Critical Peaks Redefined
$\Phi \sqcup \Psi = \top$

Nao Hirokawa[1], Julian Nagele[2], Vincent van Oostrom[2], and Michio Oyamaguchi[3]

[1] School of Information Science, JAIST, Japan
hirokawa@jaist.ac.jp
[2] Department of Computer Science, University of Innsbruck, Austria
julian.nagele@uibk.ac.at, Vincent.van-Oostrom@uibk.ac.at
[3] Nagoya University, Japan
oyamaguchi@za.ztv.ne.jp

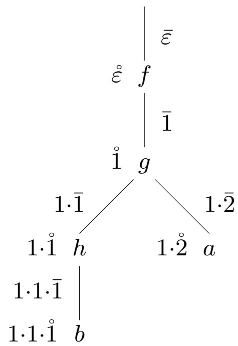
Abstract

Let a *cluster* be a term with a number of patterns occurring in it. We give two accounts of clusters, a *geometric* one as sets of (node and edge) positions, and an *inductive* one as pairs of terms with gaps (2nd order variables) and pattern-substitutions for the gaps. We show both notions of cluster and the corresponding refinement/coarsening orders on them, to be isomorphic. This equips clusters with a lattice structure which we lift to (parallel/multi) steps to yield an alternative account of the notion of critical peak.

1 Introduction

The critical pair lemma [3] is the cornerstone for proving confluence of first-order term rewrite systems. It expresses that a term rewrite system is locally confluent if and only if all its critical pairs are joinable. In case the system is moreover terminating this allows to reduce, by Newman's Lemma [4], checking confluence to checking joinability of its critical pairs, which are finitely many in the case of a finite term rewrite system. This forms the basis for Knuth–Bendix completion. The termination condition cannot be omitted without more from the critical pair lemma: On the one hand, a non-terminating TRS may fail to be confluent even in the absence of critical pairs due to non-left-linearity, as established by Klop. On the other hand, a non-terminating left-linear TRS may still fail to be confluent despite that all its critical pairs are joinable. Still, for orthogonal, i.e. left-linear and without critical pairs, TRSs confluence does hold for *geometric* reasons: redex-patterns can be contracted independently of each other, inducing a notion of residual. Starting with Church and Rosser a rich theory of residuals has been developed, but comparatively little attention has been paid to the result that lies at its basis: a strengthening of the critical pair lemma stating that any peak *either* is (a variable-instance of) a critical peak *or* can be decomposed into smaller peaks. We present such a *critical peak* lemma.

Since both for defining rewriting and for defining critical peaks the notion of *encompassment* is essential, we start off with analysing it. In particular, we call a term with a number of patterns (think of left-hand sides of rules) encompassed by it a *cluster*, and introduce two representations of clusters: a *geometric* one as sets of (node and edge) positions, and an *inductive* one as pairs of terms with gaps (2nd order variables) and pattern-substitutions for the gaps. One can think of these two representations as corresponding to the pictures respectively the formal proof of the critical pair lemma as found in e.g. [1, 5, 6]. Here we give formal accounts of *both* and of the refinement/coarsening order on them, and show them to be isomorphic. This allows one to bridge the gap between the often informal geometric intuition ('proofs by picture') at



(a) Vertex/edge positions [6]

$\emptyset$	$(f(g(h(b),a)),\ [\![\,]\!]) = \bot$
$\{\mathring{\varepsilon}\}$	$(X(g(h(b),a)),\ [\![X := f(\mathsf{v}_1)]\!])$
$\{\mathring{1}, 1{\cdot}\mathring{1}\}$	$(f(X(Y(b),a)), [\![X,Y := g(\mathsf{v}_1,\mathsf{v}_2), h(\mathsf{v}_1)]\!])$
$\{\mathring{1}, 1{\cdot}\bar{1}, 1{\cdot}\mathring{1}\}$	$(f(X(b,a)),\ [\![X := g(h(\mathsf{v}_1),\mathsf{v}_2)]\!])$
$\{\mathring{1}, 1{\cdot}\bar{2}, 1{\cdot}\mathring{2}\}$	$(f(X(h(b))),\ [\![X := g(\mathsf{v}_1, a)]\!])$
$\left\{ \begin{array}{c} \mathring{\varepsilon}, \bar{1}, \mathring{1}, 1{\cdot}\bar{1}, 1{\cdot}\mathring{1}, \\ 1{\cdot}1{\cdot}\bar{1}, 1{\cdot}1{\cdot}\mathring{1}, 1{\cdot}\bar{2}, 1{\cdot}\mathring{2} \end{array} \right\}$	$(X,\ [\![X := f(g(h(b),a))]\!]) = \top$

(b) Geometric and inductive clusters.

