SMALL REPRESENTATIONS, STRING INSTANTONS, AND FOURIER MODES OF EISENSTEIN SERIES

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WITH APPENDIX "SPECIAL UNIPOTENT REPRESENTATIONS" BY DAN CIUBOTARU AND PETER E. TRAPA

Abstract. This paper concerns some novel features of maximal parabolic Eisenstein series at certain special values of their analytic parameter, s. These series arise as coefficients in the $\mathcal{R}^4$ and $\partial^4\,\mathcal{R}^4$ interactions in the low energy expansion of the scattering amplitudes in maximally supersymmetric string theory reduced to $D = 10 - d$ dimensions on a torus, T^d $(0 \leq d \leq 7)$. For each d these amplitudes are automorphic functions on the rank $d+1$ symmetry group E_{d+1}.

Of particular significance is the orbit content of the Fourier modes of these series when expanded in three different parabolic subgroups, corresponding to certain limits of string theory. This is of interest in the classification of a variety of instantons that correspond to minimal or "next-to-minimal" BPS orbits. In the limit of decompactification from D to $D+1$ dimensions many such instantons are related to charged $\frac{1}{2}$-BPS or $\frac{1}{4}$-BPS black holes with euclidean world-lines wrapped around the large dimension. In a different limit the instantons give nonperturbative corrections to string perturbation theory, while in a third limit they describe nonperturbative contributions in eleven-dimensional supergravity.

A proof is given that these three distinct Fourier expansions have certain vanishing coefficients that are expected from string theory. In particular, the Eisenstein series for these special values of s have markedly fewer Fourier coefficients than typical maximal parabolic Eisenstein series. The corresponding mathematics involves showing that the wavefront sets of the Eisenstein series in question are supported on only a limited number of coadjoint nilpotent orbits – just the minimal and trivial orbits in the $\frac{1}{2}$-BPS case, and just the next-to-minimal, minimal and trivial orbits in the $\frac{1}{4}$-BPS case. Thus as a byproduct we demonstrate that the next-to-minimal representations occur automorphically for E_6, E_7, and E_8, and hence the first two nontrivial low energy coefficients in scattering amplitudes can be thought of as exotic θ-functions for these groups. The proof includes an appendix by Dan Ciubotaru and Peter E. Trapa which calculates wavefront sets for these and other special unipotent representations.

keywords: automorphic forms, scattering amplitudes, string theory, small representations, Eisenstein series, Fourier expansions, unipotent representations, charge lattice, BPS states, coadjoint nilpotent orbits.

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1. Introduction

String theory is expected to be invariant under a very large set of discrete symmetries ("dualities"), associated with arithmetic subgroups of a variety of reductive Lie groups. For example, maximally supersymmetric string theory (type II superstring theory), compactified on a d-torus to $D = 10-d$ space-time dimensions, is strongly conjectured to be invariant under $E_{d+1}(\mathbb{Z})$, the integral points of the rank $d+1$ split real form[1] of one of the groups in the sequence E_8, E_7, E_6, $Spin(5,5)$, $SL(5)$, $SL(3)\times SL(2)$, $SL(2)\times\mathbb{R}^+$, $SL(2)$ listed in table 1.[2]

[1]The split real forms are conventionally denoted $E_{n(n)}$, but in this paper we will truncate this to E_n except when other forms of E_n are needed.

[2] Unfortunately the literature contains some disagreement over precisely which groups $E_{d+1}(\mathbb{R})$ occur here, an ambiguity amongst the split real groups having the same Lie algebra. For example some authors have $SO(5,5,\mathbb{R})$ instead of its double cover $Spin(5,5,\mathbb{R})$; in general possible groups are related by taking quotients by a subgroup G_0 of the center of the larger group. The choices listed here, which represent the current consensus, are each the real points of an (algebraically) simply connected Chevalley group. (The real

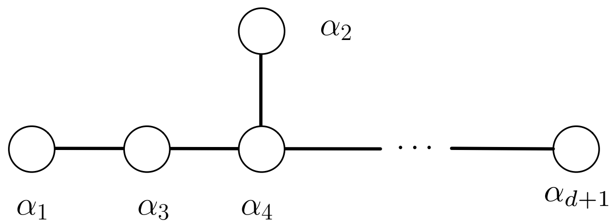


FIGURE 1. The Dynkin diagram for the rank $d+1$ Lie group E_{d+1}, which defines the symmetry group for $D = 10 - d$.

These symmetries severely constrain the dependence of string scattering amplitudes on the symmetric space coordinates (or "moduli"), ϕ_{d+1}, which parameterise the coset E_{d+1}/K_{d+1}, where the stabiliser K_{d+1} is the maximal compact subgroup of E_{d+1}. The list of these symmetry[3] groups and stabilisers is given in table 1. These moduli are scalar fields that are interpreted as coupling constants in string theory. A general consequence of the dualities is that scattering amplitudes are functions of ϕ_{d+1} that must transform as automorphic functions under the appropriate duality group $E_{d+1}(\mathbb{Z})$. It is difficult to determine the precise restrictions these dualities impose on general amplitudes, but certain exact properties have been obtained in the case of the four-graviton interactions, where a considerable amount of information has been obtained for the first three terms in the low energy (or "derivative") expansion of the four graviton scattering amplitude in [1] (and references cited therein). These are described by terms in the effective action of the form

$$\mathcal{E}^{(D)}_{(0,0)}(\phi_{d+1})\,\mathcal{R}^4\,, \quad \mathcal{E}^{(D)}_{(1,0)}(\phi_{d+1})\,\partial^4\,\mathcal{R}^4\,, \quad \text{and} \quad \mathcal{E}^{(D)}_{(0,1)}(\phi_{d+1})\,\partial^6\,\mathcal{R}^4\,, \tag{1.1}$$

where the symbol $\mathcal{R}^4$ indicates a contraction of four powers of the Riemann tensor with a standard rank 16 tensor. The coefficient functions, $\mathcal{E}^{(D)}_{(p,q)}(\phi_{d+1})$, are automorphic functions that are the main focus of our interests (the notation is taken from [1, 2] and will be reviewed later in (2.3)). More precisely we will focus on the three terms shown in (1.1) that are protected by supersymmetry, which accounts for the relatively simple form of their coefficients.

groups $E_{d+1}(\mathbb{R})$ and K_{d+1} are not topologically simply connected, except in the trivial $D = 10A$ case.)

Although we will try to be precise in our definitions, this discrepancy does not affect the results in this paper. We note, in particular, that $E_{d+1}(\mathbb{Z})$ is mathematically defined as the stabilizer of the Chevalley lattice in the Lie algebra $\mathfrak{e}_{d+1}$ under the adjoint action. Since the center acts trivially under the adjoint action, the integral points of the larger group factors as the direct product of G_0 with the integral points of the smaller group. In particular the Eisenstein series for the two groups are the same (see for example (2.13)).

[3]The continuous groups, $E_{d+1}(\mathbb{R})$, will be referred to as *symmetry groups* while the discrete arithmetic subgroups, $E_{d+1}(\mathbb{Z})$, will be referred to as *duality groups*.

D	$E_{d+1}(\mathbb{R})$	K_{d+1}	$E_{d+1}(\mathbb{Z})$
10A	$\mathbb{R}^+$	1	1
10B	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R})\times\mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z})$
8	$SL(3,\mathbb{R})\times SL(2,\mathbb{R})$	$SO(3)\times SO(2)$	$SL(3,\mathbb{Z})\times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5,\mathbb{R})$	$(Spin(5)\times Spin(5))/\mathbb{Z}_2$	$Spin(5,5,\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

TABLE 1. The symmetry groups of maximal supergravity in $D = 10-d \leq 10$ dimensions. The group $E_{d+1}(\mathbb{R})$ is a split real form of rank $d+1$, and K_{d+1} is its maximal compact subgroup. In string theory these groups are broken to the discrete subgroups, $E_{d+1}(\mathbb{Z})$, as indicated in the last column (see [3] and its updated version in [4]). The split real form $E_{d+1}(\mathbb{R})$ is determined among possible covers or quotients by its maximal compact subgroup K_{d+1}, which shares the same fundamental group. The terminology $10A$ and $10B$ in the first column refers to the two possible superstring theories (types IIA and IIB) in $D = 10$ dimensions.

The coefficients of the first two terms satisfy Laplace eigenvalue equations (2.6-2.7) and are subject to specific boundary conditions that are required for consistency with string perturbation theory and M-theory. The solutions to these equations are particular maximal parabolic Eisenstein series that were studied in [2] (for cases with rank ≤ 5) and [1] (for the E_6, E_7 and E_8 cases), and will be reviewed in the next section. The required boundary conditions in each limit amount to conditions on the constant terms in the expansion of these series in three limits associated with particular maximal[4] parabolic subgroups of relevance to the string theory analysis. Such subgroups have the form $P_\alpha = L_\alpha U_\alpha$, where α labels a simple root, U_α is the unipotent radical and $L_\alpha = GL(1)\times M_\alpha$ is the Levi factor.[5] The three subgroups of relevance here have Levi factors $L_{\alpha_1} = GL(1)\times Spin(d,d)$, $L_{\alpha_2} = GL(1)\times SL(d+1)$, and $L_{\alpha_{d+1}} = GL(1)\times E_d$, respectively. In each case the $GL(1)$ parameter, r, can be thought of as measuring the distance to the cusp[6], as will be discussed in the next section. A key feature of the boundary conditions is that they require these constant terms to have very

[4]The $D = 8$ case is degenerate and also involves non-maximal parabolics (see table 1).

[5]For clarity, we emphasize that its usage here indicates that every element of L_α can be written as an element of $GL(1)$ times an element of M_α (and not that L_α is the direct product of the two factors, which is a stronger statement).

[6]Each of the groups we are considering has a single cusp. The various limits correspond to different ways of approaching this cusp.

few components with distinct powers of the parameter r. These conditions pick out the unique solutions to the Laplace equations, which are,[7]

$$\mathcal{E}^{(10-d)}_{(0,0)} \quad = \quad 2\,\zeta(3)\,E^{E_{d+1}}_{\alpha_1;\,\frac{3}{2}}\,, \tag{1.2}$$

for the groups E_1, E_4, E_5, E_6, E_7, and E_8 [1,2] and

$$\mathcal{E}^{(10-d)}_{(1,0)} \quad = \quad \zeta(5)\,E^{E_{d+1}}_{\alpha_1;\,\frac{5}{2}}\,, \tag{1.3}$$

for the groups E_1, E_6, E_7, and E_8 [1]. Here $E^{G}_{\beta;s}$ is the maximal parabolic Eisenstein series for a parabolic subgroup $P_\beta \subset G$ that is specified by the node β of the Dynkin diagram (see (2.12) for a precise definition). This generalizes results for the $SL(2,\mathbb{Z})$ case (relevant to the ten-dimensional type IIB string theory). The functions $\mathcal{E}^{(10-d)}_{(0,0)}$ and $\mathcal{E}^{(10-d)}_{(1,0)}$ in the intermediate rank cases involve linear combinations of Eisenstein series [2], which will be discussed later in section 4. The third coefficient function, $\mathcal{E}^{(10-d)}_{(0,1)}$ satisfies an interesting inhomogeneous Laplace equation and is not an Eisenstein series [1,6]. Its constant terms in the three limits under consideration were also analysed in the earlier references but it will not be considered in this paper, which is entirely concerned with Eisenstein series.

In other words, our previous work showed that the particular Eisenstein series in (1.2) and (1.3) have strikingly sparse constant terms as required to correctly describe the coefficients of the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS interactions. But the string theory boundary conditions also determine the support of the non-zero Fourier coefficients in each of the three limits under consideration. In string theory, the non-zero Fourier modes describe instanton contributions to the amplitude. These are classified in BPS orbits obtained by acting on a representative instanton configuration with the appropriate Levi subgroup. A given instanton configuration generally depends on only a subset of the parameters of the Levi group, $L_\alpha = GL(1) \times M_\alpha$, so that a given orbit depends on the subset of the moduli that live in a coset space of the form $M_\alpha/H^{(i)}$, where $H^{(i)} \subset M_\alpha$ denotes the stabiliser of the i-th orbit. The dimension of the i-th orbit is the dimension of this coset space.

In particular, the coefficients in the $s = 3/2$ cases covered by (1.2) must be localized within the smallest possible non-trivial orbits ("minimal orbits") of the Levi actions, as required by the $\frac{1}{2}$-BPS condition. Furthermore, in the $s = 5/2$ cases covered by (1.3) the coefficients are shown to be localized within the "next-to-minimal" (NTM) orbits (see section 2.2). The role of next-to-minmal orbits was also considered in [7]. However, the specific suggestion there was based on the next-to-minimal representations of Gross and Wallach [8,9], who did not consider the split groups of relevance to the

[7] In [1,2,5] the series were indexed by the label $[1\,0\cdots0]$ of the root α_1. In the present paper, we will index the series according the labeling of the simple root in figure 1. We have as well changed the normalisations of the Eisenstein series, since our series there was instead $\mathbf{E}^{E_{d+1}}_{[10\cdots0];s} = 2\zeta(2s)E^{E_{d+1}}_{\alpha_1;s}$.

duality symmetries of type IIB string theory, which have very distinctive properties (as we shall see).

This provides motivation from string theory for the following

String motivated vanishing of Fourier modes of Eisenstein series:

(i) *The non-zero Fourier coefficients of* $E^{E_{d+1}}_{\alpha_1;\,\frac{3}{2}}$ *(*$d = 5, 6, 7$*) in any of the three parabolic subgroups of relevance are localized within the smallest possible non-trivial orbits ("minimal orbits") of the action of the Levi subgroup associated with that parabolic, as required by the* $\frac{1}{2}$*-BPS condition.*

(ii) *The non-zero Fourier coefficients of* $E^{E_{d+1}}_{\alpha_1;\,\frac{5}{2}}$ *(*$d = 5, 6, 7$*) are localized within "next-to-minimal" (NTM) orbits, as required by the* $\frac{1}{4}$*-BPS condition.*

While the special properties of the Fourier coefficients of the $s = 3/2$ series is implied by the results in [10], the corresponding properties for the NTM orbits at $s = 5/2$ is novel. One of the main mathematical contributions of this paper is to give a rigorous proof of these statements using techniques from representation theory, by connecting these automorphic forms to small representations of the split real groups E_{d+1}. The Fourier coefficients in the intermediate rank cases not covered by (1.2) and (1.3) satisfy analogous properties as we will determine by explicit calculation later in this paper.

2. Overview of scattering amplitudes and Eisenstein series

Since this paper covers topics of interest in both string theory and mathematics, this section will present a brief description of the background to these topics from both points of view followed by a detailed outline of the rest of the paper.

2.1. String theory Background.

We are concerned with exact (i.e., non-perturbative) properties of the low energy expansion of the four-graviton scattering amplitude in dimension $D = 10 - d$, which is a function of the moduli, ϕ_{d+1}, as well as of the particle momenta k_r ($r = 1, \ldots, 4$) that are null Lorentz D-vectors ($k_r^2 = k_r \cdot k_r = 0$) which are conserved ($\sum_{r=1}^4 k_r = 0$). They arise in the invariant combinations (Mandelstam invariants), $s = -(k_1 + k_2)^2$, $t = -(k_1 + k_4)^2$ and $u = -(k_1 + k_3)^2$ that satisfy $s + t + u = 0$. At low orders in the low-energy expansion the amplitude can usefully be separated into analytic and nonanalytic parts

$$A_D(s,t,u) \;=\; A_D^{analytic}(s,t,u) \;+\; A_D^{nonanalytic}(s,t,u) \qquad (2.1)$$

(where the dependence on ϕ_{d+1} has been suppressed). The analytic part of the amplitude has the form

$$A_D^{analytic}(s,t,u) \;=\; T_D(s,t,u)\, \ell_D^6\, \mathcal{R}^4\,, \qquad (2.2)$$

where ℓ_D denotes the D-dimensional Planck length scale and the factor $\mathcal{R}^4$ represents the particular contraction of four Riemann curvature tensors, $\mathrm{tr}(\mathcal{R}^4) - (\mathrm{tr}\,\mathcal{R}^2)^2/4$, that is fixed by maximal supersymmetry in a standard fashion [11]. The scalar function T_D has the expansion (in the Einstein frame[8])

$$\begin{aligned} T_D(s,t,u) &= \mathcal{E}_{(0,-1)}\,\sigma_3^{-1} + \sum_{p,q\geq 0} \mathcal{E}^{(D)}_{(p,q)}\,\sigma_2^p\,\sigma_3^q \\ &= 3\,\sigma_3^{-1} + \mathcal{E}^{(D)}_{(0,0)} + \mathcal{E}^{(D)}_{(1,0)}\,\sigma_2 + \mathcal{E}^{(D)}_{(0,1)}\,\sigma_3 + \cdots . \end{aligned} \tag{2.3}$$

Symmetry under interchange of the four gravitons implies that the Mandelstam invariants only appear in the combinations σ_2 and σ_3 with $\sigma_n = (s^n + t^n + u^n)\,(\ell_D^2/4)^n$. Since s,t,u are quadratic in momenta the successive terms in the expansion are of order $n = 2p + 3q$ in powers of $(\text{momenta})^2$. The degeneracy, $d_n = \lfloor (n+2)/2 \rfloor - \lfloor (n+2)/3 \rfloor$, of terms with power n is given by the generating function[9],

$$\frac{1}{(1-x^2)(1-x^3)} = \sum_{n=0}^{\infty} d_n\,x^n\,, \tag{2.4}$$

so $d_0 = 1$, $d_1 = 0$ and $d_n = 1$ for $2 \leq n \leq 5$.

The coefficient functions in (2.3), $\mathcal{E}^{(D)}_{(p,q)}(\phi_{d+1})$, are automorphic functions of the moduli ϕ_{d+1} appropriate to compactification on T^d. The first term on the right-hand side of (2.3) is identified with the tree-level contribution of classical supergravity and has a constant coefficient given by $\mathcal{E}^{(D)}_{(0,-1)}(\phi_{d+1}) = 3$. The terms of higher order in s, t, u represent stringy modifications of supergravity, which depend on the moduli in a manner consistent with duality invariance. This expansion is presented in the Einstein frame so the curvature, $\mathcal{R}$, is invariant under $E_{d+1}(\mathbb{Z})$ transformations, whereas it transforms nontrivially in the string frame since it is nonconstant in $\phi_{d+1} \in E_{d+1}(\mathbb{R})/K_{d+1}$.

Apart from the first term, the power series expansion in (2.3) translates into a sum of local interactions in the effective action. The first two of these have the form

$$\ell_D^{8-D} \int d^Dx\,\sqrt{-G^{(D)}}\,\mathcal{E}^{(D)}_{(0,0)}\,\mathcal{R}^4\,, \quad \ell_D^{12-D} \int d^Dx\,\sqrt{-G^{(D)}}\,\mathcal{E}^{(D)}_{(1,0)}\partial^4\mathcal{R}^4\,. \tag{2.5}$$

The three interactions with coefficient functions $\mathcal{E}^{(D)}_{(0,0)}$, $\mathcal{E}^{(D)}_{(1,0)}$ and $\mathcal{E}^{(D)}_{(0,1)}$ displayed in the second equality in (2.3) are specially simple since they are protected by supersymmetry from renormalisation beyond a given order in perturbation theory. In particular, the $\mathcal{R}^4$ interaction breaks 16 of the 32

[8]The Einstein frame is the frame in which lengths are measured in Planck units rather than string units, and is useful for discussing dualities.

[9]This is the same as the well-known dimension formula for the space of weight $2n$ holomorphic modular forms for $SL(2,\mathbb{Z})$, which are expressed as polynomials in the (holomorphic) Eisenstein series G_4 and G_6.

supersymmetries of the type II theories and is thus $\frac{1}{2}$-BPS, while the $\partial^4\mathcal{R}^4$ interaction breaks 24 supersymmetries and is $\frac{1}{4}$-BPS; likewise, the $\partial^6 R^4$ interaction breaks 28 supersymmetries and is $\frac{1}{8}$-BPS. The next interaction is the $p=2, q=0$ term in (2.3), $\mathcal{E}^{(D)}_{(2,0)}\,\partial^8\mathcal{R}^4$. Naively this interaction breaks all supersymmetries, in which case it is expected to be much more complicated, but it would be of interest to discover if supersymmetry does constrain this interaction.[10]

It was argued in [2], based on consistency under various dualities, that the coefficients $\mathcal{E}^{(D)}_{(0,0)}$, $\mathcal{E}^{(D)}_{(1,0)}$ and $\mathcal{E}^{(D)}_{(0,1)}$ satisfy the equations

$$\left(\Delta^{(D)} - \frac{3(11-D)(D-8)}{D-2}\right)\mathcal{E}^{(D)}_{(0,0)} \;=\; 6\,\pi\,\delta_{D,8}\,, \tag{2.6}$$

$$\left(\Delta^{(D)} - \frac{5(12-D)(D-7)}{D-2}\right)\mathcal{E}^{(D)}_{(1,0)} \;=\; 40\,\zeta(2)\,\delta_{D,7}\,, \tag{2.7}$$

$$\left(\Delta^{(D)} - \frac{6(14-D)(D-6)}{D-2}\right)\mathcal{E}^{(D)}_{(0,1)} \;=\; -\left(\mathcal{E}^{(D)}_{(0,0)}\right)^2 + 120\zeta(3)\delta_{D,6}\,, \tag{2.8}$$

where $\Delta^{(D)}$ is the Laplace operator on the symmetric space E_{11-D}/K_{11-D}. The discrete Kronecker δ contributions on the right-hand-side of these equations arise from anomalous behaviour and can be related to the logarithmic ultraviolet divergences of loop amplitudes in maximally supersymmetric supergravity [5].

Recall that automorphic forms for $SL(2,\mathbb{Z})$ have Fourier expansions (i.e., q-expansions) in their cusp. For higher rank groups, automorphic forms have Fourier expansions coming from any one of several maximal parabolic subgroups P_{α_r}, where the simple root α_r corresponds to node r in the Dynkin diagram for E_{d+1} in figure 1. We are particularly interested in this Fourier expansion for $r = 1$, 2, or $d+1$, because each of these expansions has a distinct string theory interpretation in terms of the contributions of instantons in the limit in which a special combination of moduli degenerate. These three limits are:

(i) *The decompactification limit* in which one circular dimension, r_d, becomes large. In this case the amplitude reduces to the $D+1$-dimensional case with $D = 10-d$. The BPS instantons of the $D = (10-d)$-dimensional theory are classified by orbits of the Levi subgroup $GL(1)\times E_d$. Apart from one exception, these instantons can be described in terms of the wrapping of the world-lines of black hole states in the decompactified $D+1$-dimensional theory around the large circular dimension (the exception will be described later). This limit is associated with the parabolic subgroup $P_{\alpha_{d+1}}$.

[10]A discussion of the properties of $\mathcal{E}^{(9)}_{(2,0)}$ in nine dimensions can be found in [12, section 4.1.1].

(ii) *The string perturbation theory limit* of small string coupling constant, in which the string coupling constant, $\sqrt{y_D}$, is small, and string perturbation theory amplitudes are reproduced. The instantons are exponentially suppressed contributions that are classified by orbits of the Levi subgroup $GL(1) \times Spin(d,d)$. This limit is associated with the parabolic subgroup P_{α_1}.

(iii) *The M-theory limit* in which the M-theory torus has large volume $\mathcal{V}_{d+1}$, and the semi-classical approximation to eleven-dimensional supergravity is valid. This involves the compactification of M-theory from 11 dimensions on the $(d+1)$-dimensional M-theory torus, where the instantons are classified by orbits of the Levi subgroup $GL(1) \times SL(d+1)$. This limit is associated with the parabolic subgroup P_{α_2}.

The special features of the constant terms that lead to consistency of all perturbative properties in these three limits appear to be highly nontrivial, and indicate particularly special mathematical properties of the Eisenstein series that define the coefficients of the $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$ interactions. The solutions to equations (2.6-2.8) satisfying requisite boundary conditions on the constant terms (zero modes) in the Fourier expansions in the limits (i), (ii), and (iii) were obtained for $7 \leq D \leq 10$ in [2], and for $3 \leq D \leq 6$ in [1]. In particular, (1.2) and (1.3) were found to be solutions for the cases with duality groups E_6, E_7 and E_8. Whereas the coefficient functions $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ are given in terms of Eisenstein series that satisfy Laplace eigenvalue equations on the moduli space, the coefficient $\mathcal{E}^{(D)}_{(0,1)}$, of the $\frac{1}{8}$-BPS interaction $\partial^6\mathcal{R}^4$, is an automorphic function that satisfies an inhomogeneous Laplace equation. Various properties of its constant terms in these three limits were also determined in [1, 2].

Whereas the earlier work concerned the zero Fourier modes of the coefficient functions, in this paper we are concerned with the non-zero modes in the Fourier expansion in any of the three limits listed above. These Fourier coefficients should have the exponentially suppressed form that is characteristic of instanton contributions. In more precise terms, the angular variables involved in the Fourier expansion with respect to a maximal parabolic subgroup P_α come from the abelianization[11] $U_\alpha/[U_\alpha, U_\alpha]$ of the unipotent radical U_α of P_α, and are conjugate to integers that define the instanton "charge lattice". Asymptotically close to a cusp a given Fourier coefficient is expected to have an exponential factor of $\exp(-S^{(p)})$, where $S^{(p)}$ is the action for an instanton of a given charge, as will be defined in section 3.1. In the case of fractional BPS instantons the leading asymptotic behaviour in the cusp is the real part of $S^{(p)}$, and is related to the charge (B.4), which enters the phase of the mode.

[11] See (4.3).

In each limit the $\frac{1}{2}$-BPS orbits are minimal orbits (i.e., smallest nontrivial orbits) while the $\frac{1}{4}$-BPS orbits are "next-to-minimal" (NTM) orbits (i.e., smallest nonminimal or nontrivial orbits). The next largest are $\frac{1}{8}$-BPS orbits, which only arise for groups of sufficiently high rank; in the E_8 case there is a further $\frac{1}{8}$-BPS orbit beyond that. These come up again as "character variety orbits", a major consideration in sections 5 and 6. They are closely related to – but not to be confused with – the minimal and next-to-minimal coadjoint nilpotent orbits that are attached to the Eisenstein series that arise in the solutions for the coefficients, $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ in (1.2) and (1.3), respectively.

Note on conventions. Following [1, Section 2.4], the parameter associated with the $GL(1)$ factor that parameterises the approach to any cusp will be called r and is normalised in a mathematically convenient manner. It translates into distinct physical parameters in each of the three limits described above, that correspond to parabolic subgroups defined at nodes $d+1$, 1 and 2, respectively, of the Dynkin diagram in figure 1. These are summarised as follows:

$$\begin{aligned}
&\text{Limit (i)} \quad && r^2 \;=\; r_d/\ell_{11-d}\,, \quad r_d \;=\; \text{radius of decompactifying circle}\,,\\
&\text{Limit (ii)} \quad && r^{-2} \;=\; \sqrt{y_D} \;=\; \text{string coupling constant}\,,\\
&\text{Limit (iii)} \quad && r^{\frac{2(1+d)}{3}} \;=\; \mathcal{V}_{d+1}/\ell_{11}^{d+1}\,, \quad \mathcal{V}_{d+1} = \text{vol. of M}-\text{theory torus}\,.
\end{aligned} \tag{2.9}$$

The D-dimensional string coupling constant is defined by $y_D = g_s^2\,\ell_s^d/V_d$, where $D = 10-d$ and g_s is either the $D=10$ IIA string coupling constant, g_A, or the IIB string coupling constant, g_B, and V_d is the volume of T^d in string units.[12] The Planck length scales in different dimensions are related to each other and to the string scale, ℓ_s, by

$$\begin{aligned}
(\ell^A_{10})^8 \;&=\; \ell_s^8\,g_A^2\,, \quad (\ell^B_{10})^8 \;=\; \ell_s^8\,g_B^2\,, \quad \ell_{11} \;=\; g_A^{\frac{1}{3}}\,\ell_s\,,\\
(\ell_D)^{D-2} \;&=\; \ell_s^{D-2}\,y_D \;=\; (\ell_{D+1})^{D-1}\,\frac{1}{r_d}\,, \quad \text{for } D\le 8\ (d\ge 2)\\
\ell_9^7 \;&=\; \ell_s^7\,y_9 = (\ell^A_{10})^8\,\frac{1}{r_A} \;=\; (\ell^B_{10})^8\,\frac{1}{r_B}\,.
\end{aligned} \tag{2.10}$$

(note the two distinct Planck lengths in the ten-dimensional case and the distinction between $r_1 = r_A$ and $r_1 = r_B$ in the two type II theories).

2.2. **Mathematics background.** Let us begin by recalling some notions from the theory of automorphic forms that are relevant to the expansion (2.3), specifically from [1, Section 2]. Let G denote the split real Lie group E_n, $n\le 8$, defined in table 1. For convenience we fix (as we may) a Chevalley basis of the Lie algebra $\mathfrak{g}$ of G, and a choice of positive roots Φ_+ for its root

[12]We will use the symbol T^d to denote the string theory d-torus while using the symbol $\mathcal{T}^{d+1}$ for the corresponding M-theory $(d+1)$-torus expressed in eleven-dimensional units.

system Φ. Letting $\Sigma \subset \Phi_+$ denote the positive simple roots, the Lie algebra $\mathfrak{g}$ has the triangular decomposition

$$\mathfrak{g} \;=\; \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{n}_- \,, \tag{2.11}$$

where $\mathfrak{n}$ (respectively, $\mathfrak{n}_-$) is spanned by the Chevalley basis root vectors X_α for $\alpha \in \Phi_+$ (respectively, $\alpha \in \Phi_-$), and $\mathfrak{a}$ is spanned by their commutators $H_\alpha = [X_\alpha, X_{-\alpha}]$. Let $N \subset G$ be the exponential of $\mathfrak{n}$; it is a maximal unipotent subgroup. Likewise $A = \exp(\mathfrak{a})$ is a maximal torus, and is isomorphic to rank(G) copies of $\mathbb{R}^+$. The group G has an Iwasawa decomposition $G = NAK$, where $K = K_n$ is the maximal compact subgroup of G listed in table 1. There thus exists a logarithm map $H : A \to \mathfrak{a}$ which is inverse to the exponential, and which extends to all $g \in G$ via its value on the A-factor of the Iwasawa decomposition of g. The integral points $G(\mathbb{Z})$ are defined as all elements $\gamma \in G$ such that the adjoint action $Ad(\gamma)$ on $\mathfrak{g}$ preserves the integral span of the Chevalley basis.

The standard maximal parabolic subgroups of G are in bijective correspondence with the positive simple roots of G. Given such a root β and a standard maximal parabolic P_β, the *maximal parabolic Eisenstein series* induced from the constant function on P_β is defined by the sum

$$E^G_{\beta;s} \;:=\; \sum_{\gamma\,\in\,(P_\beta\cap G(\mathbb{Z}))\backslash G(\mathbb{Z})} e^{2\,s\,\omega_\beta(H(\gamma g))}\,, \quad \operatorname{Re} s \,\gg\, 0\,, \tag{2.12}$$

where ω_β, the fundamental weight associated to β, is defined by the condition $\langle \omega_\beta, \alpha\rangle = \delta_{\alpha,\beta}$. These series generalize the classical nonholomorphic Eisenstein series (the case of $G = SL(2)$), and more generally the Epstein Zeta functions (the case of $G = SL(n)$ and β either the first or last node of the A_{n-1} Dynkin diagram). Because of this special case, we often refer to the $\beta = \alpha_1$ series (in the numbering of figure 1) as the *Epstein* series for a particular group, even if it is not $SL(n)$. These series are the main mathematical objects of this paper.

As we remarked in footnote 2 changing G to another Chevalley group with Lie algebra $\mathfrak{g}$ changes $G(\mathbb{Z})$ by a central subgroup, and so Eisenstein series for the cover descend to the corresponding Eisenstein series on the quotient. For example,

$$E^{Spin(d,d)}_{\beta;s}(g) \;=\; E^{SO(d,d)}_{\beta;s}(\pi(g))\,, \tag{2.13}$$

where $\pi : Spin(d,d,\mathbb{R}) \to SO(d,d,\mathbb{R})$ is the covering map. We shall sometimes refer to either as $E^{D_d}_{\beta;s}$ when we wish to emphasize that a particular statement applies to both $E^{Spin(d,d)}_{\beta;s}$ and $E^{SO(d,d)}_{\beta;s}$.

As shorthand, we often denote a root by its "root label", that is, stringing together its coefficients when written as a linear combination of the positive simple roots Σ. Thus $\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$ could be denoted $0112100\cdots$ or $[0112100\cdots]$, with brackets sometimes added for clarity. Note that Eisenstein series of the type (2.12) are parameterized by a single complex variable,

s, whereas the more general minimal parabolic series in (5.3) has rank(G) complex parameters.

The series (2.12) is initially absolutely convergent for Re s large, and has a meromorphic continuation to the entire complex plane as part of a more general analytic continuation of Eisenstein series due to Langlands. Its special value at $s = 0$ is the constant function identically equal to one. This corresponds to the trivial representation of $G(\mathbb{R})$, and clearly has no nontrivial Fourier coefficients. The main mathematical content of this paper extends this phenomenon to other special values of s which are connected to small representations of real groups (see sections 2.2.2 and 5), and which have very few nontrivial Fourier coefficients. This will be demonstrated to be in complete agreement with a number of string theoretic predictions, in particular the one stated at the end of section 1.

The main results of [1] were the identifications (1.2) and (1.3) of $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$, respectively, in terms of special values of the Epstein series, for $3 \leq D = 10-d \leq 5$. The more general automorphic function $\mathcal{E}^{(D)}_{(0,1)}$ which satisfies (2.8) was also analysed in [1], but will not be relevant in this paper. The case of $Spin(5,5)$ was also covered in [1], but is somewhat more intricate; it will be explained separately. We will show in a precise sense that these Epstein series at the special values at $s = 0$, $3/2$, and $5/2$ correspond, respectively, to the three smallest types of representations of G (see theorem 2.14) below.

2.2.1. *Coadjoint nilpotent orbits.* Let $\mathfrak{g}$ be the Lie algebra of a matrix Lie group G, whether over $\mathbb{R}$ or $\mathbb{C}$. An element of $\mathfrak{g}$ is *nilpotent* if it is nilpotent as a matrix, i.e., some power of it is zero. The group G acts on its Lie algebra $\mathfrak{g}$ by the adjoint action $Ad(g)X = gXg^{-1}$, and hence dually on linear functionals $\lambda : \mathfrak{g} \to \mathbb{C}$ through the coadjoint action given by $(Coad(g)\lambda)(X) = \lambda(Ad(g^{-1})X) = \lambda(g^{-1}Xg)$. Actually $\mathfrak{g}$ is isomorphic to its space of linear functionals via the Killing form, and so the coadjoint action is equivalent to the adjoint action. Following common usage, we thus refer to the orbits of the adjoint action of G on $\mathfrak{g}$ as *coadjoint nilpotent orbits* (even though they are, technically speaking, adjoint orbits).

The book [13] is a standard reference for the general theory of coadjoint nilpotent orbits. When G is a real or complex semisimple Lie group there are a finite number of orbits, each of which is even dimensional. The smallest of these is the trivial orbit, $\{0\}$. On the other hand, there is always an open, dense orbit, usually referred to as the *principal* or *regular orbit*. Another orbit which will be important for us is the *minimal* orbit, the smallest orbit aside from the trivial orbit. Since our groups G are all simply laced, it can be described as the orbit of any root vector X_α, for any root α.

Table 2 gives a list of some orbits that are important to us, along with their basepoints.

Group	Orbit Dimension	Basepoint
$SL(2)$	0	0
	2	X_1
$SL(3)\times SL(2)$	0	0
	2	an $SL(2)$ root
	4	an $SL(3)$ root
$SL(5)$	0	0
	8	X_{1111}
	12	$X_{1110}+X_{0111}$
$Spin(5,5)$	0	0
	14	X_{12211}
	16	$X_{11110}+X_{11101}$
	20	$X_{01111}+X_{11211}$
E_6	0	0
	22	X_{122321}
	32	$X_{111221}+X_{112211}$
	40	$X_{011221}+X_{111210}+X_{112211}$
E_7	0	0
	34	$X_{2234321}$
	52	$X_{1123321}+X_{1223221}$
	54	$X_{0112210}+X_{1112221}+X_{1122110}$
E_8	0	0
	58	$X_{23465432}$
	92	$X_{23354321}+X_{22454321}$
	112	$X_{22343221}+X_{12343321}+X_{12244321}$
	114	$X_{11232221}+X_{12233211}$

TABLE 2. Basepoints of the smallest coadjoint nilpotent orbits for the complexified E_n groups. The notation X_α denotes the Chevalley basis root vector for the simple root α, which is written here in terms of the root labels described in the text. The basepoints are given as a description of the orbit but are not otherwised used. The $SL(3)\times SL(2)$ case comes from the E_3 Dynkin diagram, which is the E_8 Dynkin diagram from figure 1 after the removal of nodes 4, 5, 6, 7, and 8. Its Lie algebra is a product of two simple Lie algebras and has a different orbit structure than the others; its smallest orbits come from the respective factors.

2.2.2. *Automorphic representations.* The right translates of an automorphic function by the group G span a vector space on which G acts. For a suitable basis of square-integrable automorphic forms and most Eisenstein series, this action furnishes an irreducible representation. As we discussed in [1, Section 2], the Eisenstein series are specializations of the larger "minimal parabolic Eisenstein series" defined in (5.3). The automorphic representations connected to the latter are generically principal series representations, an identification which can be made by comparing the infinitesimal characters (that is, the action of all G-invariant differential operators). However, at special points the principal series reduces, and the Eisenstein series is part of a smaller representation.

An irreducible representation is related to coadjoint nilpotent orbits through its *wavefront set*, also known as the "associated variety" of its "annihilator ideal". It is a theorem of Joseph [14] and Borho-Brylinski [15] that this set is always the closure of a unique coadjoint nilpotent orbit. Thus a coadjoint nilpotent orbit is attached to every irreducible representation of G.

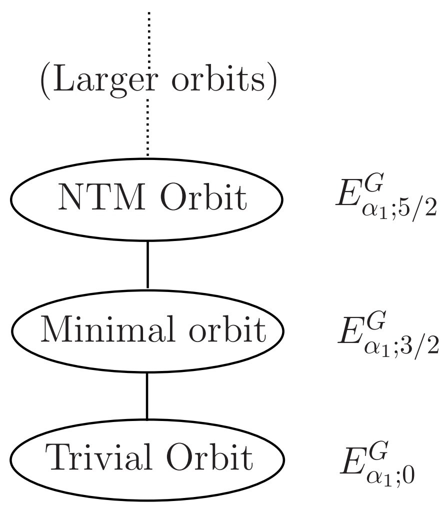


FIGURE 2. Schematic of small representations and Eisenstein special values.

Part (iii) of the following theorem is the main mathematical result of this paper, in particular the cases of E_7 and E_8. Part (i) is trivial, while part (ii) is contained in results of Ginzburg-Rallis-Soudry [10], following earlier work of Kazhdan-Savin [16].

Theorem 2.14. *Let G be one of the groups E_6, E_7, or E_8 from table 1. Then*

(i) *The wavefront set of the automorphic representation attached to the $s = 0$ Epstein series is the trivial orbit.*
(ii) *The wavefront set of the automorphic representation attached to the $s = 3/2$ Epstein series is the closure of the minimal orbit.*

(iii) *The wavefront set of the automorphic representation attached to the $s = 5/2$ Epstein series is the closure of the next-to-minimal (NTM) orbit.*

The closure of the minimal orbit is simply the union of the minimal orbit and the trivial orbit, while the closure of the next-to-minimal orbit is the union of itself, the minimal orbit, and the trivial orbit. Theorem 2.14 will be used in proving theorem 6.1, which is the mathematical proof of the statement concerning vanishing Fourier modes at the end of section 1 that was motivated by string considerations.

2.3. **Outline of paper.** This paper combines information deduced from string theory with results in number theory involving properties of Eisenstein series, which we hope will be of interest to both string theorists and number theorists. In particular, each subject is used to make nontrivial statements about the other. Sections 3–4 and appendices B–D are framed in string theory language and provide information concerning the structure expected of the non-zero Fourier modes based on instanton contributions in superstring theory and supergravity. The subsequent sections provide the mathematical foundations of these observations and generalize them significantly.

Section 3 presents the classification of the expected orbits of fractional BPS instantons in the three limits (i), (ii), and (iii) considered in section 2.1, from the point of view of string theory. The BPS constraints imply that these instantons span particular small orbits generated by the action of the Levi subgroup acting on the unipotent radical associated with the parabolic subgroup appropriate to a given limit. These orbits can be thus thought of as character variety orbits, which are discussed at the beginning of section 4.

In the rest of section 4 and appendix H we will consider explicit low-rank examples (with rank $d+1 \leq 5$) of the Fourier expansions of the functions $\mathcal{E}^{(10-d)}_{(0,0)}$ and $\mathcal{E}^{(10-d)}_{(1,0)}$ in the parabolic subgroups corresponding to each limit. In the cases with $d+1 \leq 4$ ($D \geq 7$), the definition (2.12) implies that the coefficient functions are combinations of $SL(n)$ Eisenstein series that can easily be expressed in terms of elementary lattice sums. In these cases it is straightforward to use standard Poisson summation techniques to exhibit the precise form of their Fourier modes. In particular, the non-zero Fourier modes of $\mathcal{E}^{(10-d)}_{(0,0)}$ will be determined in the three limits under consideration for the rank $d+1 \leq 4$ cases. These modes are localized within the minimal character variety orbits that contain precisely the $\frac{1}{2}$-BPS instantons that are anticipated in section 3. We will see, in particular, that in the decompactification limit (i) the precise form for each of these coefficients matches in detail with the expression determined directly from a quantum mechanical treatment of D-particle world-lines wrapped around an $S^1 \subset \mathrm{T}^d$.[13]

Explicit examples of Fourier expansions of the coefficients of the $\frac{1}{4}$-BPS interactions, $\mathcal{E}^{(D)}_{(1,0)}$, will also be presented in section 4 and appendix H. In the $D = 10B$ case (with symmetry group $SL(2)$) this function is simply equal to $\zeta(5)\, E^{SL(2)}_{\alpha_1;5/2}$, and the extension to $D = 9$ and $D = 8$ is also straightforward. But in the $D = 7$ case (with symmetry group $E_4 = SL(5)$) the coefficient function $\mathcal{E}^{(7)}_{(1,0)}$ is a sum of the regularized Epstein series $\hat{E}^{E_4}_{\alpha_1;5/2}$ and the non-Epstein Eisenstein series $\hat{E}^{E_4}_{\alpha_4;5/2}$ (coming from the third node of the Dynkin diagram). The analysis of the Fourier modes of $E^{E_4}_{\alpha_4;s}$ involves the use of several lattice summation identities that are proved in appendices E, F and G. In particular we will derive an expression for the non-Epstein Eisenstein series coming from either the second or second-to-last nodes as a Mellin transform of a certain lattice sum that is closely related to the $Spin(d,d)$ Epstein Eisenstein series, $E^{Spin(d,d)}_{\alpha_1;s-d/2}$. In appendix G we will derive a theta lift between $SL(d)$ and $Spin(d,d)$ Eisenstein series. This relation was presented in a less rigorous form in [2]. The resulting Fourier expansions contain instanton contributions localized within the minimal ($\frac{1}{2}$-BPS) character variety orbit and the next-to-minimal ($\frac{1}{4}$-BPS) character variety orbit, comprising precisely the instantons anticipated in section 3.

The coefficient $\mathcal{E}^{(6)}_{(0,0)}$ is proportional to the series $E^{Spin(5,5)}_{\alpha_1;3/2}$, which we will analyse by using the integral representation proved in proposition G.1. As expected, its non-zero Fourier modes are supported within the minimal ($\frac{1}{2}$-BPS) character variety orbits in any of the three limits. On the other hand the $\frac{1}{4}$-BPS coefficient, $\mathcal{E}^{(6)}_{(1,0)}$, involves the sum of the regularized values of $\hat{E}^{Spin(5,5)}_{\alpha_1;5/2}$ and $\hat{E}^{Spin(5,5)}_{\alpha_5;3}$. Although we have not computed the Fourier expansion of the second series, it is still possible to show that the non-zero Fourier coefficients of this sum are supported within the minimal and next-to-minimal (i.e., $\frac{1}{2}$- and $\frac{1}{4}$-BPS) character variety orbits in each of the three limits. This will be discussed at the end of section 4.

Sections 5, 6, and 7 are primarily concerned with the exceptional group cases, which correspond to $d \geq 5$ and $D \leq 5$. Since classical lattice summation techniques are difficult to apply in this context, we instead use results from representation theory to show a large number of the Fourier coefficients vanish. Indeed, avoiding explicit computations here is one of the main novelties of the paper. Section 5 discusses aspects connected to representation theory and contains a proof of theorem (2.13), which makes important use of appendix A by Ciubotaru and Trapa on special unipotent representations.

[13]The term *D-particle* refers to any point-like BPS particle state obtained by completely wrapping the spatial directions of Dp-brane states.

Section 6 then applies these results to Fourier expansions, using a detailed analysis of character variety orbits. We will see that the spectrum of instantons that are expected to vanish on the basis of string theory is precisely reproduced by the Eisenstein series in (1.2) and (1.3). For the $s = 3/2$ case (the $\frac{1}{2}$-BPS case) we will reproduce the statements in [10, 17, 18] that only the minimal orbit and the trivial orbit contribute to the Fourier expansions of the Eisenstein series. The relevance of this work to $\frac{1}{2}$-BPS states was suggested by [7, 19]. In addition, we will find that this generalizes for $s = 5/2$ (the $\frac{1}{4}$-BPS case) to the statement that no orbits larger than the next-to-minimal (NTM) orbit can contribute. The analysis in [1] showed the striking fact that the particular Eisenstein series in (1.2) and (1.3) have constant terms with very few powers of r (defined in 2.9) in their expansion around any of the three limits under consideration. The analysis in this paper demonstrates analogous special features of the orbit structure of the non-zero modes. Theorem 6.1 gives a precise statement about which Fourier modes automatically vanish because of representation theoretic reasons. This set of vanishing coefficients is exactly those that are argued to vanish for string theory reasons in section 3.

It is important to point out that our methods show the vanishing of a precise set of Fourier coefficients, but typically do not show the *nonvanishing* of the remaining Fourier coefficients. However, this is accomplished in a number of low rank cases by explicit calculations in section 4 and appendix H, and we hope to treat some of the higher rank cases in the future. Section 7 discusses square-integrability of the coefficients and conditions under which $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ are square-integrable for higher rank groups.

3. Orbits of supersymmetric instantons

From the string theory point of view our main interest is in the systematics of orbits of BPS instantons that enter the Fourier expansions of the coefficients of the low order terms in the low energy expansion of the four graviton amplitude. Before describing these orbits in sections 3.3 – 3.5 we begin with a short overview of the special features of such instantons that follow from supersymmetry. A short summary of the M-theory supersymmetry algebra and BPS particle states is given in appendix B (although this barely skims the surface of a huge subject), where the structure of the eleven-dimensional superalgebra is seen to imply the presence of an extended two-brane (the $M2$-brane) and five-brane (the $M5$-brane) in eleven dimensions. Compactification on a torus also leads to Kaluza–Klein (KK) point-like states and Kaluza–Klein monopoles (KKM), one of which is interpreted in string theory as a $D6$-brane. All the particle states in lower dimensions can be obtained by wrapping the spatial directions of these objects around cycles of the torus.

3.1. **BPS instantons.** One class of BPS instantons can be described from the eleven-dimensional semi-classical M-theory point of view by wrapping euclidean world-volumes of $M2$- and $M5$- branes around compact directions so that the brane actions are finite. These branes couple to the three-form M-theory potential and its dual, and the BPS conditions constrain their charges, $Q^{(p)}$, to be proportional to their tensions, $T^{(p)}$, where $p = 2$ or 5 (as briefly reviewed in appendix B). Wrapping the world-volume of a euclidean $M2$-brane around a 3-torus, $\mathcal{T}^3 \subset \mathcal{T}^{d+1}$, or a euclidean $M5$-brane around a 6-torus, $\mathcal{T}^6 \subset \mathcal{T}^{d+1}$, gives a $\frac{1}{2}$-BPS instanton, which has a euclidean action of the form $S^{(p)} = 2\pi\,(T^{(p)} + iQ^{(p)})$. This gives a factor in amplitude of the form $e^{-S^{(p)}}$ that has a characteristic phase determined by the charge of the brane.

In addition, the "KK instanton" is identified with the euclidean world-line of a KK charge winding around a circular dimension. The magnetic version of this is the "KKM instanton", one manifestation of which appears in string theory as a wrapped euclidean $D6$-brane. Recall that a KK monopole in eleven dimensional (super)gravity with one compactified direction labelled $x^\#$ has a metric of the form [20]

$$ds^2 = V^{-1}\,(dx^\# + \mathbf{A}\cdot\mathbf{dy})^2 + V\,\mathbf{dy}\cdot\mathbf{dy} - dt^2 + dx_6^2\,, \qquad V = 1 + \frac{R}{2|\mathbf{y}|}\,, \tag{3.1}$$

where $ds_7^2 = -dt^2 + dx_6^2$ is the seven-dimensional Minkowski metric and the other four dimensions, $x^\#$, $\mathbf{y} = (y_1, y_2, y_3)$, define a Taub–NUT space, and $|\mathbf{y}|^2 = \sum_{i=1}^3 y_i^2$. The coordinate $x^\#$ is periodic with period $2\pi R$ and the potential, $\mathbf{A}$, satisfies the equation $\nabla \times \mathbf{A} = -\nabla V = \mathbf{B}$. Poincaré duality in the ten dimensions $(t, x_6, \mathbf{y})$ relates the 1-form potential, $\mathbf{A}$, to a 7-form, i.e, $*d\mathbf{A} = dC^{(7)}$. If $x^\#$ is identified with the M-theory circle, $C^{(7)}$ couples to a $D6$-brane in the string theory limit. This gives an instanton when its world-volume is wrapped around a 7-torus. More generally, $x^\#$ can be identified with other circular dimensions of the torus $\mathcal{T}^{d+1}$, giving a further d distinct KKM's, each one of which appears as a finite action instanton when wrapped on an M-theory 8-torus, $\mathcal{T}^8$ (i.e., when $d = 7$). When describing these in the string theory parameterisation (on the string torus T^7) these will be referred to as "stringy KKM instantons". Furthermore, it is well understood how to combine wrapped branes to make $\frac{1}{2}-$, $\frac{1}{4}-$ and $\frac{1}{8}$-BPS instantons [21, 22][14] in a manner analogous to combining p-branes to make states preserving a fraction of the symmetry.

This description of instantons is directly relevant to the discussion of the semi-classical M-theory limit (case (iii)) associated with the Fourier expansion in the parabolic subgroup P_{α_2} in section 3.5. This is the large-volume limit in which eleven-dimensional supergravity is a valid approximation. Similarly, the instanton contributions in limits (i) and (ii) can be described

[14]We are concerned with compactification on tori, but more generally the BPS condition requires branes to be wrapped on special lagrangian submanifolds (SLAGs) or on holomorphic cycles [21].

by translating from the M-theory description to the string theory description of the wrapped branes. These wrapped string theory objects comprise: the fundamental string and the Neveu–Schwarz five-brane (NS5-brane) that couple to $B_{\rm NS}$; Dp-branes that couple to the Ramond–Ramond $(p+1)$-form potentials $C^{(p+1)}$ (with $-1 \le p \le 9$); and KK charges and KK monopoles that couple to modes of the metric associated with toroidal compactification on T^d.

Knowledge of this instanton spectrum is a valuable ingredient in understanding the systematics of the Fourier modes of the Eisenstein series that enter into the definitions of the coefficients of the low order interactions in the expansion of the scattering amplitude. In particular, it connects closely with the study of the Fourier expansions of specific Eisenstein series that enter into $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ (that will be discussed later in this paper), as well as with the Fourier expansion of the more general automorphic function $\mathcal{E}^{(D)}_{(p,q)}$ (that will not be discussed in this paper).

3.2. **Fourier modes and orbits of BPS charges.** The Fourier expansion associated with any parabolic subgroup, $P_\alpha = L_\alpha U_\alpha$, of E_{d+1} is a sum over integer charges that are conjugate to the angular variables that enter in its unipotent radical U_α. These determine the phases of the modes. The Levi factor is a reductive group that has the form $L_\alpha = GL(1) \times M_\alpha$, where M_α is its semisimple component.

The conjugation action on U_α of L_α – or more specifically, its intersection with the discrete duality group $L_\alpha \cap E_{d+1}(\mathbb{Z})$ – relates these charges by Fourier duality. Thus this action carves out orbits within the charge lattice, with each given orbit only covering a subset of the total charge space. This viewpoint is expanded upon in more detail in section 4.1. In this subsection we classify these orbits in cruder form, by considering the action of the continuous group L_α on the charge lattice. Indeed, since we are mainly interested in the algebraic nature of the group action, we sometimes look at the less refined action of the complexification of L_α, e.g., in order to avoid subtle issues about square roots. Though this loses information by grouping charges into broader families, those families still retain some important common features.

