# SCHUBERT POLYNOMIALS AS INTEGER POINT TRANSFORMS OF GENERALIZED PERMUTAHEDRA

ALEX FINK, KAROLA MÉSZÁROS, AND AVERY ST. DIZIER

Abstract. We show that the dual character of the flagged Weyl module of any diagram is a positively weighted integer point transform of a generalized permutahedron. In particular, Schubert and key polynomials are positively weighted integer point transforms of generalized permutahedra. This implies several recent conjectures of Monical, Tokcan and Yong.

**Keywords.** Schubert polynomial, key polynomial, dual character of the flagged Weyl module, Newton polytope, integer point transform, generalized permutahedron.

## 1. Introduction

Schubert polynomials and key polynomials are classical objects in algebraic combinatorics. Schubert polynomials, introduced by Lascoux and Schützenberger in 1982 [13], represent cohomology classes of Schubert cycles in flag varieties. Key polynomials, also known as Demazure characters, are polynomials associated to compositions. Key polynomials were first introduced by Demazure for Weyl groups [6], and studied in the context of the symmetric group by Lascoux and Schützenberger in [14, 15].

Schubert and key polynomials play an important role in algebraic combinatorics [2,3,9,11,21]. The second author and Escobar [8] showed that for permutations $1\pi'$ where $\pi'$ is dominant, Schubert polynomials are specializations of reduced forms in the subdivision algebra of flow and root polytopes. On the other hand, intimate connections of flow and root polytopes with generalized permutahedra have been exhibited by Postnikov [20], and more recently by the last two authors [18]. These works imply that for permutations $1\pi'$ where $\pi'$ is dominant, the Schubert polynomial $\mathfrak{S}_{1\pi'}(\mathbf{x})$ is equal to the integer point transform of a generalized permutahedron [18].

Using realizations of Schubert polynomials and key polynomials as dual characters of a certain module [16, 21], the main result of this paper proves that the Newton polytope of any Schubert or key polynomial is a generalized permutahedron, and that any such polynomial equals a sum over the lattice points of its Newton polytope with positive integral coefficients.

After reviewing the necessary background, we prove our main theorem and draw corollaries about Schubert and key polynomials, confirming several recent conjectures of Monical, Tokcan and Yong [19].

*Date*: May 7, 2018.

Fink is partially supported by an Engineering and Physical Sciences Research Council grant (EP/M01245X/1), and Mészáros by a National Science Foundation Grant (DMS 1501059).

## 2. Background

This section contains a collection of definitions and facts relevant to the main result of this paper. Our basic notions are Schubert polynomials, key polynomials, Newton polytopes, generalized permutahedra, (Schubert) matroids, and flagged Weyl modules.

### 2.1. Schubert polynomials.

The Schubert polynomial of the longest permutation $w_0 = n\; n{-}1\; \cdots\; 2\; 1 \in S_n$ is

$$\mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

For $w \in S_n$, $w \neq w_0$, there exists $i \in [n-1]$ such that $w(i) < w(i+1)$. For any such $i$, the **Schubert polynomial** $\mathfrak{S}_w$ ([13]) is defined as

$$\mathfrak{S}_w(x_1, \ldots, x_n) := \partial_i \mathfrak{S}_{ws_i}(x_1, \ldots, x_n),$$

where $\partial_i$ is the $i$th divided difference operator

$$\partial_i(f) := \frac{f - s_i f}{x_i - x_{i+1}} \text{ and } s_i = (i, i+1).$$

Since the $\partial_i$ satisfy the braid relations, the Schubert polynomials $\mathfrak{S}_w$ are well-defined.

