters. We then calculate the transformation of tensor $\mathcal{T}$ order-by-order in the perturbations.

In principle, when expanding the gauge generator ξ^μ one could include terms at any order possible in the parameters ϵ and η, given in the general two-parameter expansion of a tensorial quantity in Eq. (5.1). This, however, is not strictly necessary, as some orders will serve to produce new terms in the tensor $\tilde{\mathcal{T}}$ that are of no physical interest. This is the same type of problem that occurred when we expanded the sources in the field equations, for example see the discussion in Section 5.2.3. The terms we wish to retain in ξ^μ, and their orders of magnitude, are given by the following expressions:

$$
\begin{aligned}
\xi^0 &= \xi^{(1,0)0} + \xi^{(0,3)0} + \xi^{(1,2)0} + \dots \\
&\sim \epsilon L_C + \eta^3 L_N + \epsilon\eta^3 L_N + \dots
\end{aligned}
\tag{7.1}
$$

$$\begin{aligned} \xi^i &= \xi^{(1,0)i} + \xi^{(0,2)i} + \xi^{(1,1)i} + \xi^{(1,2)i} + \tfrac{1}{2}\xi^{(0,4)i} + \ldots \\ &\sim \epsilon L_C + \eta^2 L_N + \epsilon\eta^2 L_N + \eta^4 L_N + \ldots\,, \end{aligned} \tag{7.2}$$

where in the limit where post-Newtonian perturbations go to zero, $\eta \to 0$, we recover the perturbed gauge generator for single-parameter cosmological perturbation theory, given in Eqs. (3.24) and (3.25).

We now make several comments on the above two-parameter expansion of the gauge generator. Firstly, as stated previously, each of the terms $\xi^{(m,n)}$ has dimensions of length. This is because the gauge generator ξ^μ corresponds to a change in space-time coordinates x^μ and coordinates have dimensions of length. The particular length scale assigned to each term is done in the same way as described in Chapter 5, such that cosmological perturbations vary on the length scales L_C whereas post-Newtonian or mixed-order perturbations vary on length scales L_N. Secondly, one may also note that while terms of $\mathcal{O}(\epsilon L_C)$ appear similarly in both ξ^0 and ξ^i, the order of terms perturbed in the parameter η appear at different orders in ξ^0 and ξ^i, see Eqs. (7.1) and (7.2). This is, once again, because time and space derivatives on cosmologically perturbed quantities add the same order of smallness whereas they add different orders of smallness in post-Newtonian perturbation theory. The ellipses in Eqs. (7.1) and (7.2) correspond to terms that are smaller than those required to transform the field equations presented in Chapter 6.

The lowest-order-cosmological gauge generators, $\xi^{(1,0)\mu}$, are of exactly the same order as the ones used in normal cosmological perturbation theory at linear order, see Eq. (3.23). These are the parts of the gauge generator that will generate metric perturbations at order $g^{(1,0)}_{\mu\nu}$, in the usual way. This is just what we expect, as our cosmological metric perturbations are, for all intents and purposes, exactly the same as those used in standard cosmological perturbation theory (*i.e.* they have the same size, and vary in the same way in space and time). Additionally, the post-Newtonian gauge generators $\xi^{(0,3)0}$, $\xi^{(0,2)i}$ and $\xi^{(0,4)i}$ are of exactly the same order in perturbations as those that occur in usual post-Newtonian perturbation theory [141, 171]. All mixed order gauge generators are unique to our two parameter expansion, and have no counterpart in either standard cosmological perturbation theory or standard post-Newtonian theory.

We formed the above gauge generators, Eq. (7.1) and (7.2), in the same way as the perturbed sources of energy-momentum and metric, in Chapter 5, such that the gauge generator contains the minimum number of perturbations necessary for a two-parameter system. We wrote an initial ansatz gauge generator with care because of the different length scales involved. The initial ansatz was given by the sum of the gauge generators used in cosmological perturbation theory, post-Newtonian gravity and mixed order gauge generators (that are products of the lowest-order gauge generators in both the cosmological and the post-Newtonian sectors) this gives $\xi^{(1,3)0}$ and $\xi^{(1,2)i}$. However, the terms in the final ansatz metric, given in Section 5.2.2, strictly imply we require gauge generators of order $\xi^{(1,1)i}$ and $\xi^{(1,2)0}$ because we want to find and transform along *all* possible degrees of freedom[2]. Therefore, we also include gauge generators $\xi^{(1,1)i}$ and $\xi^{(1,2)0}$ in our new ansatz gauge generator, given in Eqs. (7.1) and (7.2). Now this gauge generator has the minimal number of perturbations necessary to create all necessary transformations to the metric, and energy-momentum tensor.

In order to present our results in a form that can be used for cosmology we choose to take $L_N/L_C \sim \eta$. This means that we are restricting the post-Newtonian sector of our expansion to apply on scales below about 100Mpc, which is coincidently also about the size of the homogeneity scale. This is ideal for considering the influence of galaxies, clusters and super-clusters on large-scale linear cosmological perturbations. We also choose, without loss of generality, to express our results in terms of L_N. It is necessary to relate these length scales L_N and L_C, otherwise we cannot separate the gauge transformations at different post-Newtonian, mixed and cosmologically perturbed orders, because the gauge generators are not only small in ϵ or η but vary on characteristic length scales L_C or L_N. Throughout the following chapter we will assume $L_N/L_C \sim \eta$, as is assumed in Chapter 6, but not $\epsilon \sim \eta^2$ (as this is not necessary).

To summarise, considering transformations of non-linear gravity with a two-parameter expansion, where potentials vary on different length scales and behave differently under space-time derivatives, makes this study more complex than (even) second order cosmological perturbation theory. By substitution of Eqs. (7.1) and (7.2), into Eqs. (3.22) and (3.26), we can calculate how the metric and energy-momentum tensors transform under these infinitesimal coordinate transformations, order-by-order in perturbations, we now present these results in detail.

[2] To be explicit the ij and $0i$ parts of our initial metric ansatz produced new potentials of $\mathcal{O}(\epsilon\eta)$ and $\mathcal{O}(\epsilon\eta^2)$, respectively. As explained in Section 5.2.6, we therefore included the extra metric components $g_{ij}^{(1,1)}$ and $g_{0i}^{(1,2)}$ in our new ansatz metric. The existence of these potentials then implies that we should have gauge generators of order $\xi^{(1,1)i}$ and $\xi^{(1,2)0}$, as we we want to find and transform along *all* possible degrees of freedom.

7.2. Transformation of the metric

We begin by transforming the different components of the perturbed metric, Eqs. (5.5)-(5.7), using

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu} + \tfrac{1}{2}\mathcal{L}_\xi^2 g_{\mu\nu} + \dots\,, \tag{7.3}$$

which is given from the exponential map, Eq. (3.26), and where the expansion of the gauge generator ξ^μ is given by Eqs. (7.1) and (7.2). Note that the last term in Eq. (7.3), in the transformation of the metric, is quadratic in ξ. Such terms are given explicitly in the transformation of the second-order perturbation to the metric in single-parameter cosmological perturbation theory in Eq. (3.39). All such quadratic terms are needed in order to explicitly calculate non-linear gauge transformations[3], undertaken in this chapter. These non-linear transformations are necessary to calculate the transformation of post-Newtonian potentials and had not been calculated before Refs. [97, 98].

7.2.1. Transformation of metric components

The time-time component: the perturbations of the time-time component of the metric, up to the order we wish to consider here, transform under Eq. (7.3) in the following way:

$$h_{00}^{(0,2)} \mapsto \tilde{h}_{00}^{(0,2)} = h_{00}^{(0,2)} \tag{7.4}$$

$$h_{00}^{(1,0)} \mapsto \tilde{h}_{00}^{(1,0)} = h_{00}^{(1,0)} - 2\dot{\xi}^{(1,0)0} \tag{7.5}$$

$$h_{00}^{(1,1)} \mapsto \tilde{h}_{00}^{(1,1)} = h_{00}^{(1,1)} + h_{00,i}^{(0,2)}\xi^{(1,0)i} \tag{7.6}$$

$$h_{00}^{(1,2)} \mapsto \tilde{h}_{00}^{(1,2)} = h_{00}^{(1,2)} + \dot{h}_{00}^{(0,2)}\xi^{(1,0)0} + 2h_{00}^{(0,2)}\dot{\xi}^{(1,0)0} \tag{7.7}$$

$$h_{00}^{(0,4)} \mapsto \tilde{h}_{00}^{(0,4)} = h_{00}^{(0,4)} - 4\dot{\xi}^{(0,3)0} + 2h_{00,i}^{(0,2)}\xi^{(0,2)i}\,. \tag{7.8}$$

We note that in addition to these transformations, each of which contains terms with the same order-of-magnitude, there is also a term generated from Eq. (7.3) in this component of the metric that is

$$\tfrac{1}{2}h_{00,ij}^{(0,2)}\xi^{(1,0)i}\xi^{(1,0)j}\,, \tag{7.9}$$

[3] Non-linear gauge transformations are necessary for transforming the dynamics of non-linearities Einstein's field equations.

which is of the $\mathcal{O}(\epsilon^2)$ when the length scales are taken into account appropriately. However, this term appears in the $\mathcal{O}(\eta^4 L_N^{-2})$ 00-field equation, Eq. (6.48), in the form of $R^{(2,0)}_{\mu\nu} \sim \frac{1}{2}\nabla^2(h^{(0,2)}_{00,ij}\xi^{(1,0)i}\xi^{(1,0)j}) \sim \frac{1}{2}\nabla^2 h^{(0,2)}_{00,ij}\xi^{(1,0)i}\xi^{(1,0)j} \sim \epsilon^2 L_N^{-2} \sim \eta^4 L_N^{-2}$, when $\epsilon \sim \eta^2$. We discuss how such a term cancels with another term in the field equations in Section 8.

The time-space components: the perturbations of the time-space parts of the metric transform, according to Eq. (7.3), in the following way:

$$h^{(0,3)}_{0i} \mapsto \tilde{h}^{(0,3)}_{0i} = h^{(0,3)}_{0i} - \frac{1}{a}\xi^{(0,3)0}_{,i} + a\dot{\xi}^{(0,2)}_i \tag{7.10}$$

$$h^{(1,0)}_{0i} \mapsto \tilde{h}^{(1,0)}_{0i} = h^{(1,0)}_{0i} - \frac{1}{a}\xi^{(1,0)0}_{,i} + a\dot{\xi}^{(1,0)}_i \tag{7.11}$$

$$h^{(1,2)}_{0i} \mapsto \tilde{h}^{(1,2)}_{0i} = h^{(1,2)}_{0i} - \frac{1}{a}\dot{\xi}^{(1,2)0}_{,i} + a\dot{\xi}^{(1,1)}_i + \chi^{(1,2)}_i\,, \tag{7.12}$$

where in the latter equation we define quadratic terms such that

$$\begin{aligned}\chi^{(1,2)}_i \equiv\ & \frac{1}{a}h^{(0,2)}_{00}\xi^{(1,0)0}_{,i} + a\left(h^{(0,2)}_{ij} + \xi^{(0,2)}_{(i,j)}\right)\dot{\xi}^{(1,0)j} \\ & + \left(h^{(0,3)}_{0i} - \frac{1}{2a}\xi^{(0,3)0}_{,i} + \frac{1}{2}a\dot{\xi}^{(0,2)}_i\right)_{,j}\xi^{(1,0)j} \\ & + \left(h^{(1,0)}_{0j} - \frac{1}{2a}\xi^{(1,0)0}_{,j} + \frac{1}{2}a\dot{\xi}^{(1,0)}_j\right)\xi^{(0,2)j}_{,i}\,.\end{aligned} \tag{7.13}$$

The space-space components: the transformations of the perturbations in the space-space part of the metric are more lengthy than the previous cases. They transform under the exponential map in Eq. (7.3) in the following way:

$$h^{(0,2)}_{ij} \mapsto \tilde{h}^{(0,2)}_{ij} = h^{(0,2)}_{ij} + 2\xi^{(0,2)}_{(i,j)} \tag{7.14}$$

$$h^{(1,0)}_{ij} \mapsto \tilde{h}^{(1,0)}_{ij} = h^{(1,0)}_{ij} + 2\frac{\dot{a}}{a}\xi^{(1,0)0}\delta_{ij} + 2\xi^{(1,0)}_{(i,j)} \tag{7.15}$$

$$h^{(1,1)}_{ij} \mapsto \tilde{h}^{(1,1)}_{ij} = h^{(1,1)}_{ij} + 2\xi^{(1,1)}_{(i,j)} + \chi^{(1,1)}_{ij} \tag{7.16}$$

$$h^{(1,2)}_{ij} \mapsto \tilde{h}^{(1,2)}_{ij} = h^{(1,2)}_{ij} + 2\xi^{(1,2)}_{(i,j)} + \chi^{(1,2)}_{ij} \tag{7.17}$$

$$h^{(0,4)}_{ij} \mapsto \tilde{h}^{(0,4)}_{ij} = h^{(0,4)}_{ij} + 4\frac{\dot{a}}{a}\xi^{(0,3)0}\delta_{ij} + 2\xi^{(0,4)}_{(i,j)} + \chi^{(0,4)}_{ij}\,, \tag{7.18}$$

where $\chi_{ij}^{(1,1)}$, $\chi_{ij}^{(1,2)}$ and $\chi_{ij}^{(0,4)}$ are quadratic terms defined as

$$\chi_{ij}^{(1,1)} \equiv \left(h_{ij}^{(0,2)} + 2\xi_{(i,j)}^{(0,2)}\right)_{,k} \xi^{(1,0)k} \tag{7.19}$$

$$\begin{aligned}\chi_{ij}^{(1,2)} \equiv & \left(h_{ij}^{(0,2)} + \xi_{(i,j)}^{(0,2)}\right)^{\cdot} \xi^{(1,0)0} + 2\frac{\dot{a}}{a}\left(h_{ij}^{(0,2)} + 2\xi_{(i,j)}^{(0,2)}\right)\xi^{(1,0)0} \\ & + \left(h_{ik}^{(0,2)} + \xi_{(i,k)}^{(0,2)}\right)\xi_{,j}^{(1,0)k} + \left(h_{jk}^{(0,2)} + \xi_{(j,k)}^{(0,2)}\right)\xi_{,i}^{(1,0)k} \\ & + \left(h_{ik}^{(1,0)} + \xi_{(i,k)}^{(1,0)}\right)\xi_{,j}^{(0,2)k} + \left(h_{jk}^{(1,0)} + \xi_{(j,k)}^{(1,0)}\right)\xi_{,i}^{(0,2)k}\end{aligned} \tag{7.20}$$

$$\begin{aligned}\chi_{ij}^{(0,4)} \equiv & \, 2\left(h_{ij}^{(0,2)} + \xi_{(i,j)}^{(0,2)}\right)_{,k} \xi^{(0,2)k} + 2\left(h_{ik}^{(0,2)} + \xi_{(i,k)}^{(0,2)}\right)\xi_{,j}^{(0,2)k} \\ & +2\left(h_{jk}^{(0,2)} + \xi_{(j,k)}^{(0,2)}\right)\xi_{,i}^{(0,2)k} .\end{aligned} \tag{7.21}$$

Throughout this chapter we have defined χ, omitting indices, in the same way as in Ref. [128], such that it accounts for quadratic terms in the transformation of the metric.

From these transformations we comment on the original expansion of the metric, Eqs. (5.5)-(5.7). We began with an initial ansatz by adding perturbations at orders expected from post-Newtonian gravity and cosmological perturbation theory, then we added mixed-ordered perturbations at orders which were products of perturbations in the post-Newtonian and cosmological sector of the theory. However, through the gauge transformations in Eqs. (7.6) and (7.16) we see it is necessary to also include a metric potentials $h_{00}^{(1,1)}$ and $h_{ij}^{(1,1)}$, at order $\mathcal{O}(\epsilon\eta)$, because if they were excluded, then they would be generated automatically via a general infinitesimal coordinate transformation.

Before finishing this section, let us comment on the dependence of some of these terms, in the above transformations, on the condition $L_N \sim \eta L_C$. In the time-time transformation the only terms that depend on this relation are $h_{00,i}^{(0,2)}\xi^{(1,0)i}$ and $\dot{h}_{00}^{(0,2)}\xi^{(1,0)0}$ (see Eqs. (7.6) and (7.7)), which, once length scales are taken into account properly, appear at $\mathcal{O}(\epsilon\eta)$ and $\mathcal{O}(\epsilon\eta^2)$, respectively. If a different relationship between L_N and L_C had been chosen then these terms would have appeared at a different order, and could appear in any equation of order greater than or equal to $\epsilon\eta$ and $\epsilon\eta^2$, respectively, before violating the bound in Eq. (5.30). Similarly, in the transformation of the time-space and space-space components of the metric some of the terms in $\chi_i^{(1,2)}$ and $\chi_{ij}^{(1,2)}$ (see Eqs. (7.13) and (7.20)), and terms $4\frac{\dot{a}}{a}\xi^{(0,3)0}\delta_{ij}$ and $\chi_{ij}^{(1,1)}$ (see Eqs. (7.18) and (7.19)), all depend on the relationship between L_N and L_C, and would appear at different orders if a different choice had been made for these length scales.

7.2.2. Transformation of irreducibly-decomposed potentials

Having performed the gauge transformation of our metric components, in the previous section, we will proceed to perform an invariant decomposition of these results, as was done in Section 3.3.1. This will be useful for constructing gauge invariant quantities and writing down simplified field equations.

We split the metric into scalar, divergenceless vector ($V^i{}_{,i} = 0$), and transverse and trace-free tensor ($\hat{h}^i{}_i = 0$ and $\hat{h}^{ij}{}_{,j} = 0$) parts. These are the quantities that are most often considered in cosmological perturbation theory, and that usually decouple from each other at first-order in the field equations. We decompose our metric potentials into these variables in the following way, omitting superscripts for simplicity[4]:

$$h_{00} \equiv \phi\,, \qquad h_{0i} \equiv B_{,i} + B_i \qquad \text{and} \qquad h_{ij} \equiv -\psi\delta_{ij} + E_{,ij} + F_{(i,j)} + \tfrac{1}{2}\hat{h}_{ij}\,, \quad (7.22)$$

the former is related to the lapse, the $0i$-metric potential is known as the shift and the latter corresponds to perturbations to the spatial three-metric [131]. Similarly, our two-parameter perturbed gauge generators, omitting indices, are decomposed such that

$$\xi^0 \equiv \delta t \qquad \text{and} \qquad \xi^i \equiv \delta x^{,i} + \delta x^i\,, \tag{7.23}$$

in the same way as the gauge generator in single-parameter cosmological perturbation theory, see Eqs. (3.24) and (3.25). We will now present the result of gauge transformations on each of the irreducibly decomposed objects, in each of the sectors of our perturbation theory.

Cosmological scalar, vector and tensor potentials: the gauge transformations given in Eqs. (7.5), (7.11), and (7.15) now allow us to write down the transformation of the decomposed metric components in the cosmological sector of our theory. For the scalar potentials these transformations are given by

$$\tilde{\phi}^{(1,0)} = \phi^{(1,0)} - 2\dot{\delta t}^{(1,0)} \sim \epsilon \tag{7.24}$$

$$\tilde{\psi}^{(1,0)} = \psi^{(1,0)} - 2\frac{\dot{a}}{a}\delta t^{(1,0)} \sim \epsilon \tag{7.25}$$

[4]Note that the scalar, vector and tensor decomposition of the two-parameter perturbed metric is similar to that done for single parameter cosmological perturbation theory, in Eqs. (3.5) - (3.9), but we use a change of notation and our perturbations are defined using coordinate time.

$$\tilde{B}^{(1,0)} = B^{(1,0)} + a\dot{\delta x}^{(1,0)} - \frac{1}{a}\delta t^{(1,0)} \sim \epsilon\eta^{-1}L_N \tag{7.26}$$

$$\tilde{E}^{(1,0)} = E^{(1,0)} + 2\delta x^{(1,0)} \sim \epsilon\eta^{-2}L_N^2\,, \tag{7.27}$$

for the vector potentials they are

$$\tilde{B}_i^{(1,0)} = B_i^{(1,0)} + a\dot{\delta x_i}^{(1,0)} \sim \epsilon \tag{7.28}$$

$$\tilde{F}_i^{(1,0)} = F_i^{(1,0)} + 2\delta x_i^{(1,0)} \sim \epsilon\eta^{-1}L_N\,, \tag{7.29}$$

and for the tensor potential this transformation is

$$\tilde{\hat{h}}_{ij}^{(1,0)} = \hat{h}_{ij}^{(1,0)} \sim \epsilon\,. \tag{7.30}$$

As in previous sections, the quantity after the $\sim$ sign gives the order of each of these potentials in terms of ϵ, η and any relevant length scales. We observe that the transformations of the above cosmological scalar, vector and tensor potentials in our two-parameter formalism are the same as those derived from linear cosmological perturbation theory, perturbed in a single parameter, see Eqs. (3.28)-(3.34) and Ref. [128].

Post-Newtonian scalar, vector and tensor potentials: The results given in Eqs. (7.4), (7.8), (7.10), (7.14), and (7.18) allow us to write the transformation of the decomposed post-Newtonian potentials. The scalar parts of the post-Newtonian potentials transform as

$$\tilde{\phi}^{(0,2)} = \phi^{(0,2)} \sim \eta^2 \tag{7.31}$$

$$\tilde{\phi}^{(0,4)} = \phi^{(0,4)} - 4\dot{\delta t}^{(0,3)} + 2\phi_{,i}^{(0,2)}\left(\delta x^{(0,2),i} + \delta x^{(0,2)i}\right) \sim \eta^4 \tag{7.32}$$

$$\tilde{\psi}^{(0,2)} = \psi^{(0,2)} \sim \eta^2 \tag{7.33}$$

$$\tilde{\psi}^{(0,4)} = \psi^{(0,4)} - 4\frac{\dot{a}}{a}\delta t^{(0,3)} + \frac{1}{2}\left(\nabla^{-2}\chi_{ij}^{(0,4),ij} - \chi^{(0,4)}\right) \sim \eta^4 \tag{7.34}$$

$$\tilde{B}^{(0,3)} = B^{(0,3)} + a\dot{\delta x}^{(0,2)} - \frac{1}{a}\delta t^{(0,3)} \sim \eta^3 L_N \tag{7.35}$$

$$\tilde{E}^{(0,2)} = E^{(0,2)} + 2\delta x^{(0,2)} \sim \eta^2 L_N^2 \tag{7.36}$$

$$\tilde{E}^{(0,4)} = E^{(0,4)} + 2\delta x^{(0,4)} + \frac{1}{2}\nabla^{-2}\left(3\nabla^{-2}\chi_{ij}^{(0,4),ij} - \chi^{(0,4)}\right) \sim \eta^4 L_N^2\,, \tag{7.37}$$

the vector potentials transform as

$$\tilde{B}_i^{(0,3)} = B_i^{(0,3)} + a\delta\dot{x}_i^{(0,2)} \sim \eta^3 \tag{7.38}$$

$$\tilde{F}_i^{(0,2)} = F_i^{(0,2)} + 2\delta x_i^{(0,2)} \sim \eta^2 L_N \tag{7.39}$$

$$\tilde{F}_i^{(0,4)} = F_i^{(0,4)} + 2\delta x_i^{(0,4)} + 2\nabla^{-2}\left(\chi_{ik}^{(0,4),k} - \nabla^{-2}\chi_{kj,i}^{(0,4),kj}\right) \sim \eta^4 L_N\,, \tag{7.40}$$

and the tensor potentials transform as

$$\tilde{\hat{h}}_{ij}^{(0,2)} = \hat{h}_{ij}^{(0,2)} \sim \eta^2 \tag{7.41}$$

$$\begin{aligned} \tilde{\hat{h}}_{ij}^{(0,4)} &= \hat{h}_{ij}^{(0,4)} + 2\chi_{ij}^{(0,4)} - 4\nabla^{-2}\chi_{k(i,j)}^{(0,4),k} + \left(\nabla^{-2}\chi_{kl}^{(0,4),kl} - \chi^{(0,4)}\right)\delta_{ij} \\ &\quad +\nabla^{-2}\left(\nabla^{-2}\chi_{kl}^{(0,4),kl} + \chi^{(0,4)}\right)_{,ij} \\ &\sim \eta^4\,. \end{aligned} \tag{7.42}$$

The quantity $\chi_{ij}^{(0,4)}$, defined in Eq. (7.21), in terms of irreducibly decomposed potentials, can be written as

$$\begin{aligned} \chi_{ij}^{(0,4)} &= \\ &2\left(-\psi_{,k}^{(0,2)}\delta_{ij} + E_{,ijk}^{(0,2)} + F_{(i,j)k}^{(0,2)} + \frac{1}{2}\hat{h}_{ij,k}^{(0,2)} + \delta x_{,ijk}^{(0,2)} + \delta x_{(i,j)k}^{(0,2)}\right)\left(\delta x^{(0,2),k} + \delta x^{(0,2)k}\right) \\ &+2\left(-\psi^{(0,2)}\delta_{ik} + E_{,ik}^{(0,2)} + F_{(i,k)}^{(0,2)} + \frac{1}{2}\hat{h}_{ik}^{(0,2)} + \delta x_{,ik}^{(0,2)} + \delta x_{(i,k)}^{(0,2)}\right)\left(\delta x_{,j}^{(0,2),k} + \delta x_{,j}^{(0,2)k}\right) \\ &+2\left(-\psi^{(0,2)}\delta_{jk} + E_{,jk}^{(0,2)} + F_{(j,k)}^{(0,2)} + \frac{1}{2}\hat{h}_{jk}^{(0,2)} + \delta x_{,jk}^{(0,2)} + \delta x_{(j,k)}^{(0,2)}\right)\left(\delta x_{,i}^{(0,2),k} + \delta x_{,i}^{(0,2)k}\right). \end{aligned} \tag{7.43}$$

We have also written that $\chi^{(n,m)} \equiv \delta^{ij}\chi_{ij}^{(n,m)}$.

This completes the full set of transformations in the post-Newtonian sector. We note that the lowest order post-Newtonian metric potentials $\phi^{(0,2)}$ (from $h_{00}^{(0,2)}$) and $\psi^{(0,2)}$ do not transform. This is expected from post-Newtonian gravity [171] because the Newtonian potential, associated with $\phi^{(0,2)}$, is expected to not transform, and both scalar potentials ($\phi^{(0,2)}$ and $\psi^{(0,2)}$) are related by an equivalence relation, $\phi^{(0,2)} = -\psi^{(0,2)}$ (which is expected as $\gamma = 1$ in Eq. (4.30) for the post-Newtonian limit of general relativity). Moreover, if $\phi^{(0,2)}$ were to transform (from the background field equations in Section 6.2.1), we clearly see this would also transform the background scale factor a, which is precisely not what an infinitesimal gauge transformation is designed to do, a priori it transforms perturbations (and enables us to find degeneracies in the choice of such perturbations).

As far as we are aware, the transformation of scalar, vector and tensor post-Newtonian potentials has not been calculated before. The above transformations are derived from our two-parameter formalism, but because there are only post-Newtonian (not cosmological or mixed-order) potentials, and gauge generators, in these transformations they also hold for one-parameter post-Newtonian gravity.