As will be explained in section 4.1, the action of L_α on the charge lattice is related to the adjoint representation on the Lie algebra of U_α. This representation is irreducible if and only if U_α is abelian. That is the case for the unipotent radicals we consider of every symmetry group $E_{d+1}(\mathbb{R})$ of rank $d+1 < 6$. Otherwise, the Fourier expansion is only well-defined after averaging over the commutator subgroup (see (4.3)), and hence does not capture the full content of the function. We devote the rest of this section to relating these orbits to BPS instantons in the three limits we consider. In each particular case we will explain the origin of the non-abelian nature of the unipotent radicals, which have charges that do not commute with the other brane charges. A discussion of such effects within string theory can be found, for example, in [23].

We now describe the adjoint action $V_{\hat{\alpha}}$ on the unipotent radical, where $\hat{\alpha}$ labels the node immediately adjacent to α in the Dynkin diagram (fig. 1). For the three parabolic subgroups of interest to us the representations of the unipotent radical are as follows:

(i) The maximal parabolic $P_{\alpha_{d+1}}$.

In this case $\hat{\alpha} = \alpha_d$ and $L_{\alpha_{d+1}} = GL(1) \times E_d$. The following lists the representations V_{α_d} for each value of $2 \leq d \leq 7$.

E_{d+1}	$M_{\alpha_{d+1}}$	V_{α_d}
E_8	E_7	$q^i : \mathbf{56}$, $q : \mathbf{1}$
E_7	E_6	$q^i : \mathbf{27}$
E_6	$Spin(5,5)$	$S_\alpha : \mathbf{16}$
$Spin(5,5)$	$SL(5)$	$v_{[ij]} : \mathbf{10}$
$SL(5)$	$SL(3) \times SL(2)$	$v_{ia} : \mathbf{3} \times \mathbf{2}$
$SL(3) \times SL(2)$	$SL(2) \times \mathbb{R}^+$	$vv_a : \mathbf{2}$

The notation in the last column indicates the irreducible representations are indexed by their dimensions. Both the fundamental representation and the trivial representation of E_7 occur, because the unipotent radical U_{α_8} is a Heisenberg group. The lower dimensional representations are: the fundamental representation for E_6; a spinor representation for $Spin(5,5)$; the rank 2 antisymmetric tensor representation for $SL(5)$; a bivector representation for $SL(3) \times SL(2)$; and a scalar-vector representation for $SL(2) \times \mathbb{R}^+$.

(ii) The maximal parabolic P_{α_1}.

In this case $\hat{\alpha} = \alpha_3$, which is a spinor node (following the numbering of figure 1) and $L_{\alpha_1} = GL(1) \times Spin(d,d)$. The representation $V_{\hat{\alpha}}$ always includes a spinor representation of $Spin(d,d)$. It is irreducible except in the cases of $d = 6, 7$. The case of $Spin(6,6) \subset E_7$ also includes a copy of the trivial representation, because the unipotent radical is again a Heisenberg group; the case of $Spin(7,7) \subset E_8$ also includes a copy of the standard 14-dimensional "vector" representation.

(iii) The maximal parabolic P_{α_2}.

In this case $\hat{\alpha} = \alpha_4$ and $L_{\alpha_2} = GL(1) \times SL(d+1)$. The representation $V_{\hat{\alpha}}$ always includes a rank 3 antisymmetric tensor of $SL(d+1)$, v_{ijk}, of dimension $\frac{1}{3!}(d+1)d(d-1)$. It is irreducible when the rank is less than 6 (see table 3 for the dimensions in the higher rank cases.)

In each case, the charges form a lattice within the first listed piece of $V_{\hat{\alpha}}$, that is, the irreducible subrepresentation coming from the "abelian part" of U_α. More precisely, these are the nontrivial representations in part (i), the spinor representations in part (ii), and the rank 3 antisymmetric tensors v_{ijk} in part (iii). The space $V_{\hat{\alpha}}$ is identical with the "character variety orbit" $\mathfrak{u}_{-1}$ introduced in section 4.1.

Group	first node		second node		last node	
$SL(3)\times SL(2)$	2	0	1	0	3	0
$SL(5)$	4	0	4	0	6	0
$Spin(5,5)$	8	0	10	0	10	0
E_6	16	0	20	1	16	0
E_7	32	1	35	7	27	0
E_8	64	14	56	28 + 8	56	1

TABLE 3. Dimensions of the unipotent radical U_{α_i} for the standard maximal parabolic subgroup P_{α_i} where $i=1$, $i=2$ and $i=d+1$. For each node the first column gives the dimension of the character variety $\mathfrak{u}_{-1}$ (see section 4.1), and the second column gives the dimension of the derived subgroup $[U,U]$. The sum of the two is the dimension of U. The unipotent radical U is abelian when the dimension in the second column is zero; it is a Heisenberg group when this dimension equals 1 and even more non-abelian when it is >1.

Before proceeding with the explicit list of orbits based on the counting of states and instantons in the next three subsections, we will recall basic properties of the space of nontrivial charges. Apart from the most trivial case (with duality group $SL(2,\mathbb{Z})$), the $\frac{1}{2}$-BPS orbits only fill a small fraction of the whole space. For the E_{d+1} groups with $1\le d\le 5$ the complementary space to the $\frac{1}{2}$-BPS space is filled out by $\frac{1}{4}$-BPS orbits. For E_7 and E_8 the full space is spanned by the union of $\frac{1}{2}$-, $\frac{1}{4}$- and $\frac{1}{8}$-BPS orbits. The Fourier coefficients of the BPS protected operators will have nonvanishing Fourier coefficients only associated to these nilpotent orbits. The classification of possible charge orbits only depends on the semi-classical nature of the associated BPS configurations, but does not provide any detailed information about strong quantum corrections. Such information should be encoded in the precise form of the instanton contributions to the Fourier modes.

The instanton spectrum will now be considered in each of these limits in turn. In each case we will list the single BPS instantons that form basepoints of the charge orbits. The dimension of the full spaces of charges spanned by the orbits in each case of interest is shown in table 3. For each of the three limits (i), (ii), (iii), the two columns in the table show the dimensions of the abelian and nonabelian charge spaces, respectively. Since we will be only interested in BPS (supersymmetric) orbits we will not discuss all the possible nilpotent orbits of E_7 and E_8. A complete discussion of the orbit structure is given in section 6.1.

3.3. BPS instantons in the decompactification limit: $P_{\alpha_{d+1}}$.

The parabolic subgroup of relevance to the expansion of the amplitude in $D=10-d$ dimensions when the radius r_d defined in (2.9) of one circle of the torus T^d becomes large is $P_{\alpha_{d+1}}$, which has Levi factor $L_{\alpha_{d+1}}=GL(1)\times$

E_d. In this limit there is a close correspondence between the spectrum of instantons in $D = 10 - d$ dimensions and the spectrum of black hole states in $D + 1 = 11 - d$ dimensions. This follows from the identification of the euclidean world-line of a charged black hole of mass M wrapping around a circular dimension of radius r with an instanton with action $2\pi M r$ that gives rise to an exponential factor of $e^{-2\pi M r}$ in the amplitude. In addition to instantons of this type, there can be instantons that do not decompactify to particle states in the higher dimension because their actions are singular in the large-r limit. In any dimension there are also instantons with actions independent of r that are inherited from the higher dimension in a trivial manner.

The spectrum of BPS black hole states in compactified string theory has been studied extensively. We will here follow the analysis in [24, 25], which considered the spectrum of branes wrapped on T^d. This generates charged $\frac{1}{2}$- and $\frac{1}{4}$- BPS black hole states that correspond to singular solutions in supergravity since they have zero horizon size and hence zero entropy. In addition, for E_6, E_7 and E_8 there are $\frac{1}{8}$-BPS states that correspond to black holes that have non-zero entropy (as well as states with zero entropy), the prototypes being the analysis of black holes in $D = 5$ dimensions (with E_6 duality group) in [26, 27]. The discussion of the associated nilpotent orbits was given in [28]. Our main interest is to extend the analysis in order to account for BPS instantons.

We shall, for convenience, use the M-theory description starting from eleven dimensional supergravity compactified on a $(d+1)$-torus that will be denoted $\mathcal{T}^{d+1}$. The BPS particle states in any dimension are obtained by wrapping all the spatial dimensions of the various extended objects in supergravity around the torus. These include the $M2$-brane and the $M5$-brane, together with the Kaluza–Klein modes of the metric and the magnetic dual Kaluza–Klein monopoles. The BPS instantons can be listed by completely wrapping the euclidean world-volumes of these objects on these tori.

Despite their similarities, there is a fundamental mathematical difference between the orbits of BPS states and the orbits of BPS instantons. The former are orbits under the semisimple part $M_{\alpha_{d+1}}$ of the Levi component $L_{\alpha_{d+1}} = GL(1) \times M_{\alpha_{d+1}}$, while the latter are orbits under the larger group $L_{\alpha_{d+1}}$ itself. Often these orbits coincide, but not always: the 27-dimensional orbit of E_6 and 56-dimensional orbit of E_7 are actually unions of infinitely many $M_{\alpha_{d+1}}$-orbits which are related by the $GL(1)$ action. This $GL(1)$ action is reminicent of the so-called trombone symmetry of supergravity [29]. Similar examples occur in other limits as well. The $GL(1)$ parameter, r, described in (2.9) is always normalized to act by the scalar factor of r^2 on the BPS instantons, and so never acts trivially. This action is typically compensated by a different $GL(1)$ factor in the stabilizer of a BPS instanton. When this happens we will shorten the orbit notation by canceling these two factors, even though they are mathematically different. We use a horizontal line to denote a quotient $\frac{G}{H}$ of a group G by a stabilizer H, in order to match

$D=$ $10-d$	$M_{\alpha_{d+1}}=E_d$	dim point charges $=$ dim $U_{\alpha_{d+1}}$	dim instanton charges $=\#$ +ve roots of E_d
10A	1	1	0
10B	$SL(2)$	0	1
9	$SL(2)\times\mathbb{R}^+$	3	1
8	$SL(3)\times SL(2)$	6	4
7	$SL(5)$	10	10
6	$Spin(5,5)$	16	20
5	E_6	27	36
4	E_7	56 (57)	63
3	E_8	120	120

TABLE 4. The dimensions of the spaces spanned by the BPS point-like charges and BPS instantons of maximal supergravity for the Levi subgroups in $P_{\alpha_{d+1}}$. The parentheses for $M_{\alpha_8}=E_7$ indicate that the number of BPS states is one less than the dimension of the unipotent radical, U_{α_8}, of the parabolic subgroup P_{α_8} of E_8.

orbit descriptions with those commonly found in the physics literature. We have also made an attempt to correct mathematical imprecisions in some existing descriptions. Since we do not use the explicit descriptions of these orbits this should cause no confusion.

3.3.1. *Features of $P_{\alpha_{d+1}}$ orbits.* The details of the enumeration of BPS states and instantons in the decompactification limit are reviewed in appendix C, the results of which are summarised in this subsection. These states are labelled by a set of charges that couple to components of the various tensor potentials in the theory and span a space whose dimension is given in the second-to-last column of table 4 for each Levi group, $M_{\alpha_{d+1}}$, with $0\le d\le 7$. Correspondingly, the dimension of the space of instanton charges is given in the last column. Table 5 lists the BPS orbits for each Levi group in the range $0\le d\le 7$.

Table 4 shows that, with one exception, the number of BPS instantons in dimension D equals the sum of the number of BPS particle states and the BPS instantons in dimension $D+1$, as anticipated above. The exceptional case is the parabolic subgroup with $M_{\alpha_8}=E_7$, where the number of instantons, 120, is one greater than the number of BPS states, 56, plus instantons, 63 in $D=4$. The string theory interpretation of this extra state is discussed at the end of section 3.4.1.

The BPS orbits for each value of $d=10-D$ with Levi factor $L_{\alpha_{d+1}}=GL(1)\times E_d$ are shown in table 5. The tensors v, v_a, v_{ia}, v_{ij} and the spinor S were introduced in section 3.2. I_3 and I_4 are cubic and quartic invariants of E_6 and E_7, respectively, which are defined in terms of the fundamental representation, q^i, of E_6 and E_7, as reviewed in appendices C.6 and C.7. A

$M_{\alpha_{d+1}} = E_d$	BPS	BPS condition	Orbit	Dim.
$GL(1)$	$\frac{1}{2}$	-	GL(1)	1
$SL(2)\times\mathbb{R}^+$	$\frac{1}{2}$	$v\,v_a = 0$	Union of 2 orbits	1 and 2
	$\frac{1}{4}$	$v\,v_a \neq 0$	$\frac{GL(1)\times SL(2)}{\mathbb{R}}$	3
$SL(3)\times SL(2)$	$\frac{1}{2}$	$\epsilon^{ab}\,v_{i\,a}v_{j\,b} = 0$	$\frac{SL(3)\times SL(2)}{GL(2)\ltimes\mathbb{R}^3}$	4
	$\frac{1}{4}$	$\epsilon^{ab}\,v_{i\,a}v_{j\,b} \neq 0$	$\frac{SL(3)\times SL(2)}{SL(2)\ltimes\mathbb{R}^2}$	6
$SL(5)$	$\frac{1}{2}$	$\epsilon^{ijklm}\,v_{ij}\,v_{kl} = 0$	$\frac{SL(5)}{(SL(3)\times SL(2))\ltimes\mathbb{R}^6}$	7
	$\frac{1}{4}$	$\epsilon^{ijklm}\,v_{ij}\,v_{kl} \neq 0$	$\frac{SL(5)}{Spin(2,3)\ltimes\mathbb{R}^4}$	10
$Spin(5,5)$	$\frac{1}{2}$	$(S\Gamma^m S)=0$	$\frac{Spin(5,5)}{SL(5)\ltimes\mathbb{R}^{10}}$	11
	$\frac{1}{4}$	$(S\Gamma^m S)\neq 0$	$\frac{Spin(5,5)}{Spin(3,4)\ltimes\mathbb{R}^8}$	16
E_6	$\frac{1}{2}$	$I_3=\frac{\partial I_3}{\partial q^i}=0$, and $\frac{\partial^2 I_3}{\partial q^i\partial q^j}\neq 0$.	$\frac{E_6}{Spin(5,5)\ltimes\mathbb{R}^{16}}$	17
	$\frac{1}{4}$	$I_3=0,\ \frac{\partial I_3}{\partial q^i}\neq 0$	$\frac{E_6}{Spin(4,5)\ltimes\mathbb{R}^{16}}$	26
	$\frac{1}{8}$	$I_3\neq 0$	$\frac{GL(1)\times E_6}{F_{4(4)}}$	27
E_7	$\frac{1}{2}$	$I_4=\frac{\partial I_4}{\partial q^i}=\frac{\partial^2 I_4}{\partial q^i\partial q^j}\Big\vert_{Adj_{E_7}}=0$, and $\frac{\partial^3 I_4}{\partial q^i\partial q^j\partial q^k}\neq 0$.	$\frac{E_7}{E_{6(6)}\ltimes\mathbb{R}^{27}}$	28
	$\frac{1}{4}$	$I_4=\frac{\partial I_4}{\partial q^i}=0$, and $\frac{\partial^2 I_4}{\partial q^i\partial q^j}\Big\vert_{Adj_{E_7}}\neq 0$.	$\frac{E_7}{Spin(5,6)\ltimes(\mathbb{R}^{32}\ltimes\mathbb{R})}$	45
	$\frac{1}{8}$	$I_4=0,\ \frac{\partial I_4}{\partial q^i}\neq 0$	$\frac{E_7}{F_{4(4)}\ltimes\mathbb{R}^{26}}$	55
	$\frac{1}{8}$	$I_4>0$	$\frac{\mathbb{R}^+\times E_7}{E_{6(2)}}$	56

TABLE 5. The orbits of instantons associated with the parabolic subgroup $P_{\alpha_{d+1}}$. With one exception these are orbits of charged black hole states satisfying fractional BPS conditions that are generated by the action of the Levi subgroup, $GL(1)\times E_d$, on a representative BPS state. The notation is explained in the text. The degenerate case with $d=0$ is omitted here but will be discussed in section 4.2. The information in the third and fourth columns is taken from [24] and [28], respectively. Details are provided in appendix C. Note the presence of the nonabelian 33-dimensional unipotent radical $\mathbb{R}^{32}\ltimes\mathbb{R}$ in the $\frac{1}{4}$-BPS entry for E_7.

general feature that is valid for $d > 1$ is that the $\frac{1}{2}$-BPS states fill out orbits of the form

$$\mathcal{O}_{\frac{1}{2}-BPS} = \frac{E_{d+1}}{E_d \ltimes \mathbb{R}^{n_{d+1}}}, \qquad (n_3, \ldots, n_7) = (3, 6, 10, 16, 27). \qquad (3.2)$$

The integers n_{d+1} are the dimensions of the unipotent radicals, $U_{\alpha_{d+1}}$, listed in table 3; they are also the dimensions of the spaces of BPS point charges for the symmetry groups E_{d+1} listed in table 4, apart from the case of $d = 7$ where U_{α_8} is a non-abelian Heisenberg group. As mentioned earlier, U_{α_8} has dimension 57 while the E_7 point-like states (charged black holes) are labelled by only 56 charges. The missing charge arises from the fact that among the 120 instantons in $D = 3$ dimensions (see table 4) there is one that is a wrapped KKM with $x^{\#}$ (the fibre coordinate in (3.1)) wrapped around the direction that is identified with (euclidean) time. Since particle states in $D = 4$ dimensions are obtained by identifying the decompactified direction with time, the exceptional instanton is one for which $x^{\#}$ grows in the cusp and its action becomes singular. By contrast, 56 of the $D = 3$ instantons have action proportional to r_7 and are seen as point-like states in four dimensions, and the other 63 have no r_7 dependence and decompactify to instantons in four dimensions.

It is interesting to speculate about an additional line to table 5 which we did not list, namely one for $M_{\alpha_9} = E_8$ inside the affine Kac-Moody group E_9. While this latter group is infinite dimensional, one can still make sense of the orbits in terms of the finite dimensional vector space $\mathfrak{u}_{-1}$ in (4.5). Indeed, $\mathfrak{u}_{-1}$ here is 248-dimensional and the action of E_8 is equivalent to the adjoint action on its Lie algebra. Thus the orbits there coincide with the coadjoint nilpotent orbits for E_8.

3.4. **The string perturbation theory limit: P_{α_1}.** In this limit BPS instantons give non-perturbative corrections to string perturbation theory. This involves an expansion in the parabolic subgroup P_{α_1}, with Levi factor $L_{\alpha_1} = GL(1) \times Spin(d, d)$. This limit is analogous to the limit considered in the previous subsection with the role of the decompactifying circle radius, r_d, replaced by the inverse string coupling in $D = 10 - d$ dimensions, which is denoted $1/\sqrt{y_D}$. In this case the orbits of BPS charges do not correspond to black hole charge orbits.

The BPS instantons that enter in this limit are easiest to analyse in terms of the wrapping of euclidean world-volumes of Dp-branes, the NS5-brane and stringy KKM instantons. The Dp-branes enter for all values of $d \geq 0$ and their contribution alone leads to an abelian unipotent radical, U_{α_1}. The NS5-branes contribute on tori of dimension $d \geq 6$ and the KKM instantons contribute for $d = 7$. Both these kinds of instantons render the unipotent radical nonabelian. In section 3.4.1 and appendix D we review the classification of Dp-brane instantons in terms of the classification of $Spin(d, d)$ chiral spinor orbits, which leads to the following features:

- For $d \leq 3$ there is only one non-trivial orbit, which is $\frac{1}{2}$-BPS.

- $\frac{1}{4}$-BPS orbits arise when $d \geq 4$. For $d = 4$ and 5 there is one orbit, namely the full spinor space of dimension 2^{d-1}. For $d = 6$ and $d = 7$ there is again a single $\frac{1}{4}$-BPS orbit given by constrained spinors, which has dimensions 25 and 35, respectively.
- For $d = 4$ the $\frac{1}{2}$-BPS orbit is parameterised by a spinor satisfying the $Spin(4,4)$ pure spinor constraint, $S \cdot S = 0$, while the full eight-component spinor space (with $S \cdot S \neq 0$) parameterises the $\frac{1}{4}$-BPS orbit.
- For $d = 5$ the $\frac{1}{2}$-BPS orbit is parameterised by an $Spin(5,5)$ spinor satisfying the pure spinor constraint,[15] $S\Gamma^i S = 0$, and once again the unconstrained spinor parameterises the $\frac{1}{4}$-BPS orbit.
- For $d = 6$ the $\frac{1}{2}$-BPS orbit is defined by an $Spin(6,6)$ spinor satisfying the pure spinor constraint,

$$F_2 \;:=\; \tfrac{1}{2} \sum_{i,j=1}^{12} S\Gamma^{ij} S \, dx^i \wedge dx^j \;=\; 0\,, \tag{3.3}$$

  where the $\frac{1}{4}$-BPS orbit is parameterised by a spinor satisfying the weaker constraints

$$F_2 \;\neq\; 0\;, \qquad F_2 \wedge F_2 \;=\; 0\,. \tag{3.4}$$

  In addition there is a $\frac{1}{8}$-BPS orbit which is identified with the space of a spinor satisfying

$$F_2 \wedge F_2 \;\neq\; 0\;, \qquad *F_2 \wedge F_2 \;=\; 0\,, \tag{3.5}$$

  where $*$ is the Hodge star operator, and a second $\frac{1}{8}$-BPS orbit identified with the space spanned by an unconstrained 32-component spinor.
- For $d = 7$ there are nine nontrivial orbits (in addition to the trivial orbit) that were determined by Popov [30]. The $\frac{1}{2}$-BPS case is the smallest non-trivial orbit, which is the space spanned by a spinor satisfying

$$F_3 \;:=\; \frac{1}{3!} \sum_{i,j,k=1}^{14} S\Gamma^{ijk} S \, dx^i \wedge dx^j \wedge dx^k \;=\; 0\,, \tag{3.6}$$

  where S is a $Spin(7,7)$ spinor and Γ^i ($i = 1, \ldots, 14$) are corresponding Dirac matrices. However, the description of the remaining orbits in terms of covariant constraints involving F_3 analogous to those of (3.4) and (3.5) is apparently unknown.

We now turn to a detailed description of these orbits, which draws from the information in section 6.1.

[15] The Dirac matrices Γ^i ($i = 1, \ldots 2d$) form a $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ representation of the Clifford algebra $C\ell(d,d)$. We will denote the antisymmetric product of r Dirac Γ matrices by $\Gamma^{i_1 \cdots i_r} = \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} (-)^\sigma \Gamma^{i_{\sigma(1)}} \cdots \Gamma^{i_{\sigma(r)}}$, where $(-)^\sigma$ is the signature of the permutation σ.

3.4.1. *Classification of spinor orbits.* A review of the method for classifying spinor orbits of $G = Spin(d,d)$, when viewed as the subgroup of even and invertible elements of the Clifford algebra $C\ell(d,d)$ associated with $SO(d,d)$, can be found in [31] (based on the original work in [32] for $d \leq 6$, and [30] for $d = 7$).

The following tables will summarise some facts about these orbits, which are typically cosets of the form $\mathcal{O} = Spin(d,d)/H$, H being the stabilizer of a point in the orbit; in three particular cases the quotients are actually $(GL(1)\times G)/H$ for reasons similar to those explained just above section 3.3.1. Since we do not require any specific features of these orbits we shall simplify their description by writing the real points the corresponding complex algebraic variety. For each value of d we will give a representative spinor of each orbit (labelled S^0 in column 1 and defined in appendix D), together with its stabiliser (H in column 2), its dimension ($\dim(G/H)$ in column 3) and the fraction of supersymmetry it preserves – i.e., its BPS degree $N/2^{d-1}$, which is determined by the number of linearly independent spinors N of the orbit representative S^0. In the following we will only list the BPS orbits appearing into the Fourier coefficients of the coefficients we are interested in. A more complete discussion is given in section 6.1.

The tables that follow have the following general properties:

- The bottom row is the trivial orbit and the top row is the dense orbit of a full spinor.
- The second to bottom row is the smallest non-trivial orbit, which is the $\frac{1}{2}$-BPS configuration with orbit parametrized by the coset[16]
$$\mathcal{O}_{\frac{1}{2}-BPS} = \frac{Spin(d,d)}{SL(d) \ltimes \mathbb{R}^{\frac{d(d-1)}{2}}} \tag{3.7}$$
of dimension $1+d(d-1)/2$. This is the orbit of a spinor satisfying the pure spinor constraint and can be obtained by acting on the ground state of the Fock space representation of the spinor with $SO(d,d)$ rotations.
- The third to bottom row is the second smallest non-trivial orbit (the NTM, or $\frac{1}{4}$-BPS, orbit), which arises for $d \geq 4$ and is the coset
$$\mathcal{O}_{\frac{1}{4}-BPS} \quad = \quad \frac{Spin(d,d)}{(Spin(7) \times SL(d-4)) \ltimes U_{\frac{(d-4)(d+11)}{2}}} \;, \tag{3.8}$$
where U_s is a unipotent group of dimension s (which is nonabelian for $d \geq 6$).

In more detail, the specific orbits for each $Spin(d,d)$ group are as follows:

► $Spin(1,1)$ is trivial. For $Spin(2,2)$ and $Spin(3,3)$ the action of the spin group is transitive and there are only two orbits: the trivial one of dimension

[16]Although the orbits listed in this section are over $\mathbb{R}$ or $\mathbb{C}$, the structures are largely independent of the ground field. For example, this particular orbit has the same form over any field k with characteristic different from 2, but with the $\mathbb{R}$ factor replaced by $k^{\frac{d(d-1)}{2}}$.

0, and the Weyl spinor orbit. This is in accord with the discussion in the previous subsection.

$G = Spin(2,2)$			
S^0	stabilizer H	$\dim(G/H)$	BPS
1	$SL(2) \ltimes \mathbb{R}$	2	$\frac{1}{2}$
0	$Spin(2,2)$	0	$--$

$G = Spin(3,3)$			
S^0	stabilizer H	$\dim(G/H)$	BPS
1	$SL(3) \ltimes \mathbb{R}^3$	4	$\frac{1}{2}$
0	$Spin(3,3)$	0	$--$

▶ For $d \geq 4$ the action of the spin group is not transitive and there are several non-trivial orbits represented by constrained spinors.[17] The first orbit listed in the $Spin(4,4)$ table, the full spinor orbit of dimension 8, is actually the quotient $(GL(1) \times Spin(4,4))/Spin(7)$. A similar $GL(1)$ factor occurs for the largest orbit of the groups $Spin(6,6)$ and $Spin(7,7)$, but not for $Spin(5,5)$ (see below).

$G = Spin(4,4)$			
S^0	stabilizer H	$\dim(G/H)$	BPS
$1 + e_{1234}$	$Spin(7)$	8	$\frac{1}{4}$
1	$SL(4) \ltimes \mathbb{R}^6$	7	$\frac{1}{2}$
0	$Spin(4,4)$	0	$--$

$G = Spin(5,5)$			
S^0	stabilizer H	$\dim(G/H)$	BPS
$1 + e_{1234}$	$Spin(7) \ltimes \mathbb{R}^8$	16	$\frac{1}{4}$
1	$SL(5) \ltimes \mathbb{R}^{10}$	11	$\frac{1}{2}$
0	$Spin(5,5)$	0	$--$

▶ The $Spin(6,6)$ case involves some noncommutative unipotent subgroups U_s of dimension s. The full spinor orbit of dimension 32 is $(GL(1) \times Spin(6,6))/SL(6)$.

$G = Spin(6,6)$			
S^0	stabilizer H	$\dim(G/H)$	BPS
$1 + e^*_{14} + e^*_{25} + e^*_{36}$	$SL(6)$	32	0
$1 + e^*_{14} + e^*_{25}$	$Sp(6) \ltimes \mathbb{R}^{14}$	31	$\frac{1}{8}$
$1 + e^*_{14}$	$(SL(2) \times Spin(7)) \ltimes U_{17}$	25	$\frac{1}{4}$
1	$SL(6) \ltimes \mathbb{R}^{15}$	16	$\frac{1}{2}$
0	$Spin(6,6)$	0	$--$

[17] The symbols $e_{i_1 \cdots i_r}$ and $e^*_{i_1 \cdots i_r}$ labelling the spinor S^0 are defined in appendix D.

► For $Spin(7,7)$ the full spinor orbit of dimension 64 is $(GL(1)\times Spin(7,7))/(G_2\times_{\mathbb{Z}_2} G_2)$, where G_2 is the exceptional group of rank 2 and where $H_1 \times_{\mathbb{Z}_2} H_2$ denotes the almost direct product of two groups intersecting on $\mathbb{Z}_2$. Of the total of 10 orbits obtained in [30], we quote only the ones relevant for the analysis of the Fourier modes discussed in this paper:

$G = Spin(7,7)$			
S^0	stabilizer H	$\dim(G/H)$	BPS
$1+e^*_7$	$SL(6)\ltimes \mathbb{R}^{12}$	44	$\frac{1}{8}$
$1+e^*_{147}+e^*_{257}$	$(Sp(6)\times_{\mathbb{Z}_2} GL(1))\ltimes \mathbb{R}^{26}$	43	$\frac{1}{8}$
$1+e_{1234}$	$(SL(3)\times Spin(7))\ltimes U_{27}$	35	$\frac{1}{4}$
1	$SL(7)\ltimes \mathbb{R}^{21}$	22	$\frac{1}{2}$
0	$Spin(7,7)$	0	$--$

3.4.2. *Neveu–Schwarz five-brane and stringy KKM instantons.*

The wrapped world-volume of the NS5-brane produces a new kind of instanton when $d \geq 6$, which is a source of $B_{\rm NS}$ flux. Whereas the Dp-brane instantons have actions of the form C/g_s with C independent of g_s, the wrapped NS5-brane has an action of the form C/g_s^2. This means that such NS5-instantons are suppressed by e^{-C/g_s^2}, and so, in the string perturbation theory regime they are suppressed relative to the Dp-brane instantons. The presence of the charge carried by this wrapped NS5-brane instanton leads to a non-commutativity of the unipotent radical, U_{α_1}, which is a Heisenberg group (this is analogous to the fact that the KKM instanton in $D=3$ led to non-commutativity of the unipotent radical U_{α_8} in the P_{α_8} parabolic subgroup of E_8). The non-commutativity arises because the presence of a NS5-brane charge generates a non-trivial $B_{\rm NS}$ background. This affects the definition of the D-brane charges due to the dependence on $B_{\rm NS}$ of their field-strengths, $F^{(4)} := dC^{(3)} + dB_{\rm NS}\wedge C^{(1)}$ and $*F^{(4)} = dC^{(5)} + C^{(3)}\wedge dB_{\rm NS} - dC^{(3)}\wedge B_{\rm NS}$. Since there is only one euclidean NS5-brane configuration on a 6-torus (the $D=4$ case) the non-commutative part of U_{α_1} is one-dimensional, so the unipotent radical forms a Heisenberg group.

Upon further compactification on T^7 to $D=3$ there are 7 distinct wrapped NS5-brane world-volume instantons, one for each six-cycle. In addition, there are 8 M-theory KKM instantons that are distinguished from each other in the M-theory description by identifying the coordinate $x^{\#}$ with any one of the 1-cycles, as explained earlier. In string language, one of these is the wrapped euclidean $D6$-brane that has been counted as one of the 64 components of the $SO(7,7)$ spinor space and contributes to the abelian part of the unipotent radical U_{α_1}. The other 7 are KKM instantons with $x^{\#}$ identified with a circle in one of the 7 other directions. These are T-dual to the 7 wrapped NS5-branes. The presence of the $D6$-brane and KKM instantons leads to a higher degree of non-commutativity of the unipotent radical, due for example, to the non-linear dependence of the D6-brane field strength on $B_{\rm NS}$ through $*dC^{(1)} = dC^{(7)} + \frac{1}{2}\, B_{\rm NS} \wedge dC^{(5)} - \frac{1}{2} dB_{\rm NS} \wedge C^{(5)} - \frac{1}{3} B_{\rm NS} \wedge B_{\rm NS} \wedge dC^{(3)} + \frac{1}{3} B_{\rm NS} \wedge dB_{\rm NS} \wedge dC^{(5)}$.

This counting coincides with that expected from a group theoretic analysis of the dimension of the abelian and non-abelian (i.e., derived subgroup) parts of the unipotent radical summarised in the columns labelled "first node" of table 3 on page 21.

3.5. **BPS instantons in the semi-classical M-theory limit:** P_{α_2}**.** This is the limit in which the volume, $\mathcal{V}_{d+1}$, of the M-theory torus $\mathcal{T}^{d+1}$ becomes large and semi-classical eleven-dimensional supergravity is a good approximation. The Fourier modes of interest are those associated with the maximal parabolic subgroup P_{α_2}, which has Levi subgroup $L_{\alpha_2} = GL(1) \times SL(d+1)$. The constant terms in the Fourier expansion were considered in [1] and shown to match expectations based on perturbative eleven-dimensional supergravity.

The instanton charge space can be described as follows. The wrapped KK world-lines do not give instantons in this limit since their action is independent of the volume, $\mathcal{V}_{d+1}$. Wrapped euclidean $M2$-branes appear in $D \leq 8$ dimensions (corresponding to symmetry groups with rank ≥ 3), while the wrapped euclidean $M5$-brane arises for $D \leq 5$ dimensions (corresponding to symmetry groups with rank ≥ 6) and the wrapped world-volume associated with the KKM enters first in $D = 3$ dimensions (i.e., for symmetry group E_8). These instanton actions have the exponentially suppressed form $\exp(-C\mathcal{V}_{d+1}^{a})$, where C is independent of $\mathcal{V}_{d+1}$ in the limit $\mathcal{V}_{d+1} \to \infty$, and $a = 3/(d+1)$ for the wrapped $M2$-brane, $a = 6/(d+1)$ for the wrapped $M5$-brane and $a = 7/(d+1)$ for the wrapped KKM.

The space spanned by the 3-form, $v_{[ijk]}$ that couples to $M2$-brane world-sheets wrapping 3-cycles inside the M-theory torus $\mathcal{T}^{d+1}$ has dimension

$$D_{M2}^{d+1} \;=\; \frac{(d+1)!}{3!\,(d-2)!}\,, \tag{3.9}$$

which equals 1, 4, 10, 20, 35, and 56, respectively, for tori of dimensions $d+1 = 3$, 4, 5, 6, 7, and 8 (corresponding to the duality groups $E_3, \ldots, E_8$). Similarly, the space of euclidean five-branes wrapping 6-cycles inside $\mathcal{T}^{d+1}$ has dimension

$$D_{M5}^{d+1} = \frac{(d+1)!}{6!(d-5)!}\,, \tag{3.10}$$

which equals 1, 7, and 28, respectively, for $d+1 = 6$, 7, and 8 (corresponding to duality groups E_6, E_7, and E_8). Finally, a finite action KKM instanton only exists if there are 8 circular dimensions, so it only contributes for the E_8 case. As argued earlier, there are 8 distinct objects of this kind since $x^{\#}$ is distinguished from the other circular coordinates.

Again these dimensions can be compared with those listed in section 6.1 and summarised in table 3 on page 21 under the heading "second node". The wrapped euclidean $M2$-branes contribute the dimensions of abelian part of the unipotent radical for this maximal parabolic subgroup. In fact the numbers in the left-hand column of the second node heading are equal to D_{M2}^{d+1} for all $0 \leq d \leq 7$. The $M5$-brane charge space of dimension D_{M5}^{d+1}, equals the dimension of the non-commutative part (i.e., derived subgroup) of the unipotent radical for E_6 and E_7, while for E_8 there is also a contribution of 8 from the KKM instantons. In this case the non-abelian component of the unipotent radical arises from the KKM instanton dependence on the 3-form $A^{(3)}$ configurations (analogous to the way the $B_{\rm NS}$ configurations induced the non-commutativity in the previous section).

Although we have given a list of dimensions of the space spanned by the orbits, in this case we have not analysed the BPS conditions to discover how the complete space decomposes into orbits with fractional supersymmetry. However, the latter part of this paper analyses the complete orbit structure for the subgroup P_{α_2} and the list of orbits is given in table 8 on page 60. From this we can identify, for each value of d, the minimal ($\frac{1}{2}$-BPS) and NTM ($\frac{1}{4}$-BPS) orbits, as well as many others that arise when $d \geq 5$ (i.e. for E_6, E_7 and E_8).

4. Explicit examples of Fourier modes for rank ≤ 5.

4.1. **Fourier expansions for higher rank groups.** Suppose that $\phi \in C^\infty(\Gamma\backslash G)$ is an automorphic function, and that $A \subset G$ is an abelian subgroup which is isomorphic to $\mathbb{R}^m$ for some $m > 0$. If $\Gamma \cap A$ corresponds to a lattice in $\mathbb{R}^m$ under this identification, then ϕ's restriction to A, $\phi(a)$, has a Fourier expansion. The same is true for any right translate $\phi(ag)$, for g fixed. A prime example of this is A equal to the unipotent radical U of a maximal parabolic subgroup $P = LU$ of G, when U is abelian and Γ is arithmetically defined:

$$\phi(ug) \;=\; \sum_\chi \chi(u)\phi_\chi(g)\,, \quad \phi_\chi(g) \;=\; \int_{\Gamma\cap U\backslash U} \phi(ug)\,\chi(u)^{-1}\,du\,, \tag{4.1}$$

where the sum is taken over all characters χ of U which are trivial on $\Gamma\cap U$. In particular the special case $u = e$,

$$\phi(g) \;=\; \sum_\chi \phi_\chi(g)\,, \tag{4.2}$$

reconstructs ϕ as a sum of its Fourier coefficients ϕ_χ. These Fourier coefficients are in general distinct from Whittaker functions, which are Fourier coefficients for the minimal parabolic. When U fails to be abelian the coefficients ϕ_χ defined by (4.1) still make sense, though ϕ is no longer a sum of them. Instead, it is the integral of ϕ over the commutator subgroup[18] $[U,U]$

[18]The commutator subgroup $[U,U]$ is the smallest normal subgroup of U which contains all elements of the form $[u_1,u_2]$, for $u_1,u_2 \in U$.

of U which has an expansion

$$\int_{\Gamma\cap[U,U]\backslash[U,U]} \phi(ug)\,du \;=\; \sum_{\chi} \phi_\chi(g)\,; \tag{4.3}$$

in other words, the Fourier expansion only captures a small part of ϕ's restriction to U – the part which transforms trivially under $[U,U]$.

A character on U can be viewed as a linear functional on its Lie algebra $\mathfrak{u}$ via the differential. In our case, in which U is the unipotent radical of a maximal parabolic subgroup $P = P_{\alpha_j}$ for some simple root α_j, $\mathfrak{u}$ has a graded structure

$$\mathfrak{u} \;=\; \mathfrak{u}_1 \oplus \mathfrak{u}_2 \oplus \cdots \tag{4.4}$$

where $\mathfrak{u}_k$ is the span of root vectors for roots of the form $\alpha = \sum c_k\alpha_k$, with $c_j = k$. The Killing form $B(\cdot,\cdot)$ exhibits the dual $\mathfrak{u}^*$ of $\mathfrak{u}$ as the complexification of the Lie algebra

$$\mathfrak{u}_- \;=\; \mathfrak{u}_{-1} \oplus \mathfrak{u}_{-2} \oplus \cdots\,. \tag{4.5}$$

The commutator subgroup $[U,U]$ has Lie algebra $\mathfrak{u}_2 \oplus \mathfrak{u}_3 \oplus \cdots$, so the differential of a character is sensitive only to $\mathfrak{u}_1$. Again through the bilinear pairing of the Killing form, its dual space $\mathfrak{u}_1^*$ is isomorphic to the complexification $\mathfrak{u}_{-1}\otimes\mathbb{C}$ of $\mathfrak{u}_{-1}$. The exponential of any such a linear functional is a character of U, and hence $\mathfrak{u}_{-1}\otimes\mathbb{C}$ is known as the *character variety* of U.

Now let χ be a character of U which is invariant under the discrete subgroup $\Gamma\cap U$. The above correspondence guarantees the existence of a unique

$$Y \;\in\; \mathfrak{u}_{-1}\otimes\mathbb{C} \quad \text{such that} \quad \chi(e^X) \;=\; e^{iB(Y,X)}\,. \tag{4.6}$$

The set of all such Y produced from characters χ of $(\Gamma\cap U)\backslash U$ forms the *charge lattice* in $\mathfrak{u}_{-1}$. Decompose $P = LU$, where L is the Levi component. Then formula (4.1) and the automorphy of ϕ under any $\gamma\in\Gamma\cap L$ imply that

$$\begin{aligned} \phi_\chi(\gamma g) \;&=\; \int_{\Gamma\cap U\backslash U} \phi(\gamma^{-1}u\gamma g)\,\chi(u)^{-1}\,du \\ &=\; \int_{\Gamma\cap U\backslash U} \phi(ug)\,\chi(\gamma u\gamma^{-1})^{-1}\,du\,. \end{aligned} \tag{4.7}$$

Here we have changed variables $u\mapsto\gamma u\gamma^{-1}$, which preserves the measure du because γ lies in the arithmetic subgroup $\Gamma\cap L$. In terms of (4.6)

$$\chi(\gamma e^X\gamma^{-1}) \;=\; \chi(e^{\gamma X\gamma^{-1}}) \;=\; e^{iB(Y,\gamma X\gamma^{-1})} \;=\; e^{iB(\gamma^{-1}Y\gamma,X)}\,, \tag{4.8}$$

because of the invariance of the Killing form under the adjoint action; the character in the second line of (4.7) is hence equal to the character for the Lie algebra element $\gamma^{-1}Y\gamma\in\mathfrak{u}_{-1}\otimes\mathbb{C}$.

Consequently, the Fourier coefficients (4.1) are related for characters χ which lie in the same $\Gamma\cap L$-orbit under the adjoint action on $\mathfrak{u}_{-1}\otimes\mathbb{C}$. It should be remarked that $\mathfrak{u}_{-1}$ – like each space $\mathfrak{u}_j$ – is invariant under the adjoint action of L, and in fact furnishes an irreducible representation of L (a fact which can be verified in each example using the Weyl character formula – see the tables in [33, §5], for example, for a complete list). The complexification $L_{\mathbb{C}}$ of L likewise acts on $\mathfrak{u}_{-1} \otimes \mathbb{C}$ according to an irreducible representation, and carves it up into finitely many complex character variety orbits.

Similarly, the adjoint action of $\Gamma \cap L$ on the set of characters of U which are trivial on $\Gamma \cap U$ refines these complex orbits into myriad further "integral" orbits. Those characters naturally form a lattice inside of $i\mathfrak{u}_{-1} \subset \mathfrak{u}_{-1} \otimes \mathbb{C}$, and this last action is that of a discrete subgroup of L on a lattice, e.g., the action of $GL(n, \mathbb{Z})$ on $\mathbb{Z}^n$ in a particular special case. The integral orbits are more subtle to describe because of number-theoretic reasons; indeed, even describing $\Gamma \cap L$ for a large exceptional group is quite complicated.

Each of these complex character variety orbits (and hence each of the $\Gamma \cap L$-orbits on the set of characters that are trivial on $\Gamma \cap U$) is thus contained in a single (complex) coadjoint nilpotent orbit. It therefore makes sense to categorize the complex character variety orbits by giving their basepoints and dimensions. Some of this information was provided in section 3, based on the analysis of BPS states in string theory. This analysis focused on the supersymmetric orbits and did not cover all possible orbits. A systematic and detailed analysis of the remaining orbits for the maximal parabolic subgroups we study will be given in 6.1. These have long been known for the classical groups by the study of "classical rank theory"; the paper [33] contains a listing for all maximal parabolic subgroups of exceptional groups. In addition, the integral orbits are also known in some important cases: Bhargava [34, Section 4] and Krutelevich [35] treat certain cases, with additional cases to appear in forthcoming work of Bhargava.

Note that the calculation (4.7) shows that each coefficient ϕ_χ – which is determined by its values on L – is automorphic under any γ that lies in both Γ and $\mathrm{Stab}_L(\chi)$, the stabilizer of χ within L. In terms of the differential, these are the elements of $\Gamma \cap L$ for which the adjoint action fixes the element $Y \in \mathfrak{u}_{-1} \otimes \mathbb{C}$ from (4.6). One can therefore use (4.7) to write the sum of $\phi_\chi(g)$ ranging over χ in one of the integral orbits, as the sum of left γ-translates of a fixed ϕ_χ, where γ now ranges over cosets of $\Gamma \cap L$ modulo the stabilizer of this fixed character. This shows that not all of the Fourier coefficients need to be computed individually; knowing them for orbit representatives of characters is tantamount to knowing them for all characters. Furthermore, the vanishing of any Fourier coefficient ϕ_χ as a function of L is equivalent to that of all Fourier coefficients in its orbit.

The following subsections (together with details that are presented in appendix H) concern some specific, explicit examples of the Fourier modes of the coefficient functions $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ for the low rank duality groups with $d \leq 4$ (i.e. $D \geq 6$). In these cases standard, classical techniques can be used to obtain exact expressions, including the arithmetical divisor sums that appear. These techniques have the virtue of being relatively simple in these special low rank cases; the higher rank cases of E_6, E_7 and E_8 will be discussed in the later sections, although without precise calculations – our chief contribution is to use representation theory to show that many of them vanish. The divisor sums could also be calculated using Hecke operators, though we do not do so here.

In each particular case we will explicitly identify the character χ, which lies in the lattice of characters of U that are trivial on $\Gamma \cap U$, with a tuple of integral parameters m_i, and use the notation

$$\mathcal{F}^{(D)\alpha}_{(p,q)}(\ell; m_i) \quad := \quad \Big(\mathcal{E}^{(D)}_{(p,q)}\Big)_\chi(\ell) \quad \text{and} \quad F^{G\,\alpha}_{\beta;s}(\ell; m_i) \quad := \quad \big(E^G_{\beta;s}\big)_\chi(\ell) \quad (4.9)$$

to refer to the Fourier modes of $\mathcal{E}^{(D)}_{(p,q)}$ and $E^G_{\beta;s}$, respectively. For brevity we shall sometimes drop the dependence on $\ell \in L$ from the notation.

The precise details of these Fourier coefficients could, in principle, be independently checked against an explicit evaluation of instanton contributions to the graviton scattering amplitude, but in practice such detailed verification is very difficult. However, most details of the contribution of $\frac{1}{2}$-BPS instantons to these coefficients in limit (i), the decompactification limit in which $r_d \gg 1$, can be motivated directly from string theory. This is the limit in which, for these low rank cases, the instantons are identified with wrapped world-lines of small black holes of the $(D+1)$-dimensional theory. The asymptotic behaviour can be understood by studying the fluctuations around $\frac{1}{2}$-BPS D-particle configurations in a manner that generalises the arguments of [36], leading to an expression for the modes in $D = 10 - d \leq 9$ dimensions of the form

$$\mathcal{F}^{(D)\alpha_{d+1}}_{(0,0)}(k) \quad = \quad (\tfrac{r_d}{\ell_{D+1}})^{n_D}\, \sigma_{7-D}(|k|)\, \frac{e^{-S_D(k)}}{S_D(k)^{\frac{8-D}{2}}} \, \Big(c_D + O(\tfrac{\ell_{D+1}}{r_d})\Big)\,. \quad (4.10)$$

Here c_D is a positive constant and $S_D(k) = 2\pi |k| r_d m_{\frac{1}{2}}$ is the action for the world-line of the D-particle wound around the circle of radius r_d and $m_{\frac{1}{2}}$, which is a function of the moduli, is the mass of a "minimal" $\frac{1}{2}$-BPS point-like particle state in $D+1$ dimensions – that is, a state that is related by duality to the lightest mass single-charge D-particle. Such states can form threshold bound D-particles of mass $p\, m_{\frac{1}{2}}$. The divisor sum, $\sigma_n(k) = k^n\, \sigma_{-n}(k) = \sum_{q|k} q^n$, sums over the winding number q of the world-lines of such D-particles (where $k = p \times q$) and can be identified with a matrix model partition function. The factor of $S_D(k)^{(D-8)/2}$ comes from integration over the bosonic and fermionic zero modes and n_D is a constant that depends on the dimension D. Because of the high degree of supersymmetry preserved by the $\frac{1}{2}$-BPS configuration it turns out that this approximation is exact in several cases. In $D = 6$ our results are in agreement with [7]. We have not completed an independent quantum calculation of the $\frac{1}{4}$-BPS instanton contributions, which are more subtle. We do not know a general pattern for the exponent n_D, though it is easily computable in each of the examples below.

The Fourier coefficients for different characters satisfy a number of relations between them due to (4.7). This phenomenon is particularly striking on the symmetry groups with $D \geq 7$, which are products of $SL(n)$'s. For example, the formula in (4.31) depends on p_1 and p_2 only through the combination $|p_2 + p_1\Omega|$, which is actually an instance of the principle in (4.7) (see [37] for more details). Thus these coefficient functions (aside from a substitution in their argument) are determined by the ones having $(p_1, p_2) = (1, 0)$. In general, a theorem of Piatetski-Shapiro [38] and Shalika [39] computes the Fourier expansion of an automorphic form on $SL(n)$ in terms of similarly simple Fourier coefficients. In particular, they demonstrate that the "abelian Fourier coefficients" that appear in (4.3) determine ones absent there that come from nonabelian charges.

However, this theorem is not true for groups other than $SL(n)$. Certain features still persist for the "small" automorphic representations which are the focus of this paper; see [33, section 4] for the analogous result for minimal automorphic representations of E_6 and E_7. An expression that includes contributions from non-commutative charges (which are not addressed in this paper) is presented in [7] in the case of $D = 3$. See also [40–42] for a discussion of noncommutative contributions in a different context.

4.2. $D = 10B$**:** $SL(2, \mathbb{Z})$**.** The simplest nontrivial (but very degenerate) example arises in the case of the IIB theory with $D = 10$, where the discrete duality group is $SL(2, \mathbb{Z})$.[19] In this case the $\frac{1}{2}$- and $\frac{1}{4}$-BPS interactions, $\mathcal{E}^{(10)}_{(0,0)}$ and $\mathcal{E}^{(10)}_{(1,0)}$, are given by Eisenstein series [43, 44]

$$\mathcal{E}^{(10)}_{(0,0)} = 2\zeta(3)\, E^{SL(2)}_{\frac{3}{2}}(\Omega)\,, \qquad \mathcal{E}^{(10)}_{(1,0)} = \zeta(5)\, E^{SL(2)}_{\frac{5}{2}}(\Omega)\,, \tag{4.11}$$

where $E^{SL(2)}_s(\Omega)$ is a non-holomorphic Eisenstein series and $\Omega := \Omega_1 + i\Omega_2 = C^{(0)} + i/\sqrt{y_{10}}$.

It is useful to parametrize the coset space $SL(2, \mathbb{R})/SO(2)$ (i.e., the upper half plane) associated with the continuous symmetry group, $SL(2, \mathbb{R})$, by matrix representatives of the form

$$e_2 \;=\; \begin{pmatrix} 1 & \Omega_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\Omega_2} & 0 \\ 0 & \frac{1}{\sqrt{\Omega_2}} \end{pmatrix}. \tag{4.12}$$

This matrix lies in the maximal parabolic subgroup of upper triangular matrices in $SL(2, \mathbb{R})$; its first factor is in the unipotent radical and the second factor lies in its standard Levi component. The $SL(2)$ Eisenstein series can be expressed as

$$2\zeta(2s)\, E^{SL(2)}_s(\Omega) := \sum_{M_2 \in \mathbb{Z}^2 \setminus \{0\}} (m^2_{SL(2)})^{-s} = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \frac{\Omega_2^s}{|n\Omega + m|^{2s}}\,, \tag{4.13}$$

[19] The type IIA theory has no instantons, which means that only the 0-dimensional trivial orbit contributes.

where the $SL(2,\mathbb{Z})$-invariant (mass)^2 is defined by

$$m^2_{SL(2)} := M_2 G_2 M_2^t = \frac{|n\Omega + m|^2}{\Omega_2}\,, \tag{4.14}$$

with $G_2 = e_2 e_2^t$ and $M_2 = (n\ m) \in \mathbb{Z}^2\backslash\{0\}$.

It is straightforward to determine the Fourier coefficients using the standard expansion of such series in terms of Bessel functions,

$$E_s^{SL(2)}(\Omega) = \sum_{n\in\mathbb{Z}} F_s^{SL(2)}(n)\, e^{2i\pi n\Omega_1}\,. \tag{4.15}$$

The zero Fourier mode is

$$F_s^{SL(2)}(0) = \Omega_2^s + \frac{\xi(2s-1)}{\xi(2s)}\,\Omega_2^{1-s}\,, \tag{4.16}$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. The non-zero mode with phase $e^{2i\pi n\Omega_1}$ is

$$F_s^{SL(2)}(n) = \frac{2\,\Omega_2^{\frac12}}{\xi(2s)}\frac{\sigma_{2s-1}(|n|)}{|n|^{s-\frac12}}\,K_{s-\frac12}(2\pi|n|\Omega_2)\,, \tag{4.17}$$

where $\sigma_\alpha(n) = \sum_{0<d|n} d^\alpha$ is a divisor sum. Thus the non-zero mode with frequency n is proportional to $K_{s-\frac12}$, which is a modified Bessel function of the second kind.

In this degenerate case the only limit to consider is $\Omega_2 \to \infty$, which is the limit of string perturbation theory organized as a power series in Ω_2^{-2} corresponding to the genus expansion of a closed Riemann surface. In this limit the expansion of the coefficient functions is dominated by the two power behaved constant terms in the zero mode $F_s^{SL(2)}(0)$ in (4.16), while the non-zero modes have asymptotic behaviour at large Ω_2,

$$F_s^{SL(2)}(n) = \frac{\sigma_{2s-1}(|n|)}{\xi(2s)|n|^s}\,e^{-2\pi|n|\Omega_2}\left(1 + O(\Omega_2^{-1})\right)\,, \tag{4.18}$$

where the asymptotic expansion of the Bessel function

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\left(1 + O(x^{-1})\right)\,, \quad x \gg 1\,, \tag{4.19}$$

has been used.

