CENTRAL LIMIT THEOREM FOR RECURRENT RANDOM WALKS ON A STRIP WITH BOUNDED POTENTIAL

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ABSTRACT. We prove that the recurrent random walk (RW) in random environment (RE) on a strip in bounded potential satisfies the Central Limit Theorem (CLT).

The key ingredients of the proof are the analysis of the invariant measure equation and construction of a linearly growing martingale for walks in bounded potential.

Our main result implies a complete classification of recurrent i.i.d. RWRE on the strip. Namely the walk either exhibits the Sinai behaviour in the sense that $X_t/(\ln t)^2$ converges, as $t \to \infty$, to a (random) limit (the Sinai law) or, it satisfies the CLT.

Another application of our main result is the CLT for the quasiperiodic environments with Diophantine frequencies in the recurrent case. We complement this result by proving that in the transient case the CLT holds for all uniquely ergodic environments.

We also investigate the algebraic structure of the environments satisfying the CLT. In particular, we show that there exists a collection of proper algebraic subvarieties in the space of transition probabilities such that

- If RE is stationary and ergodic and the transition probabilities are concentrated on one of subvarieties from our collection then the CLT holds;
- If the environment is i.i.d then the above condition is also necessary for the CLT.

All these results are valid for one-dimensional RWRE with bounded jumps as a particular case of the strip model.

2000 Mathematics Subject Classification: primary 60K37, 60F05; secondary 60J05, 82C44.

Keywords and Phrases: RWRE, recurrent random walks on a strip, random walks in random potential, CLT.

1. INTRODUCTION.

1.1. **Brief history of the problem.** It is well known that one dimensional RWRE exhibit features which are very different from those of classical random walks. This fact was first discovered in 1975 by Solomon ([36]) and by Kesten, Kozlov, and Spitzer ([20]) for transient random walks on $\mathbb{Z}$ for i.i.d. environments with jumps to nearest neighbours. In 1982, Sinai ([34]) found one of the most striking manifestations of that: he proved that for recurrent nearest neighbour RW in i.i.d. RE the correct scaling is $\ln^2 t$, or, more precisely, that $X_t/(\ln t)^2$ converges, as $t \to \infty$, to a (random) limit. Below, we call this phenomena the *Sinai law* or the *Sinai behaviour*.

Methods used in [34] (as well as in [36, 20]) rely heavily on the fact that the random walk is on $\mathbb{Z}$ and is allowed to jump *only* to the nearest sites. Hence the natural question asked by Sinai in [34]: would it be possible to extend his (and other) results to more general models such as RW on $\mathbb{Z}$ with bounded jumps.

In 1984, Key [24] found a recurrence criterion for RWRE on $\mathbb{Z}$ for the so called $[-l, r]$ model, where r and l are the maximal lengthes of possible jumps of the walk to the right and to the left respectively. Key's criterion was stated in terms of properties of the "middle" Lyapunov exponents of products of random matrices constructed from the parameters of the environment. This approach was developed by Letchikov [25] who in 1998 obtained a partial answer to Sinai's question. He proved that recurrent RWs on $\mathbb{Z}$ with bounded jumps in i.i.d. environment exhibit the Sinai behaviour if the probabilities of jumps of length 1 dominate the probabilities of other jumps. Further development by Brémont [5, 6, 7] of the Key-Letchikov type approach lead to a number of interesting results for the $[-l, r]$ model. Comments on the relation between the relevant Brémont's results and the results of this work will be provided later.

We turn now to RWRE on a strip. This model was introduced by Bolthausen and Goldsheid in [2] who also reduced the study of the RWRE with bounded jumps on $\mathbb{Z}$ to that of RW on a strip and proved the recurrence and transience criterion for the strip model. The technique used in [2] is completely different from that of [24, 25, 5, 6, 7].

The approach of [2] was developed in [16] where conditions for the Law of Large Numbers (LLN) and the CLT for transient RWs were provided in the quenched setting (for almost all environments). Independently, Roitershtein in [33] obtained the LLN and the annealed CLT for mixing RE.

A complete answer to Sinai's question was obtained in [3] where further development of methods from [2] and [16] allowed authors to prove that, unless the parameters of the environment belong to a certain algebraic subvariety, recurrent random walks in i.i.d. environments obey the Sinai law. The description of this subvariety is quite explicit. In particular, this description was used in [3] to show that recurrent finite range RWs in i.i.d. environments on $\mathbb{Z}$ exhibit either the Sinai behaviour or the CLT behaviour. Moreover, the CLT alternative takes place if and only if the walk on $\mathbb{Z}$ is a martingale.

Quasiperiodic environments form another class of environments where the CLT behaviour is observed. The first CLT in the nearest neighbour quasiperiodic setting (under the Diophantine conditions) is due to [1] in the transient case and due to [35] in the recurrent case. Extensions of the above results to the $[-l, r]$ model were obtained in [7]. We note that the results of [16] imply the CLT for hitting times for uniquely ergodic transient walks on the strip.

In subsection 1.2 we describe how our results allow us to obtain a complete classification of possible regimes in both i.i.d. and quasiperiodic Diophantine environments.

We remark that the papers cited above are only those closely related to our setting. In particular, since the recurrent random walks are the main subject of this work, we have mentioned only those papers on the transient case which are related to our approach to the problem. A comprehensive overview of earlier development of the subject can be found in [37, 4]. More recent results on the transient walks are contained in [29, 15, 16, 33, 9, 13, 14, 31, 10, 32].

1.2. **Motivation, goal, results, techniques.** The main motivation and goal of this paper is to answer the following question: does the Sinai/CLT alternative mentioned above hold for recurrent walks on a strip?

The principle difference between the $[-l, r]$ model and the general strip model is that, unlike in the $[-l, r]$ model, the fact that the RW in an i.i.d. RE on a strip does not obey the Sinai behaviour does not, generally, imply that this walk is a martingale. However, it does imply ([3]) that the potential of the environment is bounded (see (3.5) for the definition of the potential).

This boundedness of the potential is the main assumption under which the main result of the present work (Theorem 3.1) holds. It states that

• *random walks in stationary ergodic environments with bounded potentials satisfy the CLT.*

(The precise formulations of this and other results we discuss in this Introduction require some preparation and will be given later.)

It is important that this theorem does not use the i.i.d. property of the environment.

The main technical advance of this work is Lemma 4.4 which is the crucial ingredient in the proof of Theorem 3.1. This lemma provides a construction of an asymptotically linear solution to a martingale equation. This requires a new technique which is developed in Section 7.

Having said that, we should add that we use widely a number of both technical and principal results obtained in [2, 3, 16, 10]. Most of these results are listed in Section 2 which, on the one hand, is just necessary and on the other makes this paper more self-contained.

The above CLT criterion implies the following corollary (and answers the question which has motivated this work):

• *In recurrent i.i.d. environments on a strip there is an alternative: either the walk exhibits the Sinai behaviour or it satisfies the classical Central Limit Theorem.*

This statement largely completes the classification of possible limiting distributions in i.i.d. environments of the RWRE on the strip (complementing the results obtained in [2, 16, 3, 33, 10, 32]).

This criterion also allows us to show that

• *recurrent RWs in Diophantine quasi-periodic random environments generated by sufficiently smooth functions satisfy the CLT.*

Using a different method, we complement this statement by extending to the strip model the result which was proved in [15] for walks on $\mathbb{Z}$ with nearest neighbour jumps by proving that

• *transient RWs on a strip in environments generated by continuous uniquely ergodic transformations of a compact metric space always satisfy the CLT with positive drift.*

Note that the last two statements provide complete classification of the walks in Diophantine quasi-periodic environments. We would like to emphasize that in the transient case no smoothness of the uniquely ergodic transformation is required (in contrast to the recurrent case).

Finally as in [3], also here there is the algebraic side of the problem. We prove that there exists a collection of proper algebraic subvarieties in the space of transition probabilities such that:

• *If the RE is stationary and ergodic and the transition probabilities are concentrated on one of subvarieties from our collection then the CLT holds;*

• *If the environment is i.i.d then the above algebraic condition is also necessary for the CLT.*

Acknowledgement. The authors thank the referees for many useful comments, and, in particular, for suggesting the formulas in the Remark 2.8. The research of the first author was support by the NSF.

2. Definition of the model and some preparatory facts.

The following notations and definitions are used throughout the paper.

1 is a column vector whose components are all equal to 1.

For a vector $x = (x_i)$ and a matrix $A = (a(i, j))$ we set

$$\|x\| \stackrel{\text{def}}{=} \max_i |x_i| \text{ which implies } \|A\| = \sup_{\|x\|=1} \|Ax\| = \max_i \sum_j |a(i, j)|.$$

We say that A is strictly positive (and write $A > 0$), if all its matrix elements satisfy $a(i, j) > 0$. A is called non-negative (and we write $A \geq 0$), if all $a(i, j)$ are non negative. A similar convention applies to vectors. Note that if A is a non-negative matrix then $\|A\| = \|A\mathbf{1}\|$.

2.1. The Model.

We recall the definition of the RWRE on a strip from [2]. Consider a strip $\mathbb{S} = \mathbb{Z} \times \{1, \ldots, m\}$ and a random walk on $\mathbb{S}$. Let $L_n = \{(n, i) : 1 \leq i \leq m\}$ be layer n of the strip. In our model, the walk is allowed to jump from any point $(n, i) \in L_n$ only to points in L_{n-1}, or L_n, or L_{n+1}. To define the corresponding transition kernel consider a sequence of triples (P_n, Q_n, R_n), $-\infty < n < \infty$, of

$m \times m$ non-negative matrices such that for all $n \in \mathbb{Z}$ the sum $P_n + Q_n + R_n$ is a stochastic matrix. That is,

$$(P_n + Q_n + R_n)\mathbf{1} = \mathbf{1}, \tag{2.1}$$

The matrix elements of P_n are denoted $P_n(i,j)$, $1 \le i,j \le m$, and similar notations are used for Q_n and R_n. We now set $\omega = (\omega_n)_{n=-\infty}^{\infty} = ((P_n, Q_n, R_n))_{n=-\infty}^{\infty}$ and define

$$\mathcal{Q}_\omega(z, z_1) \stackrel{\text{def}}{=} \begin{cases} P_n(i,j) & \text{if} \quad z = (n,i),\ z_1 = (n+1,j), \\ R_n(i,j) & \text{if} \quad z = (n,i),\ z_1 = (n,j), \\ Q_n(i,j) & \text{if} \quad z = (n,i),\ z_1 = (n-1,j), \\ 0 & \text{otherwise}, \end{cases} \tag{2.2}$$

For a given ω, a random walk $\xi_t = (X_t, Y_t)$, $t \ge 0$, on $\mathbb{S}$ with transition kernel $\mathcal{Q}_\omega(\cdot,\cdot)$ is defined as follows: for any starting point $z = (n,i) \in \mathbb{S}$ the law $P_{\omega,z}$ for the Markov chain ξ is given by

$$P_{\omega,z}\left(\xi_1 = z_1, \ldots, \xi_t = z_t\right) \stackrel{\text{def}}{=} \mathcal{Q}_\omega(z, z_1)\mathcal{Q}_\omega(z_1, z_2)\cdots\mathcal{Q}_\omega(z_{t-1}, z_t). \tag{2.3}$$

From now on we suppose that each such sequence is a realization of a strictly stationary ergodic process and let $(\Omega, \mathcal{F}, \mathbb{P}, T)$ be the corresponding dynamical system with Ω denoting the space of all sequences $\omega = (\omega_n)_{n=-\infty}^{\infty} = ((P_n, Q_n, R_n))_{n=-\infty}^{\infty}$ of triples described above, $\mathcal{F}$ being the corresponding natural σ-algebra, $\mathbb{P}$ denoting the probability measure on $(\Omega, \mathcal{F})$, and T being the shift operator on Ω defined by $(T\omega)_n = \omega_{n+1}$.