Mixed-order scalar, vector and tensor potentials: the scalar parts of the mixed-order potentials, up to the order considered in the field equations presented in Chapter 6, $\mathcal{O}(\epsilon\eta^2)$, transform in the following way:

$$\tilde{\phi}^{(1,1)} = \phi^{(1,1)} + \phi^{(0,2)}_{,i}\left(\delta x^{(1,0),i} + \delta x^{(1,0)i}\right) \sim \epsilon\eta \tag{7.44}$$

$$\tilde{\phi}^{(1,2)} = \phi^{(1,2)} + \dot{\phi}^{(0,2)}\delta t^{(1,0)} + 2\phi^{(0,2)}\dot{\delta t}^{(1,0)} \sim \epsilon\eta^2 \tag{7.45}$$

$$\tilde{\psi}^{(1,1)} = \psi^{(1,1)} + \frac{1}{2}\left(\nabla^{-2}\chi^{(1,1),ij}_{ij} - \chi^{(1,1)}\right) \sim \epsilon\eta \tag{7.46}$$

$$\tilde{\psi}^{(1,2)} = \psi^{(1,2)} + \nabla^{-2}\left(\chi^{(1,2),k]l}_{k[l} + 2\mathcal{C}^{,|k]}_{k[l|,m}\mathcal{I}^{m,l}\right) \sim \epsilon\eta^2 \tag{7.47}$$

$$\tilde{B}^{(1,2)} = B^{(1,2)} + a\dot{\delta x}^{(1,1)} - \frac{1}{a}\delta t^{(1,2)} + \nabla^{-2}\chi^{(1,2),i}_{i} \sim \epsilon\eta^2 L_N \tag{7.48}$$

$$\tilde{E}^{(1,1)} = E^{(1,1)} + 2\delta x^{(1,1)} + \frac{1}{2}\nabla^{-2}(3\nabla^{-2}\chi^{(1,1),ij}_{ij} - \chi^{(1,1)}) \sim \epsilon\eta L_N^2 \tag{7.49}$$

$$\begin{aligned}\tilde{E}^{(1,2)} &= E^{(1,2)} + 2\delta x^{(1,2)} \\ &\quad + \frac{1}{2}\nabla^{-2}\left(\nabla^{-2}\left(3\chi^{(1,2),kl}_{kl} + 6\mathcal{C}^{;k}_{kl,m}\mathcal{I}^{m,l} - 2\mathcal{C}^{k}_{k,l}\mathcal{I}^{m,l}\right) - \chi^{(1,2)}\right) \\ &\sim \epsilon\eta^2 L_N^2\,,\end{aligned} \tag{7.50}$$

where we have used anti-symmetric square brackets that are defined by $2\mathcal{T}_{[ij]} \equiv \mathcal{T}_{ij} - \mathcal{T}_{ji}$. The vector parts transform as

$$\tilde{B}^{(1,2)}_i = B^{(1,2)}_i + a\dot{\delta x}^{(1,1)}_i + \chi^{(1,2)}_i - \nabla^{-2}\chi^{(1,2),j}_{j,i} \sim \epsilon\eta^2 \tag{7.51}$$

$$\tilde{F}^{(1,1)}_i = F^{(1,1)}_i + 2\delta x^{(1,1)}_i + 2\nabla^{-2}\left(\chi^{(1,1),k}_{ik} - \nabla^{-2}\chi^{(1,1),kj}_{kj,i}\right) \sim \epsilon\eta L_N \tag{7.52}$$

$$\begin{aligned}\tilde{F}^{(1,2)}_i &= F^{(1,2)}_i + 2\delta x^{(1,2)}_i \\ &\quad -2\nabla^{-2}\nabla^{-2}\left(2\chi^{(1,2),kl}_{k[i,l]} - 4\mathcal{C}^{,k}_{k[i,l]m}\mathcal{I}^{m,l} - \nabla^2\mathcal{C}_{ki,m}\mathcal{I}^{m,k} + \mathcal{C}^{;kl}_{kl,m}\mathcal{I}^{m}_{,i}\right) \\ &\sim \epsilon\eta^2 L_N\,,\end{aligned} \tag{7.53}$$

and the tensor parts transform as

$$\begin{aligned}\tilde{\hat{h}}_{ij}^{(1,1)} &= \hat{h}_{ij}^{(1,1)} + 2\chi_{ij}^{(1,1)} - 4\nabla^{-2}\chi_{k(i,j)}^{(1,1),k} + \nabla^{-2}\chi_{kl}^{(1,1),kl}\delta_{ij} - \chi^{(1,1)}\delta_{ij} \\ &\quad + \nabla^{-2}\nabla^{-2}\chi_{kl,ij}^{(1,1),kl} + \nabla^{-2}\chi_{,ij}^{(1,1)} \\ &\sim \epsilon\eta\end{aligned} \tag{7.54}$$

$$\begin{aligned}\tilde{\hat{h}}_{ij}^{(1,2)} &= \hat{h}_{ij}^{(1,2)} + 2\chi_{ij}^{(1,2)} - 4\nabla^{-2}\chi_{k(i,j)}^{(1,2),k} + \nabla^{-2}\chi_{kl}^{(1,2),kl}\delta_{ij} - \chi^{(1,2)}\delta_{ij} \\ &\quad + \nabla^{-2}\nabla^{-2}\chi_{kl,ij}^{(1,2),kl} + \nabla^{-2}\chi_{,ij}^{(1,2)} + 4\nabla^{-2}\nabla^{-2}\left(\nabla^2\mathcal{C}_{ij,mk}\mathcal{I}^{m,k}\right. \\ &\quad \left. -\nabla^2\mathcal{C}_{k(i,j)m}\mathcal{I}^{m,k} - 2\mathcal{C}_{k(i,j)klm}\mathcal{I}^{m,l} - \nabla^2\mathcal{C}_{k(i|,m}^{,k}\mathcal{I}_{,|j)}^{m} + \mathcal{C}_{kl,mn}^{,k(l|}\mathcal{I}^{m,|n)}\delta_{ij}\right) \\ &\quad + \nabla^{-2}\nabla^{-2}\left(-\nabla^2\mathcal{C}_{k,ml}^{k}\mathcal{I}^{m,l}\delta_{ij} + 2\mathcal{C}_{kl,mij}^{,k}\mathcal{I}^{m,l} + 2\mathcal{C}_{kl,m(i}^{,kl}\mathcal{I}_{,j)}^{m} + 2\mathcal{C}_{ij,mk}\mathcal{I}^{m,k}\right) \\ &\sim \epsilon\eta^2\,.\end{aligned} \tag{7.55}$$

Note that in the above equations we define $\nabla^{-2}f(\chi^{(n,m)})$ such that $\nabla^2[\nabla^{-2}f(\chi^{(n,m)})]$ is the leading order part of $f(\chi^{(n,m)})$ and no smaller, which strictly excludes higher order terms in $f(\chi^{(n,m)})$. In the above equations we have written $\chi_i^{(1,2)}$, $\chi_{ij}^{(1,2)}$ and $\chi_{ij}^{(1,1)}$ in terms of scalar, vector and tensor potentials and $\chi_{ij}^{(1,1)}$ in terms of $\mathcal{C}_{ij,m}$ and $\mathcal{I}^m$ in the following way, firstly we have

$$\begin{aligned}&\chi_i^{(1,2)} = \\ &+a\left(-\psi^{0,2}\delta_{ij} + E_{,ij}^{(0,2)} + F_{(i,j)}^{(0,2)} + \frac{1}{2}\hat{h}_{ij}^{(0,2)} + \delta x_{,ij}^{(0,2)} + \delta x_{(i,j)}^{(0,2)}\right)\left(\delta x^{(1,0),j} + \delta x^{(1,0)j}\right)^{\cdot} \\ &+\left(B_{,i}^{(0,3)} + B_i^{(0,3)} - \frac{1}{2a}\delta t_{,i}^{(0,3)} + \frac{a}{2}\left(\delta x_{,i}^{(0,2)} + \delta x_i^{(0,2)}\right)^{\cdot}\right)_{,j}\left(\delta x^{(1,0),j} + \delta x^{(1,0)j}\right) \\ &+\left(B_{,j}^{(1,0)} + B_j^{(1,0)} - \frac{1}{2a}\delta t_{,j}^{(1,0)} + \frac{a}{2}\left(\delta x_{,j}^{(1,0)} + \delta x_j^{(1,0)}\right)^{\cdot}\right)\left(\delta x^{(0,2),j} + \delta x^{(0,2)j}\right)_{,i} \\ &\frac{1}{a}\phi^{(0,2)}\delta t_{,i}^{(1,0)}\,,\end{aligned} \tag{7.56}$$

we also have

$$\begin{aligned}&\chi_{ij}^{(1,2)} = \\ &\left(-\psi^{(0,2)}\delta_{ij} + E_{,ij}^{(0,2)} + F_{(i,j)}^{(0,2)} + \frac{1}{2}\hat{h}_{ij}^{(0,2)} + \delta x_{,ij}^{(0,2)} + \delta x_{(i,j)}^{(0,2)}\right)^{\cdot}\delta t^{(1,0)} \\ &+2\frac{\dot{a}}{a}\left(-\psi^{(0,2)}\delta_{ij} + E_{,ij}^{(0,2)} + F_{(i,j)}^{(0,2)} + \frac{1}{2}\hat{h}_{ij}^{(0,2)} + 2\delta x_{,ij}^{(0,2)} + 2\delta x_{(i,j)}^{(0,2)}\right)\delta t^{(1,0)}\end{aligned} \tag{7.57}$$

$$+\left(-\psi^{(0,2)}\delta_{ik}+E^{(0,2)}_{,ik}+F^{(0,2)}_{(i,k)}+\frac{1}{2}\hat{h}^{(0,2)}_{ik}+\delta x^{(0,2)}_{,ik}+\delta x^{(0,2)}_{(i,k)}\right)\left(\delta x^{(1,0),k}+\delta x^{(1,0)k}\right)_{,j}$$
$$+\left(-\psi^{(0,2)}\delta_{jk}+E^{(0,2)}_{,jk}+F^{(0,2)}_{(j,k)}+\frac{1}{2}\hat{h}^{(0,2)}_{jk}+\delta x^{(0,2)}_{,jk}+\delta x^{(0,2)}_{(j,k)}\right)\left(\delta x^{(1,0),k}+\delta x^{(1,0)k}\right)_{,i}$$
$$+\left(-\psi^{(1,0)}\delta_{ik}+E^{(1,0)}_{,ik}+F^{(1,0)}_{(i,k)}+\frac{1}{2}\hat{h}^{(1,0)}_{ik}+\delta x^{(1,0)}_{,ik}+\delta x^{(1,0)}_{(i,k)}\right)\left(\delta x^{(0,2),k}+\delta x^{(0,2)k}\right)_{,j}$$
$$+\left(-\psi^{(1,0)}\delta_{jk}+E^{(1,0)}_{,jk}+F^{(1,0)}_{(j,k)}+\frac{1}{2}\hat{h}^{(1,0)}_{jk}+\delta x^{(1,0)}_{,jk}+\delta x^{(1,0)}_{(j,k)}\right)\left(\delta x^{(0,2),k}+\delta x^{(0,2)k}\right)_{,i}$$

and finally

$$\chi^{(1,1)}_{ij} = \mathcal{C}_{ij,k}\mathcal{I}^k\,, \tag{7.58}$$

where we have defined

$$\mathcal{C}_{ij,k} \equiv \left(-\psi^{(0,2)}\delta_{ij}+E^{(0,2)}_{,ij}+F^{(0,2)}_{(i,j)}+\frac{1}{2}\hat{h}^{(0,2)}_{ij}+\delta x^{(0,2)}_{,ij}+\delta x^{(0,2)}_{(i,j)}\right)_{,k} \tag{7.59}$$
$$\sim \eta^2 L_N^{-1}$$
$$\mathcal{I}^k \equiv \delta x^{(1,0),k}+\delta x^{(1,0)k} \sim \epsilon\eta^{-1}L_N. \tag{7.60}$$

The transformation of the above mixed-order quantities are purely a result of our two parameter formalism.

This completes our treatment of gauge transformations of the metric tensor. These transformations are original results and will be used in Section 7.5 to construct gauge invariant potentials.

7.3. Transformation of matter sources

The same freedoms, associated with infinitesimal coordinate transformations, can also be considered in the context of the total energy-momentum tensor. In the following we calculate how this tensor behaves under the gauge transformation specified in Eq. (3.26) and by the gauge generators in Eqs. (7.1) and (7.2). As before, we will first calculate the explicit transformations that apply to the components of the energy-momentum tensor, and then to their irreducibly decomposed scalar, vector and tensor parts. Again, we take $L_N \sim \eta L_C$, but not $\epsilon \sim \eta^2$.

7.3.1. Transformations of components

The transformation of $\mathbf{T}_{00}$: using the exponential map in Eq. (3.26), and the gauge generators specified in Eqs. (7.1) and (7.2), we find the transformation of T_{00} at $\mathcal{O}\left(\eta^2 L_N^{-2}\right)$ and $\mathcal{O}\left(\epsilon\eta L_N^{-2}\right)$ are given by

$$\tilde{\rho}^{(0,0)} + \tilde{\rho}^{(0,2)} = \rho^{(0,0)} + \rho^{(0,2)} \sim \frac{\eta^2}{L_N^2} \tag{7.61}$$

$$\tilde{\rho}^{(1,1)} = \rho^{(1,1)} + \rho^{(0,2)}_{,i}\xi^{(1,0)i} \sim \frac{\epsilon\eta}{L_N^2}\,. \tag{7.62}$$

At order $\mathcal{O}\left(\epsilon\eta^2 L_N^{-2}\right)$ the transformation of T_{00} is given by

$$\begin{aligned} & \tilde{\rho}^{(1,0)} + \tilde{\rho}^{(1,2)} - \tilde{h}^{(1,0)}_{00}\left(\tilde{\rho}^{(0,0)} + \tilde{\rho}^{(0,2)}\right) \\ = \; & \rho^{(1,0)} + \rho^{(1,2)} + \left(2\dot{\xi}^{(1,0)0} - h^{(1,0)}_{00}\right)\left(\rho^{(0,0)} + \rho^{(0,2)}\right) + \left(\rho^{(0,0)} + \rho^{(0,2)}\right)^{\cdot}\xi^{(1,0)0} \\ \sim \; & \frac{\epsilon\eta^2}{L_N^2}\,, \end{aligned} \tag{7.63}$$

and at $\mathcal{O}\left(\eta^4 L_N^{-2}\right)$ the transformation of T_{00} is given by

$$\begin{aligned} & \frac{1}{2}\tilde{\rho}^{(0,4)} - \tilde{h}^{(0,2)}_{00}\left(\tilde{\rho}^{(0,0)} + \tilde{\rho}^{(0,2)}\right) + \left(\tilde{\rho}^{(0,0)} + \tilde{\rho}^{(0,2)} + \tilde{p}^{(0,0)}\right)\tilde{v}^{(0,1)i}\tilde{v}^{(0,1)}_i \\ = \; & \frac{1}{2}\rho^{(0,4)} - h^{(0,2)}_{00}\left(\rho^{(0,0)} + \rho^{(0,2)}\right) + \left(\rho^{(0,0)} + \rho^{(0,2)} + p^{(0,0)}\right)v^{(0,1)i}v^{(0,1)}_i + \rho^{(0,2)}_{,i}\xi^{(0,2)i} \\ \sim \; & \frac{\eta^4}{L_N^2}\,. \end{aligned} \tag{7.64}$$

We now make several comments on the above transformations. Firstly, from Eq. (7.61) we see $\rho^{(0,0)}$ transforms in a gauge invariant way, this is consistent with intuition from the Stewart-Walker lemma [157]. We also see both $\rho^{(0,2)}$ and $\rho^{(0,0)}$ transform together because they both dominate the expansion at leading-order due to $\sim L_C^{-2} \sim \eta^2 L_N^{-2}$. This is somewhat different to cosmological perturbation theory where only the homogeneous $\rho^{(0)}(t)$ is gauge invariant.

Also, we note that one further term is generated by the gauge transformation, given by Eq. (3.26), in this part of the two-parameter perturbed energy-momentum tensor: $T^{(2,0)}_{\mu\nu} \sim \frac{1}{2}\rho^{(0,2)}_{,ij}\xi^{(1,0)i}\xi^{(1,0)j} \sim \epsilon^2 L_N^{-2}$. This term would appear in the $\mathcal{O}\left(\eta^4 L_N^{-2}\right)$ field equation along with $R^{(2,0)}_{\mu\nu} \sim \epsilon^2 L_N^{-2}$, see the term in (7.9)[5]. We explain what

[5]Such a term only appears if dust is considered. An analogous term in the gauge transformation for radiation would have no contribution because $\rho^{(0,0)}(t)_{,ij}\xi^{(1,0)i}\xi^{(1,0)j} = 0$.

happens to the terms of order $\mathcal{O}(\epsilon^2 L_N^{-2})$ in the discussion section of Chapter 8.

Finally, the expansion of the radiation energy density, given in Eq. (5.21), is such that there does not exist a term $\rho_R^{(1,1)}$. We remind ourselves that the existence of $\rho_M^{(1,1)}$ was implied by the term $\rho_{M,i}^{(0,2)}\xi^{(1,0)i}$, given in the transformation in Eq. (7.62), because if $\rho_M^{(1,1)}$ did not exist from the offset, it would be generated via any infinitesimal change in coordinates. So, the term $\rho_M^{(1,1)}$ appears purely because $\rho_M^{(0,2)}$ is a function space and time. In contrast, the corresponding term $\rho_R^{(1,1)}$ does not exist in the expansion of ρ_R because ρ_R at lowest order is only time, not space-time, dependent (see Eq. (5.25) and comments below it).

We will comment in detail on the form of the transformations of the energy density, pressure and peculiar velocities in the following section. For now, we analyse how the other components of the energy-momentum tensor transform.

The transformation of $\mathbf{T}_{0i}$: under the gauge transformation, Eq. (3.26), the time-space part of the energy-momentum tensor at $\mathcal{O}\left(\eta^3 L_N^{-2}\right)$ transforms in the following way

$$\begin{aligned} -a\left(\tilde{\rho}^{(0,0)}+\tilde{\rho}^{(0,2)}+\tilde{p}^{(0,0)}\right)\tilde{v}_i^{(0,1)} &= -a\left(\rho^{(0,0)}+\rho^{(0,2)}+p^{(0,0)}\right)v_i^{(0,1)} \\ &\sim \frac{\eta^3}{L_N^2}\,, \end{aligned} \tag{7.65}$$

and at $\mathcal{O}\left(\epsilon\eta^2 L_N^{-2}\right)$ it transforms such that

$$\begin{aligned} & -a\left(\tilde{\rho}^{(0,0)}+\tilde{\rho}^{(0,2)}\right)\left(\tilde{v}_i^{(1,0)}+\tilde{h}_{0i}^{(1,0)}\right)-a\tilde{p}^{(0,0)}\tilde{v}_i^{(1,0)}-a\tilde{\rho}^{(1,1)}\tilde{v}_i^{(0,1)} \\ = & -a\left(\rho^{(0,0)}+\rho^{(0,2)}\right)\left(v_i^{(1,0)}+h_{0i}^{(1,0)}\right)-ap^{(0,0)}v_i^{(1,0)}-a\rho^{(1,1)}v_i^{(0,1)} \\ & +\left(\rho^{(0,0)}+\rho^{(0,2)}\right)\xi_{,i}^{(1,0)0}+a^2p^{(0,0)}\dot{\xi}_i^{(1,0)}-a\rho_{,j}^{(0,2)}v_i^{(0,1)}\xi^{(1,0)j} \\ & -a\left(\rho^{(0,0)}+\rho^{(0,2)}+p^{(0,0)}\right)v_{i,j}^{(0,1)}\xi^{(1,0)j} \\ \sim & \frac{\epsilon\eta^2}{L_N^2}\,. \end{aligned} \tag{7.66}$$

The transformation of $\mathbf{T}_{ij}$: the gauge transformation of the space-space component of the energy-momentum tensor gives

$$a^2\tilde{p}^{(0,0)}\delta_{ij} = a^2\delta_{ij}p^{(0,0)} \sim \frac{\eta^2}{L_N^2}\,, \tag{7.67}$$

at $\mathcal{O}\left(\eta^2 L_N^{-2}\right)$. At order $\mathcal{O}\left(\eta^4 L_N^{-2}\right)$ the transformation is

$$
\begin{aligned}
& a^2\left(\tilde{\rho}^{(0,0)}+\tilde{\rho}^{(0,2)}+\tilde{p}^{(0,0)}\right) \tilde{v}_i^{(0,1)} \tilde{v}_j^{(0,1)}+a^2 \tilde{p}^{(0,0)} \tilde{h}_{ij}^{(0,2)}+\frac{1}{2} a^2 \tilde{p}^{(0,4)} \delta_{ij} \\
= & a^2\left(\rho^{(0,0)}+\rho^{(0,2)}+p^{(0,0)}\right) v_i^{(0,1)} v_j^{(0,1)}+a^2 p^{(0,0)} h_{ij}^{(0,2)}+\frac{1}{2} a^2 p^{(0,4)} \delta_{ij}+2 a^2 p^{(0,0)} \xi_{(i,j)}^{(0,2)} \\
\sim & \frac{\eta^4}{L_N^2},
\end{aligned}
\tag{7.68}
$$

and at order $\mathcal{O}\left(\epsilon\eta^2 L_N^{-2}\right)$ the transformation is

$$
\begin{aligned}
& a^2 \tilde{p}^{(0,0)} \tilde{h}_{ij}^{(1,0)}+a^2\left(\tilde{p}^{(1,0)}+\tilde{p}^{(1,2)}\right) \delta_{ij} \\
= & a^2 p^{(0,0)} h_{ij}^{(1,0)}+a^2\left(p^{(1,0)}+p^{(1,2)}\right) \delta_{ij}+a^2 \dot{p}^{(0,0)} \xi^{(1,0)0} \delta_{ij}+2 a^2 p^{(0,0)} \xi_{(i,j)}^{(1,0)} \\
\sim & \frac{\epsilon \eta^2}{L_N^2}.
\end{aligned}
\tag{7.69}
$$

Again we note that $p^{(0,0)}$, the lowest order contribution to the pressure that is purely due to radiation, is gauge invariant because this is the lowest-order contributions to the energy density.

The transformation of the cosmological constant: finally, using the exponential map in Eq. (3.26), and the gauge generators specified in Eqs. (7.1) and (7.2) we find that because $\Lambda^{(0,0)}$ is constant in time and space $\Lambda^{(0,0)}$ does not transform

$$
\Lambda^{(0,0)} \mapsto \tilde{\Lambda}^{(0,0)} = \Lambda^{(0,0)}\,, \tag{7.70}
$$

again this is expected from the Stewart-Walker lemma and standard cosmological perturbation theory.

7.3.2. Transformation of irreducibly-decomposed matter sources

We now irreducibly decompose the matter sources that appear on the right-hand-side of Einstein's field equations, Eq. (2.15). The cosmological constant is scalar and therefore does not need to be decomposed further, the transformation is given above in Eq. (7.70). The total energy momentum tensor is made up of scalars, with the only exception of the three-velocity, v_i. This vector can be split into a scalar and divergenceless vector part in the following way:

$$v_i \equiv v_{,i} + \hat{v}_i \,, \tag{7.71}$$

where $\hat{v}^i{}_{,i} = 0$. The scalar degrees of freedom in the metric are then given by ρ, p, v and Λ, while the only divergenceless vector is given by $\hat{v}_i$. There are no transverse and trace-free potentials in the energy-momentum tensor, as defined in Eq. (2.20).

We now consider the transformation of these scalar and vector quantities. From the above section we find that the irreducible decomposed energy density transformation, in dimensions of L_N where $L_N \sim \eta L_C$, is such that

$$\begin{aligned}
\tilde{\rho}^{(0,0)} + \tilde{\rho}^{(0,2)} &= \rho^{(0,0)} + \rho^{(0,2)} \sim \frac{\eta^2}{L_N^2} && (7.72)\\
\tilde{\rho}^{(1,1)} &= \rho^{(1,1)} + \rho^{(0,2)}_{,i}\left(\delta x^{(1,0),i} + \delta x^{(1,0)i}\right) \sim \frac{\epsilon\eta}{L_N^2} && (7.73)\\
\tilde{\rho}^{(1,0)} + \tilde{\rho}^{(1,2)} &= \rho^{(1,0)} + \rho^{(1,2)} + \left(\rho^{(0,0)} + \rho^{(0,2)}\right)^{\cdot}\delta t^{(1,0)} \sim \frac{\epsilon\eta^2}{L_N^2} && (7.74)\\
\tilde{\rho}^{(0,4)} &= \rho^{(0,4)} + 2\rho^{(0,2)}_{,i}\left(\delta x^{(0,2),i} + \delta x^{(0,2)i}\right) \sim \frac{\eta^4}{L_N^2}\,, && (7.75)
\end{aligned}$$

and the irreducible decomposed pressure transformation is such that

$$\begin{aligned}
\tilde{p}^{(0,0)} &= p^{(0,0)} \sim \frac{\eta^2}{L_N^2} && (7.76)\\
\tilde{p}^{(1,0)} + \tilde{p}^{(1,2)} &= p^{(1,0)} + p^{(1,2)} + \dot{p}^{(0,0)}\delta t^{(1,0)0} - 2\frac{\dot{a}}{a}p^{(0,0)}\delta t^{(1,0)0} \sim \frac{\epsilon\eta^2}{L_N^2} && (7.77)\\
\tilde{p}^{(0,4)} &= p^{(0,4)} \sim \frac{\eta^4}{L_N^2}\,. && (7.78)
\end{aligned}$$

We now make further comments on the form of the transformation of the energy-density and pressure which have not been observed previously. The transformations in Eqs. (7.73), (7.75) and (7.78), for matter and radiation, remain exactly the same with matter only, see Appendix C. Moreover, the post-Newtonian energy density $\rho^{(0,2)}$ and pressure $p^{(0,4)}$ are automatically gauge invariant. This is to be expected because they are the leading order post-Newtonian perturbations to the energy density and pressure, respectively. These terms describe Newtonian gravity at leading order, which transforms trivially under general coordinate transformations.

Whereas, all other transformations of the energy density and pressure change with the inclusion of radiation, compared to with matter alone. We have previously noted that the term $\rho^{(0,0)}$ transforms with the Newtonian energy density because it has magnitude $L_C^{-2} \sim \eta^2 L_N^2$. Similarly, $\rho^{(0,0)}$ appears alongside $\rho^{(0,2)}$ in the transformation in Eq. (7.74). Furthermore, as seen in Eq. (7.77), with the inclusion of radiation there exists a gauge invariant pressure term, $p^{(0,0)}$, and so the transformation of the cosmological and mixed order perturbations to the pressure are not gauge invariant, this is not the case for matter only, see Appendix C.

Furthermore, the irreducibly decomposed peculiar velocities transform in the following way

$$\tilde{v}_i^{(1,0)} = v_i^{(1,0)} - a\left(\delta x^{(1,0),i} + \delta x^{(1,0)i}\right)^{\cdot} + v_{i,j}^{(0,1)}\left(\delta x^{(1,0),j} + \delta x^{(1,0)j}\right) \sim \epsilon \quad (7.79)$$

$$\tilde{v}_i^{(0,1)} = v_i^{(0,1)} \sim \eta\,. \quad (7.80)$$

Where the transformation of the scalar part of the three-velocity, $\tilde{v}$, and the divergenceless vector part, $\tilde{\hat{v}}_i$, are derived from the divergence of Eqs. (7.79) and (7.80). We find the transformations in Eqs. (7.79) and (7.80) are the same for matter and radiation as for matter only, see Appendix C. We see Eq. (7.80) is gauge invariant, therefore both scalar and vector parts of the Newtonian three-velocity are gauge invariant. The quadratic term that appears in Eq. (7.79) shows that the small-scale Newtonian velocity is important for determining how the large-scale velocity (first-order in cosmological perturbations) transforms – this is a by-product of our two-parameter expansion and would otherwise only appear at second-order in cosmological perturbation theory.

These results differ from the quasi-static limit of cosmological perturbation theory, as space and time derivatives are treated differently, and gauge generators and velocities come in at different orders [107]. Note that, other than $\rho^{(1,1)}$, there are no more perturbations at new orders implied by the transformation of the perturbed matter sources. Therefore we have a consistent set of perturbed quantities in the matter sector. This completes our study of the gauge transformation of stress energy sources.

7.4. Allowed gauge choices

The above gauge transformations of the matter and gravity sectors of our two-parameter theory implies which gauges can be used to study cosmological perturbation theory and post-Newtonian gravity, and therefore our two-parameter theory. We find that five out of the six gauges traditionally used in studies of cosmological perturbation theory, and listed in Section 3.3.3, are not appropriate for studies of post-Newtonian gravity or the quasi-static limit of cosmological perturbation theory (as post-Newtonian gravity can be thought of as formalising the quasi-static limit), and therefore are not appropriate gauges for the post-Newtonian sector of our two-parameter theory. This can be seen directly from the transformations of the post-Newtonian sector of our theory, given in Eqs. (7.31)-(7.42), (7.72), (7.75), (7.78) and (7.80).