### 2.2. Key polynomials.

A **composition** $\alpha$ is a sequence of nonnegative integers $(\alpha_1, \alpha_2, \ldots)$ with $\sum_{k=1}^{\infty} \alpha_k < \infty$. If $\alpha$ is weakly decreasing, define the **key polynomial** $\kappa_\alpha$ ([7]) to be

$$\kappa_\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots .$$

Otherwise, set

$$\kappa_\alpha = \partial_i \left( x_i \kappa_{\hat{\alpha}} \right) \text{ where } \hat{\alpha} = (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots) \text{ and } \alpha_i < \alpha_{i+1}.$$

It is an important fact due to Lascoux and Schützenberger [14] that every Schubert polynomial is a sum of key polynomials.

### 2.3. Diagrams.

View $[n]^2$ as an $n$ by $n$ grid of boxes labeled $(i, j)$ in the same way as entries of an $n \times n$ matrix, with labels increasing as you move top to bottom along columns and left to right across rows from the upper-left corner. By a **diagram**, we mean a subset $D \subseteq [n]^2$, a collection of boxes in the $n \times n$ grid. Throughout this paper, we view $D$ as an ordered list of subsets $D = (D_1, D_2, \ldots, D_n)$ where for each $j$, $D_j = \{i : (i, j) \in D\}$ is the set of row indices of boxes of $D$ in column $j$. Two important classes of diagrams are Rothe diagrams and skyline diagrams.

**Definition 1** ([17])**.** *The **Rothe diagram** of a permutation $\pi \in S_n$ is the collection of boxes $D(\pi) = \{(i, j) : 1 \le i, j \le n,\ \pi(i) > j,\ \pi^{-1}(j) > i\}$. $D(\pi)$ can be visualized as the set of boxes left in the $n \times n$ grid after you cross out all boxes weakly below or right of $(i, \pi(i))$ for each $i \in [n]$. Let $D(\pi)_j = \{i : (i, j) \in D(\pi)\}$ for each $j$, so $D(\pi) = (D(\pi)_1, \ldots, D(\pi)_n)$.*

See Figure 1 for an example of a Rothe diagram.

**Definition 2** ([10, 19])**.** *If $\alpha = (\alpha_1, \alpha_2, \ldots)$ is a composition, let $l = \max\{i : \alpha_i \neq 0\}$ and $n = \max\{l, \alpha_1, \ldots, \alpha_l\}$. The **skyline diagram** of $\alpha$ is the diagram $D(\alpha) \subseteq [n]^2$ containing the first $\alpha_i$ boxes in row $i$ for each $i \in [n]$. More specifically, $D(\alpha) = (D(\alpha)_1, \ldots, D(\alpha)_n)$ with $D(\alpha)_j = \{j \le n : \alpha_j \ge j\}$ for each $j$.*

See Figure 2 for an example of a skyline diagram.

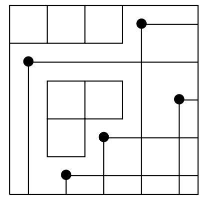

FIGURE 1. The Rothe diagram of $\pi = 41532$ is $(\{1\}, \{1,3,4\}, \{1,3\}, \emptyset, \emptyset)$.

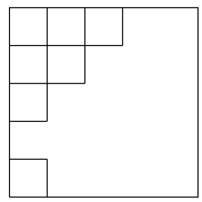

FIGURE 2. The skyline diagram of $\alpha = (3,2,1,0,1)$ is $(\{1,2,3,5\}, \{1,2\}, \{1\}, \emptyset, \emptyset)$.

2.4. **Newton polytopes and generalized permutahedra.** If $f$ is a polynomial in a polynomial ring whose variables are indexed by some set $I$, the **support** of $f$ is the lattice point set in $\mathbb{R}^I$ consisting of the exponent vectors of the monomials that have nonzero coefficient in $f$. The **Newton polytope** $\mathrm{Newton}(f) \subseteq \mathbb{R}^I$ is the convex hull of the support of $f$. Following the definition of [19], we say that a polynomial $f$ has **saturated Newton polytope** (**SNP**) if every lattice point in $\mathrm{Newton}(f)$ is a vector in the support of $f$.