The two power behaved terms in (4.16) have the interpretation of terms in string perturbation theory, which is an expansion in y_{10}, the square of the string coupling constant. Furthermore, the Eisenstein series with $s = 3/2$ and with $s = 5/2$ have the correct power-behaved terms to account precisely for the known behaviour of the $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$ terms in the low energy expansion of the four graviton amplitude in 10 dimensions. In [1] it was shown that this is in agreement with string perturbation theory and extends to the higher rank cases where the pattern of constant terms is more elaborate. Furthermore, the exponential terms in the expansion in (4.18) correspond to the expected D-instantons that arise in the $D = 10$ type IIB theory. This illustrates the fact, common to all BPS instanton processes, that the exponential decay of a Fourier mode is proportional to the charge n that determines the phase of the mode. The correction term of order Ω_2^{-1} in (4.18) indicates perturbative corrections to the instanton contribution given by an expansion in powers of the string coupling constant that corresponds to the addition of boundaries in the Riemann surface.

In this case the only instantons are $\frac{1}{2}$-BPS D-instantons – there are no $\frac{1}{4}$-BPS instantons in the ten-dimensional type IIB theory. However, it is known from string theory arguments that the Eisenstein series at $s = 3/2$ is associated with the $\frac{1}{2}$-BPS $\mathcal{R}^4$ term while the series at $s = 5/2$ is associated with the $\frac{1}{4}$-BPS $\partial^4\mathcal{R}^4$ contribution (4.11). This leaves unresolved the question as to what features of these series at special values of s encode the fraction of supersymmetry that these terms preserve? This must be encoded in the measure. Indeed in the $s = 3/2$ case it was argued in [36, 45] that the measure factor $\sigma_2(|n|)$ arises from the $\frac{1}{2}$-BPS D-instanton matrix model, which was verified in [46]. Presumably, the $s = 5/2$ measure should arise in a similar manner.

In most of the higher-rank examples that follow there is a less subtle distinction between the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS cases since in typical cases there are $\frac{1}{4}$-BPS instanton configurations that break $\frac{3}{4}$ of the supersymmetry. As will be shown in the following, these generally enter into non-zero Fourier modes of the coefficient $\mathcal{E}^{(D)}_{(1,0)}$ for $3 \le D < 10$ (although, as will also be seen later, only the $\frac{1}{2}$-BPS orbit contributes in the P_{α_1} parabolic with $D = 7, 8, 9$). The subtleties of the measure factor are not required in order to identify the fraction of supersymmetry preserved in such cases. However, there are no $\frac{1}{8}$-BPS configurations for $D > 5$. Therefore, for $D > 5$ the distinction between the coefficient $\mathcal{E}^{(D)}_{(0,1)}$ and the ones which preserve more supersymmetry is again not determined by the spectrum of instantons that contribute in the various limits under consideration. This indicates that the $\frac{1}{8}$-BPS nature of $\mathcal{E}^{(D)}_{(0,1)}$ must be encoded in the form of the measure factor.

4.3. $D = 9$: $SL(2,\mathbb{Z})$.

The coefficients of the $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$ interactions in this case are [2, 47, 48]

$$\mathcal{E}^{(9)}_{(0,0)} = 2\zeta(3)\,\nu_1^{-\frac{3}{7}}\,E^{SL(2)}_{\frac{3}{2}} + 4\zeta(2)\nu_1^{\frac{4}{7}}, \tag{4.20}$$

$$\mathcal{E}^{(9)}_{(1,0)} = \zeta(5)\nu_1^{-\frac{5}{7}}\,E^{SL(2)}_{\frac{5}{2}} + \frac{4\zeta(2)\zeta(3)}{15}\nu_1^{\frac{9}{7}}E^{SL(2)}_{\frac{3}{2}} + \frac{4\zeta(2)\zeta(3)}{15}\nu_1^{-\frac{12}{7}}\,, \tag{4.21}$$

where $\nu_1 = (\ell^B_{10}/r_B)^2 = g_A^{\frac{7}{8}}(r_A/\ell^A_{10})^{\frac{3}{2}}$ with r_B the radius of the compact dimension in the IIB theory and $r_A = \ell_s^2/r_B$ the radius in the IIA theory. The IIA string coupling, g_A, is related to that of the IIB theory by $g_A = g_B\,\ell_s/r_B$. Furthermore, the $D = 9$ theory can be viewed as the compactification of M-theory from 11 dimensions on a 2-torus, $\mathcal{T}^2$, with volume $\mathcal{V}_2 = \nu_1^{2/3}\,\ell_{11}^2$.

The limit $\nu_1 \to 0$ is the limit in which the $\mathbb{R}^+$ parameter of the continuous symmetry, $SL(2,\mathbb{R}) \times \mathbb{R}^+$, becomes infinite, which is the decompactification limit to the $D = 10$ IIB theory ($r_B \to \infty$), while the limit $\nu_1 \to \infty$ is the semi-classical M-theory limit in which $\mathcal{V}_2$, the volume of $\mathcal{T}^2$, becomes infinite. Equations (4.20) and (4.21) show that there are no non-zero modes in either of these limits. Since $\Omega_2 = g_A^{-1} r_A/\ell_s$, the perturbative IIB limit, $\Omega_2 \to \infty$, is also the $D = 10$ type IIA limit, $r_A \to \infty$. This is the limit in the parabolic subgroup $GL(1) \times U$ of the $SL(2)$ factor (given in (4.12)) in which the parameter in the $GL(1)$ Levi factor in the $SL(2)$ becomes infinite. The non-zero Fourier modes of the expression for $\mathcal{E}^{(9)}_{(0,0)}$ in (4.20) that contribute in this limit are obtained by using the mode expansion of $E_{3/2}$ given in the previous section in (4.20), giving

$$\begin{aligned} \mathcal{F}^{(9)}_{(0,0)}(k) &:= \int_{[0,1]} d\Omega_1 \, \mathcal{E}^{(9)}_{(0,0)} \, e^{-2i\pi k \Omega_1} \\ &= 8\pi \Omega_2^{\frac{1}{2}} \, \nu_1^{-\frac{3}{7}} \, \frac{\sigma_2(|k|)}{|k|} \, K_1(2\pi |k| \Omega_2) \, . \end{aligned} \tag{4.22}$$

The limit $\Omega_2 \to \infty$ in the Bessel function in the second line gives the D-instanton contribution to the coefficient of the $\mathcal{R}^4$ interaction in the type IIB perturbative string theory limit, which has the form, after reinstating the power of ℓ_9 in the effective action, (2.5),

$$\frac{1}{\ell_9} \, \mathcal{F}^{(9)}_{(0,0)}(k) = \frac{r_B}{\ell_s^2} \, \sqrt{8\pi} \, \sigma_{-2}(|k|) \, \frac{e^{-2\pi |k| \Omega_2}}{(2\pi |k| \Omega_2)^{-\frac{1}{2}}} \, (1 + O(\Omega_2^{-1})) \, , \tag{4.23}$$

where the factor of r_B/ℓ_s shows that this term survives the limit $r_B \to \infty$. Here we have used the relations $\nu_1 = (\ell_{10}/r_B)^2$, $\ell_9^7 = \ell_{10}^8/r_B$, and $\ell_{10} = \ell_s \Omega_2^{-1/4}$.

On the other hand, taking the large radius $r_A/\ell_{10} \to \infty$ limit in the IIA case gives

$$\frac{1}{\ell_9} \, \mathcal{F}^{(9)}_{(0,0)}(k) = \frac{1}{r_A} \, \sqrt{8\pi} \, \sigma_{-2}(|k|) \, \frac{e^{-2\pi |k| r_A m_{\frac{1}{2}}}}{(2\pi |k| r_A m_{\frac{1}{2}})^{-\frac{1}{2}}} \, \left(1 + O(\ell_{10}/r_A)\right) \, , \tag{4.24}$$

where $m_{\frac{1}{2}} = 1/(\ell_s g_A)$. Here we have used the relations $\Omega_2 = \frac{r_A}{\ell_s g_A}$, $\nu_1 = g_A^{1/2} r_A^{3/2} \ell_s^{-3/2}$, and $\ell_9 = g_A^{2/7} \ell_s^{8/7} r_A^{-1/7}$. This expression reproduces the asymptotic behaviour for the $\frac{1}{2}$-BPS contribution given in (4.10) with $D = 9$, $n_D = -8/7$ and $S_9(k) = 2\pi |k| r_A m_{\frac{1}{2}}$. The exponent has the interpretation of the action of the euclidean world-line of a type IIA $D0$-brane of charge p wrapped q times around the circle of radius $r_1 = r_A$, where $k = p \times q$ (and the sum over q is in $\sigma_{-2}(|k|)$).

A similar expansion of the two Eisenstein series in (4.21) gives the mode expansion of the coefficient $\mathcal{E}^{(9)}_{(1,0)}$ as the sum of two terms. The occurrence of both the $s = 3/2$ and $s = 5/2$ series demonstrates that the $\partial^4 \, \mathcal{R}^4$ interaction contains a piece that is $\frac{1}{4}$-BPS as well as a piece that is $\frac{1}{2}$-BPS. Repeating the above analysis for the $\frac{1}{4}$-BPS part of $\mathcal{E}^{(9)}_{(1,0)}$ (the $E_{5/2}$ term in (4.21)) and making use of (4.18) with $s = 5/2$ gives (after multiplying by ℓ_9^3 to reproduce the $\partial^4\mathcal{R}^4$ interaction in (2.5))

$$\ell_9^3\,\mathcal{F}^{(9)}_{(1,0)}(k)\Big|_{\frac{1}{4}-BPS} \quad\sim\quad \frac{\sqrt{2\pi}}{3}\,(\ell^A_{10})^3\,g_A^{-\frac{1}{2}}\,\left(\frac{\ell^A_{10}}{r_A}\right)^3\,\sigma_{-4}(|k|)\;\frac{e^{-S_9(k)}}{(S_9(k))^{-\frac{3}{2}}}\,. \tag{4.25}$$

As with the $D = 10$ examples, the distinction between the $s = 3/2$ and $s = 5/2$ Eisenstein series is not seen in the instanton orbits (both series contain the same 1-dimensional orbit) but must be encoded in the different measure factors, such as the divisor sum, which takes the form $\sigma_{-4}(|k|)$ when $s = 5/2$. In contrast to the $\frac{1}{2}$-BPS case we have not derived (4.25), or the analogous expressions for $D < 9$ obtained below, by explicitly evaluating the $\frac{1}{4}$-BPS instanton contributions.

4.4. $D = 8$: $SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$.

The coefficient function $\mathcal{E}^{(8)}_{(0,0)}$ is given in terms of Eisenstein series by [2,47–49]

$$\mathcal{E}^{(8)}_{(0,0)} := \lim_{\epsilon\to 0}\left(2\zeta(3+2\epsilon)\,E^{SL(3)}_{\alpha_1;\frac{3}{2}+\epsilon} + 4\zeta(2-2\epsilon)\,E^{SL(2)}_{1-\epsilon}(\mathcal{U})\right). \tag{4.26}$$

It was shown in [2] that the poles in ϵ of the individual series in parentheses cancel and the expression is analytic at $\epsilon = 0$. The coefficient function $\mathcal{E}^{(8)}_{(1,0)}$ is given by

$$\mathcal{E}^{(8)}_{(1,0)} = \zeta(5)\,E^{SL(3)}_{\alpha_1;\frac{5}{2}} + \frac{4\zeta(4)}{3}\,E^{SL(3)}_{\alpha_1;-\frac{1}{2}}\,E^{SL(2)}_{2}(\mathcal{U})\,. \tag{4.27}$$

We have suppressed the dependence of the $SL(3)$ series on the 5 parameters of the $SL(3)/SO(3)$ coset, but have indicated that the $SL(2)$ series depends on $\mathcal{U}$, the complex structure of the 2-torus, T^2 (see appendix H.1 for details).

(i) **The nonmaximal parabolic**[20] $P_{\alpha_3} = GL(1)\times SL(2)\times\mathbb{R}^+\times U_{\alpha_3}$
This is relevant for the decompactification limit $r_2/\ell_9\to\infty$. The Fourier modes, which are integrals with respect to the U_{α_3} factor (H.12), get contributions from the sum of the modes of the $SL(3)$ and $SL(2)$ Eisenstein series. The modes of $\mathcal{E}^{(8)}_{(0,0)}$ are defined by

$$\mathcal{F}^{(8)\alpha_3}_{(0,0)}(kp_1,kp_2,k') := \int_{[0,1]^3} dC^{(2)}dB_{\rm NS}d\mathcal{U}_1\,e^{-2i\pi k(p_1C^{(2)}+p_2B_{\rm NS})-2i\pi k'\mathcal{U}_1}\,\mathcal{E}^{(8)}_{(0,0)}\,, \tag{4.28}$$

[20] In this somewhat degenerate case, the decompactification limit is associated with a nonmaximal parabolic so that its Levi matches the $D = 9$ duality group.

where $\gcd(p_1,p_2) = 1$ and $C^{(2)}$, $B_{\rm NS}$ and $\mathcal{U}_1$ are the components of the unipotent radical in (H.12). Using the definition in (4.26) the Fourier modes of $\mathcal{E}^{(8)}_{(0,0)}$ are given by the sum of the Fourier modes of the $SL(3)$ and $SL(2)$ series defined in (H.13) and (H.15):

$$\mathcal{F}^{(8)\alpha_3}_{(0,0)}(kp_1,kp_2,k') = 2\zeta(3)\,F^{SL(3)\,\beta_2}_{\beta_1;\frac{3}{2}}(kp_1,kp_2) + 4\zeta(2)\,F^{SL(2)}_{1}(k')\,. \qquad (4.29)$$

We have used the notation β_1 and β_2 on the righthand side to indicate the nodes of the $SL(3)$ Dynkin diagram that correspond to α_1 and α_3 (see figure 3).

FIGURE 3. Correspondence between the labelling of the $SL(3)$ nodes in the E_3 Dynkin diagram according to figure 1 (in terms of α_1 and α_3) and the conventional labelling of the $SL(3)$ Dynkin diagram (in terms of β_1 and β_2).

Note that both contributions are nonsingular at $\varepsilon = 0$ despite the simple poles in (4.26). The reason that these Fourier coefficients do not have poles is that the residues of the series in (4.26) are constant. Using the expression (H.16) for the $SL(2)$ Fourier modes and setting $\mathcal{U}_2 = r_2/r_1 = r_2/r_B$ we obtain[21]

$$4\zeta(2)\,F^{SL(2)}_{1}(k') = 4\pi\,\sigma_{-1}(|k'|)\,e^{-2\pi\,|k'|\,r_2\times\frac{1}{r_1}}\,. \qquad (4.30)$$

The exponent can be identified with minus the action of the world-line of a $\frac{1}{2}$-BPS charge p KK state wrapped q times around a circle of radius r_2, with $p\times q = k'$. The divisor sum $\sigma_{-1}(|k'|)$ weights the different values of p with a factor of $1/p$. The expression (4.30) agrees with the general asymptotic formula (4.10), but it is notable that in this case there are no perturbative corrections.

The $SL(3)$ part is obtained from (H.14) with $s = 3/2$,

$$2\zeta(3)\,F^{SL(3)\,\beta_2}_{\beta_1;\frac{3}{2}}(kp_1,kp_2) = 4\pi\,\sigma_{-1}(|k|)\,e^{-2\pi|k|\,\frac{|p_2+p_1\Omega|}{\sqrt{\Omega_2}}\,\frac{1}{\sqrt{\nu_2}}}\,, \qquad (4.31)$$

where $\gcd(p_1,p_2) = 1$. This expression reproduces the asymptotic behaviour (which is again exact) for the $\frac{1}{2}$-BPS contribution given in (4.10) with $D = 8$. The exponent can be written as

$$-\,2\pi|k|\frac{|p_2+p_1\Omega|}{\sqrt{\Omega_2}}\,\frac{1}{\sqrt{\nu_2}} \quad = \quad -2\pi|k|r_2\,m_{p_1,p_2}\,, \qquad (4.32)$$

[21] Here, and in the following we will use the type IIB description, in which $r_1 = r_B$.

where the $k = 1$ contribution is minus the action for the world-line of a state of mass

$$m_{p_1,p_2}\,\ell_s = |p_2 + p_1\Omega|\,\frac{r_1}{\ell_s}\,, \tag{4.33}$$

wound around the circle of radius r_2. This is the mass of a (non-threshold) bound state of p_2 fundamental strings and p_1 D-strings wound around the dimension of radius r_1. In the limit $r_2/\ell_9 \to \infty$ the Fourier coefficients with different p_1's and p_2's fill out an orbit under the action of the discrete subgroup $SL(2,\mathbb{Z})$ of the Levi factor, which is the nine-dimensional duality group. This is made manifest by expressing m_{p_1,p_2} in nine-dimensional Planck units,

$$m_{p_1,p_2}\,\ell_9 = \frac{|p_2 + p_1\Omega|}{\sqrt{\Omega_2}}\,\nu_1^{-3/7}\,, \tag{4.34}$$

where $SL(2,\mathbb{Z})$ acts with the usual linear fractional transformation on Ω and leaves ν_1 invariant. When $k > 1$ (4.32) is minus the world-line action of a threshold bound state of mass $p \times m_{p_1,p_2}$ wound q times around the circle of radius r_2, where $k = p \times q$ and the divisor sum weights the contributions with a factor of $1/|q|$.

Thus, in the decompactification limit these instantons correspond to the expected contributions from the point-like $\frac{1}{2}$-BPS black hole states in nine dimensions listed in appendix C.2. The Kaluza–Klein $\frac{1}{2}$-BPS states in (4.30) are in the singlet v and the (p,q)-string bound state in (4.31) in the doublet v_a of $SL(2)$. These contributions come from separate configurations ($v = 0$, $v_a \neq 0$) and ($v \neq 0$, $v_a = 0$) so that the condition $vv_a = 0$ is satisfied.

The Fourier modes of the coefficient $\mathcal{E}^{(8)}_{(1,0)}$ in the P_{α_3} parabolic are defined as

$$\mathcal{F}^{(8)\alpha_3}_{(1,0)}(kp_1, kp_2, k') := \int_{[0,1]^3} dC^{(2)} dB_{\rm NS} d\mathcal{U}_1\, e^{-2i\pi\, k\,(p_1 C^{(2)} + p_2 B_{\rm NS}) - 2i\pi k' \mathcal{U}_1}\, \mathcal{E}^{(8)}_{(1,0)}\,, \tag{4.35}$$

where we have chosen to extract the greatest common divisor k of the coefficients of $C^{(2)}$ and $B_{\rm NS}$ so that $\gcd(p_1, p_2) = 1$. Note that, unlike in the case of $\mathcal{E}^{(8)}_{(0,0)}$, the integral does not split into the sum of two terms even though U_{α_3} is block diagonal since $\mathcal{E}^{(8)}_{(1,0)}$ contains the product of two Eisenstein series. Substituting the expression (4.27) for $\mathcal{E}^{(8)}_{(1,0)}$ (which includes a term quadratic in Eisenstein series), it is straightforward to perform the Fourier integration with the result

$$\begin{aligned} \mathcal{F}^{(8)\alpha_3}_{(1,0)}(kp_1, kp_2, k') &= \zeta(5)\, F^{SL(3)\beta_2}_{\beta_1;\frac{5}{2}}(kp_1, kp_2) \\ &+ \frac{2\pi^4}{135}\, F^{SL(3)\beta_2}_{\beta_1;-\frac{1}{2}}(kp_1, kp_2)\, F^{SL(2)}_2(k'). \end{aligned} \tag{4.36}$$

The $k = 0$ or $k' = 0$ terms are determined by $\frac{1}{2}$-BPS instantons arising from the winding of the nine-dimensional $\frac{1}{2}$-BPS states, listed in appendix C.2, around the decompactifying circle.

The $\frac{1}{4}$-BPS part is contained in the $k \neq 0$, $k' \neq 0$ modes of the second contribution in (4.36). For the physical interpretation we extract the greatest common divisor $\ell = \gcd(k, k')$, and set $k = \ell q_1$, $k' = \ell q_2$ with $\gcd(q_1, q_2) = 1$. Applying (H.14) with $s = -1/2$ and (H.16) with $s = 2$, it can be written as

$$\frac{2}{\pi}\frac{\Omega_2^{\frac{4}{3}}}{T_2^{\frac{1}{3}}}\,\sigma_{-3}(|\ell q_1|)\,\sigma_{-3}(|\ell q_2|)\,\frac{1+2\pi|\ell q_1||p_2+p_1\Omega|T_2}{|p_2+p_1\Omega|^3}\,\frac{1+2\pi|\ell q_2|\mathcal{U}_2}{\mathcal{U}_2} \\ \times\,\exp(-2\pi|\ell q_1||p_2+p_1\Omega|T_2 - 2\pi|\ell q_2|\mathcal{U}_2)\,. \tag{4.37}$$

Taking the limit $r_2/\ell_9 \to \infty$ and recalling that $T_2 = \nu_1^{-\frac{3}{7}}\,\Omega_2^{-\frac{1}{2}}\,r_2/\ell_9$ and $\mathcal{U}_2 = r_2/r_1 = \nu_1^{\frac{4}{7}}\,r_2/\ell_9$, the leading behaviour of this expression is

$$8\pi\,\frac{\ell_9^4}{\ell_8^4}\,\sigma_3(|\ell q_1|)\,\sigma_3(|\ell q_2|)\,\frac{\exp(-2\pi\,\ell\,r_2 m_{\frac{1}{4}})}{(|\ell q_1|\,\frac{|p_2+p_1\Omega|}{\sqrt{\Omega_2}}\,\nu_1^{-\frac{3}{7}})^2\times(|\ell q_2|\,\nu_1^{\frac{4}{7}})^2}\,, \tag{4.38}$$

where $r_2/\ell_9^7 = 1/\ell_8^6$ and the instanton action is described by the world-lines of the constituents (in this case bound states of F and D strings and the KK charge) of $\frac{1}{4}$-BPS bound states wound ℓ times around the circle S^1 of radius r_2. The $\frac{1}{4}$-BPS mass is given by

$$m_{\frac{1}{4}}\,\ell_9 = |q_1|\frac{|p_2+p_1\Omega|}{\sqrt{\Omega_2}}\,\nu_1^{-\frac{3}{7}} + |q_2|\,\nu_1^{\frac{4}{7}}\,, \tag{4.39}$$

or in string units

$$m_{\frac{1}{4}}\,\ell_s = |q_1||p_2+p_1\Omega|\frac{r_1}{\ell_s} + |q_2|\frac{\ell_s}{r_1}\,. \tag{4.40}$$

Much as before, the divisor sums in (4.38) encode the combinations of winding numbers and charges carried by these world-lines (although the combinatorics are here more complicated than in the $\frac{1}{2}$-BPS case and deserve further study).

(ii) **The maximal parabolic**[22] $P_{\alpha_1} = GL(1)\times Spin(2,2)\times U_{\alpha_1}$**.**
This is relevant to the string perturbation theory limit, in which the string coupling constant, y_8 gets small. The unipotent factor U_{α_1} in (H.18) is parametrized by $(C^{(2)}, \Omega_1)$. In this case the non-zero Fourier modes of $\mathcal{E}^{(8)}_{(0,0)}$ are obtained from (H.20) with $s = 3/2$,

$$\begin{aligned}\mathcal{F}^{(8)\alpha_1}_{(0,0)}(kp_1, kp_2) &:= \int_{[0,1]^2} d\Omega_1 dC^{(2)} e^{-2i\pi k\,(p_1 C^{(2)} + p_2\Omega_1)}\mathcal{E}^{(8)}_{(0,0)} \\ &= \frac{8\pi}{\sqrt{y_8}}\,\frac{\sigma_2(|k|)}{|k|}\,\frac{\sqrt{T_2}}{|p_2+p_1 T|}K_1\left(2\pi|k|\frac{|p_2+p_1 T|}{\sqrt{T_2 y_8}}\right),\end{aligned} \tag{4.41}$$

[22]Note that $Spin(2,2)$ is isomorphic to $SL(2)\times SL(2)$.

where again $\gcd(p_1, p_2) = 1$. Note that the second term in (4.26) does not contribute since it is constant in $(C^{(2)}, \Omega_1)$. Its asymptotic form for $y_8 \to 0$ is given by

$$\lim_{y_8\to 0} \mathcal{F}^{(8)\alpha_1}_{(0,0)}(kp_1, kp_2) \sim \frac{4\pi}{y_8}\, \sigma_2(|k|) \left(\frac{\sqrt{T_2 y_8}}{|k|\,|p_2 + p_1 T|}\right)^{\frac{3}{2}} e^{-2\pi|k|\frac{|p_2+p_1T|}{\sqrt{T_2 y_8}}}, \tag{4.42}$$

where $\gcd(p_1, p_2) = 1$ and the asymptotic form of the Bessel function (4.19) has been used in the last line in order to extract the leading instanton contribution in the perturbative limit, $y_8 \to 0$ with T_2 fixed [2] (recall $y_8 = (\Omega_2^2 T_2)^{-1}$ is the square of the string coupling). In this limit these non-perturbative effects behave as $e^{-C/\sqrt{y_8}}$, as expected of D-brane instantons. The $p_1 = 0$ and $p_2 \neq 0$ terms are D-instanton contributions and those with $p_1 \neq 0$ are the wrapped D-string contributions of charge (p_1, p_2) that are related by the $SL(2,\mathbb{Z})$ action on the T modulus, which is part of the perturbative T-duality symmetry.

The Fourier modes of $\mathcal{E}^{(8)}_{(1,0)}$ are given by

$$\begin{aligned}
\mathcal{F}^{(8)\alpha_1}_{(1,0)} &:= \int_{[0,1]^2} d\Omega_1 dC^{(2)}\, e^{-2i\pi k(p_1 C^{(2)} + p_2\Omega_1)} \mathcal{E}^{(8)}_{(1,0)} \\
&= \frac{16\zeta(2)}{y_8^{\frac{2}{3}}}\, \frac{\sigma_4(|k|)}{|k|^2}\, \frac{T_2}{|p_2+p_1T|^2}\, K_2\left(2\pi|k|\frac{|p_2+p_1T|}{\sqrt{T_2 y_8}}\right) \\
&+ \frac{16\zeta(4) E_2^{SL(2)}(\mathcal{U})}{\pi\, y_8^{\frac{1}{6}}}\, \frac{\sigma_2(|k|)}{|k|}\, \frac{|p_2+p_1T|}{\sqrt{T_2}}\, K_1\left(2\pi|k|\frac{|p_2+p_1T|}{\sqrt{T_2 y_8}}\right),
\end{aligned} \tag{4.43}$$

with $\gcd(p_1, p_2) = 1$. In the limit of small string coupling, $y_8 \to 0$ and recalling that $\ell_8 = \ell_s\, y_8^{1/6}$, the first line on the right-hand side behaves as

$$\frac{\ell_s^4}{\ell_8^4}\, \frac{8\zeta(2)}{y_8}\, \sigma_4(|k|) \left(\frac{\sqrt{y_8 T_2}}{|k|\,|p_2+p_1T|}\right)^{\frac{5}{2}} \exp\left(-2\pi|k|\frac{|p_2+p_1T|}{\sqrt{T_2 y_8}}\right), \tag{4.44}$$

which is characteristic of the $\frac{1}{2}$-BPS configuration due to a euclidean world-sheet of a (p_1, p_2) D-string wrapped k times around T^2.

The second line behaves in the small string coupling limit $y_8 \to 0$ as

$$\frac{\ell_s^4}{\ell_8^4}\, \frac{8\zeta(4)}{\pi} y_8\, E_2^{SL(2)}(\mathcal{U})\, \sigma_{-2}(|k|) \left(\frac{\sqrt{y_8 T_2}}{|k|\,|p_2+p_1T|}\right)^{-\frac{1}{2}} \exp\left(-2\pi|k|\frac{|p_2+p_1T|}{\sqrt{T_2 y_8}}\right), \tag{4.45}$$

which is suppressed relative to (4.44) by y_8^2 (which is itself four powers of the string coupling). As in the $D = 9$ and $D = 10$ cases, the distinction between the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS cases is not seen in the argument of the Bessel function, which determines the exponential suppression at small y_8. In other words, there are no $\frac{1}{4}$-BPS instantons so the second term on the right-hand side of (4.43) has the same exponential suppression in the $y_8 \to 0$ limit as the first term. The distinction between the $\frac{1}{2}$- and $\frac{1}{4}$-BPS contributions in

(4.43) again lies in the properties of the measure rather than in the spectrum of instantons.

(iii) **The maximal parabolic** $P_{\alpha_2} = GL(1) \times SL(3) \times U_{\alpha_2}$

This corresponds to the limit in which the volume of the M-theory 3-torus, $\mathcal{V}_3$, gets large. The unipotent factor U_{α_2} (H.21) depends only on $\mathcal{U}_1$ and the Fourier modes in this case only involve the modes of the $SL(2,\mathbb{Z})$ Eisenstein series,

$$\mathcal{F}^{(8)\alpha_2}_{(0,0)} := \int_{[0,1]} d\mathcal{U}_1 \, e^{-2i\pi\, k\mathcal{U}_1} \, \mathcal{E}^{(8)}_{(0,0)} = 4\pi\sigma_{-1}(|k|) \, e^{-2\pi|k|\mathcal{U}_2} \,. \tag{4.46}$$

Recalling [2] that $\mathcal{U}_2 = \mathcal{V}_3/\ell_P^3$ is the volume of the M-theory 3-torus, we see that these coefficients are exponentially suppressed in $\mathcal{V}_3$, and correspond to the expected contributions from euclidean $M2$-branes wrapped k times on the 3-torus.

Furthermore, the divisor sum reproduces the one derived from a direct partition function calculation in [50]. The form of this measure factor can also be seen from a simple duality argument using the fact that the wrapped $M2$-brane instanton is related to the Kaluza–Klein world-line instanton by the $SL(2,\mathbb{Z})$ part of the duality group. This duality interchanges T and $\mathcal{U}$ and, hence, the factor $\exp(-2\pi|k|/\sqrt{\Omega_2\nu_2}) = \exp(-2\pi|k|\,T_2)$ in (4.31) for $p_1 = 0$ and $p_2 = 1$ is related to $\exp(-2\pi|k|\mathcal{U}_2)$ in (4.46). This explains the fact that the measure factor, $\sigma_{-1}(|k|)$, is the same in both these equations.

4.5. $D = 7$: $SL(5,\mathbb{Z})$.

Convention on $SL(d)$ **labelling**: In the following we will consider the maximal parabolic series $E^{SL(d)}_{\beta_i;s}$ associated with a node $\beta_1,\ldots,\beta_{d-1}$ of the $SL(d)$ Dynkin diagram using its usual labeling. For example, in the particular case of $SL(5)$ this labeling is shown on the righthand side of figure 4, whereas the previous labeling (coming from the E_4 labeling in figure 1) is shown on the lefthand side. The correspondence between the two labelings is given by $\beta_1 = \alpha_1$, $\beta_4 = \alpha_2$, $\beta_2 = \alpha_3$, $\beta_3 = \alpha_4$.

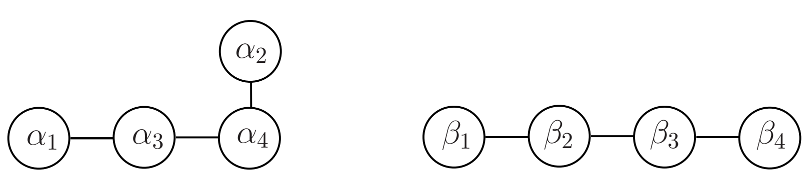


FIGURE 4. Different labelings of the A_4 Dynkin diagram.

In the case $D = 7$ the coefficient functions are given in terms of Eisenstein series by[23] [2]

$$\mathcal{E}^{(7)}_{(0,0)} = 2\zeta(3)\, E^{SL(5)}_{\beta_1;\frac{3}{2}}\,, \tag{4.47}$$

$$\mathcal{E}^{(7)}_{(1,0)} = \lim_{\epsilon\to 0}\left(\zeta(5+2\epsilon)E^{SL(5)}_{\beta_1;\frac{5}{2}+\epsilon} + \tfrac{6\zeta(4-2\epsilon)\zeta(5-2\epsilon)}{\pi^3}E^{SL(5)}_{\beta_3;\frac{5}{2}-\epsilon}\right) \tag{4.48}$$

It was shown in [2] that the poles of the individual series in the parenthesis cancel in the limit $\epsilon \to 0$ and the resulting expression is analytic at $\epsilon = 0$. The detailed analysis of properties of the Fourier modes of Epstein series $E^{SL(5)}_{\beta_1;s}$ in the three limits of interest is determined in appendix H.2.1. The modes of the non-Epstein series, $E^{SL(5)}_{\beta_3;\frac{5}{2}+\epsilon}$ in these three limits are obtained in appendix H.2.2, making use of the representation of $E^{SL(d)}_{\beta_3;s}$ as a Mellin transform of the automorphic lift of a certain lattice sum (see proposition 4.1 below).

(i) **The maximal parabolic** $P_{\alpha_4} = GL(1)\times SL(2)\times SL(3)\times U_{\alpha_4}$

This is the decompactification limit in which $r_3/\ell_8 = r^2 \to \infty$ (where r is the $GL(1)$ parameter that parameterises the approach to the cusp). Recalling the relation between the volume of the 3-torus ν_3 and the volume of the 2-torus ν_2 [2], the limit under consideration is one in which $\nu_3 = \nu_2^{\frac{5}{6}}\,(r_3/\ell_8)^{-2} \to 0$. The unipotent radical is abelian and has the form

$$U_{\alpha_4} = \left\{\begin{pmatrix} I_2 & Q_4\\ 0 & I_3\end{pmatrix}\right\}, \tag{4.49}$$

where I_n is the rank n identity matrix and Q_4 is the 2×3 matrix defined in (H.28). In the discussion of this limit in this subsection we will write the Levi component as

$$\begin{pmatrix} r^{6/5}e_2 & 0\\ 0 & r^{-4/5}e_3\end{pmatrix}\,, \tag{4.50}$$

where $e_2 \in SL(2,\mathbb{R})$ and $e_3 \in SL(3,\mathbb{R})$.

Specialising the Fourier modes of $E^{E_4}_{\alpha_1;s} = E^{SL(5)}_{\beta_1;s}$ that are given in (H.31) to the case $s = 3/2$ and using the relation between the $GL(1)$ parameter and the radius of compactification, $r^2 = r_3/\ell_8$, gives the Fourier modes of $\mathcal{E}^{(7)}_{(0,0)}$ in (4.47)

$$\begin{aligned}\mathcal{F}^{(7)\alpha_4}_{(0,0)}(k,\tilde N_4) &:= \int_{[0,1]^6} d^6Q_4\, e^{-2i\pi k\,\mathrm{tr}(\tilde N_4\cdot Q_4)}\,\mathcal{E}^{(7)}_{(0,0)}\\ &= \left(\frac{r_3}{\ell_8}\right)^{\frac{6}{5}} 8\pi\,\sigma_0(|k|)\,K_0(2\pi|k|\,r_3\,m_{\frac{1}{2}}),\end{aligned} \tag{4.51}$$

[23] In this work this non-Epstein series is related to the one in [2] by $\mathbf{E}^{SL(5)}_{\beta_3;s} = 2\zeta(2s-1)\zeta(2s)\,E^{SL(5)}_{[0010];s}$. The $SL(d)$ nodes are labelled according the natural order as indicated in figure 4.

where $\gcd(\tilde N_4)=1$ and the support of the non-vanishing Fourier coefficients is equal to the rank 1 integer-valued matrices $k\tilde N_4$ in $M_{3,2}(\mathbb{Z})$; these have the form kn^tm with $n=(n_i)\in\mathbb{Z}^3$ and $m=(m_a)\in\mathbb{Z}^2$ row vectors satisfying $\gcd(n_1,n_2,n_3)=\gcd(m_1,m_2)=1$. This factorization is unique up to signs of the three factors. The matrix $\tilde N_4=n^tm$ satisfies the relation

$$\sum_{a,b=1}^{2}\epsilon_{ab}(\tilde N_4)_i{}^a(\tilde N_4)_j{}^b=0,\qquad \forall i,j=1,2,3\,, \tag{4.52}$$

with $\epsilon_{12}=-\epsilon_{21}=1$ and $\epsilon_{11}=\epsilon_{22}=0$, which is precisely $\frac12$-BPS condition discussed in appendix C.3. The argument of the Bessel function in (4.51) is proportional to the mass of $\frac12$-BPS states, where

$$m_{\frac12}\,\ell_8\ \ :=\ \ \|me_2\|\times\|n(e_3^t)^{-1}\|\,. \tag{4.53}$$

This expression does not depend on the factorization $N_4=k\tilde N_4=kn^tm$, and transforms covariantly under the $SL(2)$ and $SL(3)$ factors of the Levi component. This is the mass of a $\frac12$-BPS bound state of fundamental strings and D-strings with Kaluza–Klein momentum. This expression is covariant under the action of the symmetry group $SL(2)\times SL(3)$ of the Levi factor. In the limit $r_3/\ell_8\to\infty$ the expression for the Fourier modes $\mathcal{F}^{(7)\alpha_4}_{(0,0)}$ takes the form

$$\mathcal{F}^{(7)\alpha_4}_{(0,0)}(k,\tilde N_4)=\left(\frac{r_3}{\ell_8}\right)^{\frac65}4\pi\,\sigma_0(|k|)\,\frac{e^{-2\pi|k|\,r_3\,m_{\frac12}}}{\sqrt{|k|r_3\,m_{\frac12}}}\,(1+O(\ell_8/r_3))\,, \tag{4.54}$$

where ℓ_8/r_3 is the inverse square of the $GL(1)$ parameter (see (H.31)). The exponent is proportional to $r_3m_{1/2}$ with $r_3\to\infty$ and $m_{1/2}$ fixed, which is in accord with the behaviour described in (4.10) with $D=7$.

The Fourier modes of $\mathcal{E}^{(7)}_{(1,0)}$ in (4.48) in this parabolic subgroup are defined as

$$\mathcal{F}^{(7)\alpha_4}_{(1,0)}(k,\tilde N_4)\ \ :=\ \ \int_{[0,1]^6}d^6Q_4\,e^{-2i\pi k\,\mathrm{tr}(\tilde N_4\cdot Q_4)}\,\mathcal{E}^{(7)}_{(1,0)}\,, \tag{4.55}$$

with $\gcd(\tilde N_4)=1$. An expression for these Fourier modes is obtained by adding (H.31) for the Epstein series $E^{SL(5)}_{\beta_1;s}$ to the modes of the non-Epstein series $E^{SL(5)}_{\beta_3;s}$ with the correct proportionality constants and setting $s=5/2$. Since each has a constant residue at $s=5/2$ we can directly use the formulas for the nonzero Fourier modes derived in appendix H.

The Fourier modes of $E^{SL(5)}_{\beta_3;s}$ are computed via appendix H.2.2 using the following proposition, which represents this series as the Mellin transform of the lattice sum

$$\mathcal{G}(\tau,X)\ \ :=\ \ \sum_{\left[\begin{smallmatrix}m\\n\end{smallmatrix}\right]\in\mathcal{M}^{(2)}_{2,d}(\mathbb{Z})}e^{-\pi\tau_2^{-1}(m+n\tau)X(m+n\bar\tau)^t}\,. \tag{4.56}$$

Here as in the usual physics notation $\tau = \tau_1 + i\tau_2 \in \mathbb{H}$ and $X = G + B$, with G a positive definite symmetric $d \times d$ matrix and B an antisymmetric $d \times d$ matrix; $\mathcal{M}^{(i)}_{2,d}$ represents $2 \times d$ matrices of rank i. This contribution is the rank 2 part of the lattice sum $\Gamma_{(d,d)}$ for even self-dual Lorentzian lattices. The properties of this sum are studied in appendix E.2, and the proof of the proposition given at the end of appendix E.3.

Proposition 4.1. For Re s large (and consequently for all $s \in \mathbb{C}$ by meromorphic continuation)

$$\begin{aligned} \int_0^\infty I(0, uG)\, u^{2s-1}\, du &= \frac{1}{2}\,\xi(2s)\xi(2s-1)E^{SL(d)}_{\beta_2;s}(e) \\ &= \frac{1}{2}\,\xi(d-2s)\xi(d-2s-1)E^{SL(d)}_{\beta_{d-2};\frac{d}{2}-s}(e)\,, \end{aligned} \tag{4.57}$$

where the function $I(s, X)$ is defined as

$$I(s, X) \quad := \quad \int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} E^{SL(2)}_s(\tau)\, \mathcal{G}(\tau, X)\, \frac{d^2\tau}{\tau_2^2}\,. \tag{4.58}$$

The equality of the two formulas on the righthand side of (4.57) represents a well-known functional equation of Eisenstein series. There is an additional functional equation between these two Eisenstein series coming from the diagram automorphism:

$$E^{SL(d)}_{\beta_2;s}(e) \quad = \quad E^{SL(d)}_{\beta_{d-2};s}(w_d(e^t)^{-1}w_d)\,, \tag{4.59}$$

where w_d is formed from the $d \times d$ identity matrix by reversing its columns. Unlike the functional equation in (4.57), the functional equation (4.59) alters the group variable $e \in SL(d, \mathbb{R})$, and consequently relates Fourier coefficients of these series in different parabolics.

The formulas for Fourier coefficients of $E^{SL(5)}_{\beta_2;s}$ in appendix H.2.2 can be adapted to $E^{SL(5)}_{\beta_3;s}$ using either functional equation, resulting in different (yet of course equivalent) formulas. Using (4.59) and (4.50) gives the identity

$$F^{SL(5)\,\beta_3}_{\beta_3;s}(r^{6/5}e_2, r^{-4/5}e_3; N_4) \quad = \quad F^{SL(5)\,\beta_2}_{\beta_2;s}(r^{4/5}\tilde{e}_3, r^{-6/5}\tilde{e}_2; -w_2N_4^tw_3)\,, \tag{4.60}$$

where $N_4 \in M_{3,2}(\mathbb{Z})$. Here we have used the "contragredient" notation $\widetilde{e}$ to represent $w_d(e^t)^{-1}w_d$ (see (F.4)), and the relation

$$\widetilde{e} \quad = \quad \begin{pmatrix} I_3 & -w_3Q^tw_2 \\ & I_2 \end{pmatrix} \begin{pmatrix} r^{4/5}\tilde{e}_3 & \\ & r^{-6/5}\tilde{e}_2 \end{pmatrix} \tag{4.61}$$

for $e = \begin{pmatrix} I_2 & Q \\ & I_3 \end{pmatrix} \begin{pmatrix} r^{6/5}e_2 & \\ & r^{-4/5}e_3 \end{pmatrix}$.

Applying (H.79) we arrive at the formula

$$
\begin{aligned}
&F^{SL(5)\,\beta_3}_{\beta_3;s}(r^{6/5}e_2, r^{-4/5}e_3; N_4) \quad = \\
&\frac{8\, r^{4+4s/5}}{\xi(2s)\xi(2s-1)} \int_{\mathbb{R}} \sum_{\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] \in SL(2,\mathbb{Z})\backslash\mathcal{M}^{(2)}_{2,3}(\mathbb{Z})} \sum_{\substack{\hat m,\hat n \in \mathbb{Z}^2 \\ \hat m p - \hat n q = -w_2 N_4^t w_3}} \left(\frac{\|(p+q\tau_1)\tilde e_3\|}{\|\tilde e_2^{-1}\hat m\|}\right)^{1/2-s} \times \\
&\qquad \left(\frac{\|q\tilde e_3\|}{\|\tilde e_2^{-1}(\hat n + \hat m\tau_1)\|}\right)^{3/2-s} K_{s-1/2}(2\pi r^2 \|(p+q\tau_1)\tilde e_3\| \|\tilde e_2^{-1}\hat m\|) \times \\
&\qquad\qquad K_{s-3/2}(2\pi r^2 \|q\tilde e_3\| \|\tilde e_2^{-1}(\hat n + \hat m\tau_1)\|)\, d\tau_1 \\
&\quad + \quad \frac{2\,\Gamma\left(s-\frac12\right)}{\xi(2s)\xi(2s-1)} r^{1+14s/5} \sum_{\substack{p\neq 0 \\ n \neq 0 \\ \hat m \perp n \\ \hat m p = -w_2 N_4^t w_3}} \left(\frac{\|\tilde e_2^{-1}\hat m\|}{\pi \|n\tilde e_2\|^2 \|p\tilde e_3\|}\right)^{s-1/2} \times \\
&\qquad\qquad K_{s-1/2}(2\pi r^2 \|\tilde e_2^{-1}\hat m\| \|p\tilde e_3\|) \qquad (4.62)
\end{aligned}
$$

(here $\hat m \in \mathbb{Z}^2$ is thought of as a column vector and $p \in \mathbb{Z}^3$ as a row vector).

Returning to (4.48), we factor $N_4 = k\tilde N_4$, where $k = \gcd(N_4)$, and furthermore factor $\tilde N_4^t$ as $\tilde N_4^t = \hat m' p'$, where $\gcd(\hat m') = \gcd(p') = 1$. This factorization is unique up to multiplication by ± 1. Fixing such a factorization, the solutions to the equation $\hat m p = -k\tilde N_4^t$ have the form $\hat m = \pm d\hat m'$ and $p = \mp \frac{k}{d} p'$ for positive divisors d of k. We now group the coefficient of the $\partial^4\mathcal{R}^4$ interaction as the sum of two contributions

$$
\mathcal{F}^{(7)\alpha_4}_{(1,0)}(k,\tilde N_4) \quad = \quad \mathcal{F}^{(7)\alpha_4}_{(1,0)\,I}(k,\tilde N_4) \; + \; \mathcal{F}^{(7)\alpha_4}_{(1,0)\,II}(k,\tilde N_4)\,, \qquad (4.63)
$$

where $\mathcal{F}^{(7)\alpha_4}_{(1,0)\,I}(k,\tilde N_4)$ comes from applying (H.31) to the first term in (4.48), and from the last line of (4.62); it is supported on rank one integer valued matrices $\tilde N_4$ (i.e., it contains the $\frac12$-BPS configurations). The second contribution $\mathcal{F}^{(7)\alpha_4}_{(1,0)\,II}(k,\tilde N_4)$ comes from the first term of (4.62) and contains the $\frac14$-BPS contributions. Using (4.53) (with the current notation where $\tilde N_4^t = \hat m' p'$) explicit formulas for these are given as

$$
\begin{aligned}
\mathcal{F}^{(7)\alpha_4}_{(1,0)\,I}(k,\tilde N_4) \quad &= \quad 8\pi^2\, r_3\, \frac{\sigma_2(|k|)}{3\,|k|}\, \frac{m_{\frac12}}{\|m' e_2\|^2}\, K_1(2\pi |k|\, r_3\, m_{\frac12}) \\
+ \quad \frac{32}{\pi}\, \frac{\sigma_4(|k|)}{k^2}\, &\frac{(r_3/\ell_8)^2\,(r_3 m_{\frac12})^2}{\|p'(e_3^t)^{-1}\|^4} K_2(2\pi|k| r_3 m_{\frac12}) \sum_{\substack{n\neq 0 \\ n\tilde N_4^t = 0}} \|n(e_2^t)^{-1}\|^{-4}\,. \qquad (4.64)
\end{aligned}
$$

The remaining contribution to (4.63) is given by the formula

$$\mathcal{F}^{(7)\alpha_4}_{(1,0)\,II}(k,\tilde N_4) \; = \; 64\pi^4 r^6 \int_{\mathbb{R}} \sum_{\substack{[{p \atop q}]\in SL(2,\mathbb{Z})\backslash\mathcal{M}^{(2)}_{2,3}(\mathbb{Z})\\ \hat m,\hat n\in\mathbb{Z}^2\\ \hat m p-\hat n q=-kw_2\tilde N_4^t w_3}} \left(\frac{\|\tilde e_2^{-1}\hat m\|}{\|(p+q\tau_1)\tilde e_3\|}\right)^2 \times \frac{\|\tilde e_2^{-1}(\hat n+\hat m\tau_1)\|}{\|q\tilde e_3\|} K_2(2\pi r^2\|(p+q\tau_1)\tilde e_3\|\|\tilde e_2^{-1}\hat m\|)\times K_1(2\pi r^2\|q\tilde e_3\|\|\tilde e_2^{-1}(\hat n+\hat m\tau_1)\|)\,d\tau_1\,. \quad (4.65)$$

We have not succeeded in simplifying the τ_1 integral in this expression and therefore the interpretation of the asymptotic behaviour as $r_3/\ell_8\to\infty$ remains obscure.

(ii) **The maximal parabolic** $P_{\alpha_1}=GL(1)\times SL(4)\times U_{\alpha_1}$

The instanton contributions to $\mathcal{E}^{(7)}_{(0,0)}$ in the perturbative string limit associated with $L_{\alpha_1}=GL(1)\times SL(4)$ are given by (H.34) upon setting $s=3/2$. The relation between the $GL(1)$ parameter and the string coupling constant in 7 dimensions is $r^{-2}=y_7^{\frac12}$ and the relation between the 7 dimension Planck length and the string length is $\ell_7=\ell_s\,y_7^{1/5}$ (cf. (2.10)). In this case the unipotent radical is abelian and has the form

$$U_{\alpha_1} \;=\; \begin{pmatrix} I_4 & Q_1\\ 0 & 1\end{pmatrix}, \quad (4.66)$$

where Q_1 is a $SL(4)$ spinor defined in (H.32).

This leads to the expression for the Fourier modes

$$\begin{aligned}\mathcal{F}^{(7)\,\alpha_1}_{(0,0)}(k,\tilde N_1) \;&:=\; \int_{[0,1]^4} d^4Q_1\, e^{-2\pi i k\,\tilde N_1 Q_1}\,\mathcal{E}^{(7)}_{(0,0)}\\ &=\; \frac{8\pi}{y_7^{\frac{7}{10}}}\,\frac{\sigma_2(|k|)}{|k|}\,\frac{K_1\left(\frac{2\pi|k|\,\|\tilde N_1 e_4\|}{\sqrt{y_7}}\right)}{\|\tilde N_1 e_4\|}\,, \end{aligned} \quad (4.67)$$

where $\tilde N_1\neq 0$ is a row vector in $\mathbb{Z}^4$ such that $\gcd(\tilde N_1)=1$. In the limit $y_7\to 0$ the right hand side of (4.67) has the exponential suppression characteristic of an instanton contribution and contributes

$$\ell_7\,\mathcal{F}^{(7)\,\alpha_1}_{(0,0)}(k,\tilde N_1)\sim \ell_s\,\frac{4\pi}{y_7}\,\sigma_2(|k|)\left(\frac{\sqrt{y_7}}{|k|\,\|\tilde N_1 e_4\|}\right)^{\frac32}\exp\left(-2\pi|k|\frac{\|\tilde N_1 e_4\|}{\sqrt{y_7}}\right) \quad (4.68)$$

to the effective $\mathcal{R}^4$ action with $D=7$ in (2.5).

Terms with $\tilde N_1=[1\,0\,0\,0]$ are D-instanton contributions. Terms with $\tilde N_1\neq[1\,0\,0\,0]$ are $\frac12$-BPS contributions due to wrapped Euclidean bound states of fundamental and D-strings. The rank 4 integer vector $k\tilde{N}_1$ is unrestricted, other than being nonzero.

The Fourier modes of $\mathcal{E}^{(7)}_{(1,0)}$ can be computed in terms of the individual Eisenstein series it is expressed from in (4.48). The modes of $E^{SL(5)}_{\beta_1;5/2}$ are given in (H.34), while the modes of $E^{SL(5)}_{\beta_3;5/2}$ can be determined from those of $E^{SL(5)}_{\beta_2;5/2}$ in (H.111) using the contragredient mechanism described in (4.60-4.61). This results in the expression

$$\begin{aligned}\mathcal{F}^{(7)\alpha_1}_{(1,0)}(k,\tilde{N}_1) &:= \int_{[0,1]^4} d^4Q_1\, e^{-2\pi i k\,\tilde{N}_1^t\cdot Q_1}\,\mathcal{E}^{(7)}_{(1,0)} \\ &= \frac{8\pi^2}{3\,y_7}\,\frac{\sigma_4(|k|)}{|k|^2}\,\frac{1}{\|\tilde{N}_1 e_4\|^2}\,K_2\left(\frac{2\pi|k|\,\|\tilde{N}_1 e_4\|}{\sqrt{y_7}}\right) \\ &+ \frac{16}{\pi\sqrt{y_7}}\times \sum_{\substack{p>0\\ n\neq 0\\ p\hat{m}=-kw_4\tilde{N}_1^t\\ n\perp w_4\tilde{N}_1^t}} \frac{\|\tilde{e}_4^{-1}\hat{m}\|}{p\,\|n\tilde{e}_4\|^4}\,K_1\left(\frac{2\pi|k|\,\|\tilde{N}_1 e_4\|}{\sqrt{y_7}}\right),\end{aligned} \tag{4.69}$$

where again $\tilde{N}_1\in\mathbb{Z}^4\backslash\{0\}$ such that $\gcd(\tilde{N}_1)=1$. Since all factorizations $p\hat{m}=-kw_4\tilde{N}_1^t$ with $p>0$ have the form $\hat{m}=-\frac{k}{p}w_4\tilde{N}_1^t$ for some divisor p of k, the second term on the righthand side can be rewritten as

$$\frac{16}{\pi\sqrt{y_7}}\,\|\tilde{N}_1 e_4\|\,|k|\,\sigma_{-2}(|k|)\,K_1\left(\frac{2\pi|k|\,\|\tilde{N}_1 e_4\|}{\sqrt{y_7}}\right)\sum_{\substack{n\neq 0\\ n\perp w_4\tilde{N}_1^t}}\|n\tilde{e}_4\|^{-4}. \tag{4.70}$$

The two contributions to the Fourier modes have the same support (i.e., in both cases the charges are labelled by the matrix $\tilde{N}_1$) because there are no $\frac{1}{4}$-BPS instantons in the expansion at node α_1 (see section 3.4.1). The different BPS nature of each contribution must be encoded in the factor multiplying the Bessel functions.