Figure 1: Positions and clusters for $f(g(h(b),a))$.

the basis of properties of residuals, and the inductive nature of ('terms and steps') of term rewriting. As a first example (we anticipate many more) we redefine in this paper the notion of critical peak in a purely lattice theoretic way, based just on the coarsening/refinement order of clusters. More precisely, we call a local peak between steps ϕ and ψ *critical* if $\phi \sqcup \psi = \top$, or in words, the redex-patterns of ϕ and ψ must be overlapping (if not, $\phi \sqcup \psi \sqsubset \top$ as the join would 'miss an edge' and comprise *two* patterns) and all symbols in the source of the peak must be part of either pattern (if not, $\phi \sqcup \psi \sqsubset \top$ as the join would 'miss a node' and not comprise *all* symbols). This definition captures the intuition behind something being *critical*: the source does encompass both redex-patterns but only just so, nothing else is encompassed.

We restrict ourselves to first-order term rewriting and to linear patterns.

2 Clusters and the refinement lattice

We introduce clusters and the refinement order on them. It is standard to represent terms as labelled trees using sequences of positive natural numbers called *positions* for the nodes of the tree. Following [6, Chapter 8] we extend this by having both *vertex* and *edge* positions, see Figure 1(a). Note that $\mathring{\imath}$ and $\bar{\imath}$ stand for $i{\cdot}\mathring{\varepsilon}$ and $i{\cdot}\bar{\varepsilon}$, respectively. Let $i{\cdot}P$ denote $\{i{\cdot}p \mid p \in P\}$.

Definition 1. *The* Tree *algebra has as carrier sets of positions, and interpretations*

$$f^{\mathsf{Tree}}(\vec{P}) = \{\bar{\varepsilon}, \mathring{\varepsilon}\} \cup \bigcup_i i{\cdot}P_i$$

The set of all positions of a term arises by assigning $\{\bar{\varepsilon}, \mathring{\varepsilon}\}$ to variables. We are interested in the *internal* positions arising from assigning $\emptyset$ to variables and removing the root edge $\bar{\varepsilon}$.

Definition 2. *A* cluster *for a given term t is a set of internal positions of t such that if an edge $p{\cdot}\bar{\imath}$ is in the set, its endpoints $p{\cdot}\mathring{\varepsilon}$ and $p{\cdot}\mathring{\imath}$ are too. Its connected components are called* patterns.

Our patterns correspond to those in [6, Chapter 8].

Example 1. *The first column of Figure 1(b) lists some clusters for the term $f(g(h(b),a))$, of which the first and third are not patterns (for the latter, $1{\cdot}\bar{1}$ is missing). Since $\{\mathring{1}, 1{\cdot}\bar{1}\}$ lacks the endpoint $1{\cdot}\mathring{1}$ of the edge $1{\cdot}\bar{1}$, it is not a cluster. Note that $\{\mathring{1}, 1{\cdot}\bar{1}, 1{\cdot}\mathring{1}\} \neq \{\mathring{1}, 1{\cdot}\mathring{1}\}$; the former is a cluster comprising a single pattern, whereas the latter comprises two patterns.*

Lemma 1. *For any term, the clusters for that term constitute a finite distributive lattice with respect to the subset order* $\subseteq$.

Proof. By terms being finite, and properties being inherited from the subset order. □

We give an alternative definition of clusters. To keep both apart, we will refer to the above notions as *geometric* and to the ones introduced below as *inductive*.

