Global Rigidity and Symmetry of Direction-length Frameworks

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Declaration

I, Katharine Clinch, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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Details of collaboration and publications

This thesis was produced under the guidance of my supervisor, Prof. Bill Jackson. The majority of the contents are my own work. The exception to this is Chapter 8, which was written in collaboration with Prof. Bill Jackson and Prof. Peter Keevash. The contents of Chapters 7 and 8 previously appeared in papers [4] and [5] respectively.

Abstract

A two-dimensional direction-length framework (G, p) consists of a multigraph $G = (V; D, L)$ whose edge set is formed of "direction" edges D and "length" edges L, and a *realisation* p of this graph in the plane. The edges of the framework represent geometric constraints: length edges fix the distance between their endvertices, whereas direction edges specify the gradient of the line through both endvertices.

In this thesis, we consider two problems for direction-length frameworks. Firstly, given a framework (G, p), is it possible to find a different realisation of G which satisfies the same direction and length constraints but cannot be obtained by translating (G, p) in the plane, and/or rotating (G, p) by 180°? If no other such realisation exists, we say (G, p) is *globally rigid.* Our main result on this topic is a characterisation of the direction-length graphs G which are globally rigid for all "generic" realisations p (where p is generic if it is algebraically independent over $\mathbb{Q}$).

Secondly, we consider direction-length frameworks (G, p) which are symmetric in the plane, and ask whether we can move the framework whilst preserving both the edge constraints and the symmetry of the framework. If the only possible motions of the framework are translations, we say the framework is *symmetry-forced rigid.* Our main result here is for frameworks with single mirror symmetry: we characterise symmetry-forced infinitesimal rigidity for such frameworks which are as generic as possible. We also obtain partial results for frameworks with rotational or dihedral symmetry.

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Part I

Background

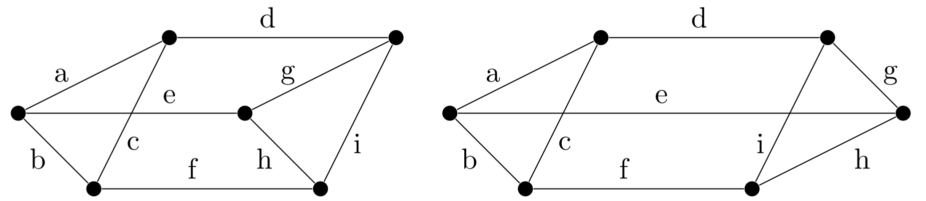
Chapter 1

Introduction

A fundamental problem in many fields is identifying whether a given structure is *rigid* or *flexible*. Such questions arise in civil engineering [31], mechanical engineering [34], computer science [12] and chemistry [38], amongst others. In each of these applications, we can often think of the structure as being formed of a collection of rigid objects which are subject to geometric constraints between them, usually a combination of fixed distances and fixed angles. We then ask whether the constraints are sufficient to make the structure rigid.

A closely related, although more subtle, question asks whether a given structure could be rebuilt differently, so that it satisfies the same geometric constraints, but the pieces are in different relative positions. A structure which can only be built in one way is said to be *globally rigid*. This property is of particular interest to molecular engineers [9]. The name "global rigidity" reflects the fact that for a structure to satisfy this property, it must first be rigid. Otherwise, we could simply move our flexible structure until two pieces are in different relative positions.

In *rigidity theory*, we model structures using frameworks. There are



(a) A flexible framework (G, p_1) with $\mathcal{P}_1 = \{\{a, g\}, \{b, h\}, \{c, i\}, \{d, e, f\}\}$.

(b) A rigid framework (G, p_2), with partition $\mathcal{P}_2 = \{\{a, h\}, \{b, g\}, \{c, i\}, \{d, e, f\}\}$.

Figure 1.1: Two bar-and-joint frameworks (G, p_i) on the same graph. In both cases, the edge set is partitioned into sets of parallel edges, $\mathcal{P}_i$. The framework in (a) is flexible; if we hold edge e in place, the triangles abc and ghi can rotate about the two endvertices of e. The framework in (b) is rigid, if we hold e in place and try to rotate the two triangles, the edge d would stretch, which is not allowed.

many different varieties of framework, but for our purposes a *framework* is a pair (G, p), where G is a graph whose vertices correspond to the rigid pieces of our structure, and whose edges correspond to geometric constraints between their endvertices. The *realisation* p maps each vertex to a list of coordinates in $\mathbb{R}^d$. We then ask whether our framework is flexible, rigid, or globally rigid.

Perhaps surprisingly, we can often determine whether a framework is rigid, solely by considering the underlying graph. However, this method does not work in general. If we place our graph in a realisation which has geometric dependencies, such as parallel lines, collinear points, or symmetry, then these dependencies will often affect the rigidity of the resulting framework. See Figure 1.1.

To avoid these problematic special cases, we restrict our consideration to *generic* frameworks (G, p), where the entries of p are algebraically independent over the rationals. This restriction is often enough to allow us to determine many properties of a generic framework by only analysing the underlying graph. A fundamental problem in rigidity theory is to identify exactly when this restriction is enough. We say a property is *generic*, if for every graph G, either all generic frameworks (G, p) satisfy the property, or none of them do. In this terminology, our first motivating question is:

Problem 1. Given a property $\mathcal{P}$, is $\mathcal{P}$ a *generic property*?

When $\mathcal{P}$ is known to be a generic property, the next challenge is to find a set of conditions on the underlying graph which determine exactly when a generic framework satisfies or fails the property. Such conditions are a *combinatorial characterisation* of $\mathcal{P}$. This is the second question we consider:

Problem 2. Can we find a combinatorial characterisation of $\mathcal{P}$?

Observe that finding a combinatorial characterisation of $\mathcal{P}$ automatically implies that the property is generic. In this thesis we consider a class of frameworks with both distance and angle-like constraints, and attempt to answer these questions for properties of such frameworks.

Systems containing angle constraints are notoriously difficult to solve, so much work in rigidity has focussed on models with only length constraints. Probably the most studied model of this type are *bar-and-joint frameworks* (G, p) where every edge in $E(G)$ fixes the distance between its endvertices. See Figure 1.1. These frameworks have a long history, going at least as far back as Maxwell [25]. In what is now a classical result of rigidity theory, Laman [24] characterised rigidity for generic bar-and-joint frameworks in the plane. More recently, Jackson and Jordán characterised global rigidity for generic bar-and-joint frameworks in the plane [16].

A second classical model is *parallel drawings* (G, p), where each edge fixes the slope of the line through its two end-vertices. Parallel drawings were once commonly used by engineers, until computer-based methods made these traditional tools redundant. Recently, there has been a renewed interest in parallel drawings and their generalisations [11, 21], as they provide a tractable way to handle angle-like constraints.

Although parallel drawings are less intuitive than bar-and-joint frameworks, in many ways they are significantly easier to work with. For example, rigidity has been characterised for generic parallel drawings in all dimensions [40], whereas for bar-and-joint frameworks, characterisations are only known in dimensions 1 and 2. Coincidentally, the requirements for a generic bar-and-joint framework or parallel drawing to be rigid in the plane are identical. Further, rigidity and global rigidity are equivalent for parallel drawings [40]. As a result, global rigidity has a simple characterisation for generic parallel drawings in all dimensions, whereas characterisations for generic bar-and-joint frameworks are again only known for 1 and 2 dimensions.

Direction-length frameworks (G, p) meld these two earlier models into one which has both length and angle-like constraints. We do this by defining our underlying graph G to have two types of edges: D for *direction edges* and L for *length edges*, and call the resulting graph $G = (V; D, L)$ a *direction-length graph*. In a direction-length framework, the direction edges impose the same slope constraints as edges in parallel drawings, and the length edges impose the distance constraints of a bar-and-joint framework. This combined model allows us to analyse a much broader class of real-world structures, and gives insight into the effects of true angle constraints. Direction-length frameworks were first introduced in [35], in which the authors characterised rigidity for generic direction-length frameworks in the plane by combining the existing planar characterisation for bar-and-joint frameworks with that for parallel drawings.

This thesis focusses on characterising two properties of direction-length frameworks in the plane. The first is global rigidity, which we consider in Part II. Ideally, we would like to approach this in a similar way to how planar rigidity was characterised by Servatius and Whiteley [35], namely, to combine the results for bar-and-joint frameworks and parallel drawings. However, this procedure is complicated by two features of the existing results. First, the requirements for a generic bar-and-joint framework or parallel drawing to be globally rigid are different, so we cannot switch so readily between these two models in our proof. And second, the requirements for a bar-and-joint framework to be globally rigid are particularly awkward to use. This made the proof in [16] quite technical, and unfortunately these complications are compounded when we add direction edges.

There is one further snag to our plan: it is not known whether global rigidity is a generic property of direction-length frameworks (Problem 1). Consequently, it is possible that for two generic realisations p and q of a direction-length graph G, the framework (G, p) is globally rigid, but (G, q) is not. Unfortunately, we cannot show that such a scenario is impossible. So instead, we provide a characterisation of when a direction-length graph G is globally rigid for *all* generic realisations.

In Part III we consider our second topic, symmetry. Many real-world structures are not generic, and so results from rigidity theory often do not apply. However, it is very common for real-world structures to exhibit symmetry. Over the past 10 years, there has been much interest in finding new methods which can characterise rigidity for *symmetric frameworks* (G, p), where the only algebraic dependencies allowed in p are those imposed by the symmetry.

In general, this is a much harder problem than in the non-generic case, so before asking whether a symmetric framework has any motions, we first ask whether it has any motions which preserve the symmetry. If not, we say it is *symmetry-forced rigid*. Symmetry-forced rigidity was recently characterised for parallel drawings in any dimension, and under any point group symmetry, by Tanigawa [37]. Once more, for bar-and-joint frameworks, the situation is more complicated. In March 2016, Jordán, Kaszanitzky and Tanigawa [22] characterised symmetry-forced rigidity for planar bar-and-joint frameworks under single reflection, rotation or odd dihedral symmetry. However, no characterisation is known for even dihedral symmetry.

These results for symmetry-forced rigidity have the same two features we highlighted when discussing global rigidity: firstly, that the requirements for bar-and-joint frameworks and parallel drawings differ, and secondly, that bar-and-joint frameworks create additional complications. Most of these complications arise when considering dihedral symmetries, so in this thesis we limit our consideration to cyclic groups. This leads to a characterisation of symmetry-forced rigidity for planar direction-length frameworks under single reflection symmetry.

Before tackling these two topics, we provide an introduction to rigidity theory in Chapter 4; focussing on the relevant results for bar-and-joint frameworks, parallel drawings, and direction-length frameworks. Our aim is to provide combinatorial characterisations of the properties we consider. To do this, we shall use ideas from graph theory and matroid theory. As such, we first provide a brief introduction to the key results and concepts from these two areas in Chapters 2 and 3 respectively.

The methods we use to construct realisations of globally rigid frameworks in Chapter 8, require some results from algebraic geometry and differential geometry. A brief introduction to these topics, and their use in rigidity theory, is included in Chapter 5.

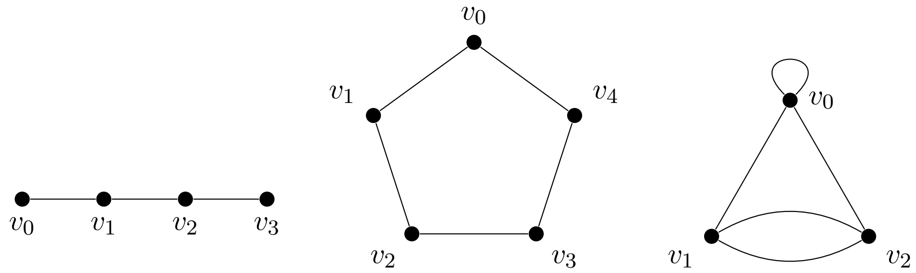
Chapter 2

Graph theory

A *graph* $G = (V, E)$ consists of a finite set of *vertices* V, together with a set of *edges* E, which are pairs of not necessarily distinct vertices. An edge e between the vertices u and v is denoted either by $\{u, v\}$, or more concisely by uv. We say e is a *uv-edge*; and that u and v are *endvertices* of e. When u is an endvertex of e we say that u and e are *incident*.

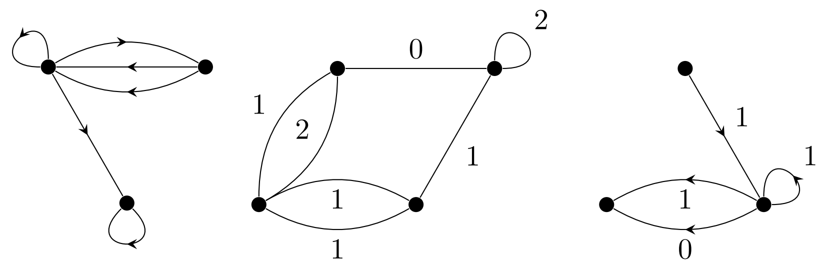
Given a graph G, we refer to its *vertex set* as $V(G)$ and *edge set* as $E(G)$. An edge $\{v, v\}$ is a called a *loop*. Two distinct edges e and f are *parallel* if e and f are both uv-edges for some $u, v \in V(G)$. A graph G is *simple* if it contains no parallel edges or loops, and is a *multigraph* otherwise. We say a multigraph is *loop-free* if it contains no loops. Graphs are often thought of as drawings, where each vertex is represented by a dot, and two dots are connected by a line or curve if there is an edge between their corresponding vertices. See Figure 2.1.

An *oriented graph* is a multigraph G, where every edge is assigned an *orientation* from one of its endvertices to the other. A uv-edge oriented from u to v is denoted by $\overrightarrow{uv}$, $\overleftarrow{vu}$, or by the ordered pair (u, v). We also use this notation for oriented loops (when $u = v$). See Figure 2.2 (a).



(a) Path on four vertices. (b) Cycle on five vertices (c) A multigraph.

Figure 2.1: Examples of graphs. Examples (a) and (b) are simple. The multigraph in (c) has a loop at v_0, and two parallel v_1v_2-edges.



(a) An oriented graph. (b) An edge-labelled graph. (c) A gain graph.

Figure 2.2: Multigraphs with oriented edges and edge-labellings. In examples (b) and (c) the edges are labelled with elements of $\mathbb{Z}_3$ and $\mathbb{Z}_2$ respectively.

An *edge-labelled* graph, is a pair (G, ψ) where G is a graph, and $\psi : E(G) \to \Gamma$ assigns an element of Γ to each edge. A *gain graph* is an edge-labelled graph (G, ψ), where Γ is a group and G is an oriented graph. For a gain graph (G, ψ), the map ψ is called the *gain function*, and we refer to the labelling $\psi(e)$ of an oriented edge e as the *gain* on e. See Figures 2.2(b) and (c). In topological graph theory, gain graphs are more commonly called *voltage graphs* [14], however the term "gain graph" is more usual in matroid theory [28].

In this thesis we consider *direction-length graphs*, which are multigraphs of the form $G = (V; D, L)$, where each edge in $E(G)$ is assigned one of two *types*: D for *direction* or L for *length*. We represent this choice by *2-colouring* the edges of our graph drawings; edges of type D are represented by dashed lines, and type L by solid lines throughout this thesis. See Figure 2.3.

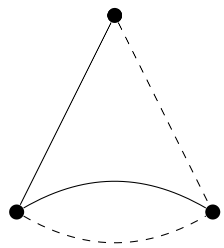


Figure 2.3: A direction-length graph.

When discussing global rigidity in Part II, we use loop-free direction-length multigraphs. When we later consider symmetry in Part III, we also use *direction-length gain graphs* (G, ψ), where $G = (V; D, L)$ is a direction-length multigraph which is oriented, and labelled with elements of some group Γ by ψ.

2.1 Graph connectivity

Given a graph $G = (V, E)$, a *subgraph* of G is a graph $H = (U, F)$ such that $U \subseteq V$ and $F \subseteq E$. Given a vertex set $X \subseteq V$, the subgraph *induced by* X in G is denoted $G[X]$, and has vertex set X and edge set $E_G(X) = \{\{u, v\} \in E(G) : u, v \in X\}$. We extend this concept to sets of edges, and say the edge set $F \subseteq E$ *induces* the subgraph $G[F]$, with edge set F and vertex set $V_G(F) = \{u \in V : \{u, v\} \in F \text{ for some } v \in V\}$. When the original graph is clear from the context, we omit these subscripts, and just refer to $E(X)$

and $V(F)$.

For vertex sets $X \subseteq V$, we follow the standard convention of letting $i(X) = |E(X)|$. When G is a direction-length graph, we adapt this notation by letting $i_D(X)$ and $i_L(X)$ denote respectively the number of direction and length edges in $G[X]$.

A graph $G = (V, E)$ is a *path* if for some ordering of the vertex set $V = \{v_0, v_1, \dots, v_t\}$ we can write $E = \{v_0v_1, v_1v_2, \dots, v_{t-1}v_t\}$. In this case, we say G is a *path from v_0 to v_t*, or simply G is a *v_0v_t-path.* If instead, there is an ordering of the vertex set which gives $E = \{v_0v_1, v_1v_2, \dots, v_{t-1}v_t, v_tv_0\}$, we say G is a *cycle.* See Figure 2.1 (a) and (b).

Given any graph G, a *walk* W in G is a sequence $v_0, e_0, v_1, e_1, v_2, \dots, v_t$ where $v_i \in V(G)$ and $e_i = v_iv_{i+1} \in E(G)$. Unlike a path, a walk may contain both repeated vertices and repeated edges. If $v_0 = v_t$, we say W is a *closed walk.*

A graph $G = (V, E)$ is *connected*, if for all distinct $u, v \in V$, G contains a uv-path. We say G is *disconnected* otherwise. If G contains no cycles, then G is a *forest*; and if G is connected with no cycles, then G is a *tree.* A subgraph H of G is a *spanning subgraph of* G if $V(H) = V(G)$. Similarly, a set of edges $F \subseteq E(G)$ or subgraph $F \subseteq G$ *spans* G when $V(F) = V(G)$. If H is a tree and spans G, we say H is a *spanning tree of* G. It is easy to see that a graph is connected if and only if it has a spanning tree.

Connectivity is key to many of our proofs, so it will often be useful to know how far our graph is from being disconnected. We say $G = (V, E)$ is *k-connected* if $|V| > k$, and for all $U \subseteq V$ with $|U| < k$, the graph $G[V - U]$ is connected. Similarly, G is *k-edge-connected* if $|E| > k$, and for all $F \subseteq E$ with $|F| < k$, the graph $G[E - F]$ is connected. A set of vertices U is called a *k-vertex-cut* of G if $|U| = k$ and $G[V - U]$ is disconnected. In the special case where $k = 1$, the unique vertex in U is called a *cut-vertex*. Again, we extend this idea to edges by saying F is a *k-edge-cut* of G if $|F| = k$ and $G[E - F]$ is disconnected.

In this terminology, G is k-connected if and only if it has no k-vertex-cuts, and is k-edge-connected if and only if it has no k-edge-cuts. When the set of edges or vertices deleted from a graph is small we shall often abuse notation by letting, for example, $G - u = G[V - \{u\}]$, $G - e = G[E - \{e\}]$ and $G - F = G[E - F]$.

A *connected component* of a graph G is a maximal subgraph H such that for all distinct vertices u and v in $V(H)$, there is a path in H from u to v. A *trivial connected component* of G consists of a single vertex, incident to no edges in G. The set of connected components of a graph G is denoted $C(G)$. For an edge set F, we let $C(F)$ denote the family of edge sets $F' \subseteq F$ such that $G[F']$ is a connected component of $G[F]$.

The following terminology and notation shall be useful when discussing connectivity. The *degree* of a vertex v in G is denoted $d_G(v)$, and counts the number of times v is incident to edges in G. Each loop at v contributes 2 to $d_G(v)$, as both ends of the edge terminate at v, whereas a non-loop edge at v contributes 1 to the degree. In the special case where v is incident to no loops, $d_G(v)$ is equal to the number of edges incident to v. When the graph is clear from the context, we use $d(v)$ instead of $d_G(v)$. The *minimum degree* of a graph G is $\delta(G) = \min_{v \in V(G)} d_G(v)$.

We extend the notion of degree to other contexts. For a graph G, and non-empty vertex sets $X, Y \subseteq V(G)$, we let $d(X, Y)$ denote the number of edges from $X - Y$ to $Y - X$ in G. We extend this further to three non-empty vertex sets $X, Y, Z \subseteq V(G)$ by letting $d(X, Y, Z)$ denote the set of edges between these three vertex sets which are not induced by any one of them. Equivalently, $d(X, Y, Z) = d(X, Y - Z) + d(Y, Z - X) + d(Z, X - Y)$.

We shall also use the related concept of the *neighbourhood* of a vertex. In a graph G, we say that two distinct vertices u and v are *neighbours* if G contains a uv-edge. The *neighbourhood* of v, denoted $N_G(v)$, is the set of all neighbours of v in G. If G contains no loops or parallel edges at v, then $|N_G(v)| = d_G(v)$; otherwise $|N_G(v)| < d_G(v)$. For a more detailed introduction to graph theory, see [3] or [10].

Chapter 3

Matroid theory

Matroids were introduced by Whitney [41], to describe the abstract rules which govern linear dependence between rows or columns in matrices.

Given a matrix M with entries in $\mathbb{R}$, a subset of the rows of M is *linearly independent* if no linear combination of these row vectors sums to zero. With this one property, many others follow: a set of rows is *linearly dependent* if they are not linearly independent, a maximal linear independent set forms a *basis* of the row space of M, and the cardinality of a basis gives the *rank* of M.

The only information about M that we used to derive these properties was its set of rows, which we denote by E, and the collection of linearly independent subsets of E, which we denote by $\mathcal{I}$. This pair $(E, \mathcal{I})$ is an example of a matroid, and in fact all matrices over fields generate a matroid in this way. But we do not need a matrix to construct a matroid, all we require is a base set E and a good definition of independence with which to build $\mathcal{I}$. This idea forms the foundation for the following definition of a matroid.

A *matroid* $\mathcal{M}$ is an ordered pair $(E, \mathcal{I})$ where E is a finite set, and $\mathcal{I}$ is a collection of subsets of E which satisfies:

(I1) $\emptyset \in \mathcal{I}$,

(I2) if $I \in \mathcal{I}$ and $I' \subseteq I$ then $I' \in \mathcal{I}$, and

(I3) if $I, I' \in \mathcal{I}$ and $|I'| < |I|$ then there exists $e \in I - I'$ such that $I' \cup \{e\} \in \mathcal{I}$.

Given a matroid $\mathcal{M} = (E, \mathcal{I})$, any set contained in $\mathcal{I}$ is said to be an *independent set* of $\mathcal{M}$. In the same way as for matrices, this one definition generates the others. A set of $S \subseteq E$ is *dependent* if $S \notin \mathcal{I}$, or is a *basis* if S is maximal in $\mathcal{I}$. Condition (I3) implies that all bases of $\mathcal{M}$ have the same cardinality. So the *rank* of $\mathcal{M}$, written $\mathrm{r}(\mathcal{M})$, is the cardinality of a basis of $\mathcal{M}$.

For matroids, we extend this last definition to all subsets of E. For any set $E' \subseteq E$, the *rank of* E', denoted $\mathrm{r}(E')$, is the cardinality of the largest independent subset of E'. Under this extended definition $\mathrm{r}(\mathcal{M}) = \mathrm{r}(E)$. The following is a trivial consequence of this definition which shall be particularly useful.

Proposition 3.0.1. *[28] Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. Then $X \in \mathcal{I}$ if and only if* $\mathrm{r}(X) = |X|$.

There are many equivalent definitions for a matroid. Common variants include defining a matroid in terms of properties of bases, dependent sets, or the rank function. The following formulation is less common, but has been used in rigidity theory [16, 22] to show that a set of conditions on a graph define a matroid. We shall use it for this purpose.

Lemma 3.0.2. *The pair $\mathcal{M} = (E, \mathcal{I})$ is a matroid if and only if*

(I1) $\emptyset \in \mathcal{I}$,

(I2) if $I \in \mathcal{I}$ and $I' \subseteq I$ then $I' \in \mathcal{I}$, and

(R3) for every $E' \subseteq E$, all maximal subsets of E' in $\mathcal{I}$ have the same cardinality.

We can use the rank function to define another function, called the closure. Conceptually, the closure can be thought of as the matroid equivalent of the span of a set of vectors. Given a matroid $\mathcal{M} = (E, \mathcal{I})$, the *closure* of a set $X \subseteq E$ is denoted $\mathrm{cl}(X)$, and is the largest set $C \subseteq E$ such that $X \subseteq C$ and $\mathrm{r}(X) = \mathrm{r}(C)$. The following properties of the closure shall be useful:

Proposition 3.0.3. *[28] Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. Let $X, Y \subseteq E$ and $x \in E$. Then*

(i) $X \subseteq \mathrm{cl}(X)$.

(ii) If $X \in \mathcal{I}$ and $X \cup \{x\} \notin \mathcal{I}$, then $x \in \mathrm{cl}(X)$.

(iii) If $X \subseteq Y \subseteq E$, then $cl(X) \subseteq \mathrm{cl}(Y)$.

(iv) If $X \subseteq \mathrm{cl}(Y)$, then $\mathrm{cl}(X \cup Y) = \mathrm{cl}(Y)$.

3.1 Matroid connectivity

We shall be mostly interested in the independent sets of a matroid. However, when considering global rigidity, many of our arguments will be structured around the minimally dependent sets, or *circuits*, of a matroid. These sets satisfy many useful properties, but in particular are key to the definition of connectivity for matroids.

A matroid $\mathcal{M} = (E, \mathcal{I})$ is *connected* if for all $e, f \in E$, either $e = f$ or there is a circuit C of $\mathcal{M}$ such that $e, f \in C$. We say $\mathcal{M}$ is *trivially connected* if $|E| = 1$. A particularly useful property of matroid connectivity is the fact it is *transitive* i.e. if for $e, f, g \in E$, there exist circuits C_1 and C_2 in $\mathcal{M}$ such that $e, f \in C_1$ and $f, g \in C_2$, then $\mathcal{M}$ also contains a circuit C_3 such that $e, g \in C_3$ [28]. We shall frequently use this property in Chapter 7.

This definition suggests we can think of a connected matroid as being formed from an intersecting collection of circuits. Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid, and $C_1, C_2, \ldots, C_m$ be a non-empty sequence of circuits in $\mathcal{M}$. Let $E_i = C_1 \cup C_2 \cup \cdots \cup C_i$ for all $1 \leq i \leq m$. The sequence $C_1, C_2, \ldots, C_m$ is a *partial ear decomposition* of $\mathcal{M}$ if for all $2 \leq i \leq m$

(E1) $C_i \cap E_{i-1} \neq \emptyset$,

(E2) $C_i - E_{i-1} \neq \emptyset$, and

(E3) no circuit C_i' satisfying (E1) and (E2) has $C_i' - E_{i-1} \subset C_i - E_{i-1}$.

When $E_m = E$, we say that $C_1, C_2, \ldots, C_m$ is an *ear decomposition of* $\mathcal{M}$. The set $\tilde{C}_i = C_i - E_{i-1}$ is the *lobe* of the circuit C_i. The following properties of ear decompositions are well known:

Lemma 3.1.1. *[8] Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid with $|E| \geq 2$ and rank function r. Then*

(i) $\mathcal{M}$ is connected if and only if $\mathcal{M}$ has an ear decomposition.

(ii) If $\mathcal{M}$ is connected, then any partial ear decomposition of $\mathcal{M}$ can be extended to an ear decomposition of $\mathcal{M}$.

(iii) If $C_1, C_2, \ldots, C_m$ is an ear decomposition of $\mathcal{M}$ then

$$r(E_i) - r(E_{i-1}) = |\tilde{C}_i| - 1 \quad \text{for } 2 \leq i \leq m.$$

Graph connectivity and matroid connectivity shall both be used frequently. The relevant definition of "connectivity" will always match the object being discussed.

Chapter 4

Rigidity theory

In Chapter 2 we defined direction-length graphs. These graphs form the basis for our main model: direction-length frameworks. A *direction-length framework* is a pair (G, p) where $G = (V; D, L)$ is a direction-length graph and $p : V \to \mathbb{R}^d$ is a *realisation* of G in $\mathbb{R}^d$ for some $d \geq 1$. In this thesis, we only consider properties of direction-length frameworks and realisations in $\mathbb{R}^2$, so for brevity, we use the terms "realisation" and "direction-length framework" to refer to the 2-dimensional case. When the broader meanings are intended, this shall be stated explicitly. See Figure 4.1 for an example of a direction-length framework.

A direction-length graph $G = (V; D, L)$ is *mixed* if both D and L are non-empty. Otherwise, we say G is *direction-pure* if $L = \emptyset$, or *length-pure* if $D = \emptyset$; and that G is *pure* if it is either direction-pure or length-pure. We apply these same definitions to edge sets, by saying that $E(G)$ is *mixed*, *direction-pure*, *length-pure* or *pure*, whenever G is. Similarly, we say a vertex v in G *mixed* if it is incident to both length and direction edges, and is *pure* otherwise. When v is incident to only length or only direction edges, then we say it is *length-pure* or *direction-pure* respectively.

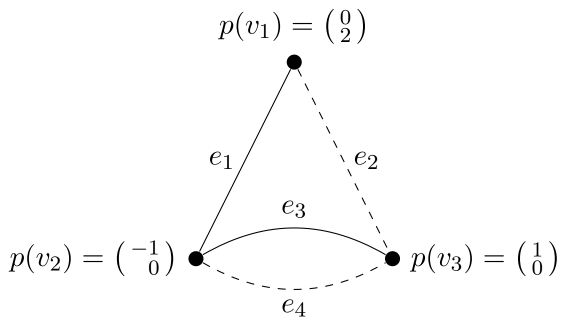


Figure 4.1: A direction-length framework (G, p). This differs from the drawing of a direction-length graph in Figure 2.3, as here the locations of the vertices are dictated by the realisation p.

In Chapter 1, we mentioned that direction and length edges correspond to edges in parallel drawings and bar-and-joint frameworks respectively. Thus a direction-length framework (G, p) is a *bar-and-joint framework* when G is length-pure, and is a *parallel drawing* when G is direction-pure. To emphasise this connection, we break with tradition and refer to bar-and-joint frameworks and parallel drawings as *length-pure frameworks* and *direction-pure frameworks* respectively. Once more, we say that a framework is *pure* if it is either direction-pure or length-pure, and is *mixed* otherwise.

4.1 Rigidity and infinitesimal rigidity

In rigidity theory, each edge of a framework corresponds to a geometric constraint. We then identify the motions of the framework which maintain these constraints. For direction-length frameworks, the type of the edge determines the type of geometric constraint imposed. Length edges fix the distance between their end-vertices, whereas direction edges fix the slope of the line through their end-vertices. Note that this means the name "direction edge" is slightly misleading, as it is the slope, not the direction, of the edge which is fixed. For a given $uv \in D$ it does not matter which order the vertices u and v appear in, we can slide one vertex past the other on a line of fixed slope without violating the constraint.

More rigorously, a *motion* of a direction-length framework (G, p) is a continuous function $p_t = P(t)$ for $0 \leq t \leq 1$ where $p_t : V(G) \to \mathbb{R}^2$ is a realisation of G which satisfies

(M1) $p_0(v) = p(v)$ for all $v \in V(G)$;

(M2) for all $\{u, v\} \in L$ and $t \in [0, 1]$, $\|p_t(u) - p_t(v)\| = \|p(u) - p(v)\|$; and

(M3) for all $\{u, v\} \in D$ and $t \in [0, 1]$ there exists a $\lambda \in \mathbb{R}$ such that $p_t(u) - p_t(v) = \lambda(p(u) - p(v))$.

A motion is *trivial* if (M2) and (M3) hold for all $u, v \in V(G)$. In other words, when (G, p_t) is a translation of (G, p) for all $0 \leq t \leq 1$. A direction-length framework is *rigid* if the only continuous motions which preserve the edge constraints are trivial. A direction-length framework which is not rigid is said to be *flexible*.

A length-pure framework has no constraints of type (M3) and so can be reflected, or continuously rotated; whereas a direction-pure framework has no constraints of type (M2), and can be continuously dilated. In the theory of bar-and-joint frameworks and parallel drawings, these give extra trivial motions which lead to modified definitions of rigidity. To distinguish these definitions from those we consider, we append the words "length-" and "direction-" accordingly. So a length-pure framework is *length-rigid* if its only motions are formed of translations and rotations, and a direction-pure framework is *direction-rigid* if its only motions are formed of translations and dilations. Note that although length-pure frameworks can be reflected, this is not a continuous motion in the plane, and thus has no impact on length-rigidity.

Every smooth motion of a framework (G, p) starts off as an *instantaneous motion* p_0, whose derivative assigns an *instantaneous velocity* vector $m_0(v)$ to every vertex v in the framework. If we concatenate these vectors, we obtain a vector $m_0 \in \mathbb{R}^{2|V(G)|}$. Since the motion preserves the edge constraints, m_0 lies in the kernel of the Jacobian matrix given by the system of equations (M2) and (M3) (i.e. the matrix of partial derivatives of p_t, see Chapter 5 for more details). However, there may be other assignments of instantaneous velocity vectors which also lie in this kernel, but cannot be extended to finite motions. This observation allows us to define a stronger version of rigidity, known as infinitesimal rigidity or first-order rigidity.

For a direction-length framework (G, p), we use the fact $p(u) - p(v)$ is fixed to obtain a dot product for each equation of type (M2) and (M3). By taking derivatives of these dot products at $t = 0$, we obtain the Jacobian matrix, $R(G, p)$, which we call the *rigidity matrix of* (G, p). This matrix has $2|V(G)|$ columns and $|E(G)|$ rows where the row corresponding to an edge $\{u, v\} \in L$ has entries $(p(u) - p(v))^T$ in the pair of columns corresponding to u, $(p(v) - p(u))^T$ in the columns corresponding to v, and 0 everywhere else. If $\{u, v\} \in D$ then the entries in the columns corresponding to u and v instead contain $((p(u) - p(v))^\perp)^T$ and $((p(v) - p(u))^\perp)^T$ respectively, where $\binom{x}{y}^\perp = \binom{y}{-x}$ and $\binom{x}{y}^T = (x, y)$.

A vector in the kernel of $R(G, p)$ is called an *infinitesimal motion* of (G, p). As described above, every instantaneous velocity of our framework is an infinitesimal motion. Since we can always translate our framework, we know $\dim(\ker R(G, p)) \geq 2$. We say (G, p) is *infinitesimally rigid* if this holds with equality, or equivalently, if all infinitesimal motions of (G, p) are trivial. Thus any infinitesimally rigid framework is rigid.

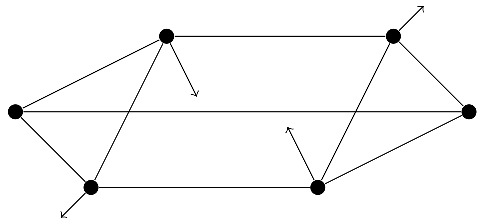


Figure 4.2: An infinitesimal motion of a length-pure framework which does not extend to a finite motion. The leftmost and rightmost vertices are assigned the zero vector.

Once again, in the theory of bar-and-joint frameworks and parallel drawings we have an analogue of infinitesimal rigidity. In these cases, we use the same rigidity matrix. However, as the underlying graph is pure, the matrix only has rows corresponding to one type of edge. When describing length-rigidity and direction-rigidity, we saw that these definitions were based on three trivial motions. This change is reflected here: a length-pure (direction-pure) framework (G, p) is *infinitesimally length-rigid* (*infinitesimally direction-rigid*) if $\dim(\ker R(G, p)) = 3$. In the same way as for mixed frameworks, these pure analogues of infinitesimal rigidity are sufficient but not necessary conditions for the corresponding pure notion of rigidity. For example, the length-pure framework in Figure 1.1(b) is length-rigid but not infinitesimally length-rigid. Figure 4.2 shows a non-trivial infinitesimal motion of this framework.

Example 4.1.1. Let (G, p) be the direction-length framework given in Figure 4.1. Then its rigidity matrix is given by

$$R(G,p) = \begin{array}{c} \begin{array}{cccccc} \overbrace{\qquad\quad}^{v_1} & & \overbrace{\qquad\quad}^{v_2} & & \overbrace{\qquad\quad}^{v_3} & \end{array} \\ \begin{array}{c} \begin{pmatrix} 1 & 2 & -1 & -2 & 0 & 0 \\ 2 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & -2 \end{pmatrix} \begin{array}{l} e_1 \\ e_2 \\ e_3 \\ e_4. \end{array} \end{array} \end{array}$$

The rows of this matrix are independent. Hence $\dim(\ker R(G,p)) = 2$ and (G,p) is infinitesimally rigid and thus also rigid.

Given a rigid direction-length framework, algebraic dependencies between the coordinates of the vertices may lead to additional row dependencies within the rigidity matrix. This was the case for the length-pure examples in Figure 1.1. This suggests there may exist rigid frameworks which are not infinitesimally rigid. To avoid this problem, we only consider frameworks where such dependencies are not allowed. We say that a realisation p or framework (G,p) is *generic* when the entries in p are algebraically independent over the rationals. For generic frameworks, rigidity and infinitesimal rigidity are equivalent:

Lemma 4.1.2. *[19, Lemma 8.1] Let (G,p) be a generic direction-length framework in $\mathbb{R}^d$. Then (G,p) is rigid if and only if (G,p) is infinitesimally rigid.*

This Lemma is the direction-length analogue of Asimow and Roth's [1] important result for bar-and-joint frameworks, which said that length-rigidity and infinitesimal length-rigidity are equivalent for generic length-pure frameworks.

Since either all generic realisations of a graph G are infinitesimally rigid, or none of them are, infinitesimal rigidity is a generic property of direction-length frameworks. Recall that identifying when a property is a *generic property* was our first motivating problem in Chapter 1. We can now define a graph G to be *infinitesimally rigid* when it has a generic realisation which is infinitesimally rigid (or equivalently, when all generic realisations are infinitesimally rigid). Lemma 4.1.2 implies rigidity is also a generic property of direction-length frameworks. Hence we have the following

Lemma 4.1.3. *Let (G, p) be a generic direction-length framework. If (G, p) is (infinitesimally) rigid, then all generic realisations of G give an (infinitesimally) rigid framework.*

This Lemma implies that for generic frameworks, our choice of realisation has no impact on the rigidity of the framework. Hence the rigidity of generic frameworks is determined solely by the structure of the underlying graph G. By identifying the necessary structure, Servatius and Whiteley obtained a combinatorial characterisation of infinitesimal rigidity, and thus also of rigidity, for generic frameworks:

Lemma 4.1.4. *[35, Theorem 4] Let (G, p) be a generic direction-length framework. Then (G, p) is infinitesimally rigid if and only if it has a spanning subgraph H which has*

(i) $|E(H)| = 2|V(H)| - 2$,

(ii) $|F| \leq 2|V(F)| - 2$ *for all mixed* $\emptyset \neq F \subseteq E(H)$, *and*

(iii) $|F| \leq 2|V(F)| - 3$ *for all pure* $\emptyset \neq F \subseteq E(H)$.

This result is the direction-length analogue of a classical result in the theory of bar-and-joint frameworks. Laman [24] showed that a generic length-pure framework (G, p) is infinitesimally length-rigid if and only if it has a spanning subgraph H with $|E(H)| = 2|V(H)| - 3$ and $|F| \leq 2|V(F)| - 3$

for all $\emptyset \neq F \subseteq E(H)$. Later, Whiteley showed that Laman's condition also characterises infinitesimal direction-rigidity for generic direction-pure frameworks [40].

4.2 The rigidity matroid

Given a generic realisation p of a direction-length graph G, any row dependencies in $R(G,p)$ are not a result of the specific choice of entries in p, as this would imply an algebraic dependency in these entries, contradicting the fact p is generic. Hence whenever a set of rows in $R(G,p)$ is independent, the corresponding set of edges in G will give an independent row set in $R(G,q)$ for all generic realisations q. Thus all generic realisations of G have the same collection $\mathcal{I}$ of independent row sets in the rigidity matrix. We use this to define a matroid $\mathcal{R}(G) = (E(G), \mathcal{I})$ on the graph, which we call the *rigidity matroid* of G.

In fact, a graph H satisfies conditions (ii) and (iii) of Lemma 4.1.4 if and only if its edge set $E(H)$ is independent in the rigidity matroid $\mathcal{R}(G)$. And for an infinitesimally rigid graph G, the bases of $\mathcal{R}(G)$ are exactly the subgraphs H which satisfy all three conditions of Lemma 4.1.4. So Lemma 4.1.4, and the definition of matroid rank, give the following characteristion of infinitesimal rigidity in terms of the rigidity matroid:

Corollary 4.2.1. *Let (G,p) be a generic direction-length framework. Then (G,p) is infinitesimally rigid if and only if* $\mathrm{r}(\mathcal{R}(G)) = 2|V(G)| - 2$.

This illustrates the fact that many properties in rigidity theory can either be written as simple conditions on the rigidity matroid, or equivalently, as more complicated conditions on the graph. Both of these viewpoints shall be helpful in this thesis.

Chapter 5

Introduction to Algebraic and Differential Geometry

Most of this thesis focuses on combinatorial results, however in Chapter 8 we shall require a handful of results from geometry. We provide a brief introduction to the necessary background here.

The two key ideas which shall be useful are, firstly, that the realisation p of a 2-dimensional framework (G, p) can be thought of as a vector with $2|V|$ entries, and thus also as a point in $2|V|$-dimensional space. And secondly, that algebraic independence is an example of matroid independence, and thus provides an *algebraic representation* of the rigidity matroid for a graph G (given a suitable choice of map).

5.1 Differential geometry and the framework space

Here we recall some basic concepts of differential geometry. We refer the reader to [23] and [26] for a more thorough introduction to this subject. Let X be a smooth manifold, $f : X \to \mathbb{R}^n$ be a smooth map, and k be the maximum rank of its derivative $df|_y$ over all $y \in X$. A point $x \in X$

is a *regular point* of f if $\operatorname{rank} df|_x = k$. The Inverse Function Theorem states that if U is open in $\mathbb{R}^k$, $f : U \to \mathbb{R}^k$ is smooth, $x \in U$, and the derivative $df|_x : \mathbb{R}^k \to \mathbb{R}^k$ is non-singular, then f maps any sufficiently small open neighbourhood of x diffeomorphically onto an open subset of $\mathbb{R}^k$. The following lemma is a simple consequence of this (see [20, Lemma 3.3]).

Lemma 5.1.1. *Let U be an open subset of $\mathbb{R}^m$, $f : U \to \mathbb{R}^n$ be a smooth map and $x \in U$ be a regular point of f. Suppose that the rank of $df|_x$ is n. Then there exists an open neighbourhood $W \subseteq U$ of x such that $f(W)$ is an open neighbourhood of $f(x)$ in $\mathbb{R}^n$.*

The following function plays an important role in rigidity theory. Let $G = (V; D, L)$. For $v_1, v_2 \in V$ with $p(v_i) = (x_i, y_i)$ let $l_p(v_1, v_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2$, and $s_p(v_1, v_2) = (y_1 - y_2)/(x_1 - x_2)$ whenever $x_1 \neq x_2$. Suppose $e = v_1v_2 \in E(G)$. We say that e is *vertical* in (G, p) if $x_1 = x_2$. The *length* of e in (G, p) is $l_p(e) = l_p(v_1, v_2)$, and the *slope* of e is $s_p(e) = s_p(v_1, v_2)$, whenever e is not vertical in (G, p). Let $V = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{e_1, e_2, \dots, e_m\}$. We view p as a point $(p(v_1), p(v_2), \dots, p(v_n))$ in $\mathbb{R}^{2n}$. Let T be the set of all points $p \in \mathbb{R}^{2n}$ such that (G, p) has no vertical direction edges. Then the *rigidity map* $f_G : T \to \mathbb{R}^m$ is given by $f_G(p) = (h(e_1), h(e_2), \dots, h(e_m))$, where $h(e_i) = l_p(e_i)$ if $e_i \in L$ and $h(e_i) = s_p(e_i)$ if $e_i \in D$.

The *Jacobian* of a vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^m$ evaluated at a point $x \in \mathbb{R}^n$, is the matrix of partial derivatives with m rows and n columns given by

$$J(f(x_1, x_2, \dots, x_n)) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

For a framework (G,p), the Jacobian of the rigidity map $f_G(p)$ has $n = 2|V(G)|$ columns and $m = |E(G)|$ rows. One can verify (see [20]) that each row in the Jacobian matrix of $f_G(p)$ is a non-zero multiple of the corresponding row in the rigidity matrix, so these matrices have the same rank. Thus the rigidity matrix achieves its maximum rank when p is a regular point of the rigidity map. In particular, this is the case when (G,p) is generic.

We say a framework (G,p), or realisation $p : V(G) \to \mathbb{R}^2$ is in *standard position* if $p(v_0) = (0,0)$ for some $v_0 \in V(G)$. The *framework space* $S_{G,p,v_0} \subseteq \mathbb{R}^{2|V|-2}$ consists of all q in standard position with respect to v_0 with (G,q) equivalent to (G,p). We identify the realisation q with the vector in $\mathbb{R}^{2|V|-2}$ obtained by concatenating the vectors $q(v)$ for $v \in V(G) - \{v_0\}$.

5.2 Field extensions and genericity

A mixed framework (G,p) is *quasi-generic* if it is a translation of a generic framework. We will be mostly concerned with quasi-generic frameworks in standard position, i.e. with one vertex positioned at the origin. Such frameworks are characterised by the following elementary lemma.

Lemma 5.2.1. *[18] Let (G,p) be a framework with vertices $\{v_1, v_2, ..., v_n\}$, $p(v_1) = (0,0)$ and $p(v_i) = (p_{2i-1}, p_{2i})$ for $2 \le i \le n$. Then (G,p) is quasi-generic if and only if $\{p_3, p_4, \dots, p_{2n}\}$ is algebraically independent over $\mathbb{Q}$.*

Given a vector $v \in \mathbb{R}^d$, $\mathbb{Q}(v)$ denotes the field extension of $\mathbb{Q}$ by the coordinates of v. We say that v is *generic* in $\mathbb{R}^d$ if the coordinates of v are algebraically independent over $\mathbb{Q}$. Note that this extends the definition of "generic" which was previously only applied to realisations. In particular, we can use this definition to describe when the rigidity map $f_G(p)$ is generic.

Given fields K, L with $K \subseteq L$ the *transcendence degree* $\mathrm{td}[L : K]$ of L over K is the size of the largest subset of L which is algebraically independent over K. A reformulation of Lemma 5.2.1 is that if (G, p) is a framework with n vertices, one of which is at the origin, then (G, p) is quasi-generic if and only if $\mathrm{td}[\mathbb{Q}(p) : \mathbb{Q}] = 2n - 2$.

Recall that $G = (V; D, L)$ is independent if $D \cup L$ is independent in the rigidity matroid of G, and that f_G denotes the rigidity map of G, which is defined at all realisations (G, p) with no vertical direction edges. The next result relates the genericity of $f_G(p)$ to the genericity of p when G is independent.

Lemma 5.2.2. *[18] Suppose that G is an independent mixed graph and (G, p) is a quasi-generic realisation of G. Then $f_G(p)$ is generic.*

It is known that field extensions define a matroid:

Proposition 5.2.3. *[28, Theorem 6.7.1] Let E be a finite subset of $\mathbb{R}$. Then the collection $\mathcal{I}$ of subsets of E which are algebraically independent over $\mathbb{Q}$ is the collection of independent sets of a matroid on E.*

Let (G, p) be a mixed graph, and let $\mathcal{M} = (E(G), \mathcal{I}_f)$ denote the matroid given by $f_G(p)$, as described in Proposition 5.2.3, where a set $\{e_1, e_2, \ldots, e_t\} \subseteq E(G)$ is independent in $\mathcal{M}$ if and only if $\mathrm{td}[\mathbb{Q}(h(e_1), h(e_2), \ldots h(e_t)) : \mathbb{Q}] = t$. Lemma 5.2.2 implies that when $G = (V, E)$ is an independent graph, the rigidity matroid $\mathcal{R}(G) = (E, \mathcal{I})$ and the matroid $\mathcal{M} = (E, \mathcal{I}_f)$ conincide. Or more formally, $f_G(p)$ is an *algebraic representation* of $\mathcal{R}(G)$ over $\mathbb{Q}$.

We use $\overline{K}$ to denote the algebraic closure of a field K. Note that $\mathrm{td}[\overline{K} : K] = 0$. We say that G is *minimally rigid* if it is rigid but $G - e$ is not rigid for any edge e; equivalently G is both rigid and independent. The following lemma relates $\overline{\mathbb{Q}(p)}$ and $\overline{\mathbb{Q}(f_G(p))}$ when G is minimally rigid.

Lemma 5.2.4. *[18] Let G be a minimally rigid mixed graph and (G, p) be a realisation of G with no vertical direction edges and with $p(v) = (0, 0)$ for some vertex v of G. If $f_G(p)$ is generic then $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(f_G(p))}$.*

Lemmas 5.2.2 and 5.2.4 imply the following result for rigid mixed graphs.

Corollary 5.2.5. *Let G be a rigid mixed graph and (G, p) be a quasi-generic realisation of G with $p(v) = (0, 0)$ for some vertex v of G. Then $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(f_G(p))}$.*

Proof. Let H be a minimally rigid spanning subgraph of G. By Lemma 5.2.2, $f_H(p)$ is generic. Hence Lemma 5.2.4 gives $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(f_H(p))}$. It is not difficult to see that $\overline{\mathbb{Q}(f_H(p))} \subseteq \overline{\mathbb{Q}(f_G(p))} \subseteq \overline{\mathbb{Q}(p)}$. Thus $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(f_G(p))}$. $\square$

Part II

Global rigidity

Chapter 6

Introduction to global rigidity

In Chapter 1, we said a framework is globally rigid if we cannot construct it differently to obtain a new framework satisfying the same constraints. Here we formalise this idea.

Given a direction-length framework (G, p), we ask whether there is another realisation q such that (G, p) and (G, q) satisfy the same constraints, i.e. where a length edge in G is assigned the same length in both frameworks, and a direction edge is assigned the same slope. If there is such a q, then (G, p) and (G, q) are *equivalent*.

Clearly, if we translate (G, p), then all the distances and slopes will be preserved, so this will give an equivalent framework. Rotating by $180°$ will also result in a framework with the same assignment of distances and slopes; although this is not a continuous motion of the framework, as any realisation q_α obtained by rotating p by an angle $0 < \alpha < 180$ will assign different slopes to the direction edges. These methods of obtaining an equivalent framework are always possible, so we say a framework (G, q) is *congruent* to (G, p) if it

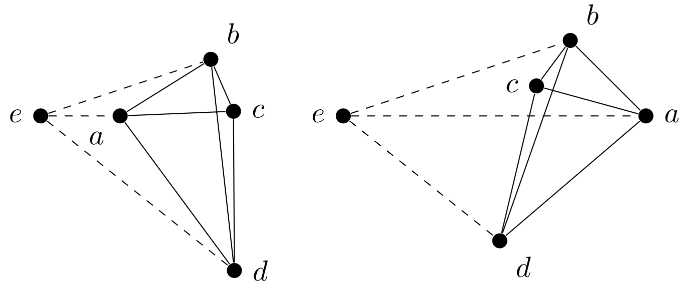


Figure 6.1: Two equivalent but non-congruent realisations of a mixed graph.

can be obtained by translating (G, p) and/or rotating it by $180°$.

With these two definitions, we can now define a direction-length framework (G, p) to be *globally rigid* when every equivalent framework (G, q), is also congruent to (G, p). Many frameworks are not globally rigid, such as those in Figure 6.1. A framework which is not globally rigid is said to be *globally flexible*.

We say a graph G is *globally rigid* if every generic realisation of G is globally rigid, and is globally flexible if every generic realisation is globally flexible. Since we do not know whether global rigidity is a generic property, it is possible that there exist graphs G which are neither globally rigid nor globally flexible i.e. for which there exist non-equivalent generic realisations p and q for which (G, p) is globally rigid, but (G, q) is globally flexible.

6.1 Pure frameworks

In the theory of bar-and-joint frameworks and parallel drawings, we have modified definitions of global rigidity. As we observed when discussing rigidity, a length-pure framework can be reflected, rotated and translated without violating the edge constraints. Thus a length-pure framework is *globally length-rigid* if all equivalent frameworks can be obtained by a combination of these moves. Similarly a direction-pure framework is *globally direction-rigid* if all equivalent frameworks can be obtained by translations and dilations. Note that dilation by -1 is equivalent to a $180°$ rotation.

Whiteley [40] showed that for all $d > 0$, direction-rigidity and global direction-rigidity are equivalent. In 2-dimensions, this implies that Laman's Theorem [24] characterises global direction-rigidity:

Theorem 6.1.1. *Let (G, p) be a generic direction-pure framework. Then (G, p) is globally direction-rigid if and only if*

(i) $|E(G)| = 2|V(G)| - 3$, *and*

(ii) $|F| \leq 2|V(F)| - 3$ *for all* $\emptyset \neq F \subseteq E(G)$.

The characterisation of global length-rigidity requires more thought. First observe that if a length-pure framework has a pair of vertices which act like a hinge in 3-dimensions, then we can reflect one half of the framework across this pair to obtain an equivalent but non-congruent realisation in the plane. To avoid this situation, the underlying graph must be 3-connected.

The second necessary property is not obvious: we require that the rigidity matroid is connected. We say that a graph G is *$\mathcal{M}$-connected*, when its rigidity matroid $\mathcal{R}(G)$ is connected. Jackson and Jordán showed that these two connectivity properties determine global length-rigidity:

Theorem 6.1.2. *[16] A generic length-pure framework (G, p) is globally length-rigid if and only if either G is a complete graph on at most 3 vertices, or G is 3-connected and $\mathcal{M}$-connected.*

A property which is very closely related to $\mathcal{M}$-connectivity is redundant rigidity. Given a direction-length framework (G, p), if $(G - e, p)$ is rigid for some $e \in E(G)$, we say the edge e is *redundant* in (G, p). If every edge in (G, p) is redundant, then (G, p) is *redundantly rigid*. Since rigidity is a generic property, redundant rigidity is too. So we say that a graph G is *redundantly rigid* if (G, p) is redundantly rigid for some generic realisation p.

A length-pure framework (G, p) is *redundantly length-rigid* if $(G - e, p)$ is length-rigid for all $e \in E(G)$. The connection between $\mathcal{M}$-connectivity and redundant length-rigidity leads to a more intuitive statement of Jackson and Jordán's result:

Theorem 6.1.3. *[16, Theorem 7.1] A generic length-pure framework (G, p) is globally length-rigid if and only if either G is a complete graph on at most 3 vertices, or G is 3-connected and redundantly length-rigid.*

This statement of Jackson and Jordán's result was the original version sought, as it proves a long-standing conjecture of Hendrickson [15]. In that paper, Hendrickson proved that $(d+1)$-connectivity and redundant length-rigidity were necessary conditions for global length-rigidity of generic length-pure frameworks in $\mathbb{R}^d$. It is not too difficult to see that these conditions are also sufficient when $d = 1$. When $d \geq 3$, there are many examples which show that these conditions are not sufficient, such as those provided by Connelly [6]. Theorem 6.1.3 proves the only remaining case: that these conditions are sufficient when $d = 2$. However, the alternative statement in Theorem 6.1.2, shall be more helpful for our purposes.

It is worth noting that Theorem 6.1.3 is not the only known characterisation of global length-rigidity. Connelly [7] showed that if a length-pure framework (G, p) in $\mathbb{R}^d$ has a "stress matrix" of maximum rank, then (G, p) is globally length-rigid. Gortler, Healy and Thurston [13] showed that this stress matrix condition is both necessary for global length-rigidity, and a generic property. Thus proving that a length-pure graph is globally length-rigid for all (or any) generic realisations in $\mathbb{R}^d$ if and only if its stress matrix has maximum rank.

As such, the stress matrix characterises global length-rigidity in a similar way to how the rigidity matrix characterises length-rigidity. However, unlike the rigidity matrix, it is not known how to write this maximum rank property in terms of properties of the graph. Currently, it is also not known whether there is an analogue of this stress matrix for direction-length graphs. Since we are interested in finding a graph theoretic characterisation of global rigidity for direction-length graphs, these two shortcomings mean that the stress matrix will be of little use to us in achieving this goal. However, it is possible that this point of view will be helpful in solving the main open problem of Chapter 8: whether global rigidity is a generic property of direction-length graphs in $\mathbb{R}^2$.

6.2 Direction-length frameworks

We aim to build on Theorems 6.1.1 and 6.1.3 to obtain a characterisation of global rigidity for generic direction-length frameworks. These earlier results suggest that our characterisation will have different conditions for direction and length edges. However, there are some simple necessary conditions which treat direction and length edges equally:

Lemma 6.2.1. *[17, Lemma 1.6] Let (G,p) be a generic direction-length framework with at least three vertices. Suppose (G,p) is globally rigid, and let $G = (V; D, L)$. Then*

(i) G is mixed,

(ii) G is rigid, and

(iii) G is 2-connected.

The first difference between Theorems 6.1.1 and 6.1.3 is whether the edges of the graph are redundant. This was a requirement for global length-rigidity but not global direction-rigidity. If our graph contains exactly one length edge, then it cannot be redundant: if we delete it, then we can dilate the remaining direction-pure graph. Servatius and Whiteley [35] showed that such graphs can still be globally rigid:

Theorem 6.2.2. *Let (G,p) be a generic direction-length framework. If $G = (V; D, L)$ is rigid and $|L| = 1$, then (G,p) is globally rigid.*

This was the first result on global rigidity for direction-length frameworks.

Much later, Jackson and Keevash [20] investigated when edge redundancy is necessary for global rigidity. In the following statement, a direction-length framework (G,p) is *unbounded* if for all $K \in \mathbb{R}$ there exists an equivalent framework (G,q) such that for some $u, v \in V(G)$, $\|q(u) - q(v)\| > K$.

Lemma 6.2.3. *[20, Theorems 1.1 and 1.3] Let (G,p) be a generic direction-length framework with at least three vertices. Suppose (G,p) is globally rigid, and let $G = (V; D, L)$.*

(i) If $|L| \geq 2$, then $G - e$ is rigid for all $e \in L$.

(ii) If $e \in D$ and $G - e$ contains a rigid subgraph on at least 2 vertices, then $G - e$ is either rigid or unbounded.

The second difference between the conditions in Theorems 6.1.1 and 6.1.3 is 3-connectivity, which is necessary for global length-rigidity but not global direction-rigidity. This is because in a length-pure framework, we can reflect a portion of our framework across a 2-vertex-cut to obtain an equivalent realisation. However, if we reflect a direction edge across a 2-vertex-cut in a generic framework, this changes its slope, thus violating the framework constraints. Hence 3-connectivity is not a necessary condition for global rigidity of direction-length frameworks: we can have 2-vertex-cuts so long as both sides of this cut contain a direction edge to make a reflection impossible.