We will proceed by turning to each of the six gauge choices from Section 3.3.3 and discuss which are not appropriate gauge choices for post-Newtonian gravity, at leading-order in perturbations, and why. We start with the spatially flat gauge, which would require $\psi^{(0,2)} = 0$. From Eq. (7.31) we observe this potential is in fact gauge-invariant and therefore there is no gauge in which it is zero. If we do make this potential zero we lose generality. Next, consider the synchronous gauge which requires $\phi^{(0,2)} = 0$, but we know this potential is gauge-invariant, from Eq. (7.33), therefore we cannot undergo any possible gauge transformation which allows it to be zero. A similar argument can be used to explain why the comoving orthogonal gauge, which would require $v^{(0,1)} = 0$, is not appropriate for post-Newtonian potentials: we cannot transform to a gauge where $v^{(0,1)} = 0$ because it does not transform under a gauge transformation (see Eq. (7.80)). The uniform density gauge requires that inhomogeneous perturbations are zero, in post-Newtonian gravity inhomogeneous perturbations are at leading-order (Newtonian-order) and therefore this gauge would require $\rho^{(0,2)} - \overline{\rho}^{(0,2)} = 0$, where $\overline{\rho}^{(0,2)}$ is the spatial average of $\rho^{(0,2)}$. We cannot undergo a gauge transformations which allows for this because the Newtonian energy-density is gauge-invariant, which is apparent from Eq. (7.72) (and its average), see Section 8.2.1.

We can undergo an infinitesimal gauge transformation which allows post-Newtonian perturbations to transform into the Newtonian (or longitudinal) gauge. This is because the Newtonian gauge requires $B^{(0,3)} = 0$, $E^{(0,2)} = 0$, and $F_i^{(0,2)} = 0$ or $B_i^{(0,3)} = 0$, and all these perturbations ($B^{(0,3)}, E^{(0,2)}, F_i^{(0,2)}$ and $B_i^{(0,3)}$) transform under a general gauge transformation (see Eqs. (7.35), (7.36), (7.39) and (7.38), respectively). Therefore we can transform to a gauge where these perturbations are zero with a specific choice of gauge generators $\delta x^{(0,2)}$, $\delta x^{(0,2)i}$ and $\delta t^{(0,3)}$. Finally, we turn to the total matter gauge, which requires $v + B = 0$. In cosmological perturbation theory, this condition holds because both v and B occur at the same orders in perturbations, and so this condition fixes one degree of freedom. For post-Newtonian gravity, however, they do not occur at the same order in perturbations therefore this condition fixes two degrees of freedom, $v^{(0,1)} = 0$ and $B^{(0,3)} = 0$, one degree of freedom more than is required. Moreover, we cannot transform to a gauge in which $v^{(0,1)} = 0$ (see Eq. (7.80)), as observed previously with regards to the comoving orthogonal gauge. One could consider the case in which $v^{(0,1)} \neq 0$, but $B^{(0,3)} = 0$ and the other conditions for the total matter gauge hold (*i.e.* $E = 0$ and $F_i = 0$). These conditions are then simply equivalent to those for the Newtonian gauge.

The spatially flat, synchronous, comoving orthogonal, total matter and uniform density gauges for post-Newtonian perturbations at leading-order are not applicable because they correspond to conditions which cannot be fulfilled by any possible infinitesimal gauge transformation. Therefore, by using the conditions necessary for these gauges at leading-order in post-Newtonian potentials one would have to lose generality by setting a physical degree of freedom to zero. Nevertheless, it has been found in the literature that such gauges have been used in either post-Newtonian gravity or the quasi-static limit of cosmological perturbation theory at leading-order [13, 166]. The implication of this finding is that N-body simulations which go beyond Newtonian theory, to include relativistic corrections due to general relativity, are limited to the Newtonian gauge for leading-order perturbations (from the list of typical gauge choices in Section 3.3.3). These other gauges may, however, be appropriate for beyond leading-order post-Newtonian gravity. For example, the synchronous gauge for post-Newtonian perturbations at order $\mathcal{O}(\eta^4)$, requires that $\phi^{(0,4)} = 0$. Unlike $\phi^{(0,2)}$, we can transform $\phi^{(0,4)}$ to a gauge where $\phi^{(0,4)} = 0$ (see Eq. (7.32)) by a specific choice of gauge generators $\delta t^{(0,3)}$, $\delta x^{(0,2)}$ and $\delta x^{(0,2)i}$.

As these gauges are not appropriate for studies of post-Newtonian gravity at leading-order, they are also not appropriate gauges for the post-Newtonian perturbations at leading-order in our two-parameter theory. Nevertheless, all six gauges in Section 3.3.3 remain valid for first order (and beyond) cosmological perturbation theory, and therefore cosmological perturbations in our two-parameter theory. Additionally, it would need to be carefully checked whether these gauges may be applied to beyond leading-order post-Newtonian perturbations in our two-parameter theory. Therefore the only gauge listed in Section 3.3.3 relevant for both leading-order perturbations in the post-Newtonian and cosmological sectors of our two-parameter theory is the Newtonian gauge.

7.5. Gauge invariant quantities

Having performed infinitesimal coordinate transformations of the metric and matter sources, we are now in a position to isolate and remove the superfluous degrees of freedom associated with diffeomorphism invariance. This will leave us with a set of quantities that represent the physical degrees of freedom in the problem only, and will remove the possibility of any interference from spurious gauge modes. The field equations written in terms of these quantities are greatly simplified, see Chapter 8.

Dealing with gauge freedoms can be done in a number of different ways, and is often approached differently in the respective literatures associated with post-Newtonian gravity [171] and cosmological perturbation theory [128]. In post-Newtonian gravity, the usual method is to make a gauge choice by setting the sum of various parts of the perturbed field equations to zero. If suitable choices are made, and if they can be shown to be self-consistent, then this method can be used to remove all gauge freedoms. This approach has the distinct benefit of allowing maximum simplification of the field equations, making these equations easier to solve, and the entire problem more tractable. However, it also has the drawback that one has to determine what is, or is not, a suitable choice of terms to eliminate from the field equations. This can sometimes be a challenge.

On the other hand, in the literature on cosmological perturbation theory a gauge choice is most usually made by irreducibly decomposing the metric and energy-momentum tensor, and then by setting some of the resulting terms to zero directly [128]. This leaves a more complicated set of field equations compared to post-Newtonian gravity, described in the previous paragraph, but does allow for the maximum possible simplification of the basic objects involved in the problem. Even in this case, however, it is still possible to leave behind residual gauge freedoms, if inappropriate choices are made. These problems were circumvented by Bardeen, who was the first to construct combinations of perturbations that remained invariant under general gauge transformations [32]. Furthermore, extensions of this have been applied to calculations of second-order gauge invariant quantities in cosmological perturbation theory [128]. This removed all ambiguity, and allowed perturbed field equations to be written down that were guaranteed to be free from all gauge freedoms.

We choose to use the latter of these two approaches, to construct gauge invariant quantities associated with the perturbations to metric and energy-momentum tensors. This involves extending the method pioneered by Bardeen to post-Newtonian perturbations, as well as using some of the extensions of this method developed for use in second-order cosmological perturbation theory [128]. By the end of this chapter we will have written down gauge-invariant quantities for all of the perturbations in our two-parameter theory. We will then write the differential equations that govern them.

7.5.1. Gauge-invariant metric perturbations

Let us begin by constructing gauge-invariant quantities from the irreducibly decomposed metric tensor. The method we will use to do this is based on that developed for single-parameter cosmological perturbation theory [128], and will be such that our gauge invariant quantities reduce to the metric perturbations in longitudinal gauge when $E = B = F_i = 0$ (we omit superscript indices here for simplicity). We note that other gauge choices are possible; we make this choice for two reasons. Firstly, it is the only gauge, out of the possible gauges traditionally used in cosmological perturbation theory, listed in Section 3.3.3, that can simultaneously be applied at leading-order to the cosmological and post-Newtonian perturbations in our two-parameter theory, see Section 7.4. Secondly, this choice means the resulting field equations look similar to those traditionally used in post-Newtonian gravity.

The procedure we will use for this will be to choose gauge generators, $\delta x, \delta x^i$ and δt, such that $\tilde{E} = \tilde{B} = \tilde{F}_i = 0$. We will then substitute these quantities back into the expressions for all of the transformed perturbations presented in Sections 7.2.2 and 7.3.2. The results now correspond to gauge invariant quantities because the original gauge transformations were written down in a completely arbitrary coordinate system. This means that newly constructed quantities cannot depend on any choice of gauge, and hence must be gauge invariant.

Below we present our results for the cosmological sector, the post-Newtonian sector, and the mixed-order sector of our expansion. All quantities have been checked, by a somewhat lengthy explicit transformation, to ensure that they are in fact gauge invariant.

Cosmological quantities: in the cosmological sector we can create several gauge invariant quantities. They are of the form of two independent scalars, one vector and one tensor. These are given by:

$$\Phi^{(1,0)} = \phi^{(1,0)} - 2a\dot{B}^{(1,0)} - 2\dot{a}B^{(1,0)} + 2\dot{a}a\dot{E}^{(1,0)} + a^2\ddot{E}^{(1,0)} \tag{7.81}$$

$$\Psi^{(1,0)} = \psi^{(1,0)} + \dot{a}a\dot{E}^{(1,0)} - 2\dot{a}B^{(1,0)} \tag{7.82}$$

$$\mathbf{B}_i^{(1,0)} = B_i^{(1,0)} - \frac{a}{2}\dot{F}_i^{(1,0)} \tag{7.83}$$

$$\mathbf{h}^{(1,0)}_{ij} = \hat{h}^{(1,0)}_{ij}\,, \tag{7.84}$$

which are all at $\mathcal{O}(\epsilon)$. The scalar gauge invariant quantities are identical to those found by Bardeen, see Eqs. (3.40) and (3.41), in the context of standard cosmological perturbation theory [32].

Post-Newtonian quantities: in the post-Newtonian sector, at $\mathcal{O}(\eta^2)$, we can create two scalar, and one tensor, gauge invariant quantities:

$$\Phi^{(0,2)} = \phi^{(0,2)} \tag{7.85}$$

$$\Psi^{(0,2)} = \psi^{(0,2)} \tag{7.86}$$

$$\mathbf{h}^{(0,2)}_{ij} = \hat{h}^{(0,2)}_{ij}\,. \tag{7.87}$$

At $\mathcal{O}(\eta^3)$ there exists one gauge invariant vector,

$$\mathbf{B}^{(0,3)}_{i} = B^{(0,3)}_{i} - \frac{a}{2}\dot{F}^{(0,2)}_{i}\,, \tag{7.88}$$

while at $\mathcal{O}(\eta^4)$ the gauge invariant quantities are two scalars and one tensor,

$$\begin{aligned}\Phi^{(0,4)} &= \phi^{(0,4)} - 4a\dot{B}^{(0,3)} - 4\dot{a}B^{(0,3)} + 4\dot{a}a\dot{E}^{(0,2)} + 2a^2\ddot{E}^{(0,2)} \\ &\quad -\phi^{(0,2)}{}_{,i}\left(E^{(0,2),i} + F^{(0,2)i}\right)\end{aligned} \tag{7.89}$$

$$\Psi^{(0,4)} = \psi^{(0,4)} - 4\dot{a}\left(B^{(0,3)} - \frac{a}{2}\dot{E}^{(0,2)}\right) + \frac{1}{2}\left(\nabla^{-2}\chi^{(0,4),ij}_{Lij} - \chi^{(0,4)}_{L}\right) \tag{7.90}$$

$$\begin{aligned}\mathbf{h}^{(0,4)}_{ij} &= \hat{h}^{(0,4)}_{ij} + 2\chi^{(0,4)}_{Lij} + \left(\nabla^{-2}\chi^{(0,4),kl}_{Lkl} - \chi^{(0,4)}_{L}\right)\delta_{ij} \\ &\quad +\nabla^{-2}\left(\nabla^{-2}\chi^{(0,4),kl}_{Lkl} + \chi^{(0,4)}_{L}\right)_{,ij} - 4\nabla^{-2}\chi^{(0,4),k}_{Lk(i\ \ j)}\,,\end{aligned} \tag{7.91}$$

where $\chi^{(0,4)}_{Lij}$ is defined such that

$$\begin{aligned}\chi^{(0,4)}_{Lij} &= -\left(-\psi^{(0,2)}_{,k}\delta_{ij} + \frac{1}{2}E^{(0,2)}_{,ijk} + \frac{1}{2}F^{(0,2)}_{(i,j)k} + \frac{1}{2}\hat{h}^{(0,2)}_{ij,k}\right)\left(E^{(0,2),k} + F^{(0,2)k}\right) \\ &\quad -\left(-\psi^{(0,2)}\delta_{ik} + \frac{1}{2}E^{(0,2)}_{,ik} + \frac{1}{2}F^{(0,2)}_{(i,k)} + \frac{1}{2}\hat{h}^{(0,2)}_{ik}\right)\left(E^{(0,2),k}_{,j} + F^{(0,2)k}_{,j}\right) \\ &\quad -\left(-\psi^{(0,2)}\delta_{jk} + \frac{1}{2}E^{(0,2)}_{,jk} + \frac{1}{2}F^{(0,2)}_{(j,k)} + \frac{1}{2}\hat{h}^{(0,2)}_{jk}\right)\left(E^{(0,2),k}_{,i} + F^{(0,2)k}_{,i}\right)\,.\end{aligned} \tag{7.92}$$

This gives a full set of gauge invariant quantities for the post-Newtonian sector of our theory, up to the order that we are considering.

Mixed-order quantities: at $\mathcal{O}(\epsilon\eta)$ the gauge invariant quantities we can construct are two scalars and one tensor:

$$\Phi^{(1,1)} = \phi^{(1,1)} - \frac{1}{2}\phi^{(0,2)}_{,i}\left(E^{(1,0),i} + F^{(1,0)i}\right) \quad (7.93)$$

$$\Psi^{(1,1)} = \psi^{(1,1)} + \frac{1}{2}\left(\nabla^{-2}\chi^{(1,1),ij}_{Lij} - \chi^{(1,1)}_{L}\right) \quad (7.94)$$

$$\begin{aligned}\mathbf{h}^{(1,1)}_{ij} = {} & \hat{h}^{(1,1)}_{ij} + 2\chi^{(1,1)}_{Lij} - 4\nabla^{-2}\chi^{(1,1),k}_{Lk(i,j)} + \nabla^{-2}\chi^{(1,1),kl}_{Lkl}\delta_{ij} - \chi^{(1,1)}_{L}\delta_{ij} \quad (7.95)\\ & +\nabla^{-2}\nabla^{-2}\chi^{(1,1),kl}_{Lkl,ij} + \nabla^{-2}\chi^{(1,1)}_{L,ij}\,,\end{aligned}$$

and at order $\mathcal{O}(\epsilon\eta^2)$ the gauge invariant quantities are two scalars, one vector and one tensor:

$$\begin{aligned}\Phi^{(1,2)} = {} & \phi^{(1,2)} + 2\phi^{(0,2)}\left(\dot{a}B^{(1,0)} + a\dot{B}^{(1,0)} - a\dot{a}\dot{E}^{(1,0)} - \frac{a^2}{2}\ddot{E}^{(1,0)}\right) \quad (7.96)\\ & +\dot{\phi}^{(0,2)}\left(aB^{(1,0)} - \frac{a^2}{2}\dot{E}^{(1,0)}\right)\end{aligned}$$

$$\Psi^{(1,2)} = \psi^{(1,2)} + \nabla^{-2}\left(\chi^{(1,2),k]l}_{Lk[l} + 2\mathcal{C}^{,|k]}_{Lk[l|,m}\mathcal{I}^{m,l}_{L}\right) \quad (7.97)$$

$$\mathbf{B}^{(1,2)}_{i} = B^{(1,2)}_{i} - \frac{a}{2}\dot{F}^{(1,1)}_{i} + \chi^{(1,2)}_{Li} - \nabla^{-2}\chi^{(1,2),j}_{Lj,i} \quad (7.98)$$

$$\begin{aligned}\mathbf{h}^{(1,2)}_{ij} = {} & \hat{h}^{(1,2)}_{ij} + 2\chi^{(1,2)}_{Lij} - 4\nabla^{-2}\chi^{(1,2),k}_{Lk(i,j)} + \nabla^{-2}\chi^{(1,2),kl}_{Lkl}\delta_{ij} - \chi^{(1,2)}_{L}\delta_{ij} \quad (7.99)\\ & +\nabla^{-2}\chi^{(1,2)}_{L,ij} + \nabla^{-2}\nabla^{-2}\chi^{(1,2),kl}_{Lkl,ij} + 4\nabla^{-2}\nabla^{-2}\Big(\nabla^{2}\mathcal{C}_{Lij,mk}\mathcal{I}^{m,k}_{L}\\ & -\nabla^{2}\mathcal{C}_{Lk(i,j)m}\mathcal{I}^{m,k}_{L} - 2\mathcal{C}_{Lk(i,j)klm}\mathcal{I}^{m,l}_{L} - \nabla^{2}\mathcal{C}^{,k}_{Lk(i|,m}\mathcal{I}^{m}_{L,|j)} + \mathcal{C}^{,k(l|}_{Lkl,mn}\mathcal{I}^{m,|n)}_{L}\delta_{ij}\Big)\\ & +\nabla^{-2}\nabla^{-2}\Big(-\nabla^{2}\mathcal{C}^{k}_{Lk,ml}\mathcal{I}^{m,l}_{L}\delta_{ij} + 2\mathcal{C}^{,k}_{Lkl,mij}\mathcal{I}^{m,l}_{L} + 2\mathcal{C}^{,kl}_{Lkl,m(i}\mathcal{I}^{m}_{L,j)}\\ & +2\mathcal{C}_{Lij,mk}\mathcal{I}^{m,k}_{L}\Big)\,.\end{aligned}$$

The definitions of $\chi^{(1,2)}_{Li}$, $\chi^{(1,2)}_{Lij}$ and $\chi^{(1,1)}_{Lij}$ are given by:

$$\begin{aligned}\chi^{(1,2)}_{Li} = {} & \phi^{(0,2)}\left(B^{(1,0)} - \frac{a}{2}\dot{E}^{(1,0)}\right)_{,i} \quad (7.100)\\ & -\frac{a}{2}\left(-\psi^{(0,2)}\delta_{ij} + \frac{1}{2}E^{(0,2)}_{,ij} + \frac{1}{2}F^{(0,2)}_{(i,j)} + \frac{1}{2}\hat{h}^{(0,2)}_{ij}\right)\left(E^{(1,0),j} + F^{(1,0)j}\right)\cdot\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{2}\left(\frac{1}{2}B^{(0,3)}_{,i} + B^{(0,3)}_{i} - \frac{a}{4}\dot{F}^{(0,2)}_{i}\right)_{,j}\left(E^{(1,0),j} + F^{(1,0)j}\right) \\
& -\frac{1}{2}\left(\frac{1}{2}B^{(1,0)}_{,j} + B^{(1,0)}_{j} - \frac{a}{4}\dot{F}^{(1,0)}_{i}\right)\left(E^{(0,2),j} + \delta F^{(0,2)j}\right)_{,i} \\
\chi^{(1,2)}_{Lij} \;=\;& 2\dot{a}\left(-\psi^{(0,2)}\delta_{ij} + \frac{1}{2}\hat{h}^{(0,2)}_{ij}\right)\left(B^{(1,0)} - \frac{a}{2}\dot{E}^{(1,0)}\right) \qquad (7.101) \\
& +a\left(-\psi^{(0,2)}\delta_{ij} + \frac{1}{2}E^{(0,2)}_{,ij} + \frac{1}{2}F^{(0,2)}_{(i,j)} + \frac{1}{2}\hat{h}^{(0,2)}_{ij}\right)^{\cdot}\left(B^{(1,0)} - \frac{a}{2}\dot{E}^{(1,0)}\right) \\
& -\frac{1}{2}\left(-\psi^{(0,2)}\delta_{ik} + \frac{1}{2}E^{(0,2)}_{,ik} + \frac{1}{2}F^{(0,2)}_{(i,k)} + \frac{1}{2}\hat{h}^{(0,2)}_{ik}\right)\left(E^{(1,0),k} + F^{(1,0)k}\right)_{,j} \\
& -\frac{1}{2}\left(-\psi^{(0,2)}\delta_{jk} + \frac{1}{2}E^{(0,2)}_{,jk} + \frac{1}{2}F^{(0,2)}_{(j,k)} + \frac{1}{2}\hat{h}^{(0,2)}_{jk}\right)\left(E^{(1,0),k} + F^{(1,0)k}\right)_{,i} \\
& -\frac{1}{2}\left(-\psi^{(1,0)}\delta_{ik} + \frac{1}{2}E^{(1,0)}_{,ik} + \frac{1}{2}F^{(1,0)}_{(i,k)} + \frac{1}{2}\hat{h}^{(1,0)}_{ik}\right)\left(E^{(0,2),k} + F^{(0,2)k}\right)_{,j} \\
& -\frac{1}{2}\left(-\psi^{(1,0)}\delta_{jk} + \frac{1}{2}E^{(1,0)}_{,jk} + \frac{1}{2}F^{(1,0)}_{(j,k)} + \frac{1}{2}\hat{h}^{(1,0)}_{jk}\right)\left(E^{(0,2),k} + F^{(0,2)k}\right)_{,i} \\
\chi^{(1,1)}_{Lij} \;=\;& \mathcal{C}_{Lij,k}\mathcal{I}^{k}_{L}\,, \qquad (7.102)
\end{aligned}
$$

where $\mathcal{C}_{Lij,k}$ and $\mathcal{I}^k_L$ are given by

$$
\mathcal{C}_{Lij,k} \equiv \left(-\psi^{(0,2)}\delta_{ij} + \frac{1}{2}E^{(0,2)}_{,ij} + \frac{1}{2}F^{(0,2)}_{(i,j)} + \frac{1}{2}\hat{h}^{(0,2)}_{ij}\right)_{,k} \qquad (7.103)
$$

$$
\mathcal{I}^{k}_{L} \equiv -\frac{1}{2}\left(E^{(1,0),k} + F^{(1,0)k}\right)\,. \qquad (7.104)
$$

This completes our study of gauge invariant quantities constructed from perturbations of the metric.

It can be seen that there are a number of perturbed quantities in our formalism that are automatically gauge-invariant. These are stated previously, but to summarise, they include the scalar Newtonian and post-Newtonian potentials $\phi^{(0,2)}$ and $\psi^{(0,2)}$, as well as the lowest-order tensor perturbations $\hat{h}^{(1,0)}_{ij}$ and $\hat{h}^{(0,2)}_{ij}$. The first two are expected as (depending on how one writes the field equations) they correspond to the gravitational potential in the Newton-Poisson equation. The last two show that the leading-order transverse and trace-free perturbations are invariant in both sectors of the theory. Comparing the form of the gauge-invariant quantities $\Phi^{(1,0)}$ and $\Phi^{(0,4)}$, it is interesting to note that they differ by a single term: $-\frac{1}{2}\phi^{(0,2)}{}_{,i}\,(E^{(0,2),i} + F^{(0,2)i})$, which is quadratic in perturbations. The cosmological gauge invariant quantity $\Phi^{(1,0)}$ cannot contain a quadratic term of this form, as this term would appear higher-order, at $\mathcal{O}(\epsilon^2)$. A number of other terms can be seen to occur in more than one of our gauge invariant quantities, and demonstrates the effect that the different length scales have on the order of perturbed quantities.

To summarise these results: here we calculate gauge invariant metric potentials $\Phi, \Psi, \mathbf{B}^i$ and $\mathbf{h}_{ij}$ (omitting superscripts for simplicity) which correspond to the cosmological, post-Newtonian, and mixed-order gauge invariant quantities. These gauge invariant quantities were constructed such that if we consider one of these terms, which we denote by $\Phi^{(n,m)}$, in the longitudinal gauge, where $E = B = F_i = 0$ (at all orders), then $\Phi^{(n,m)}$ is simply equal to the metric potential $\phi^{(n,m)}$. This construction is true for all other potentials, *i.e.* $\Psi^{(n,m)}, \mathbf{B}^{(n,m)i}$ and $\mathbf{h}_{ij}^{(n,m)}$ in the longitudinal gauge are equal to $\psi^{(n,m)}, B^{(n,m)i}$ and $\hat{h}_{ij}^{(n,m)}$, respectively.

7.5.2. Gauge invariant quantities from the matter sector

We now construct gauge invariant quantities from the transformations of the matter sources on the right-hand-side of Einstein's field equations, Eq. (2.15). Again, our gauge invariant quantities will reduce to matter sources in the longitudinal gauge when $E = B = F_i = 0$, at all orders. The invariance of all gauge invariant quantities given below has been checked through explicit transformation.

We first construct gauge-invariant scalars which correspond to energy density perturbations in the longitudinal gauge, when $E = B = F_i = 0$, they are given by

$$\boldsymbol{\rho}^{(0,0)} + \boldsymbol{\rho}^{(0,2)} = \rho^{(0,0)} + \rho^{(0,2)} \tag{7.105}$$

$$\boldsymbol{\rho}^{(1,1)} = \rho^{(1,1)} - \frac{1}{2}\rho^{(0,2)}_{,i}\left(E^{(1,0),i} + F^{(1,0)i}\right) \tag{7.106}$$

$$\boldsymbol{\rho}^{(1,0)} + \boldsymbol{\rho}^{(1,2)} = \rho^{(1,0)} + \rho^{(1,2)} + \left(\rho^{(0,0)} + \rho^{(0,2)}\right)^{\cdot}\left(aB^{(1,0)} - \frac{a^2}{2}\dot{E}^{(1,0)}\right) \tag{7.107}$$

$$\boldsymbol{\rho}^{(0,4)} = \rho^{(0,4)} - \rho^{(0,2)}_{,i}\left(E^{(0,2),i} + F^{(0,2)i}\right) . \tag{7.108}$$

The reader may note that $\rho^{(1,0)}+\rho^{(1,2)}$ transform together, because $\rho^{(1,0)}$ and $\rho^{(1,2)}$ are of the same order ($\mathcal{O}(\epsilon\eta^2 L_N^{-2})$) in our framework, even though $\rho^{(1,0)}$ is the leading-order large-scale perturbation to the energy density. They therefore form the gauge invariant quantity in Eq. (7.107). Note the gauge invariant quantity defined in Eq. (7.107) is quadratic in the inhomogeneous energy density $\rho^{(0,2)}$. This quadratic term contributing to the gauge invariant quantity corresponding to the first-order cosmological potential $\rho^{(1,0)}$, is highly unusual in linear cosmological perturbation theory. Such a term would only exist at second order in cosmological potentials.

Furthermore, we construct gauge invariant scalars which correspond to pressure perturbations in the longitudinal gauge (when $E = B = F_i = 0$) such that

$$\mathbf{p}^{(0,0)} = p^{(0,0)} \tag{7.109}$$

$$\mathbf{p}^{(1,0)} + \mathbf{p}^{(1,2)} = p^{(1,0)} + p^{(1,2)} + \left(\dot{p}^{(0,0)} - 2\frac{\dot{a}}{a}p^{(0,0)}\right)\left(aB^{(1,0)} - \frac{a^2}{2}\dot{E}^{(1,0)}\right) \tag{7.110}$$

$$\mathbf{p}^{(0,4)} = p^{(0,4)} \,. \tag{7.111}$$

The cosmological constant is constant in time and space so it does not transform, refer to Eq. (7.70), and is therefore trivially a gauge invariant quantity

$$\mathbf{\Lambda}^{(0,0)} = \Lambda^{(0,0)} \,. \tag{7.112}$$

The scalar and divergence-less vector parts of the Newtonian three-velocity are already gauge invariant, found in Eq. (7.80), and so the corresponding gauge invariant quantities are simply given by

$$\mathbf{v}^{(0,1)} = v^{(0,1)} \tag{7.113}$$

$$\hat{\mathbf{v}}_i^{(0,1)} = \hat{v}_i^{(0,1)} \,. \tag{7.114}$$

We use this to define the gauge invariant total Newtonian velocity in the following way $\mathbf{v}_i^{(0,1)} \equiv \mathbf{v}_{,i}^{(0,1)} + \hat{\mathbf{v}}_i^{(0,1)}$. This is what we use in the presentation of the field equations in the next chapter.