Definition 3. *A* skeleton *is constructed from function symbols and* 1*st and* 2*nd order variables, the latter called* gaps. *It is a* pattern-*skeleton of* arity n, *if it is not a variable and* standard*: the vector of variables occurring from left to right is* $\mathsf{v}_1, \ldots, \mathsf{v}_n$. *A* term *is a skeleton without gaps, and a* pattern *is a pattern-skeleton without gaps. A* cluster *is a pair* $(M,[\![\vec{X} := \vec{\ell}]\!])$ *with* M *a skeleton linear in the gaps, and* $[\![\vec{X} := \vec{\ell}]\!]$ *substituting patterns* $\vec{\ell}$ *for those, respecting arities. We say* $(M,[\![\vec{X} := \vec{\ell}]\!])$ *is a cluster* for *the term* $M^{[\![\vec{X} := \vec{\ell}]\!]}$.

Example 2. *The inductive clusters corresponding to the six geometric clusters of Example 1 are listed in the second column of Figure 1(b). The inequality in that example corresponds to the inequality* $(f(X(b,a)),[\![X := g(h(\mathsf{v}_1),\mathsf{v}_2)]\!]) \neq (f(X(Y(b),a)),[\![X,Y := g(\mathsf{v}_1,\mathsf{v}_2),h(\mathsf{v}_1)]\!])$.

Definition 4. *The* coarsening *order* $\sqsubseteq$ *(the* refinement *order* $\sqsupseteq$*) on inductive clusters is defined by* $(N,\beta) \sqsupseteq (M,\alpha)$ *if* $N^\gamma = M$ *and* $\beta = \alpha \circ \gamma$ *for some pattern-skeleton substitution* γ.

Example 3. *For the term* $f(a)$ *the pattern comprising both symbols may be refined to the cluster comprising two patterns:* $(Z,[\![Z := f(a)]\!]) \sqsupset (X(Y),[\![X,Y := f(\mathsf{v}_1),a]\!])$ *witnessed by the pattern-skeleton substitution* $[\![Z := X(Y)]\!]$. *Geometrically this corresponds to* $\{\mathring{\varepsilon},\bar{1},\mathring{1}\} \supset \{\mathring{\varepsilon},\mathring{1}\}$.

The main result of this section can be viewed as an instance of Birkhoff's Fundamental Theorem of Finite Distributive Lattices, expressing that *all* such lattices can be represented via downward-closed sets of join-irreducible elements ordered by subset.

Theorem 1. *For a given term, geometric clusters ordered by* $\subseteq$ *are isomorphic to inductive clusters, up to renaming of gaps, ordered by* $\sqsubseteq$. *The order is a finite distributive lattice.*

Proof. First note that we can map any inductive cluster $(M,[\![\vec{X} := \vec{\ell}]\!])$ to a geometric cluster by means of what we call a *cluster* algebra, a pair of algebras for interpreting both components:

- Shift for interpreting M: $f^{\mathsf{Shift}}(\vec{P}) = \bigcup_i i \cdot P_i$; and
- Tree for interpreting $\vec{X}$ via $\vec{\ell}$: $f^{\mathsf{Tree}}(\vec{P}) = \{\bar{\varepsilon},\mathring{\varepsilon}\} \cup f^{\mathsf{Shift}}(\vec{P})$, then removing the root edge.

This map is seen to be a bijection. That it preserves the order is seen:

(geometric ⇒ inductive) by induction on the term, simultaneously building the inductive clusters from both geometric clusters and the witnessing pattern-skeleton substitution, using that geometric clusters are preserved under left-quotienting by argument positions.

(inductive ⇒ geometric) algebraically, using a substitution lemma and that the Tree-interpretation contains the Shift-interpretation; □

Example 4. $(f(X(b,a)),[\![X := g(h(\mathsf{v}_1),\mathsf{v}_2)]\!])^{(\mathsf{Shift},\mathsf{Tree})} = 1 \cdot (g(h(\mathsf{v}_1),\mathsf{v}_2)^{\mathsf{Tree}}_{[\mathsf{v}_1,\mathsf{v}_2 \mapsto \emptyset,\emptyset]} - \{\bar{\varepsilon}\}) = 1 \cdot (\{\bar{\varepsilon},\mathring{\varepsilon},\bar{1},\mathring{1}\} - \{\bar{\varepsilon}\}) = 1 \cdot \{\mathring{\varepsilon},\bar{1},\mathring{1}\} = \{\mathring{1},1 \cdot \bar{1},1 \cdot \mathring{1}\}$.