To formalise this, let G be a graph with a k-vertex-cut X, whose removal disconnects G into $G[V_1]$ and $G[V_2]$. Let H_i be a subgraph of G with vertex set $X \cup V_i$. We say the pair (H_1, H_2) is a *k-separation* of G if $H_1 \cup H_2 = G$. The k-separation (H_1, H_2) is *edge disjoint* if $E(H_1) \cap E(H_2) = \emptyset$.

A 2-separation (H_1, H_2) of a direction-length graph G is *direction-balanced* if both $E(H_1) - E_G(X)$ and $E(H_2) - E_G(X)$ contain a direction edge, and is *length-balanced* if both of these sets contain a length edge. We say a direction-length graph G is *direction-balanced* if every 2-separation of G is direction-balanced, and is *direction-unbalanced* otherwise. We define *length-balanced* and *length-unbalanced* graphs analogously. A graph is *unbalanced* if it is either direction-unbalanced or length-unbalanced. Jackson and Jordán showed that being direction-balanced is the correct substitution for 3-connectivity:

Lemma 6.2.4. *[17, Lemma 1.6] Let (G, p) be a generic direction-length framework with at least three vertices. Suppose (G, p) is globally rigid in $\mathbb{R}^2$. Then*

(i) G is direction-balanced, and

(ii) the only 2-edge-cuts which can occur in G consist of two direction edges incident to a common vertex of degree two.

Lemma 6.2.3 implies that globally rigid frameworks may contain edges which are not redundant. However the relationship between $\mathcal{M}$-connectivity and global rigidity is less clear. A 3-connected length-pure graph G is $\mathcal{M}$-connected if and only if it is redundantly length-rigid, which made these two properties interchangeable in our characterisation of global length-rigidity (see Theorems 6.1.2 and 6.1.3). However for direction-length frameworks, an $\mathcal{M}$-connected graph may be redundantly rigid, rigid, or even flexible. So it is worth investigating the relationship between $\mathcal{M}$-connectivity and global rigidity.

Jackson and Jordán showed that being direction-balanced guarantees global rigidity for generic realisations of the simplest $\mathcal{M}$-connected graphs:

Theorem 6.2.5. *[17, Theorem 6.2] Let (G, p) be a generic realisation of a mixed graph whose rigidity matroid is a circuit. Then (G, p) is globally rigid if and only if G is direction-balanced.*

This result forms the starting point for our own work.

6.3 Summary of results

The first goal of this thesis is to extend Theorem 6.2.5 to all $\mathcal{M}$-connected direction-length graphs. To do this, we extend the methods used in [16]. This argument forms the basis of Chapter 7, and culminates in the following characterisation:

Theorem 7.6.2. *Suppose (G, p) is a generic direction-length framework and G is $\mathcal{M}$-connected. Then (G, p) is globally rigid if and only if G is direction-balanced.*

For generic length-pure frameworks, $\mathcal{M}$-connectivity is a necessary condition for global length-rigidity, and so the analogue to this approach provided a full characterisation of global length-rigidity. However, this is not true for direction-length frameworks. We need further tools to characterise global rigidity when the underlying graph is not $\mathcal{M}$-connected. We tackle this case in Chapter 8.

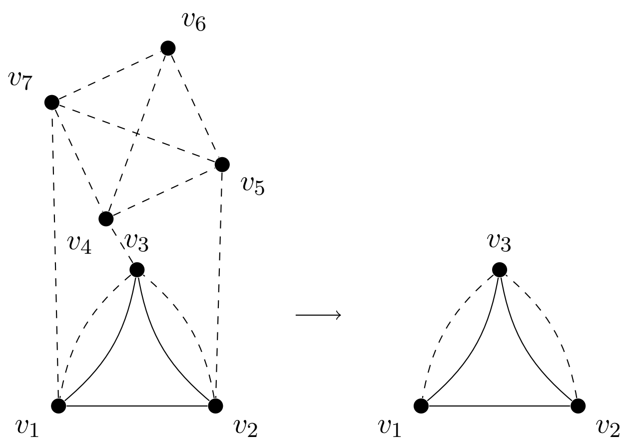


Figure 6.2: The graph G on the left is direction reducible to the subgraph H on the right in two steps. Since the direction edge v_5v_7 is contained in the direction-pure circuit induced by $\{v_4, v_5, v_6, v_7\}$ we can delete v_5v_7 by (D1). The graph we now obtain by contracting H to a single vertex is direction-pure and is the union of two edge-disjoint spanning trees so we can reduce G to H by (D2).

The main tool we use in Chapter 8 are direction reductions, where we say a graph $G = (V; D, L)$ admits a *direction reduction* to a subgraph H if either:

(D1) $H = G - e$ for some edge $e \in D$ which belongs to a direction-pure circuit in the rigidity matroid of G, or

(D2) $\emptyset \neq V(H) \subset V(G)$, and the graph G/H obtained by contracting H to a single vertex (and deleting all edges contained in H) has only direction edges and is the union of two edge-disjoint spanning trees.

If G has no direction reduction, then we say that G is *direction irreducible*. See Figure 6.2 for an example of a direction reduction.

Our first result of Chapter 8 characterises global rigidity for generic frameworks which admit a direction reduction:

Theorem 8.3.3. *Suppose (G, p) is a generic direction-length framework and*

G admits a direction reduction to a subgraph H. Then (G, p) is globally rigid if and only if $(H, p|_H)$ is globally rigid.

We then wish to show that if a generic framework is globally rigid and direction irreducible, then either it has exactly one length edge (and so is characterised by Theorem 6.2.2), or it is $\mathcal{M}$-connected (and so is characterised by Theorem 7.6.2). If this were the case, then these three Theorems would fully characterise global rigidity for generic frameworks.

Unfortunately, there is one final stumbling block: although global rigidity is a generic property in Theorems 6.2.2 and 7.6.2, we do not know whether this is true in general. It is possible that there exists a graph G which has two non-equivalent generic realisations p and q such that (G, p) is globally rigid, but (G, q) is globally flexible. As such, our methods in handling the remaining uncharacterised graphs only succeed in showing that such graphs have some generic realisation which is not globally rigid. This leads to the following statement:

Theorem 8.5.4. *Let $G = (V; D, L)$ be a direction irreducible mixed graph with $|L| \geq 2$. Then G is globally rigid for all generic realisations if and only if G is direction-balanced and $\mathcal{M}$-connected.*

We conjecture that the word "all" in the above statement can be replaced by "some". In other words, if p and q are non-equivalent generic realisations of G, and (G, p) is globally rigid, then (G, q) must also be globally rigid. If this were true, then this result would characterise all globally rigid generic frameworks where the underlying graph is direction irreducible and has at least two length edges:

Conjecture 6.3.1. *Let $G = (V; D, L)$ be a direction irreducible mixed graph with $|L| \geq 2$, and let p be a generic realisation of G. Then (G, p) is globally rigid if and only if G is direction-balanced and $\mathcal{M}$-connected.*

If this conjecture is true, it would imply that global rigidity is a generic property of all direction-length graphs. As this is still unknown, our results together with Theorem 6.2.2 of Servatius and Whiteley, instead characterise the graphs which are globally rigid for all generic realisations.

Theorem 8.5.5. *A direction-length graph $G = (V; D, L)$ is globally rigid for all generic realisations if and only if G is rigid and either $|L| = 1$ or G has a rigid, $\mathcal{M}$-connected subgraph which contains all edges in L.*

If Conjecture 6.3.1 is correct, then the conditions in Theorem 8.5.5 characterise global rigidity for all generic direction-length frameworks.

Chapter 7

Global rigidity of $\mathcal{M}$-connected frameworks

7.1 Preliminaries

Recall the definition of the rigidity matroid from Chapter 4. In Section 7.1.1 we rephrase properties of the rigidity matroid $\mathcal{R}(G)$ in terms of the graph G, and use this to obtain some simple technical results for G. Then in Section 7.1.2, we review known methods of inductively constructing globally rigid graphs.

7.1.1 Critical sets and circuits

If the edge set of the graph G is independent in the rigidity matroid $\mathcal{R}(G)$, then we say that G is *independent*. Similarly, if the edge set of G is a circuit in $\mathcal{R}(G)$, then we call G a *mixed circuit* when $E(G)$ contains both length and direction edges, and a *pure circuit* otherwise.

In Chapter 4, we noted that Lemma 4.1.4 described the conditions required for a set of edges to be a basis of $\mathcal{R}(G)$. We can easily reformulate this lemma to characterise independence instead:

Corollary 7.1.1. *[35, Theorem 4] A direction-length graph $G = (V; D, L)$ is independent if and only if for all non-empty $X \subseteq V$,*

(i) $i(X) \le 2|X| - 2$, and

(ii) $i_D(X) \le 2|X| - 3$ and $i_L(X) \le 2|X| - 3$, whenever $|X| \ge 2$.

Condition (i) implies G is loop-free; and this, together with condition (ii), implies that the pure subgraphs of G are simple. So any pair of vertices in G are connected by at most two parallel edges: one length edge, and one direction edge.

In a direction-length graph $G = (V; D, L)$, let $X \subseteq V$ with $|X| \ge 2$ and $G[X]$ independent. We call X *mixed critical* if $i(X) = 2|X| - 2$, or *pure critical* if $i(X) = 2|X| - 3$ and either $i_L(X) = 0$ or $i_D(X) = 0$ (in which case, we call X *direction critical* or *length critical* respectively). We say X is *critical* if it is either mixed or pure critical. The following results give some simple properties of critical sets. See page 25 for the definition of the function d used below.

Lemma 7.1.2. *[17, Lemma 2.4] Let $G = (V; D, L)$ be an independent mixed graph.*

(i) If X and Y are mixed critical sets with $X \cap Y \neq \emptyset$, then $X \cap Y$ and $X \cup Y$ are both mixed critical and $d(X, Y) = 0$.

(ii) If X and Y are both direction (respectively length) critical sets with $|X \cap Y| \ge 2$, then either

(a) $d(X, Y) = 0$ and $X \cap Y$, $X \cup Y$ are both direction (respectively length) critical, or

(b) $d(X,Y) = 1$, *$X \cup Y$ is mixed critical and* $i_D(X \cup Y) = 2|X \cup Y| - 3$ *(respectively* $i_L(X \cup Y) = 2|X \cup Y| - 3$*).*

(iii) If X is mixed critical and Y is pure critical with $|X \cap Y| \geq 2$, then $X \cup Y$ is mixed critical, $X \cap Y$ is pure critical and $d(X,Y) = 0$.

(iv) If X is length critical and Y is direction critical with $|X \cap Y| \geq 2$, then $X \cup Y$ is mixed critical, $|X \cap Y| = 2$ and $d(X,Y) = 0$.

Lemma 7.1.3. *[17, Lemma 2.5] Let $G = (V; D, L)$ be an independent mixed graph with mixed critical set X and pure critical sets Y and Z satisfying $|X \cap Y| = |Y \cap Z| = |X \cap Z| = 1$ and $X \cap Y \cap Z = \emptyset$. Then $X \cup Y \cup Z$ is mixed critical and $d(X,Y,Z) = 0$.*

The characterisation given in Corollary 7.1.1 of independent sets in the rigidity matroid as edge sets of sparse graphs, leads to the following results characterising circuits in the rigidity matroid:

Lemma 7.1.4. *[17, Lemma 3.1] A direction-length graph $G = (V; D, L)$ is a mixed circuit if and only if*

(i) $|D| + |L| = 2|V| - 1$ *with* $D \neq \emptyset$ *and* $L \neq \emptyset$*,*

(ii) $i(X) \leq 2|X| - 2$ *for all non-empty* $X \subset V$*, and*

(iii) $i_D(X) \leq 2|X| - 3$ *and* $i_L(X) \leq 2|X| - 3$ *for all* $X \subseteq V$ *with* $|X| \geq 2$*.*

Lemma 7.1.5. *[17, Lemma 3.2] A direction-length graph $G = (V; D, L)$ is a pure circuit if and only if*

(i) $|D| + |L| = 2|V| - 2$ *and either* $D = \emptyset$ *or* $L = \emptyset$*, and*

(ii) $i(X) \leq 2|X| - 3$ *for all* $X \subset V$ *with* $|X| \geq 2$*.*

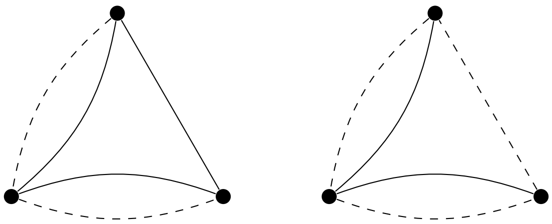


Figure 7.1: The two mixed circuits on three vertices, K_3^+ and K_3^-.

The pure circuit with fewest vertices is a pure K_4. The mixed circuits with fewest vertices are denoted by K_3^+ and K_3^-, and are obtained from a length-pure (respectively direction-pure) K_3 by adding two direction (respectively length) edges between distinct pairs of vertices, see Figure 7.1. Servatius and Whiteley [35] characterised rigidity for circuits:

Lemma 7.1.6. *[35, Theorems 2, 4] Let $G = (V; D, L)$ be a circuit. Then G is (redundantly) rigid if and only if G is mixed.*

The following result implies that the union of intersecting mixed circuits is also rigid:

Lemma 7.1.7. *Let $G = H_1 \cup H_2$ be a mixed graph, with $V(H_1) \cap V(H_2) \neq \emptyset$. If H_1 and H_2 are rigid then G is rigid.*

Proof. Let $V_i = V(H_i)$ for $i \in \{1, 2\}$, and let p be a generic realisation of G. Then $(H_1, p|_{V_1})$ and $(H_2, p|_{V_2})$ are generic realisations of H_1 and H_2 respectively.

Let $v \in V_1 \cap V_2$. Since H_1 is rigid and $p|_{V_1}$ is a generic realisation of H_1, the only motions of $(H_1, p|_{V_1})$ are translations. Hence the only motion of $(H_1, p|_{V_1})$ which fixes v is a translation of length zero, i.e. the motion which fixes all of the vertices in V_1.

Similarly, the only motion of $(H_2, p|_{V_2})$ which fixes v is the motion which fixes all of the vertices in V_2. This implies that the only motion of the entire framework (G,p) which fixes v must fix all the vertices in G. Hence (G,p) is rigid. □

We have the following results on the connectivity of critical sets and circuits.

Lemma 7.1.8. *[17, Lemma 2.3] Let $G = (V; D, L)$ be a mixed graph and let $X \subseteq V$ be a critical set. Then*

(i) $G[X]$ is 2-edge-connected unless $|X| = 2$ and $i(X) = 1$.

(ii) If (H_1, H_2) is a 1-separation of $G[X]$ then X is mixed critical and $V(H_1)$ and $V(H_2)$ are also mixed critical.

Lemma 7.1.9. *[17, Lemma 3.3] Let G be a mixed or pure circuit. Then G is 3-edge-connected and 2-connected.*

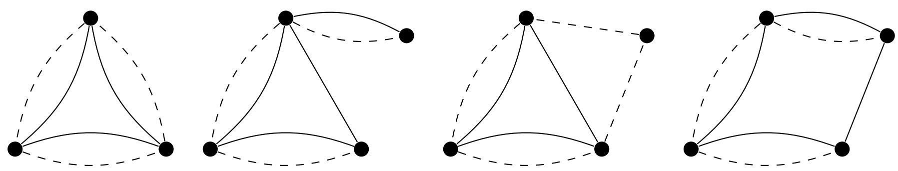
A trivial, but useful, consequence of Corollary 7.1.1 and Lemma 7.1.9 is that all circuits have the same minimum degree:

Corollary 7.1.10. *Let G be a mixed or pure circuit. Then $\delta(G) = 3$.*

7.1.2 Operations preserving global rigidity

Our goal is to characterise global rigidity for all generic direction-length frameworks with a connected rigidity matroid. We shall do this by inductively constructing all $\mathcal{M}$-connected graphs which are globally rigid for all generic realisations. To this end, we define the following three recursive operations which are known to preserve global rigidity in generic frameworks.

Given $G = (V; D, L)$, an *edge addition* adds a new edge e to G to obtain the graph $G' = G + e$. A *0-extension operation* instead adds a new vertex v to G, along with two new edges vx and vy for some $x, y \in V$, such that



(a) (b) (c) (d)

Figure 7.2: Graphs obtained from K_3^+ by an edge addition (a), 0-extension (b), direction-pure 0-extension (c), and 1-extension (d).

if $x = y$ then these edges are assigned different types. A 0-extension is *direction-pure* if both of the edges added are direction edges.

Finally, a *1-extension operation* deletes an edge $e = xy$ of G, and adds a vertex v to G, along with edges vx, vy and vz for some $z \in V$, such that at least one of these new edges is of the same type as e. A graph obtained from G in this manner is denoted G^v. See Figure 7.2 for examples of these moves.

Lemma 7.1.11. *[18, Theorems 1.2 and 1.3] Let (G, p) and (G', p') be generic direction-length frameworks. Suppose that either*

(i) (G, p) is globally rigid, and (G', p') is obtained from (G, p) by an edge addition or a direction-pure 0-extension, or

(ii) (G, p) is globally rigid, $G - e$ is rigid for some $e \in E(G)$, and (G', p') is obtained from (G, p) by a 1-extension which deletes the edge e.

Then (G', p') is globally rigid.

By Theorem 6.2.5 and Lemma 7.1.6, we know that all generic realisations of the smallest circuits, K_3^+ and K_3^-, are globally rigid and redundantly rigid. Hence any graph which can be constructed from these two graphs by the operations in Lemma 7.1.11 is also globally rigid. In particular, this implies the frameworks in Figures 7.2(a), (c) and (d) are globally rigid. The framework in Figure 7.2(b) is not globally rigid: if we rotate the new vertex by 180° about its neighbour whilst keeping the rest of the framework fixed, we obtain an equivalent but non-congruent realisation.

Since we are interested in $\mathcal{M}$-connected graphs, we need to identify which of the above operations also preserve matroid connectivity. This is the focus of the next section.

7.2 $\mathcal{M}$-Connected graphs

In Chapter 6, we noted that global rigidity has already been characterised for graphs which are circuits in the rigidity matroid (see Theorem 6.2.5). In Chapter 3, we observed that every connected matroid $\mathcal{M}$ can be constructed from a sequence of circuits of $\mathcal{M}$ by an ear decomposition. See Lemma 3.1.1, and the definition of an ear decomposition in terms of properties (E1), (E2) and (E3). As such, the key idea in this chapter is to use ear decompositions to extend the characterisation in Theorem 6.2.5 to all $\mathcal{M}$-connected graphs.

In this section, we first identify some properties of $\mathcal{M}$-connected graphs in Subsection 7.2.1. Then, in Subsections 7.2.2 and 7.2.3 we show that some of the operations introduced in Subsection 7.1.2 which preserve global rigidity, also preserve matroid connectivity. This will imply that all graphs constructed from K_3^+ or K_3^- using these moves are both $\mathcal{M}$-connected and globally rigid.

However, to characterise global rigidity for $\mathcal{M}$-connected graphs, we need to identify when an $\mathcal{M}$-connected graph can be constructed using these operations (or equivalently, deconstructed using inverse operations). This is much harder, and is the topic of Sections 7.3 to 7.6. In the final part of this section, Subsection 7.2.4, we observe some simple necessary conditions for the inverse operations to preserve global rigidity.

7.2.1 Properties of $\mathcal{M}$-connected graphs

Lemma 7.2.1. *Let G be a mixed or pure graph. If G is $\mathcal{M}$-connected then G is 2-connected.*

Proof. Assume that G has a 1-separation (H_1, H_2) and let $e \in E(H_1)$ and $f \in E(H_2)$. Since G is $\mathcal{M}$-connected, the rigidity matroid of G contains a circuit C such that $e, f \in C$. Lemma 7.1.9 implies that $G[C]$ is 2-connected. But $G[C]$ intersects both $H_1 - H_2$ and $H_2 - H_1$, which contradicts that (H_1, H_2) is a 1-separation of G. $\square$

An ear decomposition can contain both pure and mixed circuits. However, many of the properties we wish to infer for $\mathcal{M}$-connected mixed graphs hold for mixed circuits, but not pure circuits. So we need to determine when the rigidity matroid of an $\mathcal{M}$-connected graph has an ear decomposition using only mixed circuits. See Figure 7.2.1 for an example of such an ear decomposition.

Lemma 7.2.2. *Let G be an $\mathcal{M}$-connected mixed graph. Then $\mathcal{R}(G)$ has an ear decomposition into mixed circuits.*

Proof. Let l_1 be a length edge and d_1 a direction edge in $E(G)$. Since G is $\mathcal{M}$-connected, there exists a circuit C_1 in $\mathcal{R}(G)$ containing both l_1 and d_1. Clearly C_1 is a mixed circuit.

If G is a circuit, then $E(G) = C_1$ and we are done. So suppose G is not a circuit. Then by Lemma 3.1.1(ii), it is possible to extend the partial ear decomposition C_1 to a full ear decomposition $C_1, C_2, \ldots, C_m$ of $\mathcal{R}(G)$, for some $m \geq 2$. Suppose this decomposition does not consist solely of mixed circuits, and let k be the least integer such that C_k is a pure circuit. By (E2) there exists some edge e_k in the lobe of C_k.

Pick $e_1 \in \{d_1, l_1\}$ of opposite type to e_k. Since $G[\bigcup_{i=1}^{k} C_i]$ is $\mathcal{M}$-connected, there exists some circuit $C'_k \subseteq \bigcup_{i=1}^{k} C_i$ such that $e_1, e_k \in C'_k$. So C'_k is a mixed circuit. Clearly C'_k satisfies (E1) and (E2). Also, since C_k satisfies (E3) and $\tilde{C}'_k \cap \tilde{C}_k \neq \emptyset$ we must have $\tilde{C}'_k = \tilde{C}_k$, and thus C'_k also satisfies (E3).

Hence $C_1, \ldots, C_{k-1}, C'_k, C_{k+1}, \ldots, C_m$ is an ear decomposition of $\mathcal{R}(G)$ where any pure circuit C_i in the sequence must have $i > k$. By iteratively applying this argument, we generate an ear decomposition consisting of just mixed circuits. □

This result leads to the following characterisation of rigidity and redundant rigidity for $\mathcal{M}$-connected graphs:

Lemma 7.2.3. *Let $G = (V; D, L)$ be an $\mathcal{M}$-connected graph. Then G is (redundantly) rigid if and only if G is mixed.*

Proof. Suppose G is a pure graph. Then any realisation of G can either be continuously rotated (if G is length-pure) or continuously dilated (if G is direction-pure) whilst preserving the edge constraints, so G is neither rigid nor redundantly rigid.

So instead, let G be mixed. Then Lemmas 7.1.6 and 7.2.2 imply that G is a union of redundantly rigid mixed circuits $H_1, H_2, \ldots, H_m$ for some $m \geq 1$. Let $e \in E(G)$, and $F_i = H_i - e$ for all $1 \leq i \leq m$. Then $G - e$ is the union of the rigid subgraphs $F_1, F_2, \ldots, F_m$. Thus $G - e$ is rigid, by Lemma 7.1.7. Hence G is redundantly rigid. □

Lemma 7.2.4. *Let $G = (V; D, L)$ be an $\mathcal{M}$-connected mixed graph and let $H_1, H_2, \ldots, H_m$ be the subgraphs of G induced by the mixed circuits*

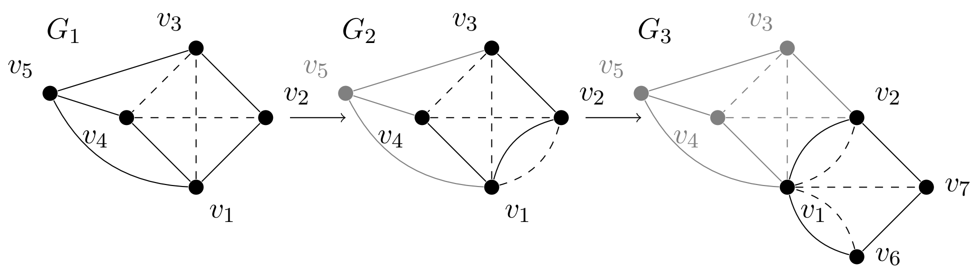


Figure 7.3: The ear decomposition of $\mathcal{R}(G_3)$ into circuits C_1, C_2, C_3 gives the above sequence of graphs. Each graph G_i is obtained from G_{i-1} by taking the union of G_{i-1} with $H_i = G[C_i]$. In each G_i, the subgraph H_i is shown in black, and $G_i - H_i$ in grey. In the terminology of Lemma 7.2.4, $m = 3$, $X = \{v_1, v_2\}$, $Y = \{v_6, v_7\}$ and $\tilde{C}_3$ is the set of 5 edges incident to either v_6 or v_7.

$C_1, \ldots, C_m$ of an ear decomposition of $\mathcal{R}(G)$, where $m \geq 2$. Let $Y = V(H_m) - \bigcup_{i=1}^{m-1} V(H_i)$ and $X = V(H_m) - Y$. Then:

(i) $|\tilde{C}_m| = 2|Y| + 1$;

(ii) either $Y = \emptyset$ and $|\tilde{C}_m| = 1$ or $Y \neq \emptyset$ and every edge $e \in \tilde{C}_m$ is incident to Y;

(iii) if $Y \neq \emptyset$, then X is mixed critical in H_m;

(iv) If $Y \neq \emptyset$ then $G[Y]$ is connected;

(v) if G is 3-connected, then $|X| \geq 3$.

Proof. Let $G_j = \bigcup_{i=1}^{j} H_i$ and $E_j = \bigcup_{i=1}^{j} C_j$. So $E(G_j) = E_j$. Lemma 3.1.1(i) implies that G_{m-1} is $\mathcal{M}$-connected. By Lemma 7.2.3, both G_{m-1} and G are rigid, so Corollary 4.2.1 implies that $\mathrm{r}(E_{m-1}) = |E_{m-1}| = 2|V - Y| - 2$ and $\mathrm{r}(E) = |E| = 2|V| - 2$. Thus, by Lemma 3.1.1(iii),

$$|\tilde{C}_m| = \mathrm{r}(E) - \mathrm{r}(E_{m-1}) + 1 = (2|V| - 2) - (2|V - Y| - 2) + 1 = 2|Y| + 1,$$

which gives part (i). Hence, when $Y = \emptyset$ we must have $|\tilde{C}_m| = 1$. Suppose $Y \neq \emptyset$, and assume that exactly k edges in $E - E_{m-1}$ have both endvertices in $V(G_{m-1})$. Since H_m is a mixed circuit, part (i) and Lemma 7.1.4 imply

$$i_{H_m}(X) = |C_m| - |\tilde{C}_m| + k = (2|X \cup Y| - 1) - (2|Y| + 1) + k = 2|X| + k - 2.$$

Since $H_m[X]$ is a proper subgraph of H_m, it must be independent. Thus $k = 0$, and X is mixed critical in H_m, proving (ii) and (iii) respectively.

We now consider part (iv). Assume $G[Y]$ is disconnected. Then $G[Y]$ consists of connected components $G[Y_1], G[Y_2], \ldots, G[Y_k]$ for some $k \geq 2$, where $Y_1, Y_2, \ldots, Y_k$ partitions Y. Since H_m is a circuit, $H_m[Y_i]$ is independent for all $1 \leq i \leq k$. Hence, by Lemma 7.1.4, each component of Y satisfies

$$i_{H_m}(X \cup Y_i) - i_{H_m}(X) \leq (2|X \cup Y_i| - 2) - (2|X| - 2) = 2|Y_i|,$$

which implies that

$$|\tilde{C}_m| = \sum_{i=1}^{k}(i_{H_m}(X \cup Y_i) - i_{H_m}(X)) \leq \sum_{i=1}^{k} 2|Y_i| = 2|Y|,$$

contradicting part (i).

Finally, we consider part (v). Suppose G is 3-connected. If $Y \neq \emptyset$, then X is a vertex-cut of G and so $|X| \geq 3$. If $Y = \emptyset$, then X is the vertex set of a mixed circuit. The smallest mixed circuits, K_3^+ and K_3^-, have 3 vertices. Hence $|X| \geq 3$. □

7.2.2 Operations preserving $\mathcal{M}$-connectivity

Here we show that two of the operations from Section 7.1.2 which preserve global rigidity: edge additions and 1-extensions, also preserve $\mathcal{M}$-connectivity for mixed graphs. We start with edge additions of mixed graphs.

Lemma 7.2.5. *Let $G = (V; D, L)$ be an $\mathcal{M}$-connected mixed graph and let G' be obtained from G by an edge addition. Then G' is $\mathcal{M}$-connected and mixed.*

Proof. Since $E(G) \subset E(G')$, G' is mixed. Denote the edge added in the edge addition by e. By Lemma 7.2.3, both G and G' are rigid on the same vertex set. Hence $\mathrm{r}(\mathcal{R}(G)) = 2|V| - 2 = \mathrm{r}(\mathcal{R}(G'))$. Let B be a maximal independent set in $E(G)$. Then $\mathrm{r}_G(B) = \mathrm{r}(\mathcal{R}(G)) = \mathrm{r}(\mathcal{R}(G'))$, so B is also maximally independent in G'. Hence $B+e$ is dependent in G', which implies $\mathcal{R}(G')$ contains a circuit C such that $e \in C \subseteq B+e$. Since $|C| \geq i(K_3^+) = 5$, we know that $C \cap E(G) \neq \emptyset$. Thus $\mathcal{R}(G')$ is connected by Lemma 3.1.1(i), and so G' is $\mathcal{M}$-connected. $\square$

Showing that 1-extensions preserve $\mathcal{M}$-connectivity requires more work. We say that a 1-extension is *pure* if all the edges added are of the same type as the edge removed, otherwise it is *mixed*. We already know that 1-extensions and edge additions preserve $\mathcal{M}$-connectivity in the following cases:

Lemma 7.2.6. *[16, Lemma 3.9] Let $G = (V, E)$ be an $\mathcal{M}$-connected pure graph and let G' be obtained from G by either a pure 1-extension or an edge addition, where in both cases the edges added are of the same type as G. Then G' is pure and $\mathcal{M}$-connected.*

Lemma 7.2.7. *[17, Lemma 3.6] Let G be a mixed circuit and G' be a 1-extension of G. Then G' is a mixed circuit.*

We shall extend these results to all $\mathcal{M}$-connected mixed graphs. But to do this, we need the following lemma, which provides a way of transferring results for pure 1-extensions to mixed 1-extensions. Recall the definition of pure and mixed vertices from page 32.

Lemma 7.2.8. *Let $G = (V; D, L)$ be a mixed circuit and let v be a pure vertex in G. Let G' be the graph obtained from G by changing the type of at most two of the edges incident to v. Then G' is a mixed circuit.*

Proof. Let $\{vx, vy\}$ be the set of edges whose type was changed. By Corollary 7.1.10, $d_G(v) \geq 3$, so at least one of the edges terminating at v was not changed. Hence $E(G')$ is mixed. Since $|E(G')| = |E(G)| = 2|V| - 1$, we know that $E(G')$ is dependent, and so there exists some set of edges $C \subseteq E(G')$ which is a circuit in the rigidity matroid.

If neither vx nor vy is contained in C, then $C \subset E(G)$, which contradicts that G is a circuit. Hence C must contain at least one of these edges. But $G'[C]$ is a circuit, so Corollary 7.1.10 implies that C contains at least 3 edges incident to v. By the construction of G' from G, this implies that C contains both a direction and a length edge. Hence C is mixed. If $C \neq E(G')$ then, by changing back the types of the edges $\{vx, vy\}$, we obtain an edge set $C' \subset E(G)$ which is dependent in G, contradicting the fact G is a circuit. Thus $C = E(G')$, and G' is a mixed circuit. □

We now show that 1-extensions preserve $\mathcal{M}$-connectivity for mixed graphs:

Lemma 7.2.9. *Let $G = (V; D, L)$ be an $\mathcal{M}$-connected mixed graph and let G' be obtained from G by a 1-extension. Then G' is mixed and $\mathcal{M}$-connected.*

Proof. Let the 1-extension used to obtain G' from G add the vertex v with neighbourhood $\{x, y, z\}$ (where potentially $x = z$) whilst removing an xy-edge e. We shall use the transitivity of matroid connectivity (see page 30) to prove that G' is $\mathcal{M}$-connected, by showing that given some $e_1 \in E(G')$, we can find a circuit containing both e_1 and e_2 for all $e_2 \in E(G') - e_1$.

Suppose $x = z$. Pick some edge $g \in E(G) - e$ of opposite type to e. Since G is $\mathcal{M}$-connected, for all $f \in E(G) - g$, there is a circuit C in $\mathcal{R}(G)$ such that $f, g \in C$. If $e \notin C$ then $C \subset E(G')$ and we are done. So instead assume $e \in C$. Then C is mixed. Since $G[C]$ contains $N_{G'}(v) = \{x, y\}$, the 1-extension which builds G' from G induces a 1-extension $G'[C']$ of $G[C]$.

By Lemma 7.2.7, $G'[C']$ is a mixed circuit and contains the edges g, vx, vy and vz, as well as the edge f (when $f \neq e$). Hence G' is $\mathcal{M}$-connected.

So instead suppose x, y and z are distinct. There are two cases to consider: when v is added to G by a pure 1-extension, and when it is added by a mixed 1-extension.

First, suppose that v is added by a pure 1-extension. Pick some edge $g \in E(G) - e$ which has z as an endvertex. Since G is $\mathcal{M}$-connected, for all $f \in E(G) - g$ there is a circuit C in the rigidity matroid of G such that $f, g \in C$. If $e \notin C$ then $C \subset E(G')$ and we are done. So suppose that $e \in C$. Since $G[C]$ contains both e and the vertices x, y and z, the 1-extension used to form G' from G, is also a pure 1-extension, $G'[C']$, of $G[C]$. Hence, using Lemma 7.2.7 if C is mixed, or Lemma 7.2.6 if C is pure, $G'[C']$ is a circuit and contains the edges vx, vy and vz as well as the edges f (when $f \neq e$) and g as required. Hence G' is $\mathcal{M}$-connected.

It remains to show that the claim holds when v is added by a mixed 1-extension. Let the graph obtained by this mixed 1-extension be denoted by G''. This graph, G'', can be obtained from the corresponding pure 1-extension, G', above, by changing the type of at most two of the edges in $\{vx, vy, vz\}$. Let C' be a mixed circuit in $E(G')$ and suppose $G'[C']$ does not contain v. Then $C' \subseteq E(G'') - \{vx, vy, vz\}$ and we are done. Otherwise $G'[C']$ contains v, and since circuits have minimum degree 3, this implies $vx, vy, vz \in C'$. We can obtain the corresponding edge set $C'' \subseteq E(G'')$ from C' by changing the type of at most two of the edges in $\{vx, vy, vz\}$, as determined above. By Lemma 7.2.8, C'' is a mixed circuit. Since G' is $\mathcal{M}$-connected, and every mixed circuit C' in G' has a corresponding mixed circuit C'' in G'', Lemma 7.2.2 implies that G'' is $\mathcal{M}$-connected. □

7.2.3 2-sums: another operation

We have seen that edge additions and 1-extensions preserve $\mathcal{M}$-connectivity. However, any graph we construct from K_3^+ or K_3^- with just these operations will be both direction-balanced and length-balanced. In Lemma 6.2.4 we saw that being direction-balanced is a necessary condition for global rigidity, but being length-balanced is not. Thus we need a third operation which will allow use to build length-unbalanced graphs whilst preserving global rigidity and $\mathcal{M}$-connectivity. A direction-pure 0-extension allows use to build length-unbalanced graphs, and in Lemma 7.1.11 we saw that this operation preserves global rigidity. But it does not preserve $\mathcal{M}$-connectivity as, from Corollary 7.1.10 and Lemma 3.1.1, $\mathcal{M}$-connected graphs have minimum degree at least 3.

Here we introduce an operation called a 2-sum. These preserve $\mathcal{M}$-connectivity and global rigidity whilst allowing us to construct graphs which are not length-balanced. We show that a 2-sum with a direction-pure K_4 is an operation which preserves both global rigidity and $\mathcal{M}$-connectivity.

Let $G_1 = (V_1; D_1, L_1)$ be a mixed graph and $G_2 = (V_2; P)$ be a direction- (respectively length-) pure graph with $V_1 \cap V_2 = \{x, y\}$ and $D_1 \cap P = \{xy\}$ (respectively $L_1 \cap P = \{xy\}$). The graph $G = (V; D, L)$ is a *2-sum of* G_1 *and* G_2, written $G = G_1 \oplus_2 G_2$, if $V = V_1 \cup V_2$, $D = (D_1 \cup P) - \{xy\}$ and $L = L_1$ (respectively $D = D_1$ and $L = (L_1 \cup P) - \{xy\}$). See Figure 7.4 for an example.

Let $G = (V; D, L)$ be a mixed or pure graph with an edge-disjoint 2-separation (H_1, H_2) on the 2-vertex-cut $\{x, y\}$ where H_2 is pure and G does not contain an xy-edge of the same type as H_2. Then the *2-cleave* of G across the pair $\{x, y\}$ adds the edge xy of the same type as H_2 to both H_1 and H_2 to form the graphs G_1 and G_2 respectively, such that $G = G_1 \oplus_2 G_2$.

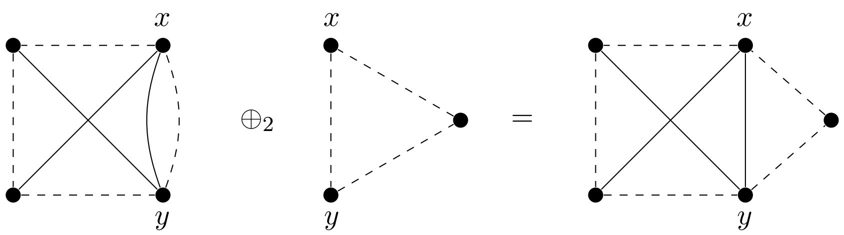


Figure 7.4: A 2-sum between a mixed graph and a direction-pure K_3.

When we make a 2-cleave, we often want to ensure we are removing the fewest possible vertices from our graph, by making $V(G_2)$ as small as possible. To aid our description of this, we introduce the following notation. Given a graph $G = (V; D, L)$ and a set $X \subset V$, the *neighbourhood* of X in G, is given by $N_G(X) = \{v \in V - X : xv \in E(G) \text{ for some } x \in X\}$. When it is clear which graph we are referring to, we simply write $N(X)$. We call $X \subset V$ an *end* of G if $|N(X)| = 2$, $V - (X \cup N(X)) \neq \emptyset$, and for all non-empty $X' \subset X$ we have $|N(X')| \geq 3$. This definition of a neighbourhood extends that given in Chapter 2, where X was a single vertex.

It is already known that 2-sums and 2-cleaves preserve $\mathcal{M}$-connectivity for pure and mixed circuits:

Lemma 7.2.10. *[2, Lemmas 4.1, 4.2] Let G be a pure graph.*

(i) Suppose G is the 2-sum of two pure graphs G_1 and G_2. If G_1 and G_2 are circuits then G is a pure circuit.

(ii) Suppose G is a pure circuit with an edge-disjoint 2-separation (H_1, H_2) on the 2-vertex-cut $\{x, y\}$. Then $d_G(x), d_G(y) \geq 4$ and xy is not an edge of G. In addition, if we 2-cleave G across $\{x, y\}$, we obtain graphs G_1 and G_2 from H_1 and H_2 respectively, such that G_1 and G_2 are both pure circuits.

Lemma 7.2.11. *[17, Lemma 3.7] Let G be a mixed graph.*

(i) Suppose G is the 2-sum of two graphs G_1 and G_2. If G_1 is a mixed circuit and G_2 is a pure circuit then G is a mixed circuit.

(ii) Suppose G is a mixed circuit and has an edge-disjoint 2-separation (H_1, H_2) on the 2-vertex-cut $\{x, y\}$ with H_2 pure. Then $d_G(x), d_G(y) \geq 4$ and G does not contain an xy-edge of the same type as H_2. In addition, if we 2-cleave G across $\{x, y\}$, we obtain the graphs G_1 and G_2 from H_1 and H_2 respectively, such that G_1 is a mixed circuit and G_2 is a pure circuit.

We extend these results to $\mathcal{M}$-connected mixed graphs:

Lemma 7.2.12. *Let G be a mixed graph.*

(i) Suppose G is the 2-sum of two graphs G_1 and G_2. If G_1 is mixed, G_2 is pure and both are $\mathcal{M}$-connected then G is an $\mathcal{M}$-connected mixed graph.

(ii) Suppose G is an $\mathcal{M}$-connected mixed graph and has an edge-disjoint 2-separation (H_1, H_2) on 2-vertex-cut $\{x, y\}$ with H_2 pure. Then $d_G(x) \geq 4$ and $d_G(y) \geq 4$. Further, if G contains an xy-edge of the same type as H_2 then $G - xy$ is $\mathcal{M}$-connected. Otherwise, we can 2-cleave G across $\{x, y\}$ to form the graphs G_1 and G_2 from H_1 and H_2 respectively, where G_1 is an $\mathcal{M}$-connected mixed graph and G_2 is an $\mathcal{M}$-connected pure graph.

Proof. First we prove part (i). Let e be the edge removed from both G_1 and G_2 by the 2-sum operation and let $f_i \in E(G_i) - \{e\}$ for $i \in \{1, 2\}$. Since G_1 and G_2 are both $\mathcal{M}$-connected, their rigidity matroids contain circuits C_1 and C_2 respectively such that $e, f_i \in C_i$ for $i \in \{1, 2\}$, C_2 is pure and C_1 is either mixed, or pure of the same type as C_2 (since it contains e).

Thus, by Lemma 7.2.10(i) or 7.2.11(i) as applicable, $C_1 \oplus_2 C_2$ is a circuit in G containing both f_1 and f_2. Hence, by the transitivity of matroid connectivity, G is $\mathcal{M}$-connected.

We shall now prove part (ii). Assume G contains the xy-edge e of the same type as H_2. Let $f_i \in E(H_i) - \{e\}$ for $i \in \{1, 2\}$. Since G is $\mathcal{M}$-connected, $\mathcal{R}(G)$ contains a circuit $C \subseteq E$ such that $f_1, f_2 \in C$. By Lemma 7.2.10(ii) or 7.2.11(ii) as relevant, $e \notin C$. Hence, by the transitivity of matroid connectivity, $G - xy$ is $\mathcal{M}$-connected. Further, by Lemma 7.1.9, vertices x and y have degree at least 3 in $G[C]$, but both x and y are also endvertices of e in G. Hence $d_G(X), d_G(y) \geq 4$.

So instead assume $e \notin E(G)$. Then e can be added to both H_1 and H_2 to form G_1 and G_2 respectively. Let $f_1 \in E(H_1)$ and $f_2 \in E(H_2)$ as before. Since G is $\mathcal{M}$-connected, $\mathcal{R}(G)$ contains a circuit C such that $f_1, f_2 \in C$. Since circuits are 2-connected, both x and y are vertices in $G[C]$. Thus by Lemma 7.2.10(ii) or 7.2.11(ii) as relevant, $C_2 = (C \cap E(H_2)) + e$ is a pure circuit in $\mathcal{R}(G_2)$ and $C_1 = (C \cap E(H_1)) + e$ is a pure (resp. mixed) circuit in $\mathcal{R}(G_1)$ when $C \cap E(H_1)$ is pure (resp. mixed). So, by the transitivity of matroid connectivity, G_1 and G_2 are both $\mathcal{M}$-connected.

Finally, by 7.1.10, we have $d_{G_i}(x) \geq 3$ and $d_{G_i}(y) \geq 3$, for $i \in \{1, 2\}$, since $\mathcal{M}$-connected graphs are the union of circuits. But xy is an edge in both G_1 and G_2. Hence $d_G(x) = (d_{G_1}(x) - 1) + (d_{G_2}(x) - 1) \geq 4$, and similarly, $d_G(y) \geq 4$. $\square$

Lemma 7.2.12 tells us that the 2-sum of an $\mathcal{M}$-connected mixed graph G with a direction-pure K_4 on an xy-edge, will be mixed and $\mathcal{M}$-connected. We need to show the move preserves global rigidity. This 2-sum is equivalent to first performing a direction-pure 0-extension on x and y, followed by a direction-pure 1-extension on xy. Since G is $\mathcal{M}$-connected and mixed, Lemma 7.2.3 implies that deleting the edge xy preserves rigidity. We also know, by Lemma 7.1.11, that both of these operations preserve global rigidity. This gives us the result we seek:

Lemma 7.2.13. *Let G be an $\mathcal{M}$-connected mixed graph which is globally rigid. Let G' be obtained from G by a 2-sum with a direction-pure K_4. Then G' is mixed, $\mathcal{M}$-connected and globally rigid.*

7.2.4 Crossing 2-vertex-cuts

Lemma 7.2.13 tells us that a 2-sum with a direction-pure K_4 preserves global rigidity, but we have not considered when the inverse operation, a 2-cleave, preserves global rigidity. In Lemma 6.2.4, we saw that being direction-balanced is a necessary condition for global rigidity, so we need to identify when 2-cleaves preserve being direction-balanced. To do this, we introduce the idea of "crossing" 2-vertex-cuts.

Let G be a mixed or pure graph with two 2-separations (H_1, H_2) and (H_1', H_2') on 2-vertex-cuts $\{x, y\}$ and $\{x', y'\}$ respectively. If x and y are in different components of $G - \{x', y'\}$ then we say that $\{x, y\}$ *crosses* $\{x', y'\}$. It is clear that if $\{x, y\}$ crosses $\{x', y'\}$, then $\{x', y'\}$ crosses $\{x, y\}$. Thus we can refer to $\{x, y\}$ and $\{x', y'\}$ as *crossing 2-vertex-cuts*, and we say that the 2-separations (H_1, H_2) and (H_1', H_2') *cross*. Further, if $\{x, y\}$ and $\{x', y'\}$ cross, then neither xy nor $x'y'$ are edges in G, so the 2-separations (H_1, H_2) and (H_1', H_2') are both edge-disjoint. See Figure 7.5.

Lemma 7.2.14. *Let G be an $\mathcal{M}$-connected mixed graph and let (H_1, H_2) and (H_1', H_2') be two 2-separations of G. If H_2 is pure then (H_1, H_2) and (H_1', H_2') do not cross.*

Proof. Assume, for a contradiction, that (H_1, H_2) and (H_1', H_2') cross, and

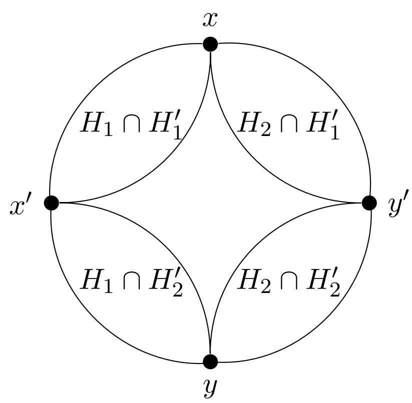


Figure 7.5: Two crossing 2-separations of a graph: (H_1, H_2) on 2-vertex-cut $\{x, y\}$, and (H_1', H_2') on $\{x', y'\}$.

let their 2-vertex-cuts be $\{x, y\}$ and $\{x', y'\}$ respectively. Since these 2-vertex-cuts cross, neither xy nor $x'y'$ are edges in G.

Let e_2 be an edge in $E(H_2)$, and e_1 be an edge of opposite type in $E(H_1)$. Since G is $\mathcal{M}$-connected, there is a mixed circuit C in $\mathcal{R}(G)$ such that $e_1, e_2 \in C$. By Lemma 7.1.9, $G[C]$ is 2-connected which implies that x, y, x' and y' are all vertices in $G[C]$. Hence

$$|C| = |C \cap E(H_1)| + |C \cap E(H_2) \cap E(H_1')| + |C \cap E(H_2) \cap E(H_2')|.$$

Let $V_C = V(G[C])$, and let $V_i = V(H_i)$ and $V_i' = V(H_i')$ for $i \in \{1, 2\}$. Since H_2 is pure, the sparsity conditions for the mixed circuit C give

$$\begin{aligned}|C| &\leq (2|V_C \cap V_1| - 2) + (2|V_C \cap V_2 \cap V_1'| - 3) + (2|V_C \cap V_2 \cap V_2'| - 3) \\ &= 2(|V_C| + |\{x, y, y'\}|) - 8 = 2|V_C| - 2\end{aligned}$$

which contradicts the edge count for a mixed circuit. Hence (H_1, H_2) and (H_1', H_2') do not cross. □

Lemma 7.2.15. *Let $G = G_1 \oplus_2 G_2$ be a mixed graph with G_2 direction-pure. Then G_1 is direction-balanced if and only if G is direction-balanced.*

Proof. The forwards direction is trivial, so we shall only prove the converse. Let $V(G_1) \cap V(G_2) = \{x, y\}$. Assume that G_1 is not direction-balanced. Then there is some end X of G_1, such that no direction edges in G_1 have an endvertex in X. If X is also an end of G, then this contradicts that G is direction-balanced. Hence $X \cap \{x, y\} \neq \emptyset$. But xy is a direction edge in G_1, which contradicts our original assumption. $\square$

7.3 Admissible nodes in mixed circuits

In the previous section we showed that performing 1-extensions, edge additions and 2-sums with pure K_4's preserves $\mathcal{M}$-connectivity. So to obtain our recursive construction of $\mathcal{M}$-connected mixed graphs, it remains show that every $\mathcal{M}$-connected mixed graph (other than K_3^+ and K_3^-) can be obtained from a smaller $\mathcal{M}$-connected mixed graph by one of these operations.

Of these three operations, it is most difficult to identify when an $\mathcal{M}$-connected mixed graph is a 1-extension of another $\mathcal{M}$-connected mixed graph. In this section, we consider the simplest case of an $\mathcal{M}$-connected graph: a circuit. We review and extend Jackson and Jordán's methods in [17] for identifying when a circuit is a 1-extension of another circuit. We will then extend these results to all $\mathcal{M}$-connected graphs in Section 7.4.

Given a mixed or pure graph $G = (V; D, L)$, any vertex of degree three in G is called a *node*, and the set of all such vertices is denoted by V_3. We call $G[V_3]$ the *node subgraph of* G. A node of G with degree at most one (exactly two, exactly three) in $G[V_3]$ is called a *leaf node* (*series node*, *branching node* respectively).

Lemma 7.3.1. *[17, Lemma 3.4] Let $G = (V; D, L)$ be a mixed circuit. Then $G[V_3]$ is a forest.*

Lemma 7.3.2. *[17, Lemma 3.5] Let $G = (V; D, L)$ be a mixed circuit and $X \subset V$ be a mixed critical set. Then G has a node in $V - X$.*

Given any node v in G, the *1-reduction operation* at v on edges vx and vy deletes v and all edges incident to v and adds a new edge xy with the proviso that if v is a pure node then xy must be of the same type as v. The graph obtained by this operation is denoted by G_v^{xy} and is called a *1-reduction* of G. The 1-reduction operation is the inverse of the 1-extension operation. A 1-reduction which adds a direction edge to G is called a *direction 1-reduction*, and similarly a *length 1-reduction* adds a length edge to G.

If G is an $\mathcal{M}$-connected mixed (pure) graph, then a 1-reduction is called *admissible* if the resulting graph is mixed (pure) and $\mathcal{M}$-connected. A node v of G is called *admissible* if there is an admissible 1-reduction at v, and is *non-admissible* otherwise. In this section, we consider the special case where G is a mixed circuit. For mixed (pure) circuits, a 1-reduction is admissible if it results in a smaller mixed (pure) circuit.

Let $G = (V; D, L)$ be a mixed circuit with a 1-reduction at v onto the edge xy. Assume G contains some critical set $Z \subset V - v$ such that $x, y \in Z$, and Z is either mixed, or pure of the same type as the xy-edge added in the 1-reduction. Then $G[Z] + xy$ is dependent in G_v^{xy}. Since $Z \neq V - v$, this implies that G_v^{xy} is not a circuit. So the existence of the critical set Z prevents this 1-reduction from being admissible. In fact, Jackson and Jordán [17] have shown that we can determine the admissibility of nodes in mixed circuits solely by the absence of such sets. However, we need to avoid different combinations of critical sets, depending on whether the node is pure or mixed.

Let G be a mixed circuit, and v be a node of G with three distinct neighbours: r, s and t. Let R, S and T be critical sets in $G - v$ with $\{s, t\} \subseteq$

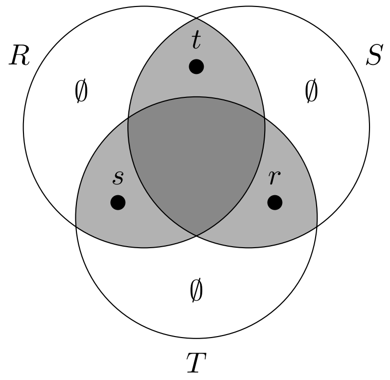


Figure 7.6: Flower formed from vertex sets R, S and T.

$R \subseteq V - \{v, r\}$, $\{r, t\} \subseteq S \subseteq V - \{v, s\}$ and $\{r, s\} \subseteq T \subseteq V - \{v, t\}$ such that either

(i) R, S and T are all mixed critical,

(ii) v is a pure node, R and S are both mixed critical, and T is pure of the same type as v, or

(iii) v is a pure node, R is mixed critical, and S and T are pure of the same type as v.

We say that the triple (R, S, T) is a *strong flower* on v if it satisfies (i), or a *weak flower* on v if it satisfies (ii), and that (R, S, T) is a *flower* if it is either a strong or a weak flower (see Figure 7.6). If instead (R, S, T) satisfies (iii), then we say (R, S, T) is a *clover* on v (see Figure 7.7). Flowers and clovers satisfy the following, very restrictive, properties:

Lemma 7.3.3. *[17, Lemma 4.2] Let $G = (V; D, L)$ be a mixed circuit and let v be a node of G. Suppose there exists a strong or weak flower (R, S, T) on v, and let $W^* = (V - v) - W$ for all $W \in \{R, S, T\}$. Then*

(i) $R \cup S = S \cup T = R \cup T = V - v$,

(ii) $R \cap S \cap T \neq \emptyset$,

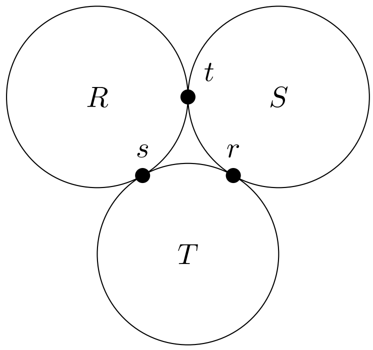


Figure 7.7: Clover formed from vertex sets R, S and T.

(iii) $d(R,S) = d(S,T) = d(R,T) = 0$, *and*

(iv) $\{R^*, S^*, T^*, R \cap S \cap T\}$ *is a partition of* $V - v$.

Lemma 7.3.4. *Let* $G = (V; D, L)$ *be a mixed circuit and let* v *be a pure node of* G *with neighbourhood* $\{r, s, t\}$. *Suppose* (R, S, T) *is a clover on* v *with* R *mixed critical. Then*

(i) $|R| \geq 3$,

(ii) $R \cup S \cup T = V - v$,

(iii) $|R \cap S| = |S \cap T| = |R \cap T| = 1$,

(iv) $R \cap S \cap T = \emptyset$,

(v) $d(R, S, T) = 0$, *and*

(vi) $(G[R], G[S \cup T + v] - E(R))$ *is an unbalanced, edge-disjoint 2-separation of* G *on 2-vertex cut* $\{s, t\}$ *with* $G[S \cup T + v] - E(R)$ *pure.*

Proof. This proof closely follows that of Lemma 4.3 in [17]. Let v be a node of type $P \in \{D, L\}$. First assume $|S \cap T| \geq 2$. Then Lemma 7.1.2(ii) implies $i_P(S \cup T) = 2|S \cup T| - 3$. Since $N_G(v) \subseteq S \cup T$, this implies $G[S \cup T + v]$

contains a pure circuit of type P, contradicting the fact G is a mixed circuit. Hence $|S \cap T| = 1$, and more specifically, $S \cap T = \{r\}$.

Instead, assume $|R \cap S| \geq 2$. Lemma 7.1.2(iii) implies $R \cup S$ is mixed critical with $d(R, S) = 0$. Since $N_G(v) \subseteq R \cup S$, this implies $G[R \cup S + v]$ contains a circuit. Hence $R \cup S = V - v$. Since $r, s \in T$ and $S \cap T = \{r\}$, we have that T intersects both $R - S$ and $S - R$, but does not intersect $R \cap S$. But nor are there any edges from $R - S$ to $S - R$, since $d(R, S) = 0$. This implies T is disconnected in G, contradicting Lemma 7.1.8(i). So our assumption is false, and $R \cap S = \{t\}$. A similar argument gives $R \cap T = \{s\}$. Thus proving parts (iii) and (iv).

Lemma 7.1.3 now implies $d(R, S, T) = 0$ (part (v)) and that $R \cup S \cup T$ is mixed critical. This in turn implies that $R \cup S \cup T + v$ is dependent, and hence $R \cup S \cup T = V - v$, thus proving part (ii).

We now consider part (i). Since G is a mixed circuit, Lemma 7.1.4 implies it contains at least two edges of opposite type to P. But S and T only induce edges of type P, and $d(R, S, T) = 0$, so this implies all such edges must be induced by R. Hence $|R| \geq 3$, as required.

Finally, since $|S \cup T| \geq 3$, $|R| \geq 3$, $R \cap (S \cup T) = \{s, t\}$ and $d(R, S, T) = 0$, we must have that $\{s, t\}$ is the 2-vertex-cut of the edge-disjoint 2-separation $(G[R], G[S \cup T + v] - E(R))$ where $G[S \cup T + v] - E(R)$ is pure, hence proving part (vi). □

Using these properties, Jackson and Jordán determined when mixed and pure nodes of a circuit are admissible:

Lemma 7.3.5. *[17, Lemma 4.5] Let $G = (V; D, L)$ be a mixed circuit such that $G \notin \{K_3^+, K_3^-\}$, and let v be a mixed node of G. Then exactly one of the following hold:*

(i) v is admissible,

(ii) v has exactly two neighbours x and y and there exists a length critical set R and a direction critical set S with $R \cap S = \{x, y\}$, $R \cup S = V - v$, $d(R, S) = 0$ and $i(R \cap S) = 0$, or

(iii) there is a strong flower on v in G.

Lemma 7.3.6. *[17, Lemmas 4.4, 4.7] Let $G = (V; D, L)$ be a mixed circuit and let v be a pure node of G. Then exactly one of the following hold:*

(i) v is admissible,

(ii) there is a strong or weak flower on v in G, or

(iii) there is a clover on v in G.