We create one further scalar, $\mathbf{v}^{(1,0)}$, and a divergence-free vector, $\hat{\mathbf{v}}_i^{(1,0)}$, which can be extracted from the divergence of the following gauge invariant quantity:

$$\begin{aligned} \mathbf{v}_i^{(1,0)} &\equiv \mathbf{v}^{(1,0)}{}_{,i} + \hat{\mathbf{v}}_i^{(1,0)} \\ &= v_i^{(1,0)} + \frac{a}{2}\left(\dot{E}_{,i}^{(1,0)} + \dot{F}_i^{(1,0)}\right) - \frac{1}{2}v_{i,j}^{(0,1)}\left(E^{(1,0),j} + F^{(1,0)j}\right) \,. \end{aligned} \tag{7.115}$$

There are no further quantities to consider in Einstein's field equations, so this gives us a full set of gauge invariant quantities in our two-parameter perturbative expansion.

At this stage we make a few comments on the gauge invariant quantities formed in this section. Most of these gauge invariant quantities are of the same form for matter only, see Appendix C.3 – this result is expected from the transformations in Section 7.3.2. However, differences in the gauge invariant quantities are also expected and are analogous to the differences found previously, in Section 7.3.2. For example the term $\rho^{(0,0)}$ appears in addition to $\rho^{(0,2)}$ in Eqs. (7.105) and (7.107) for the case of radiation and dust. Furthermore, the gauge invariant quantity in Eq. (7.107) is different for dust and radiation (compared to dust alone) because such a combination allows for a term $p^{(0,0)}$. We refer the reader to Appendix C.3 for the gauge invariant quantities analogous to those calculated in this section, but for matter only, in full. Although this is less general, it is key for readers interested in gauge invariant quantities for perturbations in the late Universe.

To summarise these results: here we calculate gauge invariant matter sources $\boldsymbol{\rho}, \mathbf{p}$ and $\mathbf{v}_i$ (omitting indices for simplicity) which correspond to the cosmological, post-Newtonian and mixed-order gauge invariant matter sources (the cosmological constant is trivially gauge invariant). These gauge invariant quantities were constructed such that if we consider one of these terms, which we denote by $\boldsymbol{\rho}^{(n,m)}$, in the longitudinal gauge, where $E = B = F_i = 0$ (at all orders), then $\boldsymbol{\rho}^{(n,m)}$ is simply equal to the perturbation $\rho^{(n,m)}$. This construction is true for all other gauge invariant quantities presented in this section, *i.e.* $\mathbf{p}^{(n,m)}$ and $\mathbf{v}_i^{(n,m)}$ in the longitudinal gauge are equal to $p^{(n,m)}$ and $v_i^{(n,m)}$. These gauge invariant quantities, when combined with the set of gauge-invariant metric potentials $\{\Phi, \Psi, \mathbf{B}^i, \mathbf{h}_{ij}\}$ constructed in Section 7.5.1, gives us a full set of gauge-invariant quantities in our two-parameter expansion. In the next chapter we present the field equations in terms of these gauge-invariant variables.

8. Dynamics of gauge invariant quantities

8.1. Field equations

We can now return to the field equations presented in Chapter 6 and write them in terms of our newly-constructed gauge invariant quantities. These equations take the same form as the field equations in the longitudinal gauge but are in fact valid in any coordinate system. Furthermore these governing equations for our gauge invariant quantities, upon specification of any particular valid gauge, should reduce to the gauge-fixed Einstein equations. As before, we write down these equations under the assumptions $\epsilon \sim \eta^2$ and $L_N/L_C \sim \eta$.

Note that we leave out both terms $R^{(2,0)}_{\mu\nu}$, in Eq. (7.9), and $T^{(2,0)}_{\mu\nu}$ from the field equations. These terms appear in the $\mathcal{O}(\eta^4 L_N^{-2})$ field equation as simply the lower order 00-field equations $\mathcal{O}(\eta^2 L_N^{-2})$ with two spatial derivatives, multiplied by two gauge generators, and so necessarily cancel and do not contribute any new dynamics to the field equations.

8.1.1. Background-order potentials

The trace-free part of the ij-equations at $\mathcal{O}(\eta^2 L_N^{-2})$ gives

$$D_{ij}\left(\Phi^{(0,2)} + \Psi^{(0,2)}\right) - \frac{1}{2}\nabla^2 \mathbf{h}^{(0,2)}_{ij} = 0\,, \tag{8.1}$$

and its divergence implies

$$\Phi^{(0,2)} = -\Psi^{(0,2)} \qquad \text{and} \qquad \mathbf{h}^{(0,2)}_{ij} = 0\,, \tag{8.2}$$

as $\mathbf{h}^{(0,2)}_{ij}$ is transverse. The 00-field equation at $\mathcal{O}(\eta^2 L_N^{-2})$ gives

$$\frac{\ddot{a}}{a} + \frac{1}{6a^2}\nabla^2\Phi^{(0,2)} = -\frac{4\pi}{3}\left(\rho^{(0,0)} + \rho^{(0,2)} + 3\mathbf{p}^{(0,0)}\right) + \frac{1}{3}\mathbf{\Lambda}\,, \tag{8.3}$$

and the trace of the ij-equation at $\mathcal{O}(\eta^2 L_N^{-2})$ gives

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{3a^2}\nabla^2\Phi^{(0,2)} = \frac{8\pi}{3}\left(\rho^{(0,0)} + \rho^{(0,2)}\right) + \frac{1}{3}\mathbf{\Lambda}\,, \tag{8.4}$$

where we have substituted in the results from Eq. (8.2). Furthermore, the results from Eq. (8.2) will be substituted into all equations in this chapter. These background equations govern the leading-order part of the gravitational field, at $\mathcal{O}(\eta^2 L_N^{-2})$.

8.1.2. Vector potentials

We now use all $0i$-field equations in Section 6.2.2. At order $\mathcal{O}(\eta^3 L_N^{-2})$, these give

$$\nabla^2\mathbf{B}_i^{(0,3)} + 2\left(a\dot{\Phi}^{(0,2)} + \dot{a}\Phi^{(0,2)}\right)_{,i} \;=\; 16\pi a^2\left(\rho^{(0,0)} + \rho^{(0,2)} + \mathbf{p}^{(0,0)}\right)\mathbf{v}_i^{(0,1)}\,. \tag{8.5}$$

Although $\mathbf{B}_i^{(0,3)}$ is purely a divergenceless vector Eq. (8.5) has a divergenceless vector and scalar part, which can be separated out with a derivative. At $\mathcal{O}(\eta^4 L_N^{-2})$ the $0i$-field equations give

$$\begin{aligned} &\nabla^2\left(\mathbf{B}_i^{(1,0)} + \mathbf{B}_i^{(1,2)}\right) + 2\left(a\left(\Phi^{(1,1)} - \Psi^{(1,0)}\right)^{\cdot} + \dot{a}\left(\Phi^{(1,1)} + \Phi^{(1,0)}\right)\right)_{,i} \\ &-2\left(2\dot{a}^2 + a\ddot{a}\right)\mathbf{B}_i^{(1,0)} - \mathbf{B}_j^{(1,0)}\Phi_{,ij}^{(0,2)} \\ =\;& 8\pi a^2\left(2\left(\rho^{(0,0)} + \rho^{(0,2)} + \mathbf{p}^{(0,0)}\right)\mathbf{v}_i^{(1,0)} + \left(\rho^{(0,0)} + \rho^{(0,2)} + 3\mathbf{p}^{(0,0)}\right)\mathbf{B}_i^{(1,0)}\right. \\ &\left.+2\rho^{(1,1)}\mathbf{v}_i^{(0,1)}\right) - 2a^2\mathbf{\Lambda}\mathbf{B}_i^{(1,0)}\,, \end{aligned} \tag{8.6}$$

which can also be split into scalar and divergenceless vector parts using a derivative. As a result, the reader may note that the quadratic term, which includes the lower-order potential $\Phi^{(0,2)}$, does not source the vector part of Eq. (8.6), although this may not be expected at first glance.

8.1.3. Higher-order scalar potentials

The 00-field equation and the trace of the ij-field equation at $\mathcal{O}(\epsilon\eta L_N^{-2})$ gives

$$\nabla^2\Phi^{(1,1)} = -8\pi a^2\rho^{(1,1)}\,, \tag{8.7}$$

and

$$\Phi^{(1,1)} = -\Psi^{(1,1)} \,. \tag{8.8}$$

The 00-field equation at $\mathcal{O}(\eta^4 L_N^{-2})$ gives

$$\begin{aligned}
&\nabla^2\left(\Phi^{(1,0)} + \frac{1}{2}\Phi^{(0,4)} + \Phi^{(1,2)}\right) + 3a^2\left(\Phi^{(0,2)} - \Psi^{(1,0)}\right)^{\cdot\cdot} + \left(\nabla\Phi^{(0,2)}\right)^2 \\
&+3a\dot{a}\left(3\Phi^{(0,2)} + \Phi^{(1,0)} - 2\Psi^{(1,0)}\right)^{\cdot} - \nabla^2\Phi^{(0,2)}\left(\Phi^{(0,2)} - \Psi^{(1,0)}\right) - \frac{1}{2}\Phi^{(0,2)}_{,ij}\mathbf{h}^{(1,0)}_{ij} \\
= \;& -8\pi a^2\left[\rho^{(1,0)} + \rho^{(1,2)} + \frac{1}{2}\rho^{(0,4)} - \left(\rho^{(0,0)} + \rho^{(0,2)} + 3\mathbf{p}^{(0,0)}\right)\left(\Phi^{(1,0)} + \Phi^{(0,2)}\right)\right. \\
&\left.+3\left(\mathbf{p}^{(1,0)} + \mathbf{p}^{(1,2)} + \frac{1}{2}\mathbf{p}^{(0,4)}\right) + 2\left(\mathbf{v}^{(0,1)}_i\right)^2\left(\rho^{(0,0)} + \rho^{(0,2)} + \mathbf{p}^{(0,0)}\right)\right] \\
&-2a^2\mathbf{\Lambda}\left(\Phi^{(0,2)} + \Phi^{(1,0)}\right)\,,
\end{aligned} \tag{8.9}$$

this field equation is analogous to the first-order 00-field equation derived from cosmological perturbation theory. Although this equation may look like what is derived from second-order cosmological perturbation theory with a quasi-static limit, because it includes many quadratic terms, this is equation is actually much larger-in-magnitude compared to the equation derived from the quasi-static limit of second-order perturbation theory. One can observe this because the cosmological potentials in Eq. (8.9) are purely linear in perturbations, $\mathcal{O}(\epsilon)$, not second-order, $\mathcal{O}(\epsilon^2)$. The above equation differs significantly to what would be derived using second-order cosmological perturbation theory because, from our expansion, the effects of non-linearities on large-scale potentials are at leading-order in ϵ, and are not sub-dominant (at second-order, order ϵ^2). Taking a precise example, we see the linear cosmological potential $\Phi^{(1,0)}$ appears in the above equation along with non-linearities. Non-linearities would normally only occur in second-order cosmological perturbation theory, alongside terms linear in $\Phi^{(2,0)}$ (not $\Phi^{(1,0)}$ – which is what occurs here).

Furthermore, the trace of the ij-field equation at $\mathcal{O}(\eta^4 L_N^{-2})$ gives

$$\begin{aligned}
&-2\nabla^2\left(\Psi^{(1,0)} + \Psi^{(1,2)} + \frac{1}{2}\Psi^{(0,4)}\right) - 3\left(2\dot{a}^2 + a\ddot{a}\right)\left(\Phi^{(1,0)} - \Psi^{(1,0)} + 2\Phi^{(0,2)}\right) \\
&+6\dot{a}a\left(\Psi^{(1,0)} - \Phi^{(0,2)}\right)^{\cdot} \\
= \;& -8\pi a^2\left[2\left(\rho^{(1,0)} + \frac{1}{2}\rho^{(0,4)} + \rho^{(1,2)}\right) + 2\left(\rho^{(0,0)} + \rho^{(0,2)} + \mathbf{p}^{(0,0)}\right)\left(\mathbf{v}^{(0,1)}_i\right)^2\right.
\end{aligned}$$

$$+\Phi^{(0,2)}\left(\rho^{(0,0)}+\rho^{(0,2)}-3\mathbf{p}^{(0,0)}\right)-\frac{1}{2}\left(\rho^{(0,0)}+\rho^{(0,2)}\right)\left(\Phi^{(1,0)}+3\Psi^{(1,0)}\right)$$
$$+\frac{3}{2}\mathbf{p}^{(0,0)}\left(\Psi^{(1,0)}-\Phi^{(1,0)}\right)\Big]-a^2\mathbf{\Lambda}\left[4\Phi^{(0,2)}+\Phi^{(1,0)}-3\Psi^{(1,0)}\right]+\mathcal{A}\,, \qquad (8.10)$$

where we have defined terms that are quadratic in metric potentials as

$$\mathcal{A} \equiv \nabla^2\Phi^{(0,2)}\left(3\Phi^{(0,2)}+\frac{1}{2}\Phi^{(1,0)}-\frac{5}{2}\Psi^{(1,0)}\right)+\frac{3}{2}\left(\nabla\Phi^{(0,2)}\right)^2+\frac{1}{2}\Phi^{(0,2)}_{,ij}\mathbf{h}^{(1,0)}_{ij}\,. \qquad (8.11)$$

These are all of the scalar equations that exist up to $\mathcal{O}(\eta^4 L_N^{-2})$.

8.1.4. Tensor potentials

The trace-free part of the ij-field equation at $\mathcal{O}(\epsilon\eta L_N^{-2})$ is

$$D_{ij}\left(\Phi^{(1,1)}+\Psi^{(1,1)}\right)-\frac{1}{2}\nabla^2\mathbf{h}^{(1,1)}_{ij}=0\,, \qquad (8.12)$$

and its divergence implies

$$\Phi^{(1,1)}=-\Psi^{(1,1)} \qquad \text{and} \qquad \mathbf{h}^{(1,1)}_{ij}=0\,, \qquad (8.13)$$

because $\mathbf{h}^{(1,1)}_{ij}$ is transverse. The reader may note that, unlike $\Psi^{(0,2)}$ and $\Phi^{(0,2)}$, the first of these conditions has already been given by the 00-field equation and the trace of the $ij-$field equations, see Eq. (8.8). We substitute the results in Eq. (8.13) into all equations in this chapter. Finally, the ij-field equation, at $\mathcal{O}(\eta^4 L_N^{-2})$, can be used to write the following trace-free equation:

$$-D_{ij}\left(\Phi^{(1,0)}+\Phi^{(1,2)}+\frac{1}{2}\Phi^{(0,4)}+\Psi^{(1,0)}+\Psi^{(1,2)}+\frac{1}{2}\Psi^{(0,4)}\right)-\frac{3}{2}a\dot{a}\dot{\mathbf{h}}^{(1,0)}_{ij}$$
$$+\frac{1}{2}\nabla^2\left(\mathbf{h}^{(1,0)}_{ij}+\mathbf{h}^{(1,2)}_{ij}+\frac{1}{2}\mathbf{h}^{(0,4)}_{ij}\right)+\frac{2}{a}\left[a^2\left(\mathbf{B}^{(0,3)}_{(i,j)}+\mathbf{B}^{(1,0)}_{(i,j)}\right)\right]^{\cdot}-\frac{1}{2}a^2\ddot{\mathbf{h}}^{(1,0)}_{ij}$$
$$-\left(2\dot{a}^2+a\ddot{a}\right)\mathbf{h}^{(1,0)}_{ij}$$
$$= -4\pi a^2\left[\left(\rho^{(0,0)}+\rho^{(0,2)}-\mathbf{p}^{(0,0)}\right)\mathbf{h}^{(1,0)}_{ij}+4\left(\rho^{(0,0)}+\rho^{(0,2)}+\mathbf{p}^{(0,0)}\right)\mathbf{v}^{(0,1)}_{\langle i}\mathbf{v}^{(0,1)}_{j\rangle}\right]$$
$$-a^2\mathbf{\Lambda}\mathbf{h}^{(1,0)}_{ij}+\mathcal{B}_{ij}\,, \qquad (8.14)$$

where we have defined terms that are quadratic in metric potentials as

$$\mathcal{B}_{ij} \equiv D_{ij}\Phi^{(0,2)}\left(2\Phi^{(0,2)}+\Phi^{(1,0)}-\Psi^{(1,0)}\right)+\Phi^{(0,2)}_{,\langle i}\Phi^{(0,2)}_{,j\rangle}-\Phi^{(0,2)}_{,k\langle i}\mathbf{h}^{(1,0)}_{j\rangle k}\,. \qquad (8.15)$$

Note that, unlike standard cosmological perturbation theory, these equations do not imply $\Phi^{(1,0)} = -\Psi^{(1,0)}$ or $\mathbf{h}_{ij}^{(1,0)} = 0$ as non-linearities act as an effective anisotropic stress, see Chapter 9. So scalar, vector and tensor modes do not decouple at linear order in cosmological perturbations because of the additional potentials in that simply do not exist in first-order cosmological perturbation theory. The fact that $\Phi^{(1,0)} \neq -\Psi^{(1,0)}$ here implies a slip between these potentials which corresponds to an effective anisotropic-stress when written as an effective fluid, refer to the next chapter. In fact, this coupling-of-modes (and non-zero slip) normally only occurs at second-order in cosmological perturbations, this effect now happens in a larger-in-magnitude field equation as a result of our two-parameter perturbed potentials, which have different characteristic length scales and vary differently in time and space. We also write the field equations for dust only, also in terms gauge invariant variables, in Appendix D. The key difference in adding dust, radiation and a cosmological constant to these field equations, rather than just including dust, is outlined in detail in Chapter 6. This completes the full set of field equations in terms of our gauge-invariant variables, up to the order in perturbations that we wish to consider here.

8.2. Discussion

In the following section we discuss the application of our two-parameter expansion to various physical situations that are of interest and comment on the resulting field equations. The first situation of which considers the field equations given previously in this chapter. Note that although the relationship between the lengths scales of non-linear structure, L_N, and linear perturbations, L_C, vary in the following discussion, in Sections 8.2.1 and 8.2.2, gravitational potentials remain small and of similar size $\epsilon \sim \eta^2$.

8.2.1. Large-scale limit: $l \sim \eta$

In this section we discuss the application of our two-parameter expansion to the largest structures that exist in the Universe: this is the case when the non-linear post-Newtonian structure, on scales L_N, compared to linear cosmological perturbations, on scales L_C, saturates the bound given in Eq. (5.30). We find that within the two-parameter formalism outlined in this thesis, the Friedmann-like equations that govern the evolution of the scale factor $a(t)$, and hence the large-scale expansion of the Universe, are *not* independent of the perturbations. This can be seen explicitly in Eqs. (8.3) and (8.4), where the Newtonian mass density and gravitational potential act as sources for the cosmological expansion.

This is in some sense a very pleasing result; the large-scale expansion is driven by the same Newtonian mass that governs the leading-order part of the gravitational field on small scales. On the other hand, it means that our "background" is not by itself an exact solution of Einstein's equations[1]. This stretches the meaning of what is usually implied by the phrase "perturbation theory" in Einstein's theory[2]. Nevertheless, both the fundamental objects being perturbed and the field equations themselves are being consistently expanded in the perturbative parameters ϵ and η, and we see no reason to expect this expansion should not converge. Indeed the present expansion seems to have much better convergence properties than the standard approach to cosmological perturbation theory, in the presence of non-linear structures [143]. Furthermore, a change of coordinates on a sub-horizon-sized region of space can be shown to be isometric to perturbed Minkowski space, with the cosmological expansion arising from boundary conditions at the edge of the region [73]. In this sense, the cosmological expansion can be considered an emergent property, and the background on small-scales could equally well be considered to be either a Friedmann model or Minkowski space (which definitely is a solution when $\epsilon = \eta = 0$).

Furthermore, in Eqs. (8.3) and (8.4) the Newtonian-mass density, the background contribution to the energy density due to radiation, the Newtonian and post-Newtonian gravitational potentials, and cosmological constant all contribute to the evolution of the scale factor. In the next-to-leading-order field equations, (8.5), (8.7) and (8.12), we have mixed-order and post-Newtonian potentials, but no quadratic lower-order terms. The latter two of these equations only exists when non-relativistic matter fluids are considered, and are both strictly zero for radiation or cosmological constant domination. Similarly, Eq. (8.1) is the same for all matter content.

[1]To emphasise this point further, perturbative expansions of post-Newtonian gravity and cosmological perturbation theory are quite different in nature. Cosmological perturbation theory is a perturbative expansion around an exact solution such that this exact solution is a good approximation to, close to in magnitude, the perturbed solution. On the other hand, post-Newtonian gravity is closer to an asymptotic expansion, a perturbative expansion which allows for small perturbations in the geometry and large perturbations to the energy density and so when these perturbations are set to zero the vacuum solution is not close to, in magnitude, the perturbed solution. Despite the difference of these perturbative expansions, both allow for a systematic treatment of accounting for relativistic effects in Einstein's field equations, *i.e.* the lowest-order field equations are solved for and substituted into higher-order field equations which are then solved for, and so on.

[2]I am grateful to Marco Bruni for a number of stimulating discussions on this point.

In the $\mathcal{O}(\eta^4)$ field equations, Eqs. (8.6), (8.9), (8.10) and (8.14), on the other hand, we find matter, radiation and a cosmological constant sourcing first order cosmological, mixed-order, Newtonian and post-Newtonian potentials. At this order in the field equations we find a combination of non-linear gravity, mode-mixing and a mixing-of-scales – which comes from the mixing of large and small scales through cosmological and post-Newtonian perturbations, respectively, sourcing one another in the same equation. For example see Eq. (8.9), where the potential for mode-mixing proposed in Chapter 6 is observed in this chapter because of the irreducible decomposition of perturbations and the mixing of scalars and transverse and traceless tensors. This means that linear-order cosmological perturbations (that usually arise as first-order corrections to the background field equations) in fact come in after two lower order field equations. These effects only arise because of the form of our two-parameter expansion, and so do not (and cannot) occur in linear-order cosmological perturbation theory.

Our expansion requires field equations to exist at orders that simply do not exist in linear cosmological perturbation theory. For example, in cosmological perturbation theory the leading-order vector mode (which contributes to frame-dragging effects) decays quickly, and so is usually taken to be zero. However, the magnitude of the second-order part of this potential has recently been found to be much bigger than one might naively estimate – between $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ [26], at about $O(\epsilon^{1.5})$. In our expansion we already have a vector potential at order $\eta^3 \sim \epsilon^{1.5}$, in Eq. (8.5), we find the gravitomagnetic potential we solve for first (that dominates) is 100 times larger than what is expected from second order cosmological perturbation theory, as found in Ref. [26]. It is clear that such a potential should exist from the post-Newtonian perturbed sector. This means that the result of Ref. [26], is at odds with cosmological perturbation theory, but fits very naturally into our framework. Our expansion also suggests that there should be field equations at $\mathcal{O}(\eta^5)$, which would correspond to a potential between the first and second order field equations in normal cosmological perturbation theory. This simply does not exist in the usual expansion, but is included if one follows the approach we have used in this thesis.

Note that cosmological perturbation theory is not recovered by simply setting $\eta \to 0$. This is because in cosmological perturbation theory the lowest order energy density is always homogeneous, whereas in the late Universe, as described by our two-parameter expansion, during matter domination, the lowest order energy density is inhomogeneous. We therefore cannot recover cosmological perturbation theory during matter domination by ignoring the post-Newtonian sources, as when $\eta \to 0$ the evolution of the scale factor in Eq. (8.3) would have no source at all. This means that post-Newtonian sector must be included, in both the equations for the background expansion and the linear-order cosmological perturbations if we wish to consider non-linear matter. Specifically, this means that standard cosmological perturbation theory is not necessarily recovered if one averages the field equations given previously in this chapter over some length scale greater than or equal to the homogeneity scale, as is usually assumed [71]. We could recover the Friedmann equations for an Einstein-de Sitter Universe, during matter domination, if there were no inhomogeneities at lowest order, such that $\boldsymbol{\rho}_M^{(0,2)}(x^\mu) \equiv \boldsymbol{\rho}_M^{(0,2)}(t)$ (or of course by adding a homogeneous background to the matter component of the energy density). On the other hand, if $\eta \to 0$ and we consider radiation or a cosmological constant, without matter, then we do recover cosmological perturbation theory essentially because there are no large density contrasts due to matter. Nevertheless, if we truly live in a two-parameter universe, then setting one parameter to zero is not physical. To compare our two-parameter expansion to cosmological perturbation theory we average the background field equations over a suitably large scale, which is what we do next.

Friedmann and Newtonian cosmological equations

We now proceed to find the simplest way in which to express the background equations that govern the large-scale expansion of space, this will enable us to compare our results to that of standard cosmological perturbation theory more easily. In order to do this we average Eq. (8.4) over a suitably large scale. We start by calculating the average mass density and radiation density on distances above the homogeneity scale, $L_{\text{hom}} \sim 100\text{Mpc}$ [104]. At leading-order these are given by

$$\overline{\boldsymbol{\rho}}_M \equiv \frac{\int_{V_{\text{hom}}} \boldsymbol{\rho}^{(0,2)} dV}{\int_{V_{\text{hom}}} dV} \qquad \text{and} \qquad \overline{\boldsymbol{\rho}}_R \equiv \frac{\int_{V_{\text{hom}}} \boldsymbol{\rho}^{(0,0)} dV}{\int_{V_{\text{hom}}} dV} = \boldsymbol{\rho}^{(0,0)}, \tag{8.16}$$

where V_{hom} indicates the spatial volume associated with the homogeneity scale. Of course, we know from Eq. (5.25), that there can be no leading-order small-scale inhomogeneities in the radiation fluid. For the matter fluid, on the other hand, small-scale fluctuations most definitely do exist and are of order unity. To accommodate these fluctuations we define

$$\delta\boldsymbol{\rho}^{(0,2)} \equiv \boldsymbol{\rho}^{(0,2)} - \overline{\boldsymbol{\rho}}_M\,, \tag{8.17}$$

one may notice that spatial derivatives acting on the quantity $\delta\boldsymbol{\rho}^{(0,2)}$ go like L_N^{-1}, not L_C^{-1}. This equation implies that the leading-order inhomogeneous part of the matter energy density, $\delta\boldsymbol{\rho}^{(0,2)}$, is formally of the same order as the background component of the matter fields, $\bar{\boldsymbol{\rho}}_M$, both being $\mathcal{O}(\eta^2 L_N^{-2})$. These quantities can now be used to write Eqs. (8.3) and (8.4) into a more useful form.

To derive a set of effective Friedmann equations we first integrate Eq. (8.4) over the volume corresponding to the homogeneity scale:

$$\int_{V_{\rm hom}} \left(3H^2 - \frac{1}{a^2}\nabla^2\Phi^{(0,2)}\right) dV \;=\; \int_{V_{\rm hom}} \left(8\pi\left(\boldsymbol{\rho}^{(0,0)} + \boldsymbol{\rho}^{(0,2)}\right) + \boldsymbol{\Lambda}\right) dV\,, \quad (8.18)$$

where $H \equiv \dot{a}/a$. Given Gauss' theorem

$$\int \nabla \cdot Y dV \;=\; \int Y dS \quad (8.19)$$

where these integrals are over the total surface S and volume V, where S contains V, and Y is a function. We now substitute Gauss' theorem into Eq. (8.18) which implies

$$3H^2 V_{\rm hom} - \frac{1}{a^2}\int_{S_{\rm hom}} \nabla\Phi^{(0,2)} \cdot dS \;=\; 8\pi\left(\overline{\boldsymbol{\rho}}_M + \overline{\boldsymbol{\rho}}_R\right) V_{\rm hom} + \boldsymbol{\Lambda} V_{\rm hom}\,. \quad (8.20)$$

If we now assume a homogeneity scale, defined such that there is no net flux of $\nabla\Phi^{(0,2)}$ into or out of the surface $S_{\rm hom}$, then the second term in Eq. (8.20) vanishes. This leaves us with

$$H^2 = \frac{8\pi}{3}\left(\overline{\boldsymbol{\rho}}_M + \overline{\boldsymbol{\rho}}_R\right) + \frac{\boldsymbol{\Lambda}}{3}\,, \quad (8.21)$$

which is exactly the same form as the standard Friedmann equation in the presence of matter, radiation and a cosmological constant. What is more, the lowest-order parts of the energy-momentum conservation equations yields the results [149]

$$\overline{\boldsymbol{\rho}}_M \propto a^{-3} \qquad \text{and} \qquad \overline{\boldsymbol{\rho}}_R \propto a^{-4}\,, \quad (8.22)$$

which are again exactly as expected from Friedmann cosmology. Finally, substituting these results back into Eq. (8.4) gives

$$\nabla^2\Phi^{(0,2)} = -8\pi a^2 \delta\boldsymbol{\rho}^{(0,2)}\,, \quad (8.23)$$

which is identical to the standard equation used in Newtonian simulations for cosmology, taken from Newtonian gravity or the lowest-order equation derived from the quasi-static limit of cosmological perturbation theory. This equation can be solved using Green's functions, N-body simulations or Fourier methods [55, 141].