(iii) **The maximal parabolic** $P_{\alpha_2}=GL(1)\times SL(4)\times U_{\alpha_2}$

Although we do not work out the details here, explicit expressions for $\mathcal{F}^{(7)\,\alpha_2}_{(0,0)}$ and $\mathcal{F}^{(7)\,\alpha_2}_{(1,0)}$ can be calculated using the expressions for the Fourier coefficients of $E^{SL(5)}_{\beta_1;s}$ and $E^{SL(5)}_{\beta_3;s}$ given in appendix H.2.2.

4.6. $D=6$: $Spin(5,5,\mathbb{Z})$.

The coefficient functions in this case are given by combinations of Eisenstein series [1],

$$\mathcal{E}^{(6)}_{(0,0)} = 2\zeta(3)\,E^{Spin(5,5)}_{\alpha_1;\frac{3}{2}}\,, \tag{4.71}$$

and

$$\mathcal{E}^{(6)}_{(1,0)} = \lim_{\epsilon\to 0}\left(\zeta(5+2\epsilon)\, E^{Spin(5,5)}_{\alpha_1;\frac{5}{2}+\epsilon} + \frac{8\zeta(6-2\varepsilon)}{45}\, E^{Spin(5,5)}_{\alpha_5;3-\epsilon}\right). \qquad (4.72)$$

It was shown in [1] that the pole of the individual series in the parentheses cancel in the limit $\epsilon \to 0$ and the resulting expression is analytic at $\epsilon = 0$. Whereas the previous cases involved $SL(n)$ Eisenstein series, which could be expressed as lattice sums that were easy to manipulate, there is much less understanding of the $Spin(5,5)$ series in terms of such explicit lattice sums. Various properties of $E^{Spin(5,5)}_{\alpha_1;s}$ were considered in [2] (where the series was denoted $(2\zeta(2s))^{-1}\,\mathbf{E}^{Spin(5,5)}_{[10000];s}$), based on the integral representation contained in the following proposition. We give a rigorous proof of it through proposition G.1 (from which it immediately follows via proposition 4.1).

Proposition 4.2. For Re s large (and consequently for all $s \in \mathbb{C}$ by meromorphic continuation)

$$\begin{aligned}&\frac{1}{4}\xi(2s)\xi(2s-1)E^{SL(d)}_{\beta_2;s}(e)\\ &= \int_0^\infty \left(u^{-d/2}\xi(d-2)\, E^{Spin(d,d)}_{\alpha_1;d/2-1}\begin{pmatrix} u^{1/2}e & \\ & u^{-1/2}\tilde{e}\end{pmatrix} + u^{-1}\,\xi(d-2)\, E^{SL(d)}_{\beta_{d-1};d/2-1}(e)\right.\\ &\qquad\qquad\left. + \,\xi(2)\right)u^{2s-1}\,du\,. \qquad (4.73)\end{aligned}$$

The convergence of this integral is not *a priori* obvious and is explained in appendix G (cf. its concluding remark). Proposition 4.2 relates $E^{SL(d)}_{\beta_2;s}(e)$ to a Mellin transform of $E^{Spin(d,d)}_{\alpha_1;s}$; note that the last two terms in (4.73) are not present in [2, 51, 52]. This integral representation will be used in appendix H.3 to obtain the Fourier modes of $E^{Spin(d,d)}_{\alpha_1;s}$. This is sufficient to discuss the Fourier modes of the coefficient $\mathcal{E}^{(6)}_{(0,0)}$, but $\mathcal{E}^{(6)}_{(1,0)}$ also involves the series $E^{Spin(5,5)}_{\alpha_5;s}$. The evaluation of its Fourier modes appears to be much more complicated and will not be performed in this paper. However, we will be able to determine its orbit content as will be discussed later.

(i) **The maximal parabolic** $P_{\alpha_5} = GL(1)\times SL(5)\times U_{\alpha_5}$

This parabolic subgroup has Levi factor $L_{\alpha_5} = GL(1)\times SL(5)$ (recalling from figure 1 that in our conventions α_5 is a spinor node of $E_5 = Spin(5,5)$). Here we will evaluate the Fourier modes using methods similar to those used in computing the constant term of the series $E^{Spin(d,d)}_{\alpha_1;s}$ in [2, appendix C]. The Fourier modes are defined as

$$F^{Spin(5,5)\alpha_5}_{\alpha_1;s}(N_2) \quad := \quad \int_{[0,1]^{10}} dQ_2\, e^{-\pi i\,\mathrm{tr}(N_2Q_2)}\, E^{Spin(5,5)}_{\alpha_1;s} \qquad (4.74)$$

where Q_2 is a 5×5 antisymmetric matrix parameterizing the abelian unipotent radical U_{α_5}, and N_2 is an antisymmetric 5×5 matrix with integer entries.

We find that the Fourier modes of the series $E^{Spin(5,5)}_{\alpha_1;s}$ are localized on the rank 1 contributions where N_2 satisfies the constraints

$$\sum_{i,j,k,l=1}^{5} \epsilon^{ijklm}(N_2)_{ij}(N_2)_{kl} = 0, \qquad \forall 1 \leq m \leq 5\,, \tag{4.75}$$

where ϵ^{ijklm} is the totally antisymmetric symbol with $\epsilon^{12345} = 1$. This constraint is the $\frac{1}{2}$-BPS condition discussed in appendix C.4. This condition can be solved as

$$N_2 = n^t m - m^t n; \qquad m, n \in \mathbb{Z}^5 - \{[0\,0\,0\,0\,0]\}\,. \tag{4.76}$$

In this case $e^{-i\pi\,\mathrm{tr}(N_2 Q_2)} = e^{-2\pi i m Q_2 n^t}$.

The Fourier modes of $\mathcal{F}^{Spin(5,5)\,\alpha_5}_{\alpha_1;s}$ are computed in (H.114) using the method of orbits for the $SL(2)$ action on τ. That formula simplifies for the special value of $s = 3/2$ to

$$\mathcal{F}^{(6)\alpha_5}_{(0,0)}(N_2) \;=\; \frac{1}{2\,\xi(3)}\left(\frac{r_4}{\ell_7}\right)^{5/2} \sum_{\substack{[\begin{smallmatrix} m' \\ n' \end{smallmatrix}] \in GL(2,\mathbb{Z})\backslash \mathcal{M}^{(2)}_{2,5}(\mathbb{Z})' \\ N_2 = k((n')^t m' - (m')^t n')}} \sigma_1(k)\, \frac{e^{-2\pi\, k\, r_4\, m_{\frac{1}{2}}}}{k\, r_4\, m_{\frac{1}{2}}}\,, \tag{4.77}$$

where

$$m^2_{\frac{1}{2}}\,\ell_7^2 \;:=\; \det([\begin{smallmatrix} m' \\ n' \end{smallmatrix}] G_5 [\begin{smallmatrix} m' \\ n' \end{smallmatrix}]^t) \;=\; \|m' e_5\|^2\, \|n' e_5\|^2 \,-\, (m' e_5 \cdot n' e_5)^2\,, \tag{4.78}$$

$k = \gcd(N_2)$, $G_5 = e_5 e_5^t$, and $\mathcal{M}^{(2)}_{2,5}(\mathbb{Z})'$ represents all possible bottom two rows of matrices in $SL(5,\mathbb{Z})$ (see (H.117)). The expression in (4.77) reproduces the asymptotic (actually exact in this case) behaviour for $\frac{1}{2}$-BPS contribution in (4.10) with $D = 6$.

The Eisenstein series $E^{Spin(5,5)}_{\alpha_1;s}$ has a single pole at $s = 5/2$ with residue proportional to the $s = 3/2$ series $E^{Spin(5,5)}_{\alpha_1;3/2}$ discussed above. The finite part of the $E^{Spin(5,5)}_{\alpha_1;s}$ series at $s = 5/2$ only receives $\frac{1}{2}$-BPS contributions (see the comment following (H.113)). The complete coefficient $\mathcal{E}^{(6)}_{(1,0)}$, defined in (4.72), also gets a $\frac{1}{4}$-BPS contribution from $E^{Spin(5,5)}_{\alpha_5;s}$, which has a pole at $s = 3$ such that the resulting combination in (4.72) is analytic as shown in [1].

(ii) **The maximal parabolic** $P_{\alpha_1} = GL(1) \times Spin(4,4) \times U_{\alpha_1}$

In this parabolic subgroup the Levi factor is $L_{\alpha_1} = GL(1) \times Spin(4,4)$. The elements of the unipotent radical are parametrized by the 4×2 matrix

$$Q_1 = \begin{pmatrix} Q_{1I} & Q_2^I \end{pmatrix}, \qquad \forall 1 \leq I \leq 4\,, \tag{4.79}$$

where $Q_1 = (u_1, u_2, u_3, u_4)$ and $Q_2 = (u_8, u_7, u_6, u_5)$ using the variables parameterizing the unipotent radical in (H.118) in appendix H.3. In the type IIA string theory description this matrix is parametrized by the four euclidean $D0$-branes wrapped on 1-cycles and four euclidean $D2$-branes wrapped on 3-cycles of T^4.

The Fourier modes of $E^{Spin(5,5)}_{\alpha_1;s}$ are defined as

$$F^{Spin(5,5)\alpha_1}_{\alpha_1;s}(N_1) := \int_{[0,1]^8} d^8Q_1 e^{-2i\pi\,\mathrm{tr}(N_1Q_1)}\, E^{Spin(5,5)}_{\alpha_1;s} \,. \tag{4.80}$$

We will write the 2×4 matrix N_1 as

$$N_1 \;:=\; \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right], \tag{4.81}$$

where the row vectors have components $M = [m^1\, m^2\, m^3\, m^4]$ and $N = [n_1\, n_2\, n_3\, n_4]$. The m^I ($I = 1, 2, 3, 4$) integers associated with the windings of the one-dimensional euclidean world-volume of a $D0$-brane on the four cycles of the 4-torus, and n_I are associated with the four distinct windings of the three dimensional euclidean world-volume of a $D2$-brane on a 4-torus[24]. This means, for example, that on a square 4-torus with radii R_I the action of a euclidean $D0$-brane is $\sum_{I=1}^4 m^I R_I/(\ell_s g_s)$ while the action of a $D2$-brane is $V_4\sum_{I=1}^4 n_I\ell_s/(R_I g_s)$, where $V_4 = R_1R_2R_3R_4/\ell_s^4$. Because of space considerations we will omit the analysis of the case when $N = [0\,0\,0\,0]$, and instead indicate how the calculations can be performed in appendix H.3. More generally, the various configurations of $(D0, D2)$ states can be classified by introducing the vector (p_L, p_R) in the even self-dual Lorentzian lattice $\Gamma_{(4,4)}$,

$$\begin{aligned} \sqrt{2}\,p_L &= (M + N(B - G_4))(e_4^t)^{-1} \\ \sqrt{2}\,p_R &= (M + N(B + G_4))(e_4^t)^{-1}\,, \end{aligned} \tag{4.82}$$

where $G_4 = e_4e_4^t$ is the metric on the torus and B is an antisymmetric 4×4 matrix. Introducing $y_6 = g_s^2/V_4$ the $GL(1)$ parameter is $r^2 = y_6^{-\frac{1}{2}}$ according to (2.9). We remark that the lattice is even because $p_L^2 - p_R^2 = 2\sum_{I=1}^4 m_I n^I \in 2\mathbb{Z}$. In terms of the modes matrix N_1 in (4.81) this is expressed as $p_L^2 - p_R^2 = \mathrm{tr}(N_1JN_1^t)$ where $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By triality the $SO(4, 4)$ vector (p_L, p_R) is equivalent to a $SO(4, 4)$ chiral spinor used for the orbit classification in section 3.4.1.

The Fourier modes are derived in (H.130) using the θ-lift representation of the $Spin(5, 5)$ Eisenstein series, yielding

$$\begin{aligned} F^{Spin(5,5)\,\alpha_1}_{\alpha_1;s}(N_1) \;&=\; \frac{1}{\xi(2s)y_6^{1/2}} \sum_{p\,|\,\gcd(N_1)} \int_0^\infty d\tau_2\, e^{-\pi\frac{p^2}{\tau_2 y_6} - \pi\frac{\tau_2}{p^2}\,(p_L^2+p_R^2)} \\ &\times \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1\, E^{SL(2)}_{s-\frac{3}{2}}(\tau)\, e^{i\pi\tau_1\,\frac{(p_L^2-p_R^2)}{p^2}}\,, \end{aligned} \tag{4.83}$$

[24]As in the earlier cases each integer should be interpreted as a product of a D-particle charge and its world volume winding number.

where we used that $\gcd(N_1) = \gcd(m^1,\dots,m^4,n_1,\dots,n_4)$. It is significant that setting $s = 3/2$ and using $E_0^{SL(2)}(\tau) = 1$, the integration over τ_1 projects onto the condition $p_L^2 - p_R^2 = 0$ which is the pure spinor condition for $SO(4,4)$. Using the triality relation between vector and spinor representation of $SO(4,4)$ this condition is the $\frac{1}{2}$-BPS (pure spinor) condition $S\cdot S = 0$ discussed in section 3.4.1. It is then straightforward to compute the integrals in (4.83) to evaluate the Fourier modes of the coefficient function $\mathcal{E}^{(6)}_{(0,0)}$, giving

$$\mathcal{F}^{(6)\,\alpha_1}_{(0,0)}(N_1) \quad = \quad \frac{4\sqrt{2}\,\pi\,\sigma_2(\gcd(N_1))}{y_6\sqrt{p_L^2}}\,K_1(2\pi y_6^{-1/2}\sqrt{2p_L^2})\,\delta_{p_L^2=p_R^2}\,, \qquad (4.84)$$

where the Kronecker δ-function localizes the contributions to $\frac{1}{2}$-BPS pure spinor locus $p_L^2 = p_R^2$ (specified by the condition $\mathrm{tr}(N_1 J N_1^t) = 0$ on the mode matrix N_1). As expected, the argument of the Bessel function is proportional to $1/\sqrt{y_6}$, the inverse of the string coupling with $D = 6$, so its asymptotic expansion is that expected from the contribution of $\frac{1}{2}$-BPS states from wrapped D-brane on the 4-torus T^4. The asymptotic form for $y_6 \to \infty$ in the weak coupling regime is given by

$$\ell_6^2\,\mathcal{F}^{(6)\,\alpha_1}_{(0,0)}(N_1) \quad \sim \quad \frac{4\pi\ell_s^2}{y_6}\,\sigma_2(\gcd(N_1))\,\frac{e^{-2\pi\,\frac{\sqrt{2p_L^2}}{\sqrt{y_6}}}}{(\sqrt{2p_L^2}\,y_6^{-\frac{1}{2}})^{\frac{3}{2}}}\,\delta_{p_L^2=p_R^2}\,, \qquad (4.85)$$

where we made use of the relation between the Planck length in six dimensions and the string scale $\ell_6 = \ell_s\,y_6^{-\frac{1}{4}}$.

When $s \neq 3/2$ the τ_1 integral in (4.83) does not impose the restriction $p_L^2 - p_R^2 = 0$ and so the solution fills a generic $Spin(4,4)$ orbit and is $\frac{1}{4}$-BPS. Although the function $\mathcal{E}^{(6)}_{(1,0)}$ in (4.72) is a linear combination of the vector Eisenstein series, $E^{Spin(5,5)}_{\alpha_1;5/2}$, and the spinor series, $E^{Spin(5,5)}_{\alpha_5;3}$, at present we know little about the explicit structure of the latter, so we will only discuss the former here. However, in this parabolic the $\frac{1}{4}$-BPS content of $\mathcal{E}^{(6)}_{(1,0)}$ is entirely contained in $E^{Spin(5,5)}_{\alpha_1;5/2}$.[25]

Therefore we can obtain the complete $\frac{1}{4}$-BPS content of (4.72) by analysing the Fourier modes of the Epstein series $E^{Spin(5,5)}_{\alpha_1;5/2}$ when p_L and p_R are assumed to satisfy $\frac{1}{4}$-BPS condition $p_L^2 - p_R^2 \neq 0$. We shall therefore assume that $p_L^2 - p_R^2 \neq 0$ for the rest of this section. Hence the $\frac{1}{4}$-BPS Fourier modes

[25]The fact that the spinor series $E^{Spin(5,5)}_{\alpha_5;3}$ contains only the $\frac{1}{2}$-BPS orbit follows from the theorem of Matumoto [53] that will be used in the context of the higher-rank groups in section 6.2.

of the first term are obtained from the $s = 5/2$ limit of (4.83),

$$\mathcal{F}^{(6)\,\alpha_1}_{(1,0)}(N_1) \;=\; \frac{\pi^{5/2}}{\Gamma(\frac{5}{2})\,y_6^{1/2}} \sum_{p\,|\,\gcd(N_1)} \int_0^\infty d\tau_2\, e^{-\pi\frac{p^2}{\tau_2 y_6} - \pi\frac{\tau_2}{p^2}(p_L^2+p_R^2)} \times \\ \times \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1\, \hat{E}_1^{SL(2)}(\tau)\, e^{i\pi\tau_1 \frac{p_L^2-p_R^2}{p^2}}\,, \tag{4.86}$$

where

$$\hat{E}_1^{SL(2)}(\tau) \;=\; \tau_2 - \tfrac{3}{\pi}\log(\tau_2 e^{-\hat{c}}) \;+\; \sum_{n\neq 0} \frac{\sigma_1(|n|)}{\xi(2)\,|n|}\, e^{-2\pi|n|\tau_2}\, e^{2\pi i n\tau_1}\,, \tag{4.87}$$

where $\hat{c} = 0.9080589548722\cdots$ (see (4.16-4.17)). Note that since the residue of $E_s^{SL(2)}$ at $s = 1$ is constant, the nonzero Fourier modes of $\hat{E}_1^{SL(2)}$ are indeed the limits of the corresponding modes of $E_s^{SL(2)}$ as $s \to 1$; these are the only coefficients relevant to the τ_1-integral in (4.86) because of the assumption $p_L^2 - p_R^2 \neq 0$. Evaluation of (4.86) gives the result

$$F^{Spin(5,5)\,\alpha_1}_{\alpha_1;\frac{5}{2}}(N_1) \;=\; \frac{16\pi}{y_6} \sum_{p\,|\,\gcd(N_1)} p^2\,\sigma_{-1}\!\left(\frac{|p_L^2-p_R^2|}{2p^2}\right) \times \\ \times\; \frac{K_1(2\pi y_6^{-\frac{1}{2}}\sqrt{p_L^2+p_R^2+|p_L^2-p_R^2|})}{\sqrt{p_L^2+p_R^2+|p_L^2-p_R^2|}}\,, \tag{4.88}$$

where the lattice momenta are such that $(p_L^2 - p_R^2)/k^2 \in 2\mathbb{Z}$. Using $SO(4,4)$ triality this corresponds to the full spinor orbit S characterizing the $\frac{1}{4}$-BPS orbits as described in section 3.4.1. In the weak coupling regime $y_6 \to \infty$ these Fourier modes take the form

$$F^{Spin(5,5)\,\alpha_1}_{\alpha_1;\frac{5}{2}}(N_1) \;\sim\; \frac{8\pi}{y_6^{\frac{3}{4}}} \sum_{p\,|\,\gcd(N_1)} p^2\,\sigma_{-1}\!\left(\frac{|p_L^2-p_R^2|}{2p^2}\right) \times \\ \times\; \frac{e^{-2\pi y_6^{-\frac{1}{2}}\sqrt{p_L^2+p_R^2+|p_L^2-p_R^2|}}}{(p_L^2+p_R^2+|p_L^2-p_R^2|)^{\frac{3}{4}}}\,. \tag{4.89}$$

In summary, the non-zero Fourier modes of $\mathcal{E}^{(6)}_{(0,0)}$ have support on the $\frac{1}{2}$-BPS orbit in limits (i), (ii) and (iii). One of the contributions to $\mathcal{E}^{(6)}_{(1,0)}$ is the regularised series $E^{Spin(5,5)}_{\alpha_1;s}$ at $s = 5/2$. This has non-zero Fourier modes with support on the $\frac{1}{2}$-BPS orbit in limits (i) and (iii), but on both the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS orbits in limit (ii). Although we have not computed the modes for the other contribution to $\mathcal{E}^{(6)}_{(1,0)}$ – the spinor series – we do know its orbit content by use of techniques similar to those in section 6.2. The result is that the non-zero Fourier modes of this series have support on the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS orbits in limits (i) and (iii), but only on the $\frac{1}{2}$-BPS orbit in limit (ii). In other words the complete coefficient $\mathcal{E}^{(6)}_{(1,0)}$ has the expected content of both the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS in its non-zero Fourier modes in all three limits.

5. The next to minimal (NTM) representation

This section contains the proof of theorem 2.14, drawing on some results in representation theory that can be found in appendix A by Ciubotaru and Trapa. As we remarked just before its statement, cases (i) and (ii) are by now well known, and so we restrict our attention to case (iii): the $s = 5/2$ series. To set some terminology, let $G = NAK$ be the Iwasawa decomposition of the split real Lie group G, B the minimal parabolic subgroup of G containing NA, and $\mathfrak{a}_{\mathbb{C}} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the Lie algebra of A. Without any loss of generality we may assume it is the complex span of the Chevalley basis vectors H_α, where α ranges over the positive simple roots. For any $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, the dual space of complex valued linear functionals on $\mathfrak{a}_{\mathbb{C}}$, define the vector space of functions on G

$$V_\lambda \;:=\; \Big\{ f : G \to \mathbb{C} \,|\, f(nag) \;=\; e^{(\lambda+\rho)(H(a))} f(g),\, \forall\; n \in N, a \in A, g \in G \Big\} . \tag{5.1}$$

The transformation law and Iwasawa decomposition show that all functions in V_λ are determined by their restriction to K. Then G acts on V_λ by the right translation operator

$$(\pi_\lambda(h) f)\,(g) \;:=\; f(gh)\,, \tag{5.2}$$

making (π_λ, V_λ) into a representation of G commonly called a *(nonunitary) principal series* representation. It is irreducible for λ in an open dense subset of $\mathfrak{a}^*_{\mathbb{C}}$, but reduces at special points with certain integrality properties – such as the ones of interest to us. The representation V_λ has a unique K-fixed vector up to scaling, namely any function whose restriction to K is constant. These are also known as the *spherical* vectors of the representation, and any representation which contains them is also called "spherical". When V_λ is reducible, it clearly can have at most one irreducible spherical subrepresentation.

The minimal parabolic Eisenstein series is defined as

$$E^G(\lambda, g) \;=\; \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} e^{(\lambda+\rho)(H(\gamma g))}\,, \tag{5.3}$$

initially for λ in Godement's range $\{\lambda | \langle \lambda, \alpha \rangle > 1 \text{ for all } \alpha \in \Sigma\}$, and then by meromorphic continuation to $\mathfrak{a}^*_{\mathbb{C}}$. When λ has the form $\lambda = 2s\omega_\beta - \rho$, it specializes to the maximal parabolic Eisenstein series (2.12). For generic λ in the range of convergence, the right translates of $E^G(\lambda, g)$ span a subspace of functions on $G(\mathbb{Z})\backslash G(\mathbb{R})$ which furnish a representation of G that is equivalent to V_λ; the group action here is also given by the right translation operator (5.2). The spherical vectors in this representation are the scalar multiplies of $E^G(\lambda, g)$, because the function $H(g)$ – the logarithm of the Iwasawa A-component – is necessarily right invariant under K. For general λ at which $E^G(\lambda, g)$ is holomorphic, its right translates span a spherical subrepresentation of V_λ, again with the group action given by the right translation operator (5.2).

As mentioned above, the principal series V_λ reduces for special values of λ. This reducibility reflects special behavior of the Eisenstein series $E^G(\lambda, g)$. This is most apparent at the point $\lambda = -\rho$, where the transformation law (5.1) indicates that the constant functions on K extend to constants on G, and hence that the trivial representation is a subrepresentation of $V_{-\rho}$. Likewise, the specialization of the minimal parabolic Eisenstein series at $\lambda = -\rho$ is the constant function identically equal to 1, a compatible fact.

The proof of theorem 2.14 rests upon special properties of the spherical constituent (i.e., Jordan-Hölder composition factor) of V_λ at the values of λ relevant to the $s = 5/2$ Epstein series. We recall that for this maximal parabolic Eisenstein series, λ has the form $\lambda = 2s\omega_{\alpha_1} - \rho$; it is characterized by having inner product $2s - 1$ with α_1, and inner product -1 with each α_j, $j \geq 2$. Write λ_{dom} for the dominant weight in the Weyl orbit of λ, i.e., one whose inner product with all positive roots is nonnegative. Table 6 gives dominant weights for the groups in Theorem 2.14 as well as its three values of $s \in \{0, 3/2, 5/2\}$, although of course only the last value is of immediate relevance in this section.

The case of $G = E_6$ is slightly easier than the others because of a low-dimensional coincidence, which in fact is mostly independent of the actual value of s in that the same statement holds for generic s. Namely, the representation V_λ we consider is part of a family of degenerate principal series representations, induced from the trivial representation on the semisimple $Spin(5,5)$ factor of the Levi component $GL(1) \times Spin(5,5)$ of the maximal parabolic subgroup P_{α_1}. These representations are indexed by the one dimensional family $\lambda = 2s\omega_{\alpha_1} - \rho$, $s \in \mathbb{C}$, which is related to the $GL(1)$ factor. Though they may reduce at particular points, their Gelfand-Kirillov dimension[26] is equal to the dimension of the unipotent radical of that parabolic, 16; likewise, any subrepresentation of it cannot have larger dimension. Since the dimension of the wavefront set of a representation is twice the Gelfand-Kirillov dimension, it is bounded by 32. For E_6, the orbits in figure 2 have dimensions 0, 22, and 32; all other orbits have larger Gelfand-Kirillov dimension. Hence the orbit attached to the $s = 5/2$ Eisenstein series for E_6 is either the trivial orbit, the minimal orbit, or the next-to-minimal orbit. It cannot be the trivial orbit, because only the trivial representation is attached to it. Likewise, Kazhdan-Savin [16] proved a uniqueness statement for the minimal orbit, that (up to Weyl equivalence) only the $s = 3/2$ series

[26]The Gelfand-Kirillov dimension is a numerical index of how "large" a representation is; it is half the dimension of the associated coadjoint nilpotent orbit (i.e., the orbit whose closure is the wavefront set of the representation). For example, finite dimensional representations have Gelfand-Kirillov dimension equal to zero.

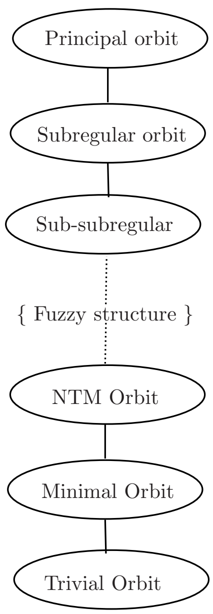
	$G = E_6$	$G = E_7$	$G = E_8$
$s = 0$			
λ_{dom} s_{GRS} z_{KS}	[1,1,1,1,1,1]	[1,1,1,1,1,1,1]	[1,1,1,1,1,1,1,1]
$s = 3/2$			
λ_{dom} s_{GRS} z_{KS}	[1,1,1,0,1,1] 1/4 7/22	[1,1,1,0,1,1,1] 5/18 11/34	[1,1,1,0,1,1,1,1] 19/58 19/58
$s = 5/2$			
λ_{dom} s_{GRS} z_{KS}	[0,1,1,0,1,1] -1/2 none	[1,1,1,0,1,0,1] 1/18 33/34	[1,1,1,0,1,0,1,1] 11/58 11/58

TABLE 6. The values of λ for the three values of s and three groups in theorem 2.14. Weights $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ are listed here in terms of their inner products with the positive simple roots as $[\langle \lambda, \alpha_1 \rangle, \langle \lambda, \alpha_2 \rangle, \ldots]$. For comparison with [10,16], we have listed the parameters s_{GRS} (the quantity s on [10, p.71]) and z_{KS} (the quantify $z(G)$ from [10, p.86]) for $s = 3/2$, as well as their corresponding generalizations for $s = 5/2$. These parameters coincide for the group E_8. The parameter z_{KS} is not defined in the $s = 5/2$ case for E_6 because the relevant Weyl orbits do not intersect (cf. [1, Section 3.1]).

is related to the minimal representation. We thus conclude it is attached to the next-to-minimal orbit.

To explain the $s = 5/2$ cases for E_7 and E_8 we need to rely on some recent results from representation theory, and some notions from there concerning *unipotent* and *special unipotent* representations (see appendix A). A striking feature from table 6 is that $\langle \lambda_{\text{dom}}, \alpha_j \rangle$ has all 1's except for a single zero for the $s = 3/2$ case, and two zeroes for the $s = 5/2$ case. This phenomenon, which came up here because of physical arguments, also arose in work on special unipotent representations. These λ_{dom} each have the property that there exists an element H of the Cartan subalgebra of $\mathfrak{g}$ such that $[H, X_\alpha] = \langle \lambda_{\text{dom}}, \alpha_j \rangle X_{\alpha_j}$ for each positive simple root α_j. Furthermore, there exists a homomorphism from $\mathfrak{sl}_2$ to $\mathfrak{g}$ carrying $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ to H, and $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ to a nilpotent element X. Thus a "dual" coadjoint nilpotent orbit, namely the one containing X, is associated to λ_{dom}. In terms of figure 5, in our three examples these related dual orbits are the top three listed, though in the reverse order. Appendix A describes a related construction for more general types of orbits beyond the ones considered in this paper.

As part of the more general result given in appendix A, corollary A.6 then asserts that the spherical constituent of each of the three principal



Marking of Orbit

E_6	E_7	E_8
[1,1,1,1,1,1]	[1,1,1,1,1,1,1]	[1,1,1,1,1,1,1,1]
[1,1,1,0,1,1]	[1,1,1,0,1,1,1]	[1,1,1,0,1,1,1,1]
[0,1,1,0,1,1]	[1,1,1,0,1,0,1]	[1,1,1,0,1,0,1,1]
[1,0,0,0,0,1] dim 32	[0,0,1,0,0,0,0] dim 52	[1,0,0,0,0,0,0,0] dim 92
[0,1,0,0,0,0] dim 22	[0,0,0,0,0,0,1] dim 34	[0,0,0,0,0,0,0,1] dim 58
[0,0,0,0,0,0] dim 0	[0,0,0,0,0,0,0] dim 0	[0,0,0,0,0,0,0,0] dim 0

FIGURE 5. The largest and smallest orbits, with markings (also known as "weightings") listed in terms of the inner products $\langle \lambda_{\text{dom}}, \alpha_j \rangle$ described in the text.

series $V_{\lambda_{\text{dom}}}$ has wavefront set equal to the closure of the dually related orbit listed in figure 5. This proves theorem 2.14 for E_7 and E_8.

6. Fourier coefficients and their vanishing

6.1. **Dimensions of orbits in the character variety.** In sections 3.3-3.5 we listed a number of explicit features of the orbits of instantons for the parabolic subgroups P_{α_1}, P_{α_2}, and $P_{\alpha_{d+1}}$ (in the numbering of figure 1). These are the character variety orbits discussed at the beginning of section 4.1. In this section we give more details, in particular basepoints and dimensions for each of the finite number of orbits under the complexification $L_{\mathbb{C}}$ of the Levi factor of the parabolic. As shorthand, we will refer to these as the "complex orbits of the Levi". We shall also use the notation Y_α to refer to the root vector $X_{-\alpha}$, in order to keep the listing of basepoints more readable.

This information is quoted from the paper [33], which lists the corresponding information for any maximal parabolic subgroup of any Chevalley group, whether classical or exceptional (see [33, §5] for more examples and details of how these are computed). We also describe the group action of the Levi in some of the cases, the rest being described in [33]. Recall that

Group	dimensions									
$SL(2)$	0	1	-	-	-	-	-	-	-	-
$SL(3)\times SL(2)$	0	2	-	-	-	-	-	-	-	-
$SL(5)$	0	4	-	-	-	-	-	-	-	-
$Spin(5,5)$	0	7	8	-	-	-	-	-	-	-
E_6	0	11	16	-	-	-	-	-	-	-
E_7	0	16	25	31	32	-	-	-	-	-
E_8	0	22	35	43	44	50	54	59	63	64

TABLE 7. Dimensions of character variety orbits for the Levi component of the parabolic formed by deleting the first node of $E_4 = SL(5)$, $E_5 = Spin(5,5)$, E_6, E_7, and E_8. A dash, $-$, signifies that there is no orbit. The character variety orbits in this parabolic subgroup are the $Spin(d,d)$ spinor orbits listed in section 3.4.1.

Group	dimensions									
$SL(2)$	0	-	-	-	-	-	-	-	-	-
$SL(3)\times SL(2)$	0	1	-	-	-	-	-	-	-	-
$SL(5)$	0	4	-	-	-	-	-	-	-	-
$Spin(5,5)$	0	7	10	-	-	-	-	-	-	-
E_6	0	10	15	19	20	-	-	-	-	-
E_7	0	13	20	21	25	26	28	31	34	35
E_8	0	16	25	28	31	32	35	38	40	$\cdots$

TABLE 8. Dimensions of character variety orbits of the Levi component for the parabolic formed by deleting the second node of $E_4 = SL(5)$, $E_5 = Spin(5,5)$, E_6, E_7, and E_8. A dash, $-$, signifies that there is no orbit. Not all E_8 orbits are listed (there are 23 in total).

the dimensions of the character varieties were given earlier in table 3 on page 21. In the following subsections, we expand upon this for the groups $E_5 = Spin(5,5)$, E_6, E_7, and E_8. For ease of reference, tables 7, 8, and 9 give the orbit dimensions for the parabolic subgroups P_{α_1}, P_{α_2}, and $P_{\alpha_{d+1}}$ of each of these groups, respectively.

6.1.1. $Spin(5,5)$. Recall that we label our $E_5 = Spin(5,5)$ Dynkin diagram according to the numbering in figure 1. This does not match the customary numbering of the $Spin(5,5)$ Dynkin diagram, but has the advantage of allowing for a uniform discussion of all of our cases of interest.

Node 1 is the so-called "vector" node, because P_{α_1} has Levi component isomorphic to $GL(1)\times Spin(4,4)$, which acts on the 8-dimensional, abelian unipotent radical by the 8-dimensional spin representation of $Spin(4,4)$. This action breaks into 3 complex orbits: the trivial orbit; a 7-dimensional

Group	dimensions				
$SL(2)$	0	-	-	-	-
$SL(3) \times SL(2)$	0	1	3	-	-
$SL(5)$	0	5	6	-	-
$Spin(5,5)$	0	7	10	-	-
E_6	0	11	16	-	-
E_7	0	17	26	27	-
E_8	0	28	45	55	56

Table 9. Dimensions of character variety orbits of the Levi component for the parabolic formed by deleting the last node of $E_4 = SL(5)$, $E_5 = Spin(5,5)$, E_6, E_7, and E_8. A dash, $-$, signifies that there is no orbit. The character variety orbits in this parabolic subgroup were also listed in table 5 based on enumeration of instanton orbits.

orbit with basepoint Y_{α_1}; and the open, dense 8-dimensional orbit with basepoint $Y_{11110} + Y_{10111}$ (see table 7).

Nodes 2 and 5 are the "spinor nodes", and have identical orbit structure (up to relabeling the nodes). Here the Levi component of P_{α_2} or P_{α_5} is now isomorphic to $GL(1) \times SL(5)$, and acts on the 10-dimensional abelian unipotent radical by the second fundamental representation, also known as the exterior square representation. In other words, the action of the $SL(5)$ piece is equivalent to that on antisymmetric 2-tensors $x \wedge y = -y \wedge x$, where x and y are 5-dimensional vectors. This action also has 3 complex orbits (which can be seen as part of a general description for abelian unipotent radicals of maximal parabolic subgroups given in [54]): the trivial orbit; a 7-dimensional orbit with basepoint Y_{α_2} in the case of node 2, and Y_{α_5} in the case of node 5; and the open, dense 10-dimensional orbit with basepoint $Y_{01121} + Y_{11111}$ (see table 8 or table 9). This last basepoint is in the open dense orbit for either P_{α_2} or P_{α_5}.

6.1.2. E_6. Nodes 1 and 6 are related by an automorphism of the Dynkin diagram, and have identical orbit structure (up to relabeling the nodes). Here the Levi component is isomorphic to $GL(1) \times Spin(5,5)$, which acts on the 16-dimensional, abelian unipotent radical by the spin representation of $Spin(5,5)$. There are three complex orbits: the trivial orbit; an 11-dimensional orbit with basepoint Y_{α_1} in the case of node 1, and Y_{α_6} in the case of node 6; and the open, dense 16-dimension orbit with basepoint $Y_{111221} + Y_{112211}$ for either nodes 1 or 6 (see table 7 or table 9).

Node 2 is the first case we encounter with a non-abelian unipotent radical. It is instead a 21-dimensional Heisenberg group, and its character variety has 5 complex orbits (another general fact for Heisenberg unipotent radicals of maximal parabolic subgroups [55]): the trivial orbit; a 10-dimensional orbit with basepoint α_2; a 15-dimensional orbit with basepoint $Y_{111221}+Y_{112211}$; a

19-dimensional orbit with basepoint $Y_{011221}+Y_{111211}+Y_{112210}$; and the open, dense 20-dimensional orbit with basepoint $Y_{010111}+Y_{112210}$ (see table 8).

6.1.3. E_7. This is the first group for which the three nodes have mathematically different structures. Node 1 has a 33-dimensional unipotent radical which is a Heisenberg group, and Levi component isomorphic to $GL(1)\times Spin(6,6)$. The action on the 32-dimensional character variety again has 5 complex orbits: the trivial orbit; a 16-dimensional orbit with basepoint Y_{α_1}; a 25-dimensional orbit with basepoint $Y_{1123321}+Y_{1223221}$; a 31-dimensional orbit with basepoint $Y_{1122221}+Y_{1123211}+Y_{1223210}$; and the open, dense 32-dimensional orbit with basepoint $Y_{1011111}+Y_{1223210}$ (see Table 7).

Node 2 has a 42-dimensional unipotent radical, and a 35-dimensional character variety. The Levi component $GL(1)\times SL(7)$ acts with 10 complex orbits: the trivial orbit; a 13-dimensional orbit with basepoint Y_{α_2}; a 20-dimensional orbit with basepoint $Y_{1122221}+Y_{1123211}$; a 21-dimensional orbit with basepoint $Y_{0112221}+Y_{1112211}+Y_{1122111}$; a 25-dimensional orbit with basepoint $Y_{1112221}+Y_{1122211}+Y_{1123210}$; a 26-dimensional orbit with basepoint $Y_{1111111}+Y_{1123210}$; a 28-dimensional orbit with basepoint $Y_{0112221}+Y_{1112211}+Y_{1122111}+Y_{1123210}$; a 31-dimensional orbit with basepoint $Y_{0112221}+Y_{1111111}+Y_{1123210}$; a 34-dimensional orbit with basepoint $Y_{0112211}+Y_{1112111}+Y_{1112210}+Y_{1122110}$; and the open, dense 35-dimensional orbit with basepoint $Y_{0112111}+Y_{0112210}+Y_{1111111}+Y_{1112110}+Y_{1122100}$ (see table 8).

Node 7 has a 27-dimensional abelian unipotent radical, and Levi component isomorphic to $GL(1)\times E_{6,6}$. The latter acts with 4 complex orbits: the trivial orbit, a 17-dimensional orbit with basepoint Y_{α_7}, a 26-dimensional orbit with basepoint $Y_{1123321}+Y_{1223221}$, and the open, dense 27-dimensional orbit with basepoint $Y_{0112221}+Y_{1112211}+Y_{1122111}$ (see Table 9).

6.1.4. E_8. This is the largest of our groups, and the unipotent radicals of its maximal parabolics are never abelian.

Node 1 has a 78-dimensional unipotent radical, and a 64-dimensional character variety. The Levi component is isomorphic to $GL(1)\times Spin(7,7)$ and acts on the character variety according to the spin representation of $Spin(7,7)$, with 10 complex orbits: the trivial orbit; a 22-dimensional orbit with basepoint Y_{α_1}; a 35-dimensional orbit with basepoint $Y_{12244321}+Y_{12343321}$; a 43-dimensional orbit with basepoint $Y_{12233321}+Y_{12243221}+Y_{12343211}$; a 44-dimensional orbit with basepoint $Y_{11122221}+Y_{12343211}$; a 50-dimensional orbit with basepoint $Y_{11233321}+Y_{12233221}+Y_{12243211}+Y_{12343210}$; a 54-dimensional orbit with basepoint $Y_{11222221}+Y_{12243211}+Y_{12343210}$; a 59-dimensional orbit with basepoint $Y_{11122221}+Y_{11233211}+Y_{12232211}+Y_{12343210}$; a 63-dimensional orbit with basepoint $Y_{11222221}+Y_{11232211}+Y_{11233210}+Y_{12232111}+Y_{12232210}$; and the open, dense 64-dimensional orbit with basepoint $Y_{11122111}+Y_{11221111}+Y_{11233210}+Y_{12232210}$ (see table 7).

Node 2 has a 92-dimensional unipotent radical, and a 56-dimensional character variety. The Levi component is isomorphic to $GL(1)\times SL(8)$

and acts according to the third fundamental representation of $SL(8)$, also known as the exterior cube representation. It acts with 23 complex orbits, the four smallest of which are: the trivial orbit; a 16-dimensional orbit with basepoint Y_{α_2}; a 25-dimensional orbit with basepoint $Y_{11232221} + Y_{11233211}$; and a 28-dimensional orbit with basepoint $Y_{11122221} + Y_{11222211} + Y_{11232111}$ (see table 8).

Node 8 has a 57 dimensional unipotent radical which is a Heisenberg group. The Levi factor is isomorphic to $GL(1) \times E_{7,7}$ and acts with 5 complex orbits on the 56-dimensional character variety: the trivial orbit; a 28-dimensional orbit with basepoint Y_{α_8}; a 45-dimensional orbit with basepoint $Y_{22454321} + Y_{23354321}$; a 55-dimensional orbit with basepoint $Y_{12244321} + Y_{12343321} + Y_{22343221}$; and the open, dense 56-dimensional orbit with basepoint $Y_{01122221} + Y_{22343211}$ (see table 9).

6.2. **Applications of Matumoto's theorem.** In Section 2.2.2 we mentioned that representations of real groups have an invariant attached to them, the wavefront set, that in a sense measures how big the representation is. Theorem A.5 indeed computes this wavefront set in many cases, including ours. There is a theorem due to Matumoto [53] that asserts, in a precise sense, that automorphic forms in small representations cannot have large Fourier coefficients. Namely, he proves that if an element $Y \in \mathfrak{u}_{-1}$ associated to the character χ from (4.6) does not lie in the wavefront set, then the Fourier coefficient ϕ_χ from (4.1) must vanish identically. We will use real group methods here in deference to the importance of the underlying symmetry groups $E_{d+1}(\mathbb{R})$, but it is notable that we could obtain the same results using p-adic methods via a vanishing result of Mœglin-Waldspurger [56]. Related information is given at the end of appendix A.

For example, the trivial representation has wavefront set $\{0\}$, and likewise the constant function does not have any nontrivial Fourier coefficients. In [33] a detailed analysis is given of the different character variety orbits for each maximal parabolic subgroup of an exceptional group, and which coadjoint nilpotent orbits they are contained in. It is then a simple matter to apply Matumoto's theorem and determine a set of Fourier coefficients which automatically vanishes because their containing coadjoint nilpotent orbits lie outside the wavefront set. In particular, it is shown in [33] that the closure of the minimal coadjoint nilpotent orbit contains the two smallest character variety orbits in each of the examples of P_{α_1}, P_{α_2}, and $P_{\alpha_{d+1}}$ for the groups E_{d+1}, $5 \leq d \leq 7$, but no others (this was known to experts, at least in special cases – see for example [10]). Likewise, it is also verified there that the closure of the next-to-minimal coadjoint nilpotent orbit contains the three smallest character variety orbits in each of these nine configurations of maximal parabolics and E_{d+1} groups, but no others.

Combining this with the characterization in Theorem 2.14 of the wavefront sets for the Epstein series at $s = 0$, $3/2$, and $5/2$, we get the following statement about the vanishing of Fourier coefficients. This gives a rigorous proof of the vanishing statements on page 6.

Theorem 6.1. *Let $5 \leq d \leq 7$ and $G = E_{d+1}$ as defined in table 1 on page 4. Then:*

(i) *All Fourier coefficients of the $s = 0$ Epstein series vanish in any of the parabolics P_{α_1}, P_{α_2}, or $P_{\alpha_{d+1}}$, with the exception of the constant terms (which were calculated in [1]).*

(ii) *All Fourier coefficients of the $s = 3/2$ Epstein series $E^G_{\alpha_1;3/2}$ vanish in any of the parabolics P_{α_1}, P_{α_2}, or $P_{\alpha_{d+1}}$, with the exceptions of the constant term and the smallest dimensional character variety orbit. This orbit has: dimension 11 for E_6 and either P_{α_1} or P_{α_6}, and dimension 10 for P_{α_2}; dimensions 16, 13, and 17 for E_7 and P_{α_1}, P_{α_2}, and P_{α_7}, respectively; and dimensions 22, 16, and 28 for E_8 and P_{α_1}, P_{α_2}, and P_{α_8}, respectively.*

(iii) *All Fourier coefficients of the $s = 5/2$ Epstein series $E^G_{\alpha_1;5/2}$ vanish in any of the parabolics P_{α_1}, P_{α_2}, or $P_{\alpha_{d+1}}$, with the exceptions of the constant term and the next two smallest dimensional character variety orbits. This additional character variety orbit is: the 16, 15, and 16-dimensional orbit for E_6 and P_{α_1}, P_{α_2}, and P_{α_6}, respectively; the 25, 20, and 26-dimensional orbit for E_7 and P_{α_1}, P_{α_2}, and P_{α_7}, respectively; and the 35, 25, and 45-dimensional orbit for E_8 and P_{α_1}, P_{α_2}, and P_{α_8}, respectively.*

7. Square integrability of special values of Eisenstein series

In this section we remark that some of the coefficient functions $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ from the expansion (2.3) provide examples of square-integrable automorphic forms on higher rank groups. In particular, we will prove this is the case for $\mathcal{E}^{(D)}_{(1,0)}$ on E_7 and E_8. In light of (1.3), this proves the associated automorphic representation is unitary, since it can be realized in the Hilbert space $L^2(E_{d+1}(\mathbb{Z})\backslash E_{d+1}(\mathbb{R}))$. This unitary can also be demonstrated by purely representation theoretic methods. It is an instance of broader conjectures of James Arthur on unitary automorphic representations, which are studied in more detail in [57]. This fact about the residual L^2 spectrum is at present more of a curiosity as far as our investigations here are concerned, since we are not aware of any particular importance for our applications. The analysis in the proof also determines the exact asymptotics of these coefficients in various limits, generalizing those studied in [1].

Theorem 7.1. *Let G denote the group E_{d+1} defined in table 1 on page 4.*

(i) *The Epstein series $E^G_{\alpha_1;0}$ is constant, and hence always square-integrable.*

(ii) *The Epstein series $E^G_{\alpha_1;3/2}$ and hence $\mathcal{E}^{(10-d)}_{(0,0)}$ is square-integrable if $4 \leq d \leq 7$.*

(iii) *The Epstein series* $E^G_{\alpha_1;5/2}$ *and hence* $\mathcal{E}^{(10-d)}_{(1,0)}$ *is square-integrable if* $6 \le d \le 7$.

Case (i) is obvious since the quotient $E_{d+1}(\mathbb{Z})\backslash E_{d+1}(\mathbb{R})$ has finite volume, while case (ii) was proven earlier by [10]. We have included them here in the statement for convenience and comparison. It should be stressed, though, that $E^G_{\alpha_1;s}$ is certainly not square integrable for *general* s. The same method treats the lower rank groups as well, though since the statements are not needed here we refer to papers [10] and [58] for $Spin(5,5)$.

Proof. Recall that the series $E^G_{\alpha_1;s}$ is a specialization of the *minimal parabolic* Eisenstein series $E^G(\lambda, g)$ from (5.3) at $\lambda = 2s\omega_1 - \rho$. This is explained in our context in [1, Section 2], where Langlands' constant term formula is also given in Theorem 2.18. The latter shows that the constant term of $E^G(\lambda, g)$ along any maximal parabolic subgroup P is a sum of other minimal parabolic Eisenstein series on its Levi component. By induction, this is also true if P is not maximal. In particular, since these Eisenstein series on smaller groups are orthogonal to all cusp forms on those groups, the constant terms are therefore orthogonal to all cusp forms on the Levi components – a meaningful statement only, of course, when the parabolic P is not the Borel subgroup B (so that the Levi is nontrivial). This means $E^G(\lambda, g)$ has "zero cuspidal component along any such P" in the sense of [59, Section 3], or equivalently that it is "concentrated" on the Borel subgroup B.

The constant term along B is explicitly given in terms of a sum over the Weyl group:

$$\int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E^G(\lambda, ng)\, dn \quad = \quad \sum_{w\,\in\,\Omega} e^{(w\lambda+\rho)(H(g))} M(w,\lambda)\,, \tag{7.2}$$

where $M(w,\lambda)$ is given by the explicit product over roots whose sign is flipped by w,

$$M(w,\lambda) \quad = \quad \prod_{\substack{\alpha\,>\,0\\ w\alpha\,<\,0}} c(\langle\lambda,\alpha\rangle)\,, \tag{7.3}$$

with

$$c(s) \quad := \quad \frac{\xi(s)}{\xi(s+1)} \qquad \text{and} \qquad \xi(s) \quad := \quad \pi^{-\frac{s}{2}}\,\Gamma(\tfrac{s}{2})\,\zeta(s) \tag{7.4}$$

(see, for example, [1, (2.16)-(2.21)]). This formula is valid for generic λ, and develops logarithmic terms at special points via meromorphic continuation. Moreover, certain coefficients $M(w,\lambda)$ may vanish, for example when $\langle\lambda,\alpha\rangle = -1$ and the respective factor in (7.4) has a zero owing to the pole of $\xi(s+1)$ at $s=-1$ (unless it is canceled by a pole from another factor). Indeed, $c(s)$ has a simple zero at $s=-1$, a simple pole at $s=1$, and is holomorphic at all other integers. Because $E^G(\lambda, g)$ is "concentrated on B", Langlands' criteria in [59, Section 5] asserts that it is square-integrable if and only if the surviving exponents $w\lambda$ have negative inner product with each fundamental weight:

$$\langle w\lambda, \omega_\alpha \rangle \; < \; 0 \quad \text{for each } \alpha > 0\,. \tag{7.5}$$

The rest of the proof involves an explicit calculation to check that for each possible value of $w\lambda$, either the sum of $e^{(w'\lambda+\rho)(H(g))}M(w',\lambda)$ over all $w' \in \Omega$ with $w'\lambda = w\lambda$ vanishes, or instead that (7.5) holds. Actually, despite the enormous size of the Weyl groups involved, $M(w,\lambda)$ vanishes for all but very few w (because of the special nature of λ).

Though the individual terms in (7.2) are frequently singular at the values of λ in question, the overall sum can be calculated explicitly by taking limits. We now present the result of this calculation. To make the condition (7.5) more transparent, we take $g = a$ to be an element of the maximal torus A (as we of course may, given that $H(g)$ depends only on the A-component of g's Iwasawa decomposition). We then furthermore parameterize a by real numbers $r_1, r_2, \ldots$ via the condition that the simple roots on a take the values

$$a^{\alpha_1} \; = \; e^{r_1}\,, \; a^{\alpha_2} \; = \; e^{r_2}\,, \; \ldots. \tag{7.6}$$

For example, for $G = E_6$ the limiting value of (7.2) as λ approaches $3\omega_1 - \rho$ can be calculated explicitly as $e^{2r_1+3r_2+4r_3+6r_4+4r_5+2r_6}$ times

$$\frac{3\zeta(3)\left(e^{2r_1+r_3} + e^{r_5+2r_6}\right) + \pi^2(e^{r_2} + e^{r_3} + e^{r_5}) + 6\pi(r_4 + \gamma - \log(4\pi))}{3\zeta(3)}\,. \tag{7.7}$$

The exponentials are all dominated by $e^{\rho(H(g))} = e^{8r_1+11r_2+15r_3+21r_4+15r_5+8r_6}$ for $r_i > 0$, that is, (7.5) holds and hence $E^G_{\alpha_1;3/2}$ is square-integrable – verifying a fact proven in [10].