Our main objects of study are the supports of Schubert and key polynomials. We prove that these polynomials have SNP and that their Newton polytopes are generalized permutahedra, which we define next.

The standard permutahedron is the polytope in $\mathbb{R}^n$ whose vertices consist of all permutations of the entries of the vector $(0,1,\ldots,n-1)$. Following the definition of [20], a **generalized permutahedron** is a deformation of the standard permutahedron obtained by translating the vertices in such a way that all edge directions and orientations are preserved (edges are allowed to degenerate to points). Generalized permutahedra are parametrized by certain collections of real numbers $\{z_I\}$ indexed by nonempty subsets $I \subseteq [n]$. Given $\{z_I\}$, the associated generalized permutahedron is given by

$$P_n^z(\{z_I\}) = \left\{ \mathbf{t} \in \mathbb{R}^n : \sum_{i \in I} t_i \geq z_I \text{ for } I \neq [n], \text{ and } \sum_{i=1}^{n} t_i = z_{[n]} \right\}.$$

Postnikov initiated the study of these fascinating polytopes in [20], and they have since been studied extensively. An important note is that the class of generalized permutahedra is closed under Minkowski sums. This follows from [1, Lemma 2.2]:

$$P_n^z(\{z_I\}) + P_n^z(\{z'_I\}) = P_n^z(\{z_I + z'_I\}).$$

2.5. **Schubert matroids.** A **matroid** $M$ is a pair $(E, \mathcal{B})$ consisting of a finite set $E$ and a nonempty collection of subsets $\mathcal{B}$ of $E$, called the **bases** of $M$. $\mathcal{B}$ is required to satisfy the **basis exchange axiom**: If $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 - B_2$, then there exists $b_2 \in B_2 - B_1$ such that $B_1 - b_1 \cup b_2 \in \mathcal{B}$. By choosing a labeling of the elements of $E$, we can always assume $E = [n]$ for some $n$.

**Definition 3** ([4], Section 2.4)**.** *Fix positive integers $1 \leq s_1 < \cdots < s_r \leq n$. The sets $\{a_1, \ldots, a_r\}$ of positive integers with $a_1 < \cdots < a_r$ such that $a_1 \leq s_1, \ldots, a_r \leq s_r$ are the bases of a matroid, called the **Schubert matroid** $SM_n(s_1, \ldots, s_r)$*

2.6. **Matroid polytopes.** Given a matroid $M = (E, \mathcal{B})$ with $E = [n]$ and a basis $B \in \mathcal{B}$, let $\zeta^B$ be the indicator vector of $B$. That is, let $\zeta^B = (\zeta_1^B, \ldots, \zeta_n^B) \in \mathbb{R}^n$ with $\zeta_i^B = 1$ if $i \in B$ and $\zeta_i^B = 0$ if $i \notin B$ for each $i$. The **matroid polytope** of $M$ is the polytope

$$P(M) = \mathrm{Conv}\{\zeta^B : B \in \mathcal{B}\}.$$

The **rank function** of $M$ is the function

$$r_M : 2^E \to \mathbb{Z}_{\geq 0}$$

defined by $r_M(S) = \max\{\#(S \cap B) : B \in \mathcal{B}\}$. The sets $S \cap B$ where $S \subseteq [n]$ and $B \in \mathcal{B}$ are called the **independent sets** of $M$.

Matroid polytopes are actually a subclass of generalized permutahedra and admit the parametrization

$$\begin{aligned} P(M) &= P_n^z\left(\{r_M(E) - r_M(E \backslash I)\}_{I \subseteq E}\right) \\ &= \left\{\mathbf{t} \in \mathbb{R}^n : \sum_{i \in I} t_i \leq r_M(I) \text{ for } I \neq E, \text{ and } \sum_{i \in E} t_i = r_M(E)\right\}. \end{aligned}$$

See [22, Corollary 40.2d] for a proof.