In the next section, we extend these results to all $\mathcal{M}$-connected mixed graphs by identifying when such a graph contains an admissible node in the last lobe of its ear decomposition. Lemma 7.2.2, and Lemma 7.2.4(ii) and (iii), ensure that the last circuit C_m in the ear decomposition of such a graph G is mixed, and that the lobe is either a single edge, or has vertex set $V(G[C_m]) - X$ for some mixed critical set X in $V(G[C_m])$. See G_2 and G_3 respectively in Figure 7.2.1 for examples of such lobes. Thus it shall be helpful to determine when a mixed circuit H, with mixed critical set $X \subset V(H)$, has an admissible node in $V(H) - X$. In the remainder of this section, we obtain a result which will help us to identify such a node.

Let $G = (V; D, L)$ be a circuit and v be a node in G with $N(v) = \{x, y, z\}$. If X is a critical set with $\{y, z\} \subseteq X \subseteq V - \{v, x\}$ then X is called a *v-critical set*. If, in addition, $d(x) \geq 4$ then X is called *v-node-critical.*

Theorem 7.3.7. *Let $G = (V; D, L)$ be a mixed circuit and let X be a mixed critical set in V. Suppose that either* *(i) there is a non-admissible series node u of G in $V - X$ with exactly one neighbour r in X, and r is a node, or*

(ii) there is a non-admissible leaf node u of G in $V - X$ with $|N(u) \cap X| \leq 1$.

Then either there exists a mixed node-critical set X^* with $X^* \supset X$, or there exists an edge-disjoint 2-separation (H_1, H_2) of G with $X \subseteq V(H_1)$ and H_2 pure.

Proof. Suppose $|N(u)| = 2$. Then Lemma 7.3.5 implies $N(u) = \{r, s\}$ is the 2-vertex-cut of an unbalanced, edge-disjoint 2-separation (H_1, H_2) of $G - u$ where H_1 is direction critical and H_2 is length critical. However X is mixed critical, so $G[X]$ contains both length and direction edges, which implies that X intersects both $V(H_1) - V(H_2)$ and $V(H_2) - V(H_1)$. By Lemma 7.1.8(i), $G[X]$ is connected, thus X intersects $V(H_1) \cap V(H_2) = \{r, s\}$. In both cases (i) and (ii), $|X \cap N(u)| \leq 1$, so X contains exactly one neighbour of u, say r. But then $\{r\}$ is the 1-vertex-cut of the unbalanced 1-separation $(G[X \cap V(H_1)], G[X \cap V(H_2)])$ of X, which contradicts Lemma 7.1.8(ii).

Thus we must have $|N(u)| = 3$. Let $N(u) = \{r, s, t\}$ and suppose condition (i) holds. Then $N(u) \cap X = \{r\}$ so, without loss of generality, both r and s are nodes but t is not. Thus $d_G(t) \geq 4$. Since u is non-admissible, Lemmas 7.3.5 and 7.3.6 imply there exists a critical set T such that $r, s \in T$ but $t, u \notin T$. We know that $G[V_3]$ is a forest by Lemma 7.3.1. So since r, s and u are nodes, and $ru, su \in E$, this implies $rs \notin E$. Thus Lemma 7.1.8(i) implies that $G[T]$ is 2-edge-connected with $|T| \geq 3$. Hence $\delta(G[T]) \geq 2$.

Since $r \in X \cap T$ is a node and $u \notin X \cup T$, in order to satisfy the minimal degree condition for $G[T]$ we must have $N(r) - \{u\} \subseteq T$. But we also know $G[X]$ is connected with $|X| \geq 2$, so some member of $N(r) - \{u\}$ must also be contained in X. Hence $|X \cap T| \geq 2$ and thus, by Lemma 7.1.2, $X^* = X \cup T$ is a mixed u-node-critical set with $X \cup T \supset X$ since $s \in T - X$.

We now consider case (ii). Since u is non-admissible with $|N(u)| = 3$, Lemmas 7.3.5 and 7.3.6 imply that there is either a strong or weak flower on u, or, if neither of these occur, then there is a clover on u.

Claim 7.3.8. *If there is a strong or a weak flower on u then there exists a mixed node-critical set X^* with $X^* \supset X$.*

Proof. Assume there exists a strong or weak flower on u with critical sets R, S and T such that $\{s,t\} \subseteq R \subseteq V - \{r,u\}$, $\{r,t\} \subseteq S \subseteq V - \{s,u\}$ and $\{r,s\} \subseteq T \subseteq V - \{t,u\}$. Since u is a leaf node, we can assume both r and t are not nodes. The definition of a a flower implies at least one of R and T is mixed critical so, relabelling if necessary, we can assume T is mixed critical.

Suppose $T \cap X = \emptyset$. Since $T \cup R = V - u \supset X$, we must have $R \supseteq X$, and hence $X^* = R$ is a mixed u-node-critical set. Further, since $R \cap N(u) = \{s,t\}$ but at most one of s and t is in X, we have $R \supset X$ as required.

We next suppose $|T \cap X| \geq 1$ and $t \notin X$. By Lemma 7.1.2(i), $X^* = T \cup X$ is a mixed u-node-critical set. Additionally, since both $r, s \in T$ and at most one of these is in X, we have that $X^* = T \cup X \supset X$ as required.

It remains to consider the case where $|T \cap X| \geq 1$ and $t \in X$. Since $|X \cap N(u)| \leq 1$, this implies $r, s \notin X$ and $|R \cap X| \geq 1$. If $|R \cap X| \geq 2$, then Lemma 7.1.2 implies $X^* = R \cup X$ is u-node-critical with $X \cup R \supset X$, since $s \in R - X$. So assume $|R \cap X| < 2$. Then $R \cap X = \{t\}$. By Lemma 7.3.3, we know $R \cup T = V - u$ and $d(R,T) = 0$. So since $t \in R - T$, these properties imply $N(t) - \{u\} \subseteq R$. Also, since $G[X]$ is connected and $t \in X$, we know X must contain some member of $N(t) - \{u\}$. Hence $|R \cap X| \geq 2$, which contradicts our assumption. □

Claim 7.3.9. *If there is a clover on u, then there exists an edge-disjoint 2-separation (H_1, H_2) of G with $X \subseteq V(H_1)$ and H_2 pure.*

Proof. Suppose there is a clover on u. Then u is pure, and by the definition of a clover, exactly one of R, S and T is mixed critical, and the other two sets are pure critical of the same type as u. Relabelling if necessary, we can assume R is mixed. Then $(G[R], G[S \cup T + u] - E(R))$ is an edge-disjoint 2-separation of G on 2-vertex-cut $\{s, t\}$ where $G[S \cup T + u] - E(R)$ is pure. Since X is mixed critical, X contains an edge e of opposite type to u. But S and T only induce edges of the same type as u, so we must have $e \in E(R)$. Hence $|X \cap R| \geq 2$.

Assume that X contains some vertex in $(S \cup T) - R$. Since X is connected, and $|X \cap N(u)| \leq 1$, this implies X contains exactly one of the vertices in the 2-vertex-cut $\{s, t\}$. But then this vertex will be a 1-vertex-cut of the 1-separation $(G[X \cap R], G[X \cap (S \cup T)])$ of $G[X]$, which contradicts Lemma 7.1.8(ii). Hence $X \subseteq R$ and $(G[R], G[S \cup T + u] - E(R))$ is the edge-disjoint 2-separation of G required. □

Claims 7.3.8 and 7.3.9, complete our proof of case (ii). □

We now use Theorem 7.3.7 to obtain our result on mixed critical sets.

Theorem 7.3.10. *Let $G = (V; D, L)$ be a mixed circuit, and X be a mixed critical set in G. Suppose $V - X$ contains a vertex which is not a node. Then either $V - X$ contains an admissible node, or there exists an edge-disjoint 2-separation (H_1, H_2) of G with $X \subseteq V(H_1)$ and H_2 pure.*

Proof. Let X' be a maximal mixed critical set in G such that $X' \supseteq X$ and $V - X'$ contains a vertex which is not a node. Lemma 7.3.2 implies that $V - X'$ contains a node. Hence, by Lemma 7.3.1, there exists some node $v \in V - X'$ such that v is a leaf in $G[V_3 - X']$.

If $d(\{v\}, X') = 3$ then $X' + v$ would break the sparsity conditions for circuits, and if $d(\{v\}, X') = 2$ then $X' + v$ would be a larger critical set such that $V - (X' \cup \{v\})$ contains a non-node, contradicting the maximality of X'. So $d(\{v\}, X') \leq 1$, and either v is a series node in G with exactly one neighbour in X', which is also a node, or v is a leaf node in G.

If v is admissible then we are done. Otherwise, Theorem 7.3.7 implies that either G has a node-critical set X^* such that $X^* \supset X'$ or G has an edge-disjoint 2-separation (H_1, H_2) with $X' \subseteq V(H_1)$ and H_2 pure. If the former case holds, then by the definition of node-critical, $V - X^*$ contains a vertex which is not a node, which contradicts the maximality of X'. So the latter case must hold, as required. $\square$

7.4 Constructing $\mathcal{M}$-connected mixed graphs

We extend the results of the previous section to show that any $\mathcal{M}$-connected mixed graph $G \notin \{K_3^+, K_3^-\}$, can be obtained from a smaller $\mathcal{M}$-connected mixed graph by an edge addition, 1-extension, or a 2-sum with a pure K_4. To do this, we consider an ear decomposition of G, and apply our results from Section 7.3 to the last circuit in the ear decomposition. Jackson and Jordán, and Berg and Jordán have shown the following results on the existence of admissible nodes in circuits:

Lemma 7.4.1. *[17, Theorem 4.11] Let $G = (V; D, L)$ be a mixed circuit with $|V| \geq 4$. Then either G can be expressed as a 2-sum of a mixed circuit with a pure K_4, or G has an admissible node.*

Lemma 7.4.2. *[2, Theorem 3.8] Let $G = (V; D, L)$ be a 3-connected pure circuit with $|V| \geq 5$. If $x, y \in V$ and xy is an edge in G, then G contains at least two admissible nodes in $V - \{x, y\}$.*

We extend the idea of admissibility to edges as well as nodes: an edge e of an $\mathcal{M}$-connected mixed or pure graph G is *admissible* if $G - e$ is $\mathcal{M}$-

connected. We know that a pure graph which is both $\mathcal{M}$-connected and 3-connected either contains an admissible edge or an admissible node:

Lemma 7.4.3. *[16, Theorem 5.4] Let $G = (V; D, L)$ be a 3-connected, $\mathcal{M}$-connected pure graph. Let $C_1, C_2, \ldots, C_m$ be an ear decomposition of $\mathcal{R}(G)$ into pure circuits. Suppose that $G - e$ is not $\mathcal{M}$-connected for all $e \in \tilde{C}_m$ and for all but at most two edges of C_m. Then $V(G[C_m]) - V(G[\bigcup_{i=1}^{m-1} C_i])$ contains an admissible node.*

In the remainder of this section, we obtain a similar result for admissible edges and admissible nodes in $\mathcal{M}$-connected mixed graphs.

Consider an ear-decomposition $C_1, C_2, \ldots, C_m$ of the rigidity matroid of an $\mathcal{M}$-connected graph G. If there is some node v which is admissible in $G[C_m]$, then, so long as v is in the lobe of $G[C_m]$, and the edge added in the 1-reduction is not already contained in $\bigcup_{i=1}^{m-1} C_i$, the node v will also be admissible in G. Actually, v needn't be admissible in C_m for this argument to work. So long as the following conditions are satisfied, v will be admissible in G:

Lemma 7.4.4. *Let $G = (V; D, L)$ be an $\mathcal{M}$-connected mixed graph, and $H_1, \ldots, H_m$ be the subgraphs of G induced by ear decomposition $C_1, \ldots, C_m$ of $\mathcal{R}(G)$ into mixed circuits, where $m \geq 2$. Let $G_{m-1} = G[\bigcup_{i=1}^{m-1} C_i]$. Let $v \in V - V(G_{m-1})$ be a node with $x, y \in N(v)$ such that we can perform a 1-reduction at v onto xy. Let C be the unique circuit in the edge set of $(H_m)_v^{xy}$. If $C \cap E(G_{m-1}) \neq \emptyset$ and $E(G_v^{xy}) - E(G_{m-1}) \subset C$ then this 1-reduction is admissible in G.*

The uniqueness of C in this statement is a consequence of circuit axiom (C3) in [28].

Proof. Since $v \notin V(G_{m-1})$, we know G_{m-1} is a mixed subgraph of G_v^{xy}. So

G_v^{xy} is mixed. Further, $\mathcal{R}(G_{m-1})$ has ear decomposition $C_1, C_2, \ldots, C_{m-1}$, so G_{m-1} is $\mathcal{M}$-connected by Lemma 3.1.1(i). Since $C \cap E(G_{m-1}) \neq \emptyset$, $C \cup E(G_{m-1})$ is connected in $\mathcal{R}(G_v^{xy})$. But $C \supset E(G_v^{xy}) - E(G_{m-1})$, so $E(G_v^{xy}) = C \cup E(G_{m-1})$. Hence G_v^{xy} is $\mathcal{M}$-connected, and so v is admissible in G. $\square$

In the special case where the node v in our last lobe has exactly two distinct neighbours, v is always admissible, so long as no edges in G are admissible:

Lemma 7.4.5. *Let $G = (V; D, L)$ be an $\mathcal{M}$-connected mixed graph, and let $H_1, \ldots, H_m$ be the subgraphs of G induced by the ear decomposition $C_1, \ldots, C_m$ of $\mathcal{R}(G)$ into mixed circuits, where $m \geq 2$. Let $Y = V(H_m) - \bigcup_{i=1}^{m-1} V(H_i)$ and $X = V(H_m) - Y$.*

Suppose no edges in G are admissible, and let $v \in Y$ be a node with $|N(v)| = 2$. Then v is admissible in G.

Proof. Let $N(v) = \{x, y\}$, and assume v is not admissible in G. Since v is a node, $d(v) = 3$, so without loss of generality, let vx be a double edge and vy a single edge. Since vx is a double edge, v must be a mixed node.

Case 1. $x, y \in X$.

Lemma 7.2.4(ii) implies that all xy-edges in G are also edges in $G_{m-1} = \bigcup_{i=1}^{m-1} H_i$. By Lemma 7.2.4(iii), X is mixed critical in H_m, so $i_{H_m}(X + v) = (2|X| - 2) + 3 = 2|X + v| - 1$, which implies $X + v = V(H_m)$ and hence $Y = \{v\}$.

Assume G contains some xy-edge, e. Since $C_1, C_2, \ldots, C_{m-1}$ is an ear decomposition of $\mathcal{R}(G_{m-1})$, we know G_{m-1} is $\mathcal{M}$-connected. Add the vertex v to G_{m-1} by a 1-extension which removes the edge e. The resulting graph, $G - e$, is $\mathcal{M}$-connected by Lemma 7.2.9. But this means e is an admissible edge in G, which contradicts our assumption. Hence G contains no xy-edges, and so we can perform a 1-reduction at v onto the pair vx, vy to form the graph $G_v^{xy} = G_{m-1} + xy$. Since G_{m-1} is $\mathcal{M}$-connected, and edge additions preserve $\mathcal{M}$-connectivity by Lemma 7.2.5, G_v^{xy} is $\mathcal{M}$-connected.

Case 2. $|\{x, y\} \cap X| \leq 1$.

By Lemma 7.2.1, G is 2-connected, so $|X| \geq 2$ and thus $|V(H_m)| \geq 4$. Hence $H_m \notin \{K_3^+, K_3^-\}$, and so xy is not a double edge in G. If v is admissible in H_m then $(H_m)_v^{xy}$ is a mixed circuit and we are done by Lemma 7.4.4. So assume that v is not admissible in H_m. Then, by Lemma 7.3.5, there exists a length critical set A and a direction critical set B with $A \cap B = \{x, y\}$, $A \cup B = V(H_m) - v$ and $d_{H_m}(A, B) = 0$. By Lemma 7.2.4(iii), X is mixed critical in H_m. Since $v \notin X$, this implies that X must intersect both $A - B$ and $B - A$. But X is connected by Lemma 7.1.8(i), so this implies $|\{x, y\} \cap X| \geq 1$.

Hence $|\{x, y\} \cap X| = 1$, and this vertex is the 1-vertex-cut of the unbalanced 1-separation $(H_m[X \cap A], H_m[X \cap B])$ of X in H_m, which contradicts Lemma 7.1.8(ii). Thus v must be admissible in H_m, and hence also in G. □

We are now in a position to prove our main result on the existence of admissible nodes and edges in $\mathcal{M}$-connected mixed graphs:

Theorem 7.4.6. *Let $G = (V; D, L)$ be an $\mathcal{M}$-connected mixed graph such that $G \notin \{K_3^+, K_3^-\}$ and G has no admissible edges. Let $H_1, H_2, \ldots, H_m$ be the subgraphs of G induced by an ear decomposition $C_1, C_2, \ldots, C_m$ of $\mathcal{R}(G)$ into mixed circuits. Let $E_j = \bigcup_{i=1}^{j} C_i$, $Y = V(H_m) - \bigcup_{i=1}^{m-1} V(H_i)$ and $X = V(H_m) - Y$. Then either Y contains an admissible node, or G can be expressed as the 2-sum of an $\mathcal{M}$-connected mixed graph and a pure K_4.*

Proof. If G is a mixed circuit, or $G = F_1 \oplus_2 F_2$ where F_2 is a pure K_4, then we are done by Lemmas 7.4.1 and 7.2.12(ii) respectively. So assume that G

is not a circuit and cannot be expressed as such a 2-sum. Since G has no admissible edges, Lemma 7.2.4 (ii) and (iii) imply $Y \neq \emptyset$ and X is mixed critical in H_m. Thus Y contains a node by Lemma 7.3.2. Since, by Lemma 7.3.1, the node-subgraph of a mixed circuit is a forest, we can find a node $u \in Y$ such that u is a leaf in $G[V_3 \cap Y]$.

If $|N(u)| = 2$, then Lemma 7.4.5 implies u is admissible in G and we are done. So assume $|N(u)| = 3$, and let $N(u) = \{r, s, t\}$.

Case 1. $N(u) \subseteq X$.
Since X is mixed critical in H_m, the sparsity conditions imply $X + u = V(H_m)$, and hence $Y = \{u\}$. In order to perform a 1-reduction on u in G, we first have to make sure that there is a pair of vertices in $N(u)$ that we can 1-reduce onto i.e. if u is pure of type $P \in \{D, L\}$, then there must be a pair of vertices in $N(u)$ which are not connected by an edge of type P; and if u is mixed, then there must exist a pair of vertices in $N(u)$ which have at most one edge between them.

Suppose u is a pure node of type P, and assume $G[N(u)]$ contains two edges, e and f, of the same type as u. By Lemma 7.2.4(ii) $e, f \in E_{m-1}$. Since e and f are not parallel, they must cover all three vertices in $N(u)$. We know $G[E_{m-1}]$ is $\mathcal{M}$-connected, so for all edges $g \in E_{m-1} - e$ there is some circuit $C_g \subseteq E_{m-1}$ such that $e, g \in C_g$. If $f \notin C_g$ then C_g is a circuit in $\mathcal{R}(G - f)$. Otherwise $f \in C_g$ and, by Lemma 7.2.6 when C_g is pure, or Lemma 7.2.7 when C_g is mixed, the 1-extension, $(C_g - f) \cup \{ur, us, ut\}$ is a circuit in $\mathcal{R}(G - f)$. So for all edges g in $G - f$, we can find a circuit in $G - f$ containing both g and e. Thus, by the transitivity of matroid connectivity, $G - f$ is $\mathcal{M}$-connected. This contradicts the hypothesis that G contains no admissible edges, so our assumption must be false, and G must contain at most one edge of type P in $G[N(u)]$.

Suppose instead that u is a mixed node, and assume that each pair of vertices in $N(u)$ is connected by a double-edge. Then $G[N(u)]$ contains two non-parallel edges e and f such that $e \in D$ and $f \in L$. By a similar argument to the pure case above, we can show that $G - f$ is $\mathcal{M}$-connected, by considering mixed circuits. Once more, this contradicts or hypothesis. So $N(u)$ contains a pair of vertices with at most one edge between them.

So regardless of whether u is mixed or pure, we can find some pair of vertices, say $\{r, s\}$, in $N(u)$ such that we can perform a 1-reduction at u onto rs. The resulting graph, $\bigcup_{i=1}^{m-1} H_i + rs$ is $\mathcal{M}$-connected, by Lemmas 3.1.1(i) and 7.2.5. Hence u is admissible in G.

Case 2. $|N(u) \cap X| = 2$.
Let $N(u) \cap X = \{r, s\}$. Since X is mixed critical in H_m, we have no admissible 1-reduction at u onto rs in H_m. If u has an admissible 1-reduction in H_m onto either rt or st, then u is also admissible in G by Lemma 7.4.4, and we are done. So suppose u is not admissible in H_m. Then Lemmas 7.3.5 and 7.3.6 imply that there exists a triple (R, S, T) which is either a flower or a clover on u in H_m. We may assume that R, S and T are minimal sets with this property. We also know that X is a mixed critical set with $r, s \in X \subseteq V(H_m) - \{t, u\}$, so by the definitions, the triple (X, R, S) is either a strong or a weak flower, or a clover on u in H_m.

Suppose (X, R, S) is either a strong or a weak flower on u in H_m. Then Lemma 7.3.3 implies $d_G(S, X) = 0$, so G contains no st-edges. Hence we can perform a 1-reduction at u onto st. Let C denote the unique circuit in $E((H_m)_u^{st})$ formed by this 1-reduction. Since R is a minimal critical set in H_m which contains both s and t, $G[C]$ must have R as its vertex set. By Lemma 7.3.3, $R \cup X = V(H_m) - u$ and $d_{H_m}(R, X) = 0$. This implies that $C \supseteq E((H_m)_u^{st}) - E_{m-1}$. Also, since $X \cap R \cap S \neq \emptyset$ and $s \in (X \cap R) - S$, we have that $|R \cap X| \geq 2$ which, by Lemma 7.1.2, implies that $R \cap X$ is critical in H_m, and hence has non-empty edge set in H_m. Thus $C \cap E_{m-1} \neq \emptyset$ and so Lemma 7.4.4 implies that u is admissible in G.

Suppose instead that (X, R, S) is a clover on u in H_m. Then Lemma 7.3.4 implies that H_m has a 2-separation $(G[X], G[R \cup S + u] - E(X))$ where $G[R \cup S + u] - E(X) = G[\tilde{C}_m]$ is pure. This implies $(G[E_{m-1}], G[\tilde{C}_m])$ is an unbalanced 2-separation of G.

Let (S_1, S_2) be a 2-separation of G with S_2 minimal such that $V(S_2) \subseteq R \cup S + u$, and let the corresponding 2-vertex-cut be $\{x, y\}$. Since $G[\tilde{C}_m]$ is pure, S_2 is also pure. If G contains an xy-edge, e, of the same type as S_2, then $G - e$ is also an $\mathcal{M}$-connected mixed graph by Lemma 7.2.12(ii), which contradicts our assumption that G contains no admissible edges. Thus $e \notin E(G)$, and so, by Lemma 7.2.12, $G = F_1 \oplus_2 F_2$ where F_1 and F_2 are $\mathcal{M}$-connected and are formed from S_1 and S_2 respectively by adding the xy-edge e. Since S_2 is minimal and $F_2 \neq K_4$, we know that F_2 must be a 3-connected, $\mathcal{M}$-connected pure graph on at least 5 vertices. Further, since $S_2 \subseteq G[\tilde{C}_m]$, Lemma 7.2.11 implies that F_2 is a pure circuit.

Hence, by Lemma 7.4.2, F_2 has at least two admissible nodes in $V(F_2) - \{x, y\}$. Let F_2' be the graph formed by an admissible 1-reduction at one of these nodes. Then $G' = F_1 \oplus_2 F_2'$ is an $\mathcal{M}$-connected mixed graph by Lemma 7.2.12(i), and is a 1-reduction of G. Hence this node is also admissible in G.

Case 3. $|N(u) \cap X| \leq 1$.

If u is admissible in H_m, then we are done by Lemma 7.4.4. So suppose u is not admissible in H_m. Since u is a leaf in $G[V_3 \cap Y]$, u has some neighbour in Y which is not a node. Thus, by Theorem 7.3.10, either Y contains an admissible node v and we are done by the above arguments; or there is a 2-separation (S_1, S_2) of H_m such that S_2 is pure and $X \subseteq V(S_1)$, in which case $(S_1 \cup G[E_{m-1}], S_2)$ is an unbalanced 2-separation of G. By the same argument used in the case where $|N(u) \cap X| = 2$ and G had an unbalanced 2-separation, we can find a 2-separation (S_1', S_2') of G on some 2-vertex-cut $\{x, y\}$, where S_2' is minimal with $S_2' \subseteq S_2$ and there is a node in $V(S_2') - \{x, y\}$ which is admissible in G. $\square$

We know, by Lemmas 7.2.5, 7.2.9 and 7.2.12(i), that the inverse of the operations used in Theorem 7.4.6 preserve $\mathcal{M}$-connectivity. Thus the following inductive construction immediately follows from this result:

Theorem 7.4.7. *Let G be a mixed graph. Then G is $\mathcal{M}$-connected if and only if G can be obtained from K_3^+ or K_3^- by a sequence of edge additions, 1-extensions and 2-sums with pure K_4's.*

7.5 Constructing direction-balanced $\mathcal{M}$-connected mixed graphs

Here we specialise our inductive construction of all $\mathcal{M}$-connected mixed graphs from the previous section, to one for all direction-balanced, $\mathcal{M}$-connected mixed graphs. Recall that a mixed circuit is globally rigid if and only if it is direction-balanced (Theorem 6.2.5). In Section 7.6, we use the construction obtained here to show this result extends to all $\mathcal{M}$-connected graphs, thus characterising global rigidity for $\mathcal{M}$-connected graphs.

To obtain this construction, we show that if a direction-balanced, $\mathcal{M}$-connected mixed graph G cannot be obtained by a 2-sum with a direction-pure K_4, then G either has an admissible node or an admissible edge, whose removal preserves being direction-balanced. As such, we define a vertex v of G to be *feasible* if there is an admissible 1-reduction at v which preserves being direction balanced, and define an edge e to be *feasible*, if it is admissible and $G - e$ is direction-balanced.

Before proving this result, we first make the following observations about graphs with unbalanced 2-separations:

Lemma 7.5.1. *Let $G = (V; D, L)$ be an $\mathcal{M}$-connected mixed graph with an edge-disjoint 2-separation (H_1, H_2) on 2-vertex-cut $\{x, y\}$ with H_2 pure. Let G' be formed from G by either an edge-deletion or a pure 1-reduction. Then $\{x, y\}$ is also a 2-vertex-cut in G'.*

Proof. This trivially holds when G' is formed from G by an edge deletion, so we only prove the case where G is formed by a 1-reduction. Suppose $G' = G_v^{rs}$ for some node $v \in V$. By Lemma 7.2.12(ii), $d_G(x), d_G(y) \geq 4$, which implies $v \notin \{x, y\}$. So without loss of generality, we can assume $v \in V(H_2) - V(H_1)$. Since $\{x, y\}$ is a 2-vertex-cut in G, this implies $N_G(v) \subset V(H_2)$. Hence $(H_1, (H_2)_v^{rs})$ is a 2-separation of G' on the 2-vertex-cut $\{x, y\}$. □

Lemma 7.5.2. *Let $G = (V; D, L)$ be a direction-balanced, $\mathcal{M}$-connected mixed graph and let v be a mixed node of G with $N(v) = \{r, s, t\}$, where potentially $t = r$. Suppose G' is the graph formed by an admissible length 1-reduction at v onto the edge $l = rs$, and that G' is not direction-balanced i.e. G' has a 2-separation (H_1, H_2) with H_2 length-pure. Suppose $l \in E(H_2)$. Then either*

(i) $\{r, s\} = V(H_1) \cap V(H_2)$. In which case $t \in V(H_2) - V(H_1)$, and G has admissible direction 1-reductions onto both rt and st; or

(ii) $\{r, s\}$ intersects $V(H_2) - V(H_1)$, and G has an admissible direction 1-reduction onto rs.

Proof. Let V_1 and V_2 denote the vertex sets of H_1 and H_2 respectively, and let $\{x, y\} = V_1 \cap V_2$. Since G is direction-balanced, $N_G(v) \cap (V_2 - V_1) \neq \emptyset$.

We first consider case (i), where $\{r, s\} = \{x, y\}$. Lemma 7.2.12(ii) implies $G' - l$ is $\mathcal{M}$-connected. Since $N_G(v) \cap (V_2 - V_1) \neq \emptyset$, we must have $t \in V_2 - V_1$. But H_2 is length-pure, so this implies neither st nor rt are direction edges in G' or G. Since edge additions preserve $\mathcal{M}$-connectivity by Lemma 7.2.5, we can add either of the direction edges st or rt to $G' - l$ to obtain G^*, which is a direction 1-reduction at v as required.

We now consider case (ii), where $\{r, s\} \neq \{x, y\}$. Since $l \in E(H_2)$, this implies at least one of the endvertices of l, say s, is contained in $V_2 - V_1$. But H_2 is length-pure, so rs is not a direction edge in G' or G. Let G^* be the graph obtained from G by a direction 1-reduction onto the edge rs, and call this direction edge d. It remains to show that G^* is $\mathcal{M}$-connected.

By construction, we have $E(G^*) = E(G') - l + d$. Let f be a direction edge in $E(G^*) \cap E(G')$. Since G' is $\mathcal{M}$-connected, we know that for all $e \in E(G') - f$, there is a circuit C' in $\mathcal{R}(G')$ such that $e, f \in C'$. If $l \notin C'$, then C' is also a circuit in $\mathcal{R}(G^*)$ and we are done. Otherwise $l \in C'$, and so C' is mixed. In which case, Lemma 7.2.8 implies that the edge set $C^* = C - l + d$ is a mixed circuit in $\mathcal{R}(G^*)$ containing f and d (and e, when $e \neq l$). Thus, by the transitivity of $\mathcal{M}$-connectivity, G^* is $\mathcal{M}$-connected. Hence the direction 1-reduction at v onto rs is admissible. $\square$

We now prove the main result of this section:

Theorem 7.5.3. *Let $G \notin \{K_3^+, K_3^-\}$ be a direction-balanced, $\mathcal{M}$-connected mixed graph. Suppose G cannot be expressed as the 2-sum of a direction-balanced mixed graph with a direction-pure K_4. Then G has either a feasible edge or a feasible vertex.*

Proof. We proceed by contradiction, i.e. by assuming there exists some mixed graph G such that all 1-reductions and edge-deletions of G which preserve $\mathcal{M}$-connectivity do not preserve being direction-balanced. Theorem 7.4.6 tells us that we are able to construct some smaller $\mathcal{M}$-connected graph G' from G, either by deleting an admissible edge, or by performing an admissible 1-reduction at some node in G. Let

$$n(G') = |\{v \in V(G') : v \in X \text{ for some length-pure end } X \text{ of } G'\}|.$$

We assume that out of all choices of admissible edges and nodes in G, the admissible move which formed G' is such that $n(G')$ is minimal. Since we are also assuming that G has no feasible moves, G' is not direction-balanced, and hence $n(G') > 0$.

Note that if G' was formed by a length 1-reduction at v onto the edge rs, and G also has an admissible direction 1-reduction at v onto rs, then every length-pure end in the graph G_d formed by the direction 1-reduction is also a length-pure end in G'. Hence $n(G_d) \leq n(G')$, and so we can assume G' was chosen to be the direction 1-reduction instead.

Let (H_1, H_2) be a 2-separation of G' on some 2-vertex-cut $\{x, y\}$, such that H_2 is length-pure, and is minimal with respect to inclusion.

Claim 7.5.4. *xy is not a length edge in G'.*

Proof. Assume xy is a length edge in G'. Then either xy is also a length edge in G, or this edge was added in a length 1-reduction of G and so $G' = G_v^{xy}$ for some $v \in V(G)$.

First, suppose xy is a length edge of G. Then Lemma 7.2.12(ii) implies $G' - xy$ is $\mathcal{M}$-connected. We can construct $G - xy$ from $G' - xy$ by performing the inverse operation to that which formed G' (an edge addition when $G' = G - e$ for some $e \in E(G)$, or a 1-extension when $G' = G_v^{rs}$ for some $v \in V(G)$). By Lemmas 7.2.5 and 7.2.9 respectively, both of these operations preserve $\mathcal{M}$-connectivity, so $G - xy$ is $\mathcal{M}$-connected. Hence xy is admissible in G.

But we assumed G had no feasible edges, so $G - xy$ must have a 2-separation (H_1', H_2') on some 2-vertex-cut $\{x', y'\}$ where H_2' is length-pure and $\{x', y'\}$ separates x and y. Lemma 7.5.1 implies $\{x', y'\}$ is also a 2-vertex-cut in $G' - xy$. Thus $G' - xy$ has two crossing 2-vertex-cuts, $\{x, y\}$ and $\{x', y'\}$, which contradicts Lemma 7.2.14.

We now consider the second case: that xy was added in a 1-reduction on v. If v is a pure node, then in order for G to be direction-balanced, v must have neighbours in both $V(H_1) - V(H_2)$ and $V(H_2) - V(H_1)$, in addition to neighbouring x and y. But this contradicts that $d_G(v) = 3$. Hence v must be a mixed node. Let $N_G(v) = \{x, y, z\}$. Lemma 7.5.2 implies $z \in V(H_2) - V(H_1)$ and that G has a different admissible 1-reduction at v, which adds the direction edge xz instead, to form the graph G^*.

By our original assumption, G^* is not direction-balanced, so it has a 2-separation (H_1^*, H_2^*) on some 2-vertex-cut $\{x^*, y^*\}$ with H_2^* length-pure and minimal with respect to inclusion. However, by the construction above, $G^* = G' - xy + xz$, and so $\{x, y\}$ is also a 2-vertex-cut in G^*. Since xz is a direction edge, the 2-separation given by $\{x, y\}$ in G^* is direction-balanced. Hence $\{x, y\} \neq \{x^*, y^*\}$. Lemma 7.2.14 now implies that these two 2-separations of G^* cannot cross. So either $V(H_2^*) \subset V(H_1)$ or $V(H_2^*) - \{x^*, y^*\} \subset V(H_2) - \{x, y, z\}$. In the former case, this implies $V(H_2^*) - \{x^*, y^*\}$ is a length-pure end in G, which contradicts the fact G is direction-balanced. In the latter case, this gives $n(G^*) < n(G')$, which contradicts our choice of G'. $\square$

Claim 7.5.4 and Lemma 7.2.12(ii) imply $G' = F_1 \oplus_2 F_2$ where F_2 is length-pure, and both F_1 and F_2 are $\mathcal{M}$-connected and are formed from H_1

and H_2 respectively by adding the length edge xy. Let

$$V^* = \begin{cases} V(F_2) - \{x, y, r, s\} & \text{if } G' = G - e \text{ and } e = rs, \\ V(F_2) - \{x, y, r, s, t\} & \text{if } G' = G_v^{rs} \text{ and } N_G(v) = \{r, s, t\}, \end{cases}$$

and

$$E^* = \begin{cases} E(F_2) - \{xy\} & \text{if } G' = G - e \text{ and } e = rs, \\ E(F_2) - \{xy, rs\} & \text{if } G' = G_v^{rs} \text{ and } N_G(v) = \{r, s, t\}. \end{cases}$$

Claim 7.5.5. *F_2 has no admissible edges $f \in E^*$, and no admissible nodes $w \in V^*$.*

Proof. We shall show that if such an admissible edge or node exists in F_2, then it is also admissible in G and contradicts our choice of G'.

Assume that F_2 has either an admissible edge $f \in E^*$, or an admissible node $w \in V^*$ with neighbourhood $\{m, n, p\}$ such that w has an admissible 1-reduction onto the edge mn. Then $F_2 - f$, respectively $(F_2)_w^{mn}$, is $\mathcal{M}$-connected and contains the length edge xy. Thus by Lemma 7.2.12(i), we can 2-sum this graph with F_1 to obtain

$$(G')^* = \begin{cases} G' - f = F_1 \oplus_2 (F_2 - f) & \text{when } f \text{ is admissible in } F_2, \text{ or} \\ (G')_w^{mn} = F_1 \oplus_2 (F_2)_w^{mn} & \text{when } w \text{ is admissible in } F_2. \end{cases}$$

Since 2-sums preserve $\mathcal{M}$-connectivity, the resulting graph $(G')^*$ is $\mathcal{M}$-connected. By the definitions of E^* and V^*, we know $(G')^*$ either contains both r and s when $G' = G - e$, or contains the vertices r, s, t and the edge rs when $G' = G_v^{rs}$. Further, when $G' = G - e$ then we cannot have $e \in E((G')^*)$, as this would imply that e was added back to G' by a 1-reduction at the length-pure node w; for this to be possible, both endvertices of e must be in H_2, which, since G was direction-balanced, implies that e is a direction edge, and thus cannot be added by such a 1-reduction.

Hence we can either add back the edge e to $(G')^*$, or perform a 1-extension on $(G')^*$ to add back the vertex v, as relevant. Both of these operations preserve $\mathcal{M}$-connectivity by Lemmas 7.2.5 and 7.2.9 respectively. So the resulting graph G^* (where $G^* = G - f$ or $G^* = G_w^{mn}$ respectively) is $\mathcal{M}$-connected. Thus f (respectively w) is admissible in G.

By our original assumption, G has no feasible nodes or edges. So G^* must have a 2-separation (H_1^*, H_2^*) on some 2-vertex-cut $\{x^*, y^*\}$ where H_2^* is length-pure, and is minimal with respect to inclusion. Since F_2 is length-pure, $f \in E^*$ is a length edge (resp. $w \in V^*$ is a length-pure node). But G is direction-balanced, so this implies that when $G^* = G - f$, the set $\{x^*, y^*\}$ separates the endvertices of f in G^*, and when $G^* = G_w^{mn}$, the set $\{x^*, y^*\}$ separates $N_G(w)$ in G^*. Since $f \in E(H_2)$ in the former case, and $\{w\} \cup N_G(w) \subseteq V(H_2)$ in the latter, this implies $\{x, y\} \neq \{x^*, y^*\}$ and that $V(H_2) - V(H_2^*) \neq \emptyset$. Lemmas 7.5.1 and 7.2.14 now imply that both $\{x, y\}$ and $\{x^*, y^*\}$ are 2-vertex-cuts in $(G')^*$ and do not cross.

When $G' = G - e$, we clearly have $V(H_2^*) \subset V(H_2)$. When instead, we have $G' = G_v^{rs}$, the fact that G is direction-balanced implies that v is either a mixed node in G, or $N_G(v)$ intersects $V(H_1) - \{x, y\}$; both of which imply that $v \notin V(H_2^*)$ and thus $V(H_2^*) \subset V(H_2)$. Hence for all choices of G' and G^*, $V(H_2^*) \subset V(H_2)$. The definition of an end thus implies $n(G^*) < n(G')$, which contradicts our choice of G'. $\square$

Claim 7.5.5 tells us that all admissible edges or nodes in F_2 must be contained in $E(F_2) - E^*$ or $V(F_2) - V^*$ respectively. In the remainder of the proof, we show that whatever the structure of F_2, we can find such an admissible move, and this move will contradict our choice of G'.

Claim 7.5.6. *$F_2 = K_4$.*

Proof. Assume $F_2 \neq K_4$. We show that we can either find a feasible move in G, thus contradicting our original assumption that G has no feasible moves; or we can find a different admissible move in G which contradicts our choice of G'. As before, there are two cases to consider: when $G' = G - e$, and when $G' = G_v^{rs}$.

Case 1. $G' = G - e$.
Since G is direction-balanced, at least one of the endvertices of e, say r, is contained in $V(H_2) - V(H_1)$, and the other endvertex, s, is either also contained in $V(H_2)$, in which case e must be a direction edge; or in $V(H_1) - V(H_2)$ and e can be either a length or a direction edge.

We know F_2 is an $\mathcal{M}$-connected length-pure graph. If F_2 is a length-pure circuit, then Lemma 7.4.2 implies F_2 contains an admissible node in $V(F_2)-\{x, y, r\}$. Otherwise, by Lemma 3.1.1(ii), we can build an ear decomposition of F_2 such that the first circuit in the ear decomposition contains both the edge xy and some edge incident to r. Lemma 7.4.3 now implies that F_2 contains either an edge other than xy which is admissible, or an admissible node in $V(F_2) - \{x, y, r\}$. Thus, in both cases, F_2 either contains an admissible edge $f \in E(F_2) - \{xy\} = E^*$, or an admissible node $w \in V(F_2) - \{x, y, r\}$. Since $V^* = V(F_2) - \{x, y, r, s\}$, and, by Claim 7.5.5, V^* and E^* contain no admissible nodes or edges respectively, it only remains to consider the case where $s \in V(H_2) - V(H_1)$ is an admissible node.

Let $N(s) = \{m, n, p\}$ in F_2, and suppose, without loss of generality, that $(F_2)_s^{mn}$ is $\mathcal{M}$-connected. We can now add the vertex s back to $(F_2)_s^{mn}$ by performing a 1-extension which deletes the edge mn and adds back the edges ms, ns and $rs = e$. The resulting graph, $F_2 - ps$, is $\mathcal{M}$-connected by Lemma 7.2.6. Hence $ps \in E^*$ is an admissible edge in F_2, contradicting Claim 7.5.5.

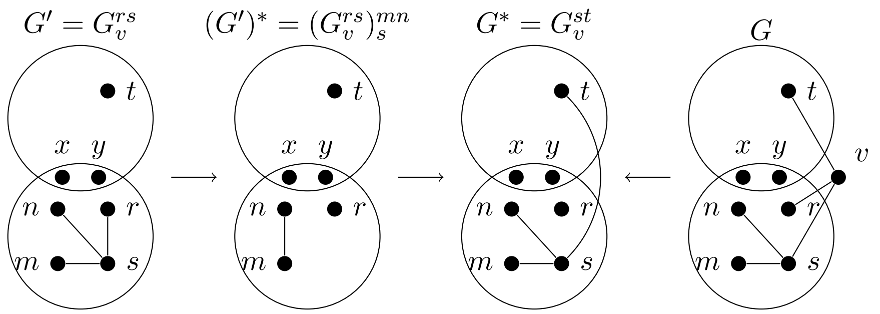
Case 2. $G' = G_v^{rs}$.

First, suppose the edge, rs, added in the 1-reduction, is contained in $E(H_1)$. By Claim 7.5.4, $\{r,s\} \neq \{x,y\}$, so without loss of generality, $r \in V(H_1) - V(H_2)$. In order for G to be direction-balanced, we must have $t \in V(H_2) - V(H_1)$. If F_2 is a circuit then, by Lemma 7.4.2, it contains an admissible vertex in $V(H_2) - \{x,y,t\} = V^*$, contradicting Claim 7.5.5. Otherwise, we can build an ear decomposition of F_2 whose first ear contains both xy and an edge incident to t. Lemma 7.4.3 then implies that there is either an admissible edge in $E(F_2) - \{xy\} = E^*$, or an admissible node in $V(H_2) - \{x,y,t\} = V^*$, once more contradicting Claim 7.5.5.

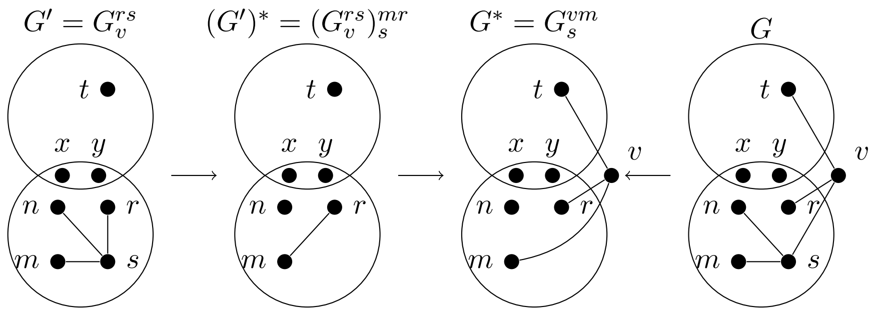
Hence $rs \in E(H_2) - E(H_1)$ and, without loss of generality, $r \in V(H_2) - V(H_1)$. Since H_2 is length-pure, rs is a length edge. If v is a mixed node in G, then Lemma 7.5.2 implies that G has a direction 1-reduction onto rs. But this contradicts our choice of G': that a length 1-reduction at v was chosen only when G had no direction 1-reduction at v onto the same pair of vertices. Hence v must be a length-pure node, and thus, since $r, s \in V(H_2)$, we must have $t \in V(H_1) - V(H_2)$ in order for G to be direction-balanced.

Suppose F_2 is not a circuit. Then we can build an ear decomposition of F_2 such that the first ear contains both the edges xy and rs. Lemma 7.4.3 now implies either $E(F_2) - \{xy, rs\} = E^*$ contains an admissible edge, or $V(F_2) - \{x,y,r,s\} = V^*$ contains an admissible node, contradicting Claim 7.5.5. Thus F_2 must be a length-pure circuit. In which case, Lemma 7.4.2 implies that F_2 contains an admissible node in $V(F_2) - \{x,y,r\} \subseteq V^* \cup \{s\}$. By Claim 7.5.5, the only possibility is that $s \in V(F_2) - \{x,y\}$ is admissible in F_2. We know $e = rs \in E(F_2)$. Let m and n denote the other two neighbours of s in F_2.

Relabelling m and n if necessary, s has an admissible 1-reduction onto either mn or mr. In both cases, we can follow this 1-reduction by a 1-



(a) Construction of $G^* = G_v^{st}$ from G' when s is admissible in G' onto the edge mn.



(b) Construction of $G^* = G_s^{vm}$ from G' when s is admissible in G' onto mr.

Figure 7.8: Constructions of G^* from $G' = G_v^{rs}$.

extension to obtain a graph G^*, where $G^* = G_v^{st}$ or $G^* = G_s^{mr}$ respectively. See Figure 7.8. Lemma 7.2.9 implies that G^* is $\mathcal{M}$-connected in both cases. Further, since the node s (respectively v) added back to obtain G^* has neighbourhood $N_{G^*}(s) = \{m, n, t\}$ (resp. $N_{G^*}(v) = \{m, r, t\}$) where $t \in V(H_1) - V(H_2)$, and either $\{m, n\} \cap (V(H_2) - V(H_1)) \neq \emptyset$ in the first case, or $r \in V(H_2) - V(H_1)$ in the second, we know $\{x, y\}$ is not a 2-vertex-cut of G^*.

By our original assumption, G^* is not direction-balanced. So let (H_1^*, H_2^*) be a 2-separation of G^* on some 2-vertex-cut $\{x^*, y^*\}$, with H_2^* length-pure and minimal with respect to inclusion. Since G is direction-balanced, the set $\{x^*, y^*\}$, must separate $\{s, t\}$ from r in G^* when $G^* = G_v^{st}$, and separate $\{v, m\}$ from n in G^* when $G^* = G_s^{vm}$. Since $m, n, r, s \in V(H_2)$, this implies $V(H_2) \cap V(H_1^*) \neq \emptyset$ in both cases. But $\{x, y\}$ and $\{x^*, y^*\}$ are distinct 2-vertex-cuts in $(G')^*$ and so cannot cross, by Lemma 7.2.14. Hence $\{x^*, y^*\} \subset V(H_2)$, which implies $V(H_2^*) \subset V(H_2)$ in both cases. The definition of an end thus gives $n(G^*) < n(G')$, which contradicts our choice of G'. □

Claim 7.5.6 tells us that the only case left to consider is when $F_2 = K_4$. We show that this cannot occur.

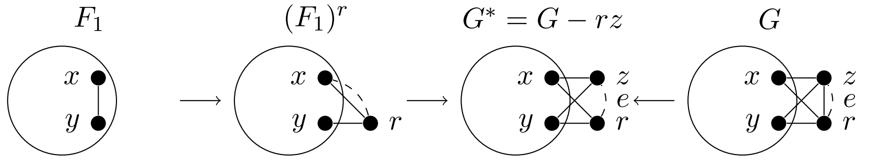
Claim 7.5.7. $F_2 \neq K_4$.

Proof. Assume $F_2 = K_4$. We shall show that we can find an admissible move in G which contradicts our choice of G'. As in Claim 7.5.6, there are two cases to consider: when $G' = G - e$ and when $G' = G_v^{rs}$.

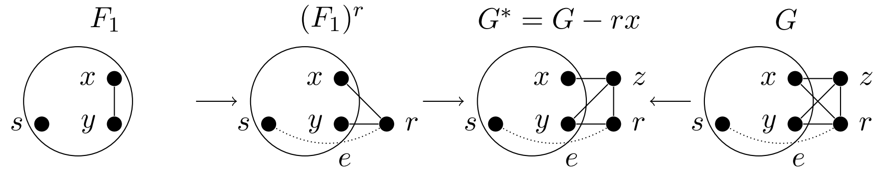
Case 1. $G' = G - e$.
Since G is direction-balanced, e must have at least one endvertex, say r, in $V(H_2) - V(H_1)$. Denote the remaining vertex in $V(H_2) - \{x, y\}$ by z.

There are two possibilities for e: either $e = rz$, in which case e is a direction edge in G. Or $e \neq rz$, in which case $e = rs$ for some $s \in V(H_1)$. In both cases we can construct a new graph, G^*, from F_1, by performing a sequence of two 1-extensions. These constructions each give $G^* = G - f$ for some $f \in E(G)$, as shown in Figure 7.9.

We know F_1 is mixed and $\mathcal{M}$-connected, and by Lemma 7.2.9, 1-extensions preserve these properties. Hence G^* is mixed and $\mathcal{M}$-connected. By the constructions, it is clear that $\{r, z\}$ is not a length-pure end in G^*, whereas it was in G'. So any length-pure end of G^* must be contained in F_1, and thus is also a length-pure end of G'. Hence $n(G^*) < n(G')$, which contradicts our choice of G'.



(a) Construction of $G^* = G - rz$ when $e = rz$ is a direction edge in G.



(b) Construction of $G^* = G - rx$ when $e = rs$ for some $s \in V(H_1)$. Where the type of an edge is not known, it is depicted by a dotted line. Note that this construction also works in the special case where $s \in \{x, y\}$.

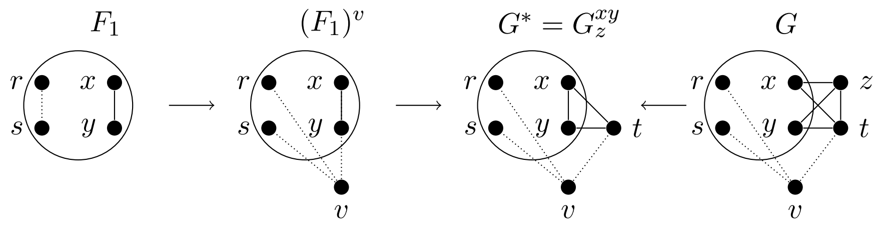
Figure 7.9: Constructions of $G^* = G - f$ from F_1.

Case 2. $G' = G_v^{rs}$.

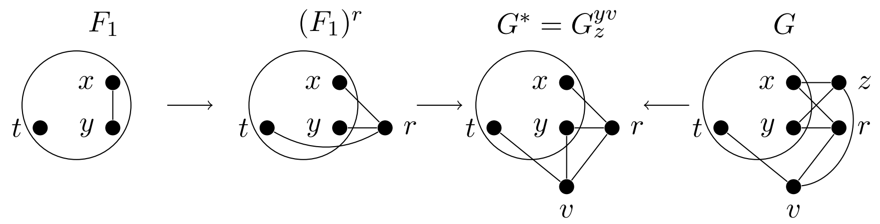
Since G is direction-balanced, v has at least one neighbour in $V(H_2) - V(H_1)$. Denote the remaining vertex in $V(H_2) - \{x, y\}$ by z. We shall show that z is admissible in G. As before, we can construct an $\mathcal{M}$-connected mixed graph G^* from F_1 by a sequence of two 1-extensions. But this time, the graph G^* obtained is a 1-reduction of G. There are three different constructions, depending on the structure G'.

First, suppose $rs \in E(H_1)$. Then we must have $t \in V(H_2) - V(H_1)$. From Claim 7.5.4, we know that $\{r, s\} \neq \{x, y\}$ so without loss of generality, $x \notin N_G(v)$. We can then obtain $G^* = G_z^{xy}$ from F_1 by the construction shown in Figure 7.10(a)

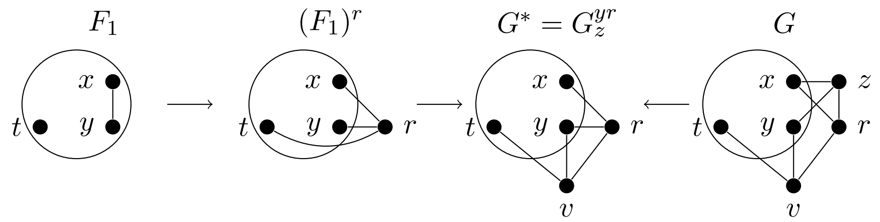
Second, suppose $rs \in E(H_2)$. If v is mixed, then Lemma 7.5.2, implies that G has an admissible direction 1-reduction at v onto rs, which contradicts our original choice of G'. Hence v is length-pure, and so, since G is direction-balanced, $t \in V(H_1) - V(H_2)$. Either both $r, s \in V(H_2) - V(H_1)$,



(a) Construction of $G^* = G_z^{xy}$ when $G' = G_v^{rs}$ and $r, s \in V(H_1)$. This construction also works when $y \in \{r, s\}$. Dotted lines depict edges of unkown type.



(b) Construction of $G^* = G_z^{yv}$ when $G' = G_v^{rs}$, $s = z$ and $r, s \in V(H_2) - V(H_1)$.



(c) Construction of $G^* = G_z^{yr}$ when $G' = G_v^{rs}$, $s = y$ and $r \in V(H_2) - V(H_1)$.

Figure 7.10: Constructions of $G^* = G_z$ from F_1.

in which case $s = z$, and we obtain $G^* = G_z^{yv}$ by the construction shown in Figure 7.10(b). Or, relabelling if necessary, $r \in V(H_2) - V(H_1)$ and $s \in V(H_1) \cap V(H_2)$, so without loss of generality $s = y$, and we construct $G^* = G_z^{yr}$ as shown in Figure 7.10(c).

In all three cases, $V(G^*) - V(F_1)$ is not a length-pure end of G^*, whereas $V(G') - V(F_1)$ was a length-pure end of G'. So any length-pure end of G^* is contained in $V(F_1)$, and must also be a length-pure end of G'. Thus $n(G^*) < n(G')$, which contradicts our choice of G'. $\square$

Clearly Claims 7.5.6 and 7.5.7 cannot both hold. Hence our original assumption is wrong, and G contains either a feasible edge or a feasible node. □

Theorem 7.5.3, together with the fact that edge additions, 1-extensions, and 2-sums with direction-pure K_4's preserve $\mathcal{M}$-connectivity (Lemmas 7.2.5, 7.2.9 and 7.2.12(i) respectively), and that these operations also preserve being direction-balanced, gives us the following inductive construction:

Theorem 7.5.8. *Let G be a mixed graph. Then G is a direction-balanced, $\mathcal{M}$-connected mixed graph if and only if G can be obtained from K_3^+ or K_3^- by a sequence of edge additions, 1-extensions and 2-sums with direction-pure K_4's.*

7.6 Characterisation of global rigidity

In order for the graphs constructed in Theorem 7.5.8 to be globally rigid, we need to know that the operations used preserve global rigidity. Lemmas 7.1.11 and 7.2.13 imply that edge additions and 2-sums with direction-pure K_4's preserve global rigidity, but 1-extensions are more troublesome. By Lemma 7.1.11, a 1-extension on a graph G which deletes an edge e will preserve global rigidity so long as $G - e$ is rigid. Fortunately, $\mathcal{M}$-connected mixed graphs are redundantly rigid, by Lemma 7.2.3, so this condition is always satisfied in our construction. Hence all graphs described in Theorem 7.5.8 are globally rigid for all generic realisations. This gives us the following result:

Lemma 7.6.1. *Let (G, p) be a generic mixed framework, and suppose G is $\mathcal{M}$-connected and direction-balanced. Then (G, p) is globally rigid.*

By Lemmas 6.2.1 and 6.2.4, all generic, globally rigid direction-length frameworks are mixed and direction-balanced. So Lemma 7.6.1 implies that global rigidity is a generic property of $\mathcal{M}$-connected direction-length frameworks. Hence characterising global rigidity for this class:

Theorem 7.6.2. *Let p be a generic realisation of an $\mathcal{M}$-connected mixed graph G. Then (G, p) is globally rigid if and only if G is direction-balanced.*

This implies that the inductive construction in Theorem 7.5.8, is also a construction of the class of $\mathcal{M}$-connected graphs which are globally rigid for all generic realisations:

Theorem 7.6.3. *Let p be a generic realisation of an $\mathcal{M}$-connected graph G. Then (G, p) is globally rigid if and only if G can be obtained from K_3^+ or K_3^- by a sequence of edge additions, 1-extensions and 2-sums with direction-pure K_4's.*

7.7 Closing remarks

Recall from page 47 that a graph G is globally rigid if all generic realisations of G are globally rigid, and is globally flexible if no generic realisation is globally rigid. Thus Theorem 7.6.3 characterises the class of globally rigid $\mathcal{M}$-connected graphs. It is also implies that global rigidity is a generic property of $\mathcal{M}$-connected direction-length graphs, or equivalently, that every $\mathcal{M}$-connected direction-length graph is either globally rigid or globally flexible.

There exist globally rigid mixed graphs whose rigidity matroid is not connected, so the results in this chapter do not characterise global rigidity for all direction-length graphs. In particular, $\mathcal{M}$-connected mixed graphs satisfy three properties which are not necessary for global rigidity: they contain at least two length edges, have minimum degree three, and every direction edge is redundant.

Servatius and Whiteley's result (Theorem 6.2.2) shows that the first and third of these properties are not necessary, whereas the fact direction-pure 0-extensions preserve global rigidity (Lemma 7.1.11(i)) shows that globally rigid graphs can have vertices of degree 2.

Since Servatius and Whiteley's result characterises global rigidity for graphs $G = (V; D, L)$ with $|L| < 2$, it remains to characterise it for graphs with $|L| \geq 2$. This chapter succeeds in doing this for a large class of such graphs, but not all of them.

In the following chapter, we build upon these results to obtain a full characterisation of globally rigid direction-length graphs. However, we do not fully characterise globally flexible direction-length graphs. There is one class of graphs remaining for which it is not known whether global rigidity is a generic property. We succeed in showing that graphs in this class have some generic realisation that is not globally rigid, thus proving that such graphs are not globally rigid. However it is not known whether these graphs are globally flexible, or equivalently, whether global rigidity is a generic property for graphs in this class.

Chapter 8

Direction reductions and irreducible frameworks

8.1 Preliminaries

Recall the definitions of direction reductions and direction irreducible graphs from Chapter 6. In this first section, we collect tools from diverse areas that we will use in our proofs.