Equipping clusters with the lattice operators $\top$, $\bot$, $\sqcap$, $\sqcup$, the theorem shows the join-irreducible elements of the refinement order can be perceived as vertices (patterns comprising a single function symbol) and edges (patterns comprising two function symbols), which can be seen as justifying having both types of positions: an edge is *more* than the join of its endpoints.

3 Critical peaks redefined

We first redefine single/parallel/multi-steps in first-order term rewriting *by second-order means* as clusters with rule symbols [6] in patterns, and next critical peaks via the clusters of its steps.

Lemma 2. *For a left-linear TRS, $t \to s$ iff $t = M^{[\![X:=\ell]\!]}$ and $M^{[\![X:=r]\!]} = s$ for some skeleton M and pattern substitution $[\![X := \ell]\!]$, for rule $\ell \to r$ with ℓ standard.*

Proof. If $t = C[\ell^\sigma]$ and $C[r^\sigma] = s$, then set $M = C[X(\vec{\mathsf{v}}^\sigma)]$ and vice versa. ☐

Turning rules into rule symbols, the lemma justifies representing a step $t \to s$ as a cluster ϕ having the rule symbol as pattern substitution. We denote it by $t \to_\phi s$.

Example 5. *The step $f(f(f(a))) \to f(g(f(a), f(a)))$ for the rule $\varrho(\mathsf{v}_1) : f(\mathsf{v}_1) \to g(\mathsf{v}_1, \mathsf{v}_1)$ can be represented by the cluster $(f(X(f(a))), [\![X := \varrho(\mathsf{v}_1)]\!])$: projecting the rule ϱ in the substitution to its left/right-hand side $[\![X := f(\mathsf{v}_1)]\!]$/ $[\![X := g(\mathsf{v}_1, \mathsf{v}_1)]\!]$ yields the step.*

We now use the lattice to measure the interaction between steps in peaks.[1] Note that the top element $\top$ for an n-ary pattern ℓ has shape $(X(\mathsf{v}_1, \dots, \mathsf{v}_n), [\![X := \ell]\!])$.

Definition 5. *A local peak $s \ {}_\phi\!\leftarrow t \to_\psi u$ is* critical *if $\phi \sqcup \psi = \top$ with t standard, where we extend the refinement order to steps via their left-hand side.*

Example 6. *Consider the (standard) rules $f(\mathsf{v}_1) \to \mathsf{v}_1$ and $f(\mathsf{v}_1) \to a$.*

- *$\mathsf{v}_1 \leftarrow f(\mathsf{v}_1) \to a$ is critical since the union of the redex-patterns is $\{\mathring{\varepsilon}\}$, the set of all internal positions of $f(\mathsf{v}_1)$;*
- *$b \leftarrow f(b) \to a$ is not critical since the union of the redex-patterns is $\{\mathring{\varepsilon}\}$, which is distinct from the set $\{\mathring{\varepsilon}, \bar{1}, \mathring{1}\}$ of all internal positions of $f(b)$; and*
- *$a \leftarrow f(f(\mathsf{v}_1)) \to f(a)$ is not critical since the union of the redex-patterns $\{\mathring{\varepsilon}, \mathring{1}\}$ misses the internal position $\bar{1}$ of $f(f(\mathsf{v}_1))$.*

Lemma 3. *The definition of critical peak for a pair of rules is equivalent to the definitions found in the literature, up to* most generalness *(unifier or common instance),* chiasmus *(1st rule–2nd rule vs. 2nd rule–1st rule),* order *(outer–inner vs. inner–outer),* renaming *(variables in the peak), and* triviality *(overlap of a rule with itself at the root).*

Proof. The definition of critical pair/peak varies along these parameters throughout the standard literature [3, 2, 1, 5, 6]. The notions in the literature *implement* our abstract notion. ☐

The above generalises to multi-steps [6] contracting a number of (non-overlapping) redex-patterns at the same time. We write $\multimap$ ($\multimap_\Phi$) for multi-step (induced by cluster Φ).