We call ω the *environment* or the *random environment* on the strip $\mathbb{S}$. Denote by Ξ_z the set of trajectories ξ starting at z. $P_{\omega,z}$ is the so called quenched probability measure on Ξ_z. The semi-direct product $\mathbb{P}(d\omega)P_{\omega,z}(d\xi)$ of $\mathbb{P}$ and $P_{\omega,z}$ is defined on the direct product $\Omega \times \Xi_z$ and is called the annealed measure. The corresponding mathematical expectations are denoted by $\mathbb{E}$ and $E_{\omega,z}$.

Remark 2.1. The study of one-dimensional RW with bounded jumps in a RE on $\mathbb{Z}$ can be reduced to the study of the above model. The explanation of this fact was given in [2] and later in [16] and [3] and shall not be repeated here.

Denote by $\mathcal{J}$ the following set of triples of $m \times m$ matrices:

$$\mathcal{J} \stackrel{\text{def}}{=} \{(P,Q,R) \,:\, P \ge 0,\ Q \ge 0,\ R \ge 0 \ \text{ and } \ (P+Q+R)\mathbf{1} = \mathbf{1}\}\,.$$

Let $\mathcal{J}_0 = \mathcal{J}_0(\mathbb{P}) \subset \mathcal{J}$ be the support of the probability distribution of the random triple (P_n, Q_n, R_n) defined above (obviously, this support does not depend on n).

Since $\Omega = \mathcal{J}^{\mathbb{Z}}$, it can be endowed by a metric (in many ways). We shall make use of the following metric. For $\omega' = \{(P_n', Q_n', R_n')\}$, $\omega'' = \{(P_n'', Q_n'', R_n'')\}$ set

$$\mathbf{d}(\omega', \omega'') = \sum_{n\in\mathbb{Z}} \frac{\|P_n' - P_n''\| + \|Q_n' - Q_n''\| + \|R_n' - R_n''\|}{2^{|n|}}. \tag{2.4}$$

Below, whenever we say that a function defined on Ω is continuous we mean that it is continuous with respect to the topology induced on Ω by the metric $\mathbf{d}(\cdot,\cdot)$.

The following two assumptions **C1** and **C2** listed below will be referred to as Condition **C** and are supposed to be satisfied throughout the paper.

Condition C:

C1: (P_n, Q_n, R_n), $-\infty < n < \infty$, is an ergodic sequence (equivalently, T is an ergodic transformation of Ω).

C2: There is an $\varepsilon > 0$ and a positive integer number $k_0 < \infty$ such that for any $(P, Q, R) \in \mathcal{J}_0$ and all i, $j \in [1, m]$

$$||R^{k_0}|| \le 1 - \varepsilon, \quad ((I-R)^{-1}P)(i,j) \ge \varepsilon, \quad ((I-R)^{-1}Q)(i,j) \ge \varepsilon. \tag{2.5}$$

Observe that $((I - R_n)^{-1}P_n)(i,j)$ is the probability that the walker starting from (n,i) arrives to $(n+1,j)$ at her first exit from the layer L_n. The meaning of $((I - R_n)^{-1}Q_n)(i,j)$ is similar.

We note that condition (2.5) is trivially satisfied if for all (i,j) we have

$$P(i,j) \ge \varepsilon, \quad Q(i,j) \ge \varepsilon, \quad R(i,j) \ge \varepsilon. \tag{2.6}$$

However (2.6) never holds for the environments on a strip generated by one dimensional walks with bounded jumps while (2.5) holds in that case under mild non-degeneracy conditions. We refer to [3] for a more detailed discussion.

2.2. **Matrices ζ_n, A_n, α_n and some related quantities.** We recall the definitions of several objects most of which were first introduced and studied in [2], [3]. In these papers, they arise naturally in the context of studying/solving equations related to different aspects of the asymptotic behaviour of the RWRE on a strip; they will play a crucial role also in this work.

For a given $\omega \in \Omega$, define a sequence of $m \times m$ stochastic matrices ζ_n as follows. Fix an integer a and a stochastic matrix ψ. For $n \ge a$ define matrices ψ_n as follows. Put $\psi_a = \psi$ and for $n > a$ define recursively

$$\psi_n = \psi_n(a,\psi) = (I - R_n - Q_n\psi_{n-1})^{-1}P_n, \quad n = a+1,\, a+2, \dots. \tag{2.7}$$

It is easy to show (see [2], Lemma 2) that matrices ψ_n are stochastic. Next, for a fixed n define

$$\zeta_n = \lim_{a \to -\infty} \psi_n. \tag{2.8}$$

As shown in [2, Theorem 1] the limit (2.8) exists and is independent of the choice of the initial matrix ψ.

Next, we define probability row-vectors $\sigma_n = \sigma_n(\omega) = (\sigma_n(\omega,1), \dots, \sigma_n(\omega,m))$ which are associated with the matrices ζ_n. Let $\tilde\sigma_a$ be an arbitrary probability row-vector (by which we mean that $\tilde\sigma_a \ge 0$ and $\sum_{i=1}^m \tilde\sigma_a(i) = 1$). Set

$$\sigma_n \overset{\text{def}}{=} \lim_{a \to -\infty} \tilde\sigma_a \zeta_a \dots \zeta_{n-1}. \tag{2.9}$$

By the standard contraction property of the product of stochastic matrices, this limit exists and does not depend on the choice of the sequence $\tilde{\sigma}_a$ (see [16, Lemma 1]). Vectors σ_n could be equivalently defined as the unique sequence of probability vectors satisfying the infinite system of equations

$$\sigma_n = \sigma_{n-1}\zeta_{n-1}, \quad n \in \mathbb{Z}. \tag{2.10}$$

Combining (2.9) with standard contracting properties of stochastic matrices ζ we obtain for $k > n$ that

$$\zeta_n \ldots \zeta_{k-1} = (\sigma_k(1)\mathbf{1}, \ldots, \sigma_k(m)\mathbf{1}) + \mathcal{O}\left(\theta^{k-n}\right), \tag{2.11}$$

where $0 \le \theta < 1$ and the implicit constant in the $\mathcal{O}(\cdot)$ term depend only on the width of the strip m, and the constants ε and k_0 from (2.5).

Define

$$\alpha_n = Q_{n+1}(I - R_n - Q_n\zeta_{n-1})^{-1}, \quad A_n = (I - R_n - Q_n\zeta_{n-1})^{-1}Q_n. \tag{2.12}$$

Note that $\alpha_n P_n = Q_{n+1}\zeta_n$ and hence

$$\alpha_n = Q_{n+1}(I - R_n - \alpha_{n-1}P_{n-1})^{-1}. \tag{2.13}$$

Remark 2.2. The above definitions imply that when $b > n$ we have

$$\alpha_{b-1}\alpha_{b-2}...\alpha_n = Q_b A_{b-1}A_{b-2}...A_{n+1}(I - R_n - Q_n\zeta_{n-1})^{-1}. \tag{2.14}$$

Products of matrices A_n and α_n arise naturally in the analysis of, respectively, the martingale equation (Section 7) and the invariant measure equation (Section 6). Even though relation (2.14) shows that their asymptotic behaviour is essentially the same, an attempt to use just A's or α's would make many of our calculations much more cumbersome. This is the main reason for introducing both of them. It should be noted that, under ellipticity conditions (2.5), matrices A have good contracting properties (see Lemma 2.3). This may not be so for α's but their products can be controlled via products of A's.

The following is a slightly modified version of Lemmas 2 and 4 from [3].

Lemma 2.3. *Suppose that matrices (P, Q, R) satisfy (2.5), ζ is a stochastic matrix, and set $\mathfrak{a} = I - R - Q\zeta$ and $A = \mathfrak{a}^{-1}Q$. Then*

$$(a)\ \|\mathfrak{a}^{-1}\| \le k_0 m^{-1}\varepsilon^{-2}, \quad (b)\ A(i,j) \ge \varepsilon \text{ for all } i,j, \quad (c)\ ||A|| \le (m\varepsilon)^{-1}. \tag{2.15}$$

Proof. Notice that $(I-R)^{-1}Q + (I-R)^{-1}P$ is a stochastic matrix and hence, due to (2.5), one has $||(I-R)^{-1}Q|| \le 1 - m\varepsilon$. Hence also

$$||(I-R)^{-1}Q\zeta|| = ||(I-R)^{-1}Q\zeta\mathbf{1}|| = ||(I-R)^{-1}Q\mathbf{1}|| \le 1 - m\varepsilon.$$

Since

$$\|(I-R)^{-1}\| \le \sum_{k=0}^{\infty}\|R\|^k = \sum_{k=0}^{\infty}\sum_{i=0}^{k_0-1}\|R\|^{kk_0+i} \le k_0\sum_{k=0}^{\infty}\|R\|^{kk_0} \le k_0\sum_{k=0}^{\infty}(1-\varepsilon)^k = k_0\varepsilon^{-1}$$

and $\mathfrak{a}^{-1} = (I-(I-R)^{-1}Q\zeta)^{-1}(I-R)^{-1} = \sum_{k=0}^{\infty}\left((I-R)^{-1}Q\zeta\right)^k (I-R)^{-1}$, we obtain

$$||\mathfrak{a}^{-1}|| \le \sum_{k=0}^{\infty} \| \left((I-R)^{-1}Q\zeta\right)^k \| \, \|(I-R)^{-1}\| \le k_0\varepsilon^{-1}\sum_{k=0}^{\infty}(1-m\varepsilon)^k = k_0 m^{-1}\varepsilon^{-2}$$

which proves (2.15) (a). Next, (2.15) (b) follows from

$$A = \left(\sum_{k=0}^{\infty}\left((I-R)^{-1}Q\zeta\right)^k\right)(I-R)^{-1}Q \ge (I-R)^{-1}Q.$$

Finally, $\|A\| \le \sum_{k=0}^{\infty}\|(I-R)^{-1}Q\zeta\|^k \le \sum_{k=0}^{\infty}(1-m\varepsilon)^k = (m\varepsilon)^{-1}$. □

Since matrices A_n have properties (2.15) (b), (c), we can set

$$v_n = \lim_{a\to-\infty}\frac{A_nA_{n-1}\dots A_{a+1}\tilde{v}_a}{\|A_nA_{n-1}\dots A_{a+1}\tilde{v}_a\|}. \tag{2.16}$$

As explained in [3, Theorem 4] this limit exists and does not depend on the choice of the sequence of vectors $\tilde{v}_a \ge 0$, $||\tilde{v}_a|| = 1$.

Remark 2.4. The components of vectors v_n are strictly positive. Moreover, $v_n \ge m\varepsilon^2\mathbf{1}$. Indeed if a vector $v \ge 0$, $||v|| = 1$ and a matrix A has properties (2.15) then

$$\frac{\min_i (Av)_i}{\|Av\|} = \frac{\min_i \sum_j A(i,j)v_j}{\max_i \sum_j A(i,j)v_j} \ge \frac{\varepsilon\sum_j v_j}{\|A\|\sum_j v_j} = \frac{\varepsilon}{\|A\|} \ge m\varepsilon^2.$$

Next, for any sequence of row-vectors $\tilde{l}_b \ge 0$, $\|\tilde{l}_b\| = 1$ such that $\tilde{l}_bQ_b \ne 0$, define

$$l_n = \lim_{b\to\infty}\frac{\tilde{l}_b\alpha_{b-1}\dots\alpha_n}{\left\|\tilde{l}_b\alpha_{b-1}\dots\alpha_n\right\|}. \tag{2.17}$$

Once again, the limit in (2.17) exists and does not depend on the choice of the sequence $\tilde{l}_b$. Vectors l_n and v_n play important roles in Sections 6 and 7.