8.1.1 Circuits and $\mathcal{M}$-components

In Chapter 7, we focused on $\mathcal{M}$-connected graphs. In this chapter, we consider graphs G which are not $\mathcal{M}$-connected. However, it shall be important to identify the subgraphs of G which are $\mathcal{M}$-connected. As such, we define a subgraph H of a direction-length graph G to be an *$\mathcal{M}$-component* of G if $\mathcal{R}(H)$ is connected, and there is no graph $H \subset H' \subseteq G$ with $\mathcal{R}(H')$ connected.

We can use the direct sum decomposition of the rigidity matroid $\mathcal{R}(G)$ to calculate its rank, which we will denote by $\mathrm{r}(G)$. Indeed, if G has $\mathcal{M}$-

components $H_1, \ldots, H_m$ then the definition of independence in a matroid implies $\mathrm{r}(G) = \sum_{i=1}^{m} \mathrm{r}(H_i)$, where $\mathrm{r}(H_i)$ is $2|V(H_i)|-3$ when H_i is pure and is $2|V(H_i)| - 2$ otherwise. We use this fact to show that $\mathcal{M}$-connectivity is equivalent to redundant rigidity when G is direction irreducible and satisfies the necessary conditions for generic global rigidity described in Chapter 6 (Lemmas 6.2.1 and 6.2.4).

Lemma 8.1.1. *Suppose G is a direction irreducible, 2-connected, direction-balanced mixed graph. Then G is $\mathcal{M}$-connected if and only if G is redundantly rigid.*

Proof. In Chapter 7, Lemma 7.2.3, we proved that redundant rigidity is a necessary condition for a mixed graph to be $\mathcal{M}$-connected. To prove sufficiency, we suppose G is redundantly rigid but not $\mathcal{M}$-connected. Let $H_1, H_2, \ldots, H_m$ be the $\mathcal{M}$-components of G. Let $V_i = V(H_i)$, $X_i = V_i - \bigcup_{j \neq i} V_j$ and $Y_i = V_i - X_i$ for all $1 \leq i \leq m$. Since G is redundantly rigid, every edge of G is contained in some circuit of $\mathcal{R}(G)$ by Lemmas 4.1.4, 7.1.4 and 7.1.5. Hence $|V_i| \geq 3$ for all $1 \leq i \leq m$. Since G is 2-connected, $|Y_i| \geq 2$ for all $1 \leq i \leq m$, and since G is direction-balanced, $|Y_i| \geq 3$ when H_i is length-pure. Since G is direction irreducible, no direction edge of G is contained in a direction-pure circuit. This implies that each of the $\mathcal{M}$-components is either mixed or length-pure. Without loss of generality, we may assume that $H_1, H_2, \ldots, H_\ell$ are length-pure for some $1 \leq \ell \leq t$, and

$H_{\ell+1}, H_{\ell+2}, \ldots, H_m$ are mixed. Then

$$\begin{aligned}
\mathrm{r}(G) &= \sum_{i=1}^{\ell}(2|V_i| - 3) + \sum_{i=\ell+1}^{m}(2|V_i| - 2) \\
&= \sum_{i=1}^{\ell}(2|X_i| + 2|Y_i| - 3) + \sum_{i=\ell+1}^{m}(2|X_i| + 2|Y_i| - 2) \\
&\geq \sum_{i=1}^{m}(2|X_i| + |Y_i|),
\end{aligned}$$

since $|Y_i| \geq 2$ for all $1 \leq i \leq m$, with strict inequality when $1 \leq i \leq \ell$. Since the X_i are all disjoint, we have $\sum_{i=1}^{m} |X_i| = |\bigcup_{i=1}^{m} X_i|$. Also, since each element of Y_i is contained in at least one other Y_j with $j \neq i$, we have $\sum_{i=1}^{m} |Y_i| \geq 2|\bigcup_{i=1}^{t} Y_i|$. Thus

$$\mathrm{r}(G) \geq 2\left(\left|\bigcup_{i=1}^{m} X_i\right| + \left|\bigcup_{i=1}^{m} Y_i\right|\right) = 2|V|.$$

This contradicts the fact that $\mathrm{r}(G) \leq 2|V| - 2$. $\square$

8.1.2 Boundedness and global rigidity

Now we recall some results from [19, 20]. A direction-length framework (G, p) is *bounded* if there exists a real number K such that $\|q(u) - q(v)\| < K$ for all $u, v \in V$ whenever (G, q) is a framework equivalent to (G, p). Our first result shows that the boundedness of (G, p) is equivalent to the rigidity of an *augmented framework*, (G^+, p):

Lemma 8.1.2. *[19, Theorem 5.1] Let (G, p) be a direction-length framework and let G^+ be obtained from G by adding a direction edge parallel to each length edge of G. Then (G, p) is bounded if and only if (G^+, p) is rigid.*

Since rigidity is a generic property (Lemma 4.1.3), Lemma 8.1.2 implies that boundedness is also a generic property. We say that a mixed graph G is *bounded* if some, or equivalently every, generic realisation of G is bounded. Note that this lemma also implies that every rigid mixed graph is bounded.

A mixed graph $G = (V; D, L)$ is *direction-independent* if D is independent in the direction-length rigidity matroid of G, i.e. the rows of the matrix $R(G, p)$ corresponding to D are linearly independent for any generic p. The fact that direction-pure circuits are redundantly direction-rigid allows us to reduce the problem of deciding if a mixed graph is bounded, to the family of direction-independent mixed graphs. The following characterisation of boundedness for direction-independent mixed graphs follows from [19, Theorem 5.1 and Corollary 4.3].

Lemma 8.1.3. *Suppose that $G = (V; D, L)$ is a direction-independent mixed graph. Then G is bounded if and only if G/L has two edge-disjoint spanning trees (where G/L is the graph obtained from G by contracting each edge in L and keeping all multiple copies of direction edges created by this contraction).*

A *bounded component* of G is a maximal bounded subgraph of G. It is shown in [19] that each edge $e \in L$ lies in a bounded component and that the vertex sets of the bounded components partition V. The following lemma is implicit in [19]; for completeness we include a short proof. We will need the well known result of Nash-Williams [27] that the edge set of a graph H can be covered by k forests if and only if every non-empty set X of vertices in H induces at most $k|X| - k$ edges of H.

Lemma 8.1.4. *Suppose $G = (V; D, L)$ is direction-independent and $\mathcal{S}$ is a set of bounded components of G with $|\mathcal{S}| \geq 2$. Then there are at most $2|\mathcal{S}| - 3$ edges of G joining distinct components in $\mathcal{S}$.*

Proof. Suppose on the contrary that there are at least $2|\mathcal{S}|-2$ edges of G that join distinct components in $\mathcal{S}$. Suppose also that $\mathcal{S}$ is minimal with respect to this property (and the condition that $|\mathcal{S}| \geq 2$). Let $G' = (V'; D', L')$ be the subgraph of G spanned by $\cup_{C_i \in \mathcal{S}} C_i$. Let H be a graph with vertex set

$\mathcal{S}$ and exactly $2|\mathcal{S}|-2$ edges, each of which corresponds to a distinct edge of G joining two different components in $\mathcal{S}$. The minimality of $\mathcal{S}$ implies that every non-empty set X of vertices of H induces at most $2|X|-2$ edges of H and hence, by the above mentioned result of Nash-Williams, H can be partitioned into two edge-disjoint spanning trees. By Lemma 8.1.3, for each bounded component $C_i = (V_i; D_i, L_i) \in \mathcal{S}$, C_i/L_i has two edge-disjoint spanning trees. We can combine the edge sets of these trees with the edge sets of the two edge-disjoint spanning trees of H to obtain two edge-disjoint spanning trees in G'/L'. Lemma 8.1.3 now implies that G' is bounded and hence is contained in a single bounded component of G. This contradicts the fact that $|\mathcal{S}| \geq 2$. $\square$

These Lemmas lead to the main result of [20], which we previously stated in Lemma 6.2.3. This result establishes when length-redundancy is a necessary condition for generic global rigidity and takes a step towards understanding when direction-redundancy is necessary.

8.1.3 Substitution

The following subgraph substitution operation is an important tool which we will use throughout this chapter. Suppose $G = (V; D, L)$ is a mixed graph, $U \subseteq V$, $H = G[U]$ is the subgraph of G induced by U, and H' is another mixed graph with vertex set U. Then the *substitution* G' of H by H' in G is obtained from G by deleting all edges of H and adding all edges of H'. We record the following properties.

Lemma 8.1.5. *If G, H and H' are rigid then G' is rigid.*

Proof. The ranks of the rigidity matroids of G and G' are both equal to the rank of the matroid of the graph obtained from G by joining all pairs of vertices of H by both a direction and a length edge. $\square$

Lemma 8.1.6. *Suppose $p : V \to \mathbb{R}^2$ is such that (G, p) and $(H', p|_U)$ are both globally rigid. Then (G', p) is globally rigid.*

Proof. Let (G', q) be an equivalent framework to (G', p). Since $(H', p|_U)$ is globally rigid, $q|_U$ is congruent to $p|_U$. In particular, $(H, q|_U)$ and $(H, p|_U)$ are equivalent. But G and G' agree on all edges not induced by U, so (G, q) and (G, p) are equivalent. Since (G, p) is globally rigid, q and p are congruent. Hence (G', p) is globally rigid. □

8.1.4 Equivalent realisations

Recall the definitions of field extensions, algebraic closures, and framework spaces from Chapter 5. The below lemmas shall be helpful when analysing frameworks which may not be globally rigid.

The proof of the following lemma is the same as that of [20, Theorem 1.3], omitting the part that proves $-p_0 \notin C$, as this is now an assumption.

Lemma 8.1.7. *Suppose (G, p) is a generic direction-length framework, e is a direction edge of G, G is rigid, and $H = G - e$ is bounded and not rigid. Let v_0 be a vertex of G, let p_0 be obtained from p by translating v_0 to the origin, and let C be the connected component of the framework space S_{H,p,v_0} containing p_0. Then C is diffeomorphic to a circle. Furthermore, if $-p_0 \notin C$ then (G, p) is not globally rigid.*

Note that since $-p_0$ is obtained from p_0 by a rotation by $180°$ about the origin, the realisations $-p_0$, p_0 and p are equivalent and congruent.

We also need the following lemma, which implies that every realisation of a rigid mixed graph which is equivalent to a generic realisation is quasi-generic.

Lemma 8.1.8. *[18] Let (G, p) be a quasi-generic realisation of a rigid mixed* *graph G. Suppose that (G, q) is equivalent to (G, p) and that $p(v) = (0, 0) = q(v)$ for some vertex v of G. Then $\overline{\mathbb{Q}(p)} = \overline{\mathbb{Q}(q)}$, so (G, q) is quasi-generic.*

8.2 Realisations of graphs with given direction constraints

Here we give a result concerning the realisation of a graph as a direction-pure framework with given directions for its edges. We need the following concepts, introduced by Whiteley in [39]. A *frame* is a graph $G = (V, E)$ together with a map $q : E \to \mathbb{R}^2$. The *incidence matrix* of the frame (G, q) is an $|E| \times 2|V|$ matrix $I(G, q)$ defined as follows. We first choose an arbitrary reference orientation for the edges of E. Each edge in E corresponds to a row of $I(G, q)$ and each vertex of V to two consecutive columns. The submatrix of $I(G, q)$ with row labeled by $e = uv \in E$ and pairs of columns labeled by $x \in V$ is $q(e)$ if $x = u$, is $-q(e)$ if $x = v$, and is the 2-dimensional zero vector otherwise. It is known (see [39]) that when q is generic, $I(G, q)$ is a linear representation of $M_2(G)$ (the matroid union of two copies of the cycle matroid of G). Thus we may use the characterisation of independence in $M_2(G)$ given by Nash-Williams [27] to determine when $I(G, q)$ has linearly independent rows.

Theorem 8.2.1. *Suppose $G = (V, E)$ is a graph and $q : E \to \mathbb{R}^2$ is generic. Then the rows of $I(G, q)$ are linearly independent if and only if $i_G(X) \leq 2|X| - 2$ for all $\emptyset \neq X \subseteq V$.*

We can use this result to show that a graph $G = (V, E)$ satisfying $i_G(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$ can be realised as a direction-pure framework with a specified algebraically independent set of slopes for its edges, and that this realisation is unique up to translation and dilation when $|E| = 2|V| - 3$. The problem of realising direction-pure frameworks was studied in detail in [36], where they are instead called "direction networks". Note that given any realisation of G, we can always translate a specified vertex z_0 to $(0,0)$ and dilate to arrange any specified distance t between a specified pair of vertices x, y.

Theorem 8.2.2. *Let $G = (V, E)$ be a graph such that $i_G(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$. Let s be an injection from E to $\mathbb{R}$ such that $\{s_e\}_{e \in E}$ is generic. Suppose $x_0, y_0, z_0 \in V$ and $t \neq 0$ is a positive real number. Then there exists an injection $p : V \to \mathbb{R}^2$ such that $\|p(x_0) - p(y_0)\| = t$, $p(z_0) = (0,0)$ and, for all $e = uv \in E$, $p(u) - p(v) \in \langle(1, s_e)\rangle$. Furthermore, if $|E| = 2|V| - 3$, then p is unique up to rotation by $180°$ about the origin.*

Proof. We will construct p as a combination of vectors in the nullspaces of certain frames. First consider a generic frame q on G such that $q(e)$ is a scalar multiple of $(-s_e, 1)$ for every $e \in E$. Then for any p in the nullspace of $I(G, q)$ and $e = uv \in E$ we have $p(u) - p(v) \in \langle(1, s_e)\rangle$. However, p need not be injective. To address this issue, we instead choose a pair of vertices $x, y \in V$, and consider the graph H obtained by adding the edge $f = xy$ to G (which may be parallel to an existing edge). Now let (H, q) be a generic frame such that $q(e)$ is a scalar multiple of $(-s_e, 1)$ for every edge e of G, and $q(f)$ is chosen arbitrarily (subject to the condition that q should be generic). For all $X \subseteq V$ with $|X| \geq 2$, we have $i_H(X) \leq i_G(X) + 1 \leq 2|X| - 2$ by hypothesis. Theorem 8.2.1 now implies that the incidence matrix $I(H, q)$ has linearly independent rows. Thus $\operatorname{rank} I(H, q) = \operatorname{rank} I(G, q|_E) + 1$. Writing Z_H for the null space of $I(H, q)$ and Z_G for the null space of $I(G, q|_E)$, we have $\dim Z_G = \dim Z_H + 1$, so we can choose $p_f \in Z_G \setminus Z_H$. Then we necessarily have $p_f(x) \neq p_f(y)$. Taking a suitable linear combination of the vectors p_f, for all possible new edges $f = xy$, for $x, y \in V$, we may construct a vector p

in Z_G with $p(x) \neq p(y)$ for all $x, y \in V$. Since $p_f(u) - p_f(v) \in \langle (1, s_e) \rangle$ for each f we also have $p(u) - p(v) \in \langle (1, s_e) \rangle$. Furthermore, as noted before the proof, we can translate and dilate to satisfy the other conditions, thus constructing the required map p.

We next show uniqueness when $|E| = 2|V| - 3$. We have $\dim Z_G = 2|V| - \operatorname{rank} I(G, q|_E) = 2|V| - |E| = 3$. Define $p_1, p_2 : V \to \mathbb{R}^2$ by $p_1(v) = (1, 0)$ and $p_2(v) = (0, 1)$ for all $v \in V$. Note that $p_1, p_2 \in Z_G$. Also, p, p_1, p_2 are linearly independent, since $p(z_0) = (0, 0)$, $p_1(z_0) = (1, 0)$ and $p_2(z_0) = (0, 1)$, so $\{p, p_1, p_2\}$ is a basis for Z_G. Now suppose that $p' : V \to \mathbb{R}^2$ has the properties described in the first part of the theorem. Then $p' \in Z_G$ so $p' = ap + bp_1 + cp_2$ for some $a, b, c \in \mathbb{R}$. Since $p'(z_0) = p(z_0) = (0, 0)$ we have $b = c = 0$. Since $\|p'(x_0) - p'(y_0)\| = t = \|p(x_0) - p(y_0)\|$ we have $p' \in \{p, -p\}$. $\square$

Given a graph G which satisfies the hypotheses of Theorem 8.2.2, and a quasi-generic realisation p of G, Lemma 5.2.2 implies that the rigidity map $f_G(p)$ is also generic. This, together with the uniqueness part of Theorem 8.2.2, gives the following two results of Whiteley, and Servatius and Whiteley.

Lemma 8.2.3. *[40] Suppose that (G, p) is a generic direction-pure framework. Then (G, p) is direction globally rigid if and only if it is direction-rigid.*

Lemma 8.2.4. *[35] Suppose that (G, p) is a generic realisation of a mixed graph $G = (V; D, L)$. If G is rigid and $|L| = 1$, then (G, p) is globally rigid.*

8.3 Direction reductions

In this section we prove the first main result of this chapter, Theorem 8.3.3, namely, that we can reduce the problem of characterising global rigidity to the class of irreducible graphs. To do this, we deal with the two direction reduction operations, (D1) and (D2), separately in the following two lemmas:

Lemma 8.3.1. *Suppose (G, p) is a generic realisation of a mixed graph $G = (V; D, L)$ and that $e = uv \in D$ belongs to a direction-pure circuit $H = (U; F, \emptyset)$ of G. Then (G, p) is globally rigid if and only if $(G - e, p)$ is globally rigid.*

Proof. If $(G - e, p)$ is globally rigid then (G, p) is clearly globally rigid. Conversely, suppose that (G, p) is globally rigid and $(G-e, q)$ is equivalent to $(G-e, p)$. Since H is a direction-pure circuit, both $(H, p|_U)$ and $(H-e, p|_U)$ are direction-rigid. Hence $(H - e, p|_U)$ is globally direction-rigid by Lemma 8.2.3. Thus $q(u)-q(v)$ is a scalar multiple of $p(u)-p(v)$, and hence (G, q) is equivalent to (G, p). Since (G, p) is globally rigid, q is congruent to p. This shows that $(G - e, p)$ is globally rigid. □

Lemma 8.3.2. *Let (G, p) be a quasi-generic realisation of a rigid mixed graph $G = (V; D, L)$. Suppose that G has a proper induced subgraph $H = (U; F, L)$ such that the graph G/H obtained by contracting H to a single vertex (deleting all edges contained in H and keeping all other edges, possibly as parallel edges) has only direction edges and is the union of two edge-disjoint spanning trees. Then (G, p) is globally rigid if and only if $(H, p|_H)$ is globally rigid.*

Proof. First suppose that $(H, p|_H)$ is globally rigid. Let G' be constructed from G by substituting H by a minimally rigid graph $H' = (U; F', \{l\})$ with exactly one length edge, l. Then G' is rigid by Lemma 8.1.5. Since G' is rigid and has exactly one length edge, (G', p) is globally rigid by Lemma 8.2.4. Thus (G, p) is globally rigid by Lemma 8.1.6.

Conversely, suppose that $(H, p|_H)$ is not globally rigid. Then there exists an equivalent but non-congruent framework $(H, \tilde{q})$. Without loss of generality we may suppose that $p(u) = (0,0) = \tilde{q}(u)$ for some $u \in V(H)$. Let $D^* = D - F$ be the set of edges of G/H and m be the number of vertices of G/H. Then $|D^*| = 2m - 2$, as G/H is the union of two edge-disjoint spanning trees. Since G is rigid we have

$$2|V| - 2 = \mathrm{r}(G) \leq |D^*| + \mathrm{r}(H) \leq 2m - 2 + 2|V(H)| - 2 = 2|V| - 2.$$

Thus equality must hold throughout. In particular, $\mathrm{r}(H) = 2|V(H))| - 2$, so H is rigid.

We again consider the rigid mixed graph $G' = (V; D', L')$ with exactly one length edge defined in the first paragraph of the proof. Since G' has $|D^*| + 2|V(H)| - 2 = 2|V| - 2$ edges, it is minimally rigid. We will construct a framework (G, q) which is equivalent to (G, p) and has $q|_H = \tilde{q}$ by applying Theorem 8.2.2 to G'.

Define $s : D' \cup L' \to \mathbb{R}$ by $s(e) = s_{\tilde{q}}(e)$ for $e \in F'$, $s(e) = l_{\tilde{q}}(e)$ for $e \in L'$, and $s(e) = s_p(e)$ for $e \in D^*$. We will use Theorem 8.2.2 to construct a framework (G', q) such that $s_q(e) = s(e)$ for all edges e of G'. To do this, we first need to show that $s|_{D'}$ is generic. We will prove the stronger result that s is generic by showing that $\mathrm{td}[\overline{\mathbb{Q}(s)} : \mathbb{Q}] = |D'| + |L'| = 2|V| - 2$. We have $\mathrm{td}[\overline{\mathbb{Q}(p)} : \mathbb{Q}] = 2|V| - 2$, as p is quasi-generic and $p(u) = (0,0)$, so it suffices to prove that $\overline{\mathbb{Q}(s)} = \overline{\mathbb{Q}(p)}$. Since G is rigid, Corollary 5.2.5 gives $\overline{\mathbb{Q}(f_G(p))} = \overline{\mathbb{Q}(p)}$. Also, s is obtained from $f_G(p)$ by replacing the values $f_H(p|_U)$ by the values $f_{H'}(\tilde{q})$, so we need to show that these generate the same algebraic closure over $\mathbb{Q}$. Since $(H, \tilde{q})$ is equivalent to $(H, p|_U)$, Lemma 8.1.8 gives $\overline{\mathbb{Q}(\tilde{q})} = \overline{\mathbb{Q}(p|_U)}$. Since $p|_U$ is quasi-generic, it follows that $\tilde{q}$ is quasi-generic. Then, since H and H' are rigid, two applications of Corollary 5.2.5, give $\overline{\mathbb{Q}(f_H(p|_U))} = \overline{\mathbb{Q}(p|_U)}$ and $\overline{\mathbb{Q}(f_{H'}(\tilde{q}))} = \overline{\mathbb{Q}(\tilde{q})}$. Putting these three equalities together gives

$$\overline{\mathbb{Q}(f_{H'}(\tilde{q}))} = \overline{\mathbb{Q}(\tilde{q})} = \overline{\mathbb{Q}(p|_U)} = \overline{\mathbb{Q}(f_H(p|_U))},$$

which is what we needed to prove $\overline{\mathbb{Q}(s)} = \overline{\mathbb{Q}(p)}$. Therefore s is generic. Now we can apply Theorem 8.2.2, with x_0y_0 equal to the unique length edge of G', to obtain a realisation (G', q) with $f_{G'}(q) = s$. By construction $(H', q|_U)$ is equivalent to $(H', \tilde{q})$. But H' is globally rigid by Lemma 8.2.4, so $q|_U$ is congruent to $\tilde{q}$. Hence we can apply a translation, and possibly a dilation by -1, to obtain $q|_U = \tilde{q}$.

Since $(H, \tilde{q})$ is equivalent to $(H, p|_H)$ and $s_q(e) = s(e) = s_p(e)$ for all $e \in D^*$, (G, q) is equivalent to (G, p) and satisfies $q|_U = \tilde{q}$. Since $(H, \tilde{q})$ is not congruent to $(H, p|_U)$, (G, q) is not congruent to (G, p). Thus (G, p) is not globally rigid. □

Our first main result in this chapter follows immediately from Lemmas 8.3.1 and 8.3.2. Namely

Theorem 8.3.3. *Suppose (G, p) is a generic direction-length framework and G admits a direction reduction to a subgraph H. Then (G, p) is globally rigid if and only if $(H, p|_H)$ is globally rigid.*

8.4 Direction irreducible mixed graphs

Theorem 8.3.3 enables us to reduce the problem of characterising globally rigid generic direction-length frameworks to the case when the underlying graph is direction irreducible. In this section we prove a structural lemma for direction irreducible mixed graphs which have a globally rigid generic realisation even though they are not redundantly rigid. This will be used in the next section to construct two equivalent but non-congruent generic realisations of a mixed graph which is direction irreducible but not redundantly rigid.

Lemma 8.4.1. *Let $G = (V; D, L)$ be a direction irreducible mixed graph which has $|L| \geq 2$ and is not redundantly rigid. Suppose that (G, p) is a globally rigid generic realisation of G. Then*

(i) $G - e$ *is bounded for all* $e \in D$*,*

(ii) $\mathrm{r}(G - e) = \mathrm{r}(G) - 1$ *for all* $e \in D$*, and*

(iii) every length edge of G *belongs to a length-pure circuit of* G*.*

Proof. (i). First note that G is direction-independent, since G is direction irreducible. Now suppose for a contradiction that $G - e$ is not bounded for some $e \in D$. We will show that G has a direction reduction. Let $H_1, H_2, \ldots, H_m$ be the bounded components of $G - e$. Then each length edge of G is contained in one of the subgraphs H_i. Let $D^* \subseteq D$ be the set of all edges of G joining distinct subgraphs H_i, and H be the graph obtained from G by contracting each H_i to a single vertex. Since G is rigid, G is bounded. Since G is direction-independent, Lemma 8.1.3 now implies that the graph G/L obtained from G by contracting each length edge has two edge-disjoint spanning trees. Since H can be obtained from G/L by contracting a (possibly empty) set of direction edges, H also has two edge-disjoint spanning trees. In particular, $|D^*| \geq 2m - 2$. On the other hand, Lemma 8.1.4 implies that $|D^* - e| \leq 2m - 3$. Thus $e \in D^*$, $|D^*| = 2m - 2$, and H is the union of two edge-disjoint spanning trees. Since G is rigid, Corollary 4.2.1 implies

$$2|V| - 2 = \mathrm{r}(G) \leq |D^*| + \sum_{i=1}^{m} \mathrm{r}(H_i) \leq 2m - 2 + \sum_{i=1}^{m} (2|V(H_i)| - 2) = 2|V| - 2.$$

Thus equality must hold throughout. In particular, $\mathrm{r}(H_i) = 2|V(H_i)| - 2$ for each i, so each subgraph H_i is rigid.

Let $G' = (V; D', L')$ be obtained from G by substituting each non-trivial subgraph H_i by a minimally rigid graph H_i' with exactly one length edge. Each framework $(H_i', p|_{H_i'})$ is globally rigid by Lemma 8.2.4. Thus repeated applications of Lemma 8.1.6 imply that (G', p) is globally rigid. On the other hand, $|D'| + |L'| = |D^*| + \sum_{i=1}^{m} \mathrm{r}(H_i) = 2m - 2 + \sum_{i=1}^{m}(2|V(H_i)| - 2) = 2|V| - 2$, so G' is minimally rigid. Theorem 6.2.3(i) now implies that G' has exactly one length edge. Since H_i' contains a length edge whenever H_i is non-trivial, $G - e$ has exactly one non-trivial bounded component, H_1 say. Since $G/H_1 = H$ and H is the union of two edge-disjoint spanning trees, G is direction reducible to H_1. This contradicts the hypothesis that G is direction irreducible.

(ii). Suppose that $\mathrm{r}(G - e) = \mathrm{r}(G)$ for some $e \in D$. Then e is contained in a circuit H of G. Since G is direction-independent, H must be a mixed circuit. Since G is not redundantly rigid, there exists some $f \in E(G)$ such that $G - f$ is not rigid. By Theorem 6.2.3(i), every such $f \in D$. Clearly f is not an edge of H and hence H is a non-trivial rigid subgraph of $G - f$. Theorem 6.2.3(ii) now implies that $G - f$ is unbounded, contradicting (i).

(iii). Choose $e \in L$. Then e belongs to a circuit H of G by Theorem 6.2.3(i). By (ii), H cannot be a mixed circuit. Hence H is length-pure. □

8.5 Characterisation of globally rigid graphs

Every generic realisation of a direction irreducible, 2-connected, direction-balanced, redundantly rigid graph is globally rigid by Theorem 7.6.2 and Lemma 8.1.1. To prove Theorem 8.5.4, we first show that these conditions are also necessary for direction irreducible graphs $G = (V; D, L)$ which are globally rigid for all generic realisations and have $|L| \geq 2$.

If G is such a graph, then G is 2-connected and direction-balanced by Lemmas 6.2.1 and 6.2.4 respectively. We complete the proof by applying Theorem 8.5.1 below to deduce that G must also be redundantly rigid. The proof idea is to show that if G is not redundantly rigid, then for any given generic realisation (G, p), we can construct a sequence of generic realisations $q_0, q_1, \ldots, q_t$ such that $t \leq |D|$ and (G, q_t) is not globally rigid. We construct this sequence from (G, p) by first reflecting (G, p) in the x-axis to obtain (G, q_0), and then recursively "correcting" the changed direction constraints back to their original value in (G, p). Every time we "correct" a direction constraint, we obtain a new realisation in our sequence.

Theorem 8.5.1. *Let $G = (V; D, L)$ be a direction irreducible mixed graph with $|L| \geq 2$ such that G is not redundantly rigid. Then some generic realisation of G is not globally rigid.*

Proof. We proceed by contradiction. Assume that all generic realisations of G are globally rigid. By Lemma 8.4.1(ii) and (iii), every length edge of G is contained in a length-pure circuit in the rigidity matroid of G, and no direction edge of G is contained in any circuit. Let $D = \{d_0, d_1, \ldots, d_k\}$, let $G_1 = (V_1; \emptyset, L_1)$ be a non-trivial $\mathcal{M}$-component of G and let $v_0 \in V_1$.

Let (G, p) be a quasi-generic realisation of G with $p(v_0) = (0, 0)$ and let (G, q_0) be the quasi-generic realisation obtained by reflecting (G, p) in the x-axis. Then $(G - D, p)$ is equivalent to $(G - D, q_0)$. In addition we have $s_{q_0}(d_i) = -s_p(d_i)$ for all $d_i \in D$, so (G, p) and (G, q_0) are not equivalent.

Claim 8.5.2. *For all $j \in \{0, 1, \ldots, k+1\}$ there exists a quasi-generic framework (G, q_j) with $q_j(v_0) = (0, 0)$, rigidity map $f_G(q_j) = (h_{q_j}(e))_{e \in E}$ given* *by*

$$h_{q_j}(e) = \begin{cases} s_{q_0}(e) & \textit{when } e \in \{d_j, d_{j+1}, \ldots, d_k\} \\ h_p(e) & \textit{otherwise,} \end{cases}$$

and with the property that that $(G_1, q_j|_{V_1})$ *can be obtained from* $(G_1, q_0|_{V_1})$ *by a rotation about the origin.*

Proof. We proceed by induction on j. If $j = 0$ then the claim holds trivially for (G, q_0). Hence suppose that the required framework (G, q_j) exists for some $0 \leq j < k + 1$. The quasi-generic framework $(G - d_j, q_j)$ is bounded but not rigid by Lemma 8.4.1(i) and (ii) (since boundedness and rigidity are generic properties). Since (G, q_j) is globally rigid by assumption, Lemma 8.1.7 implies that we can continuously move $(G - d_j, q_j)$ to form $(G - d_j, -q_j)$ whilst keeping v_0 fixed at the origin and maintaining all edge constraints. During this motion, the direction of the missing edge $d_j = u_j v_j$ changes continuously from $q_j(v_j) - q_j(u_j)$ to $-(q_j(v_j) - q_j(u_j))$, a rotation by 180°. So at some point in this motion we must pass through a realisation $(G - d_j, q_{j+1})$ at which the slope of this missing edge is $s_p(d_j)$. We can now add the edge d_j back to this realisation to obtain the desired framework (G, q_{j+1}). Note that since G_1 is a length rigid subgraph of $G - d_j$ and the motion of $(G - d_j, q_j)$ is continuous and keeps v_0 fixed at the origin, $(G_1, q_{j+1}|_{V_1})$ can be obtained from $(G_1, q_j|_{V_1})$ by a rotation about the origin.

It remains to show that (G, q_{j+1}) is quasi-generic. Let H be a minimally rigid spanning subgraph of G. Since $h_{q_{j+1}}(e) = \pm h_p(e)$ for all $e \in E(G)$ we have $\mathbb{Q}(f_H(q_{j+1})) = \mathbb{Q}(f_H(p))$. Since $f_H(p)$ is generic by Lemma 5.2.2, Lemma 5.2.4 implies that

$$\mathrm{td}[\overline{\mathbb{Q}(q_{j+1})} : \mathbb{Q}] = \mathrm{td}[\overline{\mathbb{Q}(f_H(q_{j+1}))} : \mathbb{Q}] = \mathrm{td}[\overline{\mathbb{Q}(f_H(p))} : \mathbb{Q}] = 2|V| - 2.$$

We can now use Lemma 5.2.1 to deduce that (H, q_{j+1}), and hence also

(G, q_{j+1}), are quasi-generic. □

Applying Claim 8.5.2 with $j = k+1$, we obtain a quasi-generic realisation q_{k+1} of G which is equivalent to (G, p), has $q_{k+1}(v_0) = (0, 0)$, and is such that $(G_1, q_{k+1}|_{V_1})$ can be obtained from $(G_1, q_0|_{V_1})$ by a rotation about the origin. Since q_0 was obtained from p by reflecting G across the x-axis, we have

$$q_{k+1}(v) = RZp(v) \quad \text{for all } v \in V_1$$

where R and Z are the 2×2 matrices representing this rotation and reflection. Since $(G_1, p|_{V_1})$ is a quasi-generic framework with at least four vertices and RZ acts on $\mathbb{R}^2$ as a reflection in some line through the origin, we have $q_{k+1}(v) \neq \pm p(v)$ for some $v \in V_1$. Hence $q_{k+1}|_{V_1}$ is not congruent to $p|_{V_1}$, and q_{k+1} is not congruent to p. This implies that (G, p) is not globally rigid and contradicts our initial assumption that all generic realisations of G are globally rigid. □

Theorem 8.5.1 and our above discussion immediately implies the following result:

Corollary 8.5.3. *Let $G = (V; D, L)$ be a direction irreducible mixed graph with $|L| \geq 2$. Then G is globally rigid for all generic realisations if and only if G is 2-connected, direction-balanced and redundantly rigid.*

Lemma 8.1.1 implies that we can replace redundant rigidity with $\mathcal{M}$-connectivity in the above statement. Since 2-connectivity is a property of $\mathcal{M}$-connected graphs (see Lemma 7.2.1), we can remove this condition from our statement. This gives us the result sought:

Theorem 8.5.4. *Let $G = (V; D, L)$ be a direction irreducible mixed graph with $|L| \geq 2$. Then G is globally rigid for all generic realisations if and only if G is direction-balanced and $\mathcal{M}$-connected.*

With this result, we can finally prove our characterisation of global rigidity for direction-length graphs:

Theorem 8.5.5. *A direction-length graph $G = (V; D, L)$ is globally rigid for all generic realisations if and only if G is rigid, and either $|L| = 1$ or G has a direction-balanced, $\mathcal{M}$-connected mixed subgraph which contains all edges in L.*

Proof. We first prove the forwards direction. Suppose G is globally rigid for all generic realisations. Then Lemma 6.2.1 implies G is rigid. If $|L| = 1$, we are done. So suppose $|L| \geq 2$. If G is direction irreducible, then Theorem 8.5.4 implies G is direction-balanced and $\mathcal{M}$-connected; hence G is the subgraph required. The only remaining case is when $|L| \geq 2$ and G admits a direction reduction. In this case, by definition, G has a direction irreducible subgraph H whose edge set contains L. Since G is globally rigid, Theorems 8.3.3 and 8.5.4 imply H is direction-balanced, mixed and $\mathcal{M}$-connected. Thus H is the subgraph sought, completing the proof of the forwards direction.

We now consider the backwards direction. Suppose G is rigid. If $|L| = 1$, then G is globally rigid by Lemma 6.2.2. So suppose $|L| \geq 2$, and G has a direction-balanced, $\mathcal{M}$-connected mixed subgraph containing all edges in L. Let H be a maximal subgraph of G satisfying these conditions. There are two cases to consider based on whether or not G has a direction reduction:

Case 1. G is direction irreducible.

Suppose $H \neq G$. Since $L \subseteq E(H)$, this implies $E(G) - E(H) \neq \emptyset$ is direction-pure. Let $D^* = E(G) - E(H)$, and let $d \in D^*$. Since G is direction irreducible, d is contained in no direction-pure circuit. Further, since H is maximal and $L \subseteq E(H)$, d is contained in no mixed circuits. Thus $\mathrm{r}(G) = \mathrm{r}(H) + |D^*|$. By our hypothesis and Lemma 7.2.3 respectively, G

and H are rigid. Hence

$$|D^*| = \mathrm{r}(G) - \mathrm{r}(H) = 2|V(G)| - 2|V(H)|.$$

This implies that the graph $G/H = (U; D^*, \emptyset)$ obtained from G by contracting H to a single vertex, and deleting all edges H; has $\mathrm{r}(G/H) = |D^*| = 2|U| - 2$. By [27], the edge set of G/H consists of two spanning trees. This contradicts condition (D2) of a direction irreducible graph. Hence $H = G$. Thus G is direction-balanced and $\mathcal{M}$-connected, and so, by Theorem 8.5.4, G is globally rigid.

Case 2. G admits a direction reduction.
Since G is rigid, it has a spanning, minimally rigid subgraph G'. Suppose $e \in E(G) - (E(G') \cup E(H))$. Then e is a direction edge. Since $e \notin E(G')$, e is contained in some circuit $C \subset E(G)$. If C is mixed, then $C \cap E(H) \neq \emptyset$ and $C - E(H)$ is direction-pure, which implies $H + C$ is $\mathcal{M}$-connected and direction-balanced, thus contradicting the fact H is maximal. Hence C is direction-pure, and e can be removed by (D1).

We repeat this argument for all such edges, until the graph remaining is $H \cup G'$. Since H is rigid and G' is minimally rigid, $|E(G') - E(H)| = 2|V(G) - V(H)|$. Thus, by [27], G'/H consists of two direction-pure spanning trees. Hence $G' \cup H$ can be reduced to H by (D2). So G can be reduced to H by a sequence of direction reductions. Since H is direction irreducible, it is globally rigid by the argument in Case 1. Thus, by Theorem 8.3.3, G is globally rigid. □

8.6 Closing remarks

We conjecture that the properties in Theorem 8.5.5 actually characterise global rigidity for all generic direction-length frameworks:

Conjecture 8.6.1. *A generic direction-length framework (G, p) on underlying graph $G = (V; D, L)$ is globally rigid if and only if G is rigid, and either $|L| = 1$ or G has a direction-balanced, $\mathcal{M}$-connected mixed subgraph which contains all edges in L.*

In other words, we conjecture that both global rigidity and global flexibility are generic properties of direction-length graphs. From Theorems 7.6.2 and 6.2.2, we know global rigidity is a generic property for graphs which are $\mathcal{M}$-connected or have exactly one length edge. Similarly, Lemmas 6.2.1 and 6.2.4 imply that global flexibility is a generic property of graphs which are not 2-connected, not rigid or not direction-balanced (as generic realisations of such graphs are never globally rigid). So, by Theorem 8.3.3, the only class of graphs for which we do not know whether global rigidity (or global flexibility) is a generic property, are the direction irreducible graphs which are not redundantly rigid. In other words, if we could adapt our proof of Theorem 8.5.1 to show that no generic realisation of these graphs is globally rigid, then we would have succeeded in proving that such graphs are globally flexible, thus proving our conjecture.

In the remainder of this section, we consider the special case where the set of length edges induces a length-rigid subgraph, and show that such graphs are globally flexible. First we need the following technical Lemma:

Lemma 8.6.2. *Let $G = (V; D, L)$ be a rigid mixed graph, $H = (U; \emptyset, L)$ be the length-pure subgraph induced by L, and $u \in U$. Suppose that H is length-rigid,* $\mathrm{r}(G - e) = \mathrm{r}(G) - 1$ *for all $e \in D$, and $G - e_0$ is bounded for some $e_0 \in D$. Let (G, p) be a quasi-generic framework with $p(u) = (0, 0)$ and C be the connected component of the configuration space $S_{G-e_0,p,u}$ which contains p. Then $-p \in C$.*

Proof. The idea is to rotate $(H, p|_U)$ by θ radians about $p(u) = (0, 0)$ and use Theorem 8.2.2 to show that, for almost all values of θ, we can extend the resulting framework (H, q_θ) to a framework $(G - e_0, p_\theta)$ which is equivalent to $(G - e_0, p)$. To apply Theorem 8.2.2, we construct G' from G by substituting a minimally rigid graph H' with exactly one length edge for H and then show that the required set of edge slopes for $(G' - e_0, p_\theta)$ is algebraically independent over $\mathbb{Q}$.

Let $H' = (U; D', L')$ be a minimally rigid graph on the same vertex set as H with exactly one length edge and let G' be obtained from G by replacing H by H'. We first show that $G' - e_0$ is minimally rigid. Since G is rigid, H is length-rigid and $\mathrm{r}(G - e) = \mathrm{r}(G) - 1$ for all $e \in D$, we have $|D| = 2|V| - 2 - (2|U| - 3)$ and hence $|D - e_0| = 2|V| - 2|U|$. Since H' has $2|U| - 2$ edges, this implies that G' has $2|V| - 2$ edges. It remains to show that $G' - e_0$ is rigid. Since $G - e_0$ is bounded, $(G - e_0)^+$ is rigid by Lemma 8.1.2. Since $G' - e_0$ can be obtained from $(G - e_0)^+$ by substituting H^+ with H', it is rigid by Lemma 8.1.5. Therefore $G' - e_0$ is minimally rigid.

For each $\theta \in [0, 2\pi)$ let $q_\theta : U \to \mathbb{R}^2$ be the configuration obtained by an anticlockwise rotation of $p|_U$ through θ radians about $(0, 0)$. Write $B = \{q_\theta \;:\; \theta \in [0, 2\pi)\}$, and let B^* be the set of all configurations $q_\theta \in B$ such that the set of slopes $\{s_p(e)\}_{e \in D - e_0} \cup \{s_{q_\theta}(e)\}_{e \in D'}$ is defined and is algebraically independent over $\mathbb{Q}$. We claim that B^* is a dense subset of B. First we note that $q_0 = p|_U \in B^*$, as $G' - e_0$ is independent, so Lemma 5.2.2 implies that $f_{G'}(p)$ is generic. To see the effect of a rotation by θ, consider an edge $e = v_1 v_2$ in D' and let (x_1, y_1) and (x_2, y_2) be the co-ordinates of v_1 and v_2 in p. Co-ordinates in q_θ are obtained by applying the transformation $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, so we have

$$s_{q_0}(e) = s_p(e) = \frac{y_1 - y_2}{x_1 - x_2} \text{ and } s_{q_\theta}(e) = \frac{(x_1 - x_2)\sin\theta + (y_1 - y_2)\cos\theta}{(x_1 - x_2)\cos\theta - (y_1 - y_2)\sin\theta}, \text{ so}$$

$$s_{q_\theta}(e) = r(s_p(e), \tan\theta), \text{ where } r(s,t) = \frac{t+s}{1-st}.$$

Consider any non-zero polynomial z with rational coefficients and $|D - e_0| + |D'|$ variables, labeled as $\mathbf{s} = (s_e : e \in D - e_0)$ and $\mathbf{s}' = (s'_e : e \in D')$. Substituting $\mathbf{s} = (s_p(e) : e \in D - e_0)$ and $\mathbf{s}' = (s_{q_\theta}(e) : e \in D')$ into z gives a rational function z^* in $(s_p(e) : e \in (D - e_0) \cup D')$ and $\tan\theta$. Note that z^* is not identically zero, as it is non-zero when $\theta = 0$ by the hypothesis that p is quasi-generic. Thus there are only a finite number of values of $\theta \in [0, 2\pi)$ for which z^* is zero. Furthermore, the number of such polynomials z is countable, so there are only countably many θ for which $\{s_p(e)\}_{e \in D-e_0} \cup \{s_{q_\theta}(e)\}_{e \in D'}$ is algebraically dependent over $\mathbb{Q}$. Thus $B - B^*$ is countable, so in particular B^* is a dense subset of B.

For each $q_\theta \in B^*$, we can apply Lemma 8.2.2 to obtain a configuration $p_\theta : V \to \mathbb{R}^2$ such that $l_{p_\theta}(e_1) = l_p(e_1)$, where e_1 is the unique length edge of G', $p_\theta(u) = (0,0)$, $s_{p_\theta}(e) = s_p(e)$ for $e \in D - e_0$ and $s_{p_\theta}(e) = s_{q_\theta}(e)$ for $e \in D'$. Since (H', q_θ) is globally rigid we have $p_\theta|_U \in \{q_\theta, -q_\theta\}$. Hence $(G - e_0, p_\theta)$ is equivalent to $(G - e_0, p)$. Replacing p_θ by $-p_\theta$ if necessary, we may suppose that $p_\theta|_U = q_\theta$; this determines p_θ uniquely by Lemma 8.2.2. Now note that the defining conditions of p_θ are polynomial equations with coefficients that are continuous functions of θ, except at a finite set of exceptional values for θ corresponding to vertical edges in p_θ. Since B^* is a dense subset of B, it follows that $\{p_\theta : q_\theta \in B^*\}$ all belong to the same component of the framework space $S_{G-e_0,p,u}$, which is C, since $q_0 = p|_U \in B^*$. Now note that $q_\pi \in B^*$, as $s_{q_\pi}(e) = -s_p(e)$ for $e \in D'$, so $\{s_p(e)\}_{e \in D-e_0} \cup \{s_{q_\pi}(e)\}_{e \in D'}$ generates the same extension of $\mathbb{Q}$ as $\{s_p(e)\}_{e \in D-e_0} \cup \{s_p(e)\}_{e \in D'}$. Therefore $p_\pi \in C$. Since $p_\pi = -p$ by the uniqueness property noted above, $-p \in C$. □

Theorem 8.6.3. *Let (G, p) be a globally rigid generic realisation of a di-*

rection irreducible mixed graph $G = (V; D, L)$ with at least two length edges. Suppose that $G[L]$ is length-rigid. Then (G, p) is redundantly rigid.

Proof. We proceed by contradiction. Suppose G is not redundantly rigid. Since G is direction irreducible, Lemma 8.4.1, implies that $G - e$ is bounded and $\mathrm{r}(G - e) = \mathrm{r}(G) - 1$ for all $e \in D$.

Let H denote $G[L] = (U; \emptyset, L)$. Assume there exists some $d \in D$ such that d is an edge in $G[U]$. Since H is length-rigid, this implies $H + d$ is rigid. If d is the only direction edge in G, then reflecting (G, p) across the line through d would give an equivalent but non-congruent realisation of G, contradicting the fact G is globally rigid. Hence there exists some $d' \in D - \{d\}$, and the graph $G - d'$ contains the rigid subgraph $H + d$. Lemma 6.2.3(ii) then implies that $G - d'$ is either rigid or unbounded, which contradicts Lemma 8.4.1 (ii) and (i) respectively. Hence our initial assumption was wrong, and $G[U]$ contains no direction edges. Or equivalently, $H = G[L] = G[U]$ is an induced subgraph of G.

Choose $u \in U$ and $e_0 \in D$. By translation we can replace the assumption that (G, p) is generic by the assumption that (G, p) is quasi-generic and $p(u) = (0, 0)$. Let $H' = (U; D', L')$ be a minimally rigid graph on the same vertex set as H with exactly one length edge, f, and let G' be obtained from G by substituting H by H'. We can show that G' is minimally rigid as in the proof of Lemma 8.6.2.

Let (H', q) be obtained from $(H', p|_U)$ by reflection in the x-axis. Then $s_q(e) = -s_p(e)$ for all $e \in D'$. Since $\{s_p(e)\}_{e \in D - e_0} \cup \{s_p(e)\}_{e \in D'}$ is generic, $\{s_p(e)\}_{e \in D - e_0} \cup \{s_q(e)\}_{e \in D'}$ is generic. Thus we can apply Lemma 8.2.2 to obtain $p' : V \to \mathbb{R}^2$ such that $l_{p'}(f) = l_p(f)$, $p'(v) = (0, 0)$, $s_{p'}(e) = s_p(e)$ for $e \in D - e_0$ and $s_{p'}(e) = s_q(e)$ for $e \in D'$. We have $\mathbb{Q}(f_{G' - e_0}(p')) = \mathbb{Q}(f_{G' - e_0}(p))$, so p' is quasi-generic by Lemma 5.2.4. Now consider $(G - e_0, p')$

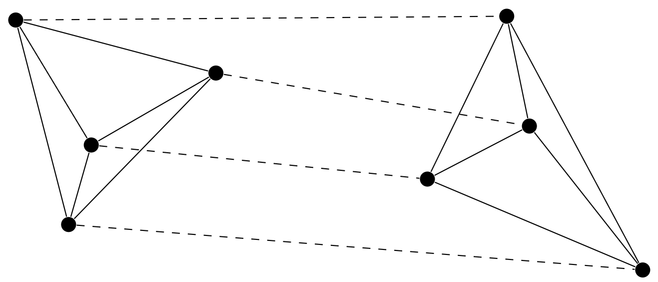


Figure 8.1: A direction irreducible mixed graph which is not redundantly rigid. We know it has a globally flexible generic realisation by Theorem 8.5.1, but do not know whether it has a globally rigid generic realisation.

and let C be the connected component of the framework space $S_{G-e_0,p',u}$ which contains p'. By Lemma 8.6.2, we have $-p' \in C$.

The remainder of the proof is similar to that of [20, Theorem 1.3]. Let $e_0 = u_0v_0$. For any $p'' \in C$ let $F(p'') = (p''(u_0) - p''(v_0))/\|p''(u_0) - p''(v_0)\|$ be the unit vector in the direction of $p''(u_0) - p''(v_0)$; this is well-defined since we never have $p''(u_0) = p''(v_0)$ by [18, Lemma 3.4]. Consider a path P in C from p' to $-p'$. Then $F(p'')$ changes continuously from $F(p')$ to $-F(p')$ along P. By the intermediate value theorem there must be some $p'' \in P$ such that $F(p'')$ is either $F(p)$ or $-F(p)$. Then (G, p'') is equivalent to (G, p). On the other hand p'' is not congruent to p since $p''|_U$ is obtained from $p|_U$ by a reflection (as well as a translation and a rotation). It follows that (G, p) is not globally rigid. □

Theorem 8.6.3 shows that if a graph satisfies the hypotheses of Theorem 8.5.1, and also satisfies the additional property that its length edges induce a length-rigid subgraph, then generic realisations of this graph are never globally rigid, or equivalently, the graph is globally flexible. This supports our conjecture that no graph satisfying the hypotheses of Theorem 8.5.1 has a globally rigid generic realisation.

The smallest graph for which we do not know whether global rigidity is a generic property is depicted in Figure 8.1. Conjecture 8.6.1 requires that generic realisations of this graph are never globally rigid.

Part III

Symmetry

Chapter 9

Introduction to symmetry-forced rigidity

In the final part of this thesis, we consider *symmetric direction-length frameworks* (G, p) in $\mathbb{R}^2$. Intuitively, these are frameworks where the positions of the vertices given by p, and the resulting locations and types of the edges in G, result in a symmetric drawing of the graph in the plane. See Figure 9.1. This is defined more formally in Chapter 10. For now, observe that since the coordinates of the vertices must obey our chosen symmetry, p cannot be generic. This means our characterisations of rigidity and global rigidity for generic frameworks may not apply. In fact, it is easy to construct symmetric frameworks which do not satisfy these generic results. For example, the graph in Figure 9.2 is rigid for generic realisations, but is flexible in this symmetric realisation.

The symmetries we shall consider are the planar point groups. Namely,

(i) the *trivial group*, $\mathcal{I} = \{I\}$, where I denotes the identity;

(ii) the *reflection group*, $\mathcal{C}_s = \langle \sigma \rangle = \{I, \sigma\}$, generated by the single reflection σ across the y-axis;

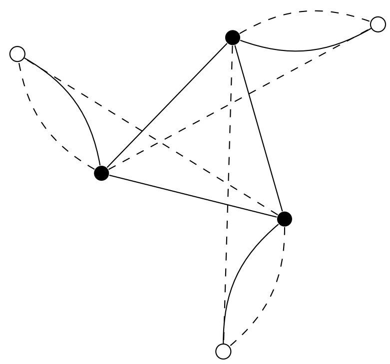


Figure 9.1: A $\mathcal{C}_3$-symmetric direction-length framework. All edges should be straight lines, however we have drawn parallel edges as curves to make them easier to see. We can always do this in such a way that it preserves the symmetry.

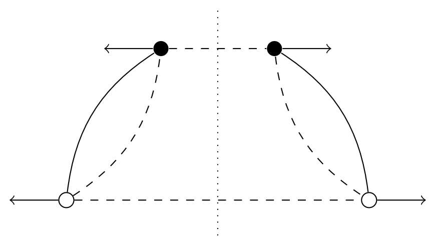


Figure 9.2: A $\mathcal{C}_s$-symmetric direction-length framework. The mirror is denoted by a dotted line. Since the two direction edges that cross the mirror are parallel, we can slide the vertices along the corresponding rays to obtain a non-trivial motion.

(iii) the *k-fold rotation group*, $\mathcal{C}_k = \langle c_k \rangle = \{c_k, c_k^2, \ldots, c_k^{k-1}, c_k^k = I\}$ where $k \geq 2$, and c_k^i denotes a rotation about the origin by $2\pi i/k$; and

(iv) the *dihedral groups*, $\mathcal{D}_k = \langle \sigma_1, c_k \rangle = \{\sigma_1, c_k, \sigma_2, c_k^2, \ldots, \sigma_k, c_k^k = I\}$ where $k \geq 2$. The element σ_i denotes reflection across the line through the origin at angle $\pi i/k$.

Technically, $\mathcal{C}_s = \mathcal{D}_1$. However, $\mathcal{C}_s$ is the only dihedral group which contains no non-trivial rotations. As a result, it behaves differently to the other dihedral groups. We use different notation to emphasise this distinction.

We say a framework is Γ-*symmetric* if it is symmetric under point group Γ. Characterising rigidity is harder for symmetric direction-length frameworks than in the generic case, so before asking whether a framework has any non-trivial motions, we first ask whether it has any non-trivial motions which preserve the symmetry. If not, we say it is *symmetry-forced rigid*. A (trivial) motion is called a *(trivial) symmetric motion* when it preserves the symmetry of the framework. When all the symmetric infinitesimal motions of a framework are trivial, we say the framework is *symmetry-forced infinitesimally rigid*. These definitions are formalised in Chapter 10.

For pure frameworks, the definitions of *symmetric direction-pure frameworks* and *symmetric length-pure frameworks* are the natural analogues of the above. The definition of a symmetric motion does not change in this context. However, remember that pure frameworks have extra trivial motions. Thus a symmetric length-pure framework is *symmetry-forced length-rigid* if its only symmetric motions are rotations and translations, whereas a symmetric direction-pure framework is *symmetry-forced direction-rigid* if its only symmetric motions are translations and dilations. *Symmetry-forced infinitesimal length-rigidity* and *symmetry-forced infinitesimal direction-rigidity* are defined analogously.

In Section 9.1, we provide a brief overview of past work in symmetry-forced rigidity, focussing on the tools used to obtain these results. Then, in Section 9.2, we explain how we extend these methods to direction-length frameworks. We also summarise our results from the following Chapters.

9.1 Symmetric pure frameworks

Schulze and Whiteley [32] showed that by exploiting the symmetry of a Γ-symmetric length-pure framework (G,p), we can delete rows and columns from its rigidity matrix $R(G,p)$ to obtain an *orbit matrix*. The vectors in the kernel of this orbit matrix correspond exactly to the symmetric infinitesimal motions of the framework. So we can use the orbit matrix to define symmetry-forced infinitesimal length-rigidity in the same way that the rigidity matrix defines infinitesimal length-rigidity.

Ross [29] introduced gain graphs (defined in Chapter 2) to the theory of periodic length-pure frameworks. These were later applied to length-pure graphs under point group symmetries, where it was found that gain graphs illustrate the combinatorics of the orbit matrix in a similar way to how the original graph reflected properties of the rigidity matrix.

In 2016, Jordán, Kaszanitzky and Tanigawa [22] used inductive constructions of gain graphs and the orbit matrix to characterise symmetry-forced infinitesimal length-rigidity in the plane under reflection and rotation symmetry. See Theorem 10.3.2. Their methods give a characterisation for Γ-generic symmetric length-pure frameworks (i.e. frameworks which are as generic as possible subject to the imposed symmetry group Γ) in terms of symmetry-adapted edge counts on the corresponding gain graph.

Characterising symmetry-forced rigidity of length-pure frameworks under dihedral symmetry is more difficult, and the methods in [22] only give a complete characterisation under odd dihedral symmetry (Theorem 10.3.3). For dihedral groups $\mathcal{D}_k$, where $k \geq 2$ is even, the edge sparsity counts in Theorem 10.3.3 are known to be necessary conditions for $\mathcal{D}_k$-generic symmetry-forced infinitesimal rigidity, but are not sufficient. Figure 9.3 shows a well-known example of a $\mathcal{D}_2$-symmetric length-pure framework which satisfies the requirements of Theorem 10.3.3, but has a symmetric motion. Until a characterisation is known for this case, it will not be possible to characterise symmetry-forced rigidity for direction-length frameworks under even dihedral symmetry. As such, we frequently restrict our consideration to just the rotation and single reflection groups in the following chapters.

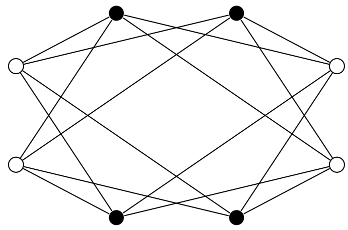


Figure 9.3: The $\mathcal{D}_2$-symmetric Bottema mechanism.

Using a different method, Tanigawa [37] characterised Γ-generic symmetry-forced infinitesimal direction-rigidity. This result covers all point group symmetries Γ in all dimensions $d \geq 1$. When we restrict our consideration to the plane, these conditions again correspond to symmetry-adapted edge counts on the corresponding gain graph (see Theorem 10.3.1). In particular, symmetry-forced rigidity under even dihedral symmetry has a simple characterisation for direction-pure frameworks. If we replace every length edge in Figure 9.3 with a direction edge, the resulting direction-pure framework has no non-trivial symmetric motions.

9.2 Summary of results

The characterisations for pure frameworks obtained in [22] and [37] give counting conditions on gain graphs. However, each paper used a completely different approach to achieve this. Here, we extend the inductive construction and orbit matrix methods used in [22], to the direction-length case.

We consider Γ-symmetric direction-length frameworks which are as generic as possible. Our main result, Theorem 12.2.1, is a characterisation of symmetry-forced infinitesimal rigidity for such frameworks when Γ is the single reflection group. We also obtain a partial result when Γ is a rotation group or odd dihedral group (Theorem 12.2.2).

As this thesis forms the first work on symmetric direction-length frameworks, before tackling these results, we must first develop all necessary theory and tools from scratch. This is the main task of Chapter 10, which culminates in a proof that the direction-length version of the orbit matrix defines symmetry-forced infinitesimal rigidity (Theorem 10.2.3). This, together with the known characterisations for pure frameworks (Theorems 10.3.1, 10.3.2 and 10.3.3), lead to our first main result: a set of necessary conditions for a Γ-symmetric direction-length framework, which is as generic as possible, to be symmetry-forced infinitesimally rigid (Theorem 10.3.4).