*Remark.* It is unconcerning that the presentation of $P(M)$ here uses $\leq$ inequalities whereas Postnikov's definition of $P_n^z\{z_I\}$ above uses $\geq$, as any inequality in the definition can be flipped by exploiting the equality $t_1 + \cdots + t_n = z_{[n]}$.

The following two lemmas about matroid polytopes are crucial to the proof of Theorem 11.

**Lemma 4.** *Let $M_1, M_2, \ldots, M_n$ be matroids and let $Q$ be the Minkowski sum $Q = P(M_1) + \cdots + P(M_n)$. Then, every integral point $q \in Q$ can be written as $q = p_1 + \cdots + p_n$, where $p_i$ is an integral point of $P(M_i)$ for each $i$.*

Lemma 4 is an easy consequence of the analogous result for integral polymatroids, see for instance [22, Corollary 46.2c]. Lemma 5 below is well-known, but we give a proof for completeness.

**Lemma 5.** *For any matroid $M$ on $E$, the only integral points of the matroid polytope $P(M)$ are its vertices.*

*Proof.* Let $P(M)$ have vertices $v_1, \ldots, v_m \in \mathbb{R}^n$ corresponding to bases $B_1, \ldots, B_m$, and assume $q = (q_1, \ldots, q_n) \in P(M)$ is an integral point that is not a vertex. Then, $q$ can be written $q = \sum_{i=1}^m \lambda_i v_i$ with $0 \leq \lambda_i \leq 1$ for all $i$ and $\sum_{i=1}^m \lambda_i = 1$. It follows from a characterization of matroid polytopes [5, Theorem 1.11.1] that every face of a matroid polytope is also a matroid polytope, so there is no loss of generality in assuming that $q$ lies in the interior of $P(M)$. In particular, this implies that all

$\lambda_i$ are positive. Note that since $q$ is integral, if $q_j \neq 0$ for some $j$ then $q_j = 1$. It follows that for any $j$, $q_j > 0$ if and only if $j \in B_i$ for all $i$. This implies that

$$\sum_{i=1}^{n} q_i = \#(B_1 \cap \cdots \cap B_m) < \#B_1 = r_M(E),$$

contradicting that $q \in P(M)$. □

2.7. **Flagged Weyl modules.** Let $G = \mathrm{GL}(n, \mathbb{C})$ be the group of $n \times n$ invertible matrices over $\mathbb{C}$ and $B$ be the subgroup of $G$ consisting of the $n \times n$ upper-triangular matrices. The flagged Weyl module is a representation $M_D$ of $B$ associated to a diagram $D$. We use the construction of $M_D$ in terms of determinants given in [16].

Denote by $Y$ the $n \times n$ matrix with indeterminants $y_{ij}$ in the upper-triangular positions $i \leq j$ and zeros elsewhere. Let $\mathbb{C}[Y]$ be the polynomial ring in the indeterminants $\{y_{ij}\}_{i \leq j}$. Note that $G$ acts on $\mathbb{C}[Y]$ on the right via left translation: if $f(\mathbf{y}) \in \mathbb{C}[Y]$, then a matrix $g \in G$ acts on $f$ by $f(\mathbf{y}) \cdot g = f(g^{-1}\mathbf{y})$. For any $R, S \subseteq [n]$, let $Y_R^S$ be the submatrix of $Y$ obtained by restricting to rows $S$ and columns $R$.

For $R, S \subseteq [n]$, we say $R \leq S$ if $\#R = \#S$ and the $k$th least element of $R$ does not exceed the $k$th least element of $S$ for each $k$. For any diagrams $C = (C_1, \ldots, C_n)$ and $D = (D_1, \ldots, D_n)$, we say $C \leq D$ if $C_j \leq D_j$ for all $j \in [n]$.