Set

$$\lambda_k = \|A_kv_{k-1}\| \text{ and } \tilde{\lambda}_k = \|l_{k+1}\alpha_k\|. \tag{2.18}$$

Then obviously

$$l_{k+1}\alpha_k = \tilde{\lambda}_k l_k, \quad A_kv_{k-1} = \lambda_k v_k \tag{2.19}$$

and for any $n \ge k$ we have

$$\|A_nA_{n-1}\dots A_kv_{k-1}\| = \lambda_n\dots\lambda_k, \quad \|l_{n+1}\alpha_n\alpha_{n-1}\dots\alpha_k\| = \tilde{\lambda}_n\dots\tilde{\lambda}_k. \tag{2.20}$$

Corollary 2.5. *If a sequence of triples* $(P_n, Q_n, R_n)_{-\infty<n<\infty}$ *satisfies* (2.5) *then*

$$\|A_nA_{n-1}\dots A_k\| \le \text{Const}\lambda_n\dots\lambda_k, \tag{2.21}$$

where $\text{Const} = 1/(m\varepsilon^2)$.

Proof. By Remark 2.4 and (2.20),
$\|A_n \dots A_k\| = \|A_n \dots A_k \mathbf{1}\| \le \|A_n \dots A_k(\text{Const}\, v_{k-1})\| = \text{Const}\, \lambda_n \dots \lambda_k.$ □

Remark 2.6. It should be emphasized that the proof provided in [2], [3] of the existence of the limits (2.8) and (2.16) in fact works for *all* (and not just almost all) sequences ω satisfying (2.5). If we define

$$\begin{aligned} &\zeta(\omega) = \zeta_0(\omega), \quad A(\omega) = A_0(\omega), \quad \alpha(\omega) = \alpha_0(\omega), \quad \sigma(\omega) = \sigma_0(\omega) \\ &v(\omega) = v_0(\omega), \quad l(\omega) = l_0(\omega) \quad \lambda(\omega) = \lambda_0(\omega), \quad \tilde\lambda(\omega) = \tilde\lambda_0(\omega) \end{aligned} \tag{2.22}$$

then

$$\begin{aligned} &\zeta_n = \zeta(T^n\omega), \quad A_n = A(T^n\omega), \quad \alpha_n = \alpha(T^n\omega), \quad \sigma_n(\omega) = \sigma(T^n\omega), \\ &v_n = v(T^n\omega), \quad l_n = l(T^n\omega), \quad \lambda_n = \lambda(T^n\omega), \quad \tilde\lambda_n = \tilde\lambda(T^n\omega). \end{aligned} \tag{2.23}$$

Moreover, the functions $\zeta(\cdot)$, $v(\cdot)$, $l(\cdot)$ are continuous in ω. The continuity of all other functions is implied by the continuity of ζ, v, and l. In fact, we have a stronger result, namely the above functions are Hölder with respect to the metric $\mathbf{d}$ defined by (2.4), see Lemma A.2. This regularity plays important role in our analysis.

Remark 2.7. Note that $m = 1$ corresponds to the random walks on $\mathbb{Z}$ with jumps to the nearest neighbours. In this case $p_n = P_\omega(\xi_{t+1} = n+1 | \xi_t = n)$ and $q_n = 1 - p_n$. The above formulae now become very simple, namely

$$\psi_n = \zeta_n = 1, \quad v_n = l_n = 1, \quad A_n = \lambda_n = \frac{q_n}{p_n}, \quad \alpha_n = \tilde\lambda_n = \frac{q_{n+1}}{p_n}.$$

Remark 2.8. Let us describe the probabilistic meaning of some of the matrices introduced above. For simplicity, we restrict ourselves to the recurrent case, referring to [2] for the discussion of the transient regime. The statements we make within this remark are not used in the sequel and because of that we only briefly explain their proofs. We believe however that they provide some intuition concerned with the behaviour of the RW in a RE with a bounded potential.

Denote $\mathfrak{t}_n = \min\{t > 0 : \xi_t \in \mathbb{L}_n\}$. Then

$$\zeta_n(i,j) = P_\omega(\xi_{\mathfrak{t}_{n+1}} = (n+1, j) |\, \xi_0 = (n,i)).$$

Thus

$$\sigma_n(j) = \lim_{a \to -\infty} P_\omega(\xi_{\mathfrak{t}_n} = (n,j) |\, \xi_0 = (a,i)).$$

That is, $\sigma_n(j)$ is the probability that the walk first enters level n at site (n,j) "provided that it starts from $-\infty$." This follows from definition (2.9) and the just mentioned meaning of ζ's.

Next, denote $\tilde R_n = R_n + Q_n\zeta_{n-1}$ and let $\mathfrak{t}_n^{(k)}$ be the k^{th} hitting time of $\mathbb{L}_n$. Note that

$$\left(Q_{n+1}\tilde R_n^k\right)(i,j) = P_\omega\left(\xi_{\mathfrak{t}_n^{(k)}} = (n,j) \text{ and } X_t \le n \text{ for all } t \le \mathfrak{t}_n^{(k)} \,|\, \xi_0 = (n+1,i)\right).$$

Then the formula $\alpha_n = Q_{n+1} \sum_{k=0}^{\infty} \tilde{R}_n^k$ shows that

$$\alpha_n(i,j) = E_\omega(\text{number of visits to } (n,j) \text{ before } \mathfrak{t}_{n+1} |\, \xi_0 = (n+1,i)). \tag{2.24}$$

The probabilistic meaning of matrices A is similar to α but it is slightly more cumbersome so we will not provide it here.

Applying (2.24) twice we get

$$\begin{aligned} & E_\omega(\text{number of visits to } (n-1,j) \text{ before } \mathfrak{t}_{n+1} |\, \xi_0 = (n+1,i)) \\ =& \sum_{k=1}^{m} [E_\omega(\text{number of visits to } (n,k) \text{ before } \mathfrak{t}_{n+1} |\, \xi_0 = (n+1,i)) \\ & \times E_\omega(\text{number of visits to } (n-1,j) \text{ before } \mathfrak{t}_n |\, \xi_0 = (n,k))] \\ =& (\alpha_n \alpha_{n-1})(i,j) \end{aligned}$$

A similar argument shows that

$$\begin{aligned} & (\alpha_n \alpha_{n-1} \dots \alpha_{n-l})(i,j) = \\ & E_\omega(\text{number of visits to } (n-l,j) \text{ before } \mathfrak{t}_{n+1} |\, \xi_0 = (n+1,i)). \end{aligned} \tag{2.25}$$

In this paper we study walks in a bounded potential (see below Definition 3.5 of the potential). If the potential $\mathcal{P}_n$ is bounded then (2.14) and Lemma 2.3 imply that also $\ln \|\alpha_n \alpha_{n-1} \dots \alpha_{n-l})\|$ is bounded. Relation (2.25) now shows that the walks in bounded potentials are characterized by the condition that there is a constant $\bar{K} > 1$ such that for each $z_1, z_2 \in \mathbb{S}$ the following property holds:

If the walk starts from z_1 then the expected number of visits to z_2 before the first return to z_1 is between $1/\bar{K}$ and $\bar{K}$.

This provides some intuition about the walks studied here.

2.3. **Recurrence and transience criteria.** The following recurrence and transience criteria were proved in [2].

Theorem 2.9 ([2], Theorem 2.)**.** *Suppose that Condition* **C** *is satisfied. Then for $\mathbb{P}$-almost all ω the following holds:*
RW is recurrent, that is $P_{\omega,z}(\liminf_{t\to\infty} X_t = -\infty$ and $\limsup_{t\to\infty} X_t = \infty) = 1$, iff $\mathbb{E}(\ln \lambda) = 0$
RW is transient to the right, that is $P_{\omega,z}(X_t \to +\infty$ as $t \to \infty) = 1$, iff $\mathbb{E}(\ln \lambda) < 0$,
RW is transient to the left, that is $P_{\omega,z}(X_t \to -\infty$ as $t \to \infty) = 1$, iff $\mathbb{E}(\ln \lambda) > 0$.

3. Statement of results

3.1. **The Central Limit Theorem.** We shall now state sufficient conditions under which the asymptotic behaviour of a recurrent RW on a strip is described by the CLT. As far as we are aware of, the only result of this kind was previously established by Brémont in [5] for the $[-l, 1]$ model which is a particular case of the

$[-l, r]$ model and the latter, in turn, reduces to the strip model (as has already been mentioned in the Introduction).

Theorem 3.1. *Consider an ergodic environment satisfying* (2.5). *Assume that there exists a function* $\beta : \Omega \to \mathbb{R}$ *such that*

$$\lambda(\omega) = \frac{\beta(T\omega)}{\beta(\omega)} \quad \textit{and} \quad \mathbb{E}(\beta^3 + \beta^{-3}) < \infty. \tag{3.1}$$

Then there is a constant $D > 0$ *such that for* $\mathbb{P}$*-almost all environments*

$$\frac{X_n}{\sqrt{n}} \Rightarrow \mathcal{N}(0, D).$$

Conditions (3.1) is related to matrices A_n. It will be convenient to have an equivalent condition related to matrices α_n. Namely, we shall prove the following

Lemma 3.2. *For ergodic environments satisfying* (2.5) *condition* (3.1) *is equivalent to the following one: there exists a function* $\tilde{\beta} : \Omega \to \mathbb{R}$ *such that*

$$\tilde{\lambda}(\omega) = \frac{\tilde{\beta}(T\omega)}{\tilde{\beta}(\omega)} \quad \textit{and} \quad \mathbb{E}(\tilde{\beta}^3 + \tilde{\beta}^{-3}) < \infty. \tag{3.2}$$

Moreover, the functions $\beta, \tilde{\beta}$ *can be chosen so that for some constant* $c > 0$

$$c^{-1}\tilde{\beta}(\omega) \le \beta(\omega) \le c\tilde{\beta}(\omega). \tag{3.3}$$

In subsection 3.2 we show how to apply the above results to independent and to quasiperiodic environments.

Remark 3.3. Due to ergodicity, the existence of β (or $\tilde{\beta}$) implies that it is unique up to a multiplication by a constant. Indeed, if say β and $\bar{\beta}$ satisfy (3.1) then $\frac{\bar{\beta}(T\omega)}{\beta(T\omega)} = \frac{\bar{\beta}(\omega)}{\beta(\omega)}$ for a.a. ω and hence $\frac{\bar{\beta}(\omega)}{\beta(\omega)} = \text{Const}$.

Remark 3.4. If conditions (3.1), (3.2) are satisfied then it follows from (2.20) that for any $n \ge k$

$$\begin{aligned} \|A_n A_{n-1} \dots A_k v_{k-1}\| &= \lambda_n \dots \lambda_k = \frac{\beta(T^{n+1}\omega)}{\beta(T^k\omega)}, \\ \|l_{n+1}\alpha_n\alpha_{n-1} \dots \alpha_k\| &= \tilde{\lambda}_n \dots \tilde{\lambda}_k = \frac{\tilde{\beta}(T^{n+1}\omega)}{\tilde{\beta}(T^k\omega)}. \end{aligned} \tag{3.4}$$

The following definition of random potential was used in [3] and is analogous to the one introduced in [35].

Definition. A *potential* is a random function of n defined by

$$\mathcal{P}_n(\omega) \equiv \mathcal{P}_n \stackrel{\text{def}}{=} \begin{cases} \ln ||A_n ... A_1|| & \text{if } n \ge 1 \\ 0 & \text{if } n = 0 \\ -\ln ||A_0 ... A_{n+1}|| & \text{if } n \le -1 \end{cases} \tag{3.5}$$

By Lemma 2.3, all matrix elements of matrices A_n are uniformly separated from 0. This, together with Corollary 2.5, implies that the map $(\omega, n) \to \mathcal{P}_n$ is bounded if and only if $\ln \|A_n \dots A_k v_{k-1}\|$ is bounded which, in turn, is equivalent to (3.1) with bounded β. In one direction, this statement is immediate due to (3.4). The other direction is implied by a well known result stated in Lemma C.1 in Appendix C.