Ideally, we wish to show that the necessary conditions given in Theorem 10.3.4 are also sufficient. The example in Figure 9.3 suggests this fails for even dihedral symmetry, but for other symmetry groups we suspect it is true. We know symmetry-forced infinitesimal rigidity is defined by orbit matrix rank, and all Γ-generic realisations of a given Γ-symmetric graph G give an orbit matrix with the same rank. This implies that row independence in the orbit matrix for Γ-generic realisations of G defines independence for an *orbit matroid* on G. So proving that the necessary conditions in Theorem 10.3.4 are also sufficient, is equivalent to proving that these conditions characterise the bases of the orbit matroid (for given $|V(G)|$ and Γ).

This is difficult to do directly. Instead, in Chapter 11 we segue into graph theory, and show that the necessary conditions from Theorem 10.3.4 define a matroid on a direction-length gain graph, which we call the *sparsity matroid.* We then show that when $\Gamma \in \{\mathcal{C}_s, \mathcal{C}_{k\geq 2}\}$, we can inductively construct all direction-length gain graphs which satisfy these conditions for our chosen Γ. This is proven in Theorems 11.2.9 and 11.3.28 respectively. This chapter is particularly technical, so we simplify our arguments by only considering rotation and reflection symmetry. We conjecture that our methods in this chapter can be extended to dihedral symmetry.

The final step to obtain the characterisation sought, is to show that when Γ is not an even dihedral group, the sparsity matroid and orbit matroid coincide. In Chapter 12, we prove this is true when $\Gamma = \mathcal{C}_s$, by showing that the inductive moves of Theorem 11.2.9 preserve row independence in the orbit matrix.

To obtain a characterisation for $\Gamma = \mathcal{C}_{k\geq 2}$, it remains to extend the methods of Chapter 12 to show that the additional inductive moves from Theorem 11.2.9 also preserve independence in the orbit matrix. We conjecture that this is possible. In comparison, to obtain a similar characterisation for odd dihedral symmetry, we must extend the methods from both Chapters 11 and 12, to odd dihedral groups. Although we conjecture that this too is possible, it is a much greater endeavour.

Chapter 10

Necessary conditions for symmetry-forced infinitesimal rigidity

10.1 Preliminaries

We start by reviewing key ideas from group theory in Section 10.1.1, before formally defining symmetric direction-length frameworks in Section 10.1.2. We then review known results for gain graphs in Section 10.1.3. Lastly, in Section 10.1.4, we extend standard definitions from symmetric rigidity theory to direction-length frameworks.

10.1.1 Symmetry

The symmetry groups we consider are the point group symmetries of $\mathbb{R}^2$ which fix the origin: $\mathcal{I}$, $\mathcal{C}_s$, $\mathcal{C}_{k\geq 2}$ and $\mathcal{D}_{k\geq 2}$. The elements of these groups each induce an action on the plane which can be represented by a matrix.

The identity element I is represented by the identity matrix:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

whereas rotation in $\mathbb{R}^2$ by θ about the origin, and reflection in $\mathbb{R}^2$ across the line at angle θ through the origin are represented by the respective matrices

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}.$$

For a general group element g, we shall denote its matrix representation in $\mathbb{R}^2$ by g_μ. The only exception to this rule is the identity matrix, which we denote by I instead of I_μ. Note that for the point groups Γ we consider, the corresponding matrices are always orthogonal. So $(g^{-1})_\mu = (g_\mu)^{-1} = (g_\mu)^T$ for all $g \in \Gamma$. Thus the matrix group $\Gamma_\mu = \{g_\mu : g \in \Gamma\}$ is a subgroup of the orthogonal group in 2-dimensions.

10.1.2 Symmetric direction-length frameworks

Let $G = (V; D, L)$ be a direction-length graph. A *graph automorphism* of G is a permutation $\Phi : V \to V$ which preserves edges. In other words, a pair of vertices $u, v \in V$ are joined by the direction edge $\{u, v\} \in D$ if and only if $\{\Phi(u), \Phi(v)\} \in D$; and similarly for length edges, $\{u, v\} \in L$ if and only if $\{\Phi(u), \Phi(v)\} \in L$.

We say that Φ *fixes* a vertex v if $\Phi(v) = v$, and *fixes* an edge $\{u, v\}$ if $\{u, v\} = \{\Phi(u), \Phi(v)\}$. The set of all graph automorphisms of G forms the *automorphism group of* G, denoted $\mathrm{Aut}(G)$.

Given a group Γ, an *action* π of Γ on a direction-length graph G is an injective group homomorphism $\pi : \Gamma \to \mathrm{Aut}(G)$. The action π is *free* if for all non-identity $g \in \Gamma$, the permutation $\pi(g)$ fixes no vertices. We simplify our notation in the same manner we used for matrix representations, by letting

g_π denote $\pi(g)$ for all $g \in \Gamma$, and $\Gamma_\pi = \{g_\pi : g \in \Gamma\}$ be the corresponding permutation group.

A direction-length graph G is (Γ, π)-*symmetric* if Γ acts on G by π. We extend this definition to frameworks: (G, p) is a (Γ, π)-*symmetric framework* if G is a (Γ, π)-symmetric direction-length graph, and for all $g \in \Gamma$,

$$p(g_\pi v) = g_\mu p(v). \tag{10.1}$$

Figure 10.1 shows some examples of symmetric frameworks under different actions.

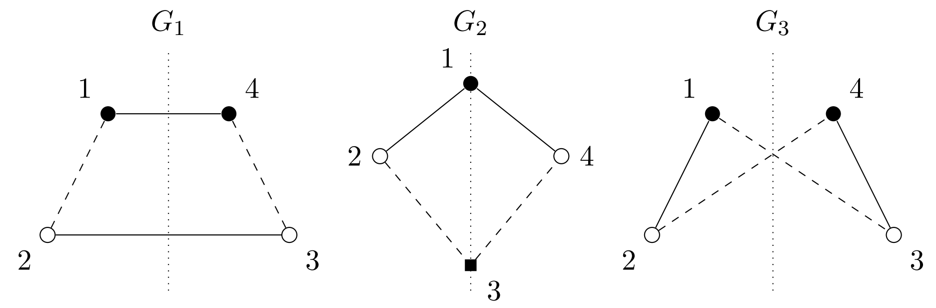


Figure 10.1: Three direction-length frameworks which are symmetric across a vertical mirror, shown as a dotted line. Each graph G_i is $(\mathcal{C}_s, \pi_i)$-symmetric, where $\pi_2(\sigma) = (24)(1)(3)$ fixes the vertices 1 and 3 of G_2; and $\pi_1(\sigma) = \pi_3(\sigma) = (14)(23)$ is a free action which fixes the edges $\{1, 4\}$ and $\{2, 3\}$ of G_1, and fixes no edges of G_3.

10.1.3 Γ-gain graphs

Recall the definitions of walks, directed graphs, edge-labellings and gain graphs from Chapter 2. One of our main tools for analysing symmetry-forced infinitesimal rigidity are direction-length gain graphs (H, ψ), where $H = (V; D, L)$ is a direction-length multigraph, whose edges are each assigned an orientation; and ψ labels each oriented edge of H with an element of some point group Γ. We call such a graph (H, ψ) a Γ-*gain graph*. Figure 10.2 shows some examples of $\mathcal{C}_3$-gain graphs. Our choice of orientation is largely arbitrary: given any edge $\{u, v\}$, the labelling $\psi(\overrightarrow{uv}) = g$ is equivalent to $\psi(\overleftarrow{uv}) = g^{-1}$.

Let (H, ψ) be a Γ-gain graph, and let $W = v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k$ be a walk in H, where $v_i \in V(H)$ and $e_i \in E(H)$. The *gain* of the walk is

$$\psi(W) = \theta(e_1) \cdot \theta(e_2) \cdots \theta(e_k),$$

where $\theta(e_i) = \psi(e_i)$ if $e_i = \overrightarrow{v_{i-1}v_i}$, and $\theta(e_i) = \psi(e_i)^{-1}$ if $e_i = \overleftarrow{v_{i-1}v_i}$.

Given $F \subseteq E(H)$ and $v \in V(F)$, we let $\langle F \rangle_{\psi,v}$ denote the group with generating set

$$\{\psi(W) : W \text{ is a closed walk starting at } v \text{ and only using edges from } F\},$$

and say that $\langle F \rangle_{\psi,v}$ is the subgroup of Γ induced by F relative to v. This definition suggests there are two choices which may change the gain of a closed walk: starting the walk at a different vertex in the sequence, or changing the gains on the edges. The first of these choices does not change the structure of the group induced by the edges:

Proposition 10.1.1. *[14, 42] Let (H, ψ) be a Γ-gain graph, and let $F \subseteq E(H)$ such that $H[F]$ is connected. Let $u, v \in V(F)$, then $\langle F \rangle_{\psi,u}$ is conjugate to $\langle F \rangle_{\psi,v}$.*

However, we need to be more careful when changing the gains on the edges. Given a vertex $v \in V(H)$ and a group element $g \in \Gamma$, a *switching operation at v with g* changes the gains at all edges incident to v to obtain a new gain function ψ' on H, where

$$\psi'(e) = \begin{cases} g\psi(e)g^{-1} & \text{if } e \text{ is a loop incident to } v, \\ \psi(e)g^{-1} & \text{if } e \text{ is not a loop, and is directed towards } v, \\ g\psi(e) & \text{if } e \text{ is not a loop, and is directed from } v, \text{ and} \\ \psi(e) & \text{otherwise.} \end{cases}$$

We say that two gain functions ψ and ψ' on a graph H are *equivalent* if ψ' can be obtained from ψ by a sequence of switching operations. Figure 10.2 shows an example of two equivalent gain functions.

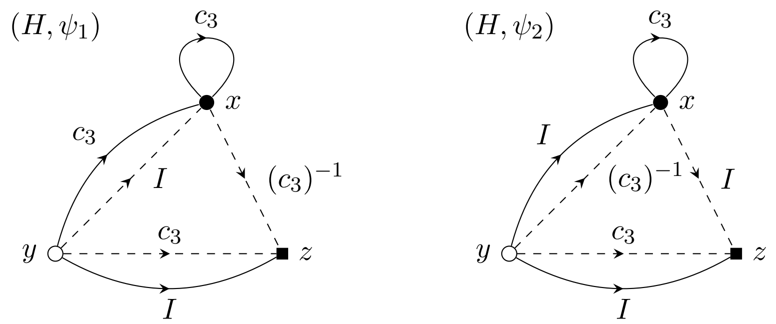


Figure 10.2: Two equivalent gain functions on a $\mathcal{C}_3$-gain graph H. The function ψ_2 is obtained from ψ_1 by a switching operation at x with gain c_3.

For a closed walk W starting at v_0, a switching operation performed at any vertex other than v_0 leaves the gain of W unchanged. This observation, together with Proposition 10.1.1 leads to the following result.

Proposition 10.1.2. *[22, Proposition 2.2] Let (H, ψ) be a Γ-gain graph, and let ψ' be equivalent to ψ. Then for any $F \subseteq E(H)$ and $v \in V(H)$, $\langle F \rangle_{\psi,v}$ is conjugate to $\langle F \rangle_{\psi',v}$.*

Let Γ be a group with subgroups S_1 and S_2 conjugate under some $g \in \Gamma$. When Γ is a 2-dimensional point group, the conjugacy transformation $T_g : S_1 \to S_2$ given by $T_g(x) = gxg^{-1}$ has $T_g(x)$ as the identity (a rotation, a reflection) in Γ if and only if x is the identity (a rotation, a reflection respectively). So for a Γ-gain graph (H, ψ) and a set of edges $F \subseteq E(H)$, Propositions 10.1.1 and 10.1.2 imply that the group $\langle F \rangle_{\psi',v}$ has the same structure for all choices of $v \in V(F)$, and all choices of ψ' equivalent to ψ. This allows us to simplify our notation, and refer to $\langle F \rangle_{\psi}$ when discussing properties which hold for all choices of $v \in V(F)$ and all equivalent ψ'.

When the gain function is clear from the context, we omit this too, and refer simply to $\langle F \rangle$.

These observations lead to the following useful results.

Proposition 10.1.3. *[22, Proposition 2.3] Let (H, ψ) be a Γ-gain graph. Then for any forest $F \subseteq E(G)$, there is a gain function ψ' equivalent to ψ such that $\psi'(e) = I$ for all $e \in F$.*

Proposition 10.1.4. *[22, Lemma 2.4] Let (H, ψ) be a connected Γ-gain graph. Let $T \subseteq E(H)$ be a spanning tree of H with $\psi(e) = I$ for all $e \in T$. Let $F \subseteq E(H)$ with $H[F]$ is connected. Then $\langle F \rangle_\psi = \langle \psi(e) : e \in F - T \rangle$.*

Since the group $\langle F \rangle_\psi$ has the same structure for all equivalent ψ', it is helpful to define this structure as a property of the underlying edge set F. We say F is *balanced* if $\langle F \rangle = \langle I \rangle = \mathcal{I}$, and F is *unbalanced* otherwise. Similarly, F is *cyclic* if $F = \langle \gamma \rangle$ for some $I \neq \gamma \in \Gamma$, and more specifically F is *reflectional* or *rotational* when γ is respectively a reflection or a non-trivial rotation. Finally, F is *dihedral* if $\langle F \rangle = \mathcal{D}_k$ for some $k \geq 2$.

Note that each non-empty edge set $F \subseteq E(H)$ of a Γ-gain graph (H, ψ) lies in exactly one of the following four categories: balanced, rotational, reflectional, or dihedral. This classification is key to the combinatorial characterisations in Section 10.3 and Chapter 12. Here it allows us to extend Propositions 10.1.3 and 10.1.4 to the following structural results:

Lemma 10.1.5. *[22, Lemma 2.5] Let Γ be a point group and (H, ψ) be a Γ-gain graph. Let $X, Y \subseteq E(H)$ with $H[X]$ and $H[Y]$ connected. Suppose the subgraph $(V(X) \cap V(Y), X \cap Y)$ is connected.*

(i) If X and Y are balanced, then $X \cup Y$ is balanced

(ii) If X is balanced, then $\langle X \cup Y \rangle = \langle Y \rangle$.

(iii) If X, Y and $X \cap Y$ are unbalanced and cyclic, then $X \cup Y$ is unbalanced and cyclic.

Lemma 10.1.6. *[22, Lemma 2.6] Let (H, ψ) be a Γ-gain graph. Let $X, Y \subseteq E(H)$ be balanced with $H[X]$ and $H[Y]$ connected. If $(V(X) \cap V(Y), X \cap Y)$ contains exactly two connected components, then $X \cup Y$ is either balanced or cyclic.*

10.1.4 Covering frameworks and quotient frameworks

Let $G = (V; D, L)$ be a (Γ, π)-symmetric direction-length graph and Γ_π be the corresponding permutation group on V. The *quotient graph* of G is the Γ-gain graph $(G/\Gamma_\pi, \psi)$, where $G/\Gamma_\pi = (V/\Gamma_\pi; D/\Gamma_\pi, L/\Gamma_\pi)$ has vertex set consisting of the vertex orbits $\Gamma_\pi v$ for $v \in V$, and edge set consisting of the edge orbits of G; and such that if $\{u, g_\pi v\} \in D$ (or $\{u, g_\pi v\} \in L$), then the quotient graph G/Γ_π contains the directed edge $(\Gamma_\pi u, \Gamma_\pi v)$ in D/Γ_π (respectively in L/Γ_π) with gain $\psi(\overrightarrow{\Gamma_\pi u, \Gamma_\pi v}) = g$.

If G is a (Γ, π)-symmetric direction-length graph and π is a free action, then for a given choice of vertex orbit representatives, the corresponding quotient graph (H, ψ) is unique. A different choice of vertex orbit representatives will give a quotient graph (H, ψ') where ψ' is equivalent to ψ. See Figure 10.3.

Conversely, given a Γ-gain graph (H, ψ), there is a unique (Γ, π)-symmetric direction-length graph G for which (H, ψ) is its quotient graph and π is a free action. We call this graph G the *covering graph* of (H, ψ).

We extend these definitions to frameworks. A Γ-*gain framework* (H, ψ, q) is a Γ-gain graph (H, ψ) together with a realisation $q : V(H) \to \mathbb{R}^2$. Given a (Γ, π)-symmetric direction-length framework (G, p) with free action π, and a choice of vertex orbit representatives $V' = \{v_1, v_2, \ldots, v_m\}$, the unique *quotient framework of* (G, p) *with respect to* V' is the Γ-gain framework $(H, \psi, \tilde{p})$, where (H, ψ) has vertex set $\{\Gamma v_1, \Gamma v_2, \ldots, \Gamma v_m\}$, and the realisation $\tilde{p}$ satisfies

$$\tilde{p}(\Gamma v_i) = p(v_i)$$

for all $1 \leq i \leq m$. See Figure 10.3 for an example of how to obtain a quotient framework from a symmetric framework.

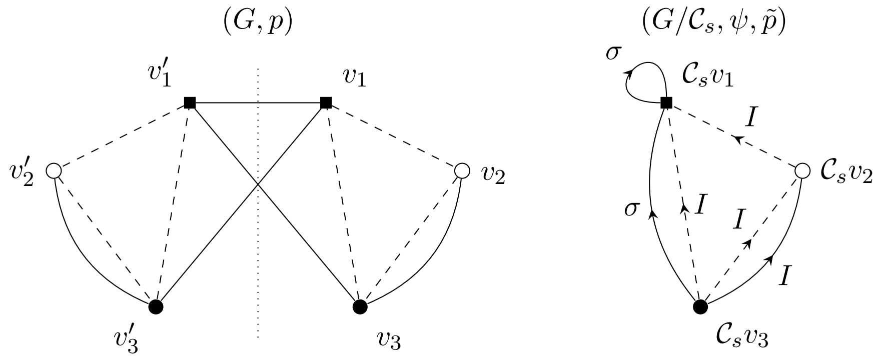


Figure 10.3: A $\mathcal{C}_s$-symmetric framework (G, p) under free action π with $\pi(\sigma) = (v_1, v_1')(v_2, v_2')(v_3, v_3')$; and its quotient framework $(G/\mathcal{C}_s, \psi, \tilde{p})$ under the choice of orbit representatives v_1, v_2, v_3.

Conversely, the *covering framework* of Γ-gain framework $(H, \psi, \tilde{p})$ is a (Γ, π)-symmetric direction-length framework (G, p) where π is a free action, G is the covering graph of (H, ψ), and for some choice of orbit representatives $u_1, u_2, \ldots, u_m$ for the respective vertices $\Gamma v_1, \Gamma v_2, \ldots, \Gamma v_m$ of H, the realisation p satisfies

$$p(g_\pi u_i) = g_\mu p(u_i) = g_\mu \tilde{p}(\Gamma v_i)$$

for all vertices $g_\pi u_i \in V(G)$.

10.2 Symmetry-forced rigidity

First, we introduce symmetric infinitesimal motions and symmetry-forced infinitesimal rigidity. Then, in Section 10.2.1, we show that we can define an *orbit matrix* for any given (Γ, π)-symmetric framework (G, p); and that this matrix and the corresponding quotient framework, are the correct tools to determine whether (G, p) is symmetry-forced infinitesimally rigid.

Given a (Γ, π)-symmetric direction-length framework (G, p), an infinitesimal motion $m : V(G) \to \mathbb{R}^2$ of our framework is *symmetric* if for all $g \in \Gamma$ and $v \in V(G)$

$$m(g_\pi v) = g_\mu m(v). \tag{10.2}$$

A *trivial symmetric infinitesimal motion* of (G, p) is a trivial infinitesimal motion which is symmetric under this definition. We denote the space of all trivial symmetric infinitesimal motions of a (Γ, π)-symmetric framework (G, p) by $\mathrm{triv}_\Gamma(G, p)$. A (Γ, π)-symmetric framework is *symmetry-forced infinitesimally rigid* if all of its symmetric infinitesimal motions are trivial.

As explained in Chapter 4, a motion of a framework must maintain the constraints imposed by the edges. This leads to the following characterisation of symmetric infinitesimal motions:

Lemma 10.2.1. *Suppose Γ is a point group. Let (G, p) be a (Γ, π)-symmetric direction-length framework with infinitesimal motion $m : V \to \mathbb{R}^2$. Then m is a symmetric infinitesimal motion of (G, p) if and only if*

(i) $\langle p(v_i) - g_\mu p(v_j), m(v_i) - g_\mu m(v_j) \rangle = 0$ *for all* $\{v_i, g_\pi v_j\} \in L$*, and*

(ii) $\langle (p(v_i) - g_\mu p(v_j))^\perp, m(v_i) - g_\mu m(v_j) \rangle = 0$ *for all* $\{v_i, g_\pi v_j\} \in D$*,*

where v_i and v_j are vertex orbit representatives of $V(G)/\Gamma_\pi$, $g \in \Gamma$ and $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)^\perp = \left(\begin{smallmatrix} y \\ -x \end{smallmatrix}\right)$.

Proof. By definition, m is an infinitesimal motion of (G, p) if and only if it lies in the kernel of the rigidity matrix $R(G, p)$. In other words,

$$\langle p(u) - p(v), m(u) - m(v)\rangle = 0 \text{ for all } \{u, v\} \in L, \text{ and}$$
$$\langle (p(u) - p(v))^{\perp}, m(u) - m(v)\rangle = 0 \text{ for all } \{u, v\} \in D.$$

Since G is (Γ, π)-symmetric, we know $u \in \Gamma_\pi v_i$ and $v \in \Gamma_\pi v_j$ for some pair of vertex orbit representatives v_i and v_j. Hence $u = f_\pi v_i$ and $v = h_\pi v_j$ for some $f, h \in \Gamma$. Further, (G, p) is a (Γ, π)-symmetric framework so the realisation p satisfies

$$p(u) = p(f_\pi v_i) = f_\mu p(v_i) \text{ and } p(v) = p(h_\pi v_j) = h_\mu p(v_j). \tag{10.3}$$

By the definition of a symmetric infinitesimal motion, m is symmetric if and only if, for all $g \in \Gamma$ and $v_k \in V(G)$, we have

$$m(g_\pi v_k) = g_\mu m(v_k). \tag{10.4}$$

Substituting (10.3) and (10.4) into the definition of an infinitesimal motion gives that m is a symmetric infinitesimal motion of (G, p) if and only if the following system of equations is satisfied:

$$\langle f_\mu p(v_i) - h_\mu p(v_j), f_\mu m(v_i) - h_\mu m(v_j)\rangle = 0 \text{ for all } \{f_\pi v_i, h_\pi v_j\} \in L, \text{ and}$$
$$\langle (f_\mu p(v_i) - h_\mu p(v_j))^{\perp}, f_\mu m(v_i) - h_\mu m(v_j)\rangle = 0 \text{ for all } \{f_\pi v_i, h_\pi v_j\} \in D.$$

Since Γ is a point group, we know that the matrices in Γ_μ are orthogonal. Hence $f_\mu^T = f_\mu^{-1}$, and these equations can be rewritten as

$$\langle p(v_i) - f_\mu^{-1} h_\mu p(v_j), m(v_i) - f_\mu^{-1} h_\mu m(v_j)\rangle = 0 \text{ for all } \{v_i, f_\pi^{-1} h_\pi v_j\} \in L,$$
$$\langle (p(v_i) - f_\mu^{-1} h_\mu p(v_j))^{\perp}, m(v_i) - f_\mu^{-1} h_\mu m(v_j)\rangle = 0 \text{ for all } \{v_i, f_\pi^{-1} h_\pi v_j\} \in D.$$

Simplifying notation by setting $g = f^{-1}h$ gives the system of equations sought. $\square$

Lemma 10.2.1 shows that when m is a symmetric infinitesimal motion of a (Γ, π)-symmetric framework (G, p), and $e \in E(G)$, every edge in the orbit $\Gamma_\pi e$ imposes an identical constraint. Thus we can replace this set of identical constraints by a single constraint for $\Gamma_\pi e$. We apply this constraint to the corresponding edge in the quotient framework:

Corollary 10.2.2. *Suppose Γ is a point group. Let (G, p) be a (Γ, π)-symmetric direction-length framework, and let V' be a set of orbit representatives of $V(G)/\Gamma_\pi$. Let $(H, \psi, \tilde{p})$ be the quotient framework of (G, p) with respect to V'. Then m is a symmetric infinitesimal motion of (G, p) if and only if whenever $e = (\Gamma_\pi v_i, \Gamma_\pi v_j) \in E(H)$ with gain $\psi(\overrightarrow{\Gamma_\pi v_i, \Gamma_\pi v_j}) = g$, we have*

(i) $\langle p(v_i) - g_\mu p(v_j), m(v_i) - g_\mu m(v_j)\rangle = 0$ if e is a length edge, and

(ii) $\langle (p(v_i) - g_\mu p(v_j))^\perp, m(v_i) - g_\mu m(v_j)\rangle = 0$ if e is a direction edge.

10.2.1 The orbit matrix

Recall that in the theory of non-symmetric frameworks, we construct a rigidity matrix from the edge constraints of our framework, and use this analyse the framework's infinitesimal motions. This was described in detail in Chapter 4. Here we adapt this method to analyse the symmetric infinitesimal motions of symmetric frameworks, by instead constructing an orbit matrix. This tool was first introduced by Schulze and Whiteley [32] in the context of length-pure frameworks.

Given a (Γ, π)-symmetric direction-length framework (G, p) and a set of vertex orbit representatives V', we first construct the quotient framework $(H, \psi, \tilde{p})$, with respect to V'. Corollary 10.2.2 then gives us a system of equations for the quotient framework, which we use to construct the *orbit* *matrix*, $O(H, \psi, \tilde{p})$. When π is a free action, this matrix has $|E(H)|$ rows and $2|V'|$ columns. If π is not free, then the number of columns in this matrix is reduced (see [32]), however we do not consider this case. For the point groups Γ we consider, the matrices in Γ_μ are orthogonal, which simplifies the matrix entries.

A non-loop edge $e = (\Gamma_\pi v_i, \Gamma_\pi v_j) \in E(H)$ with $i \neq j$ and gain g gives the following rows in the orbit matrix

$$\begin{pmatrix} 0 \cdots 0 & \overbrace{(p(v_i) - g_\mu p(v_j))^T}^{v_i} & 0 \cdots 0 & \overbrace{-(g_\mu^{-1}(p(v_i) - g_\mu p(v_j)))^T}^{v_j} & 0 \cdots 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \cdots 0 & ((p(v_i) - g_\mu p(v_j))^\perp)^T & 0 \cdots 0 & -(g_\mu^{-1}(p(v_i) - g_\mu p(v_j))^\perp)^T & 0 \cdots 0 \end{pmatrix}$$

when e is a length or direction edge respectively.

For loop edges in the quotient graph, the constraints simplify yet further. If a loop edge has identity gain, then the equations in Corollary 10.2.2 simplify to give $\langle 0, 0 \rangle = 0$, which trivially holds for all m. Hence the corresponding row in the orbit matrix has zero entries throughout. Perhaps surprisingly, we also obtain a trivial equation, $\langle 0, m(v_i) \rangle = 0$, for direction loops $(\Gamma_\pi v_i, \Gamma_\pi v_i) \in E(H)$ with reflection gain g. Such an edge restricts the motions of the symmetric graph G, to those which keep the endvertices $p(v_i)$ and $p(g_\pi v_i)$ on a line perpendicular to the mirror given by g. By definition, all symmetric infinitesimal motions satisfy this property, so this edge imposes no additional constraint.

In the remaining cases, a loop edge $l = (\Gamma_\pi v_i, \Gamma_\pi v_i) \in E(H)$ with gain g corresponds to the following rows in the orbit matrix

$$\begin{pmatrix} 0 \cdots 0 & \overbrace{((I - g_\mu^{-1})(p(v_i) - g_\mu p(v_i)))^T}^{v_i} & 0 \cdots 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \cdots 0 & ((I - g_\mu^{-1})(p(v_i) - g_\mu p(v_i))^\perp)^T & 0 \cdots 0 \end{pmatrix}$$

when l is a length edge with non-identity gain, or a direction edge with non-trivial rotation gain respectively.

Thus for point groups Γ, Corollary 10.2.2 is equivalent to saying that m is a symmetric infinitesimal motion of the (Γ, π)-symmetric direction-length framework (G, p) if and only if $m|_{V'}$ is in the kernel of $O(H, \psi, \tilde{p})$. Hence (G, p) is symmetry-forced infinitesimally rigid if and only if for some choice of vertex orbit representatives V',

$$\dim(\ker(O(G/\Gamma_\pi, \psi, \tilde{p}))) = \dim(\mathrm{triv}_\Gamma(G, p)).$$

Using the rank-nullity formula, this gives us the following characterisation of symmetry-forced infinitesimal rigidity:

Theorem 10.2.3. *Suppose Γ is a point group. Let (G, p) be a (Γ, π)-symmetric direction-length framework under free action π, and with vertex orbit representatives V'. Let $(H, \psi, \tilde{p})$ be the quotient framework of (G, p) with respect to V'. Then (G, p) is symmetry-forced infinitesimally rigid if and only if*

$$\mathrm{rank}(O(H, \psi, \tilde{p})) = 2|V'| - \dim(\mathrm{triv}_\Gamma(G, p)).$$

For direction-length frameworks, the only trivial infinitesimal motions are the translations. Hence for symmetric direction-length frameworks, the trivial symmetric infinitesimal motions are those translations which also preserve the symmetry. If $\Gamma \in \{\mathcal{C}_{k\geq 2}, \mathcal{D}_{k\geq 2}\}$, all symmetric infinitesimal motions must keep the origin fixed, and so the only possible translation is the translation by 0, which gives $\dim(\mathrm{triv}_\Gamma(G, p)) = 0$. If $\Gamma = \mathcal{C}_s$, then translating along the mirror preserves the symmetry, so $\dim(\mathrm{triv}_{\mathcal{C}_s}(G, p)) = 1$. When $\Gamma = \mathcal{I}$, no symmetric constraints are imposed on the trivial infinitesimal motions so $\dim(\mathrm{triv}_{\mathcal{I}}(G, p)) = 2$. Using these observations, we can rewrite Theorem 10.2.3 as follows:

Corollary 10.2.4. *Suppose* Γ *is a point group. Let* (G, p) *be a* (Γ, π)*-symmetric direction-length framework under free action* π*, and with vertex orbit representatives* V'*. Let* $(H, \psi, \tilde{p})$ *be the quotient framework of* (G, p) *with respect to* V'*. Then* (G, p) *is symmetry-forced infinitesimally rigid if and only if*

$$\operatorname{rank}(O(H, \psi, \tilde{p})) = 2|V'| - \begin{cases} 0 & \text{if } \Gamma \in \{\mathcal{C}_{k\geq 2}, \mathcal{D}_{k\geq 2}\}, \\ 1 & \text{if } \Gamma = \mathcal{C}_s, \text{ and} \\ 2 & \text{if } \Gamma = \mathcal{I}. \end{cases}$$

We say that a (Γ, π)-symmetric direction-length framework (G, p) is *minimally symmetry-forced infinitesimally rigid* if it is symmetry-forced infinitesimally rigid, but when we delete all edges in the edge orbit $\Gamma_\pi e$ for any $e \in E(G)$, the remaining (Γ, π)-symmetric framework $(G - \Gamma_\pi e, p)$ is not symmetry-forced infinitesimally rigid. We can rephrase this in terms of the orbit matrix:

Corollary 10.2.5. *Suppose* Γ *is a point group. Let* (G, p) *be a* (Γ, π)*-symmetric direction-length framework under free action* π*, and with vertex orbit representatives* V'*. Let* $(H, \psi, \tilde{p})$ *be the quotient framework of* (G, p) *with respect to* V'*. Then* (G, p) *is minimally symmetry-forced infinitesimally rigid if and only if*

$$|E(H)| = \operatorname{rank}(O(H, \psi, \tilde{p})) = 2|V'| - \begin{cases} 0 & \text{if } \Gamma \in \{\mathcal{C}_{k\geq 2}, \mathcal{D}_{k\geq 2}\}, \\ 1 & \text{if } \Gamma = \mathcal{C}_s, \text{ and} \\ 2 & \text{if } \Gamma = \mathcal{I}. \end{cases}$$

As with non-symmetric frameworks, algebraic dependencies in the non-zero entries of the matrix can cause additional row dependencies, and hence reduce the rank. However, unlike non-symmetric frameworks, these entries are defined in terms of both the realisation, and the matrix representation of our group. Let $\mathbb{Q}_\Gamma$ denote the smallest extension field of $\mathbb{Q}$ which contains the entries of every matrix in Γ_μ. Since we only consider point groups Γ, we only need to adjoin finitely many real numbers to $\mathbb{Q}$ to form $\mathbb{Q}_\Gamma$. We say that a realisation $\tilde{p}$ of a Γ-gain graph is *generic over* $\mathbb{Q}_\Gamma$ if the coordinates in $\tilde{p}$ are algebraically independent over $\mathbb{Q}_\Gamma$.

We say a (Γ, π)-symmetric framework (G, p) is Γ-*generic* when some (or equivalently all) quotient frameworks of (G, p) are generic over $\mathbb{Q}_\Gamma$. A (Γ, π)-symmetric graph G is *(minimally) symmetry-forced infinitesimally rigid* if all Γ-generic (Γ, π)-symmetric frameworks (G, p) are (minimally) symmetry-forced infinitesimally rigid. Corollary 10.2.5 thus implies the following necessary condition for minimal symmetry-forced infinitesimal rigidity:

Corollary 10.2.6. *Suppose Γ is a point group. Let G be a (Γ, π)-symmetric direction-length graph under free action π, and with vertex orbit representatives V'. Let (H, ψ) be the quotient graph of G with respect to V'. If G is minimally symmetry-forced infinitesimally rigid then*

$$|E(H)| = 2|V(H)| - \begin{cases} 0 & \textit{if } \Gamma \in \{\mathcal{C}_{k\geq 2}, \mathcal{D}_{k\geq 2}\}, \\ 1 & \textit{if } \Gamma = \mathcal{C}_s, \ \textit{and} \\ 2 & \textit{if } \Gamma = \mathcal{I}; \end{cases}$$

and for all $\emptyset \neq F \subseteq E(H)$

$$|F| \leq 2|V(F)| - \begin{cases} 0 & \textit{if } \langle F \rangle \in \{\mathcal{C}_{k\geq 2}, \mathcal{D}_{k\geq 2}\}, \\ 1 & \textit{if } \langle F \rangle = \mathcal{C}_s, \ \textit{and} \\ 2 & \textit{if } \langle F \rangle = \mathcal{I}. \end{cases}$$

10.3 The orbit matroid

Given Γ-gain graph (H, ψ) with realisation $\tilde{p}$ generic over $\mathbb{Q}_\Gamma$, the fact $\tilde{p}$ introduces no additional row dependencies to the orbit matrix means two things. Firstly, the rank of the orbit matrix for (H, ψ) is maximised at $\tilde{p}$; and secondly, for any other realisation $\tilde{q}$ of (H, ψ) generic over $\mathbb{Q}_\Gamma$, the orbit matrices $O(H, \psi, \tilde{p})$ and $O(H, \psi, \tilde{q})$ have the same collection of independent row sets. Thus we can define a matroid on the edge set of the graph (H, ψ), which we call the *orbit matroid of* (H, ψ), denoted $\mathcal{O}(H, \psi)$; where the *independent* sets of $\mathcal{O}(H, \psi)$ correspond to the linearly independent row sets of $O(H, \psi, \tilde{p})$.

Rephrasing Corollary 10.2.5 in matroid terminology: a Γ-symmetric graph G is minimally symmetry-forced infinitesimally rigid if and only if the matroid of its corresponding quotient graph, $\mathcal{O}(H, \psi)$, has $\text{rank}(\mathcal{O}(H, \psi)) = |E(H)|$. The inequalities in Corollary 10.2.6 provide necessary conditions for this to hold.

The orbit matrix and orbit matroid also feature in the study of symmetry-forced rigidity of pure frameworks. However, in these cases the rows in the orbit matrix all corresponded to edges of the same type, either direction or length. Thus results characterising symmetry-forced direction-rigidity and symmetry-forced length rigidity give additional necessary conditions for a pure edge set F to be independent in the orbit matroid.

In Subsection 10.3.1, we summarise these results for pure frameworks. Then, in Subsection 10.3.2, we collate these with the sparsity conditions from Corollary 10.2.6 to obtain our set of necessary conditions for independence in the orbit matroid.

10.3.1 Pure frameworks under symmetry

Recall that a length-pure framework has no constraints on its orientation, and so can be continuously rotated; whereas a direction-pure framework has no constraints on its size, and thus can be continuously dilated. When we restrict our consideration to only length-pure frameworks, or only direction-pure frameworks, these extra motions are trivial motions of the relevant pure framework. As such, our definition of infinitesimal rigidity for pure frameworks changes accordingly.

We say a (Γ, π)-symmetric direction-pure framework (length-pure framework) is *symmetry-forced infinitesimally direction-rigid* (*symmetry-forced infinitesimally length-rigid*) if its only symmetric infinitesimal motions are trivial motions for direction-pure frameworks (length-pure frameworks).

For length-pure frameworks, it is known that symmetry-forced length-rigidity and symmetry-forced infinitesimal length-rigidity are equivalent for Γ-generic realisations (see Corollary 4.8 of [30]). The analogous properties are also equivalent for direction-pure frameworks (the argument in Remark 4.1.5 of [40] is unaffected by symmetry). Thus the results here also characterise symmetry-forced length- and direction-rigidity. However, as we are interested in the consequences for the orbit matrix, we state them in terms of infinitesimal motions.

Direction-pure frameworks

In Corollary 6.4 of [37], Tanigawa characterised symmetry-forced infinitesimal direction-rigidity under point groups Γ in all dimensions $d \geq 1$. The statement of the general version of this result involves counting connected components of induced subgraphs, and evaluating the trace of the matrix g_μ for every $g \in \Gamma$. However, when we restrict our consideration to the plane, these conditions simplify to the following edge counts for reflection, rotation and dihedral symmetry:

Theorem 10.3.1. *Suppose Γ is a point group. Let (G,p) be a (Γ,π)-symmetric direction-pure framework. Suppose $p : V(G) \to \mathbb{R}^2$ is Γ-generic, and π is a free action. Then (G,p) is symmetry-forced infinitesimally direction-rigid if and only if the quotient Γ-gain graph contains a spanning subgraph (H,ψ) such that*

$$|E(H)| = 2|V(H)| - \begin{cases} 1 & \text{if } \Gamma \in \{\mathcal{C}_{k\geq 2}, \mathcal{D}_{k\geq 2}\}, \\ 2 & \text{if } \Gamma = \mathcal{C}_s, \text{ and} \\ 3 & \text{if } \Gamma = \mathcal{I}; \end{cases}$$

and for all $\emptyset \neq F \subseteq E(H)$ we have

$$|F| \leq 2|V(F)| - \begin{cases} 1 & \text{if } \langle F \rangle \in \{\mathcal{C}_{k\geq 2}, \mathcal{D}_{k\geq 2}\}, \\ 2 & \text{if } \langle F \rangle = \mathcal{C}_s, \text{ and} \\ 3 & \text{if } \langle F \rangle = \mathcal{I}. \end{cases}$$

Length-pure frameworks

For length-pure frameworks, symmetry-forced infinitesimal length-rigidity was characterised under reflection and rotation symmetry by Jordán, Kaszanitzky and Tanigawa:

Theorem 10.3.2. *[22, Theorem 6.3] Let $\Gamma \in \{\mathcal{C}_s, \mathcal{C}_k\}$ where $k \geq 2$. Let (G,p) be a Γ-generic (Γ,π)-symmetric length-pure framework in the plane with free action π. Then (G,p) is symmetry-forced infinitesimally length-rigid if and only if the quotient Γ-gain graph contains a spanning subgraph (H,ψ) such that*

$$|E(H)| = 2|V(H)| - 1,$$

and for all $\emptyset \neq F \subseteq E(H)$ *we have*

$$|F| \leq \begin{cases} 2|V(F)| - 3 & \textit{if } F \textit{ is balanced,} \\ 2|V(F)| - 1 & \textit{otherwise.} \end{cases}$$

As noted in Chapter 9, for dihedral symmetry $\mathcal{D}_{k\geq 2}$, the methods in [22] only give a complete characterisation when k is odd:

Theorem 10.3.3. *[22, Theorem 8.2] Let* $\mathcal{D}_k$ *be a dihedral group with* $k \geq 3$ *odd. Let* (G, p) *be a* $\mathcal{D}_k$*-generic* $(\mathcal{D}_k, \pi)$*-symmetric length-pure framework under free action* π*. Then* (G, p) *is symmetry-forced infinitesimally length-rigid if and only if the quotient* $\mathcal{D}_k$*-gain graph contains a spanning subgraph* (H, ψ) *such that*

$$|E(H)| = 2|V(H)|,$$

and for all $\emptyset \neq F \subseteq E(H)$ *we have*

$$|F| \leq \begin{cases} 2|V(F)| - 3 & \textit{if } F \textit{ is balanced,} \\ 2|V(F)| - 1 & \textit{if } F \textit{ is unbalanced and cyclic,} \\ 2|V(F)| & \textit{otherwise.} \end{cases}$$

For dihedral groups $\mathcal{D}_k$, where $k \geq 2$ is even, the edge sparsity counts in Theorem 10.3.3 are known to be necessary conditions for symmetry-forced infinitesimal length-rigidity, but not sufficient. See Lemma 8.1 in [22]. Thus we have the necessary conditions sought for all planar point groups, despite not having a complete characterisation in this case.

10.3.2 Necessary conditions for minimal symmetry-forced infinitesimal rigidity

The inequalities in Theorems 10.3.1, 10.3.2 and 10.3.3 give additional necessary conditions for the orbit matroid $\mathcal{O}(H, \psi)$ of the direction-length gain graph (H, ψ) to have $E(H)$ independent. Thus, they provide extra necessary conditions for minimal symmetry-forced infinitesimal rigidity. Adding these conditions to those in Corollary 10.2.6 gives us the main result of this chapter:

Theorem 10.3.4. *Suppose Γ is a point group. Let G be a (Γ, π)-symmetric direction-length graph under free action π, and with vertex orbit representatives V'. Let (H, ψ) be the quotient graph of G with respect to V'. If G is minimally symmetry-forced infinitesimally rigid then for all $\emptyset \neq F \subseteq E(H)$ we have*

$$|F| \leq 2|V(F)| - \begin{cases} 0 & \textit{if } \langle F \rangle \in \{\mathcal{C}_{k\geq 2}, \mathcal{D}_{k\geq 2}\}, \\ 1 & \textit{if } \langle F \rangle = \mathcal{C}_s, \ \textit{and} \\ 2 & \textit{if } \langle F \rangle = \mathcal{I}; \end{cases}$$

where equality holds in the above when $F = E(H)$. Further, for all direction-pure $\emptyset \neq F \subseteq E(H)$,

$$|F| \leq 2|V(F)| - \begin{cases} 1 & \textit{if } \langle F \rangle \in \{\mathcal{C}_{k\geq 2}, \mathcal{D}_{k\geq 2}\}, \\ 2 & \textit{if } \langle F \rangle = \mathcal{C}_s, \ \textit{and} \\ 3 & \textit{if } \langle F \rangle = \mathcal{I}; \end{cases}$$

and for all length-pure $\emptyset \neq F \subseteq E(H)$,

$$|F| \leq 2|V(F)| - \begin{cases} 0 & \textit{if } \langle F \rangle = \mathcal{D}_{k\geq 2}, \\ 1 & \textit{if } \langle F \rangle \in \{\mathcal{C}_{k\geq 2}, \mathcal{C}_s\}, \ \textit{and} \\ 3 & \textit{if } \langle F \rangle = \mathcal{I}. \end{cases}$$

In order for the edge sparsity conditions in this Theorem to characterise minimal symmetry-forced rigidity, they would have to be enough to guarantee the independence of $E(H)$ in the orbit matroid. If this were true, Theorem 10.3.4 would characterise the graphs (H, ψ) for which $E(H)$ is maximally independent, given $|V(H)|$ and Γ.

Unfortunately, these conditions are not sufficient when Γ is even dihedral. The length-pure Bottema mechanism from Figure 9.3 provides one counterexample. Despite having no direction edges, it satisfies the conditions of Theorem 10.3.4, but is not symmetry-forced rigid. Figure 10.4 depicts its quotient framework.

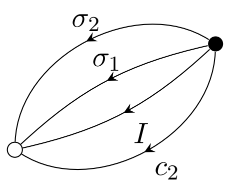


Figure 10.4: The quotient framework of the Bottema mechanism.

We can obtain a mixed counterexample by replacing the edges of the Bottema mechanism with a combination of direction and length edges. See Figure 10.5.

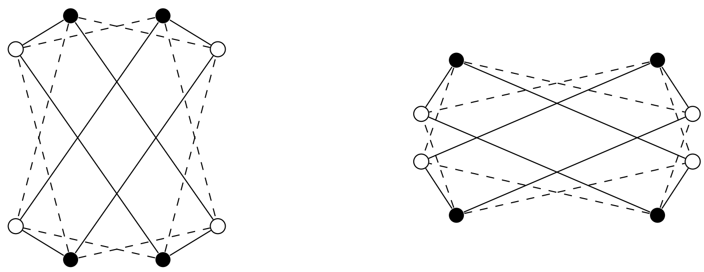


Figure 10.5: Two realisations of a $\mathcal{D}_2$-symmetric direction-length graph. There is a continuous symmetric motion from one realisation to the other.

This suggests that it is not only our count conditions for length-pure subgraphs which are insufficient under even dihedral symmetry, but also those for mixed subgraphs. So, at best, we can hope the necessary conditions in Theorem 10.3.4 are sufficient when Γ is cyclic or odd dihedral. In Chapter

11 we show that these inequalities define independence in a matroid, which we call the *sparsity matroid*, when Γ is cyclic. Then, in Chapter 12, we show that when Γ is the single reflection group, the sparsity matroid and orbit matroid coincide.

Chapter 11

The sparsity matroid

The first aim of this chapter is to show that the inequalities in Theorem 10.3.4 define independence in a matroid on the edge set of a Γ-gain graph (H, ψ), which we call the *sparsity matroid* of (H, ψ). We prove this in Section 11.1, where we first define the *sparsity function* based on these inequalities, and then use this function to construct the sparsity matroid.

Our second aim is to show that the edge set of a Γ-gain graph (H, ψ) is independent in the sparsity matroid, and maximal given $|V(H)|$ and Γ, if and only if we can inductively construct (H, ψ) from a set of base graphs using a handful of simple moves. Our methods succeed in proving this for cyclic Γ. We achieve partial results when Γ is dihedral.

In Section 11.2 we define our base graphs and inductive moves, and show that any Γ-gain graph (H, ψ) which can be constructed in this way is maximally independent in the sparsity matroid (given $|V(H)|$ and any point group Γ). It remains to prove the converse of this statement. This requires showing that if (H, ψ) has a maximally independent edge set and is not a base graph, then we can remove some vertex, v, using the inverse of one of our inductive moves. We prove this in Subsection 11.2.4 for the cases where

$d_H(v) \leq 3$. However, when $d_H(v) = 4$, this is much more complicated. In Section 11.3, we restrict to considering degree 4 vertices in $\mathcal{C}_{k\geq 2}$-gain graphs, and break this problem down into multiple subcases based on the structure of the graph.

Our main results, Theorems 11.2.9 and 11.3.28, provide a constructive characterisation of all maximally Γ-sparse graphs when Γ is respectively a reflection or rotation group.

11.1 The sparsity function and sparsity matroid

We first rephrase the sparsity conditions from Theorem 10.3.4 as a function on Γ-gain graphs. Given a point group Γ, and Γ-gain graph (H, ψ), we define the *sparsity function* s on $E(H)$ by

$$s(E') = 2|V(E')| - 3 + \alpha(E') + \beta(E')$$

for all $\emptyset \neq E' \subseteq E(H)$, where

$$\alpha(E') = \begin{cases} 0 & \text{if } \langle E' \rangle = \mathcal{I}, \\ 1 & \text{if } \langle E' \rangle = \mathcal{C}_s, \\ 2 & \text{otherwise;} \end{cases}$$

and

$$\beta(E') = \begin{cases} 0 & \text{if } E' \text{ is length-pure with } \langle E' \rangle \subseteq \mathcal{C}_k, \text{ or } E' \text{ is direction-pure,} \\ 1 & \text{otherwise.} \end{cases}$$

We define $s(\emptyset) = 0$.

An edge set F in (H, ψ) is *sparse* if $|F'| \leq s(F')$ for all $F' \subseteq F$; and is *tight* if F is sparse with $|F| = s(F)$. Further, F is *maximal tight* in (H, ψ) if there is no tight set $E' \subseteq E(H)$ with $F \subset E'$. We extend these definitions to graphs, by saying that $H[F]$ is sparse (tight, maximal tight) whenever F is. The following result follows immediately from Proposition 10.1.2.

Lemma 11.1.1. *Let (H, ψ) be a Γ-gain graph, and suppose ψ' is equivalent to ψ. Then (H, ψ) is sparse if and only if (H, ψ') is sparse.*

We aim to understand the relationship between edge sparsity in a Γ-gain graph and independence in the orbit matroid. From Theorem 10.3.4, we know that edge sparsity is a necessary condition for independence in the orbit matroid:

Corollary 11.1.2. *Suppose Γ is a point group. Let (H, ψ) be a Γ-gain graph with realisation $\tilde{p}$ generic over $\mathbb{Q}_\Gamma$. If $O(H, \psi, \tilde{p})$ is row independent, then $E(H)$ is sparse.*

However, the converse of this statement is not true in general. See Figure 9.3 for a counterexample when Γ is an even dihedral group. So instead, we restrict our consideration to when Γ is a cyclic group.

For a Γ-gain graph (H, ψ), let $\mathcal{S}(H, \psi) = (E(H), \mathcal{E})$, where $\mathcal{E}$ is the collection of sparse sets of $E(H)$. We shall prove that $\mathcal{S}(H, \psi)$ is a matroid on $E(H)$, using a method found in [16, 22] amongst others. This approach relies on partitioning sparse edge sets into families of tight sets. We first introduce some properties of tight sets.

11.1.1 Tight sets

The following basic properties of tight sets shall be used frequently. The proof of this result is almost identical to that of Lemma 7.1 in [22].

Proposition 11.1.3. *Let Γ be a point group, and let (H, ψ) be a tight Γ-gain graph with $|V(H)| \geq 2$.*

(i) Either every vertex in H is incident to at least two edges, or (H, ψ) is the pure, balanced graph consisting of a single edge.

(ii) If (H, ψ) is pure and balanced, then (H, ψ) is 2-connected.

(iii) If (H, ψ) has $|E(H)| \leq 2|V(H)| - 1$, then (H, ψ) is connected.

(iv) If (H, ψ) has $2 \leq |E(H)| \leq 2|V(H)| - 2$, then (H, ψ) is 2-edge-connected.

Proof. Let $E = E(H)$. First, consider the special case where $|E| = 1$. We know (H, ψ) is tight and $|V(H)| \geq 2$, hence (H, ψ) consists of a single edge between two distinct vertices. Thus (H, ψ) satisfies (i), (ii). and (iii).

For the remainder of the proof, suppose $|E| \geq 2$. Assume for a contradiction that there exists a vertex $v \in V(H)$ incident to exactly one edge, e. For $\gamma \in \{\alpha, \beta\}$ we know $\gamma(F') \leq \gamma(F)$ whenever $F' \subseteq F$. This, and the fact (H, ψ) is sparse give

$$|E| - 1 \leq s(E - e) \leq 2|V(H) - \{v\}| - 3 + \alpha(E) + \beta(E) = s(E) - 2.$$

Which implies $|E| < s(E)$, contradicting the fact (H, ψ) is tight. Hence (i) holds.

Suppose instead that (H, ψ) is pure and balanced. Then $\alpha(E) + \beta(E) = 0$, and H contains no loops. Assume for a contradiction that (H, ψ) is not 2-connected. Then there exists a partition of E into $\{E_1, E_2\}$ such that $|V(E_1) \cap V(E_2)| = 1$. Whenever $F \subseteq E$ and $\gamma \in \{\alpha, \beta\}$, we know $\gamma(F) \leq \gamma(E)$. This implies $\alpha(E_i) = \beta(E_i) = 0$, and hence

$$|E| = |E_1| + |E_2| \leq 2|V(E_1)| - 3 + 2|V(E_2)| - 3 = 2|V(H)| - 4 < s(E).$$

This contradicts the fact (H, ψ) is tight. Hence (ii) holds.

Next, suppose $|E| = 2|V(H)| - k$ for some $k \geq 1$. Assume for a contradiction that $V(H)$ can be partitioned into $\{V_1, V_2\}$ such that there is no path from V_1 to V_2 in H. Since (H, ψ) is sparse this implies

$$|E| = |E(V_1)| + |E(V_2)| \leq 2|V_1| - k + 2|V_2| - k = 2|V(H)| - 2k < |E|.$$

This again gives a contradiction. Hence proving (iii).

Finally, we consider part (iv). Since $2 \leq |E| \leq 2|V(H)|-2$, and (H, ψ) is Γ-tight, we know (H, ψ) is loop-free and satisfies the conditions for part (iii). Hence (H, ψ) is connected and, by part (i), every vertex is incident to at least two edges. Assume for a contradiction that (H, ψ) is not 2-edge-connected. Since H is connected, there exists a 1-edge-cut of H which partitions $V(H)$ into $\{V_1, V_2\}$. Since (H, ψ) is loop-free and $\delta(H) \geq 2$, we know $|V_1|, |V_2| \geq 2$. Let $|E| = 2|V(H)| - k$ for some $k \geq 2$. Since (H, ψ) is Γ-tight, this implies $|E(V_i)| \leq 2|V_i| - k$ for $i \in \{1, 2\}$. Hence

$$2|V(H)| - k = |E| = |E(V_1)| + |E(V_2)| + 1 \leq 2|V(H)| - 2k + 1.$$

This contradicts the fact $k \geq 2$. $\square$

A key requirement in our later proof that $\mathcal{S}(H, \psi)$ is a matroid, is that the union of intersecting tight sets is itself tight. We first prove this in some special cases.

Lemma 11.1.4. *Let Γ be a point group, and let (H, ψ) be a sparse Γ-gain graph. Let $E_1, E_2 \subseteq E(H)$ be tight edges sets with $E_1 \cap E_2 \neq \emptyset$. Suppose that either*

(i) the subgraph $(V(E_1) \cap V(E_2), E_1 \cap E_2)$ is connected, or

(ii) $\max_{i \in \{1,2\}} \alpha(E_i) + \max_{i \in \{1,2\}} \beta(E_i) = 3$.

Then $E_1 \cap E_2$ and $E_1 \cup E_2$ are tight.

Proof. Trivially $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$. Since E_1 and E_2 are tight, and $E_1 \cap E_2$ and $E_1 \cup E_2$ are sparse, this gives

$$s(E_1 \cup E_2) + s(E_1 \cap E_2) \geq |E_1 \cup E_2| + |E_1 \cap E_2| = s(E_1) + s(E_2). \quad (11.1)$$

Our condition $E_1 \cap E_2 \neq \emptyset$ ensures the above sets are non-empty. Hence, the definition of s gives

$$\begin{aligned} &\alpha(E_1 \cup E_2) + \beta(E_1 \cup E_2) + \alpha(E_1 \cap E_2) + \beta(E_1 \cap E_2) \\ &\qquad \geq \alpha(E_1) + \beta(E_1) + \alpha(E_2) + \beta(E_2), \end{aligned} \quad (11.2)$$

Note that equality holds in (11.2) if and only if equality holds in (11.1); in other words, if and only if both $E_1 \cap E_2$ and $E_1 \cup E_2$ are tight.

By the definition of s, we know

$$\gamma(E_1 \cup E_2) \geq \max_{i \in \{1,2\}} \gamma(E_i) \quad \text{for } \gamma \in \{\alpha, \beta\}. \quad (11.3)$$

We use this inequality to split the remaining proof into three cases: when (11.3) holds with equality for both α and β, when (11.3) is a strict inequality for β, and when it is strict for α but not β. Note that if hypothesis (ii) holds, then (11.3) holds with equality for both α and β, and so we are in the first of these three cases. Hence we can assume hypothesis (i) in the second and third cases.

Case 1. $\gamma(E_1 \cup E_2) = \max_{i \in \{1,2\}} \gamma(E_i)$ for $\gamma \in \{\alpha, \beta\}$.
Then (11.2) simplifies to

$$\alpha(E_1 \cap E_2) + \beta(E_1 \cap E_2) \geq \min_{i \in \{1,2\}} \alpha(E_i) + \min_{i \in \{1,2\}} \beta(E_i).$$

However, since $\gamma(F) \geq \gamma(F')$ whenever $F \supseteq F'$, we know that $\gamma(E_1 \cap E_2) \leq \min_{i \in \{1,2\}} \gamma(E_i)$, so this must hold with equality. Thus both $E_1 \cup E_2$ and $E_1 \cap E_2$ are tight.

Case 2. $\beta(E_1 \cup E_2) > \max_{i\in\{1,2\}} \beta(E_i)$, and hypothesis (i) holds.

By definition $\beta(F) \in \{0,1\}$ for all $F \subseteq E(H)$. Hence $\beta(E_1 \cup E_2) = 1$ and $\beta(E_1) = \beta(E_2) = \beta(E_1 \cap E_2) = 0$. So E_1 and E_2 are pure. Since $E_1 \cap E_2 \neq \emptyset$ and $\beta(E_1 \cup E_2) = 1$, this implies $E_1 \cup E_2$, and all of its subsets, are length-pure. By hypothesis (i) and Lemma 10.1.5, these values of β imply that E_1 and E_2 are rotational, $E_1 \cap E_2$ is balanced, and $E_1 \cup E_2$ is dihedral. Hence

$$\alpha(E_1) + \beta(E_1) + \alpha(E_2) + \beta(E_2) = 2 + 0 + 2 + 0 = 4,$$
$$\alpha(E_1 \cup E_2) + \beta(E_1 \cup E_2) + \alpha(E_1 \cap E_2) + \beta(E_1 \cap E_2) = 2 + 1 + 0 + 0 = 3.$$

These equations contradict (11.2). Hence this case cannot occur.

Case 3. $\alpha(E_1 \cup E_2) > \max_{i\in\{1,2\}} \alpha(E_i)$, $\beta(E_1 \cup E_2) = \max_{i\in\{1,2\}} \beta(E_i)$ and hypothesis (i) holds.