**Definition 6.** *For a diagram $D = (D_1, \ldots, D_n)$, the **flagged Weyl module** $M_D$ is defined by*

$$M_D = \mathrm{Span}_{\mathbb{C}} \left\{ \prod_{j=1}^{n} \det\left(Y_{D_j}^{C_j}\right) \, : \, C \leq D \right\}.$$

*$M_D$ is a $B$-module with the action inherited from the action of $B$ on $\mathbb{C}[Y]$.*

Note that since $Y$ is upper-triangular, the condition $C \leq D$ is technically unncessary since $\det\left(Y_{D_j}^{C_j}\right) = 0$ unless $C_j \leq D_j$.

Flagged Weyl modules of diagrams are related to Schubert and key polynomials through their dual characters. Let $N$ be any $B$-module, and $X \in B$ the diagonal matrix with entries $x_1, \ldots, x_n \in \mathbb{C}$. Since $X$ acts on $N$ via the $B$-action, $X$ can be viewed as a map of $\mathbb{C}$-vector spaces $X : N \to N$.

**Definition 7.** *The **character** of a $B$-module $N$ is defined as the trace of $X$ viewed as a linear map on $N$:*

$$\mathrm{char}(N)(x_1, \ldots, x_n) = \mathrm{tr}\,(X : N \to N)$$

*Similarly, the **dual character of** $N$ is the character of the dual module $N^*$:*

$$\begin{aligned} \mathrm{char}^*(N)(x_1, \ldots, x_n) &= \mathrm{tr}\,(X : N^* \to N^*) \\ &= \mathrm{char}(N)(x_1^{-1}, \ldots, x_n^{-1}). \end{aligned}$$

## 3. Newton Polytopes of Dual Characters of Flagged Weyl Modules

In this section, we prove the main results of this paper, obtaining the Newton polytopes and Schubert and key polynomials by considering the dual characters of flagged Weyl modules. We begin with the results connecting Schubert and key polynomials to flagged Weyl modules.

**Theorem 8** ([12])**.** *Let $w \in S_n$ be a permutation, $D(w)$ be the Rothe diagram of $w$, and $M_{D(w)}$ be the associated flagged Weyl module. Then,*

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \mathrm{char}^* M_{D(w)}.$$

**Theorem 9** ([7])**.** *Let $\alpha$ be a composition, $D(\alpha)$ be the skyline diagram of $\alpha$, and $M_{D(\alpha)}$ be the associated flagged Weyl module. If $l = \max\{i : \alpha_i \neq 0\}$ and $n = \max\{\alpha_1, \ldots, \alpha_l, l\}$, then*

$$\kappa_\alpha(x_1, \ldots, x_n) = \mathrm{char}^* M_{D(\alpha)}.$$

**Definition 10.** *For a diagram $D \subseteq [n]^2$, let $\chi_D = \chi_D(x_1, \ldots, x_n)$ be the dual character*

$$\chi_D = \mathrm{char}^* M_D.$$

**Theorem 11.** *Let $D = (D_1, \ldots, D_n)$ be a diagram. Then $\chi_D$ has SNP, and the Newton polytope of $\chi_D$ is the Minkowski sum of matroid polytopes*

$$\mathrm{Newton}(\chi_D) = \sum_{j=1}^{n} P(SM_n(D_j)).$$

*In particular,* $\mathrm{Newton}(\chi_D)$ *is a generalized permutahedron.*

*Proof.* Let $X \in B$ be a diagonal matrix with diagonal entries $x_1, x_2, \ldots, x_n \in \mathbb{C}$. First, note that by matrix multiplication, $y_{ij}$ is an eigenvector of $X$ with eigenvalue $x_i^{-1}$. Take a diagram $C = (C_1, \ldots, C_n)$ with $C \leq D$. Then, the element $\prod_{j=1}^n \det\left(Y_{D_j}^{C_j}\right)$ is an eigenvector of $X$ with eigenvalue