Conditions (3.1) and (3.2) may still appear artificial. In fact, as shown in [11], they are necessary and sufficient for the existence of the invariant measure on the space of environments which in turn is one of the basic ingredients of the proof of Theorem 3.1. Moreover, as will be seen in the next subsection, these conditions can be checked for some interesting classes of environments.

3.2. **Applications.** The following lemma describes one of the most important classes of environments for which conditions (3.1) and (3.2) are satisfied.

Lemma 3.5. *For i.i.d. environments satisfying* (2.5) *conditions* (3.1) *and* (3.2) *hold iff the RW is recurrent but does not exhibit the Sinai behaviour. In this case the functions* $\beta, \tilde{\beta}$ *can be chosen to be continuous.*

Corollary 3.6. *A recurrent random walk on a strip in an i.i.d. environment either exhibits the Sinai behaviour, or satisfies the CLT.*

To give more examples of environments satisfying conditions of Theorem 3.1 we need the following definition. Call a set $\Lambda \subset \mathcal{J}$ *admissible* if there exists an i.i.d. environment $\mathbb{P}$ such that the support $\mathcal{J}_0(\mathbb{P}) = \Lambda$ and the corresponding random walk is recurrent and satisfies the CLT. Note that, due to Corollary 3.6 and the continuity of functions $\beta, \tilde{\beta}$, equations (3.1) and (3.2) hold for all (not merely almost all) environments in $\Lambda^{\mathbb{Z}}$. Thus Theorem 3.1 implies the following corollary.

Corollary 3.7. *If* Λ *is admissible and* $\tilde{\mathbb{P}}$ *is a stationary ergodic measure on* $\Lambda^{\mathbb{Z}}$ *then* X_n *is recurrent and satisfies the CLT for* $\tilde{\mathbb{P}}$ *almost every* ω.

Another class of examples is described by the following result.

Lemma 3.8. *Suppose that there is a vector* $f = \{f_k\}_{k=1}^m$ *such that* $M_n = X_n + f_{Y_n}$ *is a martingale. Then* (3.1) *and* (3.2) *hold.*

Corollary 3.9. *The CLT holds for ergodic one dimensional environments where the position of the walker is a martingale.*

We have already mentioned above that the results of [3] show that the CLT behaviour of recurrent walks is exceptional for the i.i.d environments. The same need not be the case in other settings. For example, consider quasiperiodic random walks. Namely, let $\mathbb{T}^d$ be a d-dimensional torus and $\omega \in \mathbb{T}^d$. Set

$$(P_n, Q_n, R_n)(\omega) = (\bar{P}, \bar{Q}, \bar{R})(\omega + n\gamma), \tag{3.6}$$

where γ is a vector in $\mathbb{R}^d$, the sum $\omega + n\gamma$ is defined $(\mathrm{mod}\, 1)$, and $(\bar{P}, \bar{Q}, \bar{R})$ are C^r matrix valued functions on $\mathbb{T}^d$. The transformation $\omega \to (\omega + \gamma)(\mathrm{mod}\, 1)$ preserves the standard Lebesgue measure on the torus and the sequence defined by (3.6) is stationary with respect to this measure. We assume that γ is Diophantine, that is there are constants C, σ such that for each $k \in \mathbb{Z}^d, \tilde{k} \in \mathbb{Z}$

$$|\langle \gamma, k\rangle - \tilde{k}| \geq C|k|^{-\sigma}. \tag{3.7}$$

Remark 3.10. Conditions (3.7) are satisfied whenever the coordinates of the vector γ are rationally independent algebraic numbers. Additionally, they are satisfied for Lebesgue-almost all γ.

Theorem 3.11. *Assume that the matrices $(\bar{P}, \bar{Q}, \bar{R})(\omega)$ satisfy* (2.5) *for all $\omega \in \mathbb{T}^d$, the walk is recurrent, γ satisfies* (3.7), *and*

$$r > d + \sigma, \tag{3.8}$$

where r is the smoothness of the RHS of (3.6). *Then* (3.1) *holds (and hence the random walk satisfies the CLT).*

In order to obtain a complete description of RW in quasiperiodic Diophantine environments we have to consider transient RWs in these REs. To do that we extend the CLT result from [15] which applies to transient RWs on $\mathbb{Z}$ with jumps to the nearest neighbours in a uniquely ergodic environment to transient RWs on a strip in a uniquely ergodic environment. We note that quasiperiodic environments are a particular example of uniquely ergodic environments.

To formulate this extension, consider the following setting. Suppose that

$$(P_n, Q_n, R_n)(\omega) = (\bar{P}, \bar{Q}, \bar{R})(f^n\omega) \tag{3.9}$$

where f is a homeomorphism of a space Ω and $(\bar{P}, \bar{Q}, \bar{R})$ are continuous matrix valued functions on Ω. Recall that a map $f : \Omega \to \Omega$ is called *uniquely ergodic* if for any continuous real valued function Φ the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \Phi(f^n\omega) \tag{3.10}$$

exists for all $\omega \in \Omega$ and does not depend on ω. We recall ([8, Theorem 1.8.2]) that if Ω is a compact metric space then the unique ergodicity of f is equivalent to uniform in $\omega \in \Omega$ convergence of the averages (3.10) and also equivalent to the existence of a unique f-invariant measure $\mathbb{P}(d\omega)$ on Ω (with the sequence (3.9) being stationary with respect to this measure). If f in (3.9) is uniquely ergodic we call (P_n, Q_n, R_n) a uniquely ergodic environment.

The next result was proven in [15] for the one-dimensional nearest neighbour walk (the case $m = 1$). In the Appendix, we prove it for arbitrary strip.

Theorem 3.12. *A transient RW on a strip in a uniquely ergodic environment generated by a continuous $(\bar{P}, \bar{Q}, \bar{R})$ satisfies the CLT.*

A more precise statement of this result including the normalization is given in Theorem B.2.

Corollary 3.13. *CLT holds for RW on a strip in a Diophantine quasiperiodic environment satisfying* (3.8).

Proof. If the RW is recurrent the result follows from Theorems 3.1 and 3.11 and if it is transient then it follows from Theorem 3.12. □

Remark 3.14. Alili in [1] proved the CLT for RW in smooth Diophantine quasiperiodic environments with jumps to nearest neighbours on $\mathbb{Z}$. Brémont in [7] extended this result to RW with bounded jumps (the $[-l, r]$ model) in a quasiperiodic environment generated by a smooth enough function on the torus. In the recurrent regime, Brémont's result is a particular case of Theorems 3.1 and 3.11. In the transient regime, Theorem 3.12 gives a much more general result as it works for all uniquely ergodic environments and requires only continuity of the generating probabilities.

Lemma 3.5 and Corollaries 3.6 and 3.7 lead naturally to the question of characterizing the admissible sets. By Corollary 3.7 a subset of an admissible set is admissible. Recall that the Zariski closure $\bar{\mathcal{A}}$ of a set $\mathcal{A}$ is the smallest algebraic variety containing $\mathcal{A}$. The next result shows that maximal admissible sets are algebraic subvarieties.

Lemma 3.15. *The Zariski closure* $\bar{\Lambda}$ *of an admissible set* Λ *is admissible.*

3.3. **Organization of the paper.** Our main result, Theorem 3.1, is proven in Sections 4–7. Namely, Section 4 describes the main ingredients of the proof, Section 5 presents, in the case of the nearest neighbour RWs on $\mathbb{Z}$, the simplest version of the formulae for the density of the invariant measure and the martingale which play a major role in the proof of the main result. Section 6 constructs the invariant measure for the environment viewed from the particle, and Section 7 proves the existence of a martingale which is asymptotically linear with respect to the $\mathbb{Z}$-coordinate of the walk (the latter is often called the harmonic coordinate for the system). The uniqueness of the martingale is established in Section 8. Section 9 contains the proof of Lemmas 3.2 and 3.5. Lemma 3.15 is proven in Section 10. Section 11 contains the proof of Lemma 3.8. Two sections deal with quasiperiodic environments. Namely, Theorem 3.11 is proven in Section 12 and Theorem 3.12 is established in Appendix B.

4. Main ingredients in the proof of the CLT.

The proof of the main result of this paper (Theorem 3.1) explained at the end of this section follows from Lemmas 4.2, 4.3, 4.4. But first, we need the following definition.

Definition. *The environment seen by the particle* is the random process $(\bar{\omega}_n, Y_n)$, $n \geq 0$, where $\bar{\omega}_n = T^{X_n}\omega$ and $\xi_n = (X_n, Y_n)$ is the position of the walk at time n.

Denote by $\bar{\Omega} = \Omega \times \{1, 2, ..., m\}$ the phase space of the process $(\bar{\omega}_n, Y_n)$ and let $\bar{\mathbb{P}} \stackrel{\text{def}}{=} \mathbb{P} \times \{m^{-1}\}$ be the probability measure on $\bar{\Omega}$ with $\mathbb{P}$ being the measure on the set of environments Ω (as in section 2) and $\{m^{-1}\}$ denoting the uniform distribution on $\{1, ..., m\}$. This process is a Markov chain (which is a simple but important observation, see e.g. [4]).

Remark 4.1. This process was introduce by S. Kozlov [22, 23], as well as Papanicolau-Varadhan [30] and played an important role in a number of papers (see [4, 11] for futher references). The papers which are more closely related to this work are [6, 7] where the environment viewed by the particle approach played a major role in the context of the $[-l, r]$ model and [33] where it was used for the first time in the context of the RWs on a strip (but in transient regime).

Lemma 4.2. *If* (3.2) *holds then the environment seen by the particle has an invariant measure $\mu(d\omega, dy)$ on $\bar{\Omega}$ which is absolutely continuous with respect to $\bar{\mathbb{P}}$.*

Lemma 4.3. *The process $(\bar{\omega}_n, Y_n)$ is ergodic with respect to μ.*

Lemma 4.3 is a well known result. Its proof can be found in [4, Theorem 1.2].

Lemma 4.4. *If* (3.1) *holds then there is a function $M(x, y) = M_\omega(x, y)$ such that*

(1) *For almost all ω $M_n = M_\omega(X_n, Y_n)$ is a martingale;*
(2) *The increments of M are stationary and square integrable. More precisely, for any $l \in \{-1, 0, 1\}$, $Y, \hat{Y} \in \{1, \dots, m\}$ the function*

$$\delta_{l,\hat{Y}}(X, Y) = M(X + l, \hat{Y}) - M(X, Y)$$

is stationary with respect to X translations and square integrable with respect to the measure $\mu(d\omega, dy)P_{\omega,(0,y)}$;

(3) *For a.e. ω, the ratio $\frac{M_\omega(x,y)}{x} \to c,\ c \neq 0$, for all $y \in \{1 \dots m\}$ as $|x| \to \infty$.*

We note that the assumption that β and β^{-1} are in L^3 is only used in the proof of part (2) of Lemma 4.4. A weaker assumption that β and β^{-1} are in L^1 would suffice for Lemma 4.2.

Lemmas 4.2 and 4.4 imply Theorem 3.1 in a standard way which we now recall for completeness.