In summary, we find that the leading-order parts of the field equations, in the context of our two-parameter expansion, reproduce exactly the same results as standard Friedmann cosmology with dust, radiation and a cosmological constant, although the meaning of the equations is slightly different. Nevertheless, in the following chapter we will find that the same results are not derived from our two-parameter expansion and perturbed standard Friedmann cosmology when considering the beyond leading-order, non-linear, aspects of Einstein's equations, which becomes important on large scales, when $l \sim \eta$. In other words, from our two-parameter expansion, although back-reaction on the background expansion a may be small, the back-reaction from small-scale structure on large-scale perturbations are not, for example refer to Eqs. (8.9) and (8.10).

Furthermore, Eq. (8.21) provides a justification for why only the average energy density, the radiation energy density and cosmological constant source the large-scale expansion. On the other hand, only inhomogeneous matter sources the Newton-Poisson equation, Eq. (8.23). This split of the Friedmann and Newton-Poisson equation occurs even though both equations, Eqs. (8.21) and (8.23), are derived at the *same* order in perturbations, from the same equation, Eq. (8.4): the key here is the existence of a homogeneity scale at which there is no net flux in $\nabla\Phi^{(0,2)}$, which is a restrictive but necessary condition in order to derive Eqs. (8.21) and (8.23). It means that for the system to be perturbed FLRW globally with radiation and a cosmological constant we need matter to be strictly distributed such that the average energy density in *every* region is the same, which is a similar result to Ref. [148].

We can clearly see, from Eq. (8.23), that for the case where the dust component of the energy density goes to zero (or just its inhomogeneous part is zero, $\delta\boldsymbol{\rho}^{(0,2)} = 0$) then we have that $\nabla^2\Phi^{(0,2)} = 0$. Given appropriate boundary conditions this homogeneous equation has the solution $\Phi^{(0,2)} = 0$. As stated previously, this implies we can also recover the Friedmann equations for an Einstein de Sitter Universe when considering homogeneous sources. This is not the same condition as setting $\epsilon = \eta = 0$, which would correspond to an empty space within our framework. Moreover, when there does not exist inhomogeneous matter, the infinitesimal two-parameter coordinate transformation of $h_{00}^{(1,1)}$ in the metric gives $\tilde{h}_{00}^{(1,1)} = h_{00}^{(1,1)} + h_{00,i}^{(0,2)}\xi^{(1,0)i} = h_{00}^{(1,1)}$ (see Eq. (7.6)). So, not only does $h_{00}^{(1,1)} = 0$ without inhomogeneous matter, noted below Eq. (6.49), it remains zero under any infinitesimal coordinate transformation $\tilde{h}_{00}^{(1,1)} = h_{00}^{(1,1)} = 0$, or $\tilde{\Phi}^{(1,1)} = \Phi^{(1,1)} = 0$.

Not only is this important to note formally, but it shows that the Newtonian potentials are only important to the dynamics if there exists some inhomogeneous post-Newtonian matter, otherwise we expect cosmological perturbation theory to be a good approximation to the governing equations.

Finally, we comment that our two-parameter expansion was constructed such that perturbations on scales above the cut-off of 100Mpc are treated as cosmological, whereas perturbations below this cut-off are treated as post-Newtonian (see Chapter 5). This cut-off is somewhat artificial. In the real Universe there are structures, such as Baryon Acoustic Oscillations, that exist on approximately the scale of this cut-off [83]. The practical application of our two-parameter expansion to model such structures would require further thought, and perhaps some flexibility.

8.2.2. Small-scale limit: $l \ll \eta$

Let us consider what would happen if we considered a two-parameter system which described structure on the smallest scales, where inhomogeneous structure exists on scales similar to the Solar System, such that $L_N \sim L_\odot \ll \eta L_C$. Firstly, we note that long-wavelength cosmological perturbations in the energy density, $\boldsymbol{\rho}^{(1,0)}$ for example, would be relegated to very high-order field equations compared to those presented in Section 8.1, because $L_\odot \ll \eta L_C \ll L_C$. Moreover, the 'post-Newtonian-order' energy density (given by $\rho^{(4)}$ in Chapter 4) would be replaced by $\frac{1}{2}\boldsymbol{\rho}^{(0,4)} + \boldsymbol{\rho}^{(1,2)}$, given our two-parameter expansion. To disentangle $\boldsymbol{\rho}^{(1,2)}$ and $\boldsymbol{\rho}^{(0,4)}$ one would then have to use the fact that $\boldsymbol{\rho}^{(1,2)}$ has large-scale correlations, whereas $\boldsymbol{\rho}^{(0,4)}$ does not. Also note that if $l \ll \eta$ there are no potentials $\rho^{(1,1)}$, $h_{00}^{(1,1)}$ or $h_{ij}^{(1,1)}$ (see Chapter 5 where such perturbations are constructed).

However, there does remain a potential $h_{0i}^{(1,2)}$, which appears in the field equations at $\mathcal{O}(\eta^4)$ if $\epsilon \sim \eta^2$. This does not occur in usual post-Newtonian gravity, where the $0i$-field equations contain terms at $\mathcal{O}(\eta^3)$ and then at $\mathcal{O}(\eta^5)$. This means that the mixed term $h_{0i}^{(1,2)}$ would correspond to a η^4 correction to the post-Newtonian η^3 $0i$-field equation. Nevertheless, $h_{0i}^{(1,2)} \sim \eta^4$ is at higher order than anything that has so far been observed in the Solar System, as current observations have only allowed the $0i$-metric potential to be constrained to $\mathcal{O}(\eta^3)$.[3] Our formalism is therefore consistent with observed post-Newtonian gravity to date, as non-linear structure on the smallest scales is ignorant to the presence of structure on the horizon-size, up

[3]The best observational constraints on $h_{0i}^{(0,3)}$ have been made up to an accuracy of about 20% with Gravity Probe B's gyroscope precession experiment [90], and about 5% with the LAGEOS and LARES satellites [70].

to the precision of current observations. This may offer a new opportunity to test gravity at higher-orders in the future, on small-scales, as more accurate observations may one day be able to detect gravitational phenomena associated with $h_{0i}^{(1,2)}$, which couples with cosmological perturbations.

Finally, if $l \ll \eta$, the field equations will be dominated by the Newton-Poisson equation at lowest order. Cosmological terms such as $\ddot{a} \sim \rho^{(0,0)} \sim L_C^{-2}$ and $\nabla^2 h_{00}^{(1,0)} \sim \epsilon/L_C^2$ (see Eq. (6.48), will only occur at much higher order. Although the leading-order parts of post-Newtonian gravity and our two-parameter expansion are indistinguishable when applied to structure on small scales, at higher-orders (or for structures on larger scales) our formalism also includes terms that account for the sourcing of the expansion of the scale factor and large-scale cosmological potentials. These corrections simply do not appear in the usual approach to post-Newtonian gravity, where cosmological perturbations are entirely neglected. However, we recover the usual post-Newtonian expansion, for dust, in the limits $\epsilon \to 0$ and $a(t) \to 1$.

8.2.3. Other systems

Let us now consider other scenarios that one might try to model with a two-parameter approach of the type described in this thesis, that do not fall into the two cases described above, or may not satisfy $\epsilon \sim \eta^2$. The first thing that one may note for such a situation is that our two-parameter expansion simply does not allow for post-Newtonian-perturbed structures larger than the supercluster scale of 100Mpc, so great walls or voids larger than this scale cannot be considered within this expansion, see the bound in Eq. (5.30). If such situations were considered during matter domination, then the lowest order field equation would be $H^2 = 0$, which only has the solutions $a \propto t$, which corresponds to an empty universe with a Milne (not Einstein-de Sitter) solution. We note that for post-Newtonian perturbed structures smaller than supercluster scales $l < \eta$ the field equations will behave similarly to those discussed in Section 8.2.2, specifically the scale factor would be sourced at higher order, as would all terms with derivatives or units L_C, and Newtonian gravity would dominate. Note that our expansion, during radiation domination, recovers the usual solution from the Friedmann equations. Also, for cosmological constant domination our implies implies a de Sitter solution (see Section 2.4.2).

Now consider cases where $\epsilon > \eta^2$. This could be the case, for example, in a universe full of low-mass stars or large density contrast voids. In this case and for $l \sim \eta$ the evolution of the scale factor would remain in the lowest order field equation, at $\mathcal{O}(\eta^2 L_N^{-2})$, with the energy density. Long-wavelength cosmological perturbations, on the other hand, would be squeezed in somewhere between the lowest Newtonian order, $\mathcal{O}(\eta^2 L_N^{-2})$, and first post-Newtonian order, $\mathcal{O}(\eta^4 L_N^{-2})$, for 00- and ij-field equations. Nevertheless, by construction, the cosmologically perturbed energy density must be strictly less than the Newtonian perturbed energy density, see Eq. (5.32).

Finally, if $\eta^2 > \epsilon$ then the expansion around FLRW is still valid (but may start to break down if $\eta \to 1$). This holds when close to very dense compact objects, such as neutron stars and black holes. This can be seen clearly from Eq. (5.33) because if the mass of an object considered is too great, M_N, or the the length scale considered is too small, L_N, then the gravitational potentials for post-Newtonian gravity, U, may become non-perturbative, $U \sim 1$. In this case cosmological perturbations are relegated to higher-order. Of course, in the real Universe these strong gravity scenarios tend to happen on small-scales, when $L_N \ll \eta L_C$. We would also expect the scale factor to be sourced at higher order too.

As a last remark, if one were to consider a system with structure on more than two scales, say N scales, this could be modelled with an N-parameter expansion. Nevertheless, structure on supercluster scales would always remain the dominant contributor to the scale factor, as discussed throughout this section.

9. Effective fluid dynamics

In the previous chapter we derived the field equations for our two-parameter perturbative expansion in terms of gauge-invariant variables. These equations can now be applied to realistic cosmological models that contain relativistic fluids with barotropic equations of state, as well as a cosmological constant, and non-relativistic dust-like matter that can be used to model dark matter and baryons. The result is a set of equations that can be used to calculate the effect of small-scale structure on the leading-order perturbations on large-scales. These equations contain terms that are quadratic in short-scale potentials and can be written as an effective fluid, as well as mode-mixing terms that couple scalar, vector and tensor perturbations in the large-scale cosmology – this effective fluid description is what we construct in this chapter. Both of these types of terms offer exciting possibilities for testing non-linear gravity with upcoming surveys.

9.1. Perturbations

The equations presented in Section 8.1 constitute a hierarchy of field equations, where the equations from Section 8.2.1 are the leading-order parts. Once the Friedmann equation (8.21) and the Newton-Poisson equation (8.23) have been solved, then their solutions can be substituted into the remaining higher-order equations to gain a set of solutions for the leading-order cosmological perturbations. This latter set of solutions, at $\mathcal{O}(\eta^4 L_N^{-2})$, contain linear-order cosmological large-scale potentials and post-Newtonian potentials from small-scales. With this in mind, we therefore seek to recast the $\mathcal{O}(\eta^4 L_N^{-2})$ equations in the form of the equations of standard first-order cosmological perturbation theory, modified by the addition of terms related to the existence of inhomogeneity on the length scale L_N. These terms will then be form the components of an *effective fluid* on large scales, whose characteristics and behaviour is determined by the small-scale gravitational physics. Such an approach has similarities to the effective fluid approaches in, for example see Refs. [36, 64], but in our case it is also required to reduce the number of gravitational degrees of freedom to be no more than the available number of field equations.

In the end, we want to reduce to a set of six perturbed field equations for six degrees of freedom (i.e. the 10 degrees of freedom in the metric minus the four coordinate freedoms). At present Eqs. (8.5)-(8.14), from Section 8.1, contain a total of sixteen degrees of freedom: six scalars ($\Phi^{(1,0)}$, $\Phi^{(1,2)}$, $\Phi^{(0,4)}$, $\Psi^{(1,0)}$, $\Psi^{(1,2)}$ and $\Psi^{(0,4)}$), six in the tensors ($\mathbf{h}_{ij}^{(1,0)}$, $\mathbf{h}_{ij}^{(1,2)}$ and $\mathbf{h}_{ij}^{(0,4)}$) and four in the vectors ($\mathbf{B}_i^{(1,0)}$ and $\mathbf{B}_i^{(1,2)}$). Taking into account the four degrees of freedom removed by gauge fixing implies that we need to remove six degrees of freedom. This is achieved by defining new sets of variables as follows:

$$\mathrm{U} \equiv -\tfrac{1}{2}\left(\Phi^{(0,2)} + \Phi^{(1,1)}\right) \tag{9.1}$$

$$\phi \equiv -\tfrac{1}{2}\left(\Phi^{(1,0)} + \Phi^{(1,2)} + \tfrac{1}{2}\Phi^{(0,4)}\right) \tag{9.2}$$

$$\psi \equiv \tfrac{1}{2}\left(\Psi^{(1,0)} + \Psi^{(1,2)} + \tfrac{1}{2}\Psi^{(0,4)}\right) \tag{9.3}$$

$$\mathrm{S}_j \equiv -\left(\mathbf{B}_j^{(1,0)} + \mathbf{B}_j^{(0,3)} + \mathbf{B}_j^{(1,2)}\right) \tag{9.4}$$

$$\mathrm{h}_{ij} \equiv \tfrac{1}{4}\left(\mathbf{h}_{ij}^{(1,0)} + \mathbf{h}_{ij}^{(1,2)} + \tfrac{1}{2}\mathbf{h}_{ij}^{(0,4)}\right) , \tag{9.5}$$

and

$$\delta\rho_{\mathrm{N}} \equiv \delta\boldsymbol{\rho}^{(0,2)} + \boldsymbol{\rho}^{(1,1)} \tag{9.6}$$

$$\delta\rho \equiv \boldsymbol{\rho}^{(1,0)} + \boldsymbol{\rho}^{(1,2)} + \tfrac{1}{2}\boldsymbol{\rho}^{(0,4)} \tag{9.7}$$

$$\delta\mathrm{p} \equiv \mathbf{p}^{(1,0)} + \mathbf{p}^{(1,2)} + \tfrac{1}{2}\mathbf{p}^{(0,4)} \tag{9.8}$$

$$\mathrm{v}_{\mathrm{N}i} \equiv \mathbf{v}_i^{(0,1)} \tag{9.9}$$

$$\mathrm{v}_i \equiv \mathbf{v}_i^{(1,0)} , \tag{9.10}$$

which are the variables we will use in Section 9.2. A number of these new variables could be considered to be “composite quantities”, as they contain a number of different perturbative orders in the same variable. For example, the variable ψ is dominated by $\mathcal{O}(\epsilon)$ terms on cosmological length scales L_C, but contains smaller terms at $\mathcal{O}(\eta^4)$ on small-scales L_N. This is quite atypical in cosmological perturbation theory. However, the way in which these quantities arise together in the field equations (under two spatial derivatives) suggest that they should be solved for together. Note that the above composite quantities, defined in Eqs. (9.1)-(9.10), are gauge invariant, as they are formed from linear summations of gauge-invariant quantities.

Note that U has been removed from the scalar potentials ϕ and ψ because we intend it to correspond to the Newtonian gravitational potential. Furthermore, one can understand U as the leading-order part of the gravitational field produced by non-linear density contrasts. The potentials ϕ and ψ, as well as h_{ij} and S_i, contain information about both the large-scale cosmological potentials and the small-scale post-Newtonian potentials. Likewise, the Newtonian density contrast is $\delta\rho_N$, and the cosmological *and* post-Newtonian density contrast is given by $\delta\rho$. The former of these is allowed to be arbitrarily large, while the latter is required to be small. Similar comments apply to v_{Ni} and v_i.

9.2. Effective field equations

In this section we will present the perturbed field equations that result from simultaneously considering non-linear structure on small-scales and linear structure on large-scales, given in Section 8.1, but in terms of an effective fluid (in conformal time). Some of the quantities that appear in these equations will then be explained in more detail later in this section. We hope this will allow the reader to see the most physically interesting aspects of this work first.

Explicitly, the field equations we present will be expressed in terms of the following set of gauge-invariant gravitational fields:

$$\{U, \phi, \psi, S_i, h_{ij}\}\,, \tag{9.11}$$

defined in Eqs. (9.1)-(9.5), as well as a corresponding set of gauge-invariant matter perturbations:

$$\{\delta\rho_N, \delta\rho, \delta p, v_{Ni}, v_i\}\,, \tag{9.12}$$

defined in Eqs. (9.6)-(9.10), where ρ, p and v_i correspond to the total energy-density, pressure and peculiar velocity, respectively. These perturbations have been defined to be perturbations about a spatially-flat FLRW geometry, which in a particular choice of coordinates can be written as[1]

$$ds^2 = a^2(\tau)\Big[-(1+2U+2\phi)d\tau^2+\big((1-2U-2\psi)\delta_{ij}+2h_{ij}\big)dx^i dx^j-2S_i d\tau dx^i\Big]\,. \tag{9.13}$$

[1]Note that the perturbations to the metric, see the line element in Eq. (9.13), are defined in Eqs. (9.1)-(9.5), and are directly analogous to the perturbations in the cosmological perturbation theory chapter in conformal time, see Chapter 3.

In what follows we will also require the Hubble rate in conformal time, defined as $\mathcal{H} \equiv a'/a$.

After simultaneously expanding the field equations in post-Newtonian and cosmological perturbation theories, from Eqs. (8.3), (8.4) and (8.21) we find the leading-order parts are given by the effective Friedmann equations

$$\mathcal{H}^2 = \frac{8\pi a^2}{3}\bar{\rho} + \frac{1}{3}\Lambda a^2 + \mathcal{O}(\eta^4) \tag{9.14}$$

$$\mathcal{H}' = -\frac{4\pi a^2}{3}(\bar{\rho} + 3\bar{p}) + \frac{1}{3}\Lambda a^2 + \mathcal{O}(\eta^4)\,, \tag{9.15}$$

where $\bar{\rho} = \bar{\rho}_{\mathrm{M}} + \bar{\rho}_{\mathrm{R}}$, is given by Eq. (8.16), and $\bar{p} = \bar{p}_{\mathrm{R}}$ are the leading order parts of the spatial averages of the energy density and pressure, respectively. Note that they have both radiation ($\bar{\rho}_{\mathrm{R}}$ and $\bar{p}_{\mathrm{R}}$), and dark and baryonic matter ($\bar{\rho}_{\mathrm{M}}$) contributions. From Eq. (8.23) and (8.7), the Newtonian gravitational field equation occurs at the same order in our expansion, and is given by

$$\nabla^2 \mathrm{U} = 4\pi a^2 \delta\rho_{\mathrm{N}} + \mathcal{O}(\eta^4)\,. \tag{9.16}$$

Note that only dark matter and baryonic matter contribute to $\delta\rho_{\mathrm{N}}$, and not radiation.

Subsequent orders of the perturbation expansion in the field equations yield the following two equations for the scalar part of the gravitational field:

$$\begin{aligned}&\frac{1}{3}\nabla^2\phi + \mathcal{H}\phi' + \mathcal{H}\psi' + \psi'' + 2\mathcal{H}'\phi \\ =\;& \frac{4\pi a^2}{3}\big(\delta\rho + \delta\rho_{\mathrm{eff}} + 3\delta p + 3\delta p_{\mathrm{eff}}\big) + \frac{2}{3}(\mathcal{D}^{ij}\mathrm{U})\mathrm{h}_{ij} - \frac{8\pi a^2}{3}\delta\rho_{\mathrm{N}}(\psi - \phi) + \mathcal{O}(\eta^5)\end{aligned} \tag{9.17}$$

and

$$\begin{aligned}&\frac{1}{3}\nabla^2\psi - \mathcal{H}\psi' - \mathcal{H}^2\phi \\ =\;& \frac{4\pi a^2}{3}\big(\delta\rho + \delta\rho_{\mathrm{eff}}\big) + \frac{1}{3}(\mathcal{D}^{ij}\mathrm{U})\mathrm{h}_{ij} - \frac{16\pi a^2}{3}\delta\rho_N\psi + \mathcal{O}(\eta^5)\,,\end{aligned} \tag{9.18}$$

from Eqs. (8.9) and (8.10), respectively. Note that perturbations in radiation, and dark and baryonic matter contribute to both $\delta\rho$ and δp.

As Eqs. (8.9) and (8.10) contain extra terms, when compared to standard cosmological perturbation theory, so do Eqs. (9.17) and (9.18), respectively. We now comment on several of these differences. Firstly, there are effective energy density and pressure terms, $\delta\rho_{\rm eff}$ and $\delta p_{\rm eff}$. These are solely due to the presence of non-linear structures on small scales, and are given explicitly in Eqs. (9.22) and (9.23), below. In other words, by writing the field equations in an effective fluid description, one can clearly identify that small-scale non-linearities lead to, amongst other things, an effective pressure on large-scales. Secondly, in the above equations, the Newtonian potential U couples to h_{ij} and there are extra source terms on the right-hand-side of these equations that are linear in ϕ and ψ. These interaction terms do not exist in standard cosmological perturbation theory (as stated previously) and vanish in the limit in which non-linear small-scale structures vanish. In general, the interaction terms should be expected to produce mode-mixing between scalar, vector and tensor parts of the gravitational field on cosmological scales and coupling between different Fourier modes in Fourier-space.

The above equations, Eqs. (9.17) and (9.18), may have consequences for high-precision observations. For example, the large-scale potential ϕ couples to cosmological length-scales and therefore could be important in calculations of the integrated Sachs-Wolfe effect [83], where the contribution $\int \phi' d\tau$ would strictly be affected by non-linearities from Eq. (9.17). Note that the integrated Sachs-Wolfe effect is just one relativistic effect, and can be observed from cross-correlations between the galaxy density and the CMB temperature. This effect is expected to occur beyond linear order in cosmological perturbations, however, with our two-parameter framework such an effect, from non-linearities, may occur at linear order. Therefore, determining values of observables may change using our two-parameter framework compared to standard cosmological perturbation theory.

The remaining parts of the gravitational field are the vector and tensor modes. For the vectors we find that we can write the following single equation to describe S_i, accurate up to order $\mathcal{O}(\eta^5)$:

$$
\begin{aligned}
&\nabla^2 S_i + 4\partial_i\big(\psi' + \mathcal{H}\phi\big) + 16\pi a^2\big(\bar{\rho} + \bar{p} + \delta\rho_N\big)(v_i - S_i) \qquad (9.19)\\
= \;& -16\pi a^2 Q_i^{\rm eff} - 8\pi a^2 \delta\rho_N S_i - 2(\partial_j\partial_i U)S^j + \mathcal{O}(\eta^5)\,,
\end{aligned}
$$

which is derived from Eq. (8.6). We can take the leading-order part of this equation, at $\mathcal{O}(\eta^3)$, and write it as the following simple Poisson equation

$$
\nabla^2 S_i + 4\partial_i(U' + \mathcal{H}U) + 16\pi a^2\big(\bar{\rho} + \bar{p}\big)v_{Ni} \;=\; -16\pi a^2 \delta\rho_N v_{Ni} + \mathcal{O}(\eta^4)\,, \quad (9.20)
$$

from Eq. (8.5). The leading-order part of the vector gravitational field, given by the solution to Eq. (9.20), is only sourced by small-scale quantities. This is the equation that was identified in the post-Friedmann approach of Ref. [130], and solved for numerically in Ref. [159]. For the full vector equation (9.19), accurate up to $\mathcal{O}(\eta^5)$, it can be seen that there exists sources on both small and large scales and mode-mixing, which are missing from Refs. [18, 19]. For example, the term $-2(\partial_j\partial_i\mathrm{U})\mathrm{S}^j$ was missing from [18, 19]. This term was added to the relativistic N-body simulation created by the authors of [18, 19], and tentative results suggest that S_i is corrected by of order 1% – which is exactly the signal expected from our two parameter formalism. In further work we would wish to verify this fully. This equation has an effective energy flux, Q_i^{eff}, which is due to small scale potentials. It also has extra source terms on the right-hand-side that are linear in $\mathrm{S_i}$. Both of these vanish when small-scale structures are absent. The explicit expression for Q_i^{eff} is given in Eq. (9.24), below, along with the other effective fluid quantities.

We now comment on the form of our vector field equations, Eqs. (9.20) and (9.19), in comparision to the one derived in Refs. [16–19], where a quasi-static approach to cosmological perturbation theory is taken to derive the field equations. Their derived vector field equation is similar to our Eq. (9.20), which is of order η^3. However, given their book-keeping, their derived vector field equation is expected to be accurate to the same order as the scalar potential $\phi \sim \eta^4$. From our formalism, for their vector field equation to be accurate to order η^4 (see our Eq. (9.19)), one would need to include the extra (mode-mixing) term $-2(\partial_j\partial_i\mathrm{U})\mathrm{S}^j$, which does not appear in Eq. (9.20). With the inclusion of this term we expect to improve the accuracy of their calculation of $\mathrm{S_i}$ by a factor of η, which is about 1%. Furthermore, a next-to-leading-order vector equation, derived at order η^4, differentiates our book-keeping from post-Newtonian gravity, where the next-to-leading order vector equation occurs at order η^5, this implies a correction to the leading-order vector potential of about 0.1%[2], an order of magnitude smaller than what is expected from our two-parameter expansion.

The final field equations we require, in order to complete our set to the desired order, is given as follows:

$$\begin{aligned}
&\nabla^2\mathrm{h}_{ij} - \mathrm{h}''_{ij} - 2\mathcal{H}\mathrm{h}'_{ij} + \mathcal{D}_{ij}(\phi - \psi) - 2\mathcal{H}\partial_{(j}\mathrm{S}_{i)} - \partial_{(j}\mathrm{S}'_{i)} \qquad (9.21)\\
= \quad & -8\pi a^2\Pi^{\mathrm{eff}}_{ij} - 8\pi a^2\delta\rho_{\mathrm{N}}\mathrm{h}_{ij} + 4(\partial^k\partial_{\langle i}\mathrm{U})\mathrm{h}_{j\rangle k} + 2(\mathcal{D}_{ij}\mathrm{U})(\phi + \psi) + \mathcal{O}(\eta^5)\,,
\end{aligned}$$

[2] We note that, from our two-parameter expansion, the existence of a vector equation at order η^4, Eq. (9.19), is an effect generated from having two parameters. Moreover, in our expansion, the term $-2(\partial_j\partial_i\mathrm{U})\mathrm{S}^j$ is a product of small and large-scale perturbations, and as derivatives act differently these scales this term is of order η^4. Using post-Newtonian book-keeping, on the other hand, implies this term (and a vector field equation) at order η^5.

and is derived from Eq. (8.14). This equation can be used to determine the tensor part of the gravitational field, h_{ij}. It also has an effective fluid source, $\Pi^{\rm eff}_{ij}$, which this time acts as an effective anisotropic stress and is formed from the quadratic contractions of the lower-order small-scale potentials, see Eq. (9.25). Again, the non-linear structure on small-scales couples the large-scale scalar and tensor parts of the cosmological gravitational fields, and again we have additional terms on the right-hand-side that are linear in h_{ij}, resulting in mode-mixing.