We now turn to the two new cases, those of the $s = 5/2$ series for E_7 and E_8. We recall the computational method of [1, Section 2.4] to find the minimal parabolic constant terms, namely to precompute the set

$$\mathcal{S} \; := \; \{ \, w \in \Omega \; | \; w\alpha_i > 0 \quad \text{for all } \; i \neq 1 \, \}\,. \tag{7.8}$$

For $w \notin \mathcal{S}$, $M(w,\lambda)$ will include the factor $c(\langle \lambda, \alpha_i \rangle) = c(\langle 2s\omega_1 - \rho, \alpha_i \rangle) = c(-\langle \rho, \alpha_i \rangle) = c(-1) = 0$ for some $i > 1$. At the same time, at least for $\operatorname{Re} s < \frac{1}{2}$, all inner products $\langle \lambda, \alpha \rangle$ will be negative, and hence none of the other factors in (7.3) can have a pole (after all, $c(s)$ is holomorphic for $\operatorname{Re} s < 0$). Thus the term for w in (7.2) vanishes identically in s by analytic continuation, and the sum in (7.2) reduces to one over $w \in \mathcal{S}$.

For E_7 there are only 126 elements in $\mathcal{S}$ out of the 2,903,040 elements of the full Weyl group Ω. It can be calculated that all but three w of these 126 satisfy Langlands' condition (7.5), and the three that do not have the following expressions for $M(w,\lambda)$ for $s = 5/2+\varepsilon$:

$$\begin{aligned}\text{Exception 1}:\ & c(2(\epsilon-5))c(2\epsilon)^2c(2\epsilon-9)c(2\epsilon-8)^2c(2\epsilon-7)^2c(2\epsilon-6)^3\ \times\\ & \times\ c(2\epsilon-5)^3c(2\epsilon-4)^3c(2\epsilon-3)^3c(2\epsilon-2)^3c(2\epsilon-1)^3\ \times\\ & \times\ c(2\epsilon+1)^2c(2\epsilon+2)c(2\epsilon+3)c(2\epsilon+4)c(4\epsilon-7)\,,\\ \text{Exception 2}:\ & c(2\epsilon)^2c(2\epsilon-9)c(2\epsilon-8)^2c(2\epsilon-7)^2c(2\epsilon-6)^3c(2\epsilon-5)^3\ \times\\ & \times\ c(2\epsilon-4)^3c(2\epsilon-3)^3c(2\epsilon-2)^3c(2\epsilon-1)^3c(2\epsilon+1)^2\ \times\\ & \times\ c(2\epsilon+2)c(2\epsilon+3)c(2\epsilon+4)c(4\epsilon-7)\,,\\ \text{Exception 3}:\ & c(2(\epsilon-5))c(2\epsilon)^2c(2\epsilon-11)c(2\epsilon-9)c(2\epsilon-8)^2c(2\epsilon-7)^2\ \times\\ & \times\ c(2\epsilon-6)^3c(2\epsilon-5)^3c(2\epsilon-4)^3c(2\epsilon-3)^3c(2\epsilon-2)^3\ \times\\ & \times\ c(2\epsilon-1)^3c(2\epsilon+1)^2c(2\epsilon+2)c(2\epsilon+3)c(2\epsilon+4)c(4\epsilon-7)\,.\end{aligned} \tag{7.9}$$

Each of these terms is in fact zero by dint of the triple zero from the term $c(2\varepsilon-1)^3$ counterbalancing the double pole from the term $c(2\varepsilon+1)^2$ at $\epsilon = 0$. (Incidentally, the overall series $E^G_{\alpha_1;5/2}$ was shown to be non-zero in [1] for both $G = E_7$ and $G = E_8$).

For E_8 there are 2160 elements in $\mathcal{S}$ out of the 696,729,600 elements of the full Weyl group Ω. Likewise, all but 258 of these 2160 w satisfy (7.5). Again, all 258 of these terms vanish at $s = 5/2$ because their products have a triple zero (coming from three $c(s)$ factors evaluated at near $s = -1$) that counterbalance two poles (coming from two $c(s)$ factors evaluated near $s = 1$).

□

8. Discussion and future problems

In this paper we have studied the Fourier modes of the Eisenstein series that define the coefficients of the first two nontrivial interactions in the low energy expansion of the four-graviton amplitude in maximally supersymmetric string theory compactified on T^d, and verified they have certain expected features. In particular, we have shown that their non-zero Fourier coefficients contain the expected minimal and next-to-minimal ($\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS) instanton orbits for any of the symmetry groups, E_{d+1} ($0 \le d \le 7$). This extends the analysis of these functions in [1], where the constant terms of these functions were shown to reproduce all the expected features of string perturbation theory and semi-classical M-theory. Furthermore, in low rank cases we were able to present the explicit Fourier coefficients of these functions and show that they have the form expected of BPS-instanton contributions. Indeed, the form of the $\frac{1}{2}$-BPS contributions match those deduced from string theory calculations as summarised by (4.10).

For high rank cases this involved a detailed analysis of the automorphic representations connected to these coefficients. Namely, we explained that they are automorphic realizations of the smallest two types of nontrivial representations of their ambient Lie groups, and why this property automatically implies the vanishing of a slew of Fourier coefficients – precisely the Fourier coefficients that the BPS condition ought to force to vanish. We furthermore showed the most interesting cases – those of the next-to-minimal representation for E_7 and E_8 – occur in $L^2(E_{d+1}(\mathbb{Z})\backslash E_{d+1}(\mathbb{R}))$.

This raises some obviously interesting questions, both from the string theory perspective and from the mathematical perspective.

An immediately interesting mathematical direction would be the explicit computation of the non-zero Fourier modes of $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ for the high rank cases with groups E_6, E_7 and E_8, in particular to get finer information using the work of Bhargava and Krutelevich on the integral structure of the character variety orbits. In a different direction, as mentioned in section 3.3.1 it would be of interest to extend the considerations of this paper to affine E_9 and behind that to hyperbolic extensions.[27]

Another question that is natural to ask in the context of string theory is to what extent does our analysis generalise to higher order interactions in the low energy expansion, which preserve a smaller fraction of supersymmetry? Could there be a role for Eisenstein series with other special values of the index s in the description of such terms? However, the evidence is that such higher order terms involve automorphic functions that are not Eisenstein series. For example, $\mathcal{E}^{(D)}_{(0,1)}$ (the coefficient of the $\frac{1}{8}$-BPS $\partial^6\,\mathcal{R}^4$ interaction) is expected to satisfy a particular inhomogeneous Laplace eigenvalue equation [6]. Although its constant term has, to a large extent, been analysed for the relevant values of D [1], it would be most interesting to analyse the non-zero Fourier modes of $\mathcal{E}^{(D)}_{(0,1)}$, which should describe the couplings of $\frac{1}{8}$-BPS instantons in the four-graviton amplitude for low enough dimensions, D. This should reveal a rich structure. For example, the instantons that contribute in the limit of decompactification from D to $D+1$ include the $\frac{1}{8}$-BPS black holes of $D+1$ dimensions, which can have non-zero horizon size and exponential degeneracy. It is not apparent at first sight whether this degeneracy should be encoded in the solutions of the inhomogeneous equation satisfied by $\mathcal{E}^{(D)}_{(0,1)}$. Indeed, we have seen in the $\frac{1}{4}$-BPS cases that the Fourier expansion of the coefficient function $\mathcal{E}^{(D)}_{(1,0)}$ in the decompactification limit does not determine the Hagedorn-like degeneracy of $\frac{1}{4}$-BPS small black holes in $D+1$ dimensions. Rather, the divisor sums weight particular combinations of charges and windings of the wrapped world-lines of such objects.

These issues involve mathematical challenges. For example, the study of inhomogeneous Laplace equations for the group $SL(2,\mathbb{R})$ heavily relies on explicit formulas for automorphic Green functions, which do not generalize

[27] After this paper was first posted on the arXiv the paper [60] by Fleig and Kleinschmidt appeared, which makes important steps in this direction.

in an obvious manner to higher rank groups because they involve automorphic Laplace eigenfunctions which do not have moderate growth in the cusps (at present the existence of such functions is itself an open problem).

Another issue is to what extent this analysis can be extended to discuss the automorphic properties of yet higher order terms in the expansion of the four-graviton amplitude. Further afield are issues concerning the extension of these ideas to multi-particle amplitudes, to amplitudes that transform as modular forms of non-zero weight, and extensions to processes with less supersymmetry.

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Appendix A. Special unipotent representations, by Dan Ciubotaru and Peter E. Trapa

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The representations considered in Theorem 2.14 are examples of a wider class of representations which have attracted intense attention in the mathematical literature. The purpose of this appendix is to recall certain results (from a purely local point of view) which are especially relevant for the discussion of Section 5.

To begin, let G denote a real reductive group arising as the real points of a connected complex algebraic group $G_{\mathbb{C}}$. In [Ar1] and [Ar2], Arthur set forth a conjectural description of irreducible (unitary) representations contributing to the automorphic spectrum of G. In many cases, these conjectures could be reduced to a fundamental set of representations attached to (integral) "special unipotent" parameters. In the real case, Arthur's conjectures — and, in particular, the definition of the corresponding special unipotent representations — are made precise and refined in the work of Barbasch-Vogan [BV1] and, more completely, in the work of Adams-Barbasch-Vogan [ABV]. The perspective of these references is entirely local. (Of course an extensive literature approaching Arthur's conjectures by global methods exists and, for classical groups, is summarized in [Ar3].) As we now explain, the representations appearing in Theorem 2.14 are indeed special unipotent in the sense of Adams-Barbasch-Vogan.

Write $\mathfrak{g}_{\mathbb{C}}$ for the Lie algebra of $G_{\mathbb{C}}$ and fix a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ arising as the Lie algebra of a maximal torus in $G_{\mathbb{C}}$. Write Ω for the Weyl group of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. The classification of connected reductive algebraic groups naturally leads from $G_{\mathbb{C}}$ to the Langlands dual $G_{\mathbb{C}}^{\vee}$, a connected reductive complex algebraic group, e.g. [Sp]. Let $\mathfrak{g}_{\mathbb{C}}^{\vee}$ denote the Lie algebra of $G_{\mathbb{C}}$. The construction of $G_{\mathbb{C}}^{\vee}$ includes the definition of a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}^{\vee}$ which canonically identifies with the linear dual of $\mathfrak{h}_{\mathbb{C}}$,

$$\mathfrak{h}_{\mathbb{C}}^{\vee} \simeq (\mathfrak{h}_{\mathbb{C}})^{*}. \tag{A.1}$$

Let $\mathcal{N}$ denote the cone of nilpotent elements in $\mathfrak{g}_{\mathbb{C}}$, and likewise let $\mathcal{N}^{\vee}$ denote the cone of nilpotent elements in $\mathfrak{g}_{\mathbb{C}}^{\vee}$. Write $G_{\mathbb{C}}\backslash\mathcal{N}$ and $G_{\mathbb{C}}^{\vee}\backslash\mathcal{N}^{\vee}$ for the corresponding sets of adjoint orbits. These sets are partially ordered by the inclusion of closures. Spaltenstein defined an order-reversing map

$$d \,:\, G_{\mathbb{C}}^{\vee}\backslash\mathcal{N}^{\vee} \longrightarrow G_{\mathbb{C}}\backslash\mathcal{N}$$

with many remarkable properties which were refined in [BV1, Appendix]; see Theorem A.4 below.

Example A.1. Suppose the Dynkin diagram corresponding to $\mathfrak{g}_{\mathbb{C}}$ is simply laced (as is the case for the groups E_{d+1} from figure 1 and table 1). Then

$\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{g}_{\mathbb{C}}^{\vee}$ and $G_{\mathbb{C}}^{\vee}$ and $G_{\mathbb{C}}$ are isogenous. Thus $G_{\mathbb{C}}^{\vee}\backslash\mathcal{N}^{\vee}$ can be identified with $G_{\mathbb{C}}\backslash\mathcal{N}$ and d can be viewed as an order reversing map from the latter set to itself. With this in mind, consider figure 5. The map d interchanges the top three orbits with the bottom three orbits (in an order reversing way, of course). In particular d applied to the sub-subregular orbit is the next to minimal orbit. The complete calculation of d is given in [Ca].

Fix an element $\mathcal{O}^{\vee}$ of $G_{\mathbb{C}}^{\vee}\backslash\mathcal{N}^{\vee}$. According to the Jacobson-Morozov Theorem, there exists a Lie algebra homomorphism

$$\phi \;:\; \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathfrak{g}_{\mathbb{C}}^{\vee}$$

such that the image under of ϕ of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ lies in $\mathcal{O}^{\vee}$ and

$$\phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{h}_{\mathbb{C}}^{\vee} \simeq \mathfrak{h}_{\mathbb{C}}^{*}, \tag{A.2}$$

with the last isomorphism as in (A.1).

The element in (A.2) depends on the choice of ϕ. Its Weyl group orbit is well-defined however (independent of how ϕ is chosen). So define

$$\lambda(\mathcal{O}^{\vee}) := (1/2)\, \phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{h}_{\mathbb{C}}^{*}/\Omega. \tag{A.3}$$

According to the Harish-Chandra isomorphism, $\lambda(\mathcal{O}^{\vee})$ specifies a maximal ideal $Z(\mathcal{O}^{\vee})$ in the center of the enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$. Recall that an irreducible admissible representation of G is said to have infinitesimal character $\lambda(\mathcal{O}^{\vee})$ if its Harish-Chandra module is annihilated by $Z(\mathcal{O}^{\vee})$.

A result of Dixmier implies that there is a unique primitive ideal $I(\mathcal{O}^{\vee})$ in $U(\mathfrak{g}_{\mathbb{C}})$ which is maximal among all primitive ideals containing $Z(\mathcal{O}^{\vee})$. (A primitive ideal in $U(\mathfrak{g}_{\mathbb{C}})$ is, by definition, a two-sided ideal which arises as the annihilator of a simple $U(\mathfrak{g}_{\mathbb{C}})$ module.) Given any two-sided ideal I in $U(\mathfrak{g}_{\mathbb{C}})$, we can consider the associated graded ideal $gr(I)$ with respect to the canonical grading on $U(\mathfrak{g}_{\mathbb{C}})$. According to the Poincaré-Birkhoff-Witt Theorem, $gr(I)$ is an ideal in $gr(U(\mathfrak{g}_{\mathbb{C}})) \simeq S(\mathfrak{g}_{\mathbb{C}})$, the symmetric algebra of $\mathfrak{g}_{\mathbb{C}}$, and hence cuts out a subvariety (the so-called associated variety, $\mathrm{AV}(I)$, of I) of $\mathfrak{g}_{\mathbb{C}}^{*}$.

It will be convenient to identify $\mathfrak{g}_{\mathbb{C}}$ with $\mathfrak{g}_{\mathbb{C}}^{*}$ (by means of the choice of an invariant form) and view $\mathrm{AV}(I)$ as a subvariety of $\mathfrak{g}_{\mathbb{C}}$. (The choice of form is well-defined up to scalar; since $\mathrm{AV}(I)$ is a cone, $\mathrm{AV}(I)$ becomes a well-defined subvariety of $\mathfrak{g}_{\mathbb{C}}$.) A theorem of Joseph [14] and Borho-Brylinski [BoBr1] (cf. the short proof in [V2]) implies that if I is primitive, $\mathrm{AV}(I)$ is indeed the closure of a single nilpotent orbit of $G_{\mathbb{C}}$.

Theorem A.4 ([BV1, Corollary A.3])**.** *In the setting of the previous paragraph,*

$$\mathrm{AV}(I(\mathcal{O}^{\vee})) = \overline{d(\mathcal{O}^{\vee})}.$$

Example A.2. Suppose $G_{\mathbb{C}}$ is simply laced and make identifications as in Example A.1. Suppose $\mathcal{O}^\vee$ is respectively the regular, subregular, or sub-subregular, orbit in figure 5. Then $\mathrm{AV}(I(\mathcal{O}^\vee))$ is the closure respectively of the zero, minimal, or next-to-minimal orbit.

Definition A.3 (Barbasch-Vogan [BV1])**.** Fix an orbit $\mathcal{O}^\vee$ as above. Suppose further that $\mathcal{O}^\vee$ is even or, equivalently, that $\lambda(\mathcal{O}^\vee)$ is integral. An irreducible admissible representation of G is said to be *(integral) special unipotent attached to* $\mathcal{O}^\vee$ if the annihilator of its Harish-Chandra module is $I(\mathcal{O}^\vee)$.

Note that since $I(\mathcal{O}^\vee)$ is a maximal primitive ideal, special unipotent representations are, in a precise sense, as small as possible.

Theorem A.5. *Suppose G is split and π is an irreducible spherical representation with infinitesimal character $\lambda(\mathcal{O}^\vee)$ (with notation as in* (A.3)*). Suppose further that $\mathcal{O}^\vee$ is even. Then π is special unipotent in the sense of Definition A.3.*

Sketch. Chapter 27 in [ABV] defines special unipotent Arthur packets. Roughly speaking, such a packet is parametrized by a rational form of an orbit $\mathcal{O}^\vee$ in $G_{\mathbb{C}}^\vee\backslash\mathcal{N}^\vee$ ([ABV, Theorem 27.10]). In the case that $\mathcal{O}^\vee$ is even, these packets are known to consist of representations appearing in Definition A.3 ([ABV, Corollary 27.13]). As a consequence of [ABV, Definition 22.6] (see also the discussion after [ABV, Definition 1.33]), such a packet also contains a (generally nontempered) L-packet. In the case at hand, the special unipotent Arthur packet parametrize by $\mathcal{O}^\vee$ contains the L-packet consisting of the spherical representation with infinitesimal character $\lambda(\mathcal{O}^\vee)$. This completes the sketch. □

Corollary A.6. *The spherical constituents of the principal series representations $V_{\lambda_{dom}}$ from section 5 are integral special unipotent attached to $\mathcal{O}^\vee$ (Definition A.3) where $\mathcal{O}^\vee$ is, respectively, the regular, subregular, and sub-subregular nilpotent orbit (all of which are even). According to Corollary A.4 and Example A.2, the wavefront sets of these representations are, respectively, the zero, minimal, and next to minimal orbits.*

Finally, we remark that since the special unipotent representation of Definition A.3 are predicted by Arthur to appear in spaces of automorphic forms, they should be unitary.

Conjecture A.7. *Suppose π is integral special unipotent in the sense of Definition A.3. Then π is unitary.*

The representations appearing in Theorem A.5 are known to be unitary if $G_{\mathbb{C}}$ is classical or of Type G_2. This was proved by purely local methods in [V1], [V2], and [B]. For a summary of results obtained by global methods, see [Ar3].

For completeness, we discuss the analogs of these results in the p-adic case. Let F be a p-adic field, with ring of integers $\mathfrak{O}$, and finite residue field F_q. The group G is now the F-points of a connected algebraic group $G_{\overline{F}}$ defined over $\overline{F}$. We assume for simplicity that G is split and of adjoint type. Let K be the $\mathfrak{O}$-points of $G_{\overline{F}}$, a maximal compact open subgroup of G. Let I be the inverse image in K under the natural projection $K \to G_{\overline{F}}(F_q)$ of a Borel subgroup over F_q. The compact open subgroup I is called an Iwahori subgroup.

The Iwahori-Hecke algebra $\mathcal{H}(G, I)$ is the convolution algebra (with respect to a fixed Haar measure on G) of compactly supported, locally constant, I-biinvariant complex functions on G. It is a Hilbert algebra, in the sense of Dixmier, with respect to the trace function $f \mapsto f(1)$, and the $*$-operation $f^*(g) = \overline{f(g^{-1})}$, $f \in \mathcal{H}(G, I)$. Thus, there is a theory of unitary remodules of $\mathcal{H}(G, I)$ and an abstract Plancherel formula.

If (π, V) is a complex smooth G-representation, such that $V^I \neq 0$, the algebra $\mathcal{H}(G, I)$ acts on V^I via

$$\pi(f)v = \int_G f(x)\pi(x)v \; dx, \quad v \in V^I, \; f \in \mathcal{H}(G, I).$$

Theorem A.8 ([Bo])**.** *The functor $V \to V^I$ is an equivalence of categories between the category of smooth admissible G-representations and finite dimensional $\mathcal{H}(G, I)$-modules*

Borel conjectured that this functor induces a bijective correspondence of unitary representations. This conjecture was proved by Barbasch-Moy [BM1] (subject to a certain technical assumption which was later removed).

Theorem A.9 ([BM1])**.** *An irreducible smooth G-representation (π, V) is unitary if and only if V^I is a unitary $\mathcal{H}(G, I)$-module.*

The algebra $\mathcal{H}(G, I)$ contains the finite Hecke algebra $\mathcal{H}(K, I)$ of functions whose support is in K. Under the functor η, K-spherical representations of G correspond to spherical $\mathcal{H}(G, I)$-modules, i.e., modules whose restriction to $\mathcal{H}(K, I)$ contains the trivial representation of $\mathcal{H}(K, I)$.

The classification of simple $\mathcal{H}(G, I)$-modules is given by Kazhdan-Lusztig [KL].

Theorem A.10 ([KL])**.** *The simple $\mathcal{H}(G, I)$-modules are parameterized by $G^\vee_{\mathbb{C}}$-conjugacy classes of triples $(s^\vee, e^\vee, \psi^\vee)$, where:*

(i) *$s^\vee \in G^\vee_{\mathbb{C}}$ is semisimple;*
(ii) *$e^\vee \in \mathcal{N}^\vee$ such that $Ad(s)e = qe$;*
(iii) *$\psi^\vee$ is an irreducible representation of Springer type of the group of components of the mutual centralizer $Z_{G^\vee_{\mathbb{C}}}(s^\vee, e^\vee)$ of $s^\vee$ and $e^\vee$ in $G^\vee_{\mathbb{C}}$.*

Let $\pi(s^\vee, e^\vee, \psi^\vee)$ denote the simple $\mathcal{H}(G, I)$-module parametrized by $[(s^\vee, e^\vee, \psi^\vee)]$.

Example A.4. In the Kazhdan-Lusztig parametrization, the simple spherical $\mathcal{H}(G,I)$-modules correspond to the classes of triples $[(s^\vee, 0, 1)]$. Here $s^\vee$ is the Satake parameter of the corresponding irreducible spherical G-representation. On the other hand, let $\mathcal{O}^\vee$ be a fixed $G^\vee_{\mathbb{C}}$-orbit in $\mathcal{N}^\vee$, and set $s^\vee_{\mathcal{O}^\vee} = q^{\lambda_0(\mathcal{O}^\vee)}$ where $\lambda_0(\mathcal{O}^\vee)$ is any choice of representative of the element in (A.3). If $e^\vee_0$ belongs to the unique open dense orbit of $Z_{G^\vee_{\mathbb{C}}}(s^\vee)$ on $\mathfrak{g}^\vee_q = \{x \in \mathfrak{g}^\vee_q : Ad(s^\vee)x = qx\}$ (in particular $e^\vee_0 \in \mathcal{O}^\vee$), then the simple $\mathcal{H}(G,I)$-module (and the corresponding irreducible G-representation) parametrized by $[(s^\vee_{\mathcal{O}^\vee}, e^\vee_0, \psi^\vee)]$ is tempered.

The Iwahori-Hecke algebra has an algebra involution τ, called the Iwahori-Matsumoto involution, defined on the generators as in [IM]. It induces an involution on the set of simple $\mathcal{H}(G,I)$-modules, which is easily seen to map unitary modules to unitary modules. The effect of τ on the set of Kazhdan-Lusztig parameters is given by a Fourier transform of perverse sheaves [EM], and therefore it is hard to compute effectively in general, except in type A [MW]. (For a general algorithm, see [L].) However, it is easy to see that if $\pi(s^\vee_{\mathcal{O}^\vee}, 0, 1)$ is a simple spherical $\mathcal{H}(G,I)$-module, then

$$\tau(\pi(s^\vee_{\mathcal{O}^\vee}, 0, 1)) = \pi(s^\vee_{\mathcal{O}^\vee}, e^\vee_0, 1), \tag{A.11}$$

where the notation is as in Example A.4. In particular, $\pi(s^\vee_{\mathcal{O}^\vee}, 0, 1)$ is unitary. Together with Theorem A.9, this gives the following corollary (cf. Conjecture A.7).

Corollary A.12. *If π is an irreducible spherical G-representation with Satake parameter $s^\vee_{\mathcal{O}^\vee} \in G^\vee_{\mathbb{C}}$, then π is unitary.*

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Appendix B. Supersymmetry and instantons

The constraints of maximal supersymmetry are efficiently described by starting with the superalgebra generated by the 32-component Majorana spinor supercharge, $Q_\alpha = \int J^0_\alpha d^{10}x$, where J^I_α is the supercurrent (with spinor index $\alpha, \beta = 1, \ldots, 32$ and vector index $I = 0, 1, \ldots, 10$). This satisfies the anti-commutation relations,

$$\{Q_\alpha\,, Q_\beta\} = P_{I_1}\left(\Gamma^0\Gamma^{I_1}\right)_{\alpha\beta} + Z_{\alpha\beta} \tag{B.1}$$

where the central charge is

$$Z_{\alpha\beta} = Z_{I_1 I_2}\ \left(\Gamma^0\Gamma^{I_1 I_2}\right)_{\alpha\beta} + Z_{I_1\cdots I_5}\ \left(\Gamma^0\Gamma^{I_1\cdots I_5}\right)_{\alpha\beta}\ , \tag{B.2}$$

where $\Gamma^I_{\alpha\beta}$ are $SO(1,10)$ Dirac matrices[28] and P_I is the eleven-dimensional translation operator.

B.1. **BPS particle states.** Positivity of the anticommutator in (B.1) leads to the Bogomol'nyi bound that restricts the masses of states to be larger than or equal to the central charge. States saturating the bound are BPS states that form supermultiplets, the lengths of which depend on the fraction of supersymmetry broken by their presence. The shortest multiplets are $\frac{1}{2}$-BPS, with longer multiplets for smaller fractions. We refer, for instance, to [61–63] for extensive discussions of the properties of supersymmetric branes in string theory.

The presence of the 2-form component of the central charge indicates that the theory contains a membrane-like state (the $M2$-brane) carrying a

[28] $\Gamma^{I_1\cdots I_r}_{\alpha\beta}$ is the antisymmetrized product of r Gamma matrices normalised so that $\Gamma^{1\cdots r} = \Gamma^1\cdots\Gamma^r$.

conserved charge $Q^{(2)}$, while the 5-form component indicates the presence of a 5-brane state (the $M5$-brane) carrying a charge $Q^{(5)}$. The 2-form and 5–form in (B.1) are given by integration of the spatial directions of the $M2$ and $M5$ branes over 2-cycles $A_{I_1 I_2}$ or 5-cycles $A_{I_1\cdots I_5}$,

$$Z_{I_1 I_2} = Q^{(2)} \int_{A_{I_1 I_2}} d^2X \,, \quad Z_{I_1\cdots I_5} = Q^{(5)} \int_{A_{I_1\cdots I_5}} d^5X \,. \tag{B.3}$$

The $M2$ and $M5$-branes are $\frac{1}{2}$-BPS states that preserve 16 of the 32 components of supersymmetry. The 2-form charge couples to a 3-form potential ($C^{(3)}_{I_1 I_2 I_3}$), with field strength $H^{(4)} = dC^{(3)}$. This is analogous to the manner in which the Maxwell 1-form potential couples to a point-like electric charge (a 0-brane), and $H^{(4)}$ is the analogue of the Maxwell field. The analogue of the dual Maxwell-field is a 7-form field-strength, which is required by consistency with supersymmetry to take the form that $H^{(7)} = dC^{(6)} + C^{(3)} \wedge dC^{(3)}$, where $C^{(6)}$ is the 6-form potential that couples to the $M5$-brane. In other words, the $M5$-brane couples to the magnetic charge that is dual to the electric charge carried by the $M2$-brane. The BPS condition implies that the charge on the brane is equal to its tension, $T^{(r)}$,

$$Q^{(r)} = T^{(r)} \,. \tag{B.4}$$

The integrals in (B.3) are well-defined when all the spatial directions of the branes are wound around the compact cycles of the M-theory torus, $\mathcal{T}^{d+1}$, in which case the state is point-like from the point of view of the $D = 10 - d$ non-compact dimensions (so there are finite-mass point-like states due to wrapped $M2$-branes when $d \geq 1$ as well as wrapped $M5$-branes when $d \geq 4$).[29] Other kinds of $\frac{1}{2}$-BPS states also arise in the toroidal background, such as point-like Kaluza–Klein (KK) charges, which are modes of the metric that contribute for any $d \geq 0$. The magnetic dual of a KK state is a KKM, which is described by a Taub-NUT geometry in four spatial dimensions, leaving six more spatial dimensions that are interpreted as the directions on a six-brane. This has a finite mass when wrapped around $\mathcal{T}^6$, so it can arise when $d \geq 5$.

The complete spectrum of BPS states in an arbitrary toroidal compactification of type IIA or IIB string theory can be deduced by considering the toroidal compactification of the M-theory algebra (B.1) with appropriate rescalings of the moduli [64]. Combining completely wrapped branes in various combinations leads to point-like $\frac{1}{2}$-, $\frac{1}{4}$- and $\frac{1}{8}$-BPS states that are of importance in discussing the spectrum of black holes in string theory [26,27]. This spectrum is of significance in classifying the orbits of instantons that decompactify to black hole states in one higher dimension associated with

[29]There is a huge literature of far more elaborate windings of such branes around supersymmetric cycles in curved manifolds, in which case a fraction of the supersymmetry may or may not be preserved.

the parabolic subgroup $P_{\alpha_{d+1}}$. This will be sketched in the next subsection where we will make contact with the discussion of black hole orbits in [24, 25, 28].

Appendix C. Orbits of BPS instantons in the decompactification limit

A finite action instanton in $D = 10 - d$ dimensions corresponds to an embedded euclidean world-volume that can be one of three types:

(a) It has an action that does not depend on r_d as $r_d \to \infty$ and so is also an instanton of the $(D+1)$-dimensional theory – this contributes only to the constant term in this parabolic and does not appear in non-zero Fourier modes;

(b) It is a euclidean world-line of a $(D+1)$-dimensional point-like BPS black hole with mass M_{BH}, which gives a term suppressed by a factor of $e^{-2\pi\, r_d\, M_{BH}}$ in the amplitude in the limit $r_d/\ell_{D+1} \to \infty$;

(c) It has an action that grows faster than r_d/ℓ_{D+1} so it does not decompactify to give either a particle state or an instanton in $D + 1$ dimensions.

Thus, the instantons of type (b) or (c) are the ones that contribute to the character variety orbits in limit (i), which is associated with the parabolic subgroup that has Levi factor $GL(1) \times E_d(\mathbb{R})$ in $D = 10 - d$ dimensions, where the duality group is $E_{d+1}(\mathbb{Z})$.

In order to illustrate this pattern the following subsections summarise the spectrum of r_d-dependent instantons (i.e., type (b) or (c)) in each dimension in the range $3 \le D \le 10$ (i.e., $0 \le d \le 7$). Their orbits and the conditions on the charges corresponding to fractional BPS conditions are summarised in table 5 on page 24. Where appropriate we will also comment on the distinction between BPS states in dimension $D + 1$ and BPS instantons in dimension D.

C.1. BPS orbits in $D = 10$.

This degenerate case includes both $10A$ and $10B$. Although the $10A$ theory does have a decompactification limit to 11-dimensional M-theory, it has no instantons and there is no duality symmetry group. There are $\frac{1}{2}$-BPS particle states in $10A$ consisting of threshold bound states of $D0$-branes that are manifested as instantons in the $D = 9$ theory (as we will sketch in the next subsection). There is no decompactification limit for the $10B$ theory. In this case there are no BPS particle states but there is a $\frac{1}{2}$-BPS D-instanton, multiples of which only contribute to amplitudes in the string perturbation limit. There are no $\frac{1}{4}$-BPS particle states in either $10A$ or $10B$.

C.2. BPS instanton orbits in $D = 9$.

This case may be obtained by considering M-theory on a 2-torus, $\mathcal{T}^2$, where the discrete duality group $SL(2, \mathbb{Z})$ is identified with the group of large diffeomorphisms of $\mathcal{T}^2$.

There is a single type of BPS instanton that can be identified with the wrapping of the euclidean world-line of a Kaluza–Klein state formed on one cycle around the second cycle of the 2-torus; in this sense we will refer in the following to a euclidean Kaluza–Klein state wrapping a 2-cycle on $\mathcal{T}^2$. Equivalently, this instanton can be described as a wrapped euclidean world-line of a $D0$-brane of the $10A$ string theory, which is the parameterisation manifested in (4.24). In this case the unipotent radical consists of 2×2 upper triangular matrices with 1's on the diagonal, and so the one-dimensional $\frac{1}{2}$-BPS orbit is simply

$$\mathcal{O}_{\mathbf{1}} = GL(1)\,. \tag{C.1}$$

C.3. **BPS instanton orbits in $D = 8$.**

This case may be obtained by considering M-theory on a 3-torus, $\mathcal{T}^3$, where the discrete duality group is $SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$.

There is one type of instanton charge from wrapping the world-volume of the $M2$-brane around the whole of $\mathcal{T}^3$. In addition there are two types of instanton charges from Kaluza–Klein states wrapping the 2-cycles that depend on the decompactification radius r_2 (a third Kaluza–Klein state wraps the two-cycle that does not depend on r_2). This gives a total of 3 types of BPS instanton charges of type (b), which are parameterised in the same manner as the BPS particle states in $D = 9$ dimensions by a scalar v and a $SL(2)$ vector v_a. The charges of the $\frac{1}{2}$-BPS states are given by the condition $v\, v_a = 0$ and the $\frac{1}{4}$-BPS states by $v\, v_a \neq 0$.

The $\frac{1}{2}$-BPS instantons are those for which $vv_a = 0$ [24], giving the union of the orbits

$$\mathcal{O}_{\mathbf{1}} \quad = \quad GL(1) \tag{C.2}$$

for $v_a = 0$ and

$$\mathcal{O}_{\mathbf{2}} \quad = \quad \frac{SL(2)}{\mathbb{R}} \tag{C.3}$$

for $v = 0$, arising from dense open orbits in each of the two factors of the duality group $SL(2) \times SL(3)$. The bold face subscript, in this example and in the following, gives the dimensions of the coset, $\dim(\frac{G_1}{G_2}) = \dim(G_1) - \dim(G_2)$. The $\frac{1}{4}$-BPS instantons have charges satisfying $vv_a \neq 0$, giving the orbit

$$\mathcal{O}_{\mathbf{3}} \quad = \quad \frac{GL(1) \times SL(2)}{\mathbb{R}}\,. \tag{C.4}$$

C.4. **BPS instanton orbits in $D = 7$.**
Consider M-theory on a 4-torus, $\mathcal{T}^4$, with duality group $SL(5,\mathbb{Z})$.

There are 4 BPS types of instanton from euclidean $M2$-branes wrapping 3-cycles, of which 3 depend on the decompactification radius r_3, and 6 types of instanton from the Kaluza–Klein states wrapping 2-cycles, of which three depend on r_3. This gives a total of 10 types of BPS instanton charge, of which 6 depend on the decompactification radius r_3 and are of type (b). These instantons carry charges associated with the corresponding BPS states in $D = 8$ dimensions that may be parametrized by $v_{i\,a}$ transforming in the $\mathbf{3} \times \mathbf{2}$ of $SL(3) \times SL(2)$. The $\frac{1}{2}$-BPS states are given by the condition $\epsilon^{ab}\, v_{i\,a} v_{j\,b} = 0$ [24] and the $\frac{1}{4}$-BPS states by $\epsilon^{ab}\, v_{i\,a} v_{j\,b} \neq 0$. This determines two BPS instanton orbits given in [28] by

$$\tfrac{1}{2} - BPS \quad : \quad \mathcal{O}_{\mathbf{4}} = \frac{SL(3,\mathbb{R}) \times SL(2,\mathbb{R})}{GL(2,\mathbb{R}) \ltimes \mathbb{R}^3}\,, \tag{C.5}$$

$$\tfrac{1}{4} - BPS \quad : \quad \mathcal{O}_{\mathbf{6}} = \frac{SL(3,\mathbb{R}) \times SL(2,\mathbb{R})}{SL(2,\mathbb{R}) \ltimes \mathbb{R}^2}\,. \tag{C.6}$$

C.5. **BPS instanton orbits in $D = 6$.**
Consider M-theory on a 5-torus, $\mathcal{T}^5$, with duality group $Spin(5,5,\mathbb{Z})$.

There are 10 ways of wrapping the $M2$-brane world-volume around 3-cycles, of which 6 depend on the decompactification radius r_4, and 10 ways of wrapping euclidean Kaluza–Klein states on 2-cycles, of which 4 depend on r_4. This gives a total of 20 BPS instanton types of charge, of which 10 depend on r_4 (and so are of type (b)). These charges correspond to the charges of BPS states in $D = 8$ dimensions and may be parametrized by the rank-2 antisymmetric tensor $v_{[ij]}$ $(i, j = 1, \ldots, 5)$ that transforms in the $\mathbf{10}$ of $SL(5)$. The $\frac{1}{2}$-BPS states are given in [24] by the condition $\epsilon^{ijklm}\, v_{ij}\, v_{kl} = 0$ and the $\frac{1}{4}$-BPS by $\epsilon^{ijklm}\, v_{ij}\, v_{kl} \neq 0$. This determines two BPS instanton orbits given in [28] by

$$\tfrac{1}{2} - BPS \quad : \quad \mathcal{O}_{\mathbf{7}} = \frac{SL(5,\mathbb{R})}{(SL(3,\mathbb{R}) \times SL(2,\mathbb{R})) \ltimes \mathbb{R}^6}\,, \tag{C.7}$$

$$\tfrac{1}{4} - BPS \quad : \quad \mathcal{O}_{\mathbf{10}} = \frac{SL(5,\mathbb{R})}{Spin(2,3) \ltimes \mathbb{R}^4}\,. \tag{C.8}$$

C.6. **BPS instanton orbits in $D = 5$.**
Consider M-theory on a 6-torus, $\mathcal{T}^6$, with duality group $E_6(\mathbb{Z})$.

There are 20 types of instanton from the $M2$-brane world-volume wrapping 3-cycles, of which 10 depend on the decompactification radius, r_5; 15 types from Kaluza–Klein states wrapping 2-cycles, of which 5 depend on r_5; 1 type of instanton from the world-volume of the $M5$-brane world-volume wrapping the whole of $\mathcal{T}^6$. This gives a total of 36 BPS instanton charges, of which 16 depend on r_b and are of type (b).

These 16 BPS charges are parameterised by a chiral spinor S^α $(\alpha = 1, \ldots, 16)$ of $Spin(5,5)$. Such a spinor satisfies the identity $\sum_{m=1}^{10}(S\Gamma^m S) \times (S\Gamma^m S) = 0$, where Γ^m $(m = 1, \ldots, 10)$ are Dirac matrices with suppressed spinor indices. The configurations are $\frac{1}{2}$-BPS if S satisfies the pure spinor condition, $S\Gamma^m S = 0$ [24]. A standard way to analyse this condition is to decompose S into $U(5)$ representations, $\mathbf{16} = \mathbf{1}_5 \oplus \bar{\mathbf{5}}_{-3} \oplus -\mathbf{10}_1$ (where the subscripts denote the $U(1)$ charges), so it has components

$$S = (s, v_a, v^{ab}), \qquad a, b = 1, \ldots, 5\,. \tag{C.9}$$

The pure spinor ($\frac{1}{2}$-BPS) condition, $S\Gamma^m S = 0$ is $v_a = \frac{s^{-1}}{5!}\epsilon_{abcde}\, v^{bc} v^{de}$, which implies that the **5** is not independent of the other $U(5)$ representations, so the space of such spinors has dimension 11. The $\frac{1}{4}$-BPS solution is the unconstrained spinor space (excluding $S\Gamma^m S = 0$) and has dimension 16. There are two BPS orbits given in [28] by

$$\tfrac{1}{2} - BPS \quad : \quad \mathcal{O}_{\mathbf{11}} = \frac{Spin(5,5,\mathbb{R})}{SL(5,\mathbb{R}) \ltimes \mathbb{R}^{10}}\,, \tag{C.10}$$

$$\tfrac{1}{4} - BPS \quad : \quad \mathcal{O}_{\mathbf{16}} = \frac{Spin(5,5,\mathbb{R})}{Spin(3,4) \ltimes \mathbb{R}^{8}}\,. \tag{C.11}$$

C.7. **BPS instanton orbits in $D = 4$.**
Consider M-theory on a 7-torus, $\mathcal{T}^7$, with duality group $E_7(\mathbb{Z})$.

There are 35 types of instanton charge from the $M2$-brane world-volume wrapping 3-cycles, of which 15 depend on the decompactification radius r_6; 21 types of instanton charge from Kaluza–Klein states wrapping 2-cycles, of which 6 depend on r_6; 7 types of instanton charge from the $M5$-brane world-volume wrapping 6-cycles, of which 6 depend on r_6. This gives a total of 63 types of BPS instanton charge, of which 27 depend on r_6.

The distinct instanton charges are parameterised by the fundamental representation, q^i ($i = 1, \ldots, 27$), of E_6 and lead to $\frac{1}{2}$-, $\frac{1}{4}$- or $\frac{1}{8}$-BPS configurations depending on the following conditions on the E_6 cubic invariant $I_3 = \sum_{1\leq i,j,k\leq 27} (I_3)_{ijk} q^i q^j q^k$ [24]

$$\tfrac{1}{2} - BPS: \quad I_3 = 0, \qquad \frac{\partial I_3}{\partial q^i} = 0, \ \frac{\partial^2 I_3}{\partial q^i \partial q^j} \neq 0\,, \tag{C.12}$$

$$\tfrac{1}{4} - BPS: \quad I_3 = 0, \qquad \frac{\partial I_3}{\partial q^i} \neq 0\,, \tag{C.13}$$

$$\tfrac{1}{8} - BPS: \quad I_3 \neq 0\,. \tag{C.14}$$

Clearly the first of these conditions (the $\frac{1}{8}$-BPS condition) is of dimension 27. The other conditions may be analysed by decomposing the **27** of E_6 into $SO(5,5) \times U(1)$ irreducible representations, $\mathbf{27} = \mathbf{1}_4 \oplus \mathbf{10}_{-2} \oplus \mathbf{16}_1$. This means that q^i decomposes as

$$q^i = (s, v_m, S^\alpha)\,, \tag{C.15}$$

where s is a scalar, v_m is a $SO(5,5)$ vector of dimension 10 and S^α is a spinor of dimension 16 (the $U(1)$ charges have been suppressed). The cubic invariant I_3 decomposes as $I_3 = \mathbf{10}_{-2} \otimes \mathbf{10}_{-2} \otimes \mathbf{1}_4 \oplus \mathbf{16}_1 \otimes \mathbf{16}_1 \otimes \mathbf{10}_{-2}$ [24], which implies that

$$I_3 = s\, v \cdot v + (S\Gamma S) \cdot v\,, \tag{C.16}$$

where $v \cdot v$ is the $SO(5,5)$ (norm)^2 of the vector v, and $(S\Gamma S)\cdot v$ is the $SO(5,5)$ scalar product between the vector $S\Gamma^m S$ and v^m.

The $\frac{1}{4}$-BPS solution reduces to the condition

$$s\, v \cdot v + (S\Gamma S) \cdot v = 0\,, \tag{C.17}$$

with non-vanishing derivative with respect to s, v_m and S_a. Therefore the solution is given by the 26 dimensional space

$$(q^i)_{\frac{1}{4}-BPS} = (-(v\cdot v)^{-1}\,(S\Gamma S)\cdot v, v_m, S^\alpha)\,. \tag{C.18}$$

The $\frac{1}{2}$-BPS condition implies the following conditions

$$\begin{aligned} v\cdot v &= 0\,, & \text{(C.19)}\\ (S\Gamma^m S) + s\,v^m &= 0\,, & \text{(C.20)}\\ (S\Gamma^m)_a\,v^m &= 0\,, & \text{(C.21)}\end{aligned}$$

which are solved by $v^m = S\Gamma^m S$ (using the relation $(S\Gamma^m S)(S\Gamma^m S) = 0$). The $\frac{1}{2}$-BPS solution is therefore given by the 17-dimensional solution

$$(q^i)_{\frac{1}{2}-BPS} = (s, S\Gamma^m S, S_a)\,. \tag{C.22}$$

To summarise, in limit (i) the BPS instanton orbits in $D = 4$ are given in [28] by

$$\begin{aligned} &\tfrac{1}{2}-BPS \quad : \quad \mathcal{O}_{\mathbf{17}} = \frac{E_6}{Spin(5,5)\ltimes\mathbb{R}^{16}}\,, & \text{(C.23)}\\ &\tfrac{1}{4}-BPS \quad : \quad \mathcal{O}_{\mathbf{26}} = \frac{E_6}{Spin(4,5)\ltimes\mathbb{R}^{16}}\,, \qquad \text{and} & \text{(C.24)}\\ &\tfrac{1}{8}-BPS \quad : \quad \mathcal{O}_{\mathbf{27}} = \frac{GL(1)\times E_6}{F_{4(4)}}\,. & \text{(C.25)}\end{aligned}$$

The charges in the $\frac{1}{4}$-BPS orbit can be generated by applying $E_6(\mathbb{Z})$ transformations to a 2-charge instanton corresponding to a null vector in the 27 dimensional BPS state space. The charges in the $\frac{1}{8}$-BPS orbit can be generated from a 3-charge instanton corresponding to space-like or time-like vectors with $I_3 \neq 0$ in the 27 dimensional BPS state space (note that, unlike [25] we have included the scale factor $GL(1)$ in the definition of the orbit, which is of dimension 27). The last orbit of dimension 27 is the $\frac{1}{8}$-BPS orbit of black hole states with $I_3 \neq 0$, and entropy proportional to $\sqrt{|I_3|}$.

C.8. **BPS instanton orbits in $D = 3$.**
Consider M-theory on an 8-torus $\mathcal{T}^8$ with duality group $E_8(\mathbb{Z})$.

There are 56 types of instanton charge from $M2$-brane world-volumes wrapping 3-cycles, of which 21 depend on the decompactification radius, r_7; 28 types of instanton charge from Kaluza–Klein states wrapping 2-cycles, of which 7 depend on r_7; 28 types of instanton charge from $M5$-branes wrapping 6-cycles, of which 21 depend on r_7. In addition there are 8 types of instantons that depend on r_7 due to KKM world-volumes wrapping 8-cycles, which are distinguished by labelling which cycle corresponds to $x^\#$ (the fibre coordinate in (3.1)). This gives a total of 120 types of instanton charges, of which 57 depend on r_7.

The connection with the black hole states in $D = 4$ dimensions is slightly subtle. For one of the 8 KKM instantons $x^\#$ is identified with the euclidean time dimension and gives a vanishing contribution upon decompactification to $D = 4$ dimensions (the large-r_7 limit), as discussed following (3.2). It is therefore of type (c) and does not correspond to a black hole state in $D = 4$ dimensions. This accounts for the nonabelian, Heisenberg, entry in the unipotent radical for the parabolic subgroup, $GL(1) \times E_7$, of E_8. The nonzero Fourier modes in limit (i) correspond to the 56 abelian components of the unipotent radical which match the charges of BPS states in $D = 4$. These are in the fundamental representation, q^i ($i = 1, \ldots, 56$), of E_7. The $\frac{1}{2}$-, $\frac{1}{4}$- and $\frac{1}{8}$-BPS configurations are classified by the following conditions on the quartic symmetric polynomial invariant I_4 [24,65]

$$\tfrac{1}{2} - BPS: \quad I_4 = \frac{\partial I_4}{\partial q^i} = \left.\frac{\partial^2 I_4}{\partial q^i \partial q^j}\right|_{Adj_{E_7}} = 0, \quad \frac{\partial^3 I_4}{\partial q^i \partial q^j \partial q^k} \neq 0\,, \tag{C.26}$$

$$\tfrac{1}{4} - BPS: \quad I_4 = 0, \quad \frac{\partial I_4}{\partial q^i} = 0, \quad \left.\frac{\partial^2 I_4}{\partial q^i \partial q^j}\right|_{Adj_{E_7}} \neq 0\,, \tag{C.27}$$

$$\tfrac{1}{8} - BPS: \quad I_4 = 0, \quad \frac{\partial I_4}{\partial q^i} \neq 0\,, \tag{C.28}$$

$$\tfrac{1}{8} - BPS: \quad I_4 > 0\,. \tag{C.29}$$

The following is a summary of the BPS orbits [24,25,28]

$$\tfrac{1}{2} - BPS \quad : \quad \mathcal{O}_{\mathbf{28}} = \frac{E_7}{E_{6(6)} \ltimes \mathbb{R}^{27}}\,, \tag{C.30}$$

$$\tfrac{1}{4} - BPS \quad : \quad \mathcal{O}_{\mathbf{45}} = \frac{E_7}{Spin(5,6) \ltimes (\mathbb{R}^{32} \ltimes \mathbb{R})}\,, \tag{C.31}$$

$$\tfrac{1}{8} - BPS \quad : \quad \mathcal{O}_{\mathbf{55}} = \frac{E_7}{F_{4(4)} \ltimes \mathbb{R}^{26}}\,, \tag{C.32}$$

$$\tfrac{1}{8} - BPS \quad : \quad \mathcal{O}_{\mathbf{56}} = \frac{\mathbb{R}^+ \times E_7}{E_{6(2)}}\,. \tag{C.33}$$

The $\frac{1}{2}$-BPS orbit can be obtained by acting on a single charge, the $\frac{1}{4}$-BPS orbit can be obtained by acting on a 2-charge system, and the first $\frac{1}{8}$-BPS (with dimension 55) has zero entropy and can be obtained by acting on a 3-charge system. The last orbit of dimension 56 is the $\frac{1}{8}$-BPS orbit of black hole states with $I_4 > 0$, which have entropy proportional to $\sqrt{I_4}$; it can be obtained by acting on a 4-charge system in the **56** representation of E_7 as detailed in [28]. We have included the overall scale factor in the definition of the orbit. Another orbit of dimension 56 is $(\mathbb{R}^- \times E_7)/E_{6(2)}$ that has $I_4 < 0$ and does not correspond to a BPS solution at all [24,25]. All these charge orbits can be understood in terms of the superpositions of branes at angles and constructed from combinations of $(D0, D2, D4, D6)$ [66].

Note the presence of the 33-dimensional nonabelian group in the stabilizer of $\mathcal{O}_{\mathbf{45}}$. It is a Heisenberg group isomorphic to the unipotent radical of the maximal parabolic subgroup $P_{\alpha_1} = L_{\alpha_1} U_{\alpha_1}$ of E_7. This can be seen directly using the basepoint of this orbit given in [33, §5.9.8]. Different stabilizer groups of the same dimension have appeared in the physics literature listed.

Appendix D. Euclidean Dp-brane instantons.

We here sketch the background to the analysis of the euclidean Dp-brane instanton configurations that contribute in the perturbative limit of string theory discussed in section 3.4, based on an analysis of supersymmetry conditions on the embeddings of world-sheets on the string theory torus T^d. Contributions from wrapped NS5-brane world-sheets also arise for $d = 6, 7$ and KK monopoles for $d = 7$.

Wrapping a euclidean Dp-brane world-volume of either ten-dimensional type II string theory on a $(p+1)$-cycle leads to an instanton in the transverse $\mathbb{R}^{1,8-p}$ space-time. This $\frac{1}{2}$-BPS condition preserves a linear combination of the supersymmetries that act on the left-moving and right-moving modes of a closed superstring. This leads to the following constraint on the supersymmetry parameters,

$$\tilde{\varepsilon} = \prod_{i=1}^{p+1} \Gamma^i \varepsilon \tag{D.1}$$

where ε and $\tilde{\varepsilon}$ are chiral sixteen-component $SO(1,9)$ spinors parameterizing the left- and right-moving super symmetries and Γ^i are the usual $SO(1,9)$ Gamma matrices that satisfy the Clifford algebra $\{\Gamma^i, \Gamma^j\} = -2\eta^{ij}$, where η is the Minkowski metric with signature $(- + \cdots +)$.

When compactifying on a d-torus space-time becomes $\mathbb{R}^{1,9-d} \times \mathrm{T}^d$ and a $SO(1,9)$ spinor decomposes into a sum of bispinors, $\varepsilon = \hat{\varepsilon} \otimes \eta$, where $\hat{\varepsilon}$ is a $SO(1,9-d)$ spinor and η is a $SO(d)$ spinor. The condition (D.1) becomes a condition relating η and $\tilde{\eta}$. T-duality transforms the Γ matrices in (D.1) by the action of the spin group $SO(d,d)$, $R^{-1} \prod_i \Gamma^i R$. This, in general, transforms a wrapped Dp-brane into a Dq-brane so that the supersymmetry conditions

$$\tilde{\eta} = \prod_{i=1}^{q+1} \Gamma^i \eta = \prod_{i=1}^{p+1} \Gamma^i \eta \,, \tag{D.2}$$

are satisfied. As remarked in [67], this means the two spinors $\prod_{i=1}^{q+1} \Gamma^i \varepsilon$ and $\prod_{i=1}^{p+1} \Gamma^i \varepsilon$ must be in the same $Spin(d,d)$ orbit.

A euclidean Dp-brane can be wrapped over cycles of a d-torus of dimension $0 \leq p+1 \leq d$ with $p \equiv 0 \pmod 2$ for type IIA superstring theory and $p \equiv 1 \pmod 2$ for type IIB. These instanton configurations fill out a chiral spinor representation, S_A, of dimension $\sum_{p\equiv s \pmod 2} \binom{d}{p+1} = 2^{d-1}$, with $s = 0$ or 1, of the T-duality group $SO(d,d)$. The BPS condition on Dp-branes wrapping a torus in (D.2) can be interpreted as a condition on the spinor S_A. The various brane configurations are then classified by orbits of S_A under the action of the double cover $Spin(d,d)$ of the T-duality group

$SO(d,d)$. In this manner the spinor parameterizes the commuting set of instanton charges in the perturbative regime.