$$\prod_{j=1}^{n} \prod_{i \in C_j} x_i^{-1}.$$

Since $M_D$ is spanned by elements $\prod_{j=1}^n \det\left(Y_{D_j}^{C_j}\right)$ and each is an eigenvector of $D$, the monomials appearing in the dual character $\chi_D$ with nonzero coefficient are exactly

$$\left\{ \prod_{j=1}^{n} \prod_{i \in C_j} x_i : C \leq D \right\}.$$

For a diagram $C = (C_1, \ldots, C_n)$, define a vector $\xi^C = (\xi_1^C, \ldots, \xi_n^C)$ by setting $\xi_i^C = \#\{j : i \in C_j\}$ for each $i$. The exponent vector of $\prod_{j=1}^n \prod_{i \in C_j} x_i$ is exactly $\xi^C$, so the support of $\chi_D$ is precisely the set $\left\{\xi^C : C \leq D\right\}$.

However, for each $j \in [n]$, the sets $S \subseteq [n]$ with $S \leq D_j$ are exactly the bases of the Schubert matroid $SM_n(D_j)$. In particular, choosing a diagram $C \leq D$ is equivalent to picking a basis $C_j$ of $SM_n(D_j)$ for each $j \in [n]$. If $\zeta^{C_j}$ is the indicator vector of $C_j$, then comparing components shows

$$\xi^C = \sum_{j=1}^{n} \zeta^{C_j}.$$

This shows that each vector $\xi^C$ is a sum consisting of a vertex from each matroid polytope $P(SM_n(D_j))$ for $j \in [n]$. Conversely, given any sum $\sum_{j=1}^n \zeta^{B_j}$ of a vertex $\zeta^{B_j}$ from each $P(SM_n(D_j))$, let $C = (B_1, \ldots, B_n)$. Since each $B_j$ is a basis of

$SM_n(D_j)$, $C \leq D$. Thus, $\xi^C = \sum_{j=1}^n \zeta^{C_j}$ is in the support of $\chi_D$.

Consequently,

$$\text{Newton}(\chi_D) = \sum_{j=1}^{n} P(SM_n(D_j)). \tag{1}$$

To prove that $\chi_D$ has SNP, it remains to show that every integral point $q$ in $\text{Newton}(\chi_D)$ is in the support of $\chi_D$. By (1), $q$ is an integral point of a Minkowski sum of matroid polytopes, so by Lemmas 4 and 5, $q$ can be written as a sum consisting of one vertex from each $P(SM_n(D_j))$. As shown above, this is precisely what it means for $q$ to be in the support of $\chi_D$. □

**Corollary 12.** *The support of any Schubert polynomial $\mathfrak{S}_w$ or key polynomial $\kappa_\alpha$ equals the set of lattice points of a generalized permutahedron.*

This confirms Conjectures 3.10 and 5.1 of [19], namely that key polynomials and Schubert polynomials have SNP. We now confirm Conjectures 3.9 and 5.13 of [19], which give a conjectural inequality description for the Newton polytopes of Schubert and key polynomials. We state this description and match it to the Minkowski sum description proven in Theorem 11.

Let $D \subseteq [n]^2$ be any diagram with columns $D_j = \{i : (i,j) \in D\}$ for $j \in [n]$. Let $I \subseteq [n]$ be a set of row indices and $j \in [n]$ a column index. Construct a string $\text{word}_{j,I}(D)$ by reading column $j$ of the $n$ by $n$ grid from top to bottom and recording

- ( if $(i,j) \notin D$ and $i \in I$;
- ) if $(i,j) \in D$ and $i \notin I$;
- $\star$ if $(i,j) \in D$ and $i \in I$.