Finally, the effective fluid quantities in the perturbation equations above are given as follows:

$$\delta\rho_{\rm eff} = (\bar\rho + \bar{\rm p} + \delta\rho_{\rm N})({\rm v_N})^2 - \frac{1}{\pi a^2}{\rm U}\nabla^2{\rm U} + \frac{3}{4\pi a^2}\Big(\mathcal{H}^2{\rm U} + \mathcal{H}{\rm U}' - \frac{1}{2}(\nabla{\rm U})^2\Big) \tag{9.22}$$

$$\delta{\rm p}_{\rm eff} = \frac{1}{3}(\bar\rho + \bar{\rm p} + \delta\rho_{\rm N})({\rm v_N})^2 - \frac{1}{4\pi a^2}\Big({\rm U}'' + 3\mathcal{H}{\rm U}' - \frac{7}{6}(\nabla{\rm U})^2 + a^2{\rm U}(\Lambda - 8\pi\bar p)\Big) + \frac{1}{3\pi a^2}{\rm U}\nabla^2{\rm U} \tag{9.23}$$

$$Q^{\rm eff}_i = \big(\bar\rho + \bar{\rm p} + \delta\rho_{\rm N}\big){\rm v}_{{\rm N}i} + \frac{1}{4\pi a^2}\partial_i({\rm U}' + \mathcal{H}{\rm U}) \tag{9.24}$$

$$\Pi^{\rm eff}_{ij} = (\bar\rho + \bar{\rm p} + \delta\rho_{\rm N}){\rm v}_{{\rm N}\langle i}{\rm v}_{{\rm N}j\rangle} - \frac{1}{4\pi a^2}\partial_{\langle i}{\rm U}\partial_{j\rangle}{\rm U} - \frac{1}{2\pi a^2}{\rm U}\mathcal{D}_{ij}{\rm U}\,. \tag{9.25}$$

It can be seen that each of these quantities was constructed only from variables that correspond to small-scale gravitational fields, or background quantities, which have been shown to be calculated from the average of small-scale quantities, see Section 8.2.1. We therefore have a hierarchy of equations that can be solved order-by-order: firstly, the Friedmann and Newtonian equations for a and U, respectively, see Eqs. (9.14), (9.15) and (9.16). Then the equations which contain large-scale perturbations can be solved for, Eqs. (9.17)-(9.21). The former of these sets are already calculated routinely in modern N-body simulations of Friedmann cosmology. The latter are modified versions of the usual cosmological perturbation equations on large scales, and can be used to find post-Newtonian equations on small scales (as recently solved for numerically in Refs. [16–19]) using the effective fluid parameters previously calculated. The above effective quantities, in Eqs. (9.22)-(9.25), contain terms that would normally only be included in second or third order in cosmological perturbation theory. In particular, the term $\delta\rho_{\rm N}{\rm v}_{{\rm N}\langle i}{\rm v}_{{\rm N}j\rangle}$ in Eq. (9.25) would appear at third order in standard perturbation theory, but here should be expected to source a gravitational "slip" in the leading-order part of the large-scale physics.

Solving the higher-order equations in our perturbation hierarchy will inevitably be complicated by the additional "mode-mixing" terms in the cosmological perturbation equations. This will require more sophisticated techniques than at leading-order in standard cosmological perturbation theory as the usual methodology of separating equations like (9.19) and (9.21) into scalar, vector and tensor parts [128] is much more difficult to apply here. This is due to the fact terms like $(\mathcal{D}_{ij}\mathrm{U})(\phi + \psi)$ do not have scalar, vector and tensor parts that are easy to identify. This term, for example, is a scalar multiplied by a tensor, and in general should be expected to contain scalar, vector and tensor parts. This does not mean that such a separation is impossible – indeed we very much expect it to be possible. It just means that the resulting equations are very messy to write down, which is the reason why we have chosen to present these equations without such a decomposition. As highlighted in Chapter 8.1, such mode-mixing terms suggest that it may in fact be possible to generate vector and tensor modes from scalar fluctuations, which is already well known in second-order cosmological perturbation theory [127, 134], but is not usually seen at first order.

One should also note that certain terms, for example $\frac{8\pi a^2}{3}\delta\rho_{\mathrm{N}}(\psi - \phi)$ in Eq. (9.17), also mean that Fourier modes no longer decouple in a trivial way as they do in standard first-order perturbation theory, even if no mode-mixing occurs. This is because the Fourier transforms of such terms are expressible only in terms of a convolution integral over all Fourier modes. Our approach can be compared to the effective fluid approach studied previously in [36, 64], as well as the large and small wavelength split used in [99, 100]. Finally, the reader should also be warned that manipulation of these equations is considerably more difficult than in either cosmological perturbation theory or standard post-Newtonian theory. This is due to different derivative operators changing the order to the terms they operate on in different ways.

By writing the two-parameter perturbed field equations as an effective fluid we have substantially simplified the field equations given in Section 8.1 and this has allowed us to make further direct comparisons with cosmological perturbation theory more easily.

9.3. Discussion

We will discuss how our two-parameter expansion, defined in Chapter 5, with field equations in terms of gauge-invariant quantities given in Chapter 8, and in terms an effective fluid in this chapter, compares to other approaches derived in the literature.

The approaches delveloped by Milillo *et al.* [130] and Adamek *et al.* [18] all use post-Newtonian-like expansions. Instead of expanding the metric around a Minkowski metric an FLRW metric is used and spatial derivatives are large compared to time derivatives – this is equivalent to the post-Newtonian book-keeping (see Chapter 4). Baumann *et al.*, [36], uses a quasi-static limit of cosmological perturbation theory (this is problematic because it requires a priori that density contrasts remain small, which is not true in the late Universe) and write the field equations as an effective fluid. These expansions are in a single parameter, with a single characteristic length scale. On the other hand, our formalism includes two types of perturbations: post-Newtonian perturbations and cosmological perturbations, which behave differently under space-time derivatives, and vary on different length scales. This truly enables us to clearly see the effects of small-scale non-linearities on the large-scale expansion of the Universe. Our field equations are mostly in agreement with Baumann *et al.*, Adamek *et al.* [18], and Milillo *et al.*, [130]. There are, however, differences.

Baumann *et al.* include terms which should in fact be excluded if completing the quasi-static limit correctly [36]. For example, in their non-linear, next to leading-order equations, they include both leading-order and next-to-leading-order scalar potentials with two time derivatives. However, only the former term is necessary when taking the quasi-static limit. Additionally, Baumann *et al.* claim that the equations in Ref. [36] provide the dynamics of long wavelength scalar fluctuations sourced by products of short wavelength fluctuations. We believe their equations in fact show how the dynamics of short wavelength scalar fluctuations are sourced by product of short wavelength fluctuations, because they consider a single parameter expansion in the near-zone, where spatial derivatives add largeness, not in the wave-zone (where long wavelength fluctuations exist). Furthermore, the expansions in Refs. [18, 36, 130] all include a background energy density during dust domination, which does not exist in the late Universe, where matter is highly inhomogeneous, such an energy density is excluded for dust in our two-parameter expansion (see Section 5.2.3).

As stated previously, the book-keeping of the $0i-$vector potential in our two-parameter expansion is different to those derived by Adamek *et al.* and Baumann *et al.*. Their leading-order vector potential is a hundred times smaller than what is expect from our two-parameter expansion and post-Newtonian gravity. This is because both approaches expand the metric in terms of smallness $\mathcal{E}, \mathcal{E}^2$, and so on, at the offset. However, if they expanded the energy-momentum tensor fully they would find that both peculiar velocities are in fact not order $\mathcal{E}$, but order $\mathcal{E}^{\frac{1}{2}}$ for non-linear structure (see Section 5.3.1). This implies the leading-order $0i-$vector potential are of order $\mathcal{E}^{\frac{3}{2}}$, not of order $\mathcal{E}$.

In both Refs. [36] and [130] they split the perturbed field equations into linear and non-linear parts after a quasi-static expansion, which are then solved for separately even though some non-linear and linear terms may exist at the same order in perturbations. We observe this split is arbitrary as there are an infinite number of ways to split the perturbed field equations, therefore the process of doing so is not well defined. In the formalism discussed in this thesis, however, the perturbed field equations are derived and split by order of magnitude given by the strict book-keeping in Chapter 5. Note that, like Baumann *et al.*, we also write our field equations as an effective fluid, in this chapter.

Baumann *et al.* include scalar perturbations alone. The expansion outlined in this thesis, however, is completely general and includes tensor and vectors perturbations. The inclusion of these perturbations is crucial as we find there exists mode-mixing, of scalars, vectors and tensors, at orders in the field equations where non-linearities are present, and so Ref. [36] has important missing terms which contribute to the non-linear equations they are solving for. Similarly, Adamek *et al.* are missing important mode-mixing terms which, given their own book-keeping, should be included in their equations [18]. These include mode-mixing terms that appear in our Eqs. (9.17) and (9.19). Their justification for this may be empirical, from simulations, but it is not derivable from their book-keeping.

It is proposed in Ref. [18] that their expansion could be valid for modelling neutrinos, not only dust. This is possible if neutrinos are diffuse enough. If they are not diffuse, but appear at leading-order, we expect that peculiar velocities are such that $v \sim 1$. This would couple to 'cosmological' perturbations in our two-parameter framework. Therefore, at leading-order, neutrinos behave like perturbations in the wave-zone rather than the near-zone.

Finally, the quasi-static approximation of cosmological perturbation theory has been used to calculate relativistic corrections to large-scale structure observations. For example, in Ref. [161], this approximation is used to calculate the relativistic corrections to HI intensity mapping up to third order in perturbation theory. This approximation means terms given by time derivatives acting on potentials are omitted completely. On the other hand, in our two-parameter expansion these terms are strictly relegated one order higher in η for each time derivative, due to the book-keeping outlined in Chapter 5, and may appear at high-order in perturbation theory. This implies that any calculation of relativistic observables which uses the quasi-static limit, by entirely omitting terms which correspond to time derivatives on potentials, may be incomplete.

10. Conclusions and further work

In this thesis we propose and construct a two-parameter perturbation expansion around an FLRW background that simultaneously describes non-linear structures on small-scales and linear structures on large-scales. Moreover, the two-parameter formalism can model the entire evolution of the Universe by including radiation, dust and a cosmological constant, Λ. In doing so we use both cosmological and post-Newtonian perturbation theories. At lowest-order, radiation and Λ fit naturally into the cosmological sector of our theory, whereas dust fits naturally into the post-Newtonian sector of our theory. As this expansion is able to model large density contrasts and different matter components it therefore both contains the essential features of the real Universe and has a number of potential advantages over standard cosmological perturbation theory.

The book-keeping outlined in this thesis enables us to derive the two-parameter perturbed field equations valid for structure on the order of a fraction of the horizon size, the two-parameter gauge transformations of the matter and gravity sectors of our theory, and construct gauge-invariant quantities. We find that out of the gauges traditionally used in cosmological perturbation theory only the Newtonian gauge is applicable to post-Newtonian perturbations at lowest-order, and therefore also our two-parameter expansion. This may be of importance for those who use other gauges, for example, the synchronous gauge, in studies of post-Newtonian gravity [13, 167] or the quasi-static limit of standard perturbation theory. The consistency of the gauge transformations requires not only gravitational potentials and matter perturbations at the orders expected from post-Newtonian gravity and cosmological perturbation theory alone, but also a number of others at orders in perturbation which may not naively have been expected. We have therefore identified a minimal set of perturbations that are required for mathematical consistency of the problem, and written down gauge-invariant versions of the field equations that contain all such perturbations. These equations were derived to account for non-linear structure on the scales of clusters and superclusters along with ultra-large-scale cosmological perturbations, and so models the Universe on scales of a fraction of the horizon size. We also discuss the application of our formalism to a universe containing other gravitational systems. This includes a universe containing post-Newtonian structure on solar system scales, for which our field equations are consistent with post-Newtonian gravity up to the accuracy of current observations, but they differ at higher-order. In the limit of setting the cosmological expansion parameter to zero we recover standard post-Newtonian gravity. However, we do not recover standard cosmological perturbation theory during dust domination by setting the post-Newtonian expansion parameter to zero. It is recovered, however, by setting the leading-order inhomogeneous part of the Newtonian energy density to zero.

We find that the small-scale Newton-Poisson equation for the scalar gravitational potential occurs at the same order in perturbations as the Friedmann equation, but that they can be separated after the introduction of a suitable homogeneity scale. At leading order, this results in the small-scale Newton-Poisson equation sourced by the inhomogeneous part of the Newtonian energy density, and the large-scale Friedmann equations sourced by the spatial average of the leading-order parts of the energy density and pressure, and the cosmological constant. A nice feature of our equations is that a universe with dust is sourced by the average of the Newtonian rest-mass energy density, not a fictitious time-dependent background contribution.

We find later that although there is no back-reaction, from small-scale inhomogeneities, on the background expansion a (at leading order), this does not mean that the effects of small-scale structure on the large-scale cosmological perturbations are small. In fact we find in the higher-order field equations quadratic Newtonian potentials within the effective fluid terms, which source cosmological large-scale perturbations, along with post-Newtonian and mixed potentials (this mixing-of-scales is not found in cosmological perturbation theory). Critically, the two-parameter expansion allows us to clearly identify how small-scale structure can source the growth of first-order cosmological potentials on large-scales, through non-linearities, mode-mixing and mixing-of-scales in the field equations (all arising from the non-linearity of Einstein's theory). Such effects are beyond the scope of standard linear perturbation theory. We find Newtonian potentials are only important to the dynamics if our cosmology contains dust, otherwise cosmological perturbation theory should be a good approximation to the governing equations. Because our perturbation theory expansion contains the essential features of the late Universe it is advantageous over standard cosmological perturbation theory applied to epochs which include a proportion of non-linear matter. The inclusion of different fluids is important because it allows us to identify relativistic effects from our two-parameter expansion over different epochs of our Universe. It would be of interest, in further work, to calculate these potentials in relativistic N-body simulations. We expect the results to be similar to those derived in Ref. [17], but our equations do differ (*e.g.* our equations include mode-mixing and equations at new orders) so we expect their solutions to differ – they literally describe different physics. For example, our gravitomagnetic potential is a hundred times smaller than what is derived in Ref. [17], but is a result expected from Ref. [26].

Indeed, these beyond-leading-order equations contain valuable information about non-linear gravity, and could potentially be used to identify relativistic effects, which actually behave like biases in observations of large-scale structure. In the calculation of relativistic corrections to galaxy number counts the quasi-static limit of second-order cosmological perturbation theory is normally taken and, for example, the two-point correlation functions are derived [48]. Our book-keeping differs from this approach because, for example, peculiar velocities, cosmological large-scale potentials, and gravitomagnetic potentials are taken to occur at different orders compared to standard perturbation theory. Therefore the significance of each contribution to galaxy number counts, due to light travelling through an inhomogeneous universe, may differ using our approach.

These relativistic effects are of significance for the next generation of high precision surveys, such as SKA, Euclid and LSST, which will probe non-linear density contrasts on unprecedented scales – a significant fraction of our entire horizon. For example, they hope to probe primordial non-Gaussianity using observations of the late Universe. Relativistic contributions to the bispectrum, due to non-linearities (characterised in our two-parameter framework) may be degenerate with this primordial non-Gaussianity. In other words, non-linearities from Einstein's equations may contaminate signals of primordial non-Gaussianity. In further work, it would be of interest to calculate such corrections using our two-parameter framework and compare it to the frameworks described in Refs. [79, 167]. Detecting these effects would allow us to test Einstein's general relativity on unprecedented scales.

Accounting for the effects of non-linearities is also important for observations of the CMB. As photons travel from the surface of last scattering to us, their energy was effected by non-linear inhomogeneous structure (the integrated Sachs-Wolfe effect) which distorts the CMB we observe. The integrated Sachs-Wolfe effect is observed to be of order 10% [117] larger than what is expected theoretically. Our formalism could be used to calculate the integrated Sachs-Wolfe effect, determined from large-scale cosmological gravitational potentials, which are affected by quadratic Newtonian potentials. There are many other questions relating to the effects of non-linear structure on astrophysical and cosmological observables, such as large-scale magnetic fields which are much larger than predicted theoretically [84], which our expansion may also be useful for understanding.

Additionally, theoretical extensions of this work may allow for the nature of relativistic gravity to be probed more generally, as is done in parameterized-post-Newtonian gravity. This has been undertaken using parameterized post-Newtonian cosmology. In Ref. [150] four parameters are needed to characterize deviations from general relativity for conservative theories of gravity. These parameters could then be tested observationally. This would enable the determination of qualitative differences between different theories of gravity while removing leading-order degeneracies between them. Furthermore, it could help us answer how close gravity is to general relativity on the largest scales.

By presenting the higher-order field equations in terms of an effective fluid we are able to highlight the similarities and differences between our formalism and standard cosmological perturbation theory, post-Newtonian gravity and new approaches [18, 36, 130]. We expect this to aid further application of our equations by allowing some standard techniques from cosmological perturbation theory to be imported. This description also enables an easier physical interpretation of the effects of nonlinearities in the field equations, which clearly lead to, for example, a large-scale effective pressure and anisotropic stress. Since the effective fluid terms are all constructed from the solution to the short-scale Newtonian gravitational potential, their properties should be able to be determined from Newtonian N-body simulations, and the field equations can be solved for order-by-order in perturbations. Once the form of these effective fluids has been identified, one can proceed to solve the cosmological equations for the long-wavelength perturbations. This method of solution is available to us because of the hierarchical nature of the perturbation equations – short-scale fluctuations appear at lower-order compared to cosmological perturbations, and so can be solved for before cosmological perturbations. Understanding the consequences of these relativistic effects for the formation of non-linear structure in the Universe is of importance not only for removing sources of observational bias, but also because it has the potential to offer new ways of probing Einstein's theory on unprecedented scales within cosmology.

A. Energy-momentum tensor for dust

We present the stress energy tensor for dust only, $T_{\mu\nu} = T_{M\mu\nu}$, in this appendix. All the other appendices also contain calculations for dust only. This is because it is really the presence of dust that leads to non-linear small-scale dynamics, so we expect the appendices to be of most use for future applications which model non-linearities in the late Universe, for example in the calculation of relativistic corrections to Newtonian gravity in N-body simulations.

Expanding in both ϵ and η the non-vanishing components of the tensor $T_{M\mu\nu}$ are given by

$$T_{M00} = T^{(0,2)}_{M00} + T^{(1,0)}_{M00} + T^{(1,1)}_{M00} + T^{(1,2)}_{M00} + \frac{1}{2}T^{(0,4)}_{M00} + \dots \tag{A.1}$$

$$T_{M0i} = T^{(0,3)}_{M0i} + T^{(1,2)}_{M0i} + \dots \tag{A.2}$$

$$T_{Mij} = T^{(1,0)}_{Mij} + T^{(1,2)}_{Mij} + \frac{1}{2}T^{(0,4)}_{Mij} + \dots , \tag{A.3}$$

where ellipses again indicate higher-order terms that we will not consider in this thesis. The terms on the right-hand side of Eq. (A.1) are given by

$$T^{(0,2)}_{M00} = \rho^{(0,2)}_{M} \sim \frac{\eta^2}{L_N^2} \tag{A.4}$$

$$T^{(0,4)}_{M00} = \rho^{(0,4)}_{M} - 2h^{(0,2)}_{00}\rho^{(0,2)}_{M} + 2\rho^{(0,2)}_{M} v^{(0,1)i}_{M} v^{(0,1)}_{Mi} \sim \frac{\eta^4}{L_N^2} \tag{A.5}$$

$$T^{(1,0)}_{M00} = \rho^{(1,0)}_{M} \sim \frac{\epsilon}{L_C^2} \tag{A.6}$$

$$T^{(1,1)}_{M00} = \rho^{(1,1)}_{M} \sim \frac{\epsilon\eta}{L_N^2} \tag{A.7}$$

$$T^{(1,2)}_{M00} = \rho^{(1,2)}_{M} - h^{(1,0)}_{00}\rho^{(0,2)}_{M} \sim \frac{\epsilon\eta^2}{L_N^2} , \tag{A.8}$$

while the terms in Eq. (A.2) are given by

$$T^{(0,3)}_{M0i} = -a\rho^{(0,2)}_{M} v^{(0,1)}_{Mi} \sim \frac{\eta^3}{L_N^2} \tag{A.9}$$

$$\begin{aligned} T^{(1,2)}_{M0i} &= -a\left(\rho^{(0,2)}_{M} v^{(1,0)}_{Mi} + \rho^{(1,1)}_{M} v^{(0,1)}_{Mi}\right) - a\rho^{(0,2)}_{M} h^{(1,0)}_{0i} + \text{ terms of size } \left[\frac{\epsilon\eta^2}{L_C^2}\right] \\ &\sim \frac{\epsilon\eta^2}{L_N^2} + \frac{\epsilon\eta^2}{L_C^2}, \end{aligned} \tag{A.10}$$

and the terms in Eq. (A.3) are given by

$$T^{(0,4)}_{Mij} = 2a^2\rho^{(0,2)}_{M} v^{(0,1)}_{Mi} v^{(0,1)}_{Mj} + a^2 p^{(0,4)}_{M}\delta_{ij} \sim \frac{\eta^4}{L_N^2} \tag{A.11}$$

$$T^{(1,2)}_{Mij} = a^2 p^{(1,2)}_{Mij} + \text{ terms of size } \left[\frac{\epsilon\eta^2}{L_C^2}\right] \sim \frac{\epsilon\eta^2}{L_N^2} + \frac{\epsilon\eta^2}{L_C^2} \tag{A.12}$$

$$T^{(1,0)}_{Mij} = a^2 p^{(1,0)}_{M}\delta_{ij} \sim \frac{\epsilon}{L_C^2}\,. \tag{A.13}$$

All components of the energy-momentum tensor for dust only, apart from those presented in Eqs. (A.10) and (A.12), differ from the total energy-momentum tensor, presented in Section 6.1.2.

This completes the list of expanded energy-momentum tensor components for the matter fluid necessary to calculate the field equations up to the order considered in Chapter 6.

B. The field equations for dust

Using the conditions given in Eq. (6.39), *i.e.* $\epsilon \sim \eta^2$ and $L_N \sim \eta L_C$, we write the field equations for dust (see [97]).

B.1. Background-order potentials

The leading-order part of the field equations, in our formalism, comes in at $\mathcal{O}(\eta^2 L_N^{-2})$ and is given by

$$\frac{\ddot{a}}{a} + \frac{1}{6a^2}\nabla^2 h_{00}^{(0,2)} = -\frac{4\pi}{3}\rho_M^{(0,2)} \,. \tag{B.1}$$

This equation results from Eqs. (6.4), (6.5) and (A.4), and is a combination of both the Raychaudhuri equation and the Newton-Poisson equation. We can see only the rest mass density, $\rho_M^{(0,2)}$, is the source of both the Newtonian gravitational field and the large-scale acceleration equation.

At the same order of accuracy, we find that the leading-order contribution to the trace of the ij-field equations is given by

$$\frac{\dot{a}^2}{a^2} - \frac{1}{6a^2}\left(\nabla^2 h_{ii}^{(0,2)} - h_{ij,ij}^{(0,2)}\right) = \frac{8\pi}{3}\rho_M^{(0,2)} \,. \tag{B.2}$$

This equation is derived from Eqs. (6.16), (6.17) and (A.4), and is a combination of the Friedmann equation and the Newton-Poisson equation for the trace of the post-Newtonian potential $h_{ii}^{(0,2)}$.

Finally, the leading-order trace-free part of the ij-field equations is at $\mathcal{O}(\eta^2 L_N^{-2})$, and is the same as what is derived with the inclusion of non-anisotropic radiation and cosmological constant, see Eq. (6.44).

B.2. Vector potentials

Now let us consider the $0i$-field equations. The leading-order contribution to these equations comes in at $\mathcal{O}(\eta^3 L_N^{-2})$, and is given by

$$\nabla^2 h_{0i}^{(0,3)} - h_{0j,ij}^{(0,3)} - a\dot{h}_{ij,j}^{(0,2)} + a\dot{h}_{jj,i}^{(0,2)} + 2\dot{a}h_{00,i}^{(0,2)} = 16\pi a^2 \rho_M^{(0,2)} v_{Mi}^{(0,1)} \, . \tag{B.3}$$

This equation is the result of using Eqs. (6.11), (6.12) and (A.9). It can be considered as the governing equation for small-scale vector potentials and purely consists of post-Newtonian perturbations, which will source phenomena such as the Lense-Thirring effect.

At next-to-leading-order in the $0i$-field equation, at $\mathcal{O}(\eta^4 L_N^{-2})$, we find from Eqs. (6.13)-(6.15) and (6.32) that

$$\begin{aligned} &\nabla^2 \left(h_{0i}^{(1,0)} + h_{0i}^{(1,2)}\right) - \left(h_{0j}^{(1,0)} + h_{0j}^{(1,2)}\right)_{,ij} - h_{0j}^{(1,0)} h_{00,ij}^{(0,2)} - a\left(h_{ij}^{(1,0)} + h_{ij}^{(1,1)}\right)^{\cdot}_{,j} \\ &+a\left(h_{jj}^{(1,0)} + h_{jj}^{(1,1)}\right)^{\cdot}_{,i} + 2\dot{a}\left(h_{00}^{(1,0)} + h_{00}^{(1,1)}\right)_{,i} - 2h_{0i}^{(1,0)}\left(2\dot{a}^2 + a\ddot{a}\right) \\ = \;& 8\pi a^2 \left(2\rho_M^{(1,1)} v_{Mi}^{(0,1)} + \rho_M^{(0,2)}\left(h_{0i}^{(1,0)} + 2v_{Mi}^{(1,0)}\right)\right) \, . \end{aligned} \tag{B.4}$$

This equation can be thought of as the governing expression for the large-scale vector potentials at late times.

B.3. Higher-order scalar potentials

The next-to-leading-order 00-field equation is $\mathcal{O}(\eta^3 L_N^{-2})$, and is given by a Newton-Poisson equation, derived from Eqs. (6.9) and (A.7). It is sourced only by the mixed-order matter energy density $\rho_M^{(1,1)}$, and is the same with the inclusion of radiation and Λ, see Eq. (6.47).

The metric perturbations that correspond to cosmological scalar potentials are $h_{00}^{(1,0)}$ and $h_{ii}^{(1,0)}$. The governing equations for both of these perturbations occur with post-Newtonian and mixed order potentials at $\mathcal{O}(\eta^4 L_N^{-2})$. From the 00-field equation, at this order, we therefore find that

$$\begin{aligned} &\nabla^2 \left(h_{00}^{(1,0)} + h_{00}^{(1,2)} + \frac{1}{2}h_{00}^{(0,4)}\right) + \frac{1}{2}\left(\nabla h_{00}^{(0,2)}\right)^2 + a^2\left(h_{ii}^{(0,2)} + h_{ii}^{(1,0)}\right)^{\cdot\cdot} \\ &-2\left[a\left(h_{0i}^{(0,3)} + h_{0i}^{(1,0)}\right)_{,i}\right]^{\cdot} + 2a\dot{a}\left(h_{ii}^{(0,2)} + h_{ii}^{(1,0)}\right)^{\cdot} - \frac{1}{2}h_{00,i}^{(0,2)}\left(2h_{ij,j}^{(0,2)} - h_{jj,i}^{(0,2)}\right) \end{aligned} \tag{B.5}$$

$$
\begin{aligned}
&-h^{(0,2)}_{00,ij}\left(h^{(1,0)}_{ij}+h^{(0,2)}_{ij}\right)+3a\dot{a}\left(h^{(0,2)}_{00}+h^{(1,0)}_{00}\right)^{\cdot} \\
= \quad & -8\pi a^2\left[\rho^{(1,0)}_M+\rho^{(1,2)}_M+\frac{1}{2}\rho^{(0,4)}_M-\rho^{(0,2)}_M\left(h^{(1,0)}_{00}+h^{(0,2)}_{00}\right)\right. \\
&\left.+3\left(p^{(1,0)}_M+p^{(1,2)}_M+\frac{1}{2}p^{(0,4)}_M\right)+2\left(v^{(0,1)}_{Mi}\right)^2\rho^{(0,2)}_M\right],
\end{aligned}
$$

which has been derived using Eqs. (6.6)-(6.8), (6.10), (A.5), (A.6), (A.8), (A.11) and (A.13).

The ij-field equation, at $\mathcal{O}(\eta^3 L_N^{-2})$, can be split into its trace and trace-free parts. Firstly, the trace is derived from Eqs. (6.21) and (A.7) and is a Poisson equation for the trace of the mixed order potential $h^{(1,1)}_{ii}$, it is the same with the inclusion of non-anisotropic radiation and cosmological constant, see Eq. (6.49).