For $d=6$ or $d=7$ there are also contributions from NS5-branes wrapping six-cycles. Such NS5-brane configurations give contributions to the instanton charges that do not commute with those of the wrapped Dp-branes. In other words, the Dp-brane charges in the spinor representation parametrize the $\mathfrak{u}_{-1}$ component part of the unipotent radical U (the abelian part) for the standard parabolic subgroup P_{α_1} of E_{d+1} and the NS5-brane charge are in the derived subgroup $[U,U]$ component part of the unipotent radical for the standard parabolic subgroup P_{α_1} of E_{d+1} in table 3 on page 21. For $d=6$ this provides one extra charge configuration since there is a unique six-cycle. For $d=7$ there are 7 distinct six-cycles so there are 7 NS5-brane charges. In addition there are 7 stringy KKM instantons. Recall that these arise from Kaluza–Klein monopoles in ten-dimensional string theory in which the fibre direction $x^{\#}$ is identified with a circle in T^7 (whereas the $D6$-brane is seen in M-theory as a KKM formed by identifying $x^{\#}$ with the M-theory circle).

Although it is very complicated to describe how all possible compactifications of euclidean Dp-branes fit into different spinor orbits, the following discussion will indicate the procedure. For this purpose it is convenient to start in ten dimensions by defining chiral spinors of the complexified group, $SO(10,\mathbb{C})$ (complexification does not affect the BPS classification), by means of the raising and lowering operators,

$$b_{k+1} = \frac{1}{2}(\Gamma^{2k+1} - i\Gamma^{2k})\,, \qquad b^{\dagger}_{k+1} = -\frac{1}{2}(\Gamma^{2k+1} + i\Gamma^{2k})\,, \quad 0 \le k \le 4\,, \tag{D.3}$$

so that $b^k = (b^{\dagger}_k)$ and $\{b_k, b^l\} = \delta^l_k$, and $\{b_k, b_l\} = \{b^k, b^l\} = 0$. A ground state $|-----\rangle$ is defined so that $b_k|-----\rangle = 0$, for $1 \le k \le 5$. Acting with b^1 gives the state $b^1|-----\rangle = |+----\rangle$, with analogous states created by any linear combination of the b^r's, giving a total of 2^5 states with $+$ or $-$ labelling each of the 5 positions. These states are graded according to whether there an even or odd number of $+$ signs. There are therefore two chiral spinor representations of $SO(10,\mathbb{C})$ of dimension 16. Upon compactification on T^d the spinor η in (D.2) is represented as a state of the Fock space built by acting with b^i on the ground state $|-^5\rangle$. It is convenient to introduce the notation $e_{i_1\cdots i_r} := b^{i_1}\cdots b^{i_r}|-^{d/2}\rangle$ and $e^*_{i_1\cdots i_r} := b_{i_1}\cdots b_{i_r}|+^{d/2}\rangle$, which was used in section 3.4.1.

Spinors that are related by a continuous $Spin(d,d)$ transformation $\exp(\sum_{i,j} x_{ij}\gamma^{ij})$ are associated with D-brane configurations that are equivalent under T-duality. Each orbit listed in section 3.4.1 is characterized by a representative S^0. Therefore an $SO(d,d)$ pure spinor is equivalent to the ground state of the Fock space that we can denote by 1, corresponding to a pure spinor defining a D-brane wrapping a supersymmetric cycle. The notation $e_{i_1\cdots i_r}$ corresponds to a D-brane configuration wrapping the directions $\{i_1,\cdots,i_r\}$

in T^d and $e^*_{i_1\cdots i_r}$ a D-brane configuration wrapping the complementary directions to $\{i_1,\cdots,i_r\}$ in T^d.

Upon compactifying on a torus of dimension $d\leq 3$, all possible brane world-volumes are parallel, up to identification under $Spin(d,d,\mathbb{Z})$, and the condition (D.1) ensures in this case that all instanton configurations are $\frac{1}{2}$-BPS. These are $p=0$ and $p=2$ wrappings in type IIA, and $p=-1$ and $p=1$ in type IIB.

The theory compactified on a 4-torus T^4 in type IIA (for instance), includes instantons due to wrapping $D0$-brane world-lines on any of the four 1-cycles and $D2$-brane world-volumes on any of the four 3-cycles. These configurations in general fill out an eight-dimension chiral spinor representation of $SO(4,4)$, $S_A=\sum_{i=a}^4 v_a b^a+\sum_{a,b,c=1}^4 v_{abc}b^{abc}/3!$. This parametrization makes explicit the action of $SL(4)$ on v_a or $u^a=\epsilon^{abcd}v_{abc}$ (or $SU(4)$ in the complexified case).

With a single $D0$-brane or a single $D2$-brane world-volume wrapped on T^4 the condition (D.1) is always satisfied, and the configuration is $\frac{1}{2}$-BPS. However, wrapping both a $D0$-brane world-line and a $D2$-brane world-volume results in further breaking of supersymmetry unless v_a and u^a satisfy condition (D.2). It is easily seen that this condition is satisfied for all $\eta=|\pm\pm\rangle$ if $v\cdot u=0$. But if $u\cdot v\neq 0$ only $\eta=|+\pm\rangle$ satisfy the solution which is $\frac{1}{4}$-BPS. These two conditions are invariant under the action of the T-duality group $Spin(4,4)$ acting on a spinor S_A. The $\frac{1}{2}$-BPS condition corresponds to imposing the pure spinor constraint $S\cdot S=0$ while the $\frac{1}{4}$-BPS corresponds to the complementary condition, $S\cdot S\neq 0$, which defines the configuration with the $D0$-brane world-line orthogonal to the $D2$-brane world-volume.

Extensions of these arguments lead to a classification of all BPS configurations of euclidean Dp-brane world-volumes that are completely wrapped on a torus. The orbits of such configurations are obtained by imposing generalisations of the pure spinor constraint on the $SO(d,d)$ spinor that parameterizes the orbits. An orbit which preserves a smaller fraction of supersymmetry is larger and is associated with a spinor satisfying weaker constraints. The resulting orbits are described in section 3.4.1.

Appendix E. Properties of lattice sums

This appendix and appendices F and G together concern properties of lattice sums related to the Fourier expansions of certain Eisenstein series that appear in the coefficient functions for the cases $D=7$ and $D=6$ (i.e., for $SL(5)$ and $Spin(5,5)$, respectively). Those properties will later be used in section 4 and appendices H.2-H.3. The main result of the present appendix, proposition 4.1, is an integral representation for the $SL(d)$ series[30]

[30] The labelling β_i of the simple roots of $SL(d)$ here follows the conventional labelling as illustrated in figure 4 for the $SL(5)$ case.

$E^{SL(d)}_{\beta_2,s}$. The series $E^{SL(d)}_{\beta_2,s}$ will later be related to the $Spin(d,d)$ Eisenstein series $E^{Spin(d,d)}_{\alpha_1;s}$ in proposition G.1 and (G.13).

E.1. **Exponential sums of lattice norms.** Let $g \in GL(d,\mathbb{R})$ and consider the set of points

$$\{\, mg \,\in\, \mathbb{R}^d \mid m \,\in\, \mathbb{Z}^d_{\neq 0} \,\}\,, \tag{E.1}$$

where m is thought of as a row vector. This set of points is unchanged if g is replaced by γg for any $\gamma \in GL(d,\mathbb{Z})$, so we may assume that g lies in a fixed fundamental domain for $GL(d,\mathbb{Z})\backslash GL(d,\mathbb{R})$. A standard result in reduction theory asserts that every fundamental domain is contained in a Siegel set, so we may also assume that $g = nak$ where n is unit upper triangular and lies in a fixed compact set, k lies in $O(d,\mathbb{R})$, and a is a diagonal matrix with positive diagonal entries $a_1, a_2, \ldots$ such that each a_i/a_{i+1} is bounded below by an absolute constant (to be explicit, $\sqrt{\frac{3}{4}}$ [68]). Therefore $a^{-1}na$ and its inverse range over a fixed compact subset of N, which means that the operator norms of both are bounded by a constant that depends only on the dimension d. As a consequence $\|mg\| = \|mnak\| = \|mna\| = \|ma \cdot a^{-1}na\|$ is bounded above and below by multiples of $\|ma\|$:

$$c_-\,\|ma\|^2 \quad\leq\quad \|mg\|^2 \quad\leq\quad c_+\,\|ma\|^2\,, \tag{E.2}$$

where the constants c_- and c_+ depend only on d. Among other things, this implies the norms of vectors in an arbitrary lattice are crudely similar to those of a dilation of the $\mathbb{Z}^d$ lattice.

Define the θ-function

$$S(g,t) \quad:=\quad \sum_{m\,\in\,\mathbb{Z}^d_{\neq 0}} e^{-t\,\|mg\|^2}\,. \tag{E.3}$$

The first inequality in (E.2) shows this sum is absolutely convergent and bounded by

$$\sum_{m\neq 0} e^{-tc_-(m_1^2a_1^2+\cdots+m_d^2a_d^2)} \quad=\quad \theta(tc_-a_1^2)\cdots\theta(tc_-a_d^2)\,-\,1\,, \tag{E.4}$$

in terms of the Jacobi θ-function $\theta(x) = \sum_{n\in\mathbb{Z}} e^{-n^2x}$. The Jacobi θ-function satisfies the bounds $\theta(x) = 1 + O(e^{-x})$ for $x > 1$, and $\theta(x) = O(x^{-1/2})$ for $x \leq 1$. Therefore

$$S(g,t) \quad=\quad O(e^{-tc_-a_d^2})\,,\quad t \;>\; (\tfrac{4}{3})^{d/2}\,(c_-a_d^2)^{-1} \tag{E.5}$$

and

$$S(g,t) \quad=\quad O\left(\frac{t^{-d/2}}{a_1\cdots a_d}\right)\,,\quad t \;\leq\; (\tfrac{3}{4})^{d/2}\,(c_-a_1^2)^{-1}\,, \tag{E.6}$$

with implied constants that depend only on d. If g is fixed we can use the fact that $\theta'(x) < 0$ to bound the t-dependence of $S(g,t)$ by

$$S(g,t) \quad\ll\quad \begin{cases} e^{-tc_-(g)}, & t \;>\; 1\,, \\ t^{-d/2}, & t \;<\; 1\,, \end{cases} \tag{E.7}$$

where both $c_-(g)$ and the implied constant in the $\ll$-inequality depend on g.

E.2. **A constrained lattice sum over pairs.** Let $\tau = \tau_1 + i\tau_2 \in \mathbb{H}$ and define

$$\mathcal{G}(\tau, X) \quad := \sum_{\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] \in \mathcal{M}_{2,d}^{(2)}(\mathbb{Z})} e^{-\pi\tau_2^{-1}(m+n\tau)X(m+n\bar{\tau})^t}\,, \tag{E.8}$$

where in the usual physics notation $X = G + B$, with G a positive definite symmetric $d \times d$ matrix and B an antisymmetric $d \times d$ matrix, and $\mathcal{M}_{2,d}^{(i)}$ represents $2 \times d$ matrices of rank i. This contribution is the rank 2 part of the lattice sum $\Gamma_{(d,d)}$ for even self-dual Lorentzian lattices. It is necessary to use this modification of $\Gamma_{(d,d)}$ in order to resolve some convergence issues in formal calculations involving $\Gamma_{(d,d)}$. However, the constraint complicates applications of Poisson summation to it in the following appendices.

We next analyze the convergence of this sum and give estimates. Note that because $G = G^t$ and $B = -B^t$

$$(m+n\tau)X(m+n\bar{\tau})^t \quad = \quad (m+n\tau_1)G(m+n\tau_1)^t + \tau_2^2 nGn^t - 2i\tau_2 mBn^t\,. \tag{E.9}$$

Consider the sum

$$\sum_{m \in \mathbb{Z}^d} e^{-\pi\tau_2^{-1}(m+x)G(m+x)^t}\,, \tag{E.10}$$

which is absolutely convergent and represents a continuous, periodic, and hence bounded function of a row vector $x \in \mathbb{R}^d$. By Poisson summation it is equal to

$$\tau_2^{d/2}(\det G)^{-1/2} \sum_{\hat{m} \in \mathbb{Z}^d} e^{2\pi i\hat{m}\cdot x}\, e^{-\pi\tau_2 \hat{m}G^{-1}\hat{m}^t}\,, \tag{E.11}$$

where $\hat{m}$ is thought of as a row vector. Use (E.9) to write (E.8) as

$$\mathcal{G}(\tau, G+B) \quad = \quad \sum_{n \neq 0} e^{-\pi\tau_2 nGn^t} \sideset{}{'}\sum_{m} e^{-\pi\tau_2^{-1}(m+n\tau_1)G(m+n\tau_1)^t} e^{-2\pi i mBn^t}\,, \tag{E.12}$$

where the prime indicates m is not collinear to n. The interior sum is bounded by (E.10) with $x = n\tau_1$ and hence

$$\mathcal{G}(\tau, G+B) \quad \leq \quad \tau_2^{d/2}(\det G)^{-1/2} \sum_{n \neq 0} e^{-\pi\tau_2 nGn^t} \sum_{\hat{m} \in \mathbb{Z}^d} e^{-\pi\tau_2 \hat{m}G^{-1}\hat{m}^t}\,. \tag{E.13}$$

If we write $G = ee^t$, $e \in GL(d, \mathbb{R})$, then

$$\mathcal{G}(\tau, ee^t + B) \quad \leq \quad \tau_2^{d/2}(\det e)^{-1}\, S(e, \pi\tau_2)\, [1 + S((e^{-1})^t, \pi\tau_2)] \tag{E.14}$$

in terms of (E.3).

The earlier estimates (E.5-E.6) give bounds on the last two factors of (E.14). This shows that $\mathcal{G}(\tau, G+B)$ decays rapidly as $\tau_2 \to \infty$. Since replacing τ by $\tau+1$ or by $-1/\tau$ in (E.8) is tantamount to changing variables

$(m,n)\mapsto(m{+}n,n)$ or $(-n,m)$, respectively, $\mathcal{G}(\tau,G{+}B)$ is thus automorphic in τ. Consequently, the integral

$$I(s,G+B) \ \ := \ \ \int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} E_s^{SL(2)}(\tau)\,\mathcal{G}(\tau,G+B)\,\frac{d^2\tau}{\tau_2^2} \tag{E.15}$$

is well-defined as a meromorphic function of s, with poles contained among the poles of the Eisenstein series $E_s^{SL(2)}(\tau)$.

Proposition E.1. The integral $I(s,uG)$ decays rapidly as $u\to\infty$, and slower than some polynomial in $u>0$ as $u\to 0$. These estimates are uniform for G fixed and Re s ranging over a finite interval.

Proof. The Eisenstein series satisfies the bound $E_s^{SL(2)}(\tau_1+i\tau_2)\ll\tau_2^z$ over the standard fundamental domain for $SL(2,\mathbb{Z})\backslash\mathbb{H}$, where $z=\max\{\text{Re } s,\text{Re } 1-s\}\geq 1/2$ (this follows from (4.15-4.17)). Since the upper bound (E.14) is independent of τ_1,

$$\begin{aligned} &I(s,uee^t) \ \ \ll \\ &u^{-d/2}(\det e)^{-1}\int_{\frac{\sqrt{3}}{2}}^{\infty}\tau_2^{z+d/2-2}S(e,\pi\tau_2 u)\,[1+S((e^{-1})^t,\pi\tau_2 u^{-1})]\,d\tau_2\,. \end{aligned} \tag{E.16}$$

We now use the estimates in (E.7). As $u\to\infty$, $S(e,\pi\tau_2 u)$ has exponential decay in $\tau_2 u$, whereas the bracketed term is $O(u^{d/2}\tau_2^{-d/2})$. Since the range of the τ_2 integration is bounded below, the rapid decay assertion of the proposition immediately follows.

On the other hand, as $u\to 0$ the bracketed term in (E.16) is bounded. After a change of variables we are therefore left to showing that the integral

$$\begin{aligned} \int_{\frac{\sqrt{3}}{2}\pi u}^{\infty}\tau_2^{z+d/2-2}S(e,\tau_2)\,d\tau_2 \ \ &= \ \ \int_{\frac{\sqrt{3}}{2}\pi u}^{1}\tau_2^{z+d/2-2}S(e,\tau_2)\,d\tau_2 \\ &\quad + \ \int_{1}^{\infty}\tau_2^{z+d/2-2}S(e,\tau_2)\,d\tau_2 \end{aligned} \tag{E.17}$$

is bounded by a polynomial in u^{-1} as $u\to 0$. Inserting the bounds from (E.7) this is

$$\ll \ \ \int_{\frac{\sqrt{3}}{2}\pi u}^{1}\tau_2^{z-2}\,d\tau_2 \ + \ \int_{1}^{\infty}\tau_2^{z+d/2-2}\exp(-c'\tau_2)\,d\tau_2 \tag{E.18}$$

for some constant c' depending on g, and clearly bounded by a polynomial in u^{-1}.

□

E.3. **Unfolding the lattice sum at $s=0$.** The integral $I(s,G+B)$ in (E.15) is well-defined for any value of s at which the Eisenstein series $E_s^{SL(2)}(\tau)$ is holomorphic – in particular, this includes $s=0$ where $E_0(\tau)$ is identically 1.

Proposition E.2.

$$I(0,G+B) \quad = \sum_{\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] \in SL(2,\mathbb{Z})\backslash\mathcal{M}_{2,d}^{(2)}(\mathbb{Z})} e^{-2\pi i m B n^t}\,\frac{e^{-2\pi\mathcal{D}_{m,n,G}}}{\mathcal{D}_{m,n,G}}\,, \tag{E.19}$$

where

$$\mathcal{D}_{m,n,G} \quad := \quad \det(\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] G \left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]^t)^{1/2}\,. \tag{E.20}$$

Proof. Unfolding the lattice sum gives that

$$\begin{aligned} I(0,G+B) \quad &= \quad \int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} \mathcal{G}(\tau,G+B)\,\frac{d^2\tau}{\tau_2^2} \\ &= \sum_{\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] \in SL(2,\mathbb{Z})\backslash\mathcal{M}_{2,d}^{(2)}(\mathbb{Z})} \int_{\mathbb{H}} e^{-\pi\tau_2^{-1}(m+n\tau)(G+B)(m+n\bar{\tau})^t}\,\frac{d^2\tau}{\tau_2^2}\,. \end{aligned} \tag{E.21}$$

The unfolding is valid because of the absolute convergence of the series $\mathcal{G}(\tau,G+B)$ to a rapidly-decaying automorphic function in τ. The integral in the last line can be computed as

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} e^{-\pi\tau_2 nGn^t - \pi\tau_2^{-1}(mGm^t + 2\tau_1 mGn^t + \tau_1^2 nGn^t)} e^{-2\pi i m B n^t}\,d\tau_1\,\frac{d\tau_2}{\tau_2^2} \\ &\quad = \quad (nGn^t)^{-1/2} \int_0^\infty e^{-\pi\tau_2 nGn^t - \pi\tau_2^{-1}\mathcal{D}_{m,n,G}^2 (nGn^t)^{-1}} e^{-2\pi i m B n^t}\,\frac{d\tau_2}{\tau_2^{3/2}} \\ &\quad = \quad \frac{e^{-2\pi\mathcal{D}_{m,n,G}}}{\mathcal{D}_{m,n,G}} e^{-2\pi i m B n^t} \end{aligned} \tag{E.22}$$

using (E.9) and the formulas

$$\int_{\mathbb{R}} e^{-\pi\tau_2^{-1}(a+2b\tau_1+c\tau_1^2)}\,d\tau_1 \quad = \quad \sqrt{\tfrac{\tau_2}{c}}\,e^{(b^2-ac)\pi/(\tau_2 c)} \quad , \quad c > 0 \tag{E.23}$$

and

$$\int_0^\infty e^{-\pi a\tau_2 - \pi b\tau_2^{-1}}\,\frac{d\tau_2}{\tau_2^{3/2}} \quad = \quad b^{-1/2}\,e^{-2\pi\sqrt{ab}} \quad , \quad a,b > 0\,. \tag{E.24}$$

□

Therefore for Re s sufficiently large we can compute the following integral (which converges by proposition E.1) as

$$\begin{aligned} \int_0^\infty I(0,uG)\,u^{s-1}\,du \quad &= \sum_{\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] \in SL(2,\mathbb{Z})\backslash\mathcal{M}_{2,d}^{(2)}(\mathbb{Z})} \int_0^\infty \frac{e^{-2\pi u\mathcal{D}_{m,n,G}}}{\mathcal{D}_{m,n,G}}\,u^{s-2}\,du \\ &= \quad (2\pi)^{1-s}\,\Gamma(s-1) \sum_{\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] \in SL(2,\mathbb{Z})\backslash\mathcal{M}_{2,d}^{(2)}(\mathbb{Z})} \det(\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] G \left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]^t)^{-s/2}\,. \end{aligned} \tag{E.25}$$

Proposition 4.1 is now equivalent to the identification of the righthand side of (E.25) with the multiple of the $SL(d)$ Eisenstein series $E^{SL(d)}_{\beta_2;s}(e)$ given by Audrey Terras in [69, Lemma 1.1]. For completeness (and because the mechanism will be used later) we shall briefly sketch her argument. Since every relative prime vector in $\mathbb{Z}^d$ is the bottom row of a matrix in $SL(d,\mathbb{Z})$, every element $\left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right] \in \mathcal{M}^{(2)}_{2,d}(\mathbb{Z})$ can be factored as $\left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} x & a \\ 0_{d-1} & \gcd(v_2) \end{smallmatrix}\right]\gamma$ for some nonzero vector $x \in \mathbb{Z}^{d-1}$, $a \in \mathbb{Z}$, and $\gamma \in SL(d,\mathbb{Z})$, where 0_{d-1} denotes the $(d-1)$-dimensional zero vector. Thus

$$\left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]\gamma^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \gcd v_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \gcd x & 0 \\ 0 & 1 \end{smallmatrix}\right)\left[\begin{smallmatrix} x/\gcd(x) & 0 \\ 0_{d-1} & 1 \end{smallmatrix}\right]. \tag{E.26}$$

Since $x/\gcd(x)$ is a relatively prime vector in $\mathbb{Z}^{d-1}$ it is the bottom row of a matrix in $SL(d-1,\mathbb{Z})$, and so $\left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$ can be factored as the product of 2×2 upper triangular integer matrix with positive diagonal entries, times a $2\times d$ matrix which forms the bottom two rows of a matrix in $SL(d,\mathbb{Z})$. By adding integer multiples of the bottom row of this matrix to the row above it, we can reduce $b \pmod{\gcd(v_2)}$ and hence assume that this 2×2 matrix lies in the set $\mathcal{S}_+ := \{\left(\begin{smallmatrix} d_1 & b \\ 0 & d_2 \end{smallmatrix}\right) | d_1, d_2 \neq 0, 0 \le b < d_2\}$.

We now claim that the coset space $SL(2,\mathbb{Z})\backslash\mathcal{M}^{(2)}_{2,d}(\mathbb{Z})$ in the sum (E.25) is in bijection with products of the form $\gamma_1\gamma_2$, where $\gamma_1 \in \mathcal{S}_+$ and γ_2 are the bottom two rows of a fixed set of coset representatives for $P_{\beta_2}(\mathbb{Z})\backslash SL(d,\mathbb{Z})$. Recall that the latter is the quotient by $GL(2,\mathbb{Z})$ of all possible bottom two rows of matrices in $SL(d,\mathbb{Z})$. It is a standard result in the theory of Hecke operators that every right $GL(2,\mathbb{Z})$ translate of an element of $\mathcal{S}_+$ is left $SL(2,\mathbb{Z})$ equivalent to some element of $\mathcal{S}_+$ (this is because we allow for the possibility that $d_1 < 0$). Thus the previous paragraph shows that every coset in $SL(2,\mathbb{Z})\backslash\mathcal{M}^{(2)}_{2,d}(\mathbb{Z})$ has a factorization of this asserted form, and it remains to show uniqueness. After right multiplying by matrices in $SL(d,\mathbb{Z})$ it suffices to show that if

$$\left(\begin{smallmatrix} d_1 & b \\ 0 & d_2 \end{smallmatrix}\right)\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)\left(\begin{smallmatrix} d_1' & b' \\ 0 & d_2' \end{smallmatrix}\right)\left[\begin{smallmatrix} 0_{d-2} & 1 & 0 \\ 0_{d-2} & 0 & 1 \end{smallmatrix}\right] \tag{E.27}$$

for some $d_1, d_1' \neq 0$, $0 \le b < d_2$, $0 \le b' < d_2'$, $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, and $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$ which are the bottom two rows of one of these coset representatives for $P_{\beta_2}(\mathbb{Z})\backslash SL(d,\mathbb{Z})$, then $\left(\begin{smallmatrix} d_1 & b \\ 0 & d_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} d_1' & b' \\ 0 & d_2' \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0_{d-2} & 1 & 0 \\ 0_{d-2} & 0 & 1 \end{smallmatrix}\right]$. Indeed, (E.27) implies that all but the last two entries of w_1 and w_2 vanish, so that $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$ are the bottom two rows of a matrix in $P_{\beta_2}(\mathbb{Z})$ and hence equal to the representative $\left[\begin{smallmatrix} 0_{d-2} & 1 & 0 \\ 0_{d-2} & 0 & 1 \end{smallmatrix}\right]$ of its equivalence class in $P_{\beta_2}\backslash SL(d,\mathbb{Z})$. Then (E.27) reduces to the identity $\left(\begin{smallmatrix} d_1 & b \\ 0 & d_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)\left(\begin{smallmatrix} d_1' & b' \\ 0 & d_2' \end{smallmatrix}\right)$. Since $d_2, d_2' > 0$ and both sides have the same determinant, d_1 and d_1' have the same sign. Comparing the first columns then shows that $r = 0$, $p = 1$, and $d_1 = d_1'$. Consequently $s = 1$ and $d_2 = d_2'$. Finally, since $0 \le b, b' < d_2$ differ by qd_2 they must be equal and $q = 0$. This proves the claim.

Therefore the range of summation in (E.25) can be replaced by $P_{\beta_2}(\mathbb{Z})\backslash SL(d,\mathbb{Z})$, at the cost of multiplying the overall expression by $\sum_{d_1,d_2>0} d_2(d_1d_2)^{-s} = \sum_{n>0}\sigma_1(n)n^{-s} = \zeta(s)\zeta(s-1)$. This, along with definition (2.12) and standard Γ-function identities results in the first of the two equivalent formulas in (4.57).

Since the definition (E.15) of $I(0,uG)$ is an integral of a θ-function over the modular fundamental domain, proposition 4.1 indicates that the series $E^{SL(d)}_{\beta_2;s}(e)$ is the Mellin transform of a θ-lift of the constant function.

Appendix F. Identification of the $Spin(d,d)$ Epstein series with a lattice sum

In this appendix we prove that Langlands' definition of the maximal parabolic Eisenstein series $E^{Spin(d,d)}_{\alpha_1;s}$ as a sum over group cosets is equivalent to the lattice sum definition used in [51, 52]. It is easier to work directly with the group $SO(d,d,\mathbb{R})$, which is quotient of $Spin(d,d,\mathbb{R})$ by an order two subgroup of its center. We explained in (2.13) that $E^{Spin(d,d)}_{\alpha_1;s}$ is trivial on this subgroup and hence can be computed through $E^{SO(d,d)}_{\alpha_1;s}$.

Let w_n denote the anti-diagonal identity matrix obtained by reversing the columns of the $n\times n$ identity matrix. Define groups

$$\begin{aligned} G &= SO(d,d,\mathbb{R}) &= \{\, g\in SL(2d,\mathbb{R}) \mid g w_{2d} g^t = w_{2d}\,\}\,, \\ \Gamma &= SO(d,d,\mathbb{Z}) &= SO(d,d,\mathbb{R})\cap SL(2d,\mathbb{Z})\,, \end{aligned} \tag{F.1}$$

and

$$P = P_{\alpha_1} = \left\{ \left(\begin{smallmatrix} a & \star & \star \\ 0 & B & \star \\ 0 & 0 & c\end{smallmatrix}\right) \in G \mid a,c\in\mathbb{R}^*,\;\; B\in SO(d-1,d-1)(\mathbb{R}) \right\}. \tag{F.2}$$

Proposition F.1. With the above definitions

(i) If $g_1,\ g_2\in G = SO(d,d,\mathbb{R})$ have the same bottom row, then there exists some $p\in P$ such that $g_1 = pg_2$.

(ii) The bottom row v of a matrix in $G = SO(d,d,\mathbb{R})$ is orthogonal to its reverse $w_{2d}v$. In particular, if $v=[m\,n]$ then $m\perp w_d n$.

(iii) The map from a matrix to its bottom row gives a bijection from $(\Gamma\cap P)\backslash\Gamma$ to $\{v\in\mathbb{Z}^{2d}\mid \gcd(v)=1 \text{ and } v\perp w_{2d}v\}/\{\pm1\}$.

Proof. Let $e_1,\ldots,e_{2d}$ denote the standard basis vectors of $\mathbb{R}^{2d}$. In part (i), the bottom row of the matrix $g_1g_2^{-1}$ is $e_{2d}g_1g_2^{-1} = e_{2d}$, as can be seen by multiplying both sides by g_2. Thus $g_1g_2^{-1}$ has bottom row e_{2d}; membership of such a matrix in G forces its first column to equal a multiple of e_1, and so $g_1g_2^{-1}$ lies in P . Statement (ii) is a consequence of the defining property of G (since the bottom right entry of w_{2d} is zero).

Because of parts (i) and (ii) and the fact that the bottom row of a matrix in $\Gamma\cap P$ is $\pm e_{2d}$, part (iii) reduces to showing that every such vector v is the bottom row of some matrix γ in Γ. The calculation

$$\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}\begin{pmatrix} & w_d \\ w_d & \end{pmatrix}\begin{pmatrix} g_1^t & \\ & g_2^t \end{pmatrix} \;=\; \begin{pmatrix} & g_1 w_d g_2^t \\ g_2 w_d g_1^t & \end{pmatrix} \qquad \text{(F.3)}$$

shows that the matrix

$$\begin{pmatrix} \widetilde{g} & \\ & g \end{pmatrix} \;\in\; G \;\;, \quad \text{with } \; \widetilde{g} \;=\; w_d (g^t)^{-1} w_d \,, \qquad \text{(F.4)}$$

for any $g \in GL(d,\mathbb{R})$. Since g can be taken to be a matrix in $GL(d,\mathbb{Z})$ with an arbitrary relatively prime bottom row, the proposition reduces to the case when v has the form $v = [m\, n]$, where $m, n \in \mathbb{Z}^d$ and n has the special form $[0\, 0 \,\cdots\, 0\, k]$ (to see this, replace v by $v \left(\begin{smallmatrix} \widetilde{g} & \\ & g \end{smallmatrix}\right)$). The orthogonality condition in part (ii) shows that we may furthermore take m to have the form $[0\, a_2 \,\cdots\, a_d]$ for integers $a_2, \cdots, a_d$. Consider $g = \left(\begin{smallmatrix} A & b \\ & 1 \end{smallmatrix}\right) \in GL(d,\mathbb{Z})$, so that $\widetilde{g} = \left(\begin{smallmatrix} 1 & \star \\ & \widetilde{A} \end{smallmatrix}\right)$. Multiplying on the right by $\left(\begin{smallmatrix} \widetilde{g} & \\ & g \end{smallmatrix}\right)$ has the effect of replacing $[a_2\, a_3 \,\cdots\, a_d]$ by $[a_2\, a_3 \,\cdots\, a_d]\widetilde{A}$. Since $\widetilde{A}$ can be an arbitrary matrix in $GL(d-1,\mathbb{Z})$, we may arrange that $[a_2\, a_3 \,\cdots\, a_d]h = [0\, 0 \,\cdots\, 0\, r]$ for some integer r. The condition that the bottom row be relatively prime now states that $\gcd(r,k) = 1$. Such a matrix exists because of the homomorphism of $SL(2,\mathbb{R})$ into G which sends a matrix $\left(\begin{smallmatrix} a & b \\ r & k \end{smallmatrix}\right)$ to one with entries a on the 1st and d-th diagonal entries, entries k on the $d+1$-st and $2d$-th diagonal entries, $-b$ in the $(1,d+1)$ position, b in the $(d,2d)$ position, $-c$ in the $(d+1,1)$ position, and c in the $(2d,d)$ position. □

Proposition F.2. Let $h \in SO(d,d,\mathbb{R})$ and write $hh^t = \left(\begin{smallmatrix} H_1 & H_2 \\ H_2^t & H_3 \end{smallmatrix}\right)$, where H_1, H_2, and H_3 are $d \times d$ matrices. Then the maximal parabolic Eisenstein series $E^{SO(d,d)}_{\alpha_1;s}(h)$ associated to the first node (i.e., "vector node") of the D_d Dynkin diagram is given by

$$2\,\zeta(2s)\, E^{SO(d,d)}_{\alpha_1;s}(h) \;=\; \sum_{\substack{m,n\,\in\mathbb{Z}^d \\ m\,\perp\, w_d n \\ (m,n)\,\neq\,(0,0)}} (mH_1m^t + 2mH_2n^t + nH_3n^t)^{-s} \qquad \text{(F.5)}$$

for Re s large (where the sum is absolutely convergent). The same formula holds for $E^{Spin(d,d)}_{\alpha_1;s}(h')$, where $h' \in Spin(d,d,\mathbb{R})$ projects to $h \in SO(d,d)$ under the covering map.

Proof. The $SO(d,d)$ Epstein series is formed by averaging the function $f(g) = e^{2s\omega_1(H(g))}$ over $g = \gamma h$, $\gamma \in (\Gamma \cap P)\backslash\Gamma$. Recall that $f(pgk) = f(g)$ for all $p \in P$ such that each diagonal block a, B, and c in (F.2) has determinant ± 1, and for all k in the maximal compact subgroup of G. Calculation of f thus reduces via the Iwasawa decomposition to the case when g is diagonal, in which case $f(g)$ equals the $-2s$ power of the bottom right entry of g. The bottom right entry of the Iwasawa factor of g must be the norm of g's bottom row, because of these invariance properties. Hence $f(\gamma h)$ is the norm of the bottom row of γh to the $-2s$ power. If $v = [m\, n] \in \mathbb{Z}^{2d}$ is the bottom row of γ, then the norm is the squareroot of $vhh^tv^t = mH_1m^t + 2mH_2n^t + nH_3n^t$.

Thus $E^{SO(d,d)}_{\alpha_1;s}$ is given by a sum as stated, but with a gcd and modulo ± 1 condition which, when removed, results in the factor $2\zeta(2s)$ in (F.5). □

For later reference we remark that since $w_d^2 = 1$ we can present these D_d Epstein series as

$$\begin{aligned} 2\,\zeta(2s)\,E^{SO(d,d)}_{\alpha_1;s}(h') &= 2\,\zeta(2s)\,E^{SO(d,d)}_{\alpha_1;s}(h) \\ &= \sum_{\substack{m,n\in\mathbb{Z}^d \\ m\perp n \\ (m,n)\neq(0,0)}} (mH_1m^t + 2mH_2w_dn^t + nw_dH_3w_dn^t)^{-s} \end{aligned} \quad \text{(F.6)}$$

for Re s large. Also we note that the condition for a matrix of the form $\begin{pmatrix} I_d & X \\ & I_d \end{pmatrix}$ to lie in $G = SO(d,d,\mathbb{R})$ is that

$$\begin{pmatrix} Xw_d & w_d \\ w_d & \end{pmatrix} = \begin{pmatrix} I_d & X \\ & I_d \end{pmatrix}\begin{pmatrix} & w_d \\ w_d & \end{pmatrix} = \begin{pmatrix} & w_d \\ w_d & \end{pmatrix}\begin{pmatrix} I_d & \\ -X^t & I_d \end{pmatrix} = \begin{pmatrix} -w_dX^t & w_d \\ w_d & \end{pmatrix}, \quad \text{(F.7)}$$

i.e., that Xw_d is antisymmetric.

Appendix G. A theta lift between $SL(d)$ and $Spin(d,d)$ Eisenstein series

In proposition 4.1 we stated a relation between the modular integral $I(s,G)$ and the non-Epstein Eisenstein series $E^{SL(d)}_{\beta_2;s}$. In this section we see another relation (proposition G.1) to the Epstein Eisenstein series $E^{D_d}_{\alpha_1;s+d/2-1}$, where D_d is written as a shorthand for statements that apply both to $SO(d,d)$ and $Spin(d,d)$. Thus both can be thought of as θ-lifts from the usual nonholomorphic $SL(2,\mathbb{Z})$ Eisenstein series. We shall do this for Re s large, the range of absolute convergence of the Eisenstein series, and then meromorphically continue to $s \in \mathbb{C}$.

Unfolding the Eisenstein series in (E.15) gives the formula

$$I(s, G+B) = \int_0^\infty \int_0^1 \mathcal{G}(\tau_1 + i\tau_2, G+B)\, d\tau_1 \frac{d\tau_2}{\tau_2^{2-s}}\,. \quad \text{(G.1)}$$

This integral is absolutely convergent for Re s large by (E.14) and the bounds given in (E.7). We write $G = ee^t$ and introduce the notation $\|v\|^2 = v\bar{v}^t$ if v is a complex row vector. Using (E.9) we write

$$\begin{aligned} \mathcal{G}(\tau_1 + i\tau_2, ee^t + B) &= \\ &\sum_{\substack{n\neq 0 \\ m\in\mathbb{Z}^d}} \exp(-\pi\tau_2^{-1}\|(m+n\tau_1)e\|^2 - \pi\tau_2\|ne\|^2 - 2\pi i mBn^t) \\ &- \sum_{\substack{n\neq 0 \\ m\in\mathbb{Z}^d\cap\mathbb{Q}n}} \exp(-\pi\tau_2^{-1}\|(m+n\tau_1)e\|^2 - \pi\tau_2\|ne\|^2)\,, \end{aligned} \quad \text{(G.2)}$$

the second sum including all $m \in \mathbb{Z}^d$ which are parallel to n (a condition which forces $mBn^t = 0$). Accordingly break up $I(s, ee^t + B)$ as $I_1(s, ee^t + B) - I_2(s, ee^t + B)$, where $I_1(s, ee^t + B)$ and $I_2(s, ee^t + B)$ represent the integral (G.1) with $\mathcal{G}(\tau, G+B)$ replaced by the sums in the first and second lines of (G.2), respectively. Both integrals are absolutely convergent and can be interchanged with their respective summations for Re s sufficiently large. We first compute

$$\begin{aligned} I_2(s, G+B) &= \\ \sum_{n \neq 0} \int_0^\infty \int_0^1 &\sum_{m \in \mathbb{Z}^d \cap \mathbb{Q}n} \exp(-\pi \tau_2^{-1} \|(m + n\tau_1)e\|^2 - \pi \tau_2 \|ne\|^2) d\tau_1 \frac{d\tau_2}{\tau_2^{2-s}} \, . \end{aligned} \tag{G.3}$$

Since n is nonzero and m is a multiple of n, we change variables by subtracting this multiple from τ_1 to eliminate the occurrence of m in the integrand. Doing so unfolds the τ_1 integration from $[0, 1]$ to $\mathbb{R}$ by gathering together all m which are $\mathbb{Z}n$-translates of each other, though we must take into account the fact that there are gcd(n) orbits of $\{m \in \mathbb{Z}^d \cap \mathbb{Q}n\}$ under $m \mapsto m + n$:

$$\begin{aligned} I_2(s, G+B) &= \sum_{n \neq 0} \gcd(n) \int_0^\infty \int_{\mathbb{R}} \exp(-\pi(\tau_2 + \tau_1^2 \tau_2^{-1}) \|ne\|^2) d\tau_1 \frac{d\tau_2}{\tau_2^{2-s}} \\ &= \sum_{n \neq 0} \gcd(n) \int_0^\infty \exp(-\pi \tau_2 \|ne\|^2) \tfrac{\sqrt{\tau_2}}{\|ne\|} \frac{d\tau_2}{\tau_2^{2-s}} \\ &= \pi^{\frac{1}{2} - s} \, \Gamma(s - \tfrac{1}{2}) \sum_{n \neq 0} \gcd(n) \|ne\|^{-2s} \, . \end{aligned} \tag{G.4}$$

We now evaluate this last sum, writing $e = r^{1/2} e'$, where $\det e' = 1$, and $r = (\det e)^{2/d}$. Decomposing $n \in \mathbb{Z}^d_{\neq 0}$ as $n = km$, with gcd$(n) = k$ and gcd$(m) = 1$, it equals

$$r^{-s} \sum_{\substack{m \in \mathbb{Z}^d \\ \gcd(m)=1}} \sum_{k=1}^{\infty} k^{1-2s} \, \|me'\|^{-2s} \quad = \quad 2 \, r^{-s} \, \zeta(2s-1) \, E^{SL(d)}_{\beta_1;s}(e') \, , \tag{G.5}$$

so

$$I_2(s, ee^t + B) = 2 \, (\det e)^{-2s/d} \, \xi(2s-1) \, E^{SL(d)}_{\beta_1;s}(e) \, , \tag{G.6}$$

initially for Re s sufficiently large and then by meromorphic continuation to $s \in \mathbb{C}$.

Next we compute

$$I_1(s, ee^t + B) \quad = \quad \int_0^\infty \int_0^1 \sum_{\substack{n \neq 0 \\ m \in \mathbb{Z}^d}} \exp(-\pi\tau_2^{-1}\|(m+n\tau_1)e\|^2 - \pi\tau_2\|ne\|^2 - 2\pi i m B n^t) d\tau_1 \frac{d\tau_2}{\tau_2^{2-s}} \,. \tag{G.7}$$

Poisson summation allows us to rewrite

$$\begin{aligned} &\sum_{m \in \mathbb{Z}^d} \exp(-\pi\tau_2^{-1}\|(m+n\tau_1)e\|^2 - 2\pi i m B n^t) \\ = &\sum_{\hat{m} \in \mathbb{Z}^d} \exp(2\pi i \hat{m}\cdot n\tau_1) \int_{\mathbb{R}^d} \exp(-2\pi i \hat{m}\cdot m - \pi\tau_2^{-1}\|me\|^2 - 2\pi i m B n^t)\, dm \\ = &\sum_{\hat{m} \in \mathbb{Z}^d} \exp(2\pi i \hat{m}\cdot n\tau_1)(\det e)^{-1}\tau_2^{d/2} \exp(-\pi\tau_2\|(\hat{m}-nB)(e^{-1})^t\|^2)\,, \end{aligned} \tag{G.8}$$

where $\hat{m}$ is thought of as a row vector. Therefore the integration over τ_1 then forces $\hat{m} \perp n$:

$$\begin{aligned} &I_1(s, ee^t + B) \\ = \;&(\det e)^{-1} \int_0^\infty \sum_{\substack{n \neq 0 \\ \hat{m} \in \mathbb{Z}^d \\ \hat{m} \perp n}} \exp(-\pi\tau_2\|ne\|^2 - \pi\tau_2\|(\hat{m}-nB)(e^{-1})^t\|^2) \frac{d\tau_2}{\tau_2^{2-s-\frac{d}{2}}} \\ = \;&\frac{\Gamma(s+\frac{d}{2}-1)}{(\det e)\,\pi^{s+\frac{d}{2}-1}} \sum_{\substack{n \neq 0 \\ \hat{m} \in \mathbb{Z}^d \\ \hat{m} \perp n}} (nGn^t + (\hat{m}-nB)G^{-1}(\hat{m}-nB)^t)^{1-s-d/2}\,. \end{aligned} \tag{G.9}$$

Again write $e = r^{1/2}e'$ with $\det e' = 1$, and $r = (\det e)^{2/d}$ (thus $G = re'(e')^t$). Recall (F.4) and define an element $h \in SO(d,d,\mathbb{R})$ by

$$h \;=\; \begin{pmatrix} I & Bw_d \\ & I \end{pmatrix} \begin{pmatrix} r^{1/2}e' & \\ & r^{-1/2}\tilde{e}' \end{pmatrix}, \quad hh^t \;=\; \begin{pmatrix} G+BG^{-1}B^t & BG^{-1}w_d \\ w_dG^{-1}B^t & w_dG^{-1}w_d \end{pmatrix}. \tag{G.10}$$

Then the inside sum in (G.9) is computed using (F.6) as

$$\begin{aligned} &2\,\zeta(2s+d-2)\, E^{D_d}_{\alpha_1; s+d/2-1}(h) - \sum_{\hat{m} \in \mathbb{Z}^d_{\neq 0}} \|\hat{m}(e^{-1})^t\|^{2-2s-d} \\ = \;&2\,\zeta(2s+d-2)\, E^{D_d}_{\alpha_1; s+d/2-1}(h) - (\det e)^{\frac{d+2s-2}{d}} \sum_{\hat{m} \in \mathbb{Z}^d_{\neq 0}} \|\hat{m}((e')^t)^{-1}\|^{2-2s-d} \\ = \;&2\,\zeta(2s+d-2) \left(E^{D_d}_{\alpha_1; s+d/2-1}(h) - (\det e)^{\frac{d+2s-2}{d}} E^{SL(d)}_{\beta_{d-1}; s+d/2-1}(e')\right). \end{aligned} \tag{G.11}$$

Combining (G.6), (G.9), and (G.11) we conclude the following:

Proposition G.1. With h as in (G.10)

$$\begin{aligned} I(s, ee^t + B) &= 2\,(\det e)^{-1}\,\xi(2s+d-2)\,E^{D_d}_{\alpha_1;s+d/2-1}(h) \\ &\quad - 2\,(\det e)^{\frac{2s-2}{d}}\,\xi(2s+d-2)\,E^{SL(d)}_{\beta_{d-1};s+d/2-1}(e') \\ &\qquad - 2\,(\det e)^{-\frac{2s}{d}}\,\xi(2s-1)\,E^{SL(d)}_{\beta_1;s}(e')\,, \quad \text{(G.12)} \end{aligned}$$

initially for Re s large, and then to all $s \in \mathbb{C}$ by meromorphic continuation.

As before, all manipulations are valid because of the absolute convergence of the sums and integral involved and the assumption that Re s is sufficiently large. Note that only the first line on the righthand side depends nontrivially on B. In particular, if $B = 0$ and $s = 0$ the above equation provides an integral representation for $E^{D_d}_{\alpha_1;d/2-1}$,

$$\begin{aligned} 2\,u^{-d/2}\xi(d-2)\,E^{D_d}_{\alpha_1;d/2-1}\begin{pmatrix} u^{1/2}e' & \\ & u^{-1/2}\tilde{e}' \end{pmatrix} &= \\ I(0, ue'(e')^t) \,+\, 2\,u^{-1}\,\xi(d-2)\,E^{SL(d)}_{\beta_{d-1};d/2-1}(e') \,+\, 2\,\xi(2)\,, &\quad \text{(G.13)} \end{aligned}$$

for any $e' \in SL(d,\mathbb{R})$. A similar expression appeared in [2,51,52] but without that the last two terms in the second line.

Remark: According to proposition E.1 $I(0, uG)$ decays rapidly to zero as $u \to \infty$. This is not immediately obvious from (G.13), in which both other terms involving u have polynomial behavior in that limit while the remaining term is constant. However, the aggregate sum indeed does decay to zero. This can be seen explicitly through an analysis of the constant term of the $E^{D_d}_{\alpha_1;d/2-1}$ Eisenstein series in the spinor parabolic (which determines these asymptotic behaviors).

Appendix H. Fourier modes of Eisenstein series

In this appendix we will present details of the Fourier modes of Eisenstein series that enter in the expressions for the coefficients $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ when $D = 8$, $D = 7$, and $D = 6$ (although the discussion of the $D = 6$ case with symmetry group $Spin(5,5)$ is incomplete). This summarises and extends the string theory results in [2] (see [7, 51, 52, 70, 71] for related investigations).

H.1. **The $SL(3)\times SL(2)$ case.** The results of this subsection are used in section 4.4 in the text. The coefficients are functions of both the $SL(2)/SO(2)$ symmetric space, which depends on $\mathcal{U} = \mathcal{U}_1 + i\mathcal{U}_2$ (the complex structure of the 2-torus, T^2), and the $SL(3)/SO(3)$ space, which depends on 5 parameters. We will parametrise the $SL(2)/SO(2)$ coset by (4.12) (with Ω replaced by $\mathcal{U}$) while the $SL(3)/SO(3)$ coset will be parameterised by the string fluxes as

$$e_3 = \begin{pmatrix} 1 & B_{\rm NS} & C^{(2)} + \Omega_1 B_{\rm NS} \\ 0 & 1 & \Omega_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \nu_2^{-\frac{1}{3}} & 0 & 0 \\ 0 & \nu_2^{\frac{1}{6}}\sqrt{\Omega_2} & 0 \\ 0 & 0 & \frac{\nu_2^{\frac{1}{6}}}{\sqrt{\Omega_2}} \end{pmatrix}, \tag{H.1}$$

where $\nu_2^{-\frac{1}{2}} = r_1 r_2/\ell_{10}^2 = \sqrt{\Omega_2}T_2$ is the volume of the 2-torus in 10 dimensional Planck units and $T_2 = r_1 r_2/\ell_s^2$ is the volume in string units. The five parameters of the coset are packaged into $(\Omega, T, C^{(2)})$, where $\Omega = \Omega_1 + i\Omega_2$ and $T = T_1 + iT_2$ (where $T_1 = B_{\rm NS}$). We shall also make use of the combination $y_8^{-1} = \Omega_2^2 T_2$, which is the square of the inverse string coupling. The complex parameter T is interpreted as the Kähler structure of T^2.

The coefficient functions $\mathcal{E}^{(8)}_{(0,0)}$ and $\mathcal{E}^{(8)}_{(1,0)}$ are solutions of (2.6) and (2.7) with $D = 8$ [2, 48],

$$\Delta^{(8)}\, \mathcal{E}^{(8)}_{(0,0)} = 6\pi \tag{H.2}$$

$$(\Delta^{(8)} - \frac{10}{3})\, \mathcal{E}^{(8)}_{(1,0)} = 0\,, \tag{H.3}$$

where the $SL(3) \times SL(2)$ Laplace operator is defined in terms of the parameters introduced above by

$$\Delta^{(8)} := \Delta^{SL(3)} + 2\Delta_{U}^{SL(2)}\,, \tag{H.4}$$

with

$$\Delta^{SL(3)} = \Delta_\Omega + \frac{|\partial_{B_{\rm NS}} - \Omega\partial_{C^{(2)}}|^2}{\nu_2\Omega_2} + 3\partial_{\nu_2}(\nu_2^2\partial_{\nu_2}) \tag{H.5}$$

$$\text{and} \quad \Delta_Z^{SL(2)} = Z_2^2\,(\partial_{Z_1}^2 + \partial_{Z_2}^2)\,, \tag{H.6}$$

where $Z = Z_1 + iZ_2$ and $Z = \Omega$ or $\mathcal{U}$. The fact that the eigenvalue in (H.2) vanishes, together with the presence of the 6π on the right-hand side, is related to the presence of a 1-loop ultraviolet divergence in eight-dimensional maximally supersymmetric supergravity [5].