Proof of Theorem 3.1. Observe that Lemma 4.4 implies that

$$\frac{X_n}{\sqrt{n}} = \frac{M_n}{c\sqrt{n}}(1 + o(1)) + o(1) \text{ as } n \to \infty. \tag{4.1}$$

Indeed, if $|X_n| \geq n^{1/4}$ then (4.1) holds due to Lemma 4.4(3) while if $|X_n| \leq n^{1/4}$ then (4.1) holds since both the RHS and the LHS are $o(1)$. Due to (4.1) it suffices to prove the CLT for M_n. By Corollary 3.1 on page 58 of [18], it suffices to show that $\frac{D_n}{n}$ converges for $\mathbb{P}$-almost all ω to a non-random limit, where

$$D_n = \sum_{k=0}^{n-1} E_\omega \left([M(X_{k+1}, Y_{k+1}) - M(X_k, Y_k)]^2 \,|(X_0, Y_0) \dots (X_k, Y_k) \right)$$
$$= \sum_{k=0}^{n-1} E_\omega \left([M(X_{k+1}, Y_{k+1}) - M(X_k, Y_k)]^2 \,|(X_k, Y_k) \right).$$

Using the ergodicity of the $(\bar{\omega}_n, Y_n)$ process and stationarity of the increments of M we obtain that

$$\lim_{n\to\infty} \frac{D_n}{n} = \int E_{\omega,(0,y)} \left([M_1 - M_0]^2 \right) \mu(d\omega, dy)$$

completing the proof of the theorem. □

5. Nearest neighbour walks on $\mathbb{Z}$.

Below we present proofs of Lemmas 4.2 and 4.4 in the case of the nearest neighbour walks on $\mathbb{Z}$ where the formulae for ρ_n and M_n are simple. They may seem to be a result of a guess rather than a derivation. In fact, we borrow the form of ρ_n from [35] and the formula for M_n results from the analysis of a solution to (5.1) considered, for example, in [12]. (Of course, they could also be obtained as simplified versions of formulae for ρ_n and M_n we derive in Sections 6 and 7.)

Note that in the case of walks on $\mathbb{Z}$ (see Remark 2.7), condition (3.1) takes the form

$$A_n = \frac{q_n}{p_n} = \lambda_n = \frac{\beta_{n+1}}{\beta_n}, \text{ where } \beta_n = \beta(T^n\omega).$$

Proof of Lemma 4.2 for $\mathbb{Z}$. Let ρ be the density of the invariant measure and $\rho_n(\omega) = \rho(T^n\omega)$. Then ρ satisfies

$$\rho_n = p_{n-1}\rho_{n-1} + q_{n+1}\rho_{n+1}.$$

We claim that this equation has a solution of the form $\rho_n = \frac{1}{\beta_n q_n}$. Indeed

$$p_{n-1}\rho_{n-1}+q_{n+1}\rho_{n+1} = \frac{p_{n-1}}{q_{n-1}\beta_{n-1}}+\frac{1}{\beta_{n+1}} = \frac{1}{\beta_n}+\frac{p_n}{q_n\beta_n} = \frac{1}{\beta_n}\left(1+\frac{p_n}{q_n}\right) = \frac{1}{q_n\beta_n} = \rho_n.$$

□

Proof of Lemma 4.4 for $\mathbb{Z}$. If $X_t,\ t \geq 0$, is the nearest neighbour walk on $\mathbb{Z}$ in random environment ω then $M_\omega(X_t)$ is a martingale if the sequence $\{M_n = M_\omega(n),\ n \in \mathbb{Z}\}$ satisfies the equation

$$M_n = p_n M_{n+1} + q_n M_{n-1}. \tag{5.1}$$

The space of solutions to (5.1) is two-dimensional and we claim that a solution linearly independent of $M_n \equiv 1$ has the form

$$M_n = \begin{cases} \sum_{j=1}^{n} \beta_j & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \\ -\sum_{j=n+1}^{0} \beta_j & \text{if } n \leq -1. \end{cases}$$

Let us check this claim, say for $n \geq 1$. In this case

$$p_n M_{n+1} + q_n M_{n-1} = p_n(M_n + \beta_{n+1}) + q_n(M_n - \beta_n) = M_n + p_n\beta_{n+1} - q_n\beta_n = M_n.$$

□

6. Environment seen by the particle.

Proof of Lemma 4.2. We will construct the density $\rho : \Omega \times [1, \ldots, m] \to \mathbb{R}$ as a solution to (6.1) below. Denote by $\rho = \rho(\omega)$ the row-vector with components $\rho(\omega, i)$ and let $\rho_n = \rho(T^n\omega)$ be a vector with components $\rho_n(i) = \rho(T^n\omega, i)$. For ρ to be a density of the invariant measure of the Markov chain $(T^{X_t}\omega, Y_t)$, $t \geq 0$, the corresponding vectors ρ_n should satisfy the system of equations

$$\rho_n = \rho_{n+1}Q_{n+1} + \rho_n R_n + \rho_{n-1}P_{n-1}, \quad -\infty < n < \infty. \tag{6.1}$$

The restriction of this equation to a finite strip $a \leq n \leq b$ was analyzed in [2, section 3]. The solution found there satisfies certain (reflecting) boundary conditions and has a meaning different from the one we are interested in here.

However, we borrow from [2] the following fact. For any m-dimensional vector h set $\rho_b^h = h$ and define ρ_n^h for $n \leq b-1$ by the recursion $\rho_n^h = \rho_{n+1}^h \alpha_n$, where the matrices α_n are defined in (2.12). Then the vectors ρ_n^h solve (6.1) for all $n \leq b-1$. For the sake of completeness, we shall check this statement. Obviously, if $n \leq b-1$ then

$$\rho_n^h = h\alpha_{b-1} \ldots \alpha_n \tag{6.2}$$

and hence

$$\begin{aligned} \rho_{n+1}^h Q_{n+1} + \rho_n^h R_n + \rho_{n-1}^h P_{n-1} &= h\alpha_{b-1} \ldots \alpha_{n+1}(Q_{n+1} + \alpha_n R_n + \alpha_n\alpha_{n-1}P_{n-1}) \\ &\overset{(*)}{=} h\alpha_{b-1} \ldots \alpha_{n+1}\alpha_n = \rho_n^h, \end{aligned}$$

where $(*)$ follows from the relation $\alpha_n = Q_{n+1} + \alpha_n R_n + \alpha_n\alpha_{n-1}P_{n-1}$ which in turn is equivalent to (2.13).

Next, note that for vectors l_n defined in (2.17) it follows from (2.19) and condition (3.2) that

$$l_{n+1}\alpha_n = \tilde{\lambda}_n l_n = \frac{\tilde{\beta}(T^{n+1}\omega)}{\tilde{\beta}(T^n\omega)} l_n \text{ and so } \frac{1}{\tilde{\beta}(T^{n+1}\omega)} l_{n+1}\alpha_n = \frac{1}{\tilde{\beta}(T^n\omega)} l_n. \tag{6.3}$$

Remember that $l_n = l(T^n\omega)$. Set

$$\rho(\omega) = \frac{1}{Z\tilde{\beta}(\omega)} l(\omega), \text{ where } Z = \mathbb{E}\left[\frac{1}{\tilde{\beta}(\omega)}\sum_{i=1}^{m} l(\omega, i)\right]. \tag{6.4}$$

Then the second equation in (6.3) has the form $\rho_n = \rho_{n+1}\alpha_n$, where $\rho_n = \rho(T^n\omega)$ for all $n \in \mathbb{Z}$. Hence, the ρ_n, $n \in \mathbb{Z}$, solve (6.1) which means that ρ defined by (6.4) is the density of the invariant measure of our Markov chain. □

7. Construction of the martingale.

In this section we prove Lemma 4.4. The idea behind the proof is the following one. Let $M(\cdot)$ be a martingale with the properties listed in Lemma 4.4. Consider $z = (x, y) \in \mathbb{S}$ and $a, b \in \mathbb{Z}$ such that $x - a \gg 1, b - x \gg 1$ and let $\tau_{a,b}$ be the first time the walker reaches $\mathbb{L}_a$ or $\mathbb{L}_b$. Set

$$\mathfrak{p}_{a,b}(z) = P_\omega(X_{\tau_{a,b}} = a|\xi_0 = z)$$

(we recall the notation $\xi_t = (X_t, Y_t)$). By the Optional Stopping Theorem

$$M(z) = E_{\omega,z}(M(\xi_{\tau_{a,b}})) = E_{\omega,z}(M(\xi_{\tau_{a,b}})1_{X_{\tau_{a,b}}=a}) + E_{\omega,z}(M(\xi_{\tau_{a,b}})1_{X_{\tau_{a,b}}=b}).$$

If z is far from both $\mathbb{L}_a$ and $\mathbb{L}_b$ then the distributions of $Y_{\tau_{a,b}}1_{X_{\tau_{a,b}}=a}$ in $\mathbb{L}_a$ and $Y_{\tau_{a,b}}1_{X_{\tau_{a,b}}=b}$ in $\mathbb{L}_b$ is approximately given by $\mathfrak{p}_{a,b}(z)\sigma_a^-$ and $(1 - \mathfrak{p}_{a,b}(z))\sigma_b$ respectively. So we expect that

$$E_{\omega,z}(M(\xi_{\tau_{a,b}})1_{X_{\tau_{a,b}}=a}) \approx \mathfrak{p}_{a,b}(z)M_a, \quad E_{\omega,z}(M(\xi_{\tau_{a,b}})1_{X_{\tau_{a,b}}=b}) \approx (1 - \mathfrak{p}_{a,b}(z))M_b,$$

where $M_b = \sum_{j=1}^m \sigma_b(j)M(b, j)$, $M_a = \sum_{j=1}^m \sigma_a^-(j)M(a, j)$ (see (2.9) and Remark 2.8 explaining the meaning of σ_b). This would give

$$M(z) = \mathfrak{p}_{a,b}(z)(M_a - M_b) + M_b,$$

that is $M(z)$ is obtained from $\mathfrak{p}_{a,b}(z)$ by an affine transformation. In the proof below we will show using the formula for $\mathfrak{p}_{a,b}(z)$ obtained in [2], that a proper rescaling of $\mathfrak{p}_{a,b}$ indeed gives a linearly growing martingale.

Proof. Let $\mathfrak{m}_n$ denote a vector with components $\mathfrak{m}_n(i) = M(n, i)$. For the process $M(X_t, Y_t)$, $t \geq 0$, to be a martingale with respect to the measure $P_{\omega,z}$, the vectors $\mathfrak{m}_n$ should satisfy the equation

$$\mathfrak{m}_n = P_n\mathfrak{m}_{n+1} + R_n\mathfrak{m}_n + Q_n\mathfrak{m}_{n-1}. \tag{7.1}$$

The analysis of the solution to this equation on a finite part of the strip, $a \leq n \leq b$, has played a crucial role in [2]. Inevitably, some calculations are similar to those in [2] but here the analysis goes in a very different direction. We would like to emphasize that, apart of the fact stated in (7.7) and the preceding comment, the proof presented below is self-contained.

As in [2], define a sequence of $m \times m$ matrices φ_n, $n \geq a+1$ by setting $\varphi_a = 0$ and computing φ_n recursively

$$\varphi_n = (I - R_n - Q_n\varphi_{n-1})^{-1}P_n, \quad \text{if } n \geq a. \tag{7.2}$$

The solutions to (7.1) with boundary conditions $\mathfrak{m}_a = 0$, $\mathfrak{m}_b = \mathfrak{f}$ can be presented in the following form:

$$\mathfrak{m}_n = \varphi_n\varphi_{n+1}\dots\varphi_{b-1}\mathfrak{f}, \quad a \leq n \leq b. \tag{7.3}$$

For $n = a$ or $n = b$ this statement is obvious and for $a < n < b$ it can be verified by substituting the right hand side of (7.3) into (7.1).