Let $\theta_D^j(I) = \#\text{paired ()'s in } \text{word}_{j,I}(D) + \#\star\text{'s in } \text{word}_{j,I}(D)$, and set

$$\theta_D(I) = \sum_{j=1}^{n} \theta_D^j(I).$$

**Definition 13** ([19])**.** *For any diagram $D \subseteq [n]^2$, define the Schubitope $\mathcal{S}_D$ by*

$$\mathcal{S}_D = \left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^{n} \alpha_i = \#D \text{ and } \sum_{i \in I} \alpha_i \leq \theta_D(I) \text{ for all } I \subseteq [n] \right\}.$$

**Theorem 14.** *Let $D$ be a diagram $D \subseteq [n]^2$ with columns $D_j = \{i : (i,j) \in D\}$ for each $j \in [n]$. The Schubitope $\mathcal{S}_D$ equals the Minkowski sum of matroid polytopes*

$$\mathcal{S}_D = \sum_{j=1}^{n} P\left(SM_n\left(D_j\right)\right).$$

*Proof.* Let $r_j$ be the rank function of the matroid $SM_n(D_j)$. By [1, Lemma 2.2], the Minkowski sum $\sum_{j=1}^n P\left(SM_n\left(D_j\right)\right)$ equals

$$\left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{\geq 0} : \sum_{i \in [n]} \alpha_i = \sum_{j=1}^{n} r_j([n]) \text{ and } \sum_{i \in I} \alpha_i \leq \sum_{j=1}^{n} r_j(I) \text{ for all } I \subseteq [n] \right\}.$$

Thus, it is sufficient to prove that $\theta_D^j(I) = r_j(I)$ for each $j \in [n]$ and $I \subseteq [n]$. Fix $I$ and $j$, and let $\text{word}_{j,I}(D)$ have $p$ paired ()'s and $q$ $\star$'s.

First, note that $D_j$ is a basis of $SM_n(D_j)$. Let $B$ be any basis of $SM_n(D_j)$ and pick elements $r_1$ and $r_2$ with $r_1 \notin B$, $r_2 \in B$, and $r_1 < r_2$. Consider the set $B' = B\backslash\{r_2\} \cup \{r_1\}$. Then $B' \leq B \leq D_j$, so $B'$ is also a basis of $SM_n(D_j)$. Using this observation, we build a decreasing sequence of bases $D_j \geq B_1 \geq \cdots \geq B_p$.

Order the set of paired ()'s in $\text{word}_{j,I}(D)$ from 1 to $p$. For the first pair, we get two grid squares $(r_1, j)$ and $(r_2, j)$ with $r_1 < r_2$, $r_1 \in I\backslash D_j$, and $r_2 \in D_j\backslash I$. Define $B_1$ to be the basis $D_j\backslash\{r_2\} \cup \{r_1\}$.

Inductively, the $i$th set of paired ()'s in $\text{word}_{j,I}(D)$ gives two grid squares $(r_1, j)$ and $(r_2, j)$ with $r_1 < r_2$, $r_1 \in I\backslash B_{i-1}$, and $r_2 \in (B_{i-1} \cap D_j)\backslash I$. Define $B_i$ to be the basis $B_{i-1}\backslash\{r_2\} \cup \{r_1\}$.

By construction, $\#(I \cap B_p) = p + \#(I \cap D_j) = p + q$. The proof will be complete if we can show $I \cap B_p$ is a maximal independent subset of $I$. If not, there is some $k \in I\backslash B_p$ and $l \in (B_p \cap D_j)\backslash I$ such that $B_p\backslash\{l\} \cup \{k\}$ is a basis. If $k < l$, then $k$ and $l$ correspond to a () in $\text{word}_{j,S}(D)$, so $k \in B_p$ already, a contradiction. If $k > l$, then in $\text{word}_{j,S}(D)$, $k$ and $l$ correspond to a subword )( where neither parenthesis was paired. Then, the position of $l$ in $B_p$ is the same as the original position of $l$ in $D_j$, since it cannot have been changed by any of the swaps. In this case, $k > l$ implies $B\backslash\{l\} \cup \{k\}$ is not a basis. □

Theorem 14 confirms Conjectures 3.9 and 5.13 of [19].