The solutions to these equations are given in terms of $SL(2)$ and $SL(3)$ Eisenstein series. The $SL(2)$ series is given by (4.13) while the $SL(3)$ (Epstein) Eisenstein series is given by

$$2\zeta(2s)\, E^{SL(3)}_{\alpha_1;s}(e_3) = \sum_{M_3 \in \mathbb{Z}^3 \backslash \{0\}} (m^2_{SL(3)})^{-s}\,, \tag{H.7}$$

where, setting $M_3 = (m_1\, m_2\, m_3) \in \mathbb{Z}^3$, the mass squared is given by

$$\begin{aligned} m^2_{SL(3)} &:= M_3 G_3 M_3^t \\ &= \frac{\nu_2^{\frac{1}{3}}}{\Omega_2}\left(|m_3 + m_2\Omega + \mathcal{B}m_1|^2 + (m_1\Omega_2 T_2)^2\right) \end{aligned} \tag{H.8}$$

with

$$G_3 := e_3 e_3^t = \nu_2^{\frac{1}{3}} \begin{pmatrix} \nu_2^{-1} + (G_2)_{ab} B^a B^b & (G_2)_{ab} B^b \\ (G_2)_{ab} B^a & (G_2)_{ab} \end{pmatrix}, \tag{H.9}$$

$$G_2 := \frac{1}{\Omega_2} \begin{pmatrix} |\Omega|^2 & \Omega_1 \\ \Omega_1 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} B_{\rm NS} \\ C^{(2)} \end{pmatrix}, \quad \text{and} \quad \mathcal{B} := C^{(2)} + \Omega B_{\rm NS}. \tag{H.10}$$

The Eisenstein series $E^{SL(3)}_{\alpha_1;s}$ is related to $E^{SL(3)}_{\alpha_2;s}$ by the functional relation

$$\xi(2s)\ E^{SL(3)}_{\alpha_1;s}(e_3) = \xi(3-2s)\, E^{SL(3)}_{\alpha_2;\frac{3}{2}-s}(e_3). \tag{H.11}$$

The Fourier modes of the coefficient functions can now be considered in each of the three parabolic subgroups of interest, after putting the $SL(3,\mathbb{Z})$ part together with the $SL(2,\mathbb{Z})$ part. The unipotent radicals in these three cases are given by:

(i) **The unipotent radical U_{α_3} of the nonmaximal parabolic $P_{\alpha_3} = GL(1) \times SL(2) \times \mathbb{R}^+ \times U_{\alpha_3}$.** As noted earlier, the relevant parabolic is nonmaximal in order to match the $D = 9$ duality group associated with the decompactification limit. The unipotent radical is parametrized by $(C^{(2)}, B_{\rm NS})$ and takes the block diagonal form,

$$U_{\alpha_3} = \begin{pmatrix} \begin{pmatrix} 1 & B_{\rm NS} & C^{(2)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & \mathcal{U}_1 \\ 0 & 1 \end{pmatrix} \end{pmatrix}. \tag{H.12}$$

In the maximal parabolic subgroup of $SL(3)$ determined by $B_{\rm NS}$ and $C^{(2)}$ the Fourier coefficients of the $SL(3)$ Eisenstein series in (H.7) are defined by[31]

$$F^{SL(3)\beta_2}_{\beta_1;s}(kp_1, kp_2) := \int_{[0,1]^2} dB_{\rm NS} dC^{(2)}\, e^{-2i\pi k(p_1 C^{(2)} + p_2 B_{\rm NS})}\, E^{SL(3)}_{\alpha_1;s}, \tag{H.13}$$

with $\gcd(p_1, p_2) = 1$. Extending the constant term computation in [2, Appendix B.4], the Fourier coefficients for $k \neq 0$ are

$$F^{SL(3)\,\beta_2}_{\beta_1;s}(kp_1, kp_2) = \frac{2}{\xi(2s)}\, \Omega_2^{1-\frac{2s}{3}} T_2^{1-\frac{s}{3}}\, \frac{\sigma_{2s-2}(k)}{|k|^{s-1}}\, \frac{K_{s-1}(2\pi |k|\, |p_2 + p_1\Omega| T_2)}{|p_2 + p_1\Omega|^{1-s}}. \tag{H.14}$$

The Fourier modes of the $SL(2)$ series are defined as

$$F^{SL(2)}_s(k') := \int_{[0,1]} d\mathcal{U}_1\, e^{-2i\pi k' \mathcal{U}_1}\, E^{SL(2)}_s(\mathcal{U}), \tag{H.15}$$

where

$$F^{SL(2)}_s(k') = \frac{2\sqrt{\mathcal{U}_2}}{\xi(2s)}\, \frac{\sigma_{2s-1}(|k'|)}{|k'|^{s-\frac{1}{2}}}\, K_{s-\frac{1}{2}}(2\pi |k'| \mathcal{U}_2) \tag{H.16}$$

for $k' \neq 0$ (cf. (4.17)).

[31] The labelling of the simple roots β_1 and β_2 on these Fourier coefficients uses the conventional labelling of the $SL(3)$ Dynkin diagram according the convention in figure 3.

Putting this together, the Fourier modes of the product of the $SL(3)$ and the $SL(2)$ series are given by

$$\begin{aligned} F^{SL(3)\times SL(2)\alpha_3}_{\alpha_1;s,s'}(kp_1,kp_2,k') &:= \int_{[0,1]^2} dB_{\rm NS} dC^{(2)}\, e^{-2i\pi k(p_1 C^{(2)}+p_2 B_{\rm NS})}\, E^{SL(3)}_{\beta_1;s} \\ &\quad \times \int_{[0,1]} d\mathcal{U}_1\, e^{-2i\pi k'\mathcal{U}_1}\, E^{SL(2)}_{s'}(\mathcal{U}) \\ &= \mathcal{F}^{SL(3)\beta_2}_{\beta_1;s}(kp_1,kp_2)\,\mathcal{F}^{SL(2)\alpha_3}_{s'}(k')\,, \end{aligned} \tag{H.17}$$

with $\gcd(p_1,p_2)=1$. These results are used in (4.28) and (4.29), where we provide a physical interpretation of the Fourier modes in the decompactification regime (limit (i) in the notation of (2.9)).

(ii) **The unipotent radical U_{α_1} of the maximal parabolic subgroup** $P_{\alpha_1} = GL(1)\times Spin(2,2)\times U_{\alpha_1}$ associated with the string perturbation regime is parametrized by $(\Omega_1, C^{(2)})$ and takes the form,

$$U_{\alpha_1} = \begin{pmatrix} \begin{pmatrix} 1 & 0 & C^{(2)} \\ 0 & 1 & \Omega_1 \\ 0 & 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}. \tag{H.18}$$

In this maximal parabolic only the $SL(3)$ series have non-vanishing Fourier coefficients, which are defined by

$$F^{SL(3)\beta_1}_{\beta_1;s}(kp_1,kp_2) := \int_{[0,1]^2} d\Omega_1 dC^{(2)}\, e^{-2i\pi k(p_1 C^{(2)}+p_2\Omega_1)}\, E^{SL(3)}_{\alpha_1;s} \tag{H.19}$$

with $\gcd(p_1,p_2)=1$. Extending the constant term calculation in [2, Appendix B.4] leads to

$$F^{SL(3)\,\beta_1}_{\beta_1;s}(kp_1,kp_2) = \frac{2}{\xi(2s)}\, T_2^{\frac{2s}{3}}\, \Omega_2^{\frac12+\frac{s}{3}}\, \frac{\sigma_{2s-1}(k)}{|k|^{s-\frac12}}\, \frac{K_{s-\frac12}(2\pi|k|\,|p_1T+p_2|\Omega_2)}{|p_1T+p_2|^{s-\frac12}}\,. \tag{H.20}$$

These results are used in (4.41) and (4.43), where we provide a physical interpretation of the Fourier modes in the perturbative regime (limit (ii) in the notation of (2.9))

(iii) **The unipotent radical U_{α_2} of the maximal parabolic** $P_{\alpha_2} = GL(1)\times SL(3)\times U_{\alpha_2}$ associated with the semi-classical M-theory limit is parametrized by $\mathcal{U}_1$ and takes the form

$$U_{\alpha_2} = \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & \mathcal{U}_1 \\ 0 & 1 \end{pmatrix} \end{pmatrix}, \tag{H.21}$$

In this maximal parabolic subgroup only the $SL(2)$ series has non-vanishing Fourier coefficients, which are given in (H.15-H.16). The evaluation of the non-zero Fourier coefficients of $\mathcal{E}^{(8)}_{(0,0)}$ and $\mathcal{E}^{(8)}_{(1,0)}$ in each of the three limits of interest is straightforwardly obtained by using the above expressions, and is discussed in section 4.4.

H.2. **The $SL(5)$ case.** Here we consider the Fourier modes of the Eisenstein series that enter the expressions for the coefficients $\mathcal{E}^{(7)}_{(0,0)}$ and $\mathcal{E}^{(7)}_{(1,0)}$ that are used in section 4.5 in the text.

In $D=7$ dimensions the coefficient functions are automorphic under the action of the duality group $SL(5,\mathbb{Z})$ and are functions on the 14-dimensional coset space $SL(5)/SO(5)$, which is parametrized, using the notation that arises from string theory, by

$$e_5 = \begin{pmatrix} & B^1_{\rm NS} & C^{(2)\,1}+\Omega_1 B^1_{\rm NS}\\ u_3 & B^2_{\rm NS} & C^{(2)\,2}+\Omega_1 B^2_{\rm NS}\\ & B^3_{\rm NS} & C^{(2)\,3}+\Omega_1 B^3_{\rm NS}\\ 0 & 1 & \Omega_1\\ 0 & 0 & 1\end{pmatrix} \begin{pmatrix} & \nu_3^{-2/15} D_3 & & 0 & 0\\ & & & 0 & 0\\ & & & 0 & 0\\ 0 & 0 & 0 & \nu_3^{\frac15}\sqrt{\Omega_2} & 0\\ 0 & 0 & 0 & 0 & \frac{\nu_3^{\frac15}}{\sqrt{\Omega_2}}\end{pmatrix}, \tag{H.22}$$

where Ω_2 is the inverse string coupling constant, Ω_1 is the type IIB RR pseudoscalar, and $B^i_{\rm NS}$ and $C^{(2)\,i}$ $(i=1,\dots,3)$ the NS and RR charges. The quantity u_3 is a 3×3 unit upper triangular matrix and D_3 is a 3×3 diagonal matrix with $\det D_3 = 1$. These are defined so that $\tilde e_3 = u_3 D_3$ or equivalently $\tilde G_3 = \tilde e_3\cdot \tilde e_3^t$ parametrizes the coset $SL(3)/SO(3)$ describing the perturbative string compactified on a three-torus. We will make use of the following combinations,

$$\nu_3^{-1} = \left(\frac{r_1r_2r_3}{\ell_{10}^3}\right)^2 = \Omega_2^{\frac32}\left(\frac{r_1r_2r_3}{\ell_s^3}\right)^2, \qquad y_7^{-1} = \Omega_2^2\,\frac{r_1r_2r_3}{\ell_s^3}, \tag{H.23}$$

where r_1, r_2 and r_3 are the radii of T^3 and y_7 is the 7-dimensional string coupling. Note that ν_3 is invariant under the action of $SL(3)\times SL(2)$.

The coset space $SL(5)/SO(5)$ is parametrized by the metric $G_5 = e_5 e_5^t$,

$$G_5 = \nu_3^{\frac25}\begin{pmatrix}\nu_3^{-\frac23}(\tilde G_3)_{ij} + (G_2)_{ab}B^a_iB^b_j & (G_2)_{ab}B^b_j\\ (G_2)_{ab}B^a_j & (G_2)_{ab}\end{pmatrix}, \tag{H.24}$$

where again

$$G_2 = \frac{1}{\Omega_2}\begin{pmatrix}|\Omega|^2 & \Omega_1\\ \Omega_1 & 1\end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix}B^1_{\rm NS} & B^2_{\rm NS} & B^3_{\rm NS}\\ C^{(2)\,1} & C^{(2)\,2} & C^{(2)\,3}\end{pmatrix}. \tag{H.25}$$

The $SL(5)$ mass squared is given by the quadratic form

$$\begin{aligned} m^2_{SL(5)} &:= M_5 G_5 M_5^t \qquad\qquad (\mathrm{H.26})\\ &= \nu_3^{\frac25}\,\frac{|m_1+m_2\Omega + n^t\cdot(C^{(2)}+\Omega B_{\rm NS})|^2}{\Omega_2} + \frac{n\tilde G_3 n^t}{\nu_3^{\frac{4}{15}}}, \end{aligned}$$

where $M_5 := [n_1\, n_2\, n_3\, m_2\, m_1] \in \mathbb{Z}^5 \backslash \{0\}$, $n := [n_1\, n_2\, n_3]$, and $B_{\rm NS}$ and $C^{(2)}$ are the first and second rows of the matrix B, respectively. This expression will later be useful for describing the $SL(5)$ Eisenstein series.

The $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS coefficients, $\mathcal{E}^{(7)}_{(0,0)}$ and $\mathcal{E}^{(7)}_{(1,0)}$, that solve (2.6) and (2.7) together with the appropriate boundary conditions are given[32] in [2] by linear combinations of the $E^{SL(5)}_{\beta_1;s}$ and $E^{SL(5)}_{\beta_3;s}$ Eisenstein series as described in (4.47-4.48). The definitions and Fourier expansions of the Eisenstein series in this expression will now be reviewed.

H.2.1. *Fourier modes of the series* $E^{SL(5)}_{\beta_1;s}$.

The $E^{SL(5)}_{\beta_1;s}$ series may be written using (H.26) in the form

$$2\zeta(2s)\, E^{SL(5)}_{\beta_1;s} = \sum_{M_5 \in \mathbb{Z}^5 \backslash \{0\}} (M_5 G_5 M_5^t)^{-s}\,. \tag{H.27}$$

The constant terms with respect to the maximal parabolic subgroups P_{β_3}, P_{β_1}, and P_{β_4} (corresponding to limits (i), (ii), and (iii), respectively) were evaluated in [2]. Note that in terms of our matrix identification used in (H.27), P_{α_1} corresponds to the subgroup of $SL(5)$ whose bottom row has the form $(0\,0\,0\,0\,\star)$.

(i) **The parabolic** $P_{\beta_3} = GL(1) \times SL(2) \times SL(3) \times U_{\beta_3}$.

The unipotent radical for this parabolic subgroup is abelian and is given by

$$U_{\beta_3} = \begin{pmatrix} I_2 & Q_4 \\ 0 & I_3 \end{pmatrix}\,, \quad \text{with} \quad Q_4 = \begin{pmatrix} G_{13} & B^1_{\rm NS} & C^{(2)\,1} + \Omega_1 B^1_{\rm NS} \\ G_{23} & B^2_{\rm NS} & C^{(2)\,2} + \Omega_1 B^2_{\rm NS} \end{pmatrix}. \tag{H.28}$$

The Fourier modes are defined by

$$F^{SL(5)\beta_3}_{\beta_1;s}(N_4) := \int_{[0,1]^6} d^6 Q_4\, e^{-2i\pi\ {\rm tr}(N_4 Q_4)}\, E^{SL(5)}_{\beta_1;s}\,, \tag{H.29}$$

where $N_4 \in M(3,2;\mathbb{Z})$.

For all values of s the Fourier modes are only non-zero when N_4 has rank 1. Such a matrix can be written as $N_4 = k\,\tilde N_4$, with $\gcd(\tilde N_4) = 1$ and

$$\tilde N_4 = n^t m = \begin{pmatrix} m_1 n_1 & m_2 n_1 \\ m_1 n_2 & m_2 n_2 \\ m_1 n_3 & m_2 n_3 \end{pmatrix}\,, \quad n = (n_i) \in \mathbb{Z}^3,\, m = (m_a) \in \mathbb{Z}^2\,. \tag{H.30}$$

The decomposition $N_4 = k n^t m$ of the rank one matrix N_4 is unique up to signs of the factors. Moreover, $\gcd(n_1, n_2, n_3) = \gcd(m_1, m_2) = 1$.

Poisson resummation on two integers, keeping the off-diagonal terms in the parametrisation of [2, section B.5.2], results in the following formula for

[32] In [2] these series were defined as $\mathbf{E}^{SL(5)}_{[1000];s} = 2\zeta(2s) E^{SL(5)}_{\beta_1;s}$ and $\mathbf{E}^{SL(5)}_{[0010];s} = 4\zeta(2s)\zeta(2s-1) E^{SL(5)}_{\beta_3;s}$

the Fourier coefficients:

$$F^{SL(5)\,\beta_3}_{\beta_1;s}(k,\tilde N_4) \;=\; \frac{2}{\xi(2s)}\, r^{3-\frac{2s}{5}}\, \frac{\sigma_{2s-3}(|k|)}{|k|^{s-\frac32}} \left(\frac{\|n(e_3^t)^{-1}\|}{\|me_2\|}\right)^{s-\frac32} \times$$
$$K_{s-\frac32}\left(2\pi|k|\,r^2\,\|me_2\|\,\|n(e_3^t)^{-1}\|\right)\,, \quad \text{(H.31)}$$

where e_2 and e_3 are the $SL(2)$ and $SL(3)$ components, respectively, of the semisimple part of the Levi component of P_{β_3}, and $\tilde e_3$ refers to the contragredient defined in (F.4). Note $\|me_2\|$ and $\|n\tilde e_3\|$ are independent of the choice of factorization of $\tilde N_4 = n^t m$. The matrix $\tilde N_4$ is transformed by the action of $SL(3,\mathbb{Z})$ on the left and by the action of $SL(2,\mathbb{Z})$ on the right. Because $\tilde N_4$ has rank 1, it therefore satisfies the $\frac12$-BPS conditions $\epsilon_{ab}(N_4)_i{}^a(N_4)_j{}^b=0$ of section C.3. In other words, for any value of s, the Fourier modes fill out $\frac12$-BPS orbits – one for each value of k.

(ii) **The parabolic** $P_{\beta_1} = GL(1)\times SL(4)\times U_{\beta_1}$**.**

The unipotent radical for this parabolic is abelian and is given in our parameterisation by

$$U_{\beta_1} = \begin{pmatrix} I_4 & Q_1\\ 0 & 1\end{pmatrix}\,, \quad \text{with} \quad Q_1 = \begin{pmatrix} C^{(2)\,1}+\Omega_1 B^1_{\rm NS}\\ C^{(2)\,2}+\Omega_1 B^2_{\rm NS}\\ C^{(2)\,3}+\Omega_1 B^3_{\rm NS}\\ \Omega_1\end{pmatrix}\,, \qquad \text{(H.32)}$$

where I_4 is the 4×4 identity matrix and Q_1 is a four-dimensional vector that can also be thought of as a spinor for $Spin(3,3)$.

The Fourier modes are defined by

$$F^{SL(5)\beta_1}_{\beta_1;s}(k,N_1) := \int_{[0,1]^4} d^4Q_1\, e^{-2i\pi\, kN_1Q_1}\, E^{SL(5)}_{\beta_1;s}\,, \qquad \text{(H.33)}$$

where the row vector $N_1\in\mathbb{Z}^4$ is such that $\gcd(N_1)=1$. These Fourier modes are evaluated by a straightforward extension of the expansion given in [2, section B.5.1], which computed only the constant terms (for which it is sufficient to set $Q_1=0$). The result is

$$F^{SL(5)\,\beta_1}_{\beta_1;s}(k,N_1) = \frac{2}{\xi(2s)}\, r^{1+\frac{6s}{5}}\, \frac{\sigma_{2s-1}(|k|)}{|k|^{s-\frac12}}\, \frac{K_{s-\frac12}\left(2\pi|k|\,r^2\,\|N_1e_4\|\right)}{\|N_1e_4\|^{s-\frac12}}\,. \quad \text{(H.34)}$$

(iii) **The parabolic** $P_{\beta_4} = GL(1)\times SL(4)\times U_{\beta_4}$

The unipotent radical is abelian and given by

$$U_{\beta_4} = \begin{pmatrix} 1 & Q_2\\ 0 & I_4\end{pmatrix}\,, \qquad Q_2 = \begin{pmatrix} C_{123} & C_{124} & C_{234} & C_{134}\end{pmatrix}\,, \qquad \text{(H.35)}$$

where Q_2 is again a $SL(4)$ (row) vector. The notation indicates that it is parametrized by the 3-form flux of the $M2$-brane world-volume wrapped on the M-theory 4-torus, $\mathcal{T}^4$. This translates into the NS components of flux, $B_{\rm NS\,12}, B_{\rm NS\,23}, B_{\rm NS\,13}$, and the RR $D2$-brane flux, $C^{(3)}_{123}$. In type IIB language these components become the NS flux $B_{\rm NS\,12}$, the RR D-string flux $C^{(2)}_{12}$ and the Kaluza–Klein momenta from the components of the metric $g_{i\,3}$ with $i=1,2$.

The Fourier coefficients in this parabolic are indexed by $k\in\mathbb{Z}$ and $N_4\in\mathbb{Z}^4$ with $\gcd(N_4)=1$ by the formula

$$F^{SL(5)\beta_4}_{\beta_1;s}(k,N_4) := \int_{[0,1]^4} d^4Q_2\, e^{-2i\pi\, k\, N_4^t\cdot Q_2}\, E^{SL(5)}_{\beta_1;s}\,. \tag{H.36}$$

These coefficients can again be evaluated by an extension of the computation of [2, section B.5.1], keeping the off-diagonal terms, which gives

$$\begin{aligned} F^{SL(5)\,\beta_4}_{\beta_1;s}(k,N_4) \quad &= \\ &\frac{2}{\xi(2s)}\, r^{4-\frac{6s}{5}}\, \frac{\sigma_{2s-4}(|k|)}{|k|^{s-2}}\, \|N_4 e_4^{-1}\|^{s-2}\, K_{s-2}\left(2\pi |k|\, r^2\, \|N_4 e_4^{-1}\|\right)\,, \end{aligned} \tag{H.37}$$

where $r=\mathcal{V}_4^{3/8}\ell_{11}^{-3/2}$, and again $\gcd(N_4)=1$.

H.2.2. *Fourier modes of the series* $E^{SL(5)}_{\beta_2;s}$.

Our method of determining the Fourier modes of the non-Epstein $SL(5)$ series $E^{SL(5)}_{\beta_3;s}$ is based on the integral representation described in proposition 4.1. For computational reasons it is easier to work with the series $E^{SL(5)}_{\beta_2;s}$, which is related both by the functional equation in (4.57) and the contragredient map $g\mapsto\widetilde{g}$ defined in (F.4). Here we shall compute its nonzero Fourier modes in each of the four standard maximal parabolic subgroups P_{β_1}, P_{β_2}, P_{β_3}, and P_{β_4} of $SL(5)$; the relevant Fourier modes for $E^{SL(5)}_{\beta_3;s}$ will be derived from this in section 4.5.

H.2.2.1. **The parabolic** $P_{\beta_1}=SL(4)\times GL(1)\times U_{\beta_1}$.

In this case e has the special form $\left(\begin{smallmatrix} I_4 & Q\\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} e_4 & \\ & e_1\end{smallmatrix}\right)$, where $Q\in M_{4,1}(\mathbb{R})$, $e_1\neq 0$, and $e_4\in GL(4,\mathbb{R})$. Note that we do not assume that $\det e=1$, so that we can later utilize proposition 4.1. The sum (E.8) can be written as

$$\mathcal{G}(\tau,ee^t) \quad := \sum_{\left[\begin{smallmatrix} p & m\\ q & n\end{smallmatrix}\right]\in\mathcal{M}^{(2)}_{2,5}(\mathbb{Z})} e^{-\pi\tau_2^{-1}\|[p+q\tau\,,\,m+n\tau]e\|^2}\,, \tag{H.38}$$

where $p,q\in\mathbb{Z}^4$ and $m,n\in\mathbb{Z}$. For emphasis we have used commas to separate the entries of the row vectors. The exponent is

$$-\pi\tau_2^{-1}\|(p+q\tau)e_4\|^2 \;-\; \pi\tau_2^{-1}e_1^2|(p+q\tau)Q+m+n\tau|^2\,. \tag{H.39}$$

This is independent of Q if both $p=q=0$. Hence the nonzero Fourier coefficients of (H.38) come from terms where $\left[\begin{smallmatrix} p\\ q\end{smallmatrix}\right]$ has rank 1 or 2. We thus separate these contributions and write

$$\mathcal{G}(\tau,ee^t) \;=\; \mathcal{G}_1(\tau,ee^t) \;+\; \mathcal{G}_2(\tau,ee^t)\,, \tag{H.40}$$

where

$$\mathcal{G}_i(\tau, ee^t) \quad := \sum_{\substack{\text{rank}[\begin{smallmatrix} p \\ q \end{smallmatrix}] = i \\ [\begin{smallmatrix} p & m \\ q & n \end{smallmatrix}] \in \mathcal{M}_{2,5}^{(2)}(\mathbb{Z})}} e^{-\pi\tau_2^{-1} \|[p+q\tau\,,\, m+n\tau]e\|^2} . \tag{H.41}$$

Let us first consider $\mathcal{G}_2(\tau, ee^t)$. Changing τ to $\tau+1$ is equivalent to changing (p,q,m,n) to $(p+q,q,m+n,n)$, while changing τ to $-\tau^{-1}$ is equivalent to changing (p,q,m,n) to $(-q,p,-n,m)$. Thus the sum is modular invariant and can be written as a sum over $SL(2,\mathbb{Z})$ cosets:

$$\mathcal{G}_2(\tau, ee^t) \quad = \sum_{\gamma \in SL(2,\mathbb{Z})} \mathcal{G}_2^{\circ}(\gamma\tau, ee^t)\,, \tag{H.42}$$

where

$$\mathcal{G}_2^{\circ}(\tau, ee^t) \;= \sum_{\substack{[\begin{smallmatrix} p \\ q \end{smallmatrix}] \in SL(2,\mathbb{Z})\backslash \mathcal{M}_{2,4}^{(2)}(\mathbb{Z}) \\ m,n \in \mathbb{Z}}} e^{-\pi\tau_2^{-1}\|(p+q\tau)e_4\|^2 - \pi\tau_2^{-1} e_1^2 |(p+q\tau)Q + m + n\tau|^2} \tag{H.43}$$

(here we have used that $\text{rank}[\begin{smallmatrix} p \\ q \end{smallmatrix}] = 2$ implies that $\text{rank}[\begin{smallmatrix} p & m \\ q & n \end{smallmatrix}] = 2$). Applying Poisson summation over m and n this is

$$\mathcal{G}_2^{\circ}(\tau, ee^t) \quad = \sum_{\substack{[\begin{smallmatrix} p \\ q \end{smallmatrix}] \in SL(2,\mathbb{Z})\backslash \mathcal{M}_{2,4}^{(2)}(\mathbb{Z}) \\ \hat{m},\hat{n} \in \mathbb{Z}}} e^{-\pi\tau_2^{-1}\|(p+q\tau)e_4\|^2} e^{2\pi i(\hat{m}p + \hat{n}q)Q} \times$$
$$\int_{\mathbb{R}^2} e^{-2\pi i(\hat{m}m + \hat{n}n)} e^{-\pi\tau_2^{-1} e_1^2 |m+n\tau|^2}\, dm\, dn\,. \tag{H.44}$$

Thus its Fourier coefficient for $e^{2\pi i N_1 Q}$, $N_1 \in \mathbb{Z}^4$, is equal to

$$\mathcal{F}\mathcal{G}_2^{\circ}(\tau, e_1, e_4; N_1) \quad = \sum_{\substack{[\begin{smallmatrix} p \\ q \end{smallmatrix}] \in SL(2,\mathbb{Z})\backslash \mathcal{M}_{2,4}^{(2)}(\mathbb{Z}) \\ \hat{m},\hat{n} \in \mathbb{Z} \\ \hat{n}q - \hat{m}p = N_1}} e^{-\pi\tau_2^{-1}\|(p+q\tau)e_4\|^2} \times$$
$$\int_{\mathbb{R}^2} e^{2\pi i \hat{m}m - (\hat{n}+\hat{m}\tau_1)n} e^{-\pi\tau_2^{-1} e_1^2 (m^2 + n^2\tau_2^2)}\, dm\, dn$$
$$= \quad e_1^{-2} \sum_{\substack{[\begin{smallmatrix} p \\ q \end{smallmatrix}] \in SL(2,\mathbb{Z})\backslash \mathcal{M}_{2,4}^{(2)}(\mathbb{Z}) \\ \hat{m},\hat{n} \in \mathbb{Z} \\ \hat{n}q - \hat{m}p = N_1}} e^{-\pi\tau_2^{-1}\|(p+q\tau)e_4\|^2} e^{-\pi\tau_2^{-1} e_1^{-2} |\hat{n}+\hat{m}\tau|^2} . \tag{H.45}$$

Analogously to (H.42)

$$\mathcal{G}_1(\tau, ee^t) \quad = \sum_{\gamma \in \{\pm\Gamma_\infty\}\backslash SL(2,\mathbb{Z})} \mathcal{G}_1^{\circ}(\gamma\tau, ee^t)\,, \tag{H.46}$$

where $\Gamma_\infty = \{(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix})\,|n\in\mathbb{Z}\}$ and

$$\mathcal{G}_1^\circ(\tau, ee^t) \quad := \quad \sum_{\substack{p\neq 0\\ m\in\mathbb{Z}\\ n\neq 0}} e^{-\pi\tau_2^{-1}\|pe_4\|^2-\pi\tau_2^{-1}e_1^2|pQ+m+n\tau|^2} \tag{H.47}$$

(this parametrization is due to the fact that any $SL(2,\mathbb{Z})$ orbit in $\mathcal{M}_{2,4}^{(1)}(\mathbb{Z})$ has an element with bottom row equal to zero, and that the rank 2 condition is then equivalent to the bottom right entry, n, being nonzero). Applying Poisson summation over m gives the formula

$$\begin{aligned}\mathcal{G}_1^\circ(\tau, ee^t) \quad &= \quad \sum_{\substack{p\neq 0\\ \hat m\in\mathbb{Z}\\ n\neq 0}} e^{-\pi\tau_2^{-1}\|pe_4\|^2}\int_{\mathbb{R}} e^{-2\pi i\hat m m}\, e^{-\pi\tau_2^{-1}e_1^2|pQ+m+n\tau|^2}\,dm \\ &= \tau_2^{\frac12}\, e_1^{-1}\sum_{\substack{p\neq 0\\ \hat m\in\mathbb{Z}\\ n\neq 0}} e^{2\pi i\hat m(pQ+n\tau_1)}e^{-\pi\tau_2^{-1}\|pe_4\|^2-\pi\tau_2 e_1^{-2}\hat m^2-\pi\tau_2 e_1^2 n^2}\,. \end{aligned} \tag{H.48}$$

Since $N_1\neq 0$ its Fourier mode for $e^{2\pi i N_1 Q}$ is thus

$$\begin{aligned}\mathcal{F}\mathcal{G}_1^\circ(\tau, e_1, e_4; N_1) \quad &= \\ &\tau_2^{\frac12}\, e_1^{-1}\sum_{\substack{\hat m p = N_1\\ n\neq 0}} e^{2\pi i\hat m n\tau_1}e^{-\pi\tau_2^{-1}\|pe_4\|^2-\pi\tau_2 e_1^{-2}\hat m^2-\pi\tau_2 e_1^2 n^2}\,. \end{aligned} \tag{H.49}$$

It follows using proposition 4.1 that the nonzero Fourier modes of $F_{\beta_2;s}^{SL(5)\,\beta_1}$ are given by

$$\begin{aligned}\tfrac12\xi(2s)\xi(2s-1)F_{\beta_2;s}^{SL(5)\,\beta_1}(N_1) \quad &= \\ & 2\int_0^\infty\int_{\mathbb{H}} \mathcal{F}\mathcal{G}_2^\circ(\tau, u^{\frac12}e_1, u^{\frac12}e_4; N_1)\,\frac{d^2\tau}{\tau_2^2}\,\frac{du}{u^{1-2s}} \\ &+ \int_0^\infty\int_{\Gamma_\infty\backslash\mathbb{H}} \mathcal{F}\mathcal{G}_1^\circ(\tau, u^{\frac12}e_1, u^{\frac12}e_4; N_1)\,\frac{d^2\tau}{\tau_2^2}\,\frac{du}{u^{1-2s}}\,, \end{aligned} \tag{H.50}$$

the factor of 2 coming from unfolding pairs of elements $\pm\gamma\in SL(2,\mathbb{Z})$ that have identical actions on $\mathbb{H}$. By integrating the expression given in (H.49) for $\mathcal{F}\mathcal{G}_1^\circ(\tau, e_1, e_4; N_1)$ over the strip $\Gamma_\infty\backslash\mathbb{H}$, the τ_1-integration over $[0,1]$ forces $\hat m n$ to vanish. Since $n\neq 0$ this means $N_1 = 0$, and hence there are no nontrivial Fourier contributions from $\mathcal{G}_1$.

The contribution from the modes $\mathcal{F}\mathcal{G}_2^\circ$ is given by

$$2\,e_1^{-2}\int_0^\infty\int_{\mathbb{H}}\sum_{\substack{[\begin{smallmatrix}p\\q\end{smallmatrix}]\in SL(2,\mathbb{Z})\backslash\mathcal{M}_{2,4}^{(2)}(\mathbb{Z})\\ \hat m,\hat n\in\mathbb{Z}\\ \hat n q-\hat m p=N_1}} e^{-\pi\tau_2^{-1}u\|(p+q\tau)e_4\|^2-\pi\tau_2^{-1}u^{-1}e_1^{-2}|\hat n+\hat m\tau|^2}\,\frac{d^2\tau}{\tau_2^2}\,\frac{du}{u^{2-2s}}\,. \quad \text{(H.51)}$$

Changing variables to $x=u/\tau_2$ and $y=\tau_2 u$, so that $u=\sqrt{xy}$, $\tau_2=\sqrt{y/x}$, and $d\tau_2du=\frac{dxdy}{2x}$, yields

$$e_1^{-2}\int_0^\infty\sum_{\substack{[\begin{smallmatrix}p\\q\end{smallmatrix}]\in SL(2,\mathbb{Z})\backslash\mathcal{M}_{2,4}^{(2)}(\mathbb{Z})\\ \hat m,\hat n\in\mathbb{Z}\\ \hat n q-\hat m p=N_1}}\int_0^\infty e^{-\pi x\|(p+q\tau_1)e_4\|^2-\pi x^{-1}e_1^{-2}\hat m^2}\,\frac{dx}{x^{1-s}}$$
$$\times\int_0^\infty e^{-\pi y\|qe_4\|^2-\pi y^{-1}e_1^{-2}(\hat n+\hat m\tau_1)^2}\,\frac{dy}{y^{2-s}}d\tau_1$$
$$=\ 4\,e_1^{-2}\int_0^\infty\sum_{\substack{[\begin{smallmatrix}p\\q\end{smallmatrix}]\in SL(2,\mathbb{Z})\backslash\mathcal{M}_{2,4}^{(2)}(\mathbb{Z})\\ \hat m,\hat n\in\mathbb{Z}\\ \hat n q-\hat m p=N_1}}\left(\frac{|\hat m|}{\|e_1(p+q\tau_1)e_4\|}\right)^s\left(\frac{|\hat n+\hat m\tau_1|}{\|e_1qe_4\|}\right)^{s-1}\times$$
$$K_s(\,2\pi\,|\hat m|\,\|e_1^{-1}(p+q\tau_1)e_4\|\,)\,K_{s-1}(\,2\pi\,|\hat n+\hat m\tau_1|\,\|e_1^{-1}qe_4\|\,)d\tau_1\,. \quad \text{(H.52)}$$

H.2.2.2. **The parabolic** $P_{\beta_2}=GL(1)\times SL(3)\times SL(2)\times U_{\beta_2}$. We may rewrite (E.8) in the case of $d=5$ as

$$\mathcal{G}(\tau,ee^t)\ :=\ \sum_{[\begin{smallmatrix}p&m\\q&n\end{smallmatrix}]\in\mathcal{M}_{2,5}^{(2)}(\mathbb{Z})} e^{-\pi\tau_2^{-1}\|[p+q\tau\,,\,m+n\tau]e\|^2}\,, \quad \text{(H.53)}$$

where $p,q\in\mathbb{Z}^3$ and $m,n\in\mathbb{Z}^2$. Let us further take e to have the special form $e=\left(\begin{smallmatrix}I_3&Q\\&I_2\end{smallmatrix}\right)\left(\begin{smallmatrix}e_3&\\&e_2\end{smallmatrix}\right)$, where $Q\in M_{3,2}(\mathbb{R})$, $e_2\in GL(2,\mathbb{R})$, and $e_3\in GL(3,\mathbb{R})$. We will be interested in Fourier coefficients in Q for the Fourier modes $Q\mapsto e^{2\pi i\,\mathrm{tr}\,NQ}$, where $N\in M_{2,3}(\mathbb{Z})$. Break up the sum as

$$\mathcal{G}(\tau,ee^t)\ =\ \mathcal{G}_0(\tau,ee^t)\ +\ \mathcal{G}_1(\tau,ee^t)\ +\ \mathcal{G}_2(\tau,ee^t)\,, \quad \text{(H.54)}$$

where

$$\mathcal{G}_i(\tau,ee^t)\ :=\ \sum_{\substack{\mathrm{rank}[\begin{smallmatrix}p\\q\end{smallmatrix}]=i\\ [\begin{smallmatrix}p&m\\q&n\end{smallmatrix}]\in\mathcal{M}_{2,5}^{(2)}(\mathbb{Z})}} e^{-\pi\tau_2^{-1}\|[p+q\tau\,,\,m+n\tau]e\|^2}\,. \quad \text{(H.55)}$$

If $\mathrm{rank}[\begin{smallmatrix} p \\ q \end{smallmatrix}] = 2$, then $[\begin{smallmatrix} p & m \\ q & n \end{smallmatrix}]$ automatically has rank 2. Thus

$$\mathcal{G}_2(\tau, ee^t) \quad := \sum_{\substack{\mathrm{rank}[\begin{smallmatrix} p \\ q \end{smallmatrix}] = 2 \\ m,n \in \mathbb{Z}^2}} e^{-\pi\tau_2^{-1}\|[p+q\tau\,,\,m+n\tau]e\|^2}\,. \tag{H.56}$$

Using the method of orbits we may write this as an average over $SL(2,\mathbb{Z})$:

$$\mathcal{G}_2(\tau, ee^t) \quad = \sum_{\gamma \in SL(2,\mathbb{Z})} \mathcal{G}_2^\circ(\gamma\tau, ee^t)\,, \tag{H.57}$$

where

$$\mathcal{G}_2^\circ(\tau, ee^t) \quad = \sum_{\substack{[\begin{smallmatrix} p \\ q \end{smallmatrix}] \in SL(2,\mathbb{Z})\backslash\mathcal{M}_{2,3}^{(2)}(\mathbb{Z}) \\ m,n \in \mathbb{Z}^2}} e^{-\pi\tau_2^{-1}\|[p+q\tau\,,\,m+n\tau]e\|^2}\,. \tag{H.58}$$

Poisson summation over the inner $m,n \in \mathbb{Z}^2$ sum gives

$$\mathcal{G}_2^\circ(\tau, ee^t) \quad = \sum_{\substack{[\begin{smallmatrix} p \\ q \end{smallmatrix}] \in SL(2,\mathbb{Z})\backslash\mathcal{M}_{2,3}^{(2)}(\mathbb{Z}) \\ \hat{m},\hat{n} \in \mathbb{Z}^2}} \int_{\mathbb{R}^4} e^{-2\pi i(m\hat{m}-n\hat{n})} e^{-\pi\tau_2^{-1}\|[p+q\tau\,,\,m+n\tau]e\|^2}\,dm\,dn\,, \tag{H.59}$$

where $\hat{m},\hat{n} \in \mathbb{Z}^2$ are column vectors. With the particular form $e = \begin{pmatrix} I_3 & Q \\ & I_2 \end{pmatrix}\begin{pmatrix} e_3 & \\ & e_2 \end{pmatrix}$ the exponent of the second factor is

$$-\pi\tau_2^{-1}[p+q\tau\,,\,(p+q\tau)Q+m+n\tau]\begin{pmatrix} e_3e_3^t & \\ & e_2e_2^t \end{pmatrix}[p+q\tau\,,\,(p+q\bar{\tau})Q+m+n\bar{\tau}]^t. \tag{H.60}$$

Thus after changing variables $m \mapsto m - pQ$, $n \mapsto n - qQ$ (H.59) becomes

$$\begin{aligned}\mathcal{G}_2^\circ(\tau, ee^t) \quad = \sum_{[\begin{smallmatrix} p \\ q \end{smallmatrix}] \in SL(2,\mathbb{Z})\backslash\mathcal{M}_{2,3}^{(2)}(\mathbb{Z})} e^{-\pi\tau_2^{-1}\|(p+q\tau)e_3\|^2} \sum_{\hat{m},\hat{n} \in \mathbb{Z}^2} e^{2\pi i(pQ\hat{m}-qQ\hat{n})} \times \\ \int_{\mathbb{R}^4} e^{-2\pi i(m\hat{m}-n\hat{n})} e^{-\pi\tau_2^{-1}\|(m+n\tau)e_2\|^2}\,dm\,dn\,.\end{aligned} \tag{H.61}$$

To compute this integral we change variables $m \mapsto me_2^{-1}$, $n \mapsto ne_2^{-1}$, which has the effect of dividing both dm and dn each by $\det e_2$: the integral equals $(\det e_2)^{-2}$ times

$$\begin{aligned}\int_{\mathbb{R}^4} e^{-2\pi i(me_2^{-1}\hat{m}-ne_2^{-1}\hat{n})} e^{-\pi\tau_2^{-1}\|(m+n\tau)\|^2}\,dm\,dn \quad = \\ \int_{\mathbb{R}^4} e^{-2\pi i(me_2^{-1}\hat{m}-ne_2^{-1}(\hat{n}+\tau_1\hat{m}))} e^{-\pi\tau_2^{-1}\|m\|^2-\pi\tau_2\|n\|^2}\,dm\,dn\end{aligned} \tag{H.62}$$

after changing variables $m \mapsto m - n\tau_1$ in the last step. This then factors as two Fourier transforms of Gaussians and (H.61) is equal to

$$\mathcal{G}_2^\circ(\tau, ee^t) = (\det e_2)^{-2} \sum_{\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] \in SL(2,\mathbb{Z})\backslash \mathcal{M}_{2,3}^{(2)}(\mathbb{Z})} e^{-\pi\tau_2^{-1}\|(p+q\tau)e_3\|^2} \sum_{\hat{m},\hat{n} \in \mathbb{Z}^2} e^{2\pi i(pQ\hat{m}-qQ\hat{n})} e^{-\pi\tau_2\|e_2^{-1}\hat{m}\|^2 - \pi\tau_2^{-1}\|e_2^{-1}(\hat{n}+\hat{m}\tau_1)\|^2} . \quad \text{(H.63)}$$

The dependence on Q is manifest in the exponential factors in the sum, and hence taking Fourier coefficients in Q amounts to restricting p, q, $\hat{m}$, and $\hat{n}$. In particular the Fourier coefficient for $N_4 \in M_{2,3}(\mathbb{Z})$ is equal to

$$\mathcal{F}\mathcal{G}_2^\circ(\tau, e_2, e_3; N_4) = (\det e_2)^{-2} \sum_{\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] \in SL(2,\mathbb{Z})\backslash \mathcal{M}_{2,3}^{(2)}(\mathbb{Z})} \sum_{\substack{\hat{m},\hat{n} \in \mathbb{Z}^2 \\ \hat{m}p - \hat{n}q = N_4}} e^{-\pi\tau_2^{-1}\|(p+q\tau)e_3\|^2 - \pi\tau_2\|e_2^{-1}\hat{m}\|^2 - \pi\tau_2^{-1}\|e_2^{-1}(\hat{n}+\hat{m}\tau_1)\|^2} . \quad \text{(H.64)}$$

Let us now consider $\mathcal{G}_1(\tau, ee^t)$, which has the contributions for $p, q \in \mathbb{Z}^3$ such that $\mathrm{rank}\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] = 1$:

$$\mathcal{G}_1(\tau, ee^t) := \sum_{\substack{\mathrm{rank}\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] = 1 \\ \left[\begin{smallmatrix} p & m \\ q & n \end{smallmatrix}\right] \in \mathcal{M}_{2,5}^{(2)}(\mathbb{Z})}} e^{-\pi\tau_2^{-1}\|[p+q\tau\,,\, m+n\tau]e\|^2} . \quad \text{(H.65)}$$

We may write this as an average over $\{\pm\Gamma_\infty\}\backslash SL(2,\mathbb{Z})$:

$$\mathcal{G}_1(\tau, ee^t) = \sum_{\gamma \in \{\pm\Gamma_\infty\}\backslash SL(2,\mathbb{Z})} \mathcal{G}_1^\circ(\gamma\tau, ee^t) , \quad \text{(H.66)}$$

where

$$\begin{aligned} \mathcal{G}_1^\circ(\tau, ee^t) &:= \sum_{\substack{p \neq 0 \\ \left[\begin{smallmatrix} p & m \\ 0 & n \end{smallmatrix}\right] \in \mathcal{M}_{2,5}^{(2)}(\mathbb{Z})}} e^{-\pi\tau_2^{-1}\|[p,\, m+n\tau]e\|^2} \\ &= \sum_{\substack{p \neq 0 \\ n \neq 0 \\ m \in \mathbb{Z}^2}} e^{-\pi\tau_2^{-1}\|pe_3\|^2 - \pi\tau_2^{-1}\|(pQ+m+n\tau_1)e_2\|^2 - \pi\tau_2\|ne_2\|^2} . \end{aligned} \quad \text{(H.67)}$$

Here we used that the matrix $\left[\begin{smallmatrix} p & m \\ 0 & n \end{smallmatrix}\right]$ has rank 2 if and only if $n \neq 0$ (since $p \neq 0$). Poisson sum over m then gives the formula

$$\mathcal{G}_1^\circ(\tau, ee^t) = \frac{\tau_2}{\det e_2} \sum_{\substack{p \neq 0 \\ n \neq 0 \\ \hat{m} \in \mathbb{Z}^2}} e^{2\pi i(pQ+n\tau_1)\hat{m}} e^{-\pi\tau_2^{-1}\|pe_3\|^2 - \pi\tau_2\|ne_2\|^2 - \pi\tau_2\|e_2^{-1}\hat{m}\|^2} \quad \text{(H.68)}$$

for (H.67), where again $\hat{m} \in \mathbb{Z}^2$ is a column vector.

We conclude that the Fourier coefficient of $\mathcal{G}_1^\circ(\tau, ee^t)$ for N_4 is equal to

$$\mathcal{F}\mathcal{G}_1^\circ(\tau, e_2, e_3; N_4) = \frac{\tau_2}{\det e_2} \sum_{\substack{p\neq 0\\ n\neq 0\\ \hat{m}p=N_4}} e^{2\pi i\tau_1 n\hat{m}} e^{-\pi\tau_2^{-1}\|pe_3\|^2-\pi\tau_2\|ne_2\|^2-\pi\tau_2\|e_2^{-1}\hat{m}\|^2}. \quad \text{(H.69)}$$

Note that $\mathcal{F}\mathcal{G}_1(\tau, e_2, e_3; N_4) \equiv 0$ if $\operatorname{rank}(N_4) = 2$. Finally since $[0\,0\,0\,\star\,\star]\begin{pmatrix} I_3 & Q \\ & I_2\end{pmatrix}$ is independent of Q, so too is $\mathcal{G}_0(\tau, ee^t)$, the sum over terms with $p = q = [0\,0\,0]$. It therefore has no nontrivial Fourier coefficients.

We now return to the identity of proposition 4.1,

$$\frac{1}{2}\xi(2s)\xi(2s-1)\, E^{SL(5)}_{\beta_2;s}(e) = \int_0^\infty \int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} \mathcal{G}(\tau, uee^t)\, \frac{d^2\tau}{\tau_2^2}\, \frac{du}{u^{1-2s}}, \quad \text{(H.70)}$$

with the specialization that $e \in SL(d,\mathbb{R})$ has the form $e = \begin{pmatrix} I_3 & Q \\ & I_2\end{pmatrix}\begin{pmatrix} e_3 & \\ & e_2\end{pmatrix}$. Its Fourier coefficient for N_4 can be written as

$$\begin{aligned}\frac{1}{2}\xi(2s)\xi(2s-1)F^{SL(5)\,\beta_2}_{\beta_2;s}(e_2, e_3; N_4) &= \\ 2\int_0^\infty \int_{\mathbb{H}} &\mathcal{F}\mathcal{G}_2^\circ(\tau, u^{\frac12}e_2, u^{\frac12}e_3; N_4)\, \frac{d^2\tau}{\tau_2^2}\, \frac{du}{u^{1-2s}} \\ + \int_0^\infty \int_{\Gamma_\infty\backslash\mathbb{H}} &\mathcal{F}\mathcal{G}_1^\circ(\tau, u^{\frac12}e_2, u^{\frac12}e_3; N_4)\, \frac{d^2\tau}{\tau_2^2}\, \frac{du}{u^{1-2s}}. \quad \text{(H.71)}\end{aligned}$$

Let us consider the first integral,

$$\begin{aligned}2\int_0^\infty \int_{\mathbb{H}} (\det e_2)^{-2} &\sum_{\left[\begin{smallmatrix} p\\ q\end{smallmatrix}\right]\in SL(2,\mathbb{Z})\backslash\mathcal{M}^{(2)}_{2,3}(\mathbb{Z})} e^{-\pi\tau_2^{-1}u\|(p+q\tau)e_3\|^2} \\ &\sum_{\substack{\hat{m},\hat{n}\in\mathbb{Z}^2\\ \hat{m}p-\hat{n}q=N_4}} e^{-\pi\tau_2 u^{-1}\|e_2^{-1}\hat{m}\|^2-\pi\tau_2^{-1}u^{-1}\|e_2^{-1}(\hat{n}+\hat{m}\tau_1)\|^2}\, \frac{d^2\tau}{\tau_2^2}\, \frac{du}{u^{3-2s}}. \quad \text{(H.72)}\end{aligned}$$

Changing variables to $x = u/\tau_2$ and $y = \tau_2 u$, so that $u = \sqrt{xy}$, $\tau_2 = \sqrt{y/x}$ and $d\tau_2 du = \frac{dxdy}{2x}$ the integral becomes

$$\begin{aligned}(\det e_2)^{-2}\int_{\mathbb{R}}\int_0^\infty\int_0^\infty &\sum_{\left[\begin{smallmatrix} p\\ q\end{smallmatrix}\right]\in SL(2,\mathbb{Z})\backslash\mathcal{M}^{(2)}_{2,3}(\mathbb{Z})} \sum_{\substack{\hat{m},\hat{n}\in\mathbb{Z}^2\\ \hat{m}p-\hat{n}q=N_4}} \\ e^{-\pi x\|(p+q\tau_1)e_3\|^2-\pi x^{-1}\|e_2^{-1}\hat{m}\|^2} &e^{-\pi y\|qe_3\|^2-\pi y^{-1}\|e_2^{-1}(\hat{n}+\hat{m}\tau_1)\|^2}\, \frac{dx}{x^{3/2-s}}\, \frac{dy}{y^{5/2-s}} d\tau_1. \quad \text{(H.73)}\end{aligned}$$

Integrating over x and y yields

$$4\,(\det e_2)^{-2}\int_{\mathbb{R}}\sum_{\left[\begin{smallmatrix}p\\q\end{smallmatrix}\right]\in SL(2,\mathbb{Z})\backslash\mathcal{M}^{(2)}_{2,3}(\mathbb{Z})}\ \sum_{\substack{\hat m,\hat n\in\mathbb{Z}^2\\ \hat m p-\hat n q=N_4}}\left(\frac{\|(p+q\tau_1)e_3\|}{\|e_2^{-1}\hat m\|}\right)^{1/2-s}\times$$
$$\left(\frac{\|qe_3\|}{\|e_2^{-1}(\hat n+\hat m\tau_1)\|}\right)^{3/2-s}K_{s-1/2}(2\pi\|(p+q\tau_1)e_3\|\|e_2^{-1}\hat m\|)\times$$
$$K_{s-3/2}(2\pi\|qe_3\|\|e_2^{-1}(\hat n+\hat m\tau_1)\|)d\tau_1 \quad \text{(H.74)}$$

for the first line on the righthand side of (H.71).