In order to construct a linearly growing solution of (7.1) we consider the solution $\mathfrak{m}_n$ corresponding to $\mathfrak{f} = \mathbf{1}$ (in which case $\mathfrak{m}_n(i) = \mathfrak{p}_{a,b}((n,i))$) and study some related limits of this solution as $a \to -\infty, b \to \infty$ so that $|a| \gg b$. So, from now on and to the end of this section our $b > 0$ and $a < -b$.

Set $\Delta_n = \zeta_n - \varphi_n$, where ζ_n are matrices defined in (2.7), (2.8). Following [2], we present this difference as

$$\begin{aligned}\Delta_n &= (I - R_n - Q_n\zeta_{n-1})^{-1}P_n - (I - R_n - Q_n\varphi_{n-1})^{-1}P_n \\ &= (I - R_n - Q_n\zeta_{n-1})^{-1}Q_n\Delta_{n-1}(I - R_n - Q_n\varphi_{n-1})^{-1}P_n = A_n\Delta_{n-1}\varphi_n.\end{aligned} \tag{7.4}$$

Iterating the last relation gives, (cf. [2, equation (2.13)]) that if $|n| < b$ then

$$\Delta_n = A_n \dots A_{-b+1}\Delta_{-b}\varphi_{-b+1}\dots\varphi_n. \tag{7.5}$$

The immediate corollary from here is the inequality $\|\Delta_n\| \leq \|A_n \dots A_{-b+1}\| \, \|\Delta_{-b}\|$ which in turn, together with (2.21) and (3.1), gives

$$\|\Delta_n\| \leq \text{Const}\, \lambda_n \dots \lambda_{-b+1} \, \|\Delta_{-b}\| \leq H(\omega, b) \, \|\Delta_{-b}\| \tag{7.6}$$

where

$$H(\omega, b) = \text{Const}\, \frac{\max_{|n|\leq b} \beta(T^{n+1}\omega)}{\beta(T^{-b+1}\omega)}.$$

We note that in order to complete the argument the precise form of the RHS of (7.6) is not important, we just need that it is linear in $\|\Delta_{-b}\|$ and the prefactor H is uniform in n and a.

Next, it follows from (3.1) that $\mathbb{E}(\ln\lambda) = 0$ and hence by Theorem 2.9 the walk is recurrent. Recall (see [2, formula (2.3)]) that $\varphi_n(i,j)$ is the $P_{\omega,(n,i)}$-probability that a RW starting from (n,i) reaches layer $n+1$ before layer a and that it hits layer $n+1$ at $(n+1,j)$. So, due to recurrence, we have that for any i

$$\sum_{j=1}^{m} \varphi_n(i,j) = P_{\omega,(n,i)}\{\text{reach layer } n+1 \text{ before } a\} \to 1 \text{ as } a \to -\infty. \tag{7.7}$$

Since $\Delta_a = \zeta_a$, (7.4) implies that $\Delta_n \geq 0$. Therefore

$$\|\Delta_n\| = \|\Delta_n\mathbf{1}\| = \|(\zeta_n - \varphi_n)\mathbf{1}\| = 1 - \min_i \sum_{j=1}^{m} \varphi_n(i,j) \to 0 \text{ as } a \to -\infty.$$

Define $\varepsilon_b(a) = ||\Delta_{-b}||$. Obviously

$$\varepsilon_b(a) \to 0 \text{ as } a \to -\infty \text{ and } b \text{ is fixed.} \tag{7.8}$$

Next, (7.4) also gives

$$\Delta_n = A_n\Delta_{n-1}(\zeta_n - \Delta_n) = A_n\Delta_{n-1}\zeta_n - A_n\Delta_{n-1}\Delta_n. \tag{7.9}$$

Substituting $\Delta_{n-1} = A_{n-1}\Delta_{n-2}\zeta_{n-1} - A_{n-1}\Delta_{n-2}\Delta_{n-1}$ only in the term $A_n\Delta_{n-1}\zeta_n$ we obtain

$$\Delta_n = A_nA_{n-1}\Delta_{n-2}\zeta_{n-1}\zeta_n - A_nA_{n-1}\Delta_{n-2}\Delta_{n-1}\zeta_n - A_n\Delta_{n-1}\Delta_n.$$

Continuing this process we obtain

$$\Delta_n = A_n \dots A_{-b+1}\Delta_{-b}\zeta_{-b+1}\dots\zeta_n - \sum_{k=-b}^{n-1} A_n\dots A_{k+1}\Delta_k\Delta_{k+1}\zeta_{k+2}\dots\zeta_n, \tag{7.10}$$

where by convention $A_n \dots A_{k+1} = I$ if $k+1 < n$ and $\zeta_{k+2}\dots\zeta_n = I$ if $k+2 > n$. Equality (7.10) together with (7.6) implies that

$$\Delta_n = A_n\dots A_{-b+1}\Delta_{-b}\zeta_{-b+1}\dots\zeta_n + \mathcal{O}(\varepsilon_b^2(a)H(b,\omega)^2 b). \tag{7.11}$$

Applying similar reasoning to (7.3) with $\mathfrak{f} = \mathbf{1}$ and $\varphi_j = \zeta_j - \Delta_j$ gives

$$\begin{aligned}\mathfrak{m}_n &= \mathbf{1} - \sum_{n\le k\le b-1} \zeta_n\dots\zeta_{k-1}\Delta_k\zeta_{k+1}\dots\zeta_{b-1}\mathbf{1} + \mathcal{O}(\varepsilon_b^2(a)H(\omega,b)^2b^2)\\ &= \mathbf{1} - \sum_{n\le k\le b-1} \zeta_n\dots\zeta_{k-1}\Delta_k\mathbf{1} + \mathcal{O}(\varepsilon_b^2(a)H(\omega,b)^2b^2).\end{aligned}$$

Substituting (7.11) into the last equation gives

$$\mathfrak{m}_n = \mathbf{1} - \sum_{n\le k\le b-1} \zeta_n\dots\zeta_{k-1}A_k\dots A_{-b+1}w_{-b} + \mathcal{O}(\varepsilon_b^2(a)H(\omega,b)^2b^2)$$

where $w_{-b} = \Delta_{-b}\mathbf{1}$. Now set

$$\bar{\mathfrak{m}}_n^{a,b} = \frac{\mathbf{1} - \mathfrak{m}_n}{||w_{-b}||}.$$

$\bar{\mathfrak{m}}_n^{a,b}$ satisfies (7.1) since it is a linear combination of two solutions. Note that

$$\bar{\mathfrak{m}}_n^{a,b} = \sum_{n\le k\le b-1} \zeta_n\dots\zeta_{k-1}A_k\dots A_{-b+1}u_{-b} + \mathcal{O}(\varepsilon_b(a)H(\omega,b)^2b^2), \tag{7.12}$$

where $u_{-b} = w_{-b}/||w_{-b}||$.

We shall now compute the limit of $\bar{\mathfrak{m}}_n^{a,b}$ as $a \to -\infty$. To this end note that

$$\lim_{a\to-\infty} u_{-b} = \lim_{a\to-\infty} \frac{A_{-b}\dots A_{a+1}\zeta_a\varphi_{a+1}\dots\varphi_{-b}\mathbf{1}}{\|A_{-b}\dots A_{a+1}\zeta_a\varphi_{a+1}\dots\varphi_{-b}\mathbf{1}\|} = v_{-b}, \tag{7.13}$$

where we first use (7.5) and then proceed as in (2.16) with $\tilde{v}_a = \frac{\zeta_a \varphi_{a+1} \cdots \varphi_{-b} \mathbf{1}}{\|\zeta_a \varphi_{a+1} \cdots \varphi_{-b} \mathbf{1}\|}$. Passing to the limit $a \to -\infty$ in (7.12) and using (7.8) we obtain the following solution on $(-b, b)$:

$$\bar{\mathfrak{m}}_n^b = \sum_{n \le k \le b-1} \zeta_n \ldots \zeta_{k-1} A_k \ldots A_{-b+1} v_{-b}.$$

By (2.20) and (3.4) we have

$$A_k \ldots A_{-b+1} v_{-b} = \lambda_k \ldots \lambda_{-b+1} v_k = \frac{\beta(T^{k+1}\omega)}{\beta(T^{-b+1}\omega)} v_k \tag{7.14}$$

and by (2.11)

$$\zeta_n \ldots \zeta_{k-1} v_k = (\sigma_k(1)\mathbf{1}, \ldots, \sigma_k(m)\mathbf{1}) v_k + \mathcal{O}\left(\theta^{k-n}\right) = (\sigma_k v_k)\mathbf{1} + \mathcal{O}\left(\theta^{k-n}\right), \tag{7.15}$$

where here and below we denote $(\sigma_k v_k) \overset{\text{def}}{=} \sum_{i=1}^m \sigma_k(i) v_k(i)$. We thus see that

$$\beta(T^{-b+1}\omega)\bar{\mathfrak{m}}_n^b = \sum_{k=n}^{b-1} \beta(T^{k+1}\omega)(\sigma_k v_k)\mathbf{1} + \sum_{k=n}^{b-1} \beta(T^{k+1}\omega)\mathcal{O}\left(\theta^{k-n}\right)$$

is also a solution to (7.1) on $(-b, b)$ and so is

$$\begin{aligned} \hat{\mathfrak{m}}_n^b &\overset{\text{def}}{=} \beta(T^{-b+1}\omega)\bar{\mathfrak{m}}_n^b - \sum_{k=0}^{b-1} \beta(T^{k+1}\omega)(\sigma_k v_k)\mathbf{1} \\ &= -\sum_{k=0}^{n-1} \beta(T^{k+1}\omega)(\sigma_k v_k)\mathbf{1} + \sum_{k=n}^{b-1} \beta(T^{k+1}\omega)\mathcal{O}\left(\theta^{k-n}\right) \end{aligned}$$

The series $\sum_{k=n}^{\infty} \beta(T^{k+1}\omega)\mathcal{O}\left(\theta^{k-n}\right)$ converges absolutely because of (2.11) (note that the terms of the last sum do not depend on b). Hence setting $M(x, \cdot) = \lim_{b\to\infty} \hat{\mathfrak{m}}_x^b$ we obtain a solution

$$M(x, \cdot) = \sum_{k=0}^{x-1} \beta(T^{k+1}\omega)(\sigma_k v_k)\mathbf{1} + \mathcal{B}(T^x\omega), \tag{7.16}$$

where

$$\mathcal{B}(\omega) = \sum_{k=0}^{\infty} \beta(T^{k+1}\omega)(\zeta_0 \ldots \zeta_{k-1} v_k - (\sigma_k v_k)\mathbf{1}).$$

It remains to check statements (2) and (3) of Lemma 4.4.

Denote by $\mathbb{E}^\mu$ the expectation with respect to the measure μ. To check that (2) holds, we have to show that

$$\mathcal{D} \overset{\text{def}}{=} \mathbb{E}^\mu \left(E_\omega \left(M_\omega(X_{t+1}, Y_{t+1}) - M_\omega(X_t, Y_t)\right)^2\right) \tag{7.17}$$

$$= \mathbb{E}^\mu \left(E_\omega \left(M_\omega(X_1, Y_1) - M_\omega(X_0, Y_0)\right)^2\right) < \infty. \tag{7.18}$$

Note that equality (7.18) holds since μ is an invariant measure of the Markov chain $(T^{X_t}\omega, Y_t),\ t \geq 0$. Now $\mathcal{D}$ can be presented as

$$\mathcal{D} = \mathbb{E}\left(\sum_{i=1}^{m} \rho(\omega, i) \sum_{s=0,\pm 1; 1\leq j\leq m} \mathcal{Q}_\omega((0,i),(s,j))(M_\omega(0,i) - M_\omega(s,j))^2\right),$$

where $\mathcal{Q}_\omega((0,i),(s,j))$ is defined by (2.2). Equation (7.16) implies that

$$|M_\omega(0,i) - M_\omega(s,j)| \leq C\sum_{k=0}^{\infty} \theta^k \beta(T^k\omega),$$

where, as before, C and θ depend only on the ε from (2.5). This inequality, together with (6.4) and (3.3), implies

$$\mathcal{D} \leq C\mathbb{E}\left(\sum_{k\geq 0, j\geq 0} \theta^{k+j}\beta^{-1}(\omega)\beta(T^k\omega)\beta(T^j\omega)\right).$$

But

$$\mathbb{E}(\beta^{-1}(\omega)\beta(T^k\omega)\beta(T^j\omega)) \leq \frac{1}{3}\mathbb{E}(\beta^{-3}(\omega)+\beta^3(T^k\omega)+\beta^3(T^j\omega)) = \frac{1}{3}\mathbb{E}(\beta^{-3}(\omega)+2\beta^3(\omega))$$

finishing the proof of property (2).