Then $\alpha(E_i) \leq 1$, so E_1 and E_2 are either balanced or reflectional. Since hypothesis (i) holds and $\alpha(E_1 \cup E_2) > \alpha(E_i)$, Lemma 10.1.5(ii) implies both E_i are reflectional. Hence, by Lemma 10.1.5(iii), $E_1 \cap E_2$ is balanced and $E_1 \cup E_2$ is dihedral. This gives

$$\alpha(E_1) + \alpha(E_2) = 1 + 1 = 0 + 2 = \alpha(E_1 \cap E_2) + \alpha(E_1 \cup E_2).$$

This, and the fact $\beta(E_1 \cup E_2) = \max_{i\in\{1,2\}} \beta(E_i)$, reduces (11.2) to

$$\beta(E_1 \cap E_2) \geq \min_{i\in\{1,2\}} \beta(E_i).$$

By definition, $\beta(E_1 \cap E_2) \leq \min_{i\in\{1,2\}} \beta(E_i)$. Thus (11.2) holds with equality, and so both $E_1 \cup E_2$ and $E_1 \cap E_2$ are tight. $\square$

We now extend the above result to all intersecting tight edge sets.

Lemma 11.1.5. *Let Γ be a point group, and let (H, ψ) be a sparse Γ-gain graph. Let $E_1, E_2 \subseteq E(H)$ be tight edge sets with $E_1 \cap E_2 \neq \emptyset$. Then $E_1 \cup E_2$ is tight.*

Proof. If $\max_{i\in\{1,2\}} \alpha(E_i) + \max_{i\in\{1,2\}} \beta(E_i) = 3$, or the subgraph $(V(E_1) \cap V(E_2), E_1 \cap E_2)$ is connected, then we are done by Lemma 11.1.4. So suppose neither of these hold. Let $\{F_1, \ldots, F_t\}$ be a partition of $E_1 \cap E_2$ such that $H[F_j]$ is a connected component of $H[E_1 \cap E_2]$. Since H is sparse, we know $0 < |F_j| \leq s(F_j)$. Hence

$$|E_1 \cap E_2| = \sum_{j=1}^{t} |F_j| \leq \sum_{j=1}^{t} (2|V(F_j)| - 3 + \alpha(F_j) + \beta(F_j)).$$

We know $\gamma(F_j) \leq \gamma(E_1 \cap E_2)$ for $\gamma \in \{\alpha, \beta\}$. Let U denote the set of isolated vertices in $(V(E_1) \cap V(E_2), E_1 \cap E_2)$. Then $\{U, V(F_1), V(F_2), \ldots, V(F_t)\}$ is a partition of $V(E_1) \cap V(E_2)$. Using these two facts in the above inequality, we obtain

$$|E_1 \cap E_2| \leq 2|V(E_1) \cap V(E_2)| - 2|U| - t(3 - \alpha(E_1 \cap E_2) - \beta(E_1 \cap E_2)).$$

Substituting this into $|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$, and using the fact E_1 and E_2 are tight, gives

$$\begin{aligned} |E_1 \cup E_2| \geq{}& 2|V(E_1) \cup V(E_2)| - 6 + \alpha(E_1) + \beta(E_1) + \alpha(E_2) + \beta(E_2) \\ &+ 2|U| + t(3 - \alpha(E_1 \cap E_2) - \beta(E_1 \cap E_2)). \end{aligned} \tag{11.4}$$

However, (H, ψ) is sparse, so $2|V(E_1) \cup V(E_2)| \geq |E_1 \cup E_2|$. Clearly $\gamma(E_i) \geq \gamma(E_1 \cap E_2)$ for $\gamma \in \{\alpha, \beta\}$. These observations reduce (11.4) to the simpler inequality

$$0 \geq 2|U| + (t - 2)(3 - \alpha(E_1 \cap E_2) - \beta(E_1 \cap E_2)). \tag{11.5}$$

Since E_1 and E_2 do not satisfy condition (ii) of Lemma 11.1.4, we know $3 - \alpha(E_1 \cap E_2) - \beta(E_1 \cap E_2) > 0$. So for (11.5) to hold, we must have $t \leq 2$. Since $E_1 \cap E_2 \neq \emptyset$ and $(V(E_1) \cap V(E_2), E_1 \cap E_2)$ is not connected, this implies $t \geq 1$ and $|U| + t \geq 2$. So there are exactly two cases to consider: either $t = 2$ and $|U| = 0$, or $t = |U| = 1$.

First suppose $t = 2$ and $|U| = 0$. Since $\gamma(E_i) \geq \gamma(E_1 \cap E_2)$ for $\gamma \in \{\alpha, \beta\}$, (11.4) implies $|E_1 \cup E_2| \geq 2|V(E_1) \cup V(E_2)| \geq s(E_1 \cup E_2)$. But $E(H)$ is sparse, so this must hold with equality. Hence $E_1 \cup E_2$ is tight.

Suppose instead $t = |U| = 1$. Since $\gamma(E_i) \geq \gamma(E_1 \cap E_2)$ for $\gamma \in \{\alpha, \beta\}$, (11.4) implies

$$|E_1 \cup E_2| \geq 2|V(E_1) \cup V(E_2)| - 1 + \max_{i \in \{1,2\}} \alpha(E_i) + \max_{i \in \{1,2\}} \beta(E_i). \quad (11.6)$$

Hence $\max_{i \in \{1,2\}} \alpha(E_i) + \max_{i \in \{1,2\}} \beta(E_i) \leq 1$; and if this holds with equality, then $E_1 \cup E_2$ is tight. Suppose $\max_{i \in \{1,2\}} \alpha(E_i) + \max_{i \in \{1,2\}} \beta(E_i) = 0$. Then both E_1 and E_2 are balanced and pure. Since $E_1 \cap E_2 \neq \emptyset$, $E_1 \cup E_2$ is also pure, and by Lemma 10.1.6, $\langle E_1 \cup E_2 \rangle$ is either balanced or cyclic. Hence $s(E_1 \cup E_2) \leq 2|V(E_1) \cup V(E_2)| - 1$. Since H is sparse, this, together with (11.6) implies $s(E_1 \cup E_2) = |E_1 \cup E_2| = 2|V(E_1) \cup V(E_2)| - 1$. Thus $E_1 \cup E_2$ is tight, as required. □

We require one further structural result.

Lemma 11.1.6. *Let Γ be a point group, and let (H, ψ) be a Γ-gain graph. Let E_1, E_2 be edge sets with $\emptyset \neq E_1 \subseteq E_2 \subseteq E(H)$, such that for each $i \in \{1, 2\}$, either $s(E_i) = 2|V(E_i)|$, or $H[E_i]$ is connected. Let $e \in E(H) - E_2$. If $s(E_1) = s(E_1 + e)$, then $s(E_2) = s(E_2 + e)$.*

Proof. Since $s(E_1) = s(E_1 + e)$, the definition of s implies $V(E_1 + e) = V(E_1)$, $\alpha(E_1 + e) = \alpha(E_1)$ and $\beta(E_1 + e) = \beta(E_1)$. As both endvertices of e are contained in $V(E_1)$ and $V(E_1) \subseteq V(E_2)$, this gives $V(E_2 + e) = V(E_2)$. So to prove that $s(E_2 + e) = s(E_2)$, it suffices to show

$$\gamma(E_2 + e) = \gamma(E_2) \quad \text{for } \gamma \in \{\alpha, \beta\}. \quad (11.7)$$

The functions α and β are non-decreasing, so if they obtain their maximal values on E_2, then (11.7) holds. So it only remains to consider the cases (11.7) where $\alpha(E_2) + \beta(E_2) \leq 2$.

Suppose $\alpha(E_2) + \beta(E_2) \leq 2$. Since $E_1 \subseteq E_2$, this implies $\alpha(E_1) + \beta(E_1) \leq \alpha(E_2) + \beta(E_2) \leq 2$. Hence, by our hypothesis, $H[E_1]$ and $H[E_2]$ are connected. Let T_1 be a spanning tree of $H[E_1]$, and extend this to a spanning tree T_2 of E_2. By Propositions 10.1.3 and 10.1.4, there is a gain function ψ' equivalent to ψ such that for all $F \subseteq E_2 + e$ we have

$$\langle F\rangle = \langle \psi'(f) : f \in F - T_2\rangle. \tag{11.8}$$

First, suppose $\alpha(E_2) \leq 1$. Since $E_1 \subseteq E_2$, this implies $\alpha(E_1) = \alpha(E_1 + e) \leq 1$. Thus E_1 and $E_1 + e$ are either both balanced or both reflectional, giving $\langle E_1\rangle = \langle E_1{+}e\rangle$. Hence, by the definition of ψ', we have $\psi'(e) \in \langle E_1\rangle \subseteq \langle E_2\rangle$, which in turn implies $\langle E_2 + e\rangle = \langle E_2\rangle$. Hence $\alpha(E_2 + e) = \alpha(E_2)$, as required. If $\beta(E_2) = 1$, then trivially $\beta(E_2 + e) = \beta(E_2)$. So suppose $\beta(E_2) = 0$. Then $\beta(E_1 + e) = \beta(E_1) = 0$. Thus E_1, E_2 and e are pure of the same type. Since $\langle E_2 + e\rangle = \langle E_2\rangle$, this implies $\beta(E_2 + e) = \beta(E_2)$.

Instead, suppose $\alpha(E_2) = 2$ and $\beta(E_2) = 0$. Then trivially $\alpha(E_2 + e) = \alpha(E_2) = 2$, so it just remains to show $\beta(E_2 + e) = \beta(E_2)$. Since $E_1 \subseteq E_2$ and β is increasing, we know $\beta(E_2) = \beta(E_1) = \beta(E_1 + e) = 0$. Hence E_1, E_2 and e are pure of the same type. If $E_2 + e$ is direction-pure, then $\beta(E_2 + e) = \beta(E_2) = 0$ and we are done. So suppose $E_2 + e$ is length-pure. Since $\alpha(E_2) = 2$ and $\beta(E_2) = 0$, E_2 is rotational. Thus E_1 is either rotational or balanced. Since $\beta(E_1 + e) = \beta(E_1)$, (11.8) implies $\psi'(e)$ is either a rotation or the identity. Hence, applying (11.8) once more, $E_2 + e$ is rotational, and so $\beta(E_2 + e) = 0 = \beta(E_2)$ as required. □

11.1.2 The sparsity matroid

Now we know the technical results hold, the proof of the following Theorem is identical to that of Theorem 7.4 in [22]. We include it for completeness. There is one further definition we require. For a Γ-gain graph (H, ψ), and edge set $E' \subseteq E(H)$, we say F is *maximal sparse in* E' if F is sparse, $F \subseteq E'$, and there exists no sparse set F' with $F \subset F' \subseteq E'$.

Theorem 11.1.7. *Let* Γ *be a point group, and* (H, ψ) *a* Γ*-gain graph. Then* $\mathcal{S}(H, \psi) = (E(H), \mathcal{E})$ *is a matroid with rank function given by*

$$\operatorname{rank}(E') = \min \left\{ \sum_{i=1}^{t} s(E'_i) : \{E'_1, \ldots, E'_t\} \text{ is a partition of } E' \right\}$$

for all $\emptyset \neq E' \subseteq E(H)$.

Proof. Recall $\mathcal{E}$ is the family of sparse sets in $E(H)$. To prove $(E(H), \mathcal{E})$ is a matroid, it suffices to show it satisfies the three matroid axioms: (I1), (I2) and (R3), of Lemma 3.0.2.

The definition of s immediately gives (I1) and (I2), so it only remains to prove (R3). Let $E' \subseteq E(H)$, and let F be a maximal sparse set in E'.

A non-loop edge $e \in E(H)$ always forms a tight, pure, balanced set $\{e\}$. Since F is sparse, each loop edge $e \in F$ is either a length edge with non-trivial gain, or a direction edge with reflection gain; in both cases $\{e\}$ is a tight, pure, cyclic set. Hence $\{e\}$ is tight for all $e \in F$. By Lemma 11.1.5, maximal tight sets do not intersect. So, by taking unions of these tight, single edge sets, we obtain a partition $\{F_1, F_2, \ldots, F_t\}$ of F into maximal tight sets. Hence $|F| = \sum_{i=1}^{t} |F_i| = \sum_{i=1}^{t} s(F_i)$.

We extend this partition to a partition of E'. Let $e \in E' - F$. Since F is a maximal sparse set in E', we know $F + e$ is not sparse. So there exists a tight set $X \subseteq F$ such that $X + e \notin \mathcal{E}$. Since s is non-decreasing, this gives $|X + e| = |X| + 1 > s(X + e) \geq s(X) = |X|$, and thus $s(X + e) = s(X)$. By Lemma 11.1.5, $X \subseteq F_i$ for some $1 \leq i \leq t$. Since X is tight, either $s(X) = 2|X|$, or $H[X]$ is connected by Proposition 11.1.3 (iii). Similarly, either $s(F_i) = 2|V(F_i)|$, or $H[F_i]$ is connected. Thus, X and F_i satisfy the hypotheses of Lemma 11.1.6, which implies $s(F_i + e) = s(F_i) = |F_i|$. By repeating this argument for all $e \in E' - F$, we obtain a partition $\{E_1, E_2, \ldots, E_t\}$ of E' such that $E_i \supseteq F_i$ and $s(E_i) = s(F_i)$ for all $1 \leq i \leq t$. Hence $|F| = \sum_{i=1}^{t} s(E_i)$. But F is sparse, so

$$|F| \leq \min \left\{ \sum_{i=1}^{t'} s(E'_i) : \{E'_i\} \text{ is a partition of } E' \right\} \leq \sum_{i=1}^{t} s(E_i).$$

Since equality holds throughout, this implies

$$|F| = \min \left\{ \sum_{i=1}^{t'} s(E'_i) : \{E'_i\} \text{ is a partition of E'} \right\},$$

thus proving (R3). □

We refer to $\mathcal{S}(H, \psi)$, as the *sparsity matroid of* (H, ψ), or simply as the *sparsity matroid* when the relevant graph is clear from the context. Following standard matroid terminology, a set $E' \subseteq E$ is *independent* in $(E, \mathcal{E})$ if $E' \in \mathcal{E}$ and *dependent* otherwise. So Theorem 11.1.7 proves that sparsity is equivalent to independence in the sparsity matroid.

11.2 Inductive constructions

In the previous section, we introduced the sparsity matroid for Γ-gain graphs. Here we consider the class of sparse graphs (H, ψ), which have the maximum number of edges given $|V(H)|$ and Γ. We call such graphs Γ-*tight*.

More precisely, given $n \in \mathbb{N}$ and a point group Γ, let $(H_{\Gamma,n}, \psi')$ be the Γ-gain graph on n vertices where for every pair of vertices $u, v \in V(H_{\Gamma,n})$, and every $g \in \Gamma$, $H_{\Gamma,n}$ contains both a direction and a length $\overrightarrow{uv}$-edge with gain g. A Γ-gain graph (H, ψ) on n vertices is Γ-*tight* if $E(H)$ is tight with $\langle E(H) \rangle \subseteq \Gamma$ and $s(E(H)) = s(E(H_{\Gamma,n}))$. Note this is different from the definition of maximal tight sets of the previous section.

We wish to show that, for a given Γ, we can inductively construct all Γ-tight graphs from a handful of base graphs, using a small set of moves.

In this Chapter, we shall obtain such inductive constructions when Γ is a reflection group or a rotation group, and obtain some partial results for dihedral groups.

We first introduce the base graphs for each Γ in Subsection 11.2.1. We then introduce our six inductive graph moves, and show that these preserve sparsity in Subsection 11.2.2. Finally, we consider when a Γ-tight graph can be obtained from a smaller Γ-tight graph using one of our inductive moves. For three of our moves, this is quite involved, and so these cases are dealt with separately in Section 11.3.

The remaining three moves are considered in Subsection 11.2.4, leading to a characterisation of the $\mathcal{C}_s$-tight graphs in Subsection 11.2.5, and of the $\mathcal{C}_k$-tight graphs in Subsection 11.3.5.

11.2.1 Base graphs

For a point group Γ, the Γ-*base graphs* are the Γ-tight graphs on exactly one vertex. The reflection group $\mathcal{C}_s$ has a unique base graph, see Figure 11.1(a).

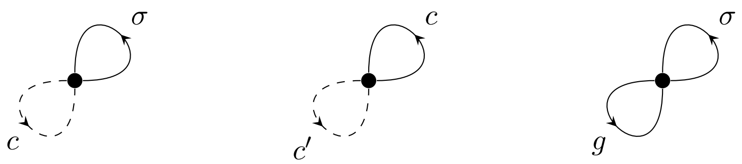
For rotation groups, $\mathcal{C}_{k\geq 2}$, the base graph is unique up to the choice of gains, see Figure 11.1(b). A $\mathcal{C}_k$-tight gain graph need not be connected, however its connected components must be $\mathcal{C}_k$-tight. So we instead construct each $\mathcal{C}_k$-tight graph from a graph whose connected components are each a $\mathcal{C}_k$-base graph.

A $\mathcal{D}_{k\geq 2}$-tight gain graph can also be disconnected, in which case it is not constructible from a single $\mathcal{D}_k$-base graph. So we must consider building such graphs from one whose connected components are each $\mathcal{D}_k$-base graphs. Figure 11.2 shows the three different types of $\mathcal{D}_k$-base graph. Note that the edge set of these base graphs need not induce a dihedral group (Figure 11.2(b)), and could be length-pure (Figure 11.2(c)).

(a) The unique $\mathcal{C}_s$-base graph; $\langle\sigma\rangle = \mathcal{C}_s$.

(b) A $\mathcal{C}_k$-base graph. The gains c and c' are non-trivial rotations in $\mathcal{C}_k$.

Figure 11.1: Base graphs for $\mathcal{C}_s$ and $\mathcal{C}_k$.



(a) A mixed $\mathcal{D}_k$-base graph which induces a dihedral group.

(b) A mixed $\mathcal{D}_k$-base graph which induces a rotational group.

(c) A length-pure $\mathcal{D}_k$-base graph which induces a dihedral group.

Figure 11.2: The three types of $\mathcal{D}_k$-base graphs for $k \geq 2$. The gains c and c' are non-trivial rotations, σ is a reflection, and $g \neq \sigma$ is either a reflection or a non-trivial rotation in $\mathcal{D}_k$.

11.2.2 Extensions

Here we define a set of moves which each add one vertex to a Γ-tight gain graph. In Subsection 11.2.3, we shall prove that these moves preserve sparsity. The definition of the sparsity function implies the following result:

Proposition 11.2.1. *Let Γ be a point group, and (H,ψ) be a Γ-tight gain graph.*

(i) If (H,ψ) is $\mathcal{C}_s$-tight, then H has a vertex of degree 2 or 3.

(ii) If (H,ψ) is $\mathcal{C}_{k\geq 2}$-tight or $\mathcal{D}_{k\geq 2}$-tight, then H has a vertex of degree 2, 3 or 4.

This Proposition implies that in order to construct all Γ-tight gain graphs, we need only consider inductive moves which add a vertex of degree 2, 3 or 4. Let (H, ψ) be a Γ-gain graph. The first three moves we define add a vertex of degree 2 or 3.

A *0-extension* adds a single vertex v to (H, ψ), incident to two non-loop edges $e_1 = \overrightarrow{vv_1}$ and $e_2 = \overrightarrow{vv_2}$ for some $v_1, v_2 \in V(H)$. See Figure 11.3. Sparse graphs cannot have duplicate edges, so if $v_1 = v_2$ and the edges e_i are the same type, our gain assignment ψ must satisfy $\psi(e_1) \neq \psi(e_2)$. This caveat (that parallel edges must either have different gains or different types) applies to every extension we define. However, for brevity, we shall not repeat it.

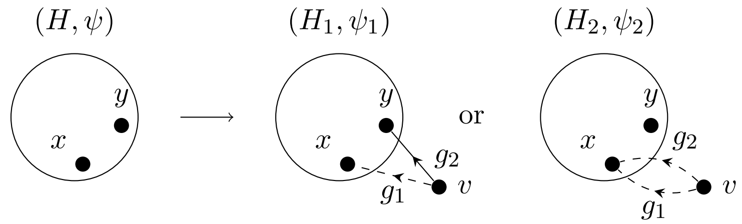


Figure 11.3: Two different 0-extensions which construct (H_i, ψ_i) from (H, ψ). In (H_2, ψ_2), the 0-extension adds two parallel direction edges, so $g_1 \neq g_2$.

A *loop 0-extension* adds a vertex v to (H, ψ) incident to two edges: a loop l with non-identity gain, and some non-loop edge $\overrightarrow{vv_1}$ where $v_1 \in V(H)$. If l is a direction loop, then its gain is a non-trivial rotation. See Figure 11.4.

A *1-extension* on (H, ψ) first deletes some edge $e = \overrightarrow{v_1v_2} \in E(H)$ with gain g, and then adds a vertex v incident to three non-loop edges $\overrightarrow{vv_1}$, $\overrightarrow{vv_2}$ and $\overrightarrow{vv_3}$ for some $v_3 \in V(H)$. At least one of these edges is of the same type as e, and the gain assignment ψ satisfies $\psi(\overrightarrow{v_1v})\psi(\overrightarrow{vv_2}) = \psi(\overrightarrow{v_1v_2})$. See Figures 11.5 and 11.6.

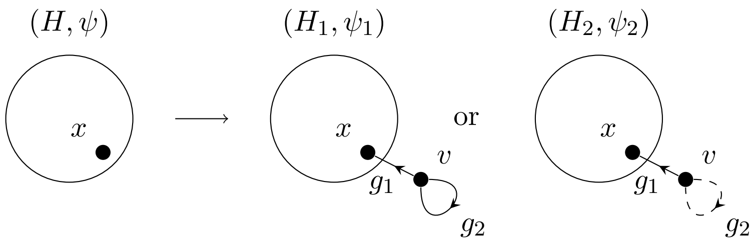


Figure 11.4: Two different loop 0-extensions which construct (H_i, ψ_i) from (H, ψ). In (H_2, ψ_2), the loop is a direction edge, so g_2 is a non-trivial rotation.

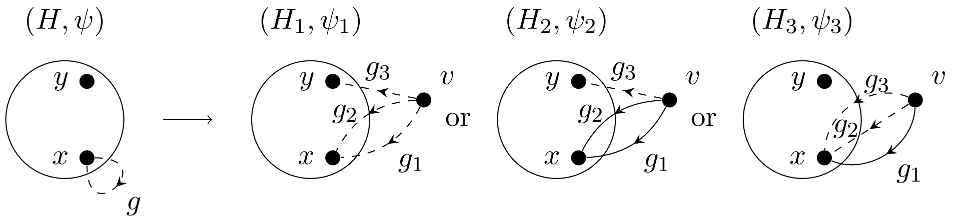


Figure 11.5: Three different 1-extensions on a loop edge. In each of these extensions $g_1^{-1} g_2 = g$. Our parallel gain condition requires $g_1 \neq g_2$ in (H_1, ψ_1) and (H_2, ψ_2), and $g_2 \neq g_3$ in (H_3, ψ_3).

The final three moves each add a vertex of degree 4.

A *loop 1-extension* is identical to a 1-extension, except that instead of adding a third non-loop edge $\overrightarrow{vv_3}$, we add a loop at v. The gain on this loop satisfies the usual rules: it is non-trivial on length loops, and a non-trivial rotation on direction loops. See Figure 11.7.

A *loop-to-loop extension* deletes a loop with gain g at some vertex v_1, then adds a vertex v by a 0-extension, incident to two edges $\overrightarrow{vv_1}$ and $\overrightarrow{vv_2}$ for some $v_2 \in V(H)$ with gains g_1 and g_2 respectively. Finally, it adds a loop at v with gain $g_1 g g_1^{-1}$. As with a 1-extension, at least one of the three edges added must be the same type as the deleted loop. Additionally, if g is a reflection, then the loop added at v must be a length loop (since $g_1 g g_1^{-1}$ is a reflection if and only if g is). See Figure 11.8. Observe that the graphs

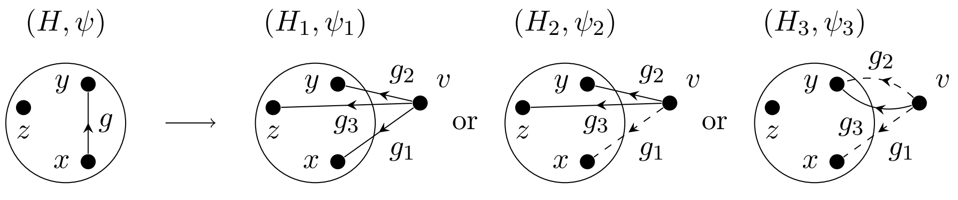


Figure 11.6: Three different 1-extensions on a non-loop edge. In each (H_i,ψ_i) we have $g_1^{-1}g_2 = g$.

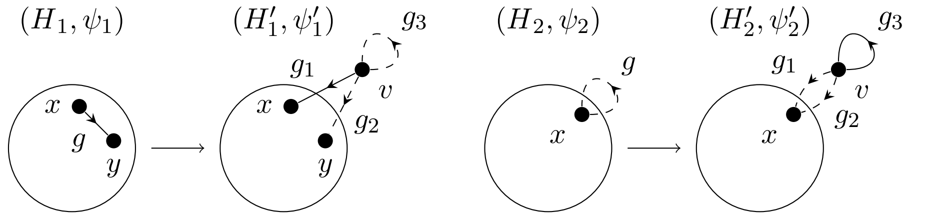


Figure 11.7: Two loop 1-extensions which form the graphs (H_i',ψ_i') from (H_i,ψ_i) by deleting a non-loop edge and a loop respectively. In both cases $g_1^{-1}g_2 = g$; and in (H_2',ψ_2'), we have $g_1 \neq g_2$.

(H_i',ψ_i') in Figures 11.7 and 11.8, differ only in their gain assignment.

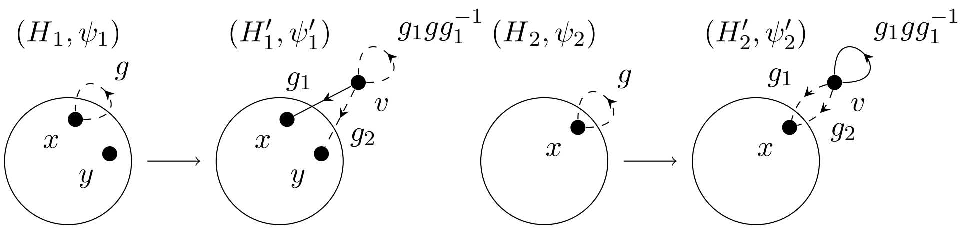


Figure 11.8: Two examples of loop-to-loop extensions. The only restriction on g_1 and g_2 is that when they appear on parallel edges of the same type, as in (H_2',ψ_2'), we have $g_1 \neq g_2$.

Finally, a *2-extension* deletes two edges $e = \overrightarrow{v_1v_2}$ and $f = \overrightarrow{v_3v_4}$ in (H,ψ), and then adds a vertex v incident to four edges: $e_i = \overrightarrow{vv_i}$ for $1 \le i \le 4$. The gains are assigned so that $\psi^{-1}(e_1)\psi(e_2) = \psi(e)$ and $\psi^{-1}(e_3)\psi(e_4) = \psi(f)$. We also restrict the types of the added edges: either at least one of e_1, e_2 is assigned the same type as e, and at least one of e_3, e_4 is the same type as f;

or e_1, e_2 and f are all the same type, and e_3, e_4 and e are all the other. This peculiar constraint on edge types results from the simpler restriction in our definition of a 1-extension. Namely, to guarantee our 2-extension maintains sparsity, we must ensure that for every choice of three added edges, the corresponding 1-extension is valid. For example, for the choice e_1, e_3, e_4; the corresponding 1-extension is that which deletes f, and so at least one of e_1, e_3, e_4 must be the same type as f. Figure 11.9 illustrates possible 2-extensions given different relative positions and types of the deleted edges e and f.

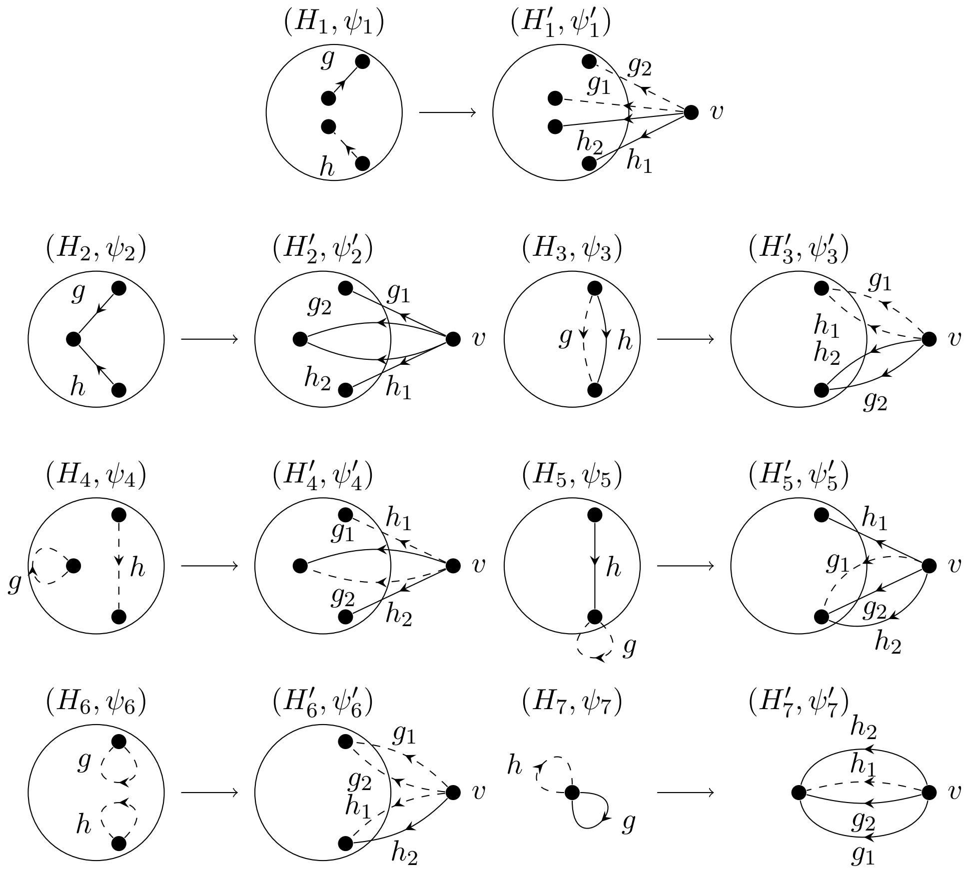


Figure 11.9: Seven different 2-extensions. In each case, $h_1^{-1}h_2 = h$ and $g_1^{-1}g_2 = g$.

To the best of our knowledge, the loop-to-loop extension is a new addition to the literature, and allows us to significantly reduce the number of base graphs we consider. The other moves are direction-length analogues of ones seen in [22], although we use a slightly different naming convention. Here the number corresponds to the number of edges deleted during the extension, and the addition or omission of the word "loop" identifies whether or not we add a loop incident to our new vertex. We refer to these moves collectively as *extensions*.

11.2.3 Extensions preserve sparsity

The following Lemma shows that all of our defined extensions preserve sparsity. Note that when (H, ψ) is a Γ-gain graph, we have $\langle E(H) \rangle_\psi \subseteq \Gamma$. In other words, the edge set of the graph may not induce the entire group Γ.

Lemma 11.2.2. *Let Γ be a point group, and (H, ψ) be a sparse Γ-gain graph. Performing an extension on (H, ψ), which adds edges with gains in Γ, results in a sparse Γ-gain graph (H', ψ') with $|V(H')| = |V(H)| + 1$ and $|E(H')| = |E(H)| + 2$.*

Proof. Each (loop) k-extension adds a vertex v to (H, ψ) incident to $k + 2$ edges, and then deletes k edges between the neighbours of v. When $k = 1$ this also describes a loop-to-loop extension. So the definition of our extensions immediately gives $|V(H')| = |V(H)| + 1$ and $|E(H')| = |E(H)| + 2$. Since all added edges have gains in Γ, we also know $\langle E(H') \rangle_{\psi'} \subseteq \Gamma$, and hence (H', ψ') is a Γ-gain graph. It remains to prove that (H', ψ') is sparse.

Let $\emptyset \neq F' \subseteq E(H')$. We define a corresponding edge set $F \in E(H)$ as follows: when (H', ψ') is formed by a (loop) 0-extension then $F = F' \cap E(H)$. When (H', ψ') is formed by a (loop) 1-extension or a loop-to-loop extension which deletes an edge e, and adds the edges e_1, e_2, e_3 then

$$F = \begin{cases} (F' \cap E(H)) \cup \{e\} & \text{if } F' \supseteq \{e_1, e_2, e_3\}, \\ F' \cap E(H) & \text{otherwise.} \end{cases}$$

Finally, when (H', ψ') is formed by a 2-extension which deletes e and f, and replaces them with the respective pairs of edges $\{e_1, e_2\}$ and $\{f_1, f_2\}$, then

$$F = \begin{cases} (F' \cap E(H)) \cup \{e, f\} & \text{if } F' \supseteq \{e_1, e_2, f_1, f_2\}, \\ (F' \cap E(H)) \cup \{e\} & \text{if } e_1, e_2 \in F' \text{ and } |\{f_1, f_2\} \cap F'| = 1, \\ (F' \cap E(H)) \cup \{f\} & \text{if } f_1, f_2 \in F' \text{ and } |\{e_1, e_2\} \cap F'| = 1, \\ F' \cap E(H) & \text{otherwise.} \end{cases}$$

Claim 11.2.3. $\langle F' \rangle_{\psi'} \supseteq \langle F \rangle_{\psi}$.

Proof. By the definitions above, we know $0 \leq |F - F'| \leq 2$. First, suppose $|F - F'| = 0$. Then $F \subseteq F'$, and so trivially $\langle F \rangle_{\psi} \subseteq \langle F' \rangle_{\psi'}$.

Instead, suppose $|F - F'| = 1$. Let $e = \overrightarrow{v_1 v_2}$ denote the unique edge in $F - F'$, where v_1, v_2 need not be distinct. Suppose (H', ψ') was formed by a loop-to-loop extension. Then e is a loop with gain g, and our choice of F ensures that $F' - F \supseteq \{e_1, e_2, e_3\}$, where e_1 is a $\overrightarrow{vv_1}$-edge with gain g_1 and e_3 is a loop at v with gain $g_1 g g_1^{-1}$. Let C be a closed walk in F containing e, and let C' be the corresponding closed walk in F' which replaces every occurrence of $v_1 e v_1$ with the walk $v_1 e_1 v e_3 v e_1 v_1$. Our choice of gains gives $\psi'(C') = \psi(C)$. Hence $\langle F' \rangle_{\psi'} \supseteq \langle F \rangle_{\psi}$. In all other types of extension, our choice of F ensures H contains a pair of edges $e_1 = \overrightarrow{vv_1}$ and $e_2 = \overrightarrow{vv_2}$ such that $(\psi')^{-1}(e_1)\psi'(e_2) = \psi(e)$. Let C be a closed walk in $E(H)$ containing e, and let C' be the corresponding closed walk in $E(H')$ which replaces every occurrence of $v_1 e v_2$ with the path $v_1 e_1 v e_2 v_2$. Our choice of gains again gives $\psi'(C') = \psi(C)$. Hence $\langle F' \rangle_{\psi'} \supseteq \langle F \rangle_{\psi}$.

Finally, suppose $|F-F'|=2$. Then (H',ψ') was formed by a 2-extension, which replaced $e=\overrightarrow{v_1v_2}$ and $f=\overrightarrow{u_1u_2}$ with the respective pairs of edges $e_i=\overrightarrow{vv_i}$, and $f_i=\overrightarrow{vu_i}$ for $i\in\{1,2\}$, such that $\psi(e)=(\psi')^{-1}(e_1)\psi'(e_2)$ and $\psi(f)=(\psi')^{-1}(f_1)\psi'(f_2)$. Our choice of F ensures $e_1,e_2,f_1,f_2\in F'$ and $e,f\in F$. Let C be a closed walk in $E(H)$. We form a corresponding closed walk C' in $E(H')$ by replacing every occurrence of v_1ev_2 in C with $v_1e_1ve_2v_2$, and every occurrence of u_1fu_2 with $u_1f_1vf_2u_2$. Our choice of gains on these edges ensures $\psi'(C')=\psi(C)$. Hence $\langle F'\rangle_{\psi'}\supseteq\langle F\rangle_{\psi}$. □

Assume for a contradiction that F' is not sparse. If F' contains no edges incident to v, then $F'\subseteq E(H)$, and so F' is sparse since (H,ψ) is. Hence $v\in V(F')$. The definition of F then implies $V(F)=V(F')-\{v\}$. Since F' is not sparse, we have

$$|F'|>s(F')=2|V(F')|-3+\alpha(F')+\beta(F').$$

Claim 11.2.3 implies $\alpha(F')\geq\alpha(F)$. The restrictions on the types of edges added in extensions, and the definition of F, ensures that whenever $F-F'$ contains a length (direction) edge, then $F'-F$ also contains a length (direction) edge. Hence $\beta(F')\geq\beta(F)$. Substituting these bounds into the above inequality gives

$$|F'|>2|V(F')|-3+\alpha(F)+\beta(F).$$

Since (H,ψ) is sparse, we know $s(F)\geq|F|$. Substituting this into the above inequality, and using the fact $|V(F')|=|V(F)|+1$ gives $|F'|>|F|+2$, which contradicts our choice of F. Hence F' is sparse for all $\emptyset\neq F'\subseteq E(H')$, and so (H',ψ') is also sparse. □

The following is an immediate consequence of Lemma 11.2.2.

Corollary 11.2.4. *Let Γ be a point group. Let (H, ψ) and (H', ψ') be Γ-gain graphs such that (H', ψ') was formed from (H, ψ) by an extension which adds edges with gains in Γ. If (H, ψ) is Γ-tight, then (H', ψ') is Γ-tight.*

It is more difficult to show the converse of this result. Namely, that a maximal Γ-tight gain graph can be deconstructed using these moves until only disjoint Γ-base graphs remain. This is the topic of the remainder of this Chapter.

11.2.4 Reductions at vertices of degree 2 or 3

Let (H, ψ) and (H', ψ') be two Γ-gain graphs for some point group Γ, and suppose (H', ψ') was obtained from (H, ψ) by an extension. Then we can reconstruct (H, ψ) from (H', ψ') by performing the inverse of this move. We refer to the inverse of an extension as a *reduction*. More specifically, the inverse of a loop k-extension, k-extension, and loop-to-loop extension is respectively a *loop k-reduction*, *k-reduction* and *loop-to-loop reduction*. A reduction of a sparse Γ-gain graph is said to be *admissible* if it results in another sparse Γ-gain graph. Otherwise, it is *inadmissible*.

If (H', ψ') is obtained from a Γ-gain graph (H, ψ) by a 0-reduction or loop 0-reduction, then $(H', \psi') \subseteq (H, \psi)$, and so (H', ψ') is also Γ-sparse. This implies the following result:

Proposition 11.2.5. *Let Γ be a point group, and (H, ψ) a sparse Γ-gain graph. Let $v \in V(H)$ be incident to exactly two edges.*

(i) If neither of these edges are loops, then v can be removed by an admissible 0-reduction.

(ii) If exactly one of these edges is a loop, then v can be removed by an admissible loop 0-reduction.

The argument for 1-reductions is more involved and uses properties of the matroid closure from Chapter 3.

Lemma 11.2.6. *Let Γ be a point group, and (H, ψ) a sparse Γ-gain graph. Let $v \in V(H)$ be a vertex of degree 3, which is not incident to any loop. Then there is an admissible 1-reduction at v.*

Proof. Denote the three edges incident to v by $e_i = vv_i$ where $i \in \{1, 2, 3\}$. Note the vertices v_1, v_2 and v_3 need not be distinct. Without loss of generality, we can assume each e_i is oriented towards v, and $\psi(\overrightarrow{v_i v}) = g_i$ for some $g_i \in \Gamma$. Each possible 1-reduction at v deletes v and all edges incident to it, and then adds exactly one edge $\overrightarrow{v_i v_j}$ with gain $\psi(\overrightarrow{v_i v_j}) = g_i g_j^{-1}$ where $i, j \in \{1, 2, 3\}$. If v is pure, then the added edge $\overrightarrow{v_i v_j}$ must be of the same type as the edges deleted, otherwise $\overrightarrow{v_i v_j}$ could be either a direction or a length edge.

Let Y denote the set of all possible edges that could be added in a 1-reduction at v, and let $Y^* = Y \cup \{e_1, e_2, e_3\}$. From the definition of a 1-extension, the gains on the edges in Y are dictated by the gains on e_1, e_2 and e_3: if $e \in Y$ is a $\overrightarrow{v_i v_j}$-edge which replaces the pair $\{e_i, e_j\}$, then the gain on e is $g_i g_j^{-1}$.

Let (H^*, ψ^*) denote the Γ-gain graph obtained from (H, ψ) by adding all of the edges in Y. Recall that the sparsity matroid $\mathcal{S}(H^*, \psi^*) = (E^*, \mathcal{E}^*)$ has $E^* = E(H^*)$, and $\mathcal{E}^*$ as the collection of sparse edge sets in E^*. Assume for a contradiction that no 1-reduction at v is admissible. Then for all $e \in Y$, the set $E(H - v) \cup \{e\}$ is not sparse, and so by Proposition 3.0.3(ii) and (iv), $Y \subseteq \mathrm{cl}(E(H - v))$. We shall use the properties of matroid closure to obtain our contradiction.

Claim 11.2.7. *If v is pure, then there exists a sparse set $Y' \subseteq Y^* - \{e_1\}$ such that Y^* contains no larger sparse set X with $X \supset Y'$.*

Proof. Since v is pure, $Y = \{e_{12}, e_{23}, e_{31}\}$ where $e_{ij} = \overrightarrow{v_i v_j}$ is an edge of the same type as those incident to v, with gain $g_i g_j^{-1}$.

Suppose $|N(v)| = 3$. Then, by Proposition 10.1.3 and Lemma 11.1.1, we may assume $\psi(e_1) = \psi(e_2) = \psi(e_3) = I$, which in turn implies $\psi(e_{ij}) = I$ for all $e_{ij} \in Y$. Hence Y^* is a pure, balanced K_4, and thus $Y' = Y^* - \{e_1\}$.

Suppose $|N(v)| = 2$. Without loss of generality, let $v_2 = v_3$. By Proposition 10.1.3 we can assume $\psi(e_1) = \psi(e_2) = I$ and $\psi(e_3) = g$ for some non-identity $g \in \Gamma$. Hence $\psi(e_{12}) = I$, $\psi(e_{31}) = g$ and e_{23} is a loop with $\psi(e_{23}) = g$. This implies $\langle Y^* \rangle = \langle g \rangle$, and so Y^* is pure and cyclic. If Y^* is direction-pure and reflectional then $Y' = Y^* - \{e_{23}, e_1\}$, otherwise $Y' = Y^* - \{e_1\}$.

Suppose $|N(v)| = 1$. Then all three edges at v are parallel. Since v is pure, each e_i has a different gain. Hence $\{e_1, e_2, e_3\}$ is either rotational or dihedral. By Proposition 10.1.3, we can assume $\psi(e_1) = I$. If $\langle e_1, e_2, e_3 \rangle$ is dihedral and v is length-pure, then $Y' = \{e_2, e_3, e_{1,2}, e_{3,1}\}$; otherwise there exists some i, j for which $g_i g_j^{-1}$ is a rotation, and we can take $Y' = \{e_2, e_3, e_{ij}\}$. $\square$

Claim 11.2.8. *If v is mixed, then there exists a set $Y' \subseteq Y^* - \{e_1\}$ which is maximal sparse in Y^*.*

Proof. Since v is mixed, $Y = \{d_{12}, d_{23}, d_{31}, l_{12}, l_{23}, l_{31}\}$ where d_{ij} and l_{ij} are respectively a direction and length $\overrightarrow{v_i v_j}$ edge with gain $g_i g_j^{-1}$.

Suppose $|N(v)| = 3$. As in the pure case above, Proposition 10.1.3 means we can assume every edge in Y^* has identity gain. Hence Y^* is mixed and balanced, which gives $Y' = \{e_2, e_3, l_{12}, l_{23}, d_{12}, d_{23}\}$.

Suppose $|N(v)| = 2$, and without loss of generality $v_2 = v_3$. Then by Proposition 10.1.3 we can assume $\psi(e_1) = \psi(e_2) = \psi(l_{12}) = \psi(d_{12}) = I$ and all other edges in Y^* have gain g. This implies $\langle Y^* \rangle = \langle g \rangle$ for some

$g \in \Gamma$. If $g = I$, then Y^* is balanced and mixed, which gives $Y' = \{e_2, e_3, d_{12}, l_{12}\}$. So suppose $g \neq I$. Then Y^* is mixed and cyclic. If Y^* is reflectional, then $Y' = \{e_2, e_3, d_{12}, d_{13}, l_{23}\}$; otherwise Y^* is rotational and $Y' = \{e_2, e_3, d_{12}, d_{13}, l_{12}, l_{23}\}$.

Suppose $|N(v)| = 1$. Then e_1, e_2 and e_3 are parallel, so at least one of these edges has non-identity gain. Thus Y^* is unbalanced. By Proposition 10.1.3, we can assume $\psi(e_1) = I$. There are three cases to consider. Firstly, suppose $\langle Y^* \rangle$ is reflectional. Then for some pair i, j, the gain $g_i g_j^{-1}$ is a reflection, and we choose $Y' = \{e_2, e_3, l_{ij}\}$. Secondly, suppose $\langle Y^* \rangle$ is dihedral and both e_2 and e_3 have reflection gain. Then $g_2 \neq g_3$, and thus $g_2 g_3^{-1}$ is a non-trivial rotation. Hence $Y' = \{e_2, e_3, l_{23}, d_{23}\}$. Finally, suppose neither of these cases hold. Then some $e_j \in \{e_2, e_3\}$ has non-trivial rotational gain. In which case, $Y' = \{e_2, e_3, l_{1j}, d_{1j}\}$. □

Claims 11.2.7 and 11.2.8 imply that in all cases there exists some $Y' \subseteq Y^* - \{e_1\}$ such that Y' is sparse but $Y' \cup \{e_1\}$ is not. Hence by Proposition 3.0.3(ii) and (iii) respectively, $e_1 \in \mathrm{cl}(Y') \subseteq \mathrm{cl}(Y^* - \{e_1\}) \subseteq \mathrm{cl}(E(H^*) - \{e_1\})$. Our original assumption that no 1-reduction at v is admissible, and Proposition 3.0.3(iii), give $Y \subseteq \mathrm{cl}(E(H) - \{e_1, e_2, e_3\}) \subseteq \mathrm{cl}(E(H) - \{e_1\})$. Since $E(H^*) - \{e_1\} = (E(H) - \{e_1\}) \cup Y$, Proposition 3.0.3(iv) implies $\mathrm{cl}(E(H^*) - \{e_1\}) = \mathrm{cl}(E(H) - \{e_1\})$. Hence $e_1 \in \mathrm{cl}(E(H) - \{e_1\})$. The definition of closure thus implies that $\mathrm{rank}(E(H) - \{e_1\}) = \mathrm{rank}(E(H))$. Since (H, ψ) is sparse, this implies $|E(H) - \{e_1\}| = |E(H)|$, giving the contradiction sought. □

11.2.5 Construction of $\mathcal{C}_s$-tight graphs

The results in this section lead to a characterisation of $\mathcal{C}_s$-tight graphs:

Theorem 11.2.9. *Let (H, ψ) be a $\mathcal{C}_s$-gain graph. Then (H, ψ) is a $\mathcal{C}_s$-tight graph if and only if (H, ψ) can be constructed from the $\mathcal{C}_s$-base graph by a sequence of 0-extensions, loop 0-extensions and 1-extensions which add edges with gains in $\mathcal{C}_s$.*

Proof. We first prove the forwards direction. Suppose (H, ψ) is a $\mathcal{C}_s$-tight graph with $|V(H)| = n$. Let $H_n = H$. We shall show that we can construct a sequence of $\mathcal{C}_s$-tight graphs (H_i, ψ_i) for $1 \leq i \leq n-1$ such that (H_{i+1}, ψ_{i+1}) is obtained from (H_i, ψ_i) by one of these extensions, $|V(H_i)| = i$, and (H_1, ψ_1) is the unique $\mathcal{C}_s$-base graph.

If (H, ψ) is the unique $\mathcal{C}_s$-base graph, then we are done, so suppose not. Let (H_i, ψ_i) be a $\mathcal{C}_s$-tight gain graph with $i \geq 2$. Propositions 11.1.3 and 11.2.1(i) imply that H_i has a vertex v of degree 2 or 3 which is incident to at least one non-loop edge. If v is incident to exactly two edges, then Proposition 11.2.5 implies that v can be removed by either a 0-reduction or loop 0-reduction to give a $\mathcal{C}_s$-tight graph (H_{i-1}, ψ_{i-1}). Otherwise, $d_H(v) = 3$, and v is incident to exactly 3 non-loop edges, and so by Lemma 11.2.6 we can remove v by a 1-reduction to give a $\mathcal{C}_s$-tight graph (H_{i-1}, ψ_{i-1}). If $i - 1 = 1$, then since (H_{i-1}, ψ_{i-1}) is $\mathcal{C}_s$-tight it must be the unique $\mathcal{C}_s$-base graph. Otherwise, we repeat this argument for (H_{i-1}, ψ_{i-1}).

We now prove the converse. Suppose $\{(H_i, \psi_i)\}_{1 \leq i \leq n}$ is a set of graphs such that (H_1, ψ_1) is the unique $\mathcal{C}_s$-base graph, H_i is obtained from H_{i-1} by a (loop) 0-extension or 1-extension, and $(H_n, \psi_n) = (H, \psi)$. We wish to show (H_i, ψ_i) is $\mathcal{C}_s$-tight for all i. Clearly this holds when $i = 1$. Suppose (H_{i-1}, ψ_{i-1}) is $\mathcal{C}_s$-tight. Then Lemma 11.2.2 implies that (H_i, ψ_i) is sparse with $|E(H_i)| = 2|V(H_i)| - 1$. Further, our restriction on edge gains ensures $\psi_i(e) \in \mathcal{C}_s$ for all $e \in E(H_i)$. Thus every closed walk C in H_i has $\psi_i(C) \in \mathcal{C}_s$, which implies $\langle E(H_i) \rangle \subseteq \mathcal{C}_s$. Hence (H_i, ψ_i) is $\mathcal{C}_s$-tight for all i. In particular, this holds when $i = n$, thus proving our result. □

Graphs which are $\mathcal{C}_{k\geq 2}$-tight or $\mathcal{D}_{k\geq 2}$-tight may contain no vertices of degree 2 or 3. So to obtain a similar construction for graphs in these classes, we must prove that we can find a reduction on some vertex of degree 4 in such a graph. This is the topic of the next section.

11.3 Reductions in 4-regular graphs

In the previous section, we showed that if a Γ-tight graph has a vertex of degree 2 or 3, then there is an admissible reduction at this vertex. However, $\mathcal{C}_{k\geq 2}$ and $\mathcal{D}_{k\geq 2}$-tight graphs may have no such vertices. In these cases, the definition of the sparsity function implies that our graphs satisfy the hypotheses of the following graph-theoretic result:

Proposition 11.3.1. *Let $H = (V, E)$ be a multi-graph with $|E| = 2|V|$, and $i_H(X) \leq 2|X|$ for all $X \subseteq V$. If H has no vertices of degree 2 or 3, then every vertex has degree 4.*

A graph in which every vertex has degree 4 is said to be *4-regular*. So to find an inductive construction of all $\mathcal{C}_k$ and $\mathcal{D}_k$-tight graphs, we only need to consider reductions at degree 4 vertices in 4-regular graphs. The hand-shaking lemma implies the following useful relationship:

Proposition 11.3.2. *Let $H = (V, E)$ and $X \subseteq V$. Then $\sum_{x\in X} d_H(x) = 2i_H(X) + d_H(X, V - X)$.*

Proposition 11.3.2 immediately implies the following result for 4-regular graphs.

Corollary 11.3.3. *Let $H = (V, E)$ be 4-regular. Then $i_H(X) = 2|X| - \frac{1}{2}d_H(X, V - X)$ for all $\emptyset \neq X \subseteq V$. In addition, if H is connected and $X \neq V$, then $i_H(X) \leq 2|X| - 1$.*

Using these properties, we first show, in Subsection 11.3.1, that if a 4-regular graph contains a loop, then it has an admissible reduction. It is more difficult to identify when a 2-reduction is admissible. To simplify our arguments, we restrict to considering 2-reductions in 4-regular, loop-free, $\mathcal{C}_k$-gain graphs. In Subsection 11.3.2 we state some necessary conditions for a 2-reduction to be admissible. In Section 11.3.3, we show that when the graphs we consider are close to being disconnected, we can always find an admissible 2-reduction. Finally, in Subsection 11.3.4, we prove that we can find an admissible 2-reduction in the remaining cases.

11.3.1 4-regular graphs with loops

Lemma 11.3.4. *Let $\Gamma = \mathcal{C}_k$ or $\Gamma = \mathcal{D}_k$ for some $k \geq 2$. Let (H, ψ) be a connected, 4-regular, Γ-tight graph. Let $v \in V(H)$ be incident to exactly one loop, and two non-loop edges. Then there is either an admissible loop 1-reduction at v, or an admissible loop-to-loop reduction at v, and the graph obtained by this admissible reduction is connected.*

Proof. Denote the non-loop edges incident to v by $e_i = \overrightarrow{vv_i}$ for $i \in \{1, 2\}$, where potentially $v_1 = v_2$. By Proposition 10.1.3, we can assume $\psi(e_1) = I$. Let $\psi(e_2) = g_2$, and let the loop incident to v be denoted by e and have gain $g \neq I$.

A loop 1-reduction at v adds a $\overrightarrow{v_1v_2}$ edge to $(H - v, \psi)$ with gain g_2. We denote this edge by l_{12} or d_{12} when it is respectively a length or direction edge. A loop-to-loop reduction at v adds a loop to $H - v$ which is either incident to v_1 with gain g, or incident to v_2 with gain $g_2 g g_2^{-1}$. When this loop is incident to v_i, we denote it by l_i or d_i when it is respectively a length or a direction loop. Let Y denote the set of all edges that could be added in either a loop 1-reduction or loop-to-loop reduction at v, and let

$Y^* = Y \cup \{e_1, e_2, e\}$. Let (H^*, ψ^*) be the graph obtained from (H, ψ) by adding every edge in Y.

Assume, for a contradiction, that none of these reductions results in a sparse graph. Then for all $f \in Y$, the set $E(H - v) \cup \{f\}$ is not sparse. So in the sparsity matroid $\mathcal{S}(H^*, \psi^*) = (E^*, \mathcal{E}^*)$, we have $E(H - v) \in \mathcal{E}^*$ but $E(H - v) \cup \{f\} \notin \mathcal{E}^*$. Hence, by Proposition 3.0.3 (ii), we have $Y \subseteq \mathrm{cl}(E(H-v)) = \mathrm{cl}(E(H) - \{e, e_1, e_2\})$. We shall use properties of the closure to obtain a contradiction.

Claim 11.3.5. *If v is pure, then there exists $Y' \subseteq Y^* - \{e\}$ which is maximal sparse in Y^*.*

Proof. Let $t = d$ or $t = l$ when v is respectively direction or length pure. Then $Y = \{t_{12}, t_1, t_2\}$. Note that if v is direction-pure, then e is a direction loop with gain g, and so g is a rotation.

First consider the case where $v_1 = v_2$. Then Y consists of three loops at v_1. Since e_1 and e_2 are parallel edges of the same type we must have $g_2 \neq I$. Suppose $\langle g, g_2 \rangle$ is cyclic. Then either v is length-pure, or v is direction-pure with $\langle g, g_2 \rangle$ rotational. In both cases $Y' = \{e_1, e_2, t_1\}$. Suppose instead that $\langle g, g_2 \rangle$ is dihedral. Then $g \neq g_2$ and $Y' = \{e_1, e_2, t_1\}$ when v is direction-pure, and $Y' = \{e_1, e_2, t_1, t_{12}\}$ when v is length-pure.

Now consider the case where $v_1 \neq v_2$. By performing a switching operation at v_2, we can assume $g_2 = I$, which implies $g_2 g g_2^{-1} = g$. Hence $Y' = \{e_1, e_2, t_1, t_2, t_{12}\}$ in maximal sparse in Y^*. □

Claim 11.3.6. *If v is mixed, then there exists $Y' \subseteq Y^* - \{e\}$ which is maximal sparse in Y^*.*

Proof. First consider the case where $v_1 = v_2$. Then $Y = \{d_{12}, l_{12}, d_1, l_1, d_2, l_2\}$, and every edge in Y is a loop at v_1. Since $g \neq I$, $\langle g, g_2 \rangle$ is unbalanced. Suppose $\langle g, g_2\rangle$ is reflectional. Then $g_2 \in \{I, g\}$ and e is a length loop. Since v is mixed, $\{e_1, e_2\}$ contains a direction edge. Hence $Y' = \{e_1, e_2, l_1\}$ is mixed and maximal sparse in Y^*. Suppose instead that $\langle g, g_2\rangle$ is rotational, then $Y' = \{e_1, e_2, d_1, l_1\}$ is mixed and maximal sparse. Finally, suppose $\langle g, g_2\rangle$ is dihedral. If g is a reflection, then e is a length loop, and $\{e_1, e_2\}$ contains a direction edge; hence $Y' = \{e_1, e_2, l_1, l_{12}\}$. If g is rotation, then $Y' = \{e_1, e_2, d_1, l_{12}\}$.

Now consider the case where $v_1 \neq v_2$. Then $Y = \{d_{12}, l_{12}, d_1, l_1, d_2, l_2\}$. By performing a switching operation at v_2, we can assume $g_2 = I$, and thus $g_2 g g_2^{-1} = g$. Hence $\langle g, g_2\rangle$ is non-trivial cyclic. If g is a reflection, then $Y' = \{e_1, e_2, l_{12}, d_{12}, l_1\}$. If instead g is a rotation, then $Y' = \{e_1, e_2, l_{12}, d_{12}, l_1, d_2\}$. □

Claims 11.3.5 and 11.3.6 imply that in all cases there exists some $Y' \subseteq Y^* - \{e\}$ which is maximal sparse in Y^*. A similar argument to that in Lemma 11.2.6 gives $e \in \mathrm{cl}(E(H) - \{e\})$. So, by the definition of closure, $\mathrm{rank}(E(H)) = \mathrm{rank}(E(H) - \{e\})$. Since (H, ψ) is sparse, Proposition 3.0.1 gives $|E(H) - \{e\}| = |E(H)|$, a contradiction.