Next we analyze the second integral in (H.71), in which we assume N_4 has rank 1 (since it vanishes if it has rank 2):

$$\frac{1}{\det e_2}\int_0^\infty\int_{\Gamma_\infty\backslash\mathbb{H}}\sum_{\substack{p\neq 0\\ n\neq 0\\ \hat m p=N_4}} e^{2\pi i\hat m\cdot n\tau_1}\times$$
$$e^{-\pi\tau_2^{-1}u\|pe_3\|^2-\pi\tau_2 u\|ne_2\|^2-\pi\tau_2 u^{-1}\|e_2^{-1}\hat m\|^2}\,\frac{d^2\tau}{\tau_2}\,\frac{du}{u^{2-2s}}\,. \quad \text{(H.75)}$$

The τ_1 integration over $[0,1]$ enforces the condition that $n\perp\hat m$ (which implies $n\perp N_4$): (H.75) equals

$$\frac{1}{\det e_2}\int_0^\infty\int_0^\infty\sum_{\substack{p\neq 0\\ n\neq 0\\ \hat m\perp n\\ \hat m p=N_4}} e^{-\pi\tau_2^{-1}u\|pe_3\|^2-\pi\tau_2 u\|ne_2\|^2-\pi\tau_2 u^{-1}\|e_2^{-1}\hat m\|^2}\,\frac{d\tau_2}{\tau_2}\,\frac{du}{u^{2-2s}}\,. \quad \text{(H.76)}$$

As before, change variables $x=u/\tau_2$ and $y=\tau_2 u$ so that (H.76) becomes

$$\frac{1}{2(\det e_2)}\sum_{\substack{p\neq 0\\ n\neq 0\\ \hat m\perp n\\ \hat m p=N_4}}\int_0^\infty\int_0^\infty e^{-\pi x\|pe_3\|^2-\pi x^{-1}\|e_2^{-1}\hat m\|^2}e^{-\pi y\|ne_2\|^2}\,\frac{dx}{x^{\frac{3}{2}-s}}\,\frac{dy}{y^{\frac{3}{2}-s}}$$
$$=\ \frac{\Gamma\left(s-\frac{1}{2}\right)}{(\det e_2)}\sum_{\substack{p\neq 0\\ n\neq 0\\ \hat m\perp n\\ \hat m p=N_4}}\left(\frac{\|e_2^{-1}\hat m\|}{\pi\|ne_2\|^2\|pe_3\|}\right)^{s-1/2}K_{s-1/2}(2\pi\|e_2^{-1}\hat m\|\|pe_3\|)\,. \quad \text{(H.77)}$$

The matrices e_2 and e_3 in the above argument are unconstrained except for the condition that $\det(e_2)\det(e_3)=1$. For our application in section 4.5 it will be helpful to restate these calculations using the $GL(1)$ parameter r

from (2.9). We set

$$\begin{pmatrix} e_3 & \\ & e_2 \end{pmatrix} \;=\; \begin{pmatrix} r^{4/5}e_3' & \\ & r^{-6/5}e_2' \end{pmatrix}, \tag{H.78}$$

where $e_2' \in SL(2,\mathbb{R})$ and $e_3' \in SL(3,\mathbb{R})$. Then after inserting (H.74), and (H.77) we may restate (H.71) as

$$\begin{aligned} & F^{SL(5)\,\beta_2}_{\beta_2;s}(r^{-6/5}e_2', r^{4/5}e_3'; N_4) \;= \\ & \frac{8\,r^{4+4s/5}}{\xi(2s)\xi(2s-1)} \int_{\mathbb{R}} \sum_{\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] \in SL(2,\mathbb{Z})\backslash\mathcal{M}^{(2)}_{2,3}(\mathbb{Z})} \; \sum_{\substack{\hat m, \hat n \in \mathbb{Z}^2 \\ \hat m p - \hat n q = N_4}} \left(\frac{\|(p+q\tau_1)e_3'\|}{\|e_2'^{-1}\hat m\|} \right)^{1/2-s} \times \\ & \quad \left(\frac{\|q e_3'\|}{\|e_2'^{-1}(\hat n + \hat m \tau_1)\|} \right)^{3/2-s} K_{s-1/2}(2\pi r^2 \|(p+q\tau_1)e_3'\| \|e_2'^{-1}\hat m\|) \times \\ & \qquad K_{s-3/2}(2\pi r^2 \|q e_3'\| \|e_2'^{-1}(\hat n + \hat m\tau_1)\|)\, d\tau_1 \\ & \quad + \; \frac{2\,\Gamma\left(s-\frac{1}{2}\right)}{\xi(2s)\xi(2s-1)} r^{1+14s/5} \sum_{\substack{p \neq 0 \\ n \neq 0 \\ \hat m \perp n \\ \hat m p = N_4}} \left(\frac{\|e_2'^{-1}\hat m\|}{\pi \|n e_2'\|^2 \|p e_3'\|} \right)^{s-1/2} \times \\ & \qquad K_{s-1/2}(2\pi r^2 \|e_2'^{-1}\hat m\| \|p e_3'\|)\,. \end{aligned} \tag{H.79}$$

H.2.2.3. **The parabolic** $P_{\beta_3} = GL(1) \times SL(2) \times SL(3) \times U_{\beta_3}$. We may rewrite (E.8) in the case of $d = 5$ as

$$\mathcal{G}(\tau, ee^t) \;:=\; \sum_{\left[\begin{smallmatrix} p & m \\ q & n \end{smallmatrix}\right] \in \mathcal{M}^{(2)}_{2,5}(\mathbb{Z})} e^{-\pi\tau_2^{-1}\|[p+q\tau\,,\,m+n\tau]e\|^2}, \tag{H.80}$$

where $p, q \in \mathbb{Z}^2$ and $m, n \in \mathbb{Z}^3$. We take e to have the special form $e = \begin{pmatrix} I_2 & Q \\ & I_3 \end{pmatrix} \begin{pmatrix} e_2 & \\ & e_3 \end{pmatrix}$, where $Q \in M_{2\times 3}(\mathbb{R})$, $e_2 \in GL(2,\mathbb{R})$, and $e_3 \in GL(3,\mathbb{R})$. We will be interested in Fourier coefficients in Q for the modes $Q \mapsto e^{2\pi i\,\mathrm{tr}\,NQ}$, where $N \in M_{3,2}(\mathbb{Z})$. Break up the sum as

$$\mathcal{G}(\tau, ee^t) \;=\; \mathcal{G}_0(\tau, ee^t) \;+\; \mathcal{G}_1(\tau, ee^t) \;+\; \mathcal{G}_2(\tau, ee^t)\,, \tag{H.81}$$

where

$$\mathcal{G}_i(\tau, ee^t) \;:=\; \sum_{\substack{\mathrm{rank}\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] = i \\ \left[\begin{smallmatrix} p & m \\ q & n \end{smallmatrix}\right] \in \mathcal{M}^{(2)}_{2,5}(\mathbb{Z})}} e^{-\pi\tau_2^{-1}\|[p+q\tau\,,\,m+n\tau]e\|^2}\,. \tag{H.82}$$

If $\mathrm{rank}\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] = 2$, then $\left[\begin{smallmatrix} p & m \\ q & n \end{smallmatrix}\right]$ automatically has rank 2. Thus

$$\mathcal{G}_2(\tau, ee^t) \;:=\; \sum_{\substack{\mathrm{rank}\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] = 2 \\ m,n \in \mathbb{Z}^3}} e^{-\pi\tau_2^{-1}\|[p+q\tau\,,\,m+n\tau]e\|^2}\,. \tag{H.83}$$

Again we use modular invariance to write

$$\mathcal{G}_2(\tau, ee^t) \;=\; \sum_{\gamma\in\{\pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\}\backslash SL(2,\mathbb{Z})} \mathcal{G}_2^\circ(\gamma\tau, ee^t)\,, \tag{H.84}$$

where

$$\mathcal{G}_2^\circ(\tau, ee^t) \;=\; \sum_{\substack{p=[p_1\,p_2]\\ q=[0\,q_2]\\ p_1>0,\,0\le p_2<|q_2|\\ m,n\in\mathbb{Z}^3}} e^{-\pi\tau_2^{-1}\|[p+q\tau\,,\,m+n\tau]e\|^2}\,. \tag{H.85}$$

Poisson summation over the inner $m,n\in\mathbb{Z}^3$ sum gives

$$\mathcal{G}_2^\circ(\tau, ee^t) \;=\; \sum_{\substack{p=[p_1\,p_2]\\ q=[0\,q_2]\\ p_1>0,\,0\le p_2<|q_2|\\ \hat m,\hat n\in\mathbb{Z}^3}} \int_{\mathbb{R}^6} e^{-2\pi i(m\hat m-n\hat n)}e^{-\pi\tau_2^{-1}\|[p+q\tau\,,\,m+n\tau]e\|^2}\,dm\,dn\,, \tag{H.86}$$

where $\hat m,\hat n\in\mathbb{Z}^3$ are column vectors. With the particular form $e=\left(\begin{smallmatrix}I_2&Q\\&I_3\end{smallmatrix}\right)\left(\begin{smallmatrix}e_2&\\&e_3\end{smallmatrix}\right)$ the exponent of the second factor is

$$-\pi\tau_2^{-1}[p+q\tau\,,\,(p+q\tau)Q+m+n\tau]\begin{pmatrix}e_2e_2^t&\\&e_3e_3^t\end{pmatrix}[p+q\tau\,,\,(p+q\bar\tau)Q+m+n\bar\tau]^t\,. \tag{H.87}$$

Thus after changing variables $m\mapsto m-pQ$, $n\mapsto n-qQ$ (H.86) becomes

$$\mathcal{G}_2^\circ(\tau, ee^t) \;=\; \sum_{\substack{p=[p_1\,p_2]\\ q=[0\,q_2]\\ p_1>0,\,0\le p_2<|q_2|}} e^{-\pi\tau_2^{-1}\|(p+q\tau)e_2\|^2}\sum_{\hat m,\hat n\in\mathbb{Z}^3} e^{2\pi i(pQ\hat m-qQ\hat n)}\times$$
$$\int_{\mathbb{R}^6} e^{-2\pi i(m\hat m-n\hat n)}e^{-\pi\tau_2^{-1}\|(m+n\tau)e_3\|^2}\,dm\,dn\,. \tag{H.88}$$

To compute this integral we change variables $m\mapsto me_3^{-1}$, $n\mapsto ne_3^{-1}$, which has the effect of dividing both dm and dn each by $\det e_3$: the integral equals $(\det e_3)^{-2}$ times

$$\int_{\mathbb{R}^6} e^{-2\pi i(me_3^{-1}\hat m-ne_3^{-1}\hat n)}e^{-\pi\tau_2^{-1}\|(m+n\tau)\|^2}\,dm\,dn \;=\;$$
$$\int_{\mathbb{R}^6} e^{-2\pi i(me_3^{-1}\hat m-ne_3^{-1}(\hat n+\tau_1\hat m))}e^{-\pi\tau_2^{-1}\|m\|^2-\pi\tau_2\|n\|^2}\,dm\,dn \tag{H.89}$$

and (H.88) is equal to

$$\mathcal{G}_2^\circ(\tau, ee^t) = (\det e_3)^{-2} \sum_{\substack{p=[p_1\,p_2] \\ q=[0\,q_2] \\ p_1>0,\, 0\le p_2<|q_2|}} e^{-\pi\tau_2^{-1}\|(p+q\tau)e_2\|^2} \sum_{\hat m,\hat n\in\mathbb{Z}^3} e^{2\pi i(pQ\hat m - qQ\hat n)} e^{-\pi\tau_2\|e_3^{-1}\hat m\|^2 - \pi\tau_2^{-1}\|e_3^{-1}(\hat n+\hat m\tau_1)\|^2}. \tag{H.90}$$

The Fourier coefficient for $N_4 \in M_{3,2}(\mathbb{Z})$ is equal to

$$\mathcal{F}\mathcal{G}_2^\circ(\tau, e_2, e_3; N_4) = (\det e_3)^{-2} \sum_{\substack{p=[p_1\,p_2] \\ q=[0\,q_2] \\ p_1>0,\, 0\le p_2<|q_2|}} \sum_{\substack{\hat m,\hat n\in\mathbb{Z}^3 \\ \hat m p - \hat n q = N_4}} e^{-\pi\tau_2^{-1}\|(p+q\tau)e_2\|^2 - \pi\tau_2\|e_3^{-1}\hat m\|^2 - \pi\tau_2^{-1}\|e_3^{-1}(\hat n+\hat m\tau_1)\|^2}. \tag{H.91}$$

Let us now consider $\mathcal{G}_1(\tau, ee^t)$, which has the contributions for $p, q \in \mathbb{Z}^2$ such that $\mathrm{rank}[\begin{smallmatrix} p \\ q \end{smallmatrix}] = 1$:

$$\mathcal{G}_1(\tau, ee^t) := \sum_{\substack{\mathrm{rank}[\begin{smallmatrix} p \\ q \end{smallmatrix}]=1 \\ [\begin{smallmatrix} p & m \\ q & n \end{smallmatrix}]\in\mathcal{M}^{(2)}_{2,5}(\mathbb{Z})}} e^{-\pi\tau_2^{-1}\|[p+q\tau\,,\, m+n\tau]e\|^2}. \tag{H.92}$$

We may write this as an average over $\{\pm\Gamma_\infty\}\backslash SL(2,\mathbb{Z})$:

$$\mathcal{G}_1(\tau, ee^t) = \sum_{\gamma\in\{\pm\Gamma_\infty\}\backslash SL(2,\mathbb{Z})} \mathcal{G}_1^\circ(\gamma\tau, ee^t), \tag{H.93}$$

where

$$\begin{aligned}\mathcal{G}_1^\circ(\tau, ee^t) &:= \sum_{\substack{p\neq 0 \\ [\begin{smallmatrix} p & m \\ 0 & n \end{smallmatrix}]\in\mathcal{M}^{(2)}_{2,5}(\mathbb{Z})}} e^{-\pi\tau_2^{-1}\|[p, m+n\tau]e\|^2} \\ &= \sum_{\substack{p\neq 0 \\ n\neq 0 \\ m\in\mathbb{Z}^3}} e^{-\pi\tau_2^{-1}\|pe_2\|^2 - \pi\tau_2^{-1}\|(pQ+m+n\tau_1)e_3\|^2 - \pi\tau_2\|ne_3\|^2}. \end{aligned} \tag{H.94}$$

Here we used that the matrix $[\begin{smallmatrix} p & m \\ 0 & n \end{smallmatrix}]$ has rank 2 if and only if $n \neq 0$ (since $p \neq 0$). Poisson sum over m gives the formula

$$\mathcal{G}_1^\circ(\tau, ee^t) = \frac{\tau_2^{\frac{3}{2}}}{\det e_3} \sum_{\substack{p\neq 0 \\ n\neq 0 \\ \hat m\in\mathbb{Z}^3}} e^{2\pi i(pQ+n\tau_1)\hat m} e^{-\pi\tau_2^{-1}\|pe_2\|^2 - \pi\tau_2\|ne_3\|^2 - \pi\tau_2\|e_3^{-1}\hat m\|^2} \tag{H.95}$$

for (H.94), where $\hat m$ is a column vector.

We conclude that the Fourier coefficient of $\mathcal{G}_1^\circ(\tau, ee^t)$ for N_4 is equal to

$$\mathcal{F}\mathcal{G}_1^\circ(\tau, e_2, e_3; N_4) = \frac{\tau_2^{\frac{3}{2}}}{\det e_3} \sum_{\substack{p\neq 0\\ n\neq 0\\ \hat{m}p=N_4}} e^{2\pi i\tau_1 n\hat{m}} e^{-\pi\tau_2^{-1}\|pe_2\|^2-\pi\tau_2\|ne_3\|^2-\pi\tau_2\|e_3^{-1}\hat{m}\|^2} . \quad \text{(H.96)}$$

Observe that $\mathcal{F}\mathcal{G}_1(\tau, e_2, e_3; N_4) \equiv 0$ if $\operatorname{rank}(N_4) = 2$, and again that $\mathcal{G}_0(\tau, ee^t)$ has no nonzero Fourier coefficients because $[0\,0\star\star\star]\begin{pmatrix} I_2 & Q \\ & I_3\end{pmatrix}$ is independent of Q.

Proposition 4.1 states that

$$\frac{1}{2}\xi(2s)\xi(2s-1)\, E^{SL(5)}_{\beta_2;s}(e) = \int_0^\infty \int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} \mathcal{G}(\tau, uee^t)\, \frac{d^2\tau}{\tau_2^2}\, \frac{du}{u^{1-2s}} . \quad \text{(H.97)}$$

Since we have specialized $e \in SL(d,\mathbb{R})$ to have the form $e = \begin{pmatrix} I_2 & Q \\ & I_3\end{pmatrix}\begin{pmatrix} e_2 & \\ & e_3\end{pmatrix}$ the Fourier coefficient for N_4 can be written as

$$\begin{aligned}\frac{1}{2}\xi(2s)\xi(2s-1)F^{SL(5)\,\beta_3}_{\beta_2;s}(N_4) &= \\ &\int_0^\infty \int_{\mathbb{H}} \mathcal{F}\mathcal{G}_2^\circ(\tau, u^{\frac{1}{2}}e_2, u^{\frac{1}{2}}e_3; N_4)\, \frac{d^2\tau}{\tau_2^2}\, \frac{du}{u^{1-2s}} \\ &+ \int_0^\infty \int_{\Gamma_\infty\backslash\mathbb{H}} \mathcal{F}\mathcal{G}_1^\circ(\tau, u^{\frac{1}{2}}e_2, u^{\frac{1}{2}}e_3; N_4)\, \frac{d^2\tau}{\tau_2^2}\, \frac{du}{u^{1-2s}} . \quad \text{(H.98)}\end{aligned}$$

Let us consider the first integral,

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{H}} (\det e_3)^{-2} \sum_{\substack{p=[p_1\,p_2]\\ q=[0\,q_2]\\ p_1>0,\,0\leq p_2<|q_2|}} e^{-\pi\tau_2^{-1}u\|(p+q\tau)e_2\|^2} \\ &\qquad \sum_{\substack{\hat{m},\hat{n}\in\mathbb{Z}^3\\ \hat{m}p-\hat{n}q=N_4}} e^{-\pi\tau_2 u^{-1}\|e_3^{-1}\hat{m}\|^2-\pi\tau_2^{-1}u^{-1}\|e_3^{-1}(\hat{n}+\hat{m}\tau_1)\|^2}\, \frac{d^2\tau}{\tau_2^2}\, \frac{du}{u^{4-2s}} . \quad \text{(H.99)}\end{aligned}$$

Changing variables to $x = u/\tau_2$ and $y = \tau_2 u$, so that $u = \sqrt{xy}$, $\tau_2 = \sqrt{y/x}$ and $d\tau_2 du = \frac{dxdy}{2x}$ the integral becomes

$$\frac{1}{2}(\det e_3)^{-2}\int_{\mathbb{R}}\int_0^\infty\int_0^\infty \sum_{\substack{p=[p_1\,,\,p_2]\\ q=[0\,,\,q_2]\\ p_1>0\\ 0\le p_2<|q_2|}}\ \sum_{\substack{\hat m,\hat n\,\in\,\mathbb{Z}^3\\ \hat m p-\hat n q=N_4}}$$
$$e^{-\pi x\|(p+q\tau_1)e_2\|^2-\pi x^{-1}\|e_3^{-1}\hat m\|^2}e^{-\pi y\|qe_2\|^2-\pi y^{-1}\|e_3^{-1}(\hat n+\hat m\tau_1)\|^2}\,\frac{dx}{x^{2-s}}\,\frac{dy}{y^{3-s}}d\tau_1\,. \tag{H.100}$$

Integrating over x and y yields

$$2(\det e_3)^{-2}\int_{\mathbb{R}} \sum_{\substack{p=[p_1\,,\,p_2]\\ q=[0\,,\,q_2]\\ p_1>0\\ 0\le p_2<|q_2|}}\ \sum_{\substack{\hat m,\hat n\,\in\,\mathbb{Z}^3\\ \hat m p-\hat n q=N_4}} \left(\frac{\|(p+q\tau_1)e_2\|}{\|e_3^{-1}\hat m\|}\right)^{1-s}\ \times$$
$$\left(\frac{\|qe_2\|}{\|e_3^{-1}(\hat n+\hat m\tau_1)\|}\right)^{2-s} K_{s-1}(2\pi\|(p+q\tau_1)e_2\|\|e_3^{-1}\hat m\|)\ \times$$
$$K_{s-2}(2\pi\|qe_2\|\|e_3^{-1}(\hat n+\hat m\tau_1)\|)d\tau_1\,. \tag{H.101}$$

Next we analyze the second integral in (H.98), in which we assume N_4 has rank 1 (since it vanishes if it has rank 2):

$$\tfrac{1}{\det e_3}\int_0^\infty\int_{\Gamma_\infty\backslash\mathbb{H}} \sum_{\substack{p\,\neq\,0\\ n\,\neq\,0\\ \hat m p=N_4}} e^{2\pi i\tau_1 n\hat m}\ \times$$
$$e^{-\pi\tau_2^{-1}u\|pe_2\|^2-\pi\tau_2 u\|ne_3\|^2-\pi\tau_2 u^{-1}\|e_3^{-1}\hat m\|^2}\,\frac{d^2\tau}{\tau_2^{\frac12}}\,\frac{du}{u^{\frac52-2s}}\,. \tag{H.102}$$

The τ_1 integration over $[0,1]$ enforces the condition that $n\perp\hat m$ (which implies $n\perp N_4$):

$$\tfrac{1}{\det e_3}\int_0^\infty\int_0^\infty \sum_{\substack{p\,\neq\,0\\ n\,\neq\,0\\ \hat m\,\perp\,n\\ \hat m p=N_4}} e^{-\pi\tau_2^{-1}u\|pe_2\|^2-\pi\tau_2 u\|ne_3\|^2-\pi\tau_2 u^{-1}\|e_3^{-1}\hat m\|^2}\,\frac{d\tau_2}{\tau_2^{\frac12}}\,\frac{du}{u^{\frac52-2s}}\,. \tag{H.103}$$

As before, change variables $x = u/\tau_2$ and $y = \tau_2 u$ so that (H.103) becomes

$$\frac{1}{2(\det e_3)} \sum_{\substack{p\neq 0\\ n\neq 0\\ \hat m\perp n\\ \hat m p=N_4}} \int_0^\infty\int_0^\infty e^{-\pi x\|pe_2\|^2-\pi x^{-1}\|e_3^{-1}\hat m\|^2} e^{-\pi y\|ne_3\|^2}\,\frac{dx}{x^{2-s}}\,\frac{dy}{y^{\frac32-s}}$$

$$= \frac{\Gamma\left(s-\frac12\right)}{(\det e_3)} \sum_{\substack{p\neq 0\\ n\neq 0\\ \hat m\perp n\\ \hat m p=N_4}} (\pi\|ne_3\|^2)^{\frac12-s}\left(\frac{\|e_3^{-1}\hat m\|}{\|pe_2\|}\right)^{s-1} K_{s-1}(2\pi\|e_3^{-1}\hat m\|\|pe_2\|)\,. \tag{H.104}$$

H.2.2.4. **The parabolic** $P_{\beta_4} = GL(1)\times SL(4)\times U_{\beta_4}$.

In this case $e\in GL(5,\mathbb{R})$ has the special form $\begin{pmatrix} I_1 & Q\\ & I_4\end{pmatrix}\left(\begin{smallmatrix} e_1 & \\ & e_4\end{smallmatrix}\right)$, where Q is a 4-dimensional row vector, e_1 is a nonzero real number, and $e_4\in GL(4,\mathbb{R})$. We work with a sum of the form (H.80) but now instead $p,q\in\mathbb{Z}$ and $m,n\in\mathbb{Z}^4$. Then the exponent (H.87) becomes

$$-\pi\tau_2^{-1}e_1^2|p+q\tau|^2 \;-\; \pi\tau_2^{-1}\|((p+q\tau)Q+m+n\tau)e_4\|^2\,. \tag{H.105}$$

If $p=q=0$ then the exponent and hence $\mathcal{G}(\tau,ee^t)$ is independent of Q. To get nontrivial Fourier modes in Q, we must thus assume to the contrary that $\mathrm{rank}\left[\begin{smallmatrix}p\\ q\end{smallmatrix}\right]=1$. We write the contributions of these rank one terms as

$$\mathcal{G}_1(\tau,ee^t) \;=\; \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\mathcal{G}_1^\circ(\gamma\tau,ee^t)\,, \tag{H.106}$$

where

$$\mathcal{G}_1^\circ(\tau,ee^t) \;:=\; \sum_{\substack{p>0\\ m,n\in\mathbb{Z}^4\\ n\neq 0}} e^{-\pi\tau_2^{-1}e_1^2p^2-\pi\tau_2^{-1}\|(pQ+m+n\tau)e_4\|^2} \tag{H.107}$$

(this uses the fact that the $SL(2,\mathbb{Z})$ orbits of rank one integer matrices $\left[\begin{smallmatrix}p\\ q\end{smallmatrix}\right]$ each have representatives with $p>0$ and $q=0$, and that the rank 2 condition for $\left[\begin{smallmatrix}p & m\\ 0 & n\end{smallmatrix}\right]$ is that $n\neq 0$). Applying Poisson summation over $m\in\mathbb{Z}^4$ results in the expression

$$\mathcal{G}_1^\circ(\tau,ee^t) \;=\; \sum_{\substack{p>0\\ \hat m,n\in\mathbb{Z}^4\\ n\neq 0}} e^{-\pi\tau_2^{-1}e_1^2p^2-\pi\tau_2\|ne_4\|^2}e^{2\pi i(pQ\hat m+\tau_1 n\hat m)}$$

$$\times\int_{\mathbb{R}^4} e^{-2\pi i m\hat m}e^{-\pi\tau_2^{-1}\|me_4\|^2}\,dm\,. \tag{H.108}$$

Here we think of $\hat{m}$ as a column vector. Thus the Fourier coefficient for $e^{2\pi i Q N_4}$, when the column vector $N_4 \in \mathbb{Z}^4$ is not zero, is equal to

$$\mathcal{F}\mathcal{G}_1^\circ(\tau, ee^t) = \tau_2^2(\det e_4)^{-1} \sum_{\substack{p>0 \\ n\neq 0 \\ p\hat{m}=N_4}} e^{2\pi i \tau_1 n\hat{m}} e^{-\pi\tau_2^{-1}e_1^2p^2 - \pi\tau_2\|ne_4\|^2 - \pi\tau_2\|e_4^{-1}\hat{m}\|^2} . \quad \text{(H.109)}$$

Using proposition 4.1 the N_4-th Fourier coefficient of $\frac{1}{2}\xi(2s)\xi(2s-1)E^{SL(5)}_{\beta_2;s}(e)$ is

$$\frac{1}{\det e_4}\int_0^\infty\int_0^\infty\int_0^1 e^{2\pi i\tau_1 n\hat{m}} \times \sum_{\substack{p>0 \\ n\neq 0 \\ p\hat{m}=N_4}} e^{-\pi\tau_2^{-1}ue_1^2p^2 - \pi\tau_2 u\|ne_4\|^2 - \pi\tau_2 u^{-1}\|e_4^{-1}\hat{m}\|^2} d\tau_1\, d\tau_2 \frac{du}{u^{3-2s}}$$
$$= \frac{1}{\det e_4}\int_0^\infty\int_0^\infty \sum_{\substack{p>0 \\ n\neq 0 \\ p\hat{m}=N_4 \\ n\perp N_4}} e^{-\pi\tau_2^{-1}ue_1^2p^2 - \pi\tau_2 u\|ne_4\|^2} \times e^{-\pi\tau_2 u^{-1}\|e_4^{-1}\hat{m}\|^2}\, d\tau_2\, \frac{du}{u^{3-2s}} . \quad \text{(H.110)}$$

Changing variables to $x = u/\tau_2$ and $y = \tau_2 u$, so that $u = \sqrt{xy}$, $\tau_2 = \sqrt{y/x}$ and $d\tau_2 du = \frac{dxdy}{2x}$ (H.110) equals

$$\frac{1}{2\det e_4}\sum_{\substack{p>0 \\ n\neq 0 \\ p\hat{m}=N_4 \\ n\perp N_4}} \int_0^\infty\int_0^\infty e^{-\pi x e_1^2p^2 - \pi x^{-1}\|e_4^{-1}\hat{m}\|^2 - \pi y\|ne_4\|^2}\, \frac{dx}{x^{\frac{5}{2}-s}}\, \frac{dy}{y^{\frac{3}{2}-s}} =$$
$$\frac{\Gamma\left(s-\frac{1}{2}\right)}{\pi^{s-\frac{1}{2}}(\det e_4)} \sum_{\substack{p>0 \\ n\neq 0 \\ p\hat{m}=N_4 \\ n\perp N_4}} \left(\tfrac{\|e_4^{-1}\hat{m}\|}{p|e_1|}\right)^{s-\frac{3}{2}}\|ne_4\|^{1-2s}K_{s-\frac{3}{2}}(2\pi e_1 p\|e_4^{-1}\hat{m}\|) . \quad \text{(H.111)}$$

H.3. **The $Spin(5,5)$ case.** Here we analyze the Fourier modes of the series $E^{Spin(5,5)}_{\alpha_1;s}$, which is one of the two Eisenstein series appropriate to the $D = 6$ case. The results are summarized in section 4.6. Here we shall use the expressions (E.15) and (G.12), which for $d=5$ imply

$$E^{SO(5,5)}_{\alpha_1;s+3/2}\left(\begin{pmatrix} I & Bw_5 \\ & I\end{pmatrix}\begin{pmatrix} v^{1/2}e & \\ & v^{-1/2}\tilde e\end{pmatrix}\right) = \frac{v^{5/2}}{2\,\xi(2s+3)}\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} E^{SL(2)}_s(\tau)\,\mathcal{G}(\tau, vee^t+B)\,\frac{d^2\tau}{\tau_2^2} + v^{s+3/2}\,E^{SL(5)}_{\beta_4;s+3/2}(e) + v^{5/2-s}\,\frac{\xi(2s-1)}{\xi(2s+3)}\,E^{SL(5)}_{\beta_1;s}(e)\,, \quad (H.112)$$

where $v>0$ and $e\in SL(5,\mathbb{R})$. Formula (2.13) shows that the same formula is valid for $E^{Spin(d,d)}_{\alpha_1;s+3/2}(h')$, where $h'\in Spin(d,d,\mathbb{R})$ is any element which projects onto $\begin{pmatrix} I & Bw_5 \\ & I\end{pmatrix}\begin{pmatrix} v^{1/2}e & \\ & v^{-1/2}\tilde e\end{pmatrix}$ via the covering map $Spin(d,d,\mathbb{R})\to SO(d,d,\mathbb{R})$.

(i) **The parabolic** $P_{\alpha_5}=GL(1)\times SL(5)\times U_{\alpha_5}$

The analysis in this section also covers limit (iii), since both parabolics come from spinor nodes. Formula (H.112) shows that the nontrivial spinor parabolic Fourier coefficients (in B) all come from the integral on the right-hand side. In limit (i) the parameter v plays the role of the parameter r^2 from (2.9), and so we set $v=r^2$. Substituting the formula (E.12) for $\mathcal{G}(\tau,r^2ee^t+B)$ we see that the contribution to the nonzero Fourier modes of (H.112) is given by

$$\frac{r^5}{2\,\xi(2s+3)}\int_{SL(2,\mathbb{Z})\backslash\mathbb{H}} E^{SL(2)}_s(\tau)\sum_{\left[\begin{smallmatrix} m\\ n\end{smallmatrix}\right]\in\mathcal{M}^{(2)}_{2,5}(\mathbb{Z})} e^{-\pi\tau_2^{-1}\,r^2\,\|(m+n\tau)e\|^2}e^{-2\pi i mBn^t}\,\frac{d^2\tau}{\tau_2^2}\,. \quad (H.113)$$

Note that all nonzero Fourier modes have the form $B\mapsto e^{2\pi i mBn^t}$, which is precisely the $\frac{1}{2}$-BPS condition from (4.76).

We conclude that for $N_2\in\mathcal{M}_{5,5}(\mathbb{Z})$ the Fourier coefficient of $E^{SO(5,5)}_{\alpha_1;s}\left(\left(\begin{smallmatrix} I & Bw_d \\ & I\end{smallmatrix}\right)\left(\begin{smallmatrix} re & \\ & r^{-1}\tilde e\end{smallmatrix}\right)\right)$ for the character $B\mapsto e^{i\pi(\operatorname{tr}N_2B)}$ is equal to

$$\frac{r^5}{2\,\xi(2s+3)}\int_{\mathbb{H}} E^{SL(2)}_s(\tau)\sum_{\substack{\left[\begin{smallmatrix} m\\ n\end{smallmatrix}\right]\in SL(2,\mathbb{Z})\backslash\mathcal{M}^{(2)}_{2,5}(\mathbb{Z})\\ N_2=n^tm-m^tn}} e^{-\pi\tau_2^{-1}r^2\,\|(m+n\tau)e\|^2}\,\frac{d^2\tau}{\tau_2^2} \quad (H.114)$$

with $r^2=r_4/\ell_7$ according the identification of the parameters in [1] recalled in (2.9).

In the case of interest in section 4.6 the parameter s is equal to zero, and the integral was computed in (E.22) as

$$\frac{r^3}{2\,\xi(3)}\sum_{\substack{\left[\begin{smallmatrix} m\\ n\end{smallmatrix}\right]\in SL(2,\mathbb{Z})\backslash\mathcal{M}^{(2)}_{2,5}(\mathbb{Z})\\ N_2=n^tm-m^tn}}\frac{e^{-2\pi r^2\det(\left[\begin{smallmatrix} m\\ n\end{smallmatrix}\right]ee^t\left[\begin{smallmatrix} m\\ n\end{smallmatrix}\right]^t)^{1/2}}}{\det(\left[\begin{smallmatrix} m\\ n\end{smallmatrix}\right]ee^t\left[\begin{smallmatrix} m\\ n\end{smallmatrix}\right]^t)^{1/2}}\,. \quad (H.115)$$

In the claim following (E.25) we saw that the $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]$ in this sum can be parametrized as $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right] = \left(\begin{smallmatrix} d_1 & b \\ 0 & d_2 \end{smallmatrix}\right)\left[\begin{smallmatrix} m' \\ n' \end{smallmatrix}\right]$, where $d_1 \neq 0$, $0 \leq b < d_2$, and $\left[\begin{smallmatrix} m' \\ n' \end{smallmatrix}\right]$ ranges over left $GL(2,\mathbb{Z})$-cosets of $\mathcal{M}^{(2)}_{2,5}(\mathbb{Z})' := \{$all possible bottom two rows of matrices in $SL(5,\mathbb{Z})\}$. (This coset space is in bijective correspondence with $P_{\beta_2}(\mathbb{Z})\backslash SL(5,\mathbb{Z})$.) The constraint $N_2 = n^t m - m^t n$ then reads $N_2 = d_1 d_2((n')^t m' - (m')^t n')$. As a consequence we can rewrite (H.115) as

$$\frac{r^3}{2\,\xi(3)} \sum_{\substack{\left[\begin{smallmatrix} m' \\ n' \end{smallmatrix}\right] \in GL(2,\mathbb{Z})\backslash \mathcal{M}^{(2)}_{2,5}(\mathbb{Z})' \\ N_2 = d_1 d_2((n')^t m' - (m')^t n')}} \frac{d_2}{d_1 d_2} \, \frac{e^{-2\pi r^2 d_1 d_2 \det(\left[\begin{smallmatrix} m' \\ n' \end{smallmatrix}\right] e e^t \left[\begin{smallmatrix} m' \\ n' \end{smallmatrix}\right]^t)^{1/2}}}{\det(\left[\begin{smallmatrix} m' \\ n' \end{smallmatrix}\right] e e^t \left[\begin{smallmatrix} m' \\ n' \end{smallmatrix}\right]^t)^{1/2}} . \quad \text{(H.116)}$$

The product $d_1 d_2$ obviously divides each entry of N_2, but the entries of $N_2 = n^t m - m^t n$ can have a nontrivial common factor even if $\gcd(m) = \gcd(n) = 1$. On the other hand, the $\binom{5}{2} = 10$ minors of the two bottom rows $\left[\begin{smallmatrix} m' \\ n' \end{smallmatrix}\right]$ must be relatively prime, since the determinant of the $SL(5,\mathbb{Z})$ matrix (i.e., 1) is an integral linear combination of them. These minors are the entries of N_2, up to sign. We conclude that $d_1 d_2 = \gcd(N_2)$ and that (H.115) is equal to

$$\frac{r^3}{2\,\xi(3)} \sum_{\substack{\left[\begin{smallmatrix} m' \\ n' \end{smallmatrix}\right] \in GL(2,\mathbb{Z})\backslash \mathcal{M}^{(2)}_{2,5}(\mathbb{Z})' \\ N_2 = \gcd(N_2)((n')^t m' - (m')^t n')}} \frac{\sigma_1(\gcd(N_2))}{\gcd(N_2)} \, \frac{e^{-2\pi r^2 \gcd(N_2) \det(\left[\begin{smallmatrix} m' \\ n' \end{smallmatrix}\right] e e^t \left[\begin{smallmatrix} m' \\ n' \end{smallmatrix}\right]^t)^{1/2}}}{\det(\left[\begin{smallmatrix} m' \\ n' \end{smallmatrix}\right] e e^t \left[\begin{smallmatrix} m' \\ n' \end{smallmatrix}\right]^t)^{1/2}} . \quad \text{(H.117)}$$

Again, (2.13) shows that this formula is also valid for $F^{Spin(d,d)}_{\alpha_1;s}(h')$ and any $h' \in Spin(d,d,\mathbb{R})$ which projects onto $\left(\begin{smallmatrix} I & BW_d \\ & I \end{smallmatrix}\right)\left(\begin{smallmatrix} re & \\ & r^{-1}\tilde{e} \end{smallmatrix}\right)$ via the covering map $Spin(d,d,\mathbb{R}) \to SO(d,d,\mathbb{R})$.

(ii) **The parabolic** $P_{\alpha_1} = GL(1) \times Spin(4,4) \times U_{\alpha_1}$

We shall use (H.112) to compute the nonzero Fourier modes of $E^{SO(5,5)}_{\alpha_1;s}$ and hence $E^{Spin(5,5)}_{\alpha_1;s}$. Before beginning the calculation, it is helpful to explicitly write out the groups and characters involved. The unipotent radical $U = U_{\alpha_1}$ of P_{α_1} is an abelian group isomorphic to $\mathbb{R}^8$ under the map

$$u_1, u_2, \ldots, u_8 \quad \mapsto \quad \begin{pmatrix} 1 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & -u_1u_8-u_2u_7-u_3u_6-u_4u_5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u_8 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -u_7 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -u_6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -u_5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -u_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -u_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -u_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -u_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{(H.118)}$$

and $\Gamma \cap U$ is isomorphic to $\mathbb{Z}^8$ under this identification. The general Fourier mode is indexed $N_1 = \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right] = \left[\begin{smallmatrix} m^1 & m^2 & m^3 & m^4 \\ n_1 & n_2 & n_3 & n_4 \end{smallmatrix}\right] \in M_{2,4}(\mathbb{Z})$ from (4.81), and is given by the character

$$\chi_{N_1}(u) \;:=\; e^{2\pi i(m^1u_1+m^2u_2+m^3u_3+m^4u_4+n_1u_8+n_2u_7+n_3u_6+n_4u_5)}\,. \qquad (H.119)$$

The Fourier coefficient (4.80) is given by

$$F^{SO(5,5)\,\alpha_1}_{\alpha_1;s}(N_1) \;=\; \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E^{SO(5,5)}_{\alpha_1;s}(uh)\,\chi_{N_1}(u)^{-1}\,du\,. \qquad (H.120)$$

The general element h of the Levi component has the form

$$h \;=\; h(a,h_4) \;=\; \begin{pmatrix} a & 0 & 0 \\ 0 & h_4 & 0 \\ 0 & 0 & 1/a \end{pmatrix}, \qquad (H.121)$$

where $a \neq 0$ and $h_4 \in SO(4,4)(\mathbb{R})$.

Given the structure of the last two terms in (H.112) (which are insensitive to u_5, u_6, u_7, u_8) it makes sense to treat the cases $N \neq [0\,0\,0\,0]$ and $N = [0\,0\,0\,0]$ separately. Since $E^{SO(5,5)}_{\alpha_1;s}$ is invariant under the Weyl group element $h(1,w_8)$ (w_8 denoting the reversed-8×8 identity matrix) and conjugating the matrix (H.118) by $h(1,w_8)$ reverses the order of the u_i, the Fourier coefficient $F^{SO(5,5)\,\alpha_1}_{\alpha_1;s}([\begin{smallmatrix} M \\ N\end{smallmatrix}])$ evaluated at $h(a,h_4)$ equals $F^{SO(5,5)\,\alpha_1}_{\alpha_1;s}([\begin{smallmatrix} N \\ M\end{smallmatrix}])$ evaluated at $h(a,w_8h_4)$. Since we are studying nontrivial Fourier coefficients at least one entry of the matrix N_1 is nonzero. Thus the determination of these coefficients for N_1 of the form $[\begin{smallmatrix} M \\ 0\end{smallmatrix}]$ reduces to those of the form $[\begin{smallmatrix} 0 \\ N\end{smallmatrix}]$. Therefore in performing these computations we can assume that $N \neq [0\,0\,0\,0]$, and then convert afterwards to $N = [0\,0\,0\,0]$ using this w_8-mechanism. For reasons of space we will not carry out this conversion here, and instead limit our discussion in section 4.6 to the case when $N \neq [0\,0\,0\,0]$. Thus for the remainder of the paper we assume $N \neq [0\,0\,0\,0]$.

Suppose $e \in SL(5,\mathbb{R})$ has the form $e = \left(\begin{smallmatrix} 1 & Q \\ & I_4\end{smallmatrix}\right)\left(\begin{smallmatrix} v^{-1/2}r^2 & \\ & e_4\end{smallmatrix}\right) = \left(\begin{smallmatrix} v^{-1/2}r^2 & Qe_4 \\ & e_4\end{smallmatrix}\right)$, with $Q = [q_1\,q_2\,q_3\,q_4]$ and $e_4 \in GL(4,\mathbb{R})$ a matrix with determinant $v^{1/2}r^{-2}$. The reason for writing e this way is ensure that r plays the same role it does in (2.9). Furthermore suppose

$$Bw_5 \;=\; \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & 0 \\ b_5 & b_6 & b_7 & 0 & -b_4 \\ b_8 & b_9 & 0 & -b_7 & -b_3 \\ b_{10} & 0 & -b_9 & -b_6 & -b_2 \\ 0 & -b_{10} & -b_8 & -b_5 & -b_1 \end{pmatrix}. \qquad (H.122)$$

Then the argument $\left(\begin{smallmatrix} I & Bw_5 \\ & I\end{smallmatrix}\right)\left(\begin{smallmatrix} v^{1/2}e & \\ & v^{-1/2}\tilde{e}\end{smallmatrix}\right)$ of the first line of (H.112) lies in P_{α_1} (recall that the parameter v determines the determinant, $v^{5/2}$, of the upper left 5×5 block of this matrix). This product also has the factorization uh, where u is the matrix (H.118) with $(u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8) = (q_1,q_2,q_3,q_4,b_1-b_5q_1-b_8q_2-b_{10}q_3, b_2-b_6q_1-b_9q_2+b_{10}q_4, b_3-b_7q_1+b_9q_3+b_8q_4, b_4+b_7q_2+b_6q_3+b_5q_4)$ and $h = h(r^2,h_4)$, where $h_4 = \left(\begin{smallmatrix} I_4 & B'w_4 \\ & I_4\end{smallmatrix}\right)\left(\begin{smallmatrix} v^{1/2}e_4 & \\ & v^{-1/2}\tilde{e}_4\end{smallmatrix}\right)$. Thus the character $\chi_{N_1}(u) = \exp(2\pi i(m^1 - n_4b_5 - n_3b_6 - n_2b_7)q_1 + (m^2 -$

$n_4b_8 - n_3b_9 + n_1b_7)q_2 + (m^3 - n_4b_{10} + n_2b_9 + n_1b_6)q_3 + (m^4 + n_3b_{10} + n_2b_8 + n_1b_5)q_4 + n_4b_1 + n_3b_2 + n_2b_3 + n_1b_4)$.

Recall (E.12), which states

$$\mathcal{G}(\tau, vee^t + B) \;=\; \sum_{\left[\begin{smallmatrix} p & m_1 \\ q & m_2 \end{smallmatrix}\right] \in \mathcal{M}^{(2)}_{2,5}(\mathbb{Z})} e^{-\pi\tau_2 v\|[q\, m_2]e\|^2 - \pi\tau_2^{-1} v\|[p+q\tau_1 \;\; m_1+m_2\tau_1]e\|^2} e^{-2\pi i [p\, m_1] B [q\, m_2]^t} \tag{H.123}$$

after the elements of $\mathbb{Z}^5$ are grouped together as an integer p or q and a vector $m_1 = [m_{12}\, m_{13}\, m_{14}\, m_{15}]$ or $m_2 = [m_{22}\, m_{23}\, m_{24}\, m_{25}] \in \mathbb{Z}^4$. At this point identify the variables $b_1 = u_5$, $b_2 = u_6$, $b_3 = u_7$, and $b_4 = u_8$. Then $-[p\, m_1]B[q\, m_2]^t = -p(u_8m_{22} + u_7m_{23} + u_6m_{24} + u_5m_{25}) + q(u_8m_{12} + u_7m_{13} + u_6m_{14} + u_5m_{15}) - m_1B'm_2^t$. Hence the $\left[\begin{smallmatrix} p & m_1 \\ q & m_2 \end{smallmatrix}\right]$ which contribute to the Fourier mode $(u_5, u_6, u_7, u_8) \mapsto e^{2\pi i(n_4u_5+n_3u_6+n_2u_7+n_1u_8)}$ are those having $pm_{22} - qm_{12} = -n_1$, $pm_{23} - qm_{13} = -n_2$, $pm_{24} - qm_{14} = -n_3$, and $pm_{25} - qm_{15} = -n_4$. This condition on the minors of the 2×5 matrix $\left[\begin{smallmatrix} p & m_1 \\ q & m_2 \end{smallmatrix}\right]$ is $SL(2,\mathbb{Z})$-invariant. Each $SL(2,\mathbb{Z})$ orbit has an element with $q = 0$ and $p > 0$, at which the conditions simplify to

$$pm_2 \;=\; p[m_{22}\, m_{23}\, m_{24}\, m_{25}] \;=\; -\,[n_1\, n_2\, n_3\, n_4] \;=\; -N\,, \tag{H.124}$$

which cannot be zero because $\left[\begin{smallmatrix} p & m_1 \\ q & m_2 \end{smallmatrix}\right]$ has rank 2.

For the rest of the paper we shall assume that $N \neq 0$. Under this assumption all contributions to the Fourier coefficient come from the second line of (H.112). Thus the terms in (H.123) which contribute to the Fourier coefficient can be written as

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathcal{G}^0_a(\gamma\tau, vee^t + B)\,, \tag{H.125}$$

where

$$\mathcal{G}^0_a(\tau, vee^t + B) \;:=\; \sum_{\substack{pm_2 = -N \\ m_1 \in \mathbb{Z}^4}} e^{-\pi\tau_2 v\|[0\, m_2]e\|^2 - \pi\tau_2^{-1} v\|[p \;\; m_1+m_2\tau_1]e\|^2} e^{-2\pi i [p\, m_1] B [0\, m_2]^t}\,. \tag{H.126}$$

Using the facts that $e^{-2\pi i [p\, m_1] B [0\, m_2]^t} = e^{2\pi i(n_4u_5+n_3u_6+n_2u_7+n_1u_8)} e^{-2\pi i m_1 B' m_2^t}$ and $[0\, m_2]e = m_2e_4$ we now execute Poisson summation over $m_1 \in \mathbb{Z}^4$ in

(H.126):

$$\begin{aligned} e^{-2\pi i(n_4u_5+n_3u_6+n_2u_7+n_1u_8)}\mathcal{G}_a^0(\tau, vee^t+B) \quad &= \\ \sum_{\substack{pm_2=-N\\ \hat{m}_1\in\mathbb{Z}^4}} e^{-\pi\tau_2 v\|m_2e_4\|^2}\int_{\mathbb{R}^4} e^{2\pi i(m_2B'-\hat{m}_1)\cdot m_1} e^{-\pi\tau_2^{-1}v\|[p\;\; m_1+m_2\tau_1]e\|^2}\,dm_1 \\ = \sum_{\substack{pm_2=-N\\ \hat{m}_1\in\mathbb{Z}^4}} e^{2\pi i(\hat{m}_1-m_2B')\cdot m_2\tau_1} e^{-\pi\tau_2 v\|m_2e_4\|^2} \times \\ \int_{\mathbb{R}^4} e^{2\pi i(m_2B'-\hat{m}_1)\cdot m_1} e^{-\pi\tau_2^{-1}v\|[p\, m_1]e\|^2}\,dm_1\,. \quad \text{(H.127)} \end{aligned}$$

Again using the special form $e=\begin{pmatrix}1 & Q\\ & I_4\end{pmatrix}\begin{pmatrix}v^{-1/2}r^2 & \\ & e_4\end{pmatrix}=\begin{pmatrix}v^{-1/2}r^2 & Qe_4\\ & e_4\end{pmatrix}$ (so that $[p\, m_1]e=[v^{-1/2}r^2p\quad (pQ+m_1)e_4])$, this equals

$$\begin{aligned} = \sum_{\substack{pm_2=-N\\ \hat{m}_1\in\mathbb{Z}^4}} & e^{2\pi i(\hat{m}_1-m_2B')\cdot m_2\tau_1} e^{-\pi\tau_2 v\|m_2e_4\|^2} \\ & \times \int_{\mathbb{R}^4} e^{2\pi i(m_2B'-\hat{m}_1)\cdot m_1} e^{-\pi\tau_2^{-1}p^2r^4-\pi\tau_2^{-1}v\|(pQ+m_1)e_4\|^2}\,dm_1 \\ = \sum_{\substack{pm_2=-N\\ \hat{m}_1\in\mathbb{Z}^4}} & e^{2\pi i(\hat{m}_1-m_2B')\cdot(m_2\tau_1+pQ)} e^{-\pi\tau_2 v\|m_2e_4\|^2-\pi\tau_2^{-1}p^2r^4} \\ & \times \int_{\mathbb{R}^4} e^{2\pi i(m_2B'-\hat{m}_1)\cdot m_1} e^{-\pi\tau_2^{-1}v\|m_1e_4\|^2}\,dm_1 \\ & = \frac{\tau_2^2\, r^2}{v^{5/2}} \sum_{\substack{pm_2=-N\\ \hat{m}_1\in\mathbb{Z}^4}} e^{2\pi i(\hat{m}_1-m_2B')\cdot(m_2\tau_1+pQ)} \\ & \times e^{-\pi\tau_2 v\|m_2e_4\|^2-\pi\tau_2^{-1}p^2r^4-\pi\tau_2 v^{-1}\|(m_2B'-\hat{m}_1)(e_4^t)^{-1}\|^2}\,, \quad \text{(H.128)} \end{aligned}$$

where we have used that $\det e_4 = v^{1/2}r^{-2}$.

We now use (H.128) to determine the remaining Fourier dependence on (u_1,u_2,u_3,u_4). The dependence on $Q=[q_1\,q_2\,q_3\,q_4]$ here is from $e^{2\pi i(\hat{m}_1p-m_2B'p)Q^t} = e^{2\pi i(p\hat{m}_1+NB')Q^t}$. Writing $\hat{m}_1=[\hat{m}_{12}\,\hat{m}_{13}\,\hat{m}_{14}\,\hat{m}_{15}]$ the argument here is $(p\hat{m}_{12}-n_2b_7-n_3b_6-n_4b_5)q_1+(p\hat{m}_{13}+n_1b_7-n_3b_9-n_4b_8)q_2+(p\hat{m}_{14}+n_1b_6+n_2b_9-n_4b_{10})q_3+(p\hat{m}_{15}+n_1b_5+n_2b_8+n_3b_{10})q_4$. The character describing the Fourier mode above was $\chi_{N_1}(u)=\exp(2\pi i(m^1-n_4b_5-n_3b_6-n_2b_7)q_1+(m^2-n_4b_8-n_3b_9+n_1b_7)q_2+(m^3-n_4b_{10}+n_2b_9+n_1b_6)q_3+(m^4+n_3b_{10}+n_2b_8+n_1b_5)q_4+n_4b_1+n_3b_2+n_2b_3+n_1b_4)$. The condition that these match is thus that $p\hat{m}_1=p[\hat{m}_{12}\,\hat{m}_{13}\,\hat{m}_{14}\,\hat{m}_{15}]=[m^1\,m^2\,m^3\,m^4]=M$. Then the relevant Fourier coefficient $\mathcal{F}\mathcal{G}_a^0(\tau, vee^t + B; [\,{}^M_N\,])$ of $\mathcal{G}_a^0(\tau, vee^t + B)$ is

$$\begin{aligned} &= \frac{\tau_2^2\, r^2}{v^{5/2}} \sum_{\substack{p\hat{m}_1 = M \\ pm_2 = -N}} e^{2\pi i \hat{m}_1\cdot m_2 \tau_1} e^{-\pi\tau_2 v\|m_2 e_4\|^2} \times \\ &\qquad\qquad e^{-\pi\tau_2^{-1}p^2r^4 - \pi\tau_2 v^{-1}\|(m_2B' - \hat{m}_1)(e_4^t)^{-1}\|^2} \\ &= \frac{\tau_2^2\, r^2}{v^{5/2}} \sum_{p\,|\,\gcd(m^1,\ldots,n_4)} e^{-\pi\tau_2^{-1}p^2r^4} e^{-2\pi i\, p^{-2}\,\tau_1 M\cdot N} \times \\ &\qquad\qquad e^{-\pi\, p^{-2}\,\tau_2 v\|Ne_4\|^2 - \pi\, p^{-2}\,\tau_2 v^{-1}\|(NB'+M)(e_4^t)^{-1}\|^2}, \end{aligned} \tag{H.129}$$

the sum being over the positive common divisors p of $m^1, \ldots, n_4$.

Finally, we insert (H.129) into the second line of (H.112), and unfold to the strip. In terms of (H.120), this gives the Fourier coefficient $F^{SO(5,5)\,\alpha_1}_{\alpha_1;s+3/2}$ at $h = h(r^2, h_4)$, where $h_4 = \begin{pmatrix} I_4 & B'w_4 \\ & I_4 \end{pmatrix}\begin{pmatrix} v^{1/2}e_4 & \\ & v^{-1/2}\tilde{e}_4 \end{pmatrix}$:

$$\begin{aligned} F^{SO(5,5)\,\alpha_1}_{\alpha_1;s+3/2}(h(r^2,h_4);[\,{}^M_N\,]) \;\;=& \\ &\frac{r^2}{\xi(2s+3)} \int_{\Gamma_\infty\backslash\mathbb{H}} E_s^{SL(2)}(\tau)\, d^2\tau \sum_{p\,|\,\gcd(m^1,\ldots,n_4)} e^{-2\pi i\tau_1 \frac{M\cdot N}{p^2}} \\ &\times\; e^{-\pi\tau_2^{-1}p^2r^4 - \pi\, p^{-2}\,\tau_2 v\|Ne_4\|^2 - \pi\, p^{-2}\,\tau_2 v^{-1}\|(NB'+M)(e_4^t)^{-1}\|^2}. \end{aligned} \tag{H.130}$$

The matrix e_4 here is normalized differently than in (4.82), where it corresponds to the $SO(4,4)$ semisimple part of the Levi component. In our setting that is instead $v^{1/2}e_4$, so that $G_4 = ve_4e_4^t$. Here B' plays the role of the antisymmetric matrix B and so (4.82) reads

$$\begin{aligned} \sqrt{2}\,p_L &= v^{-1/2}(M + NB')(e_4^t)^{-1} - v^{1/2}Ne_4 \\ \sqrt{2}\,p_R &= v^{-1/2}(M + NB')(e_4^t)^{-1} + v^{1/2}Ne_4\,. \end{aligned} \tag{H.131}$$

It follows that

$$p_L^2 + p_R^2 \;=\; v^{-1}\|(M + NB')(e_4^t)^{-1}\|^2 + v\|Ne_4\|^2 \tag{H.132}$$

while

$$p_L^2 - p_R^2 \;=\; -2\,(M + NB')(e_4^t)^{-1}(Ne_4)^t \;=\; -2\,M\cdot N\,. \tag{H.133}$$

With these substitutions and replacing s by $s - 3/2$, (2.13) and (H.130) lead to (4.83).

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