Remark 7.1. Note that in the case of a RWRE on $\mathbb{Z}$ with nearest neighbour jumps condition (3.1) can be replaced by $\mathbb{E}(\beta(\omega) + \beta^{-1}(\omega)) < \infty$. On a strip, we need the stronger requirement (3.1) because of the term $\mathcal{B}$ in (7.16).

Finally, property (3) follows from the ergodic theorem. In fact, for the martingale constructed above, $c > 0$ since β, σ_k and v_k in the RHS of (7.16) are all positive. □

8. The Liouville Theorem.

The construction of the martingale in the previous section was based on a choice of two particular solutions of the martingale equation on finite intervals. The following lemma shows that the final result is essentially unique. And even though this lemma is not used in the rest of the paper, it provides an important contribution to the understanding of the whole picture.

Let $\mathfrak{M}$ denote the space of martingales satisfying conditions (1)–(2) of Lemma 4.4 and such that if $M(\cdot,\cdot) \in \mathfrak{M}$ then

$$\lim_{x\to\pm\infty} \frac{M(x,y)}{x} = 1.$$

(Clearly, we can scale the martingale from Lemma 4.4 to achieve this condition.)

Lemma 8.1. *If $M_1, M_2 \in \mathfrak{M}$ then* $M_1 - M_2 = \text{Const}$.

Proof. Let $\bar{M}(x,y) = M_1(x,y) - M_2(x,y)$. Then Theorem 3.1 implies that, for almost all ω, $\frac{X_n}{\sqrt{n}}$ is tight. Since $\bar{M}(x,y)$ grows sublinearly, for almost all ω, $\frac{\bar{M}(X_n,Y_n)}{\sqrt{n}} \to 0$ in probability with respect to the $P_{\omega,z}$ measure on the space of trajectories. On the other hand the proof of Theorem 3.1 shows that $\frac{\bar{M}(X_n,Y_n)}{\sqrt{n}} \to 0$ iff $\frac{\bar{D}_n}{n} \to 0$ where

$$\bar{D}_n = \sum_{k=0}^{n-1} \mathbb{E}\left(E_\omega\left(\left[\bar{M}(X_{k+1},Y_{k+1}) - \bar{M}(X_k,Y_k)\right]^2 |X_k\right)\right).$$

By Ergodic Theorem,

$$\lim_{n\to\infty} \frac{\bar{D}_n}{n} = \int E_{\omega,(0,y)}\left[\left(\bar{M}_1 - \bar{M}_0\right)^2\right] \mu(d\omega, dy)$$

and this expression vanishes iff $\bar{M}_1 \equiv \bar{M}_0$ which implies, due to the stationarity, that $\bar{M}$ is a constant. □

9. Equivalent conditions for boundedness of the potential.

Proof of Lemma 3.2. Suppose that $\beta(\cdot)$ satisfying (3.1) exists. Define $\mathfrak{a}_n = I - R_n - Q_n\zeta_{n-1}$. In these notations, we have $A_n = \mathfrak{a}_n^{-1}Q_n$ and $\alpha_{n-1} = Q_n\mathfrak{a}_{n-1}^{-1}$ and hence $\mathfrak{a}_nA_n = \alpha_{n-1}\mathfrak{a}_{n-1} = Q_n$. Multiplying all parts of this equality by vectors l_n and v_{n-1} we obtain

$$l_n\mathfrak{a}_nA_nv_{n-1} = l_n\alpha_{n-1}\mathfrak{a}_{n-1}v_{n-1} = l_nQ_nv_{n-1}$$

and this, by (2.18) and (2.19), gives

$$(l_n\mathfrak{a}_nv_n)\,\lambda_n = (l_{n-1}\mathfrak{a}_{n-1}v_{n-1})\tilde{\lambda}_{n-1} = l_nQ_nv_{n-1}.$$

Since $l_n > 0$ and $v_n > 0$ for all n and since Q_n has no zero columns (because of (2.5)), also $l_nQ_nv_{n-1} > 0$ and therefore $(l_n\mathfrak{a}_nv_n) > 0$ for all n. We thus can write

$$\tilde{\lambda}_{n-1} = \frac{(l_n\mathfrak{a}_nv_n)}{(l_{n-1}\mathfrak{a}_{n-1}v_{n-1})}\lambda_n = \frac{(l_n\mathfrak{a}_nv_n)\,\beta(T^{n+1}\omega)}{(l_{n-1}\mathfrak{a}_{n-1}v_{n-1})\beta(T^n\omega)}. \tag{9.1}$$

We now set

$$\tilde{\beta}(\omega) = (l(\omega)\mathfrak{a}(\omega)v(\omega))\beta(T\omega) = (l(\omega)\mathfrak{a}(\omega)v(\omega)\lambda(\omega))\beta(\omega). \tag{9.2}$$

With this definition of $\tilde{\beta}(\omega)$, equation (9.1) reads $\tilde{\lambda}_{n-1} = \frac{\tilde{\beta}(T^n\omega)}{\tilde{\beta}(T^{n-1}\omega)}$ which in particular proves that (3.2) holds.

Similarly, (3.2) implies (3.1).

It remains to notice that the factor $l(\cdot)\mathfrak{a}(\cdot)v(\cdot)\lambda(\cdot)$ in (9.2) is a continuous function of ω and this, together with strictly positivity of this function and compactness of the space of environments satisfying (2.5), implies (3.3). □

Proof of Lemma 3.5. Suppose that the walk is recurrent but does not exhibit the Sinai behaviour. It is proven on pages 273–274 of [3], that in this case condition (iii) of Theorem 6 of [3] holds. This condition says that there exists a function F defined on the space of pairs (ϕ, w), where ϕ is a stochastic matrix and w is a unit vector such that for each triple $(P, Q, R) \in \mathcal{J}_0$

$$\ln ||Bw|| = F(\phi, w) - F\left((I - R - Q\phi)^{-1}P, \frac{Bw}{||Bw||}\right), \tag{9.3}$$

where $B = (I - R - Q\phi)^{-1}Q$. To obtain (9.3) from equation (2.19) of [3] we observe that, due to recurrence, Theorem 2.9 tells us that $\mathbb{E}(\ln \lambda) = 0$ (note that this expectation is denoted by λ in [3]).

Applying (9.3) to $(P, Q, R) = (P_n, Q_n, R_n)$ $(\phi, w) = (\zeta_{n-1}, v_{n-1})$ we obtain

$$\ln \lambda_n = F(\zeta_{n-1}, v_{n-1}) - F(\zeta_n, v_n).$$

This proves (3.1) with

$$\beta(\omega) = e^{-F(\zeta(T^{-1}\omega), v(T^{-1}\omega))}.$$

It remains to note that β is continuous due to continuity of F which is evident from the explicit formula for this function, namely formula (4.11) from [3].

Conversely if (3.1) and (3.2) hold then the RW does not exhibit the Sinai behaviour by Theorem 3.1. □

10. Periodic boundary conditions.

Here we describe a criterion for recurrence and the CLT in terms of periodic approximations to our random environment. We remark that the results below are analogous to the Livsic theory for hyperbolic dynamical systems (cf. [27, 28]).

Given N let $\pi^N(n, y)$ denote the invariant measure for the random walk on $[0, N-1] \times [1 \dots m]$ with periodic boundary conditions. Let π_n^N denote the vector with components $\pi_n^N(y) = \pi^N(n, y)$.

Proposition 10.1. *Suppose that Condition* (2.5) *is satisfied and that for any* $N \geq 1$ *the support of the measure* $\mathbb{P}$ *contains all periodic sequences generated by periodic repetition of finite sequences of the form* $((P_n, Q_n, R_n))_{n=0}^{N-1} \in \mathcal{J}_0^N$.

Then condition (3.1) *holds for all* $\omega \in \Omega$ *with* $|\ln \beta|$ *bounded if and only if for each* N *and for each* $((P_n, Q_n, R_n))_{n=0}^{N-1} \in \mathcal{J}_0^N$ *the following identity holds*

$$\pi_0^N Q_0 \mathbf{1} = \pi_{N-1}^N P_{N-1} \mathbf{1}. \tag{10.1}$$

The proof consists of two steps.

Lemma 10.2. (3.1) *holds with* $|\ln \beta|$ *bounded if and only if for each* N *and for each environment* ω *such that* $T^N\omega = \omega$ *we have*

$$\lambda_0 \lambda_1 \dots \lambda_{N-1} = 1. \tag{10.2}$$

Proof. By Lemma C.1 we need to show that (10.2) is equivalent to

$$\sum_{j=0}^{n-1} \ln \lambda(T^j\omega) \tag{10.3}$$

being uniformly bounded in $\omega \in \Omega$ and $n \in \mathbb{N}$.

(a) If (10.3) is bounded for each ω it is in particular bounded for periodic ω and hence

$$\sum_{j=0}^{kN-1} \ln \lambda(T^j\omega) = k\left[\sum_{j=0}^{N-1} \ln \lambda_j\right]$$

is uniformly bounded in k which is only possible if (10.2) holds.

(b) Suppose that (10.2) holds. Given ω, N let $\tilde{\omega}$ be the environment such that $\tilde{\omega}_j = \omega_j$ for $j \in \{0, \dots N-1\}$ and such that $\tilde{\omega}$ is periodic with period N. Then due to Lemma A.2

$$\left|\sum_{j=0}^{N-1} \ln \lambda(T^j\omega)\right| = \left|\sum_{j=0}^{N-1} \ln \lambda(T^j\omega) - \sum_{j=0}^{N-1} \ln \lambda(T^j\tilde{\omega})\right| = \left|\sum_{j=0}^{N-1} \left[\ln \lambda(T^j\omega) - \ln \lambda(T^j\tilde{\omega})\right]\right|$$

$$\leq \text{Const} \sum_{j=0}^{N-1} \mathbf{d}^{\mathbf{s}}(T^j\omega, T^j\tilde{\omega}) \leq \text{Const} \sum_{j=0}^{N-1} 2^{-\alpha(\min(j,N-j))} \leq \text{Const}$$

where $\mathbf{s}$ is the Hölder exponent of $\ln \lambda$ given by Lemma A.2. It follows that (10.3) is bounded. □

Lemma 10.3. *For each periodic environment* (10.1) *and* (10.2) *are equivalent.*

Proof. Since periodic environments are stationary and ergodic, Theorem 2.9 implies that in this case recurrence is equivalent to 1 being the top eigenvalue of any of the products $A_{N+k-1}...A_k$, which is what (10.2) says.