The trace of the ij-field equation, at $\mathcal{O}(\eta^4 L_N^{-2})$, gives

$$
\begin{aligned}
&\left(\delta_{ij}\nabla^2-\partial_i\partial_j\right)\left(h^{(1,0)}_{ij}+h^{(1,2)}_{ij}+\frac{1}{2}h^{(0,4)}_{ij}\right)+4\dot{a}\left(h^{(1,0)}_{0i}+h^{(0,3)}_{0i}\right)_{,i} \\
&-\left(2\dot{a}^2+a\ddot{a}\right)\left(h^{(1,0)}_{ii}+h^{(0,2)}_{ii}+3h^{(1,0)}_{00}+3h^{(0,2)}_{00}\right)-2a\dot{a}\left(h^{(1,0)}_{ii}+h^{(0,2)}_{ii}\right)^{\cdot} \\
= \quad & -4\pi a^2\left[\rho^{(0,2)}_M\left(h^{(1,0)}_{ii}+h^{(0,2)}_{ii}-h^{(1,0)}_{00}-h^{(0,2)}_{00}+4\left(v^{(0,1)}_{Mi}\right)^2\right)\right. \\
&\left.+4\left(\rho^{(1,0)}_M+\rho^{(1,2)}_M+\frac{1}{2}\rho^{(0,4)}_M\right)\right]+\mathcal{A}\,, \qquad \text{(B.6)}
\end{aligned}
$$

where the trace-free part will be given in the next section and we have simplified this expression using Eq. (6.48) multiplied by a factor of a^2. These expressions result from Eqs. (6.18)-(6.20), (6.22), (A.5), (A.6), (A.8), (A.11) and (A.13). The $\mathcal{A}$ in Eq. (B.6) represents the sum of all terms that are quadratic in lower-order potentials, and is given defined Eq. (6.51). Note that both $\mathcal{A}$ and the left-hand-side of Eq. (B.6) are the same as those in Eq. (6.50). This is simply because the perturbed metric, given by Eqs. (5.5)-(5.7), is the same for dust only and for dust, radiation and a cosmological constant.

B.4. Tensor potentials

The next-to-leading-order trace-free ij-field equation is at $\mathcal{O}(\eta^3 L_N^{-2})$, and is given by Eqs. (6.21) and (A.7). It is exactly the same as Eq. (6.52), derived with the inclusion of the non-anisotropic fluid of radiation and a cosmological constant, and has the same form as the lowest order trace-free ij-field equation, given in Eq. (6.46).

The remaining part of the field equations that we wish to consider is the trace-free part of the ij-component. At $\mathcal{O}(\eta^4 L_N^{-2})$ we find that this equation is given by

$$\nabla^2\left(h^{(1,0)}_{\langle ij\rangle}+h^{(1,2)}_{\langle ij\rangle}+\frac{1}{2}h^{(0,4)}_{\langle ij\rangle}\right)-2\left(h^{(1,0)}_{k\langle i}+h^{(1,2)}_{k\langle i}+\frac{1}{2}h^{(0,4)}_{k\langle i}\right)_{,j\rangle k} \tag{B.7}$$
$$-D_{ij}\left(h^{(1,0)}_{00}+h^{(1,2)}_{00}+\frac{1}{2}h^{(0,4)}_{00}-h^{(1,0)}_{kk}-h^{(1,2)}_{kk}-\frac{1}{2}h^{(0,4)}_{kk}\right)-a^2\left(h^{(1,0)}_{\langle ij\rangle}+h^{(0,2)}_{\langle ij\rangle}\right)^{\cdot\cdot}$$
$$-2\left(2\dot{a}^2+a\ddot{a}\right)\left(h^{(1,0)}_{\langle ij\rangle}+h^{(0,2)}_{\langle ij\rangle}\right)-3a\dot{a}\left(h^{(1,0)}_{\langle ij\rangle}+h^{(0,2)}_{\langle ij\rangle}\right)^{\cdot}+\frac{2}{a}\left[a^2\left(h^{(1,0)}_{0\langle i}+h^{(0,3)}_{0\langle i}\right)\right]^{\cdot}_{,j\rangle}$$
$$=-8\pi a^2\rho^{(0,2)}_M\left[h^{(1,0)}_{\langle ij\rangle}+h^{(0,2)}_{\langle ij\rangle}+2v^{(0,1)}_{M\langle i}v^{(0,1)}_{Mj\rangle}\right]+\mathcal{B}_{ij}\,,$$

where we used $\mathcal{B}_{ij}$ to denote the summation of all terms that are quadratic in lower-order potentials, and is defined in Eq. (6.54). Again, $\mathcal{B}_{ij}$ and the left-hand-side of Eq. (B.7) are the same as those in Eq. (B.7) because the perturbed metric, given by Eqs. (5.5)-(5.7), is the same for dust only and for dust, radiation and a cosmological constant. This expressions results from Eqs. (6.18)-(6.20), (6.22), (A.5), (A.6), (A.8), (A.11) and (A.13).

C. Two-parameter gauge transformations for dust

We perform the two-parameter transformation, given by Eq. (3.26), on the perturbed energy-momentum tensor for dust given in Appendix A. We then irreducibly decompose these transformations into scalar and divergence-less vector parts – there is no anisotropic stress and therefore no tensor part. Throughout this section we assume $L_N/L_C \sim \eta$, but not $\epsilon \sim \eta^2$.

C.1. Transformation of the energy-momentum tensor for dust

The transformation of $\mathbf{T}_{00}$: using the exponential map in Eq. (3.26), and the gauge generators specified in Eqs. (7.1) and (7.2), we find the following transformation at lowest order

$$\tilde{\rho}_M^{(0,2)} = \rho_M^{(0,2)} \sim \frac{\eta^2}{L_N^2}\,, \tag{C.1}$$

and at higher-order we have

$$\begin{aligned} &\tilde{\rho}_M^{(1,0)} + \tilde{\rho}_M^{(1,2)} - \tilde{h}_{00}^{(1,0)}\tilde{\rho}_M^{(0,2)} \\ = \quad &\rho_M^{(1,0)} + \rho_M^{(1,2)} - h_{00}^{(1,0)}\rho_M^{(0,2)} + \dot{\rho}_M^{(0,2)}\xi^{(1,0)0} + 2\rho_M^{(0,2)}\dot{\xi}^{(1,0)0} \\ \sim \quad &\frac{\epsilon\eta^2}{L_N^2}\,, \end{aligned} \tag{C.2}$$

and

$$\begin{aligned} &\frac{1}{2}\tilde{\rho}_M^{(0,4)} - \tilde{h}_{00}^{(0,2)}\tilde{\rho}_M^{(0,2)} + \tilde{\rho}_M^{(0,2)}\tilde{v}_M^{(0,1)i}\tilde{v}_{Mi}^{(0,1)} \\ = \quad &\frac{1}{2}\rho_M^{(0,4)} - h_{00}^{(0,2)}\rho_M^{(0,2)} + \rho_M^{(0,2)}v_M^{(0,1)i}v_{Mi}^{(0,1)} + \rho_{M,i}^{(0,2)}\xi^{(0,2)i} \end{aligned} \tag{C.3}$$

$$\sim \quad \frac{\eta^4}{L_N^2}\,,$$

The transformation of $T_{M00}^{(1,1)}$ is the same as the transformation of the total energy-momentum tensor $T_{00}^{(1,1)}$ given in Eq. (7.62). We note that the Stewart-Walker lemma tells us $\rho_M^{(0,2)}$ is gauge invariant because this is the only lowest-order contribution to the energy-density for dust [157].

The transformation of $\mathbf{T}_{0i}$: the same gauge transformations give the following results for the time-space components of the energy-momentum tensor:

$$-a\tilde{\rho}_M^{(0,2)}\tilde{v}_{Mi}^{(0,1)} = -a\rho_M^{(0,2)}v_{Mi}^{(0,1)} \sim \frac{\eta^3}{L_N^2} \tag{C.4}$$

and

$$\begin{aligned} & -a\tilde{\rho}_M^{(0,2)}\tilde{v}_{Mi}^{(1,0)} - a\tilde{\rho}_M^{(1,1)}\tilde{v}_{Mi}^{(0,1)} - a\tilde{\rho}_M^{(0,2)}\tilde{h}_{0i}^{(1,0)} \\ = \; & -a\rho_M^{(0,2)}v_{Mi}^{(1,0)} - a\rho_M^{(1,1)}v_{Mi}^{(0,1)} - a\rho_M^{(0,2)}h_{0i}^{(1,0)} + \rho_M^{(0,2)}\xi_{,i}^{(1,0)0} - a\left(\rho_M^{(0,2)}v_{Mi}^{(0,1)}\right)_{,j}\xi^{(1,0)j} \\ \sim \; & \frac{\epsilon\eta^2}{L_N^2}\,. \end{aligned} \tag{C.5}$$

The transformation of $\mathbf{T}_{ij}$: finally, the gauge transformation of the space-space components of the energy-momentum tensor gives

$$a^2\tilde{\rho}_M^{(0,2)}\tilde{v}_{Mi}^{(0,1)}\tilde{v}_{Mj}^{(0,1)} + \frac{1}{2}a^2\tilde{p}_M^{(0,4)}\delta_{ij} = a^2\rho_M^{(0,2)}v_{Mi}^{(0,1)}v_{Mj}^{(0,1)} + \frac{1}{2}a^2p_M^{(0,4)}\delta_{ij} \sim \frac{\eta^4}{L_N^2} \tag{C.6}$$

$$a^2\left(\tilde{p}_M^{(1,0)} + \tilde{p}_M^{(1,2)}\right)\delta_{ij} = a^2\left(p_M^{(1,0)} + p_M^{(1,2)}\right)\delta_{ij} \sim \frac{\epsilon\eta^2}{L_N^2}\,. \tag{C.7}$$

We note $p_M^{(1,0)}+p_M^{(1,2)}$ is gauge invariant because there is no homogeneous (or constant) background pressure. This is because at late times the Universe is dust dominated, but we allow for a small cosmological source of pressure.

C.2. Transformation of irreducibly-decomposed sources for dust

The irreducible decomposition of the quantities that appear in the energy-momentum tensor for dust are simplified by the fact that they are all three-scalars, with the exception of the three-velocity, v_{Mi}. This vector can be split into scalar and divergenceless vector parts:

$$v_{Mi} \equiv v_{M,i} + \hat{v}_{Mi}\,, \tag{C.8}$$

where $\hat{v}^{i}_{M,i} = 0$. The scalar degrees of freedom are given by ρ_M, p_M and v_M, while the only divergenceless vector is given by $\hat{v}_{Mi}$. There are no transverse and trace-free tensorial terms in the energy-momentum tensor for, as defined in Eq. (2.20).

Cosmological and mixed-order scalar and vector sources: using Eq. (C.1)-(C.7), we find that the irreducibly decomposed scalars transform according to

$$\tilde{\rho}^{(1,0)}_M + \tilde{\rho}^{(1,2)}_M = \rho^{(1,0)}_M + \rho^{(1,2)}_M + \dot{\rho}^{(0,2)}_M \delta t^{(1,0)} \tag{C.9}$$

$$\tilde{p}^{(1,0)}_M + \tilde{p}^{(1,2)}_M = p^{(1,0)}_M + p^{(1,2)}_M\,, \tag{C.10}$$

and the transformation of $\rho^{(1,1)}_M$ in terms of three-scalars and vectors is given in Eq. (7.73). The transformation of the scalar part of the three-velocity, $v^{(1,0)}_M$, and the divergenceless vector part, $\hat{v}^{(1,0)}_{Mi}$, are the same as those for the total three-velocity, derived from the divergence of Eq. (7.79).

Post-Newtonian scalar and vector sources: Eqs. (C.1)-(C.7) can also be used to find the transformation of the scalar and vector parts of the post-Newtonian sector of our theory, this gives

$$\tilde{\rho}^{(0,2)}_M = \rho^{(0,2)}_M\,, \tag{C.11}$$

the transformation of $\rho^{(0,4)}_M$ and $p^{(0,4)}_M$ are the same for the total energy density and pressure, see Eqs. (7.75) and (7.78), respectively. Furthermore, the transformation of the scalar part of the three-velocity, $v^{(0,1)}_M$, and the divergenceless vector part, $\hat{v}^{(0,1)}_{Mi}$, are the same as those for the total three-velocity, derived from the divergence of Eq. (7.80).

The leading-order parts of the post-Newtonian three-velocity, energy density and pressure are automatically gauge invariant for dust. This is to be expected, as these equations describe Newtonian gravity at leading order, which transforms trivially under general coordinate transformations. These results differ from the quasi-static limit of cosmological perturbation theory, as space and time derivatives are treated differently and velocities come in at different orders [107]. This completes our study of the gauge transformations of this tensor.

C.3. Gauge invariant quantities for dust

We construct gauge invariant quantities from perturbations of the energy-momentum tensor for dust. Our gauge invariant quantities will reduce to the matter sources of energy-momentum in the longitudinal gauge (when $E = B = F_i = 0$). We will do this first for the cosmological sector, and then for the post-Newtonian sector.

Cosmological and mixed-order quantities: we can construct the following three gauge-invariant scalars, corresponding to the mixed-order and cosmological energy density and pressure:

$$\boldsymbol{\rho}_M^{(1,0)} + \boldsymbol{\rho}_M^{(1,2)} = \rho_M^{(1,0)} + \rho_M^{(1,2)} + \dot{\rho}_M^{(0,2)} \left(aB^{(1,0)} - \frac{a^2}{2} \dot{E}^{(1,0)} \right) \tag{C.12}$$

$$\mathbf{p}_M^{(1,0)} + \mathbf{p}_M^{(1,2)} = p_M^{(1,0)} + p_M^{(1,2)} , \tag{C.13}$$

and the gauge invariant quantity $\boldsymbol{\rho}_M^{(1,1)}$ is equivalent to the total energy density gauge invariant quantity $\boldsymbol{\rho}^{(1,1)}$, see Eq. (7.106)

One further scalar, $\mathbf{v}_M^{(1,0)}$, and a divergence-free vector, $\hat{\mathbf{v}}_{Mi}^{(1,0)}$, can be extracted from the divergence of the gauge invariant quantity defined in Eq. (7.115). These are all of the gauge invariant quantities that can be constructed from the energy-momentum tensor for dust, in the cosmological and mixed-order sector of our theory.

Post-Newtonian quantities: in the post-Newtonian sector we have the gauge invariant quantity

$$\boldsymbol{\rho}_M^{(0,2)} = \rho_M^{(0,2)} , \tag{C.14}$$

we also have $\boldsymbol{\rho}_M^{(0,4)}, \mathbf{p}_M^{(0,4)}, \mathbf{v}_M^{(0,1)}$ and $\hat{\mathbf{v}}_{Mi}^{(0,1)}$, given in Eqs (7.108), (7.111), (7.113) and (7.114), respectively. We note that there is a strong similarity between the post-Newtonian gauge invariant quantities for dust and radiation, compared to dust alone. The fact that many of the post-Newtonian perturbations are themselves gauge invariant is unsurprising, as many of these objects appear in the Newtonian equations of hydrodynamics.

These gauge invariant quantities derived from the transformation of the energy-momentum for dust are all that are needed to write the field equations (up to the order we wish to consider) in terms of gauge invariant quantities, see the following appendix.

D. Dynamics of gauge invariant quantities for dust

With the gauge invariant quantities constructed in Appendix C.3 and Section 7.5.1, and the field equations in Appendix B, we can write the field equations for dust in terms of gauge invariant quantities. These equations take the same form as the field equations in the longitudinal gauge but are in fact valid in any coordinate system. Furthermore, these equations can be used to write down the governing equations for our gauge invariant quantities, which, upon specification of any particular gauge, reduce to the gauge-fixed Einstein equations. As before, we write down these equations under the assumptions $\epsilon \sim \eta^2$ and $L_N/L_C \sim \eta$.

D.1. Background-order potentials

The background-order 00-field equation can be used to write

$$\frac{\ddot{a}}{a} + \frac{1}{6a^2}\nabla^2\Phi^{(0,2)} = -\frac{4\pi}{3}\boldsymbol{\rho}_M^{(0,2)}\,, \tag{D.1}$$

while the trace of the background-order ij-equation gives

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{3a^2}\nabla^2\Phi^{(0,2)} = \frac{8\pi}{3}\boldsymbol{\rho}_M^{(0,2)}\,, \tag{D.2}$$

The background order trace-free ij-equation gives Eq. (8.1) and its derivative implies the conditions in Eq. (8.2). Note that all equations in this appendix are written with the substitution of the results in Eq. (8.2), because dust and radiation matter sources are non-anisotropic. These equations govern the leading-order part of the gravitational fields for non-relativistic matter, at $\mathcal{O}(\eta^2 L_N^{-2})$.

D.2. Vector potentials

We now use all $0i$-field equations. At order $\mathcal{O}(\eta^3 L_N^{-2})$, these give

$$\nabla^2 \mathbf{B}_i^{(0,3)} + 2\left(a\dot{\Phi}^{(0,2)} + \dot{a}\Phi^{(0,2)}\right)_{,i} = 16\pi a^2 \boldsymbol{\rho}_M^{(0,2)} \mathbf{v}_{Mi}^{(0,1)} \,. \tag{D.3}$$

Although $\mathbf{B}_i^{(0,3)}$ is a divergenceless vector, Eq. (8.5), has a divergenceless vector and scalar part, which can be separated out with a derivative. The same goes for the $\mathcal{O}(\eta^4 L_N^{-2})$ $0i$-field equation, which gives

$$\begin{aligned} &\nabla^2\left(\mathbf{B}_i^{(1,0)} + \mathbf{B}_i^{(1,2)}\right) + 2\left(a\left(\Phi^{(1,1)} - \Psi^{(1,0)}\right)^{\cdot} + \dot{a}\left(\Phi^{(1,1)} + \Phi^{(1,0)}\right)\right)_{,i} \\ &-2\left(2\dot{a}^2 + a\ddot{a}\right)\mathbf{B}_i^{(1,0)} - \mathbf{B}_j^{(1,0)}\Phi_{,ij}^{(0,2)} \\ =\ & 8\pi a^2\left(2\boldsymbol{\rho}_M^{(1,1)}\mathbf{v}_{Mi}^{(0,1)} + \boldsymbol{\rho}_M^{(0,2)}\left(\mathbf{B}_i^{(1,0)} + 2\mathbf{v}_{Mi}^{(1,0)}\right)\right) \,. \end{aligned} \tag{D.4}$$

D.3. Higher-order scalar potentials

The 00- and ij-trace field equation at $\mathcal{O}(\epsilon\eta L_N^{-2})$ gives exactly the same equation as Eq. (8.7) because $\rho^{(1,1)}$ by definition only has a contribution from matter perturbations, so $\boldsymbol{\rho}^{(1,1)} \equiv \boldsymbol{\rho}_M^{(1,1)}$. The derivative of Eq. (8.7) implies the condition in Eq. (8.8), which is substituted in throughout this section.

Using the 00-field equation, at $\mathcal{O}(\eta^4 L_N^{-2})$, we find

$$\begin{aligned} &\nabla^2\left(\Phi^{(1,0)} + \frac{1}{2}\Phi^{(0,4)} + \Phi^{(1,2)}\right) + \left(\nabla\Phi^{(0,2)}\right)^2 + 3a\dot{a}\left(3\Phi^{(0,2)} + \Phi^{(1,0)} - 2\Psi^{(1,0)}\right)^{\cdot} \\ &+3a^2\left(\Phi^{(0,2)} - \Psi^{(1,0)}\right)^{\cdot\cdot} + 6a\ddot{a}\left(\Phi^{(0,2)} - \Psi^{(1,0)}\right) - \frac{1}{2}\Phi_{,ij}^{(0,2)}\mathbf{h}_{ij}^{(1,0)} \\ =\ & -8\pi a^2\left(\boldsymbol{\rho}_M^{(1,0)} + \boldsymbol{\rho}_M^{(1,2)} + \frac{1}{2}\boldsymbol{\rho}_M^{(0,4)} + 3\left(\mathbf{p}_M^{(1,0)} + \mathbf{p}_M^{(1,2)} + \frac{1}{2}\mathbf{p}_M^{(0,4)}\right)\right. \\ &\left.-\boldsymbol{\rho}_M^{(0,2)}\left(\Phi^{(1,0)} + \Psi^{(1,0)} - 2\left(\mathbf{v}_{Mi}^{(0,1)}\right)^2\right)\right) \,. \end{aligned} \tag{D.5}$$

The trace of the ij-field equation gives, at $\mathcal{O}(\eta^4 L_N^{-2})$,

$$\begin{aligned} &-2\nabla^2\left(\Psi^{(1,0)} + \Psi^{(1,2)} + \frac{1}{2}\Psi^{(0,4)}\right) - 3\left(2\dot{a}^2 + a\ddot{a}\right)\left(\Phi^{(1,0)} - \Psi^{(1,0)} + 2\Phi^{(0,2)}\right) \\ &+6\dot{a}a\left(\Psi^{(1,0)} - \Phi^{(0,2)}\right)^{\cdot} \end{aligned}$$

$$= \quad -4\pi a^2 \left(4\left(\boldsymbol{\rho}_M^{(1,0)} + \boldsymbol{\rho}_M^{(1,2)} + \frac{1}{2}\boldsymbol{\rho}_M^{(0,4)}\right) + \boldsymbol{\rho}_M^{(0,2)}\left(2\Phi^{(0,2)} - \Phi^{(1,0)} - 3\Psi^{(1,0)}\right.\right.$$
$$\left.\left. +4\left(\mathbf{v}_{Mi}^{(0,1)}\right)^2\right)\right) + \mathcal{A}\,, \tag{D.6}$$

where we have defined terms that are quadratic in metric potentials as $\mathcal{A}$, given in Eq. (8.11). These are all of the scalar equations that exist at this order.

D.4. Tensor potentials

The trace-free ij-field $\mathcal{O}(\epsilon\eta L_N^{-2})$ equation is given by Eq. (8.12), as it is the same with the inclusion of non-relativistic matter, radiation and a cosmological constant. This is because we are considering non-anisotropic radiation and a cosmological constant. The derivative of Eq. (8.12) implies the conditions in Eq. (8.13). We substitute the results in Eq. (8.13) into all equations in this appendix.

Finally, the ij-field equation, at $\mathcal{O}(\eta^4 L_N^{-2})$, can be used to write the following trace-free equation:

$$-D_{ij}\left(\Phi^{(1,0)} + \Phi^{(1,2)} + \frac{1}{2}\Phi^{(0,4)} + \Psi^{(1,0)} + \Psi^{(1,2)} + \frac{1}{2}\Psi^{(0,4)}\right) - \left(2\dot{a}^2 + a\ddot{a}\right)\mathbf{h}_{ij}^{(1,0)}$$
$$+\frac{1}{2}\nabla^2\left(\mathbf{h}_{ij}^{(1,0)} + \mathbf{h}_{ij}^{(1,2)} + \frac{1}{2}\mathbf{h}_{ij}^{(0,4)}\right) + 4\dot{a}\left(\mathbf{B}_{(i,j)}^{(0,3)} + \mathbf{B}_{(i,j)}^{(1,0)}\right) + 2a\left(\mathbf{B}_{(i,j)}^{(0,3)} + \mathbf{B}_{(i,j)}^{(1,0)}\right)^{\cdot}$$
$$-\frac{3}{2}a\dot{a}\dot{\mathbf{h}}_{ij}^{(1,0)} - \frac{1}{2}a^2\ddot{\mathbf{h}}_{ij}^{(1,0)}$$
$$= \quad -8\pi a^2\boldsymbol{\rho}_M^{(0,2)}\left(\frac{1}{2}\mathbf{h}_{ij}^{(1,0)} + 2\mathbf{v}_{M\langle i}^{(0,1)}\mathbf{v}_{Mj\rangle}^{(0,1)}\right) + \mathcal{B}_{ij}\,, \tag{D.7}$$

where we have defined terms that are quadratic in metric potentials as $\mathcal{B}_{ij}$, given in Eq. (8.15). Importantly, we observe that, unlike in linear cosmological perturbation theory, our expansion scheme does not imply $\Phi^{(1,0)} = -\Psi^{(1,0)}$ or $\mathbf{h}_{ij}^{(1,0)} = 0$ during matter domination, because of the additional non-linearities in Eq. (D.7) that do not exist in first-order cosmological perturbation theory. This completes the full set of equations for dust in terms of gauge-invariant variables, up to the order in perturbations that we wish to consider here.

Bibliography

[1] **SKA** Collaboration. www.skatelescope.org.

[2] **Euclid** Collaboration. http://sci.esa.int/euclid/.

[3] **LSST** Collaboration. www.lsst.org.

[4] *Computer algebra package.* http://www.xact.es.

[5] *Computer algebra package.* http://www.xact.es/xPand.

[6] A. Abate and O. Lahav, *The Three Faces of Ω_m: Testing Gravity with Low and High Redshift SN Ia Surveys*, *Mon. Not. Roy. Astron. Soc.* **389** (2008) 47, [arXiv:0805.3160].

[7] **Virgo, LIGO Scientific** Collaboration, B. P. Abbott et al., *GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence*, *Phys. Rev. Lett.* **116** (2016) 241103, [arXiv:1606.04855].

[8] **Virgo, LIGO Scientific** Collaboration, B. P. Abbott et al., *Observation of Gravitational Waves from a Binary Black Hole Merger*, *Phys. Rev. Lett.* **116** (2016) 061102, [arXiv:1602.03837].

[9] **VIRGO, LIGO Scientific** Collaboration, B. P. Abbott et al., *GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2*, *Phys. Rev. Lett.* **118** (2017) 221101, [arXiv:1706.01812].

[10] **DES** Collaboration, T. Abbott et al., *Cosmology from cosmic shear with Dark Energy Survey Science Verification data*, *Phys. Rev.* **D94** (2016) 022001, [arXiv:1507.05552].

[11] **LSST Science, LSST Project** Collaboration, P. A. Abell et al., *LSST Science Book, Version 2.0*, arXiv:0912.0201.

[12] **Planck** Collaboration, R. Adam et al., *Planck 2015 results. I. Overview of products and scientific results*, *Astron. Astrophys.* **594** (2016) A1, [arXiv:1502.01582].

[13] J. Adamek, C. Clarkson, D. Daverio, R. Durrer, and M. Kunz, *Safely smoothing spacetime: backreaction in relativistic cosmological simulations*, arXiv:1706.09309.

[14] J. Adamek, C. Clarkson, R. Durrer, and M. Kunz, *Does small scale structure significantly affect cosmological dynamics?*, *Phys.Rev.Lett.* **114** (2015) 051302.

[15] J. Adamek, D. Daverio, R. Durrer, and M. Kunz, *General Relativistic n-body simulations in the weak field limit*, *Phys.Rev.* **D88** (2013) 103527.

[16] J. Adamek, D. Daverio, R. Durrer, and M. Kunz, *General relativity and cosmic structure formation*, *Nature Phys.* **12** (2016) 346–349.

[17] J. Adamek, D. Daverio, R. Durrer, and M. Kunz, *gevolution: a cosmological n-body code based on General Relativity*, *JCAP* **1607** (2016) 053.

[18] J. Adamek, R. Durrer, and M. Kunz, *N-body methods for relativistic cosmology*, *Class.Quant.Grav.* **31** (2014) 234006.

[19] J. Adamek, M. Gosenca, and S. Hotchkiss, *Spherically Symmetric N-body Simulations with General Relativistic Dynamics*, *Phys.Rev.* **D93** (2016) 023526.

[20] **Planck** Collaboration, P. A. R. Ade et al., *Planck 2015 results. XIII. Cosmological parameters*, *Astron. Astrophys.* **594** (2016) A13, [arXiv:1502.01589].

[21] **BOSS** Collaboration, S. Alam et al., *The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: cosmological analysis of the DR12 galaxy sample*, *Submitted to: Mon. Not. Roy. Astron. Soc.* (2016) [arXiv:1607.03155].

[22] **Euclid Theory Working Group** Collaboration, L. Amendola et al., *Cosmology and fundamental physics with the Euclid satellite*, *Living Rev. Rel.* **16** (2013) 6, [arXiv:1206.1225].

[23] L. Amendola et al., *Cosmology and Fundamental Physics with the Euclid Satellite*, arXiv:1606.00180.

[24] M. A. Amin, R. V. Wagoner, and R. D. Blandford, *A sub-horizon framework for probing the relationship between the cosmological matter distribution and metric perturbations*, *Mon. Not. Roy. Astron. Soc.* **390** (2008) 131–142, [arXiv:0708.1793].

[25] **BOSS** Collaboration, L. Anderson et al., *The clustering of galaxies in the SDSS-III Baryon Oscillation Spectroscopic Survey: baryon acoustic oscillations in the Data Releases 10 and 11 Galaxy samples*, *Mon. Not. Roy. Astron. Soc.* **441** (2014) 24–62, [arXiv:1312.4877].