## Acknowledgements

We thank Balázs Elek and Allen Knutson for inspiring conversations.

## References

[1] F. Ardila, C. Benedetti, and J. Doker. Matroid polytopes and their volumes. *Discrete & Computational Geometry*, 43(4):841–854, 2010.

[2] N. Bergeron and S. Billey. Rc-graphs and schubert polynomials. *Experimental Mathematics*, 2(4):257–269, 1993.

[3] S. Billey, W. Jockusch, and R. P. Stanley. Some combinatorial properties of Schubert polynomials. *Journal of Algebraic Combinatorics*, 2(4):345–374, 1993.

[4] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler. *Oriented Matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1999.

[5] A. Borovik, I. Gelfand, and N. White. *Coxeter Matroids*. Birkhuser Basel, 2003.

[6] M. Demazure. Key polynomials and a flagged Littlewood–Richardson rule. *Bull. Sci. Math.*, 98:163–172, 1974.

[7] M. Demazure. Une nouvelle formule des caracteres. *Journal of Combinatorial Theory, Series A*, 70(1):107–143, 1995.

[8] L. Escobar and K. Mészáros. Subword complexes via triangulations of root polytopes. 2015. arXiv:1502.03997.

[9] S. Fomin and A. N. Kirillov. The Yang-Baxter equation, symmetric functions, and Schubert polynomials. *Discrete Mathematics*, 153(1):123–143, 1996. Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics.

[10] J. Haglund, M. Haiman, and N. Loehr. A combinatorial formula for nonsymmetric macdonald polynomials. *American Journal of Mathematics*, 130(2):359–383, 2008.

[11] A. Knutson and E. Miller. Subword complexes in Coxeter groups. *Adv. Math.*, 184(1):161–176, 2004.

[12] W. Kraśkiewicz and P. Pragacz. Foncteurs de Schubert. *C. R. Acad. Sci. Paris Sér. I Math.*, 304(9):209–211, 1987.

[13] A. Lascoux and M.-P. Schützenberger. Polynômes de Schubert. *C. R. Acad. Sci. Paris Sér. I Math.*, 294(13):447–450, 1982.

[14] A. Lascoux and M.-P. Schützenberger. Tableaux and noncommutative Schubert polynomials. *Funct. Anal. Appl.*, 23(3):223–225, 1989.

[15] A. Lascoux and M.-P. Schützenberger. Keys and Standard Bases. *Institute for Mathematics and Its Applications*, 19:125, 1990.

[16] P. Magyar. Schubert polynomials and Bott-Samelson varieties. *Comment. Math. Helv.*, 73(4):603–636, 1998.

[17] L. Manivel. *Symmetric functions, Schubert polynomials and degeneracy loci*, volume 6 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001. Translated from the 1998 French original by John R. Swallow, Cours Spécialisés [Specialized Courses], 3.

[18] K. Mészáros and A. St. Dizier. From generalized permutahedra to Grothendieck polynomials via flow polytopes. 2017. arXiv:1705.02418.

[19] C. Monical, N. Tokcan, and A. Yong. Newton polytopes in algebraic combinatorics. arXiv:1703.02583, 2017.

[20] A. Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, (6):1026–1106, 2009.

[21] V. Reiner and M. Shimozono. Key polynomials and a flagged Littlewood–Richardson rule. *Journal of Combinatorial Theory, Series A*, 70(1):107–143, 1995.

[22] A. Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. B*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003. Matroids, trees, stable sets, Chapters 39–69.

Alex Fink, School of Mathematical Sciences, Queen Mary University of London, UK, E1 4NS. a.fink@qmul.ac.uk

Karola Mészáros, Department of Mathematics, Cornell University, Ithaca NY 14853. karola@math.cornell.edu

Avery St. Dizier, Department of Mathematics, Cornell University, Ithaca NY 14853. ajs624@cornell.edu