On the other hand, in the periodic environment, the recurrence holds if and only if the walker has zero speed. Let $h(x)$ denote the integer part of x/N. Then the speed is zero if and only if

$$\lim_{t\to\infty} \frac{h(X(t))}{t} = 0.$$

But $h(X(t+1))$ may differ from $h(X(t))$ only if $X(t)$ is comparable to either 0 or to $N-1$ mod N. Therefore by the Ergodic Theorem for Markov chains

$$\lim_{t\to\infty} \frac{h(X(t))}{t} = \pi^N_{N-1} P_{N-1}\mathbf{1} - \pi^N_0 Q_0 \mathbf{1},$$

so the walk is recurrent iff (10.1) holds. □

Proof of Lemma 3.15. For given matrices

$$(P_0, Q_0, R_0), (P_1, Q_1, R_1), \dots, (P_{N-1}, Q_{N-1}, R_{N-1})$$

the entries $\pi^N(n,y)$ are rational functions of the coefficients. Accordingly equation (10.1) can be written as

$$\mathbb{F}_N((P_0,Q_0,R_0),(P_1,Q_1,R_1)\dots(P_{N-1},Q_{N-1},R_{N-1}))=0$$

where $\mathbb{F}_N$ is a certain polynomial. In other words (3.1) holds if and only if for each N, $\mathbb{F}_N$ vanishes on Λ^N. But then it also vanishes on $\bar{\Lambda}^N$ and hence $\bar{\Lambda}$ is also admissible. □

11. Stationary case.

Proof of Lemma 3.8. The condition that $M_n = X_n + f_{Y_n}$ is a martingale is equivalent to

$$f = (P+R+Q)f + (P-Q)\mathbf{1} \quad \text{for all} \quad (P,Q,R)\in\mathcal{J}_0. \tag{11.1}$$

Let $\mathcal{J}_{\varepsilon,f}$ be the set of all triples $(P,Q,R)\in\mathcal{J}$ satisfying (2.5) and (11.1). Consider the random environment where (P_n,R_n,Q_n) are iid and are uniformly distributed on $\mathcal{J}_{\varepsilon,f}$. Then by [18, Theorem 4.1] given $\bar{\varepsilon}$ there exists $\delta>0$ such that

$$\mathbb{P}(|X_n|>\delta\sqrt{n})>1-\bar{\varepsilon}$$

for large n. Accordingly, X_n does not exhibit the Sinai behaviour. Therefore by Lemma 3.5, (3.1) and (3.2) are satisfied for all environments in $(\mathcal{J}_{\varepsilon,f})^{\mathbb{Z}}$. □

12. Quasiperiodic case: proof of Theorem 3.11

We turn now to the quasiperiodic case with the sequence (P_n,Q_n,R_n) defined by (3.6). Note that by stationarity there exist functions $\bar{\zeta}$, $\bar{A}$, $\bar{\alpha}$, $\bar{v}$, $\bar{l}$, $\bar{\lambda}$, $\bar{\tilde{\lambda}}$ on $\mathbb{T}^d$ such that

$$\zeta_n(\omega)=\bar{\zeta}(\omega+n\gamma),\quad A_n(\omega)=\bar{A}(\omega+n\gamma),\quad \alpha_n(\omega)=\bar{\alpha}(\omega+n\gamma),$$

$$v_n(\omega)=\bar{v}(\omega+n\gamma),\quad l_n(\omega)=\bar{l}(\omega+n\gamma),\quad \lambda_n(\omega)=\bar{\lambda}(\omega+n\gamma),\quad \tilde{\lambda}_n(\omega)=\bar{\tilde{\lambda}}(\omega+n\gamma).$$

Lemma 12.1. *The functions $\bar{\zeta},\bar{A},\bar{\alpha},\bar{v},\bar{l},\bar{\lambda}$ and $\bar{\tilde{\lambda}}$ are C^r smooth.*

This lemma is proven in Appendix A.

Next, by Theorem 2.9 ([2, Theorem 2]) recurrence is equivalent to

$$\int_{\mathbb{T}^d}\ln\bar{\lambda}(\omega)d\omega=0 \tag{12.1}$$

Now [21] tells us that if $\Phi\in C^r(\mathbb{T}^d)$ has zero mean and (3.7) and (3.8) are satisfied then there is $\tilde{\Phi}\in C^0(\mathbb{T}^d)$ such that

$$\Phi(\omega)=\tilde{\Phi}(\omega+\gamma)-\tilde{\Phi}(\omega) \text{ and hence} \sum_{k=0}^{n-1}\Phi(\omega+k\gamma)=\tilde{\Phi}(\omega+n\gamma)-\tilde{\Phi}(\omega). \tag{12.2}$$

Applying (12.2) with $\Phi=\ln\bar{\lambda}$ we obtain (3.1).

Appendix A. The Invariant Section Theorem

The following result is useful for ascertaining the regularity of auxiliary sequences of matrices considered in this paper.

Let $\mathbf{X}$ and $\mathbf{Y}$ be complete metric spaces.

Consider a skew product transformation $F : \mathbf{X}\times\mathbf{Y} \to \mathbf{X}\times\mathbf{Y}$ given by $F(x, y) = (f(x), g(x, y))$ and such that

(1) F is a continuous transformation;
(2) $f : \mathbf{X} \to \mathbf{X}$ is a homeomorphism;
(3) $g(x, \cdot) : \mathbf{Y} \to \mathbf{Y}$ is a fiber contraction, that is, there exists $\theta < 1$ such that

$$d(g(x, y'), g(x, y'')) \leq \theta d(y', y'').$$

Proposition A.1. *(*[19, Theorem 3.5]*) (a) F admits an invariant section. That is, there exists a map $\Gamma : \mathbf{X} \to \mathbf{Y}$ such that $g(x, \Gamma(x)) = \Gamma(f(x))$.*

(b) If F is C^δ, f and f^{-1} are Lipshitz, and $\theta[Lip(f^{-1})]^\delta < 1$ then Γ belongs to a Hölder space C^δ.

(c) If $\mathbf{X}$ is a manifold and $\mathbf{Y}$ is a manifold with boundary and $g(x, \cdot) : \mathbf{Y} \to Int(\mathbf{Y})$ for each $x \in \mathbf{X}$ and if F is a C^r diffeomorphism such that

$$\theta[\mathrm{Lip}(f^{-1})]^r < 1 \tag{A.1}$$

then Γ is C^r smooth.

Lemma A.2. *The maps*

$$\omega \to \zeta(\omega),\ \omega \to A(\omega),\ \omega \to \alpha(\omega),\ \omega \to v(\omega), \omega \to l(\omega),\ \omega \to \lambda(\omega),\ \omega \to \tilde{\lambda}(\omega)$$

defined by (2.22) *are Hölder continuous with respect to the metric* $\mathbf{d}$ *defined by* (2.4).

Proof. We start with the smoothness of ζ. To this end we apply Proposition A.1 to the map F_1 defined on the product of $\Omega \times Z$, where Z is the space of stochastic matrices by the formula

$$F_1(\omega, \zeta) = (T\omega, (I - Q(\omega)\zeta - R(\omega))^{-1}P(\omega)).$$

Thus f is a shift T and so $\mathrm{Lip}(T^{-1}) = 2$. On the other hand due to [10, Proposition D.1], there are constants $\bar{K} > 0, \bar{\theta} < 1$ which depend only on the width of the strip m and on ε in (2.5) such that

$$d(F_1^n(\omega, \zeta'), F_1^n(\omega, \zeta'')) \leq \bar{K}\bar{\theta}^n d(\zeta', \zeta'').$$

Applying Proposition A.1 to $F_1^{n_0}$ where n_0 is such that $\bar{K}\bar{\theta}^{n_0} < 1$ we get that ζ is C^δ where δ is such that

$$2^{\delta n_0}\bar{K}\bar{\theta}^{n_0} < 1.$$

(Since n_0 can be arbitrarily large we can optimize with respect to n_0 and conclude that $\omega \to \zeta(\omega)$ is C^δ provided that $2^\delta\bar{\theta} < 1$.)

Since $\omega \to \zeta(\omega)$ is C^δ, (2.12) shows that $\omega \to A(\omega)$ and $\omega \to \delta(\omega)$ is C^δ as well.

Next, $A(\omega)$ are positive matrices and therefore preserve the positive cone in $\mathbb{R}^m$. Moreover they act as contractions in the so called Hilbert metric (see e.g [26]). Consider now the map F_2 acting on $\Omega \times \mathbb{S}_+^{m-1}$ by the formula

$$F_2(\omega, v) = \left(T\omega, \frac{A(\omega)v}{||A(\omega)v||}\right),$$

where $\mathbb{S}_+^{m-1}$ is the set of unit vectors with positive coordinates. This map is a fiber contraction in the metric induced on $\mathbb{S}_+^{m-1}$ in a natural way by the Hilbert metric. Thus Proposition A.1 implies that $\omega \to v(\omega)$ is C^δ. The Hölder property of $\omega \to l(\omega)$ is established similarly by looking at the projective action of α.

Finally the Hölder property of $\lambda(\omega)$ follows from the Hölder property of A and v, and the Hölder property of $\tilde{\lambda}(\omega)$ follows from the Hölder property of α and l. □

Proof of Lemma 12.1. The proof of Lemma 12.1 is similar to the proof of Lemma A.2 except that now we apply Proposition A.1 to skew products with the base map being toral translation $f(\omega) = \omega+\gamma$ rather than the shift of Ω. Thus $f^{-1}(\omega) = \omega-\gamma$ is an isometry and thus $\text{Lip}(f^{-1}) = 1$. Accordingly, (A.1) holds for all r implying that $\bar{\zeta}, \bar{A}, \bar{\alpha}, \bar{v}, \bar{l}, \bar{\lambda}$ and $\bar{\tilde{\lambda}}$ are C^r smooth. □

Appendix B. CLT for transient uniquely ergodic environments.

In this section we consider uniquely ergodic environments defined by (3.9). Below, whenever there is no danger of confusion, we write, with a slight abuse of notation, $f^{-1}\omega$ for $f^{-1}(\omega)$ and, more generally, $f^n\omega$ for $f^n(\omega)$.

By stationarity there exist functions $\bar{\zeta}, \bar{A}, \bar{v}, \bar{\lambda}(\omega) = ||\bar{A}(\omega)\bar{v}(f^{-1}\omega)||$ on Ω such that

$$\zeta_n(\omega) = \bar{\zeta}(f^n\omega), \quad A_n(\omega) = \bar{A}(f^n\omega), \quad v_n(\omega) = \bar{v}(f^n\omega), \quad \lambda_n(\omega) = \bar{\lambda}(f^n\omega).$$

Applying C^0 Invariant Section Theorem (Proposition A.1(a) and Lemma A.2) we conclude similarly to Section 12 that the above functions $\bar{\zeta}$, $\bar{A}$, $\bar{v}$ and hence also $\bar{\lambda}$ are continuous.

Without loss of generality we assume that $X_t \to +\infty$ as $t \to \infty$ and hence $\boldsymbol{\lambda} = \mathbb{E}(\ln \bar{\lambda}) < 0$. We recall the general results proven in [16] for ergodic environments satisfying the following assumption:

(B.1) $$\mathbb{E}\left(||A_n(\omega)\dots A_2(\omega)A_1(\omega)v_0(\omega)||^2\right) = \mathbb{E}\left(\left[\bar{\lambda}(f^{n-1}\omega)\dots\bar{\lambda}(f\omega)\bar{\lambda}(\omega)\right]^2\right)$$
decays exponentially as $n \to \infty$.

In our case (B.1) is satisfied. Indeed, due to the unique ergodicity

(B.2) $$\frac{\sum_{i=0}^{n-1} \ln \bar{\lambda}(f^j\omega)}{n} \to \boldsymbol{\lambda} \text{ as } n \to \infty \text{ uniformly in } \omega.$$

Hence for any $\varepsilon > 0$ there is N_ε such that for all $n > N_\varepsilon$ and all $\omega \in \Omega$ there is