Thus at least one of these reductions results in a sparse graph (H', ψ'). Suppose for a contradiction that (H', ψ') is disconnected. Then there exists a partition of $V(H')$ into V_1, V_2 such that $v_i \in V_i$. But then $d_H(V_i, V(H) - V_i) = 1$, so Corollary 11.3.3 implies $i_H(V_i) = 2|V_i| - \frac{1}{2}$, which is clearly impossible as $i_H(V_i)$ is an integer. Hence (H', ψ') is connected. □

The only remaining step in obtaining our constructive characterisation for Γ-tight graphs, is to show that 4-regular, loop-free graphs have an admissible 2-reduction. Unfortunately, this is not always true when $\Gamma = \mathcal{D}_{k\geq 2}$. For an example, see Figure 11.10 which illustrates an example from [22]. Thus, in the remainder of this section, we only consider 2-reductions of $\mathcal{C}_{k\geq 2}$-gain

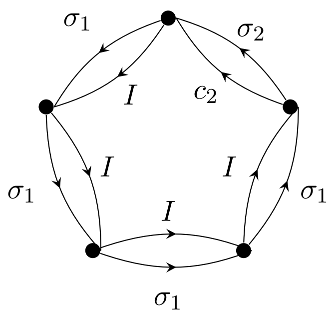


Figure 11.10: A length-pure $\mathcal{D}_2$-tight graph with no admissible 2-reductions. Every 2-reduction either results in a pair of parallel edges with the same gain, or in a graph (H', ψ') with $\langle E(H') \rangle = \langle \sigma_1 \rangle$. The gains σ_1 and σ_2 are reflections across perpendicular mirrors, and c_2 is the rotation by π.

graphs.

Let $\mathcal{F}_k$ denote the class of $\mathcal{C}_k$-tight graphs (H, ψ) where H is connected, 4-regular and loop-free. We shall show there is always some vertex in H at which we can perform an admissible 2-reduction to give a connected graph. First we prove some structural results.

11.3.2 Tight and near-tight sets

A reduction is inadmissible if the resulting Γ-gain graph is no longer sparse. With a 2-reduction, there are two ways this can occur: by adding an edge to a tight subgraph, or by adding two edges e_1, e_2 to a subgraph (V', E') such that $E' + e_i$ is tight for some i. In the latter case, we say that the graph (V', E') and the edge set E' are *near-tight*, as adding a single edge gives a tight graph.

Note that an edge set may be both tight and near-tight. For example, suppose we have a mixed edge set F which is balanced, tight, and contains exactly one direction edge d. Deleting d gives the near-tight edge set $F - d$; however, this set is also length-pure and balanced with $|F - d| = |F| - 1 =$

$2|V(F)| - 3$, making it tight.

The following is an immediate consequence of Proposition 11.1.3(iv).

Corollary 11.3.7. *Let Γ be a point group. Let (H, ψ) be Γ-tight, and $e \in E(H)$. If $s(E(H)) \leq 2|V(H)| - 2$ and $E(H - e) \neq \emptyset$, then the near-tight graph $(H - e, \psi)$ is connected.*

The next result formalises the relationship between sets which are tight or near-tight, and inadmissible 2-reductions. Recall the definition of $\mathcal{F}_k$ from page 194.

Lemma 11.3.8. *Let (H', ψ') be connected and formed from some $(H, \psi) \in \mathcal{F}_k$ by a 2-reduction which removes a vertex $v \in V(H)$ and adds the edges $e_1 = v_1v_2$ and $e_2 = v_3v_4$. Suppose this 2-reduction is inadmissible. Then there exists $F \subseteq E(H')$ such that $|F| > s(F)$, and either F is balanced with*

(i) $\{e_1, e_2\} \cap F = \{e_1\}$ and $F - e_1$ is tight,

(ii) $\{e_1, e_2\} \cap F = \{e_2\}$ and $F - e_2$ is tight, or

(iii) $\{e_1, e_2\} \subseteq F$ and $F - \{e_1, e_2\}$ is near-tight;

or

(iv) $F = E(H')$ is pure.

Proof. Since v is inadmissible, (H', ψ') is not sparse. So there exists some minimal $F \subseteq E(H')$ with $|F| > s(F)$. Since $E(H)$ is sparse, we know $F \cap \{e_1, e_2\} \neq \emptyset$.

Suppose $|F \cap \{e_1, e_2\}| = 1$, and consider the case where $F \cap \{e_1, e_2\} = \{e_1\}$. Then $v_1, v_2 \in V(F)$. Since $F - e_1$ is sparse and F is not, we have $|F| > s(F) \geq s(F - e_1) \geq |F - e_1| = |F| - 1$. Thus $F - e_1$ is tight. Since $d_H(\{v\}, V(F)) \geq 2$, Corollary 11.3.3 implies $|F| \leq 2|V(F)| - 1$. Thus

$s(F) \leq 2|V(F)| - 2$. Since $\Gamma = \mathcal{C}_k$, this implies F is balanced. Hence F satisfies (i). If $F \cap \{e_1, e_2\} = \{e_2\}$, then a similar argument implies F satisfies (ii).

Suppose instead that $F \supseteq \{e_1, e_2\}$. By our choice of F, every proper subset of F is sparse. In particular, $|F - e_1| = |F| - 1 \leq s(F - e_1) \leq s(F) < |F|$. Hence $F - e_1$ is tight, and so, by definition, $F - \{e_1, e_2\}$ is near-tight. First consider the case where $|F| \leq 2|V(F)| - 1$. Then $F - e_1$ is tight with $|F - e_1| \leq 2|V(F)| - 2$. Thus $F - e_1$ is balanced. Since $|F| > s(F)$, the definition of the sparsity function implies F is also balanced. Hence F satisfies (iii). So instead, consider the case where $|F| = 2|V(F)|$. Since F is minimal with $|F| > s(F)$, this implies F is pure and cyclic. Since $e_1, e_2 \in F$, we know $N_H(v) \subseteq V(F)$ and $|F - \{e_1, e_2\}| = 2|V(F)| - 2$. Hence Corollary 11.3.3 implies $F = E(H')$, giving case (iv). $\square$

To simplify our later arguments, we refer to an edge set F in (H', ψ') which satisfies (i) or (ii) of Lemma 11.3.8 as a *I-block of* (H', ψ') *on* e_1 or e_2 respectively. If F satisfies (iii), we say it is a *II-block of* (H', ψ') *on* $\{e_1, e_2\}$. We refer to I-blocks and II-blocks on edges in $\{e_1, e_2\}$ collectively as *blocks of* (H', ψ') *on* $\{e_1, e_2\}$, or more generally, as *blocks*. Note that a block is always balanced.

In this terminology, Lemma 11.3.8 says that if a 2-reduction of (H, ψ) which adds some pair of edges $\{e_1, e_2\}$ is inadmissible, then either the resulting graph is pure, or it contains a block on $\{e_1, e_2\}$.

11.3.3 Admissible 2-reductions for graphs in $\mathcal{F}_k$ with low connectivity

Unlike the reduction proofs in Section 11.2.4, our argument for 2-reductions varies considerably with the structure of (H, ψ) and the choice of v. Here we show that if a graph in $\mathcal{F}_k$ is close to being disconnected, then it contains a vertex with an admissible 2-reduction.

Recall the ideas of k-connectivity and k-edge-connectivity from Chapter 2. We shall show that if a graph $(H, \psi) \in \mathcal{F}_k$ has a cut-vertex or a 2-edge-cut, then it contains a vertex at which we can perform an admissible 2-reduction.

Lemma 11.3.9. *Let $(H, \psi) \in \mathcal{F}_k$ be connected but not 2-connected, and suppose $|V(H)| \geq 2$. Then there exists some $v \in V(H)$ at which we can perform an admissible 2-reduction such that the resulting graph is connected.*

Proof. We shall show that there is some cut-vertex of H at which we can perform an admissible 2-reduction. We first prove some properties of cut-vertices.

Claim 11.3.10. *Suppose u is a cut-vertex of H, and let $\{X, Y\}$ be a partition of $V(H) - \{u\}$ such that $H - u$ contains no edge from X to Y. Then*

(i) $H[X]$ *and* $H[Y]$ *are connected,*

(ii) $d_H(\{u\}, X) = d_H(\{u\}, Y) = 2$*, and*

(iii) $|X|, |Y| \geq 2$.

Proof. Since H is connected, Corollary 11.3.3 implies that for every connected component $H[U]$ in $H - u$, $d_H(U, \{u\})$ is a positive even number. Since $d_H(u) = 4$, this implies $d_H(\{u\}, X) = d_H(\{u\}, Y) = 2$, and so both $H[X]$ and $H[Y]$ are connected. Assume $|X| = 1$. Then Corollary 11.3.3 implies $i_H(X) = 1$, and so the single vertex in X is incident to a loop. This contradicts the fact H is loop-free. Hence $|X| \geq 2$, and similarly, $|Y| \geq 2$. □

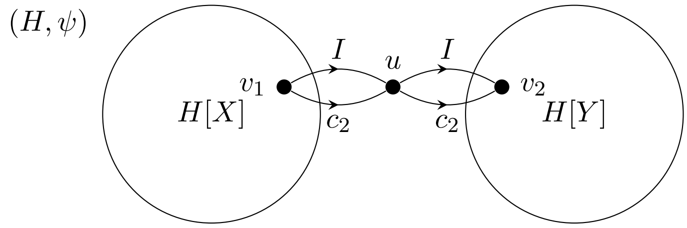


Figure 11.11: A $\mathcal{C}_2$-gain graph with cut-vertex u. Parallel edges of the same type cannot be assigned the same gain, so we cannot perform a 2-reduction at u which adds a pair of parallel $\overrightarrow{v_1v_2}$ edges. So the only possible 2-reduction at u adds a pair of length loops at v_1 and v_2, thus disconnecting the graph.

If a vertex is pure and incident to two sets of parallel edges, it is possible that every admissible 2-reduction which removes this vertex also disconnects the graph. See Figure 11.11. We next show that H contains a cut-vertex which does not fit this description.

Claim 11.3.11. *There exists a cut-vertex u of H such that $|N_H(u)| \geq 3$.*

Proof. Let u be a cut-vertex in H which partitions $V(H) - \{u\}$ into X and Y such that $d_H(X, Y) = 0$ and X is minimal with respect to inclusion. Assume for a contradiction that u does not satisfy the Claim. By Claim 11.3.10, u sends exactly two edges to each of X and Y. Since $|N_H(u)| \leq 2$, u has exactly one neighbour in each of X and Y, and sends a pair of parallel edges to each of these neighbours. But this implies that the neighbour $x \in X$ of u is a cut-vertex of H which partitions $V(H) - \{x\}$ into $X - \{x\}$ and $Y \cup \{u\}$. This contradicts the fact X was chosen to be minimal. Hence u satisfies the Claim. $\square$

By Claim 11.3.11, H contains a cut-vertex v with $|N_H(v)| \geq 3$. Let X and Y denote the vertex sets of the two connected components of $H - v$, and label the four edges incident to v by $e_i = \overrightarrow{vv_i}$ with gain g_i for $1 \leq i \leq 4$; $v_1, v_2 \in X$; and $v_3, v_4 \in Y$. Without loss of generality v_2, v_3 and v_4 are distinct. Let (H', ψ') be formed by a 2-reduction at v which adds the edges $f_1 = \overrightarrow{v_1v_3}$ and $f_2 = \overrightarrow{v_2v_4}$. If v is a mixed vertex, we choose $\{f_1, f_2\}$ to be mixed.

Suppose this 2-reduction is inadmissible. By Claim 11.3.10, and our positioning of f_1 and f_2, (H', ψ') is connected. If v is pure, then the sparsity function implies $H - v$ contains an edge of opposite type to v; if v is mixed, we choose $\{f_1, f_2\}$ to be mixed. In both cases, this implies (H', ψ') is mixed. Hence Lemma 11.3.8 implies (H', ψ') contains a block F on $\{f_1, f_2\}$. Since $H - v$ is disconnected, $F \cap E(H) = (F \cap E(X)) \cup (F \cap E(Y))$. Further, by the definition of a block, $F - \{f_i\}$ is tight for some i. Hence, by Proposition 11.1.3(i), every vertex in $V(F)$ is incident to at least two edges in $F - \{f_i\}$. Since f_i has an endvertex in each of X and Y, this implies $F \cap E(X) \neq \emptyset$ and $F \cap E(Y) \neq \emptyset$. Hence

$$|F \cap E(H)| \leq s(F \cap E(X)) + s(F \cap E(Y)) \leq 2|V(F)| - 6 + 2\beta(F).$$

Since $\beta(F) \leq 1$ and $|F - E(H)| \leq 2$, this gives

$$|F| \leq |F \cap E(H)| + 2 \leq 2|V(F)| - 3 + \beta(F) = s(F),$$

a contradiction. □

Lemma 11.3.12. *Let $(H, \psi) \in \mathcal{F}_k$ and suppose H is 2-connected but not 3-edge-connected. Then there exists some $v \in V(H)$ at which we can perform an admissible 2-reduction such that the resulting graph is connected.*

Proof. Since H is 2-connected, it is 2-edge-connected. As H is not 3-edge-connected, there exists a partition $\{X, Y\}$ of $V(H)$ such that $d_H(X, Y) = 2$. Since H is 2-edge-connected, $H[X]$ and $H[Y]$ are connected. Let $e_1 = \overrightarrow{x_1y_1}$ and $e_2 = \overrightarrow{x_2y_2}$ denote the two edges with $x_i \in X$ and $y_i \in Y$. Since H has no cut-vertices, we know $x_1 \neq x_2$ and $y_1 \neq y_2$. Since $H \in \mathcal{F}_k$, the definition of the sparsity function implies H is mixed. So at least one of the graphs $H-x_1$, $H-x_2$, $H-y_1$ or $H-y_2$ is mixed. Relabelling if necessary, suppose $H-x_1$ is mixed. We shall show that there is an admissible 2-reduction at x_1.

Since H is 4-regular and loop-free, x_1 is incident to exactly four edges. One of these is $e_1 = \overrightarrow{x_1y_1}$. Denote the other three edges by $e_{i1} = \overrightarrow{v_ix_1}$ for $i \in \{1,2,3\}$. Note these v_i need not be distinct. However, since $\{e_1, e_2\}$ is a 2-edge-cut, we know $v_i \in X$ for all i. Hence $y_1 \notin \{v_1, v_2, v_3\}$.

Consider the graph $(H - e_1, \psi)$. In this graph, x_1 in incident to exactly three non-loop edges. So, by Lemma 11.2.6, there exists an admissible 1-reduction at x_1. Without loss of generality, suppose this 1-reduction adds the edge $f_1 = \overrightarrow{v_1v_2}$. To extend this to a 2-reduction, (H', ψ') of H, it remains to add an edge $f_2 = \overrightarrow{v_3y_1}$ with gain $\psi'(f_2) = \psi(\overrightarrow{e_{31}})\psi(e_1)$.

Suppose for a contradiction that this 2-reduction is not admissible. Since $(H' - f_2, \psi')$ was formed by an admissible 1-reduction, it is sparse and connected. So (H', ψ') is connected; and since $H - x_1$ is mixed, so is H'. Thus Lemma 11.3.8 implies that f_2 is contained in some block F of (H', ψ'). Since $F - \{f_2\}$ is tight, Proposition 11.1.3(i) implies every vertex in $V(F)$ is incident to at least two edges in $F - \{f_2\}$. Since $\{f_2, e_2\}$ is a 2-edge-cut in (H', ψ'), this implies $F \cap E_{H'}(X) \neq \emptyset$ and $F \cap E_{H'}(Y) \neq \emptyset$. Hence

$$\begin{aligned}
|F| &= |F \cap E'_H(X)| + |F \cap E'_H(Y)| + |F \cap \{e_2, f_2\}| \\
&\le (2|V(F) \cap X| - 3 + \beta(F)) + (2|V(F) \cap Y| - 3 + \beta(F)) + 2 \\
&\le 2|V(F)| - 4 + 2\beta(F).
\end{aligned}$$

By definition $\beta(F) \le 1$. So this gives $|F| \le s(F)$, a contradiction. $\square$

11.3.4 Admissible 2-reductions for graphs in $\mathcal{F}_k$

Corollary 11.3.3 implies that a minimal edge-cut in a 4-regular graph must contain an even number of edges. So any 3-edge-connected, 4-regular graph is also 4-edge-connected. This fact, together with Lemmas 11.3.9 and 11.3.12 mean it only remains to find admissible 2-reductions in 2-connected, 4-edge-connected graphs $(H, \psi) \in \mathcal{F}_k$. We do this by considering a mixed vertex $v \in V(H)$, and then proving that for each value of $|N_H(v)|$, we can find an admissible 2-reduction of (H, ψ). We first prove some properties of tight sets in such graphs.

Lemma 11.3.13. *Let $(H, \psi) \in \mathcal{F}_k$ be 2-connected and 4-edge-connected, and let $v \in V$ have at least two distinct neighbours v_1, v_2. Let (H_1, ψ_1) and (H_2, ψ_2) be formed from (H, ψ) by inadmissible 2-reductions at v which add the edge pairs $\{e_1, f_1\}$ and $\{e_2, f_2\}$ respectively. Suppose f_1 and f_2 are both $\overrightarrow{v_1v_2}$-edges, and that for $i \in \{1, 2\}$, (H_i, ψ_i) contains a I-block F_i on f_i. Let $\psi_i(f_i) = g_i$. Then*

(i) $g_1 = g_2$, and

(ii) if $\{f_1, f_2\}$ is mixed, then there exists a set $E' \subseteq E(H - v)$ such that $E' + f_i$ is a mixed I-block of (H_i, ψ_i) on f_i for $i \in \{1, 2\}$.

Proof. Suppose F_1 and F_2 are the minimal such sets which satisfy these hypotheses. For ease of notation, let $E_i = F_i \cap E(H - v)$, and $V_i = V(E_i)$.

Assume for a contradiction that $g_1 \neq g_2$. Since F_i is a I-block, E_i is tight. So by Proposition 11.1.3(iii), each E_i is connected, and thus contains a path P_i from v_1 to v_2. Since $F_i \supseteq E_i$ is balanced and $f_i \in F_i$, we know $\psi(\overrightarrow{v_1 P_i v_2}) = g_i$. Hence $E_1 \cup E_2$ is unbalanced, as it contains the closed walk $v_1 P_1 v_2 P_2 v_1$ with gain $g_1 g_2^{-1} \neq I$. Lemma 10.1.5 thus implies $(V_1 \cap V_2, E_1 \cap E_2)$ is not connected. If $E_1 \cap E_2 \neq \emptyset$, then either $(V_1 \cap V_2, E_1 \cap E_2)$ contains an isolated vertex, or, by Proposition 11.1.3(iii) and Corollary 11.3.7, $E_1 \cap E_2$ is neither tight nor near-tight. In both cases this implies $|E_1 \cap E_2| \leq 2|V_1 \cap V_2| - 4$. Trivially, this inequality also holds when $E_1 \cap E_2 = \emptyset$. Hence, in all cases we have

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2| \geq 2|V_1 \cup V_2| - 2 + \beta(E_1) + \beta(E_2). \quad (11.9)$$

Suppose E_i is mixed for some i. Then $\beta(E_i) = 1$, and (11.9) implies $|E_1 \cup E_2| \geq 2|V_1 \cup V_2| - 1$. Since $v \notin V_1 \cup V_2$, Corollary 11.3.3 implies this holds with equality, and $d_H(V_1 \cup V_2, V(H) - (V_1 \cup V_2)) = 2$. But this contradicts the fact H is 4-edge-connected. Hence both E_i are pure, $\beta(E_i) = 0$, and (11.9) becomes

$$|E_1 \cup E_2| \geq 2|V_1 \cup V_2| - 2. \quad (11.10)$$

Since $v \notin V_1 \cup V_2$ and H is 4-edge-connected, Corollary 11.3.3 again implies (11.10) holds with equality. In particular, this gives $|E_1 \cap E_2| = 2|V_1 \cap V_2| - 4$. Since $(V_1 \cap V_2, E_1 \cap E_2)$ is disconnected, pure, balanced and sparse, this implies $E_1 \cap E_2 = \emptyset$ and $V_1 \cap V_2 = \{v_1, v_2\}$. Since H is 4-regular and $v_1, v_2 \in N_H(v)$, we know $d_{H[E_1]}(v_i) + d_{H[E_2]}(v_i) \leq 3$ for both v_i. But this implies that either $d_{H[E_1]}(v_i) \leq 1$ or $d_{H[E_2]}(v_i) \leq 1$, which contradicts Proposition 11.1.3(i). Hence part (i) holds.

We now prove part (ii). Suppose $\{f_1, f_2\}$ is mixed. Then without loss of generality f_1 and f_2 are respectively a length and a direction edge. If some E_i is mixed, then $E' = E_i$ and we are done. So suppose this is not the case. Then E_1 is length-pure and E_2 direction-pure. Hence $E_1 \cap E_2 = \emptyset$, and so

$$|E_1 \cup E_2| = |E_1| + |E_2| = (2|V_1 \cup V_2| - 2) + (2|V_1 \cap V_2| - 4) \geq 2|V_1 \cup V_2| - 2.$$

Since H is 4-edge-connected and $v \notin V_1 \cup V_2$, Corollary 11.3.3 implies this holds with equality, and so $V_1 \cap V_2 = \{v_1, v_2\}$. Both F_1 and F_2 are balanced, so every path from v_1 to v_2 in E_1 and E_2 has gain $g_1 = g_2$. Thus $F_1 \cup F_2$ is balanced, and so $E' = E_1 \cup E_2$ satisfies (ii). $\square$

Lemma 11.3.14. *Let $(H, \psi) \in \mathcal{F}_k$, and let $v \in V$ have at least three distinct neighbours v_1, v_2, v_3. For $i \in \{1, 2\}$ let (H_i, ψ_i) be formed from (H, ψ) by an admissible 2-reduction at v which adds a pair of edges $\{e_i, f_i\}$ where $f_i = \overrightarrow{v_i v_3}$. Suppose each (H_i, ψ_i) contains a I-block F_i on f_i. Let $E_i = F_i \cap E(H - v)$.*

(i) If $E_1 \cap E_2 \neq \emptyset$, then $E_1 \cup E_2$ is balanced and tight.

(ii) If $E_1 \cap E_2 = \emptyset$, then for some $i \in \{1, 2\}$, $V(E_i) = \{v_i, v_3\}$ and E_i consists of a single $\overrightarrow{v_i v_3}$-edge with the same gain and type as f_i.

Proof. Let $V_i = V(E_i)$. Since F_1 and F_2 are I-blocks, both E_i are tight and balanced, so

$$|E_1 \cup E_2| + |E_1 \cap E_2| = (2|V_1 \cup V_2| - 3 + \beta(E_1)) + (2|V_1 \cap V_2| - 3 + \beta(E_2)).$$

Since $d_H(V_1 \cup V_2, \{v\}) \geq 3$, Corollary 11.3.3 implies $|E_1 \cup E_2| < 2|V_1 \cup V_2| - 1$. Combining this with the above equation gives

$$|E_1 \cap E_2| > 2|V_1 \cap V_2| - 5 + \beta(E_1) + \beta(E_2). \tag{11.11}$$

First suppose $E_1 \cap E_2 \neq \emptyset$. Assume for a contradiction that $(V_1 \cap V_2, E_1 \cap E_2)$ is disconnected. Then either $(V_1 \cap V_2, E_1 \cap E_2)$ contains an isolated vertex, or $H[E_1 \cap E_2]$ is disconnected. In both cases, this gives $|E_1 \cap E_2| \leq 2|V_1 \cap V_2| - 5 + \beta(E_2)$ which contradicts (11.11). Thus $(V_1 \cap V_2, E_1 \cap E_2)$ is connected, and Lemmas 10.1.5(i) and 11.1.5 imply $E_1 \cup E_2$ is balanced and tight, hence proving (i).

So instead, suppose $E_1 \cap E_2 = \emptyset$. We know $v_3 \in V_1 \cap V_2$, and since H is 4-regular, $d_{H-v}(v_3) \leq 3$. If $|V_i| \geq 3$ for some $i \in \{1, 2\}$, then Proposition

11.1.3(i) implies $d_{H[E_i]}(v_3) \geq 2$. Since $E_1 \cap E_2 = \emptyset$, this cannot hold for both E_1 and E_2. Hence for some $j \in \{1,2\}$, $V_j = \{v_j, v_3\}$. Since F_j is balanced, it is loop-free; so this implies F_j consists of two parallel $\overrightarrow{v_j v_3}$-edges of the same gain and type, thus proving (ii). □

We use these two structural Lemmas to show that, given a 2-connected, 4-edge-connected graph $(H, \psi) \in \mathcal{F}_k$, and a mixed vertex $v \in V(H)$, we can always perform an admissible 2-reduction on some vertex in (H, ψ). Our argument uses the structure of the graph around v, and so we consider each value of $|N_H(v)|$ separately, starting with $|N_H(v)| = 1$.

Lemma 11.3.15. *Let $(H, \psi) \in \mathcal{F}_k$ be connected, and let v be a mixed vertex in H with $|N_H(v)| = 1$. Then there is an admissible 2-reduction at v such that the resulting graph is also connected.*

Proof. Since H is connected and 4-regular, the only possibility is that $V(H)$ consists of exactly two vertices, u and v, with four parallel edges between them. We label these $\overrightarrow{uv}$ edges by e_i where $1 \leq i \leq 4$, and $\psi(e_i) = g_i$. Since v is mixed, a 2-reduction adds both a direction loop and a length loop at u with respective gains $g_D = g_i g_j^{-1}$, and $g_L = g_k g_l^{-1}$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. See (H_7, ψ_7) in Figure 11.9. For this to be a 2-reduction we must have $g_D, g_L \neq I$, so it just remains to show that such an assignment is always possible.

First suppose $g_i = g_j$ for some $i \neq j$. By performing switching operations we can assume $g_i = g_j = I$. Since parallel edges of the same gain have different type, one of these is a length edge, and one a direction edge. The same observation implies the two remaining edges, e_k and e_l, have non-identity gain. So the choice $g_D = g_k$ and $g_L = g_l$ gives an admissible 2-reduction.

Instead, suppose g_1, g_2, g_3 and g_4 are distinct. Then $g_i g_j^{-1} \neq I$ for all $i \neq j$. Since v is mixed, we can assume, relabelling if necessary, that e_1 is a direction edge and e_2 is a length edge. In which case, setting $g_D = g_1 g_3^{-1}$ and $g_L = g_2 g_4^{-1}$ gives an admissible 2-reduction. $\square$

Lemma 11.3.16. *Let $(H, \psi) \in \mathcal{F}_k$ be 2-connected and 4-edge-connected. Let v be a mixed vertex in H with $|N_H(v)| = 2$. Then there is an admissible 2-reduction at v such that the resulting graph is also connected.*

Proof. Since H is 2-connected, $H - v$ is connected. So all 2-reductions result in a connected graph. It remains to show that some 2-reduction also preserves sparsity.

Denote the neighbours of v by v_1 and v_2. Then v is incident to some pair of edges $e_1 = \overrightarrow{vv_1}$ and $e_2 = \overrightarrow{vv_2}$. By Lemma 10.1.3, we can assume $\psi(e_1) = \psi(e_2) = I$. Since H is 4-regular and loop-free, there are two other edges incident to v, e_3 and e_4, with gains $\psi(e_3) = g_3$ and $\psi(e_4) = g_4$. We assume both of these edges are oriented away from v. Since $|N_H(v)| = 2$, there are two cases to consider: either e_3 and e_4 are parallel to e_1 and e_2 respectively, or, relabelling if necessary, both e_3 and e_4 are parallel to e_2.

Suppose the latter case holds. Since H is 4-regular, and contains three $\overrightarrow{vv_2}$-edges, v_2 is incident to exactly one further edge f whose other end-vertex is in $V(H) - \{v, v_2\}$. But then the pair $\{e_1, f\}$ forms a 2-edge-cut of H, which disconnects $\{v, v_2\}$ from $V(H) - \{v, v_2\}$. This contradicts the fact H is 4-edge-connected. Hence the former case must hold: e_1 is parallel to e_3, and e_2 is parallel to e_4.

First, suppose $g_3 = g_4 = I$. Since loops cannot have identity gain, the only possible 2-reduction at v is that which adds two $\overrightarrow{v_1v_2}$ edges f_1 and f_2, both with identity gain, to $(H - v, \psi)$ to form (H', ψ'). Hence $\{f_1, f_2\}$ is mixed. Assume for a contradiction that this 2-reduction is inadmissible.

Then, by Lemma 11.3.8, (H', ψ') has some block F on $\{f_1, f_2\}$. Let $E' = F \cap E(H - v)$. Then $V(E') \supseteq \{v_1, v_2\}$. If F is a I-block, then E' is tight. Otherwise, F is a II-block, and since $\{f_1, f_2\}$ is mixed, E' is near-tight with $|F| = 2|V(F)| - 3$. In both cases, this implies (H, ψ) contains the balanced set $F' = E' \cup \{e_1, e_2, e_3, e_4\}$ with $|F'| \geq 2|V(F')| - 1 > s(F')$. This contradicts the fact (H, ψ) is sparse. Hence this 2-reduction is admissible.

Second, consider the case where neither g_3 nor g_4 is the identity. We conider the 2-reduction which forms (H'', ψ'') from $(H - v, \psi)$ by adding loops f_1 and f_2 at v_1 and v_2 with gains g_3 and g_4 respectively. Since v is mixed, and (H, ψ) is rotational, we can choose these loops to be of different types. Since balanced sets cannot contain loops, (H'', ψ'') has no block on $\{f_1, f_2\}$. Hence Lemma 11.3.8 implies this 2-reduction is admissible.

Finally, consider the case where $g_3 \neq g_4$, and exactly one of these gains is the identity. Then any 2-reduction at v must add two $\overrightarrow{v_1v_2}$ edges, f_1 and f_2, with $\psi(f_1) = g_3$ and $\psi(f_2) = g_4$. When f_i is a length or direction edge, we refer to it as l_i or d_i respectively. Let (H_1, ψ_1) and (H_2, ψ_2) denote the graphs obtained from $(H - v, \psi)$ by adding the respective edge pairs $R_1 = \{l_1, d_2\}$ and $R_2 = \{d_1, l_2\}$.

Assume for a contradiction that neither of these 2-reductions is admissible. Since in both cases R_i is mixed and unbalanced, Lemma 11.3.8 implies that for $i \in \{1, 2\}$, (H_i, ψ_i) has a I-block F_i on R_i. Since l_1, d_1, l_2 and d_2 are distinct, Lemma 11.3.13(ii) implies that there exists a set $E' \subseteq E(H - v)$ such that for some $i \in \{1, 2\}$, $E' + l_i$ and $E' + d_i$ are mixed I-blocks in their respective graphs. Relabelling if necessary, suppose this holds for $i = 1$. If $\psi(f_1) = I$, then (H, ψ) contains the balanced edge set $F' = E' \cup \{e_1, e_2, e_3\}$. Otherwise, we have $\psi(f_2) = I$, and the set $F' = E' \cup \{e_1, e_2, e_4\}$ is balanced in (H, ψ). In either case, $E(H)$ contains a set F' with $|F'| = 2|V(F')| - 1 >$

$s(F')$, a contradiction. Hence at least one of these 2-reductions is admissible. $\square$

Lemma 11.3.17. *Let $(H, \psi) \in \mathcal{F}_k$ be 2-connected and 4-edge-connected. Let v be a mixed vertex in H with $|N_H(v)| = 3$. Then there exists a vertex in $\{v\} \cup N_H(v)$ which can be removed by an admissible 2-reduction, such that the resulting graph is connected.*

Proof. Since (H, ψ) is 2-connected, $H - u$ is connected for all $u \in V(H)$. So every 2-reduction results in a connected graph. It remains to find an admissible 2-reduction.

Denote the neighbours of v by v_1, v_2 and v_3, and let $e_i = \overrightarrow{vv_i}$. By Proposition 10.1.3, we can assume $\psi(e_1) = \psi(e_2) = \psi(e_3) = I$. Since H is 4-regular, there is one further edge e_4 at v with gain g. Relabelling if necessary, $e_4 = \overrightarrow{vv_2}$. In the 2-reductions we consider, let f_{ij} and f'_{ij} denote a $\overrightarrow{v_iv_j}$ edge with gain I or g respectively. When f_{ij} is a length or direction edge, we denote it by l_{ij} or d_{ij} respectively. Similarly, f'_{ij} becomes l'_{ij} or d'_{ij}.

Claim 11.3.18. *If $g = I$, then there is an admissible 2-reduction at either v or v_3.*

Proof. Since e_2 and e_4 are parallel with the same gain, they must consist of a direction and a length edge. Let $R_1 = \{l_{12}, d_{32}\}$ and $R_2 = \{d_{12}, l_{32}\}$. For $i \in \{1, 2\}$, let (H_i, ψ_i) be the graph formed from (H, ψ) by the 2-reduction which deletes v then adds the edge set R_i.

If one of these (H_i, ψ_i) is sparse, we are done. So suppose not. Since both R_i are mixed, Lemma 11.3.8 implies that for both i, (H_i, ψ_i) contains a block F_i on R_i. Let $E_i = F_i \cap E(H - v)$.

Suppose that for some i, F_i is a II-block on R_i. Then E_i is balanced and near-tight. Since R_i is mixed, this implies $|E_i| = 2|V(E_i)| - 3$. Hence the set $F' = E_i \cup \{e_1, e_2, e_3, e_4\}$ in $E(H)$ is balanced with $|F'| = 2|V(F')| - 1 > s(F')$, a contradiction. Suppose instead that for some i, F_i is a mixed I-block on l_{j2} (and thus also on d_{j2}) where $j \in \{1, 3\}$. Then (H, ψ) contains the balanced edge set $F' = E_i \cup \{e_j, e_2, e_4\}$ with $|F'| = 2|V(F')| - 1 > s(F')$, again a contradiction. Hence, neither (H_1, ψ_1) nor (H_2, ψ_2) contains a II-block or mixed I-block on the corresponding R_i. Thus F_1 and F_2 are pure I-blocks.

By Lemma 11.3.13(ii), one of $\{F_1, F_2\}$ is a block on a $\overrightarrow{v_1v_2}$ edge, and the other on a $\overrightarrow{v_2v_3}$ edge. Suppose $E_1 \cap E_2 \neq \emptyset$. Then Lemma 11.3.14(i) implies $E_1 \cup E_2$ is pure, balanced and tight. Hence $E_1 \cup E_2 \cup R_1$ and $E_1 \cup E_2 \cup R_2$ are II-blocks in (H_1, ψ_1) and (H_2, ψ_2) respectively, which, as we saw in the previous paragraph, cannot occur. Thus $E_1 \cap E_2 = \emptyset$. Since $v_2 \in V(E_1) \cap V(E_2)$ and $d_{H-v}(v_2) = 2$, Proposition 11.1.3(i) implies either $E_1 \cup E_2 = \{l_{12}, l_{23}\}$, or $E_1 \cup E_2 = \{d_{12}, d_{23}\}$. In both cases, this implies $E_1 \cup E_2 \cup \{e_1, e_2, e_3, e_4\}$ is mixed, balanced and tight. We shall show that we can remove v_3 by an admissible 2-reduction.

We know v_3 is incident to both e_3, and some edge $f_{23} \in \{l_{23}, d_{23}\}$ with identity gain. Since H is 4-regular and loop-free, v_3 is incident to two further edges, $e_1^* = \overrightarrow{v_3u_1}$ and $e_2^* = \overrightarrow{v_3u_2}$ in H. We know v is incident to e_1, e_2, e_3 and e_4; and v_2 is incident to e_2, e_4, and the pair of edges in $E_1 \cup E_2$. Since H is 4-regular, this implies $u_1, u_2 \in V(H) - \{v, v_2, v_3\}$, although these u_i need not be distinct. Let $\psi(e_i^*) = g_i$. Let $R_3 = \{f_1^*, f_2^*\}$ where $f_1^* = \overrightarrow{vu_1}$ has gain g_1, $f_2^* = \overrightarrow{v_2u_2}$ has gain g_2; and if v_3 is mixed, we choose $\{f_1^*, f_2^*\}$ to be mixed. Let (H_3, ψ_3) denote the graph formed from (H, ψ) by the 2-reduction which deletes v_3 and adds the edge set R_3. If (H_3, ψ_3) is sparse, we are done. So assume not.

Since $E(H - v_3)$ contains the mixed set $\{e_2, e_4\}$, Lemma 11.3.8 implies

(H_3, ψ_3) has some block, F_3, on $\{f_1^*, f_2^*\}$. Let $E_3 = F_3 \cap E(H)$. By Proposition 11.1.3(iii) and Corollary 11.3.7, E_3 is connected. Since H is 4-regular and 4-edge-connected, either $u_1, u_2 \in V(H) - \{v, v_1, v_2, v_3\}$, or we are in the special case where $u_1 = u_2 = v_1$ and $V(H) = \{v, v_1, v_2, v_3\}$. In the latter case, since $E_1 \cup E_2 \cup \{e_1, e_2, e_3, e_4\}$ is mixed, balanced and tight, and (H, ψ) is sparse, the two $\overrightarrow{v_3v_1}$-edges, e_1^* and e_2^*, must have non-identity gain. See Figure 11.12. Hence $g_1, g_2 \neq I$, and so the 2-reduction which adds $f_1^* = vv_1$ and $f_2^* = v_2v_1$ is admissible.

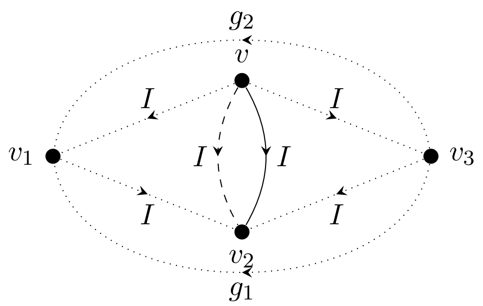


Figure 11.12: The graph (H, ψ) in the special case where $u_1 = u_2 = v$. Dotted lines depict edges which could be of either type.

So suppose instead that $u_1, u_2 \in V(H) - \{v, v_1, v_2, v_3\}$. Then $\{v_1, v_3\}$ is a 2-vertex-cut of H, and $H - \{v_1, v_3\}$ has connected components $H[X]$ and $H[Y]$, where $X = \{v, v_2\}$ and $Y = V(H) - \{v, v_1, v_2, v_3\} \supseteq \{u_1, u_2\}$. Since $V(E_3)$ intersects both X and Y, and E_3 is connected, v_1 is a cut-vertex of $H[E_3]$. Hence $|V(E_3)| \geq 3$.

Suppose F_3 is a I-block on some f_i^* in (H_3, ψ_3). Then E_3 is balanced and tight. The fact v_1 is a cut-vertex of $H[E_3]$ implies E_3 is mixed by Proposition 11.1.3(ii). Hence

$$\begin{aligned}|E_3| &\leq |E_H(X) \cap E_3| + |E_H(Y) \cap E_3| + d_H(v_1)\\ &\leq (2|V_3 \cap X| - 2) + (2|V_3 \cap Y| - 2) + 4 = 2|V_3| - 2.\end{aligned}$$

Since E_3 is tight, equality holds throughout. In particular, this implies E_3 contains $E_H(X)$ and all four edges incident to v_1 in (H, ψ). So $E_3 \supset \{e_1, e_2, e_4, f_{12}\}$. Hence H contains the balanced set $F' = E_3 \cup \{e_3, f_{23}, e_i^*\}$ with $|F'| = 2|V(F')| - 1 > s(F')$, a contradiction.

Finally, suppose F_3 is a II-block on R_3. Then $E_3 \cup f_i^*$ is balanced and tight for some $f_i^* \in R_3$. So Proposition 11.1.3(i) implies that E_3 contains edges incident to each of v and v_2. Hence E_3 contains a path with identity gain from v to v_2 (either ve_2v_2, ve_4v_2 or $ve_1v_1f_{12}v_2$). Thus $F' = E_3 \cup \{e_3, f_{23}, e_1^*, e_2^*\}$ is balanced. Since we chose $\{f_1^*, f_2^*\}$ to be mixed whenever v_3 was, this implies $|F'| = 2|V(F')| - 2 + \beta(F_3) > s(F')$, a contradiction. Hence our assumption was wrong, and at least one of (H_1, ψ_1), (H_2, ψ_2) and (H_3, ψ_3) is sparse. □

For the remainder of the proof, we proceed by contradiction. Assume (H, ψ) has no admissible 2-reduction at v. Then Claim 11.3.18 implies $g \neq I$. We consider three different 2-reductions at v. These form the graphs (H_1, ψ_1), (H_2, ψ_2) and (H_3, ψ_3) by adding the respective mixed edge pairs $R_1 = \{f_{13}, f'_{22}\}$, $R_2 = \{f'_{12}, f_{32}\}$ and $R_3 = \{f_{12}, f'_{32}\}$ to $H - v$. Recall f_{ij} and f'_{ij} denote a $\overrightarrow{v_iv_j}$ edge with gain I or g respectively.

By our assumption, all of these 2-reductions are inadmissible. Since H_1, H_2 and H_3 are mixed, Lemma 11.3.8 implies each (H_i, ψ_i) contains a block F_i on R_i. Let $E_i = F_i \cap E(H - v)$. Since R_1 contains a loop, F_1 is a I-block on f_{13}; whereas since R_2 and R_3 are mixed, F_2 and F_3 are either I-blocks or mixed II-blocks. Note that $V(F_i) = V(E_i)$, in the rest of the proof we denote this set by V_i.

Claim 11.3.19. *Let $i, j \in \{1, 2, 3\}$ be distinct. If F_i is a I-block and F_j is a II-block, then $E_i \cup E_j = E(H - v)$, $E_i \cap E_j = \emptyset$ and $|V_i \cap V_j| = 2$.*

Proof. Let $f = \overrightarrow{xy}$ denote the non-loop edge blocked by F_i. By Proposition 11.1.3(iii) and Corollary 11.3.7 respectively, both E_i and E_j are connected. Since F_j is a II-block, we know it is mixed and $N_H(v) \subseteq V_j$. Hence $x, y \in V_i \cap V_j$. Since E_i and E_j are connected, they contain $\overrightarrow{xy}$-paths P_i and P_j respectively, with $\psi(P_i) = \psi_i(f) \neq \psi(P_j)$. This implies $E_i \cup E_j$ is unbalanced, and so, by Lemma 10.1.5, $(V_i \cap V_j, E_i \cap E_j)$ is not connected. Proposition 11.1.3(iii) and Corollary 11.3.7 thus imply either $|E_i \cap E_j| \leq s(E_i \cap E_j) - 2$, or $E_i \cap E_j = \emptyset$ and $V_1 \cap V_2 = \{x, y\}$. Suppose the former holds. Since $E_1 \cap E_2$ is balanced, $s(E_i \cap E_j) \leq 2|V_i \cap V_j| - 3 + \beta(E_i)$. Hence

$$|E_i \cup E_j| = |E_i| + |E_j| - |E_i \cap E_j| \geq 2|V_i \cup V_j| - 1.$$

This contradicts Corollary 11.3.3 as $d_H(V_j, \{v\}) = 4$. So instead, we must have $V_i \cap V_j = \{x, y\}$ and $E_i \cap E_j = \emptyset$. This implies

$$|E_i \cup E_j| = |E_i| + |E_j| = 2|V_i \cup V_j| - 2 + \beta(E_i).$$

As $d_H(V_j, \{v\}) = 4$, Corollary 11.3.3 implies $\beta(E_i) = 0$ and $V(H) - \{v\} = V_i \cup V_j$. Hence $E_i \cup E_j = E(H - v)$. □

Claim 11.3.20. *F_1, F_2 and F_3 are I-blocks.*

Proof. Recall that if F_2 and F_3 are not I-blocks, then they are mixed II-blocks. Also recall that F_1 is a I-block on f_{13}.

First suppose both F_2 and F_3 are II-blocks. By Corollary 11.3.7, each E_i contains a $\overrightarrow{v_1 v_2}$ path P_i. Since $E_2 \cup \{f'_{12}\}$ and $E_3 \cup \{f_{12}\}$ are both balanced, $\psi(P_2) = g \neq I = \psi(P_3)$. Hence $E_2 \cup E_3$ is not balanced, and so Lemma 10.1.5(i) implies $(V_2 \cap V_3, E_2 \cap E_3)$ is not connected. Since $|V_2 \cap V_3| \geq 3$, and $E_1 \cap E_2$ is balanced, this gives $|E_2 \cap E_3| \leq 2|V_2 \cap V_3| - 4$. Using this, and the fact $|E_i| = 2|V_i| - 3$ for $i \in \{2, 3\}$, gives $|E_2 \cup E_3| \geq 2|V_2 \cup V_3| - 2$. However, since $d_H(V_2 \cup V_3, \{v\}) = 4$, Corollary 11.3.3 implies $E_2 \cup E_3 = E(H - v)$. But $E_1 \neq \emptyset$, so $E_1 \cap (E_2 \cup E_3) \neq \emptyset$, which contradicts Claim 11.3.19.

Instead, suppose exactly one of F_2 and F_3 is a I-block. Relabelling if necessary, F_2 is a I-block on R_2 and F_3 is a II-block on R_3. Claim 11.3.19 then implies $E_1 \cup E_3 = E_2 \cup E_3 = E(H - v)$, $E_1 \cap E_3 = E_2 \cap E_3 = \emptyset$ and $|V_1 \cap V_3| = 2$. But this implies $E_1 = E_2$, which in turn gives $V_1 = V_2 \supseteq \{v_1, v_2, v_3\}$, contradicting the fact $|V_1 \cap V_3| = 2$. $\square$

Claim 11.3.21. *F_1, F_2 and F_3 are pure I-blocks, and if $i \neq j$, then the pair of edges blocked by F_i and F_j are not parallel.*

Proof. Claim 11.3.20 implies F_1, F_2 and F_3 are I-blocks. By definition of the edge sets R_i, F_1 blocks a $\overrightarrow{v_1v_3}$-edge, which is not parallel to any edge in $R_2 \cup R_3$. Since $g \neq I$, Lemma 11.3.13 implies F_2 and F_3 are not blocks on a pair of parallel edges. So either F_2 blocks a $\overrightarrow{v_1v_2}$-edge, and F_3 blocks a $\overrightarrow{v_3v_2}$-edge or vice versa; and both of these blocked edges have the same gain. It remains to show that all three of these blocks are pure.

Suppose that for some $i \in \{1, 2, 3\}$, F_i is mixed. If $E_i \cap E_j \neq \emptyset$ for some $j \neq i$, then Lemma 11.3.14(i) implies that $E' = E_i \cup E_j$ is mixed, balanced and tight. Otherwise, $E_i \cap E_j = \emptyset$ for both $j \neq i$, and Lemma 11.3.14(ii) implies $E' = E_1 \cup E_2 \cup E_3$ is mixed, balanced and tight. The edges blocked by F_2 and F_3 both either have gain I or g. And so $E(H)$ contains a mixed balanced set F' where either $F' = E' \cup \{e_1, e_3, e_2\}$ or $F' = E' \cup \{e_1, e_3, e_4\}$, when the sets F_2 and F_3 block a pair of edges with gain I or g respectively. But $|F'| = 2|V(E') \cup \{v\}| - 1$, thus contradicting the fact (H, ψ) is sparse. $\square$

Claim 11.3.22. *F_1, F_2 and F_3 are pure I-blocks of the same type, $P \in \{D, L\}$. The vertex v is incident to exactly three edges of type P in (H, ψ): e_1, e_2 and e_3.*

Proof. Claim 11.3.21 implies F_1, F_2 and F_3 are pure I-blocks, and for $i \in \{1,2,3\}$, no (H_i, ψ_i) contains a mixed I-block on any edge in R_i. Lemma 11.3.13 thus implies that for every distinct pair $v_i, v_j \in \{v_1, v_2, v_3\}$, all blocks on a $\overrightarrow{v_i v_j}$-edge are pure and have the same gain on $\overrightarrow{v_i v_j}$. In particular, this implies that each (H_i, ψ_i) contains I-blocks on exactly one of the edges f_i in R_i, and these blocks are pure.

Let R_i^* denote the pure set formed from R_i by replacing f_i with the parallel edge f_i^* which has the same gain as f_i, but opposite type. For $i \in \{1,2,3\}$ let (H_i^*, ψ_i^*) denote the graph formed from $(H - v, \psi)$ by adding the edge set R_i^*. Note that the move which forms (H_i^*, ψ_i^*) from (H, ψ) may not be a 2-reduction, as the types of the edges in R_i^* may not satisfy the requirements in the definition.

Since E_i and R_i^* are of opposite type, and $E_i \subseteq E(H - v)$, we know $E(H - v) \cup R_i^*$ is mixed. Since $E_1, E_2, E_3 \subseteq E(H - v)$, Claim 11.3.21 and Lemma 11.3.13 implies that, for all $i \in \{1,2,3\}$, (H_i^*, ψ_i^*) has no block on the edges in R_i^*. If the move which forms (H_i^*, ψ_i^*) from (H, ψ) was a 2-reduction, then Lemma 11.3.8 would imply this 2-reduction is admissible, contradicting our assumption. Thus, for all $i \in \{1,2,3\}$, the types of the edges in R_i^* do not satisfy the requirements for a 2-reduction. This implies that the definition of a 2-reduction allowed us to add the edge f_i, but not the edge f_i^*. Since the only difference between these two edges is their type, this implies v is incident to exactly three edges, e_1, e_3 and e_j (where $j \in \{2,4\}$), of the same type, $P \in \{D, L\}$. By performing a switching operation at v_2, and relabelling, we can assume $j = 2$. This implies our I-blocks F_1, F_2 and F_3 are of type P on the respective edges f_{13}, f_{32} and f_{12}, also of type P, and with identity gain. □

Suppose $E_i \cap E_j \neq \emptyset$ for some distinct $i, j \in \{1,2,3\}$. Then Lemma

11.3.14(i) implies $E_i \cup E_j$ is pure, balanced and tight. Hence $F' = E_i \cup E_j \cup \{e_1, e_2, e_3\}$ is pure and balanced with $|F'| = 2|V_1 \cup V_2 \cup \{v\}| - 2 > s(F')$, contradicting the fact (H, ψ) is sparse.

So instead, suppose $E_i \cap E_j = \emptyset$ for all distinct $i, j \in \{1, 2, 3\}$. Then Lemma 11.3.14(ii) implies that at least two of these E_i consist of a single edge with identity gain, and so $E_1 \cup E_2 \cup E_3$ is pure, balanced and tight. Hence $F' = E_1 \cup E_2 \cup E_3 \cup \{e_1, e_2, e_3\}$ is pure and balanced with $|F'| = 2|V(F')| - 2 > s(F')$, a contradiction. Thus our assumption is false, and H contains an admissible 2-reduction. □

Lemma 11.3.23. *Let $(H, \psi) \in \mathcal{F}_k$ be 2-connected and 4-edge-connected. Let v be a mixed vertex in H with $|N_H(v)| = 4$. Then there is an admissible 2-reduction at v such that the resulting graph is connected.*

Proof. Let $N_H(v) = \{v_1, v_2, v_3, v_4\}$, and let $e_i = \overrightarrow{vv_i}$. By Proposition 10.1.3, we can assume that $E(H)$ contains a spanning tree with edge set T such that $\psi(t) = I$ for all $t \in T$, and $e_1, e_2, e_3, e_4 \in T$. Let f_{ij} denote an edge $\overrightarrow{v_iv_j}$ with identity gain. We consider the three 2-reductions at v which form the graphs (H_1, ψ_1), (H_2, ψ_2) and (H_3, ψ_3) from $(H - v, \psi)$ by adding the respective edge pairs $R_1 = \{f_{23}, f_{14}\}$, $R_2 = \{f_{13}, f_{24}\}$ and $R_3 = \{f_{12}, f_{34}\}$. Since v is mixed, we can, and do, choose each R_i to be mixed. Since $H - v$ is connected, this implies every H_i is both mixed and connected. Assume for a contradiction that (H, ψ) has no admissible 2-reduction at v. Then Lemma 11.3.8 implies that every (H_i, ψ_i) has a block F_i on R_i. Let $E_i = F_i \cap E(H)$, and $V_i = V(E_i)$. Note that $V(E_i) = V(F_i)$.

Claim 11.3.24. *For $i \in \{1, 2, 3\}$, (H_i, ψ_i) has no II-blocks on R_i.*

Proof. Suppose this claim is false. Then, relabelling if necessary, we can assume F_3 is a II-block on R_3. Since R_3 is mixed, this implies E_3 is balanced with $|E_3| = 2|V_3| - 3$. But $V_3 \supseteq N_H(v)$, so $i(V_3 \cup \{v\}) \geq |E_3| + 4 = 2|V_3 \cup \{v\}| - 1$. Since H is 4-edge-connected, Corollary 11.3.3 implies $V_3 \cup \{v\} = V(H)$, and $E(H - v) = E_3 \cup \{e\}$ for some edge e. Note that since H is 4-edge-connected, we can assume $e \notin T$.

Hence the graph $(V - \{v\}, T \cap E_3)$ has exactly four connected components (U_j, T_j) for $j \in \{1, 2, 3, 4\}$, such that $v_j \in U_j$, and either $H[T_j]$ is a tree, or $T_j = \emptyset$ and $U_j = \{v_j\}$. Consider the partition $\{X_{12}, X_{34}\}$ of $V - \{v\}$, where $X_{jk} = U_j \cup U_k$. Since $F_3 = E_3 \cup \{f_{12}, f_{34}\}$ is balanced, and every edge in $T \cup \{f_{12}, f_{34}\}$ has identity gain, we know that every edge $\overrightarrow{xy} \in E_3$ has identity gain, unless $x \in X_{12}$ and $y \in X_{34}$, in which case $\psi(\overrightarrow{xy}) = g$ for some fixed $g \in \mathcal{C}_k$.

If $g = I$ then $F' = E_3 \cup \{e_1, e_2, e_3, e_4\}$ is balanced in (H, ψ) with $|F'| = 2|V(F')| - 1 > s(F')$, a contradiction. Hence $g \neq I$. By the definition of R_1 and R_2, there exists some $j \in \{1, 2\}$ such that e is not parallel to any edge in R_j. Since F_j is a block, Proposition 11.1.3(iii) or Corollary 11.3.7 imply that E_j is connected. Hence E_j contains a path P from X_{12} to X_{34} with $\psi(P) = I$. Since every edge from X_{12} to X_{34} in $E(H - v) - \{e\}$ has gain $g \neq I$, this implies $e \in P$ is an edge from X_{12} to X_{34} with $\psi(e) = I$. But F_j is balanced, so the set of all edges in F_j from X_{12} to X_{34} with identity gain form an edge-cut of $H[F_j]$, which partitions V_j into $\{X', Y'\}$. Since e is not parallel to any edge in R_j, and is the only edge with identity gain from X_{12} to X_{34} in $(H - v, \psi)$, Proposition 11.1.3(i) implies $|X'|, |Y'| \geq 2$. Hence, since E_j is balanced and sparse, this gives

$$\begin{aligned} |F_j| &= |E_j \cap E_H(X')| + |E_j \cap E_H(Y')| + |F_j \cap R_j| + |F_j \cap \{e\}| \\ &\leq 2|V_j| - 5 + 2\beta(F_j) + |R_j \cap F_j|. \end{aligned}$$

Since F_j is a block, we must have $s(F_j) < |F_j|$; so this implies $\beta(F_j) = 1$ and $|R_j \cap F_j| = 2$. Hence F_j is a mixed II-block. Since E_j is connected and every edge in $H[X_{12}] \cup H[X_{34}]$ has identity gain, $H[E_j]$ contains a $\overrightarrow{v_k v_l}$-path with identity gain for all distinct $v_k, v_l \in N_H(v)$. Hence $F' = E_j \cup \{e_1, e_2, e_3, e_4\}$ is balanced with $|F'| = 2|V(F')| - 1 > s(F')$, contradicting the fact (H, ψ) is sparse. □

Claim 11.3.25. *For $i \in \{1, 2, 3\}$, (H_i, ψ_i) has no mixed I-block on any edge in R_i.*

Proof. By Claim 11.3.24, we know F_1, F_2 and F_3 are I-blocks. Suppose for a contradiction that the claim is false. Then without loss of generality, F_1 is a mixed I-block.

Suppose $E_1 \cap E_i \neq \emptyset$ for some $i \in \{2, 3\}$. Then Lemma 11.3.14(i) implies that $E_1 \cup E_i$ is mixed, balanced and tight, and $E_1 \cup E_i \cup \{f_{jk}, f_{kl}\}$ is balanced for some distinct $j, k, l \in \{1, 2, 3, 4\}$. Hence $F' = E_1 \cup E_i \cup \{e_j, e_k, e_l\}$ is balanced in (H, ψ) with $|F'| = 2|V(F')| - 1 > s(F')$, contradiction.

So instead suppose $E_1 \cap E_2 = E_1 \cap E_3 = \emptyset$. Since F_1 is mixed, Lemma 11.3.14(ii) implies $|E_2| = |E_3| = 1$ and the unique edge in each of these sets is identical in gain, type and location to some edge $f_{jk} \in R_2$ and $f_{kl} \in R_3$ respectively. Hence $V_2 = \{v_j, v_k\}$ and $V_3 = \{v_k, v_l\}$. If $v_k \in V_1$, then Proposition 11.1.3(i) implies E_1 contains at least two edges incident to v_k. But $d_{H-v}(v_k) = 3$, so this implies E_1 intersects $E_2 \cup E_3$, a contradiction. Hence $v_k \notin V_1$, and so F_1 is a block on $f_{jl} \in R_1$. Hence (H, ψ) contains the balanced set $F' = E_1 \cup E_2 \cup E_3 \cup \{e_j, e_k, e_l\}$ on vertex set $V' = V_1 \cup \{v, v_k\}$ with $|F'| = 2|V'| - 1 > s(F')$, a contradiction. □

Claims 11.3.24 and 11.3.25, imply that F_1, F_2 and F_3 are pure I-blocks. Suppose F_i blocks the edge $f_i \in R_i$, and let f_i' denote the edge obtained from f_i by swapping its type. Let $R_i' = R_i - f_i + f_i'$, and let (H_i', ψ_i') be the graph formed from $(H - v, \psi)$ by adding R_i'. Note that unlike (H_i, ψ_i), the graph (H_i', ψ_i') is not necessarily formed by a 2-reduction of (H, ψ).

Claim 11.3.26. *For $i \in \{1, 2, 3\}$, (H_i', ψ_i') has no II-blocks on R_i'.*

Proof. Assume for a contradiction that for some i, F_i' is a II-block on R_i'. Then $|F_i'| > s(F_i')$. Let $E_i' = F_i' \cap E(H)$ and $V_i' = V(E_i')$.

Claim 11.3.24 implies that $|E_i' \cup R_i| \leq s(E_i' \cup R_i)$, and thus $E_i' \cup R_i'$ is pure. Hence $|E_i'| = 2|V_i'| - 4$. By Claim 11.3.25, E_i and E_i' are both pure of opposite type. Hence $E_i \cap E_i' = \emptyset$, and so

$$|E_i \cup E_i'| = (2|V_i \cup V_i'| - 3) + (2|V_i \cap V_i'| - 4).$$

Since F_i' is a II-block, $|V_i \cap V_i'| \geq 2$. If this inequality is strict, then $|E_i \cup E_i'| \geq 2|V_i \cup V_i'| - 1$, which contradicts Corollary 11.3.3 as $d_H(v, V_i \cup V_i') = 4$. Hence $|V_i \cap V_i'| = 2$, and so $|E_i \cup E_i'| = 2|V_i \cup V_i'| - 3$.