Lemma 4. *For a left-linear TRS, $t \multimap s$ iff $t = M^{[\![\vec{X}:=\vec{\ell}]\!]}$ and $M^{[\![\vec{X}:=\vec{r}]\!]} = s$, for some skeleton M and pattern substitution $[\![\vec{X} := \vec{\ell}]\!]$, for rules $\overrightarrow{\ell \to r}$ with $\vec{\ell}$ standard.*

Definition 6. *A peak $s \ {}_\Phi\!\multimapinv t \multimap_\Psi u$ is* critical *if $\Phi \sqcup \Psi = \top$ with t standard, where we extend the refinement order to multi-steps via their left-hand side.*

That is, the same concise (5 symbols) definition of critical peak as before allows us to capture all the notions of parallel critical peaks and development critical peaks (having definitions of up to 2 pages), due to Gramlich, Toyama, Okui, and Felgenhauer from the literature.

[1] The lattice structure on clusters does, in itself, not give rise to a lattice structure on rules/steps/reductions.

Example 7. *The following are critical peaks in our sense:*

- *the parallel (one–parallel) critical peak* $c \leftarrow f(a,a) \mathrel{+\!\!\!\!\!\longrightarrow} f(b,b)$ *for rules* $f(a,a) \to c$, $a \to b$;
- *the development (one–multi) critical peak* $g(c) \leftarrow g(f(a)) \mathrel{-\!\!\!\circ\!\!\!\to} b$ *for rules* $f(a) \to c$, $g(f(\mathsf{v}_1)) \to \mathsf{v}_1$, $a \to b$; *and*
- *the multi–multi critical peaks* $f(g^n(\mathsf{v}_1)) \leftarrow\!\!\circ\!\!- f^{2n+1}(\mathsf{v}_1) -\!\!\circ\!\!\to g^n(f(\mathsf{v}_1))$ *for* $f(f(\mathsf{v}_1)) \to g(\mathsf{v}_1)$.

Lemma 5 (Critical peak)**.** *If* $s \;{}_{\Phi}\!\leftarrow\!\!\circ\!\!- t -\!\!\circ\!\!\to_{\Psi} u$ *having more than one redex-pattern (in total),*

- $\Phi \sqcup \Psi = \top$*: the peak is a variable-substitution instance of a* critical *peak;*[2] *or*
- $\Phi \sqcup \Psi \neq \top$*:* $\Phi = \Phi_0^{[x:=\Phi_1]}$ *and* $\Psi = \Psi_0^{[x:=\Psi_1]}$*, for peaks* $s_i \;{}_{\Phi_i}\!\leftarrow\!\!\circ\!\!- t_i -\!\!\circ\!\!\to_{\Psi_i} u_i$ *with* $i \in \{0,1\}$*, having* smaller *skeletons.*

We sketch how the lemma allows to prove many confluence-by-critical-pair-analysis results by induction on the size of the skeleton and splitting in these two cases.

Example 8.

- *the critical pair lemma for (left-linear) TRSs follows in the first case by the assumption that critical pairs are joinable, and in the second case by the induction hypothesis (twice) and then using that reduction is closed under substitution to* recompose*;*
- *that orthogonal TRSs are confluent follows by proving that multi-steps have the diamond property, with the first case being trivial since each critical peak is trivial by orthogonality, and concluding in the second case by the induction hypothesis (twice) and then using that multi-steps are closed under substitution to* recompose *the multi-steps; and*
- *that development-closed TRSs are confluent is proven by refining the proof of the previous item by an extra (outer) induction on the* amount *of overlap between the multi-steps. For that it is essential that the refinement order is a* distributive *lattice, as that allows to express the amount of overlap between two multi-steps as the sum of the amounts of overlap (the sizes of the meets) between their constituting redex-patterns.*

We expect the above extends to non-left-linear,[3] higher-order pattern, and graph rewriting.

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[2] Note that if $\varsigma \sqcup \zeta = \top$, then $\varsigma, \zeta \neq \bot$ iff $\varsigma \sqcap \zeta \neq \bot$ by connectedness and downward closedness of clusters.

[3] Although the definition of critical peak remains the same, we then do not get a distributive lattice.