[26] S. Andrianomena, C. Clarkson, P. Patel, O. Umeh, and J.-P. Uzan, *Non-linear relativistic contributions to the cosmological weak-lensing convergence*, *JCAP* **1406** (2014) 023.

[27] A. Ashtekar, B. Berger, J. Isenberg, and M. MacCallum, *General Relativity and Gravitation: A Centennial Perspective.* Cambridge University Press, 2015.

[28] É. Aubourg et al., *Cosmological implications of baryon acoustic oscillation measurements*, *Phys. Rev.* **D92** (2015) 123516, [arXiv:1411.1074].

[29] A. Avilez-Lopez, A. Padilla, P. M. Saffin, and C. Skordis, *The Parametrized Post-Newtonian-Vainshteinian Formalism*, *JCAP* **1506** (2015) 044, [arXiv:1501.01985].

[30] T. Baker, P. G. Ferreira, and C. Skordis, *The Parameterized Post-Friedmann framework for theories of modified gravity: concepts, formalism and examples*, *Phys. Rev.* **D87** (2013) 024015, [arXiv:1209.2117].

[31] T. Baker, P. G. Ferreira, C. Skordis, and J. Zuntz, *Towards a fully consistent parameterization of modified gravity*, *Phys. Rev.* **D84** (2011) 124018, [arXiv:1107.0491].

[32] J. M. Bardeen, *Gauge Invariant Cosmological Perturbations*, *Phys.Rev.* **D22** (1980) 1882–1905.

[33] R. A. Battye and J. A. Pearson, *Effective action approach to cosmological perturbations in dark energy and modified gravity*, *JCAP* **1207** (2012) 019, [arXiv:1203.0398].

[34] R. A. Battye and J. A. Pearson, *Massive gravity, the elasticity of space-time and perturbations in the dark sector*, *Phys. Rev.* **D88** (2013) 084004, [arXiv:1301.5042].

[35] L. Baudis, *Dark matter searches*, *Annalen Phys.* **528** (2016) 74–83, [arXiv:1509.00869].

[36] D. Baumann, A. Nicolis, L. Senatore, and M. Zaldarriaga, *Cosmological Non-Linearities as an Effective Fluid*, *JCAP* **1207** (2012) 051, [arXiv:1004.2488].

[37] E. Bellini and I. Sawicki, *Maximal freedom at minimum cost: linear large-scale structure in general modifications of gravity*, *JCAP* **1407** (2014) 050, [arXiv:1404.3713].

[38] I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier, and G. Veneziano, *Average and dispersion of the luminosity-redshift relation in the concordance model*, *JCAP* **1306** (2013) 002.

[39] I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier, and G. Veneziano, *Do stochastic inhomogeneities affect dark -energy precision measurements?*, *Phys.Rev.Lett.* **110** (2013) 021301.

[40] I. Ben-Dayan, G. Marozzi, F. Nugier, and G. Veneziano, *The second-order luminosity-redshift relation in a generic inhomogeneous cosmology*, *JCAP* **1211** (2012) 045, [arXiv:1209.4326].

[41] C. L. Bennett, A. Banday, K. M. Gorski, G. Hinshaw, P. Jackson, P. Keegstra, A. Kogut, G. F. Smoot, D. T. Wilkinson, and E. L. Wright, *Four year COBE DMR cosmic microwave background observations: Maps and basic results*, *Astrophys.J.* **464** (1996) L1–L4.

[42] D. Bertacca, R. Maartens, A. Raccanelli, and C. Clarkson, *Beyond the plane-parallel and Newtonian approach: Wide-angle redshift distortions and convergence in general relativity*, *JCAP* **1210** (2012) 025, [arXiv:1205.5221].

[43] G. Bertone, D. Hooper, and J. Silk, *Particle dark matter: Evidence, candidates and constraints*, *Phys. Rept.* **405** (2005) 279–390, [hep-ph/0404175].

[44] B. Bertotti, L. Iess, and P. Tortora, *A test of general relativity using radio links with the Cassini spacecraft*, *Nature* **425** (2003) 374–376.

[45] **SDSS** Collaboration, M. Betoule et al., *Improved cosmological constraints from a joint analysis of the SDSS-II and SNLS supernova samples*, *Astron. Astrophys.* **568** (2014) A22, [arXiv:1401.4064].

[46] A. Biswas and K. R. S. Mani, *Relativistic Perihelion Precession of Orbits of Venus and the Earth*, *Central Eur. J. Phys.* **6** (2008) 754, [arXiv:0802.0176].

[47] J. K. Bloomfield, É. É. Flanagan, M. Park, and S. Watson, *Dark energy or modified gravity? An effective field theory approach*, *JCAP* **1308** (2013) 010, [arXiv:1211.7054].

[48] C. Bonvin, *Isolating relativistic effects in large-scale structure*, *Class.Quant.Grav.* **31** (2014) 234002.

[49] C. Bonvin, S. Andrianomena, D. Bacon, C. Clarkson, R. Maartens, T. Moloi, and P. Bull, *Dipolar modulation in the size of galaxies: The effect of Doppler magnification*, *Mon. Not. Roy. Astron. Soc.* **472** (2017) 3936, [arXiv:1610.05946].

[50] C. Bonvin, C. Clarkson, R. Durrer, R. Maartens, and O. Umeh, *Cosmological ensemble and directional averages of observables*, *JCAP* **1507** (2015) 040.

[51] C. Bonvin, C. Clarkson, R. Durrer, R. Maartens, and O. Umeh, *Do we care about the distance to the CMB? Clarifying the impact of second-order lensing*, *JCAP* **1506** (2015) 050.

[52] C. Bonvin and R. Durrer, *What galaxy surveys really measure*, *Phys. Rev.* **D84** (2011) 063505, [arXiv:1105.5280].

[53] C. Bonvin, R. Durrer, and M. A. Gasparini, *Fluctuations of the luminosity distance*, *Phys. Rev.* **D73** (2006) 023523, [astro-ph/0511183]. [Erratum: Phys. Rev.D85,029901(2012)].

[54] C. Bonvin, R. Durrer, and M. Kunz, *The dipole of the luminosity distance: a direct measure of H(z)*, *Phys. Rev. Lett.* **96** (2006) 191302, [astro-ph/0603240].

[55] M. Boylan-Kolchin, V. Springel, S. D. M. White, A. Jenkins, and G. Lemson, *Resolving Cosmic Structure Formation with the Millennium-II Simulation*, *Mon. Not. Roy. Astron. Soc.* **398** (2009) 1150, [arXiv:0903.3041].

[56] I. A. Brown, A. A. Coley, D. L. Herman, and J. Latta, *Physical effects of consistent cosmological averaging*, *Phys.Rev.* **D88** (2013) 083523.

[57] M. Bruni, L. Gualtieri, and C. F. Sopuerta, *Two parameter nonlinear space-time perturbations: Gauge transformations and gauge invariance*, *Class.Quant.Grav.* **20** (2003) 535–556.

[58] M. Bruni, J. C. Hidalgo, and D. Wands, *Einstein's signature in cosmological large-scale structure*, *Astrophys. J.* **794** (2014) L11, [arXiv:1405.7006].

[59] M. Bruni, S. Matarrese, S. Mollerach, and S. Sonego, *Perturbations of space-time: Gauge transformations and gauge invariance at second order and beyond*, *Class. Quant. Grav.* **14** (1997) 2585–2606, [gr-qc/9609040].

[60] T. Buchert, *On average properties of inhomogeneous fluids in general relativity. 1. Dust cosmologies*, *Gen.Rel.Grav.* **32** (2000) 105–125.

[61] T. Buchert, *On average properties of inhomogeneous fluids in general relativity: Perfect fluid cosmologies*, *Gen.Rel.Grav.* **33** (2001) 1381–1405.

[62] T. Buchert and S. Räsänen, *Backreaction in late-time cosmology*, *Ann. Rev. Nucl. Part. Sci.* **62** (2012) 57–79, [arXiv:1112.5335].

[63] C. L. Carilli and S. Rawlings, *Science with the Square Kilometer Array: Motivation, key science projects, standards and assumptions*, *New Astron. Rev.* **48** (2004) 979, [astro-ph/0409274].

[64] J. J. M. Carrasco, M. P. Hertzberg, and L. Senatore, *The Effective Field Theory of Cosmological Large Scale Structures*, *JHEP* **09** (2012) 082, [arXiv:1206.2926].

[65] A. Challinor and A. Lewis, *The linear power spectrum of observed source number counts*, *Phys. Rev.* **D84** (2011) 043516, [arXiv:1105.5292].

[66] S. Chandrasekhar, *The Post-Newtonian Equations of Hydrodynamics in General Relativity.*, *Astrophys. J.* **142** (1965) 1488–1512.

[67] N. E. Chisari and M. Zaldarriaga, *Connection between Newtonian simulations and general relativity*, *Phys.Rev.* **D83** (2011) 123505.

[68] G. Chon, H. Böhringer, and S. Zaroubi, *On the definition of superclusters*, *Astron.Astrophys.* **575** (2015) L14.

[69] A. J. Christopherson, J. C. Hidalgo, C. Rampf, and K. A. Malik, *Second-order cosmological perturbation theory and initial conditions for n-body simulations*, *Phys.Rev.* **D93** (2016) 043539.

[70] I. Ciufolini, A. Paolozzi, E. C. Pavlis, R. Koenig, J. Ries, V. Gurzadyan, R. Matzner, R. Penrose, G. Sindoni, C. Paris, H. Khachatryan, and S. Mirzoyan, *A test of general relativity using the lares and lageos satellites and a grace earth gravity model*, *The European Physical Journal C* **76** (2016) 120.

[71] C. Clarkson, G. Ellis, J. Larena, and O. Umeh, *Does the growth of structure affect our dynamical models of the universe? The averaging, backreaction and fitting problems in cosmology*, *Rept. Prog. Phys.* **74** (2011) 112901, [arXiv:1109.2314].

[72] C. Clarkson and O. Umeh, *Is backreaction really small within concordance cosmology?*, *Class.Quant.Grav.* **28** (2011) 164010.

[73] T. Clifton, *Cosmology Without Averaging*, *Class.Quant.Grav.* **28** (2011) 164011.

[74] T. Clifton, *Back-reaction in relativistic cosmology*, *International Journal of Modern Physics D* **22** (2013) 1330004.

[75] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, *Modified Gravity and Cosmology*, *Phys. Rept.* **513** (2012) 1–189, [arXiv:1106.2476].

[76] E. J. Copeland, M. Sami, and S. Tsujikawa, *Dynamics of dark energy*, *Int. J. Mod. Phys.* **D15** (2006) 1753–1936, [hep-th/0603057].

[77] A. Cox, *Allen's astrophysical quantities.* Springer, 2000.

[78] **Boomerang** Collaboration, P. de Bernardis et al., *A Flat universe from high resolution maps of the cosmic microwave background radiation*, *Nature* **404** (2000) 955–959, [astro-ph/0004404].

[79] E. Di Dio, H. Perrier, R. Durrer, G. Marozzi, A. M. Dizgah, J. Noreña, and A. Riotto, *Non-Gaussianities due to Relativistic Corrections to the Observed Galaxy Bispectrum*, *JCAP* **1703** (2017) 006, [arXiv:1611.03720].

[80] R. H. Dicke, *The Theoretical Significance of Experimental Relativity.* London, 1964.

[81] A. M. Dizgah and R. Durrer, *Lensing corrections to the $E_g(z)$ statistics from large scale structure*, *JCAP* **1609** (2016) 035, [arXiv:1604.08914].

[82] S. Dodelson, *Modern cosmology.* Academic Press, San Diego, CA, 2003.

[83] R. Durrer, *The Cosmic Microwave Background.* Cambridge University Press, 2008.

[84] R. Durrer and A. Neronov, *Cosmological Magnetic Fields: Their Generation, Evolution and Observation*, *Astron. Astrophys. Rev.* **21** (2013) 62, [arXiv:1303.7121].

[85] R. Durrer and V. Tansella, *Vector perturbations of galaxy number counts*, *JCAP* **1607** (2016) 037, [arXiv:1605.05974].

[86] A. Einstein, *The Field Equations of Gravitation*, *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1915** (1915) 844–847.

[87] G. F. R. Ellis and M. Bruni, *Covariant and gauge-invariant approach to cosmological density fluctuations*, *Phys. Rev. D* **40** (1989) 1804–1818.

[88] G. F. R. Ellis, *Inhomogeneity effects in Cosmology*, *Class.Quant.Grav.* **28** (2011) 164001.

[89] C. W. F. Everitt et al., *Gravity Probe B: Final Results of a Space Experiment to Test General Relativity*, *Phys. Rev. Lett.* **106** (2011) 221101, [arXiv:1105.3456].

[90] C. W. F. Everitt et al., *The Gravity Probe B test of general relativity*, *Class.Quant.Grav.* **32** (2015) 224001.

[91] C. Fidler, C. Rampf, T. Tram, R. Crittenden, K. Koyama, and D. Wands, *General relativistic corrections to N-body simulations and the Zel'dovich approximation*, *Phys. Rev.* **D92** (2015) 123517, [arXiv:1505.04756].

[92] S. F. Flender and D. J. Schwarz, *Newtonian versus relativistic cosmology*, *Phys.Rev.* **D86** (2012) 063527.

[93] P. Fleury, C. Clarkson, and R. Maartens, *How does the cosmic large-scale structure bias the Hubble diagram?*, *JCAP* **1703** (2017) 062, [arXiv:1612.03726].

[94] P. Fleury, H. Dupuy, and J.-P. Uzan, *Can all cosmological observations be accurately interpreted with a unique geometry?*, *Phys. Rev. Lett.* **111** (2013) 091302, [arXiv:1304.7791].

[95] J. Fonseca, S. Camera, M. Santos, and R. Maartens, *Hunting down horizon-scale effects with multi-wavelength surveys*, *Astrophys. J.* **812** (2015) L22, [arXiv:1507.04605].

[96] J. Fonseca, R. Maartens, and M. G. Santos, *Probing the primordial Universe with MeerKAT and DES*, *Mon. Not. Roy. Astron. Soc.* **466** (2017) 2780–2786, [arXiv:1611.01322].

[97] S. R. Goldberg, T. Clifton, and K. A. Malik, *Cosmology on all scales: a two-parameter perturbation expansion*, *Phys. Rev.* **D95** (2017) 043503, [arXiv:1610.08882].

[98] S. R. Goldberg, C. Gallagher, and T. Clifton, *Perturbation theory for cosmologies with nonlinear structure*, *Phys. Rev.* **D96** (2017) 103508, [arXiv:1707.01042].

[99] S. R. Green and R. M. Wald, *A new framework for analyzing the effects of small scale inhomogeneities in cosmology*, *Phys. Rev.* **D83** (2011) 084020, [arXiv:1011.4920].

[100] S. R. Green and R. M. Wald, *Newtonian and Relativistic Cosmologies*, *Phys. Rev.* **D85** (2012) 063512, [arXiv:1111.2997].

[101] S. R. Green and R. M. Wald, *Newtonian and Relativistic Cosmologies*, *Phys.Rev.* **D85** (2012) 063512.

[102] G. Gubitosi, F. Piazza, and F. Vernizzi, *The Effective Field Theory of Dark Energy*, *JCAP* **1302** (2013) 032, [arXiv:1210.0201].

[103] S. W. Hawking, *Perturbations of an expanding universe*, *Astrophys. J.* **145** (1966) 544–554.

[104] D. W. Hogg, D. J. Eisenstein, M. R. Blanton, N. A. Bahcall, J. Brinkmann, J. E. Gunn, and D. P. Schneider, *Cosmic homogeneity demonstrated with luminous red galaxies*, *Astrophys.J.* **624** (2005) 54–58.

[105] W. Hu and I. Sawicki, *A Parameterized Post-Friedmann Framework for Modified Gravity*, *Phys. Rev.* **D76** (2007) 104043, [arXiv:0708.1190].

[106] I. Huston and K. A. Malik, *Numerical calculation of second order perturbations*, *JCAP* **0909** (2009) 019, [arXiv:0907.2917].

[107] P. J. E. Peebles, *Principles of Physical Cosmology.* Princeton Theories in Physics. Princeton University Press, 1993.

[108] M. Jarvis, D. Bacon, C. Blake, M. Brown, S. Lindsay, A. Raccanelli, M. Santos, and D. J. Schwarz, *Cosmology with SKA Radio Continuum Surveys*, *PoS* **AASKA14** (2015) 018.

[109] S. Jolicoeur, O. Umeh, R. Maartens, and C. Clarkson, *Imprints of local lightcone projection effects on the galaxy bispectrum. II*, arXiv:1703.09630.

[110] A. Joyce, B. Jain, J. Khoury, and M. Trodden, *Beyond the Cosmological Standard Model*, *Phys. Rept.* **568** (2015) 1–98, [arXiv:1407.0059].

[111] T. Kitching, D. Bacon, M. Brown, P. Bull, J. Mcewen, M. Oguri, R. Scaramella, K. Takahashi, K. Wu, and D. Yamauchi, *Euclid & SKA Synergies*, *PoS* **AASKA14** (2015) 146.

[112] H. Kodama and M. Sasaki, *Cosmological Perturbation Theory*, *Prog. Theor. Phys. Suppl.* **78** (1984) 1–166.

[113] L. Koenigsberger, *Hermann von Helmholtz.* Clarendon Press, 1906.

[114] Y. Kojima and M. Hosonuma, *The r-mode oscillations in relativistic rotating stars*, *The Astrophysical Journal* **520** (1999) 788.

[115] E. W. Kolb, S. Matarrese, A. Notari, and A. Riotto, *The Effect of inhomogeneities on the expansion rate of the universe*, *Phys.Rev.* **D71** (2005) 023524.

[116] J. Kovac, E. M. Leitch, C. Pryke, J. E. Carlstrom, N. W. Halverson, and W. L. Holzapfel, *Detection of polarization in the cosmic microwave background using DASI*, *Nature* **420** (2002) 772–787.

[117] **DES** Collaboration, A. Kovács et al., *Imprint of DES superstructures on the cosmic microwave background*, *Mon. Not. Roy. Astron. Soc.* **465** (2017) 4166–4179, [arXiv:1610.00637].

[118] **Supernova Cosmology Project** Collaboration, M. Kowalski et al., *Improved Cosmological Constraints from New, Old and Combined Supernova Datasets*, *Astrophys. J.* **686** (2008) 749–778, [arXiv:0804.4142].

[119] M. Lagos, T. Baker, P. G. Ferreira, and J. Noller, *A general theory of linear cosmological perturbations: scalar-tensor and vector-tensor theories*, *JCAP* **1608** (2016) 007, [arXiv:1604.01396].

[120] L. Landau and E. Lifshitz, *The Classical Theory of Fields.* Course of theoretical physics. Butterworth-Heinemann, 1975.

[121] **EUCLID** Collaboration, R. Laureijs et al., *Euclid Definition Study Report*, arXiv:1110.3193.

[122] B. Li, G.-B. Zhao, R. Teyssier, and K. Koyama, *ECOSMOG: An Efficient Code for Simulating Modified Gravity*, *JCAP* **1201** (2012) 051, [arXiv:1110.1379].

[123] Y.-C. Li and Y.-Z. Ma, *Constraints on Primordial non-Gaussianity from Future HI Intensity Mapping Experiments*, arXiv:1701.00221.

[124] C. Llinares, D. F. Mota, and H. A. Winther, *ISIS: a new N-body cosmological code with scalar fields based on RAMSES. Code presentation and application to the shapes of clusters*, *Astron. Astrophys.* **562** (2014) A78, [arXiv:1307.6748].

[125] K. H. Lockitch, N. Andersson, and J. L. Friedman, *The Rotational modes of relativistic stars. 1. Analytic results*, *Phys. Rev.* **D63** (2001) 024019, [gr-qc/0008019].

[126] D. Lovelock, *The four-dimensionality of space and the Einstein tensor*, *J. Math. Phys.* **13** (1972) 874–876.

[127] K. A. Malik and D. Wands, *Evolution of second-order cosmological perturbations*, *Class. Quant. Grav.* **21** (2004) L65–L72, [astro-ph/0307055].

[128] K. A. Malik and D. Wands, *Cosmological perturbations*, *Phys. Rept.* **475** (2009) 1–51, [arXiv:0809.4944].

[129] V. Marra, L. Amendola, I. Sawicki, and W. Valkenburg, *Cosmic variance and the measurement of the local Hubble parameter*, *Phys. Rev. Lett.* **110** (2013) 241305, [arXiv:1303.3121].

[130] I. Milillo, D. Bertacca, M. Bruni, and A. Maselli, *Missing link: A nonlinear post-Friedmann framework for small and large scales*, *Phys.Rev.* **D92** (2015) 023519.

[131] C. Misner, K. Thorne, and J. Wheeler, *Gravitation.* No. pt. 3 in Gravitation. W. H. Freeman, 1973.

[132] V. F. Mukhanov, L. R. W. Abramo, and R. H. Brandenberger, *On the Back reaction problem for gravitational perturbations*, *Phys. Rev. Lett.* **78** (1997) 1624–1627, [gr-qc/9609026].

[133] M. Nakahara, *Geometry, Topology and Physics, Second Edition.* Taylor & Francis, 2003.

[134] K. Nakamura, *Gauge-invariant formulation of the second-order cosmological perturbations*, *Phys. Rev.* **D74** (2006) 101301, [gr-qc/0605107].

[135] J. T. Nielsen and R. Durrer, *Higher order relativistic galaxy number counts: dominating terms*, *JCAP* **1703** (2017) 010.

[136] D. Parkinson et al., *The WiggleZ Dark Energy Survey: Final data release and cosmological results*, *Phys. Rev.* **D86** (2012) 103518, [arXiv:1210.2130].

[137] P. Peter and J.-P. Uzan, *Primordial cosmology.* Oxford Graduate Texts. Oxford Univ. Press, Oxford, 2009.

[138] F. Piazza and F. Vernizzi, *Effective Field Theory of Cosmological Perturbations*, *Class. Quant. Grav.* **30** (2013) 214007, [arXiv:1307.4350].

[139] C. Pitrou, X. Roy, and O. Umeh, *xPand: An algorithm for perturbing homogeneous cosmologies*, *Class.Quant.Grav.* **30** (2013) 165002.

[140] S. Planelles, D. R. G. Schleicher, and A. M. Bykov, *Large-Scale Structure Formation: from the first non-linear objects to massive galaxy clusters*, *Space Sci. Rev.* **188** (2015) 93–139, [arXiv:1404.3956].

[141] E. Poisson and C. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic.* Cambridge University Press, 2014.

[142] E. Puchwein, M. Baldi, and V. Springel, *Modified Gravity-GADGET: A new code for cosmological hydrodynamical simulations of modified gravity models*, *Mon. Not. Roy. Astron. Soc.* **436** (2013) 348, [arXiv:1305.2418].

[143] S. Räsänen, *Applicability of the linearly perturbed FRW metric and Newtonian cosmology*, *Phys.Rev.* **D81** (2010) 103512.

[144] M. Rees, *Polarization and spectrum of the primeval radiation in an anisotropic universe*, *Astrophys. J.* **153** (1968) L1.

[145] G. Rigopoulos and W. Valkenburg, *On the accuracy of N-body simulations at very large scales*, *Mon.Not.Roy.* **446** (2015) 677–682.

[146] J. Ruoff, A. Stavridis, and K. D. Kokkotas, *Evolution equations for the perturbations of slowly rotating relativistic stars*, *Mon. Not. Roy. Astron. Soc.* **332** (2002) 676, [gr-qc/0109065].

[147] L. Samushia, W. J. Percival, and A. Raccanelli, *Interpreting large-scale redshift-space distortion measurements*, *Mon. Not. Roy. Astron. Soc.* **420** (2012) 2102–2119, [arXiv:1102.1014].

[148] V. A. A. Sanghai and T. Clifton, *Post-Newtonian Cosmological Modelling*, *Phys.Rev.* **D91** (2015) 103532.

[149] V. A. A. Sanghai and T. Clifton, *Cosmological backreaction in the presence of radiation and a cosmological constant*, *Phys.Rev.* **D94** (2016) 023505.

[150] V. A. A. Sanghai and T. Clifton, *Parameterized Post-Newtonian Cosmology*, *Class. Quant. Grav.* **34** (2017) 065003, [arXiv:1610.08039].

[151] F. Schmidt, *Self-Consistent Cosmological Simulations of DGP Braneworld Gravity*, *Phys. Rev.* **D80** (2009) 043001, [arXiv:0905.0858].

[152] D. J. Schwarz, D. Bacon, S. Chen, C. Clarkson, D. Huterer, M. Kunz, R. Maartens, A. Raccanelli, M. Rubart, and J.-L. Starck, *Testing foundations of modern cosmology with SKA all-sky surveys*, *PoS* **AASKA14** (2015) 032, [arXiv:1501.03820].

[153] C. Skordis, *Consistent cosmological modifications to the Einstein equations*, *Phys. Rev.* **D79** (2009) 123527, [arXiv:0806.1238].

[154] C. F. Sopuerta, M. Bruni, and L. Gualtieri, *Nonlinear N parameter space-time perturbations: Gauge transformations*, *Phys.Rev.* **D70** (2004) 064002.

[155] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2009.

[156] J. Stewart and J. Stewart, *Advanced General Relativity*. Cambridge Monographs on Mathem. Cambridge University Press, 1993.

[157] J. M. Stewart and M. Walker, *Perturbations of spacetimes in general relativity*, *Proc.Roy.Soc.Lond.* **A341** (1974) 49–74.

[158] J. H. Taylor and J. M. Weisberg, *Further experimental tests of relativistic gravity using the binary pulsar PSR 1913+16*, *Astrophys. J.* **345** (1989) 434–450.

[159] D. B. Thomas, M. Bruni, and D. Wands, *The fully non-linear post-Friedmann frame-dragging vector potential: Magnitude and time evolution from N-body simulations*, *Mon.Not.Roy.Astron.Soc.* **452** (2015) 1727–1742.

[160] R. B. Tully, H. Courtois, Y. Hoffman, and D. Pomarede, *The laniakea supercluster of galaxies*, *Nature* **513** (2014) 71–73. Letter.

[161] O. Umeh, *Imprint of non-linear effects on HI intensity mapping on large scales*, *JCAP* **1706** (2017) 005, [arXiv:1611.04963].

[162] O. Umeh, C. Clarkson, and R. Maartens, *Nonlinear relativistic corrections to cosmological distances, redshift and gravitational lensing magnification: I. Key results*, *Class. Quant. Grav.* **31** (2014) 202001, [arXiv:1207.2109].

[163] O. Umeh, S. Jolicoeur, R. Maartens, and C. Clarkson, *A general relativistic signature in the galaxy bispectrum: the local effects of observing on the lightcone*, *JCAP* **1703** (2017) 034, [arXiv:1610.03351].

[164] W. Valkenburg, M. Kunz, and V. Marra, *Intrinsic uncertainty on the nature of dark energy*, *Phys. Dark Univ.* **2** (2013) 219–223, [arXiv:1302.6588].

[165] R. J. van den Hoogen, *Averaging Spacetime: Where do we go from here?*, in *On recent developments in theoretical and experimental general relativity, astrophysics and relativistic field theories. Proceedings, 12th Marcel Grossmann Meeting on General Relativity, Paris, France, July 12-18, 2009. Vol. 1-3*, pp. 578–588, 2010. arXiv:1003.4020.

[166] E. Villa, S. Matarrese, and D. Maino, *Cosmological dynamics: from the Eulerian to the Lagrangian frame. Part I. Newtonian approximation*, *JCAP* **1406** (2014) 041, [arXiv:1403.6806].

[167] E. Villa, L. Verde, and S. Matarrese, *General relativistic corrections and non-Gaussianity in large scale structure*, *Class. Quant. Grav.* **31** (2014) 234005, [arXiv:1409.4738].

[168] R. Wald, *General Relativity*. University of Chicago Press, 1984.

[169] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* Wiley, New York, NY, 1972.

[170] C. M. Will, *The Confrontation between General Relativity and Experiment*, *Living Rev. Rel.* **17** (2014) 4, [arXiv:1403.7377].

[171] C. Will, *Theory and Experiment in Gravitational Physics.* Cambridge University Press, 1993.