Recall $R_i' = R_i - f_i + f_i'$. Suppose f_i is an $\overrightarrow{xy}$-edge. Then $V_i \cap V_i' = \{x, y\}$, and every $\overrightarrow{xy}$-path in E_i has identity gain. Since F_i' blocks the $\overrightarrow{xy}$-edge f_i' with identity gain, the same holds for E_i'. Thus $F_i \cup E_i'$ is balanced and a II-block on R_i, contradicting Claim 11.3.24. □

We are now in a position to prove our result. Since v is a mixed vertex, it is incident to both direction and length edges. We use this to split the remaining argument into two cases. In what follows, d_{ij} and l_{ij} denote respectively a direction and length edge $\overrightarrow{v_i v_j}$ with identity gain.

Case 1. v is incident to exactly two length edges and two direction edges.

Since v is incident to exactly two edges of each type, there are no restrictions on the types of edges we can add in a 2-reduction. Consider the set of 2-reductions of (H, ψ) which add a $\overrightarrow{v_1 v_2}$ and $\overrightarrow{v_3 v_4}$ edge. Namely, these are the 2-reductions which add $R_1^* = \{d_{12}, d_{34}\}$, $R_2^* = \{d_{12}, l_{34}\}$, $R_3^* = \{l_{12}, d_{34}\}$ and $R_4^* = \{l_{12}, l_{34}\}$. We let (H_i^*, ψ_i^*) denote the graph formed by the 2-reduction which adds R_i^*.

By Claims 11.3.24 and 11.3.25, no (H_i^*, ψ_i^*) contains a mixed block on R_i^*. And so, by Lemma 11.3.13(ii), we cannot have both a length-pure I-block and a direction-pure I-block on any $\overrightarrow{v_iv_j}$. So in order for the 2-reductions which add $R_2^* = \{d_{12}, l_{34}\}$ and $R_3^* = \{l_{12}, d_{34}\}$ to be inadmissible, Lemma 11.3.8 implies that the corresponding graphs (H_i^*, ψ_i^*) contain pure I-blocks F_i^* of the same type: one of which blocks a $\overrightarrow{v_1v_2}$-edge, and the other blocks a $\overrightarrow{v_3v_4}$-edge. Without loss of generality, suppose F_2^* and F_3^* are direction-pure I-blocks on d_{12} and d_{34} respectively. Then $H - v$ contains direction edges. Suppose the 2-reduction which adds $R_4^* = \{l_{12}, l_{34}\}$ is inadmissible. Then Lemma 11.3.8 implies (H_4^*, ψ_4^*) either contains a length-pure II-block on this set, which contradicts Claim 11.3.26, or it contains a length-pure I-block on one of $\overrightarrow{v_1v_2}$ or $\overrightarrow{v_3v_4}$, which contradicts Claim 11.3.25 by Lemma 11.3.13. Hence for some $i \in \{1, 2, 3, 4\}$, (H_i^*, ψ_i^*) is formed by an admissible 2-reduction.

Case 2. v is incident to exactly three edges of the same type.
Without loss of generality, suppose e_1, e_2 and e_3 are length edges, and e_4 is a direction edge. Then any 2-reduction at v adds two edges $\overrightarrow{v_iv_j}$, and $\overrightarrow{v_kv_4}$ to $(H - v, \psi)$, where $\{i, j, k\} = \{1, 2, 3\}$. In order to fit our definition of a 2-reduction, the added $\overrightarrow{v_iv_j}$-edge must be a length edge, whereas the $\overrightarrow{v_kv_4}$-edge can be of either type.

Consider the three 2-reductions introduced at the start of the proof. These created the graphs (H_1, ψ_1), (H_2, ψ_2) and (H_3, ψ_3) by adding the respective mixed edge sets R_1, R_2 and R_3 to $(H - v, \psi)$. Our observation on the types of edges added in 2-reductions implies $R_1 = \{l_{23}, d_{14}\}$, $R_2 = \{l_{13}, d_{24}\}$ and $R_3 = \{l_{12}, d_{34}\}$. Recall that each (H_i, ψ_i) has a pure I-block, F_i, on some edge in R_i.

Suppose that for some $i \in \{1, 2, 3\}$, F_i blocks the edge $d_{i4} \in R_i$. Then

F_i is a direction-pure I-block. Consider the graph (H_i', ψ_i') obtained from (H, ψ) by a 2-reduction which deletes v and adds the edges $R_i' = \{l_{jk}, l_{i4}\}$, where $\{i, j, k\} = \{1, 2, 3\}$. Since $E(H_i') \supseteq E_i \cup R_i'$, it is mixed; so Lemma 11.3.8 implies that (H_i', ψ_i') has a block F_i' on R_i'. Claims 11.3.24, 11.3.25 and 11.3.26 imply F_i' is a length-pure I-block. Further, since F_i is a direction-pure I-block on d_{i4}, Lemma 11.3.13 and Claim 11.3.25 imply that F_i' is a length-pure I-block on the edge l_{jk}.

Hence for all i, either F_i or F_i' is a length-pure I-block on l_{jk} where $\{i, j, k\} = \{1, 2, 3\}$. For ease of notation, let F_i^* denote this block for each i, and let $E_i^* = F_i^* \cap E(H)$.

Suppose $E_i^* \cap E_j^* \neq \emptyset$ for some $i \neq j$. Then $E_i^* \cup E_j^*$ is pure, balanced and tight by Lemma 11.3.14(i). Since $F_i^* \cup F_j^* = E_i^* \cup E_j^* \cup \{l_{jk}, l_{ik}\}$ is balanced, the set $F' = E_i^* \cup E_j^* \cup \{e_1, e_2, e_3\}$ is length-pure and balanced in (H, ψ) with $|F'| = |E_i^* \cup E_j^*| + 3 = 2|V(F')| - 2 > s(F')$, a contradiction.

So suppose instead that $E_1^* \cap E_2^* = E_1^* \cap E_3^* = E_2^* \cap E_3^* = \emptyset$. Then Lemma 11.3.14(ii) implies that at least two of these E_i^* consist of a single length edge $\overrightarrow{v_j v_k}$ with identity gain, where $\{i, j, k\} = \{1, 2, 3\}$. Without loss of generality, suppose this occurs for $i = 1$ and $i = 2$. Proposition 11.1.3(i) implies that either E_3^* consists of a single $\overrightarrow{v_1 v_2}$ length edge with identity gain, or every vertex in V_3^* is incident to at least two edges in E_3^*. In the former case, this implies $E_1^* \cup E_2^* \cup E_3^*$ is a length-pure cycle on $\{v_1, v_2, v_3\}$, which is balanced and tight. In the latter case, since $d_{H-v}(v_3) = 3$, and E_3^* is disjoint from the pair of edges in $E_1^* \cup E_2^*$ incident to v_3, this implies $v_3 \notin V_3^*$. Hence $E_1^* \cup E_2^* \cup E_3^*$ is length-pure, balanced and tight on vertex set $V_3^* \cup \{v_3\}$.

In both cases, this implies $E_1^* \cup E_2^* \cup E_3^* \cup \{l_{12}, l_{23}, l_{13}\}$ is balanced. Hence, in both cases (H, ψ) contains the length-pure, balanced set $F' =$

$E_1^* \cup E_2^* \cup E_3^* \cup \{e_1, e_2, e_3\}$ with $|F'| = 2|V(F')| - 2 > s(F')$, a contradiction. Thus for some $i \in \{1, 2, 3\}$, at least one (H_i, ψ_i) and (H_i', ψ_i') is formed by an admissible 2-reduction. □

11.3.5 Construction of $\mathcal{C}_k$-tight graphs

Here we use the results from this and the previous Section to obtain an inductive construction which characterises the $\mathcal{C}_k$-tight graphs. We first prove a special case.

Lemma 11.3.27. *Let (H, ψ) be a connected, 4-regular, $\mathcal{C}_k$-tight graph. If $|V(H)| \geq 2$, then there is a vertex in $V(H)$ which can be removed by an admissible reduction such that the resulting graph is connected.*

Proof. If H contains a vertex v incident to a loop, then Lemma 11.3.4 implies we can remove v by either a loop 1-reduction or a loop-to-loop reduction such that the resulting graph is connected and $\mathcal{C}_k$-tight.

Suppose instead that H is 4-regular and loop-free. If H contains either a cut-vertex or a 2-edge-cut, then Lemmas 11.3.9 and 11.3.12 imply that there is some 2-reduction of H which results in a connected, $\mathcal{C}_k$-tight graph.

Suppose neither of these cases hold. Then H is loop-free, 2-connected and 4-edge-connected, by Corollary 11.3.3; and the sparsity counts imply H is mixed. Thus H contains a mixed vertex v incident to exactly four edges. In which case, Lemmas 11.3.15, 11.3.16, 11.3.17 and 11.3.23 imply that there is a 2-reduction at v which results in a connected, $\mathcal{C}_k$-tight graph. □

Theorem 11.3.28. *Let (H, ψ) be a $\mathcal{C}_k$-gain graph. Then (H, ψ) is $\mathcal{C}_k$-tight if and only if it can be constructed from a graph (H_m, ψ_m) on m vertices whose connected components are $\mathcal{C}_k$-base graphs, by a sequence of 0-extensions, loop 0-extensions, 1-extensions, loop 1-extensions, loop-to-loop extensions and 2-extensions; which add edges with gains in $\mathcal{C}_k$.*

Proof. We first prove the forwards direction. Suppose $|V(H)| = n$, and let $(H, \psi) = (H_n, \psi_n)$. We shall show that we can construct a sequence of $\mathcal{C}_k$-tight graphs (H_i, ψ_i) for $m \leq i \leq n-1$ such that (H_{i+1}, ψ_{i+1}) is obtained from (H_i, ψ_i) by an extension. We do this in reverse, by performing admissible reductions.

If the connected components of (H_i, ψ_i) are $\mathcal{C}_k$-base graphs, then we are done; so suppose not. Then H_i contains some connected component, $H_i[F]$, on at least two vertices. Propositions 11.2.1(ii) and 11.3.1 imply that either $V(F)$ contains a vertex v of degree 2 or 3 in H_i, or $H_i[F]$ is 4-regular. In the former case, Proposition 11.2.5 and Lemma 11.2.6 imply that we can remove v from (H_i, ψ_i) by either a 0-reduction, loop 0-reduction or 1-reduction to obtain a $\mathcal{C}_k$-tight graph (H_{i-1}, ψ_{i-1}). Otherwise, $H_i[F]$ is 4-regular, and Lemma 11.3.27 implies that (H_i, ψ_i) has an admissible reduction at some vertex in $V(F)$ which forms a $\mathcal{C}_k$-tight graph (H_{i-1}, ψ_{i-1}).

We now prove the converse. Suppose $\{(H_i, \psi_i) : m \leq i \leq n\}$ is a set of graphs such that $|V(H_i)| = i$, (H_i, ψ_i) is obtained from (H_{i-1}, ψ_{i-1}) by an extension, $(H_n, \psi_n) = (H, \psi)$, and the connected components of (H_m, ψ_m) are $\mathcal{C}_k$-base graphs. We wish to show (H_i, ψ_i) is $\mathcal{C}_k$-tight for all i. Clearly this holds when $i = m$. Suppose (H_{i-1}, ψ_{i-1}) is $\mathcal{C}_k$-tight. Then Lemma 11.2.2 implies that (H_i, ψ_i) is sparse with $|E(H_i)| = 2|V(H_i)|$. Further, our restriction on edge gains ensures $\psi_i(e) \in \mathcal{C}_k$ for all $e \in E(H_i)$. Thus every closed walk C in H_i has $\psi_i(C) \in \mathcal{C}_k$, which implies $\langle E(H_i) \rangle \subseteq \mathcal{C}_k$. Hence (H_i, ψ_i) is $\mathcal{C}_k$-tight, thus proving our result. $\square$

One shortcoming of Theorem 11.3.28, is that it does not give a value for m. Lemma 11.3.27 implies that whenever a 4-regular $\mathcal{C}_k$-tight graph has an admissible reduction, it also has an admissible reduction which does not increase the number of connected components. Unfortunately, this is

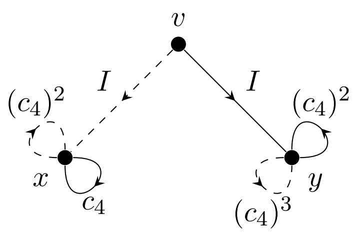


Figure 11.13: A $\mathcal{C}_4$-tight graph with no reduction which preserves connectivity.

not true in general. For example, consider the graph in Figure 11.13. The only admissible reduction here is a 0-reduction at v, but this increases the number of connected components by 1.

We suspect that 0-reductions are the only moves where it may be unavoidable to increase the number of connected components. It is trivial that loop 0-reductions can never increase the number of connected components, and Lemma 11.3.27 implies that in 4-regular graphs, we can always find a move which does not increase this number. So to prove this conjecture, it only remains to extend Lemma 11.2.6 to prove that such graphs always have an admissible 1-reduction which does not increase the number of connected components. If this were true, then the number of 0-extensions used in the construction of (H, ψ) in Theorem 11.3.28 would provide an upper bound for $m + 1$.

Chapter 12

Characterisation of symmetry-forced rigidity for $\mathcal{C}_s$

In Section 12.1, we show that our three simplest inductive moves: 0-extensions, loop 0-extensions and 1-extensions, preserve the dimension of the kernel of the orbit matrix for all point groups Γ we consider. These are the only moves required to inductively construct all $\mathcal{C}_s$-tight graphs (Theorem 11.2.9). In Section 12.2 we use this construction to obtain a characterisation of minimal symmetry-forced rigidity for $\mathcal{C}_s$-symmetric frameworks which are as generic as possible (Theorem 12.2.1). These three moves are not sufficient to construct all $\mathcal{C}_{k\geq 2}$-tight or $\mathcal{D}_{k\geq 2}$-tight graphs, as observed in Chapter 11, so we obtain a partial result for these cases. In Section 12.3 we explain the steps required to extend this partial result.

12.1 Extensions preserve row independence in the orbit matrix

We first prove some preliminary results. Namely, we show that switching operations on Γ-gain graphs, and reorienting edges and inverting the gain, both preserve the rank of the orbit matrix. This means that when we consider our inductive moves later in this section, we can reduce the problem to the case where the edge gains are easiest to deal with.

Note that for any $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{R}^2$, we can write $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = I\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)^{\perp} = P\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ where I is the 2-dimensional identity matrix and $P = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ corresponds to a rotation by $-\pi/2$ about the origin. This allows us to simplify the orbit matrix entries in the following proofs.

Lemma 12.1.1. *Let Γ be a point group, and let (H, ψ, q) be a Γ-gain framework. Suppose (H, ψ', q) is obtained from (H, ψ, q) by reversing some edge e of H and inverting the gain. Then* $\ker(O(H, \psi, q)) = \ker(O(H, \psi', q))$.

Proof. Let $e = \overrightarrow{v_1 v_2}$ for some $v_1, v_2 \in V(H)$, and suppose $\psi(\overrightarrow{v_1 v_2}) = g$. Then $\psi'(\overleftarrow{v_1 v_2}) = g^{-1}$. Let $T_0 = I$ or $T_0 = P$ when e is respectively a length or a direction edge. First suppose e is not a loop. Then the row corresponding to e in $O(H, \psi, q)$ is given in block matrix form by

$$\begin{array}{ccccc} & v_1 & & v_2 & \\ \Big(0 & \Big| \; (T_0(q(v_1) - g_\mu q(v_2)))^T & \Big| \; 0 \; \Big| & -(g_\mu^{-1} T_0(q(v_1) - g_\mu q(v_2))^T & \Big| \; 0\Big); \end{array}$$

whereas the corresponding row in $O(H, \psi', q)$ is

$$\begin{array}{ccccc} & v_1 & & v_2 & \\ \Big(0 & \Big| \; -(g_\mu T_0(q(v_2) - g_\mu^{-1} q(v_1)))^T & \Big| \; 0 \; \Big| & (T_0(q(v_2) - g_\mu^{-1} q(v_1)))^T & \Big| \; 0\Big), \end{array}$$

which can be rewritten as

$$\begin{array}{ccccccccc} & & v_1 & & & & v_2 & & \\ \Big(0 & \Big| & (g_\mu T_0 g_\mu^{-1}(q(v_1) - g_\mu q(v_2)))^T & \Big| & 0 & \Big| & -(T_0 g_\mu^{-1}(q(v_1) - g_\mu q(v_2)))^T & \Big| & 0 \Big) \end{array}.$$

When $T_0 = I$, it is clear that this row is identical to the original row in $O(H, \psi, q)$. However, when $T_0 = P$, we have to be slightly more careful. If g is a rotation or the identity, then $Pg_\mu^{-1} = g_\mu^{-1}P$, and so once more, these rows are identical. If instead, g is a reflection, then $Pg_\mu^{-1} = -g_\mu^{-1}P$, and so the original row in $O(H, \psi, q)$ can be obtained by multiplying the row in $O(H, \psi', q')$ by -1. This operation does not change the kernel of the matrix. Hence $\ker(O(H, \psi, q)) = \ker(O(H, \psi', q))$.

When e is a loop, we apply the same argument. In this case if e is a direction loop with reflection gain, then the corresponding rows in $O(H, \psi, q)$ and $O(H, \psi', q)$ are the zero vector. In all other cases $T_0 g_\mu = g_\mu T_0$, and so the corresponding rows are identical. Hence, once more, $\ker(O(H, \psi, q)) = \ker(O(H, \psi', q))$. $\square$

Lemma 12.1.2. *Let Γ be a point group, and let (H, ψ, q) be a Γ-gain framework. Suppose (H, ψ') is obtained from (H, ψ) by a switching operation with gain h at some vertex v, and let q' be a realisation of (H, ψ') such that $q'(u) = q(u)$ for all $u \in V(H) - \{v\}$, and $q'(v) = h_\mu q(v)$. Then* $\text{rank}(O(H, \psi, q)) = \text{rank}(O(H, \psi', q'))$.

Proof. Since $q'(u) = q(u)$ for all $u \in V(H) - \{v\}$, and the switching operation only changes the gains on edges incident to v, the matrices $O(H, \psi, q)$ and $O(H, \psi', q')$ differ only in the rows corresponding to edges incident to v. Such edges fall into three classes: loops at v, non-loops oriented towards v, and non-loops oriented away from v. By Lemma 12.1.1, we can assume that all non-loop edges are oriented towards v. Thus we only need to consider the first two cases.

First suppose $\overrightarrow{uv}$ is a non-loop edge, with $\psi(\overrightarrow{uv}) = g$. Then $\psi'(\overrightarrow{uv}) = gh^{-1}$, and the corresponding row in $O(H, \psi', q')$ in block matrix form is

$$\begin{array}{ccccccccc} & & u & & & & v & & \\ \Big(0 & \Big| & (T_1(q'(u) - g_\mu h_\mu^{-1} q'(v)))^T & \Big| & 0 & \Big| & -((g_\mu h_\mu^{-1})^{-1} T_1(q'(u) - g_\mu h_\mu^{-1} q'(v)))^T & \Big| & 0\Big) \end{array}$$

where $T_1 \in \{I, P\}$. Using the fact $q'(u) = q(u)$ and $q'(v) = h_\mu q(v)$, we can rewrite this as

$$\begin{array}{ccccccccc} & & u & & & & v & & \\ \Big(0 & \Big| & (T_1(q(u) - g_\mu q(v)))^T & \Big| & 0 & \Big| & -(h_\mu g_\mu^{-1} T_1(q(u) - g_\mu q(v)))^T & \Big| & 0\Big) \end{array}.$$

This row only differs from the corresponding row in $O(H, \psi, q)$, in that the pair of columns for v are multiplied on the right by h_μ^T.

Next suppose l is a loop at v with gain $\psi(l) = f$. After switching, we have $\psi'(l) = hfh^{-1}$. So the corresponding row in $O(H, \psi', q')$ is

$$\begin{array}{ccccc} & & v & & \\ \Big(0 & \Big| & ((I - (hfh^{-1})_\mu^{-1}) T_2 (I - (hfh^{-1})_\mu) q'(v))^T & \Big| & 0\Big) \end{array}$$

where $T_2 \in \{I, P\}$. Once more, using $q'(v) = h_\mu q(v)$, we can rewrite this as

$$\begin{array}{ccccc} & & v & & \\ \Big(0 & \Big| & (h_\mu (I - f_\mu^{-1}) h_\mu^{-1} T_2 h_\mu (I - f_\mu) q(v))^T & \Big| & 0\Big) \end{array}.$$

When $T_2 = P$ and h is a reflection we have $T_2 h_\mu = -h_\mu T_2$, otherwise $T_2 h_\mu = h_\mu T_2$. Hence $h_\mu^{-1} T_2 h_\mu$ is $-T_2$ or T_2 respectively, and so we can obtain this row from the corresponding row in $O(H, \psi, q)$ by multiplying on the right by $-h_\mu^T$ or h_μ^T respectively.

Thus we can obtain $O(H, \psi', q')$ from $O(H, \psi, q)$ by multiplying every row vector in the pair of columns corresponding to v by h_μ^T on the right, and then, if h is a reflection, multiplying the rows corresponding to any direction loops at v by -1. This is equivalent to performing a sequence of column and row operations. Since such operations preserve rank, we have $\mathrm{rank}(O(H, \psi', q')) = \mathrm{rank}(O(H, \psi, q))$. □

In Lemma 12.1.2, q' is algebraically independent over $\mathbb{Q}_\Gamma$ if and only if q is. Further, for a given Γ-gain graph, the orbit matrix attains maximum rank at realisations which are generic over $\mathbb{Q}_\Gamma$. Hence Lemma 12.1.2 implies that the switching operation preserves the rank of such frameworks:

Corollary 12.1.3. *Let Γ be a point group, and let (H, ψ, q) be a Γ-gain framework with q generic over $\mathbb{Q}_\Gamma$. If (H, ψ') is obtained from (H, ψ) by a sequence of switching operations, then* $\mathrm{rank}(O(H, \psi, q)) = \mathrm{rank}(O(H, \psi', q))$.

12.1.1 0-extensions and loop 0-extensions

We now show that 0-extensions and loop 0-extensions preserve row independence in the orbit matrix:

Theorem 12.1.4. *Let Γ be a point group, and let (H, ψ) and (H', ψ') be Γ-gain graphs such that (H', ψ') is obtained from (H, ψ) by either a 0-extension or a loop 0-extension which adds a vertex v incident to some set of vertices $X \subseteq V(H)$.*

Suppose q is a realisation of (H, ψ) such that $O(H, \psi, q)$ is row independent and q is generic over $\mathbb{Q}_\Gamma$. Let q' be a realisation of (H', ψ') which is generic over $\mathbb{Q}_\Gamma$ with $q'|_{V(H)} = q$. Then $O(H', \psi', q')$ is row independent.

Proof. By Lemma 12.1.1, we can assume our two new edges are oriented away from v. We split the remainder of the proof into two cases depending on whether the inductive move is a 0-extension or loop 0-extension.

Case 1. v is added by a 0-extension.

Suppose the edges added are $\overrightarrow{vv_1}$ and $\overrightarrow{vv_2}$ for some $\{v_1, v_2\} \subseteq V(H)$ which may not be distinct, and let $\psi'(\overrightarrow{vv_1}) = g$ and $\psi'(\overrightarrow{vv_2}) = h$. Then, reordering the rows and columns as necessary, the orbit matrix of (H', ψ', q') can be written in block form as

$$\begin{array}{c} \\ \overrightarrow{vv_1} \\ \overrightarrow{vv_2} \\ E(H) \end{array} \begin{array}{c} \begin{array}{cc} \qquad\qquad v \qquad\qquad & \qquad V(H) \qquad \end{array} \\ \left(\begin{array}{c|c} (T_1(q'(v) - g_\mu q'(v_1)))^T & * \\ (T_2(q'(v) - h_\mu q'(v_2)))^T & * \\ \hline 0 & O(H, \psi, q) \end{array}\right) \end{array}$$

where $T_1, T_2 \in \{I, P\}$. Since $O(H, \psi, q)$ is row independent, and the first two columns of $O(H', \psi', q')$ have zeroes in all rows except the first two, $O(H', \psi', q')$ will be row independent if and only if the vectors $T_1(q'(v) - g_\mu q'(v_1))$ and $T_2(q'(v) - h_\mu q'(v_2))$ are linearly independent.

Assume for contradiction that these vectors are linearly dependent. Then there exists some non-zero $\lambda \in \mathbb{R}$ such that

$$(T_1 - \lambda T_2)q'(v) - T_1 g_\mu q'(v_1) + \lambda T_2 h_\mu q'(v_2) = 0.$$

Since q' is generic over $\mathbb{Q}_\Gamma$, this can only hold when the coefficients of the coordinates for distinct vertices is zero. Since v is distinct from $\{v_1, v_2\}$, this implies $\lambda = 1$ and $T_1 = T_2$. Which in turn implies $h = g$ and $v_1 = v_2$. In other words $\overrightarrow{vv_1}$ and $\overrightarrow{vv_2}$ are parallel edges of the same type and with the same gain. This contradicts the definition of a 0-extension. Hence $O(H', \psi', q')$ is row independent.

Case 2. v is added by a loop 0-extension.

Suppose the non-loop edge added terminates at $v_1 \in V(H)$ and suppose the gains on the two new edges are $\psi'(\overrightarrow{vv}) = g$ and $\psi'(\overrightarrow{vv_1}) = h$ for some

$g, h \in \Gamma$. Then the orbit matrix for (H', ψ', q') can be written in block form as

$$\begin{array}{c} \\ \overrightarrow{vv_1} \\ \overrightarrow{vv} \\ E(H) \end{array} \begin{array}{c} \begin{array}{cc} \quad v \quad & \quad V(H) \end{array} \\ \left(\begin{array}{c|c} (T_1(q'(v) - h_\mu q'(v_1)))^T & * \\ ((I - g_\mu^{-1})T_2(I - g_\mu)q'(v))^T & 0 \\ \hline 0 & O(H, \psi, q) \end{array}\right) \end{array}$$

where $T_1, T_2 \in \{I, P\}$. As in Case 1, $O(H', \psi', q')$ is row independent if and only if the vectors in the first two columns of the first two rows are independent. Assume for a contradiction that they are not. Then there exists some non-zero $\lambda \in \mathbb{R}$ such that

$$(T_1 + \lambda(I - g_\mu^{-1})T_2(I - g_\mu))q'(v) - T_1 h_\mu q'(v_1) = 0.$$

Since q' is generic over $\mathbb{Q}_\Gamma$ and $v \neq v_1$, for this to hold the coefficients of $q'(v)$ and $q'(v_1)$ must both be zero. But this is impossible for $q'(v_1)$. Hence $O(H', \psi', q')$ is row independent. $\square$

12.1.2 1-extensions

Before we tackle 1-extensions, there is one further tool we require, which is that of limit matrices:

Proposition 12.1.5. *[33] Let $m_{i,j} : \mathbb{R} \to \mathbb{R}$ be a continuous function for all integers $1 \leq i \leq n$ and $1 \leq j \leq m$, and let M_t be an $n \times m$ matrix with i, j entry given by $m_{i,j}(t)$. Suppose $\lim_{t \to \infty} M_t = M$. Then there exists some N such that for all $t \geq N$, $\operatorname{rank}(M_t) \geq \operatorname{rank}(M)$.*

The following argument for 1-extensions is based on that in [35], but extended to deal with symmetry. Roughly the argument takes the following form: if our 1-extension adds a vertex v by deleting some edge $e = \overrightarrow{v_1 v_2}$ and adding edges $e_i = \overrightarrow{vv_i}$ for $i \in \{1, 2, 3\}$, we first perform a 0-extension to add edges e_2 and e_3, and then show that we can find a special position for v such that the rows indexed by e, e_1 and e_2 are linearly dependent. This allows us to delete e and add e_1 whilst preserving the rank.

Theorem 12.1.6. *Let Γ be a point group. Let (H, ψ) and (H', ψ') be Γ-gain graphs such that (H', ψ') is obtained from (H, ψ) by a 1-extension which deletes some edge $e \in E(H)$ and adds a vertex v.*

Suppose q is a realisation of (H, ψ) such that $O(H, \psi, q)$ is row independent and q is generic over $\mathbb{Q}_\Gamma$. Then there exists a realisation q' of (H', ψ') with $q'|_{V(H)} = q$ such that $O(H', \psi', q')$ is row independent.

Proof. Let $e = \overrightarrow{v_1v_2}$ have gain g. By Lemma 12.1.1, we can assume the extension adds three edges $e_i = \overrightarrow{vv_i}$ where the vertices $v_1, v_2, v_3 \in V(H)$ are not necessarily distinct. Let g_i denote $\psi'(e_i)$. The definition of a 1-extension requires $g = g_1^{-1}g_2$. By Corollary 12.1.3, it suffices to find a realisation q' when $g_1 = I$ and $g_2 = g$.

Let (H^*, ψ^*) be the graph obtained from (H, ψ) by a 0-extension which adds the vertex v and the edges e_2 and e_3 with gains g and g_3 respectively. To construct (H', ψ) from (H^*, ψ^*), it only remains to delete the edge e and add the edge e_1 with gain $\psi'(e_1) = I$. So to prove our result, it suffices to find a realisation q' for which $O(H^*, \psi^*, q')$ is row independent, and replacing the row for e with that for e_1 maintains row independence.

When e is not a loop, the row corresponding to e in $O(H, \psi, q')$ is given by

$$\begin{array}{ccc} v_1 & v_2 & V(H) - \{v_1, v_2\} \\ \Big((T_0(q'(v_1) - g_\mu q'(v_2)))^T & \Big| \; -(g_\mu^{-1}T_0(q'(v_1) - g_\mu q'(v_2)))^T \; \Big| & 0 \Big), \end{array}$$

where $T_0 = I$ or $T_0 = P$ when e is respectively a length or a direction edge.

If e is a loop, then $v_1 = v_2$, and the block corresponding to this vertex is the sum of the above blocks for v_1 and v_2.

Similarly, the row in the orbit matrix $O(H', \psi', q')$ corresponding to each non-loop edge e_i incident to v is given by

$$\begin{array}{ccc} v & v_i & V(H) - \{v_i\} \\ \multicolumn{3}{c}{\left(\begin{array}{c|c|c} (T_i(q'(v) - (g_i)_\mu q'(v_i)))^T & -((g_i)_\mu^{-1} T_i(q'(v) - (g_i)_\mu q'(v_i)))^T & 0 \end{array} \right)} \end{array}$$

where $T_i = I$ or $T_i = P$ when e_i is respectively a length or direction edge.

For clarity, we allocate distinct columns in the block matrices to each of v, v_1, v_2 and v_3. This implicitly assumes that v_1, v_2 and v_3 are distinct, which is not always true. If any of combination of these vertices are identified, then the correct matrix can be obtained by summing the corresponding column blocks. In the following two claims, we prove some useful properties of these block matrices.

Claim 12.1.7. *There is no $k \in \mathbb{R}$ such that*

$$q(v_1) - g_\mu q(v_2) = k(q(v_1) - (g_3)_\mu q(v_3)). \tag{12.1}$$

Proof. Assume for a contradiction that equation (12.1) holds for some k. Then the points $q(v_1)$, $g_\mu q(v_2)$ and $(g_3)_\mu q(v_3)$ are collinear. Since Γ is a point group and q is algebraically independent over $\mathbb{Q}_\Gamma$, this implies $v_1 = v_2 = v_3$ and $g \in \{I, g_3\}$. If $g = I$, then e is a loop with identity gain, whereas if $g = g_3$, then e_2 and e_3 are parallel edges with the same gain and same type. In both cases, this contradicts the definition of a direction-length Γ-gain graph. □

Claim 12.1.8. *Suppose $T_2 \neq T_3$, and that there is some $k \in \mathbb{R}$ such that*

$$T_2(q(v_1) - g_\mu q(v_2)) = kT_3(q(v_1) - (g_3)_\mu q(v_3)). \tag{12.2}$$

Then $k \neq 0$, $v_1 = v_2 = v_3$ and $g_\mu = -(g_3)_\mu$.

Proof. Since q is generic over $\mathbb{Q}_\Gamma$, for (12.2) to hold, the coefficients of distinct $q(v_i)$ must be zero. This implies that v_2 is not distinct from $\{v_1, v_3\}$, and, since $T_2 \neq T_3$, v_1 is also not distinct from $\{v_2, v_3\}$. Hence $v_1 = v_2$.

Suppose $k = 0$. Then (12.2) gives $g_\mu = I$, which implies e is a loop with identity gain, hence contradicting our definition of a Γ-gain graph. Thus $k \neq 0$, which in turn implies $v_1 = v_2 = v_3$. Since $\{T_2, T_3\} = \{I, P\}$, we know $T_2^{-1}T_3 \in \{\pm P\}$. So we can rewrite (12.2) as

$$(q(v_1) - g_\mu q(v_1)) = \pm k(q(v_1) - (g_3)_\mu q(v_1))^\perp.$$

Since $k \neq 0$, (12.2) has a solution if and only if the two vectors $q(v_1) - g_\mu q(v_1)$ and $q(v_1) - (g_3)_\mu q(v_1)$ are perpendicular at $q(v_1)$. Since Γ is a point group, this implies $(g_3)_\mu q(v_1)$ can be obtained from $g_\mu q(v_1)$ by a rotation of $180°$ about the origin. In other words $(g_3)_\mu = P^2 g_\mu = -g_\mu$ as required. □

In the remainder of the proof, our choice of realisation is dependent on the values of T_0, T_1 and T_2, so we consider each of the following three cases separately: $T_0 = T_1 = T_2$ (i.e. both e_1 and e_2 are of the same type as e), $T_1 \neq T_2$ (e_1 is of different type to e_2), and $T_0 \neq T_1 = T_2$ (neither e_1 nor e_2 is of the same type as e). Note that the definition of a 1-extension requires that at least one of e_1, e_2 and e_3 is of the same type as e, so in this third case, e_3 must be of the same type as e. To simplify our block matrix entries, let $A = q(v_1) - g_\mu q(v_2)$ and $B = q(v_1) - (g_3)_\mu q(v_3)$.

Case 1. $T_0 = T_1 = T_2$.

Let $\lambda \in \mathbb{R}$, and q_λ be a realisation of (H^*, ψ^*) with $q_\lambda|_{V(H)} = q$ and

$$q_\lambda(v) = (1 - \lambda)q(v_1) + \lambda g_\mu q(v_2).$$

Let $(H^* + e_1, \psi^*)$ denote the graph formed from (H^*, ψ^*) by adding e_1 and extending ψ^* by allocating $\psi^*(e_1) = I$. The orbit matrix for $(H^* + e_1, \psi^*, q_\lambda)$

is given in block form by:

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \quad v \quad & \quad v_1 \quad & \quad v_2 \quad & \quad v_3 \quad & \end{array} \\
\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e \\ \\ \end{array} &
\left(\begin{array}{c|c|c|c|c}
-\lambda(T_0A)^T & \lambda(T_0A)^T & 0 & 0 & 0 \\
(1-\lambda)(T_0A)^T & 0 & -(1-\lambda)(g_\mu^{-1}T_0A)^T & 0 & 0 \\
(T_3(B-\lambda A))^T & 0 & 0 & -((g_3)_\mu^{-1}T_3(B-\lambda A))^T & 0 \\
0 & (T_0A)^T & -(g_\mu^{-1}T_0A)^T & 0 & 0 \\
\hline
0 & \multicolumn{4}{c}{O(H-e,\psi,q)}
\end{array}\right)
\end{array}
$$

We wish to find a realisation q_λ such that $O(H^*, \psi^*, q_\lambda)$ is row independent. This is guaranteed when the blocks corresponding to v for the edges e_2 and e_3 are linearly independent. If $T_0 = T_3$, then this holds by Claim 12.1.7. So suppose instead that $T_0 \neq T_3$. Then $T_3^{-1}T_0 \in \{\pm P\}$ and so these blocks are dependent if and only if there exists some $k \in \mathbb{R}$ such that

$$B = \lambda A + kA^\perp.$$

Since A and $A^\perp$ are linearly independent, there are unique values $\lambda = \lambda^*$ and $k = k^*$ which satisfy this equation. Assume that our choice of q_λ is such that $\lambda \neq \lambda^*$. Then these rows are independent and $\text{rank}(O(H^* + e_1, \psi^*, q_\lambda)) \geq \text{rank}(O(H^*, \psi^*, q_\lambda)) = |E(H)| + 2$.

If we multiply the rows for e_1, e_2 and e by $\frac{1}{\lambda}$, $\frac{1}{1-\lambda}$ and -1 respectively, then the sum of the resulting rows gives the zero vector. Hence deleting the row corresponding to e from $O(H^* + e_1, \psi^*, q_\lambda)$ does not reduce the rank. Deleting this row gives the orbit matrix of (H', ψ', q_λ), so this implies that for all $\lambda \in \mathbb{R} - \{0, 1, \lambda^*\}$,

$$\text{rank}(O(H', \psi', q_\lambda)) = \text{rank}(O(H^* + e_1, \psi^*, q_\lambda)) = |E(H)| + 2 = |E(H')|.$$

Hence $O(H', \psi', q_\lambda)$ is row independent for all $\lambda \in \mathbb{R} - \{0, 1, \lambda^*\}$.

Case 2. $T_1 \neq T_2$.

By Lemma 12.1.1 reorienting e and inverting its gain does not change the rank of the orbit matrix. So without loss of generality, we can assume $T_0 = T_2$. This case is the most subtle of the three, as it is possible for T_3B and T_0A to be linearly dependent. As such, we split our argument into two subcases.

Subcase 2(a). There is no $k \in \mathbb{R}$ such that $T_3B = kT_0A$.

For $\lambda \in \mathbb{R}$, let q_λ be a realisation of (H^*, ψ^*) with $q_\lambda|_{V(H)} = q$ and

$$q_\lambda(v) = (I - \lambda P)q(v_1) + \lambda P g_\mu q(v_2).$$

Letting $\overline{A} = \lambda PA$, the orbit matrix for $(H^* + e_1, \psi^*, q_\lambda)$ is given in block form by

$$\begin{array}{cc}
 & \begin{array}{cccc} v & v_1 & v_2 & v_3 \end{array}\\
\begin{array}{c} e_1\\ e_2\\ e_3\\ e\\ \\ \end{array} & \left(\begin{array}{c|c|c|c|c}
-(T_1\overline{A})^T & (T_1\overline{A})^T & 0 & 0 & 0\\
(T_0(A-\overline{A}))^T & 0 & -(g_\mu^{-1}T_0(A-\overline{A}))^T & 0 & 0\\
(T_3(B-\overline{A}))^T & 0 & 0 & -((g_3)_\mu^{-1}T_3(B-\overline{A}))^T & 0\\
0 & (T_0A)^T & -(g_\mu^{-1}T_0A)^T & 0 & 0\\
\hline
0 & \multicolumn{4}{c}{O(H-e,\psi,q)}
\end{array}\right)
\end{array}$$

Let M_λ denote the matrix formed from $O(H^* + e_1, \psi^*, q_\lambda)$ by multiplying the row corresponding to e_1 by $\frac{1}{\lambda}$. Then for all $\lambda \neq 0$, $\text{rank}(M_\lambda) = \text{rank}(O(H^* + e_1, \psi^*, q_\lambda))$. Let $M = \lim_{\lambda \to 0} M_\lambda$, then M is given in block form by

$$\begin{array}{cc}
 & \begin{array}{cccc} v & v_1 & v_2 & v_3 \end{array}\\
\begin{array}{c} e_1\\ e_2\\ e_3\\ e\\ \\ \end{array} & \left(\begin{array}{c|c|c|c|c}
-(T_1PA)^T & (T_1PA)^T & 0 & 0 & 0\\
(T_0A)^T & 0 & -(g_\mu^{-1}T_0A)^T & 0 & 0\\
(T_3B)^T & 0 & 0 & -((g_3)_\mu^{-1}T_3B)^T & 0\\
0 & (T_0A)^T & -(g_\mu^{-1}T_0A)^T & 0 & 0\\
\hline
0 & \multicolumn{4}{c}{O(H-e,\psi,q)}
\end{array}\right)
\end{array}$$

By our assumption, the rows in M for the edges e_2 and e_3 are independent in the columns corresponding to v. Thus $\text{rank}(M) \geq \text{rank}(O(H, \psi, q)) +$

$2 = |E(H')|$. Since $\{T_0, T_1\} = \{I, P\}$, we know $T_1 P = \pm T_0$. Hence the rows of M indexed by e_1, e_2 and e are linearly dependent. So deleting the row corresponding to e does not change the rank. Let M^* and M^*_λ denote the matrices formed from M and M_λ respectively by deleting the row corresponding to e. Then Proposition 12.1.5 implies that for small enough λ

$$\mathrm{rank}(O(H^*{+}e_1{-}e, \psi^*, q_\lambda)) = \mathrm{rank}(M^*_\lambda) \geq \mathrm{rank}(M^*) = \mathrm{rank}(M) \geq |E(H')|.$$

But $(H^* + e_1 - e, \psi^*) = (H', \psi')$, and $|E(H')|$ is the number of rows of $O(H', \psi', q_\lambda)$. So this holds with equality, and thus $O(H', \psi', q_\lambda)$ is row independent for small enough λ.

Subcase 2(b). There is some $k \in \mathbb{R}$ such that $T_3 B = k T_0 A$.
By Claim 12.1.7 we must have $T_3 \neq T_0$. Thus $T_1 = T_3 \neq T_2 = T_0$. Claim 12.1.8 now implies that $k \neq 0$, $v_1 = v_2 = v_3$, and $(g_3)_\mu = -g_\mu$. We define q_λ as we did in Case 1, by letting $q_\lambda|_{V(H)} = q$ and

$$q_\lambda(v) = (1-\lambda)q(v_1) + \lambda g_\mu q(v_2).$$

The orbit matrix for $(H^* + e_1, \psi^*, q_\lambda)$ is given by

$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccc} \quad v \quad & \qquad v_1 \qquad & V(H)-v_1 \end{array}\\
\left(\begin{array}{c|c|c}
-\lambda(T_1A)^T & \lambda(T_1A)^T & 0 \\
(1-\lambda)(T_0A)^T & -(1-\lambda)(g_\mu^{-1}T_0A)^T & 0 \\
(T_1(B-\lambda A))^T & (g_\mu^{-1}T_1(B-\lambda A))^T & 0 \\
0 & ((I-g_\mu^{-1})T_0A)^T & 0 \\
\hline
0 & \multicolumn{2}{c}{O(H-e,\psi,q)}
\end{array}\right)
\end{array}$$

where $A = (I - g_\mu)q(v_1)$ and $B = (I + g_\mu)q(v_1)$.

Let M_λ be the matrix obtained by multiplying the rows for e_1 and e_2 by $\frac{1}{\lambda}$ and $\frac{1}{1-\lambda}$ respectively. Our argument now diverges depending on whether g is a reflection or a rotation. When g is a reflection, we consider the matrix

$M = \lim_{\lambda \to 0} M_\lambda$. When g is a rotation, we multiply the row for e_3 in M_λ by $\frac{1}{\lambda}$ to obtain the matrix N_λ, and then consider the matrix $N = \lim_{\lambda \to \infty} N_\lambda$.

First suppose g is a reflection. Then $g^{-1} = g$, and e is a length loop. Hence $T_0 = I$ and $T_1 = P$. This implies $g_\mu^{-1} T_0 A = g_\mu^{-1}(I - g_\mu)q(v_1) = -A$, and $(I - g_\mu^{-1})T_0 A = (2I - g_\mu - g_\mu^{-1})q(v_1) = 2A$. Using these properties, and the fact that $T_1 B = kT_0 A$, we can write M as

$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccc} \quad v \quad & \quad v_1 \quad & V(H) - v_1 \end{array} \\
\left(\begin{array}{c|c|c}
-(PA)^T & (PA)^T & 0 \\
A^T & A^T & 0 \\
A^T & -A^T & 0 \\
0 & 2A^T & 0 \\
\hline
0 & \multicolumn{2}{c}{O(H-e,\psi,q)}
\end{array}\right)
\end{array}$$

Since PA is perpendicular to A, the rows for e_1 and e_3 are linearly independent in the columns corresponding to v. Hence $\mathrm{rank}(M) \geq \mathrm{rank}(O(H, \psi, q)) + 2 = |E(H')|$. Further, the rows for e_2, e_3 and e are linearly dependent, so deleting the row for e does not change the rank. Let M^* and M^*_λ be the matrices obtained from M and M_λ respectively by deleting the row corresponding to e. Then by Proposition 12.1.5, for small enough λ

$$\mathrm{rank}(O(H', \psi', q_\lambda)) = \mathrm{rank}(M^*_\lambda) \geq \mathrm{rank}(M^*) = |E(H')|.$$

Hence $O(H', \psi', q_\lambda)$ is row independent for such λ.

Suppose instead that g is a rotation. Then $T_0 g_\mu = g_\mu T_0$, and, using the structure of a rotation matrix given on page 143, we have $g_\mu + g_\mu^{-1} = 2\cos(\theta) I$ when g is a rotation by θ about the origin. Hence

$$(I - g_\mu^{-1})T_0 A = (2I - g_\mu - g_\mu^{-1})T_0 q(v_1) = 2(1 - \cos(\theta))T_0 q(v_1).$$

Also, since $T_1 B = kT_0 A$, $\{T_0, T_1\} = \{I, P\}$ and $P^{-1} = -P$, we have that

$kT_1A = -T_0B$. Using these properties, we can write N as

$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} \quad v \quad & \qquad v_1 \qquad & V(H) - v_1 \end{array} \\ \left(\begin{array}{c|c|c} (T_0B)^T & -(T_0B)^T & 0 \\ (T_0A)^T & -(g_\mu^{-1}T_0A)^T & 0 \\ (T_0B)^T & (g_\mu^{-1}T_0B)^T & 0 \\ 0 & (T_0q(v_1))^T & 0 \\ \hline 0 & \multicolumn{2}{c}{O(H-e,\psi,q)} \end{array}\right) \end{array}$$

Since $T_1B = kT_0A$, the vectors T_0A and T_0B are perpendicular, hence the rows for edges e_2 and e_3 are linearly independent in the columns corresponding to v. Thus $\text{rank}(N) \geq \text{rank}(O(H,\psi,q)) + 2 = |E(H')|$. However, if we subtract the row for e_1 from the row for e_3, the columns corresponding to v cancel, and in the columns corresponding to v_1 we obtain

$$(I + g_\mu^{-1})T_0B = (2I + g_\mu + g_\mu^{-1})T_0q(v_1) = 2(1+\cos(\theta))T_0q(v_1).$$

Hence the rows for e_1, e_3 and e form a linearly dependent set, and deleting the row corresponding to e from N does not change the rank. Let N^* denote the matrix formed from N when this row is deleted, then by Proposition 12.1.5, for large enough λ

$$\text{rank}(O(H',\psi',q_\lambda)) = \text{rank}(N^*_\lambda) \geq \text{rank}(N^*) = |E(H')|$$

as required.

Case 3. $T_1 = T_2 \neq T_0 = T_3$.

Let $\lambda \in \mathbb{R}$, and q_λ be a realisation of (H^*,ψ^*) with $q_\lambda|_{V(H)} = q$ and

$$q_\lambda(v) = (I - \lambda P)q(v_1) + \lambda P g_\mu q(v_2).$$

Substituting these equations into the orbit matrix $O(H^* + e_1, \psi^*, q_\lambda)$, gives exactly the same rows for e and e_1 as we had in Subcase 2(a), whereas for e_2 we need to replace T_0 with T_1, and for e_3 we replace T_3 by T_0 throughout.

Let M_λ denote the matrix formed by taking $O(H^* + e_1, \psi^*, q_\lambda)$, and multiplying the rows for e_1, e_2 and e_3 by $-\frac{1}{\lambda}$, $\frac{1}{\lambda}$ and $\frac{1}{\lambda}$ respectively, and let $M = \lim_{\lambda\to\infty} M_\lambda$. The limit matrix M is given in block form by

$$\begin{array}{c c}
 & \begin{array}{ccccc} \quad v \quad & \quad v_1 \quad & \quad v_2 \quad & \quad v_3 \quad & \end{array} \\
\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e \\ \\ \end{array} &
\left(\begin{array}{c|c|c|c|c}
(T_1PA)^T & -(T_1PA)^T & 0 & 0 & 0 \\
-(T_1PA)^T & 0 & (g_\mu^{-1}T_1PA)^T & 0 & 0 \\
-(T_0PA)^T & 0 & 0 & -((g_3)_\mu^{-1}T_0PA)^T & 0 \\
0 & (T_0A)^T & -(g_\mu^{-1}T_0A)^T & 0 & 0 \\
\hline
0 & \multicolumn{4}{c}{O(H-e,\psi,q)}
\end{array}\right)
\end{array}$$

Since $\{T_1, T_0\} = \{I, P\}$ and $A \neq 0$, the vectors T_1PA and T_0PA are non-zero and orthogonal, and hence the rows corresponding to e_2 and e_3 in M are linearly independent. Thus

$$\operatorname{rank}(M) \geq \operatorname{rank}(O(H-e,\psi,q)) + 2 = |E(H')|.$$

Further, since $T_1P = \pm T_0$, the rows in M indexed by e_1, e_2 and e are linearly dependent. Hence deleting the row for e does not reduce the rank. Let M^* and M^*_λ denote the matrices formed from M and M_λ respectively by deleting the row corresponding to e. Deleting this same row from $O(H^* + e_1, \psi^*, q_\lambda)$ gives $O(H', \psi', q_\lambda)$. Proposition 12.1.5 implies that for large enough λ, we have

$$\operatorname{rank}(O(H', \psi', q_\lambda)) = \operatorname{rank}(M^*_\lambda) \geq \operatorname{rank}(M^*) = |E(H')|.$$

But $O(H', \psi', q_\lambda)$ has exactly $|E(H')|$ rows, so this must hold with equality, and thus $O(H', \psi', q_\lambda)$ is row independent for such λ. $\square$

12.2 Characterisation for $\mathcal{C}_s$

The results from the preceding chapters culminate in the following result:

Theorem 12.2.1. *Let (G, p) be a $\mathcal{C}_s$-generic $(\mathcal{C}_s, \pi)$-symmetric direction-length framework under free action π. Let V' be a set of vertex orbit representatives of G, and let $(H, \psi, \tilde{p})$ be the quotient framework of (G, p) with respect to V'. Then (G, p) is minimally symmetry-forced infinitesimally rigid if and only if (H, ψ) is $\mathcal{C}_s$-tight.*

Proof. The forwards direction is given by Theorem 10.3.4. So suppose (H, ψ) is $\mathcal{C}_s$-tight. Then Theorem 11.2.9 implies that (H, ψ) can be inductively constructed from the unique $\mathcal{C}_s$-base graph shown in Figure 12.1 by a sequence of 0-extensions, loop 0-extensions and 1-extensions.

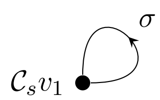


Figure 12.1: The unique $\mathcal{C}_s$-base graph.

Let (H_i, ψ_i) denote the graph on i vertices which appears in this construction sequence, and consider the realisation $\tilde{p}|_{V(H_i)}$ of this graph. Since $\tilde{p}(\mathcal{C}_s v_1) = p(v_1)$, the orbit matrix for the realisation $\tilde{p}|_{\{\mathcal{C}_s v_1\}}$ of the base graph (H_1, ψ_1) is

$$O(H_1, \psi_1, \tilde{p}|_{\{\mathcal{C}_s v_1\}}) = (2(I - \sigma_\mu)p(v_1))^T.$$

Since p is $\mathcal{C}_s$-generic, the unique row of this matrix is non-zero. Hence $O(H_1, \psi_1)$ is row independent. Theorems 12.1.4 and 12.1.6 thus imply that for every graph (H_i, ψ_i) in our construction sequence, the orbit matrix $O(H_i, \psi_i, \tilde{p}|_{V(H_i)})$ is row independent. In particular, when $i = |V(H)|$, this implies $O(H, \psi, \tilde{p})$ is row independent. Since (H, ψ) is $\mathcal{C}_s$-tight, this means $|E(H)| = 2|V(H)| - 1 = \text{rank}(O(H, \psi, \tilde{p}))$. Thus (G, p) is minimally symmetry-forced infinitesimally rigid by Corollary 10.2.5. $\square$

We also obtain the following partial result for rotational and dihedral symmetry:

Theorem 12.2.2. *Let $\Gamma \in \{\mathcal{C}_{k\geq 2}, \mathcal{D}_{k\geq 2}\}$. Let (G,p) be a Γ-generic (Γ,π)-symmetric direction-length framework on connected graph G under free action π. Let V' be a set of vertex orbit representatives of G, and $(H,\psi,\tilde{p})$ be the quotient framework of (G,p) with respect to V'. If (H,ψ) can be constructed from a Γ-base graph by a sequence of 0-extensions, loop 0-extensions and 1-extensions, then (G,p) is minimally symmetry-forced infinitesimally rigid.*

Proof. Let $m = |V(H)|$, and $\{(H_i,\psi_i)\}_{1\leq i\leq m}$ denote the sequence of Γ-gain graphs in the construction, where $i = |V(H_i)|$, (H_1,ψ_1) is a Γ-base graph and $(H_m,\psi_m) = (H,\psi)$. The possible Γ-base graphs (B_j,ϕ_j) where $1 \leq j \leq 4$ are shown in Figure 12.2.

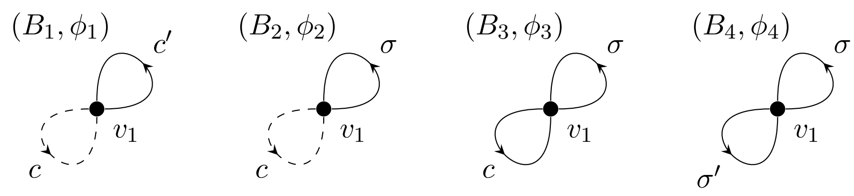


Figure 12.2: The four Γ-base graphs when Γ is a rotational or dihedral group. Here c and c' denote non-trivial rotations which need not be distinct, and σ and σ' denote distinct reflections.

Claim 12.2.3. *If $p' : \{v_i\} \to \mathbb{R}^2$ is generic over $\mathbb{Q}_\Gamma$, then $O(B_j,\phi_j,p')$ is row independent for $1 \leq j \leq 4$.*

Proof. Using properties of the matrix representatives for reflections and rotations, and row operations on the orbit matrices, we can rewrite each orbit matrix $O(B_j, \phi_j, p')$ as the matrix O_j, where

$$O_1 = \begin{pmatrix} p'(v_1)^T \\ p'(v_1)^{\perp T} \end{pmatrix}, \ O_2 = \begin{pmatrix} ((I-\sigma_\mu)p'(v_1))^T \\ p'(v_1)^{\perp T} \end{pmatrix}, \ O_3 = \begin{pmatrix} ((I-\sigma_\mu)p'(v_1))^T \\ p'(v_1)^T \end{pmatrix}$$

$$\text{and } O_4 = \begin{pmatrix} ((I-\sigma_\mu)p'(v_1))^T \\ ((I-\sigma'_\mu)p'(v_1))^T \end{pmatrix}.$$

The rows of O_1 are two perpendicular non-zero vectors, so O_1 is clearly row independent. Suppose O_2 is not row independent. Then the line through $p'(v_1)$ and $\sigma_\mu p'(v_1)$ is perpendicular to the line through $p'(v_1)$ and the origin. Since Γ is a point group and $p'(v_1) \neq \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$, this implies $\sigma = I$, a contradiction. Similarly, if the rows of O_3 are linearly dependent then $p'(v_1)$, $\sigma_\mu p'(v_1)$ and the origin are collinear, implying $\sigma_\mu \in \{I, c_2\}$, a contradiction. Finally suppose the rows of O_4 are linearly dependent. Then $p'(v_1)$, $\sigma_\mu p'(v_1)$ and $\sigma'_\mu p'(v_1)$ are collinear, which implies $\sigma = \sigma'$, again a contradiction. □

In our inductive construction, our first framework $(H_1, \psi_1, \tilde{p}|_{V(H_1)})$ has $(H_1, \psi_1) \in \{(B_j, \phi_j)\}_{1 \leq j \leq 4}$. Since $\tilde{p}$ is generic over $\mathbb{Q}_\Gamma$, Claim 12.2.3 implies $O(H_1, \psi_1, \tilde{p}|_{V(H_1)})$ is row independent. Theorems 12.1.4 and 12.1.6, then imply $O(H_i, \psi_i, \tilde{p}|_{V(H_i)})$ is row independent for all $1 \leq i \leq m$. Thus by Lemma 11.2.2, $\text{rank}(O(H, \psi, \tilde{p})) = |E(H)| = 2|V(H)|$. Hence (G, p) is minimally symmetry-forced infinitesimally rigid by Corollary 10.2.5. □

12.3 Further work

For rotational symmetry, the only remaining step required to extend Theorem 12.2.2 to a full characterisation is to show that the inductive moves which add degree 4 vertices: loop 1-extensions, loop-to-loop extensions and 2-extensions; preserve row independence in the orbit matrix, when the gains on any added edges are from $\mathcal{C}_k$. This would require building on our methods from Section 12.1.

Dihedral symmetry requires more work. As well as extending the orbit matrix arguments from Section 12.1, we also need to prove an equivalent result to Theorem 11.3.28 for dihedral groups (that for all k, we can characterise the class of $\mathcal{D}_k$-tight graphs by a inductive construction). For the orbit matrix arguments, examples like the Bottema mechanism (see Figures 9.3 and 10.5) suggest that 2-extensions do not always preserve row independence when the gains are from an even dihedral group, even when the realisation is as generic as possible. So we could only hope to extend the arguments from Section 12.1 to extensions which add degree four vertices when the symmetry group is odd dihedral.

In Chapter 11, we have already completed many of the steps required to characterise the class of $\mathcal{D}_k$-tight graphs in terms of an inductive construction. However, it remains to extend the results of Section 11.3 to the dihedral case. In other words, to show that when a $\mathcal{D}_k$-tight gain graph is 4-regular and loop-free, then it has a vertex which can be removed by an admissible 2-reduction. In Section 11.3 we tackled this problem for $\mathcal{C}_k$-tight graphs by introducing the idea of blocks, and showing that a 2-reduction is admissible if the resulting graph is mixed and has no blocks (Lemma 11.3.8). For $\mathcal{C}_k$-tight graphs these blocks were always balanced, however for $\mathcal{D}_k$-tight graphs we need to consider unbalanced blocks too. A inadmissible 2-reduction in a $\mathcal{D}_k$-tight graph could create a subset of edges which is direction-pure, reflectional, and not sparse. So to extend the arguments of Section 11.3 to dihedral groups, we would need to adapt our proofs to account for direction-pure, reflectional blocks. This would make our arguments significantly more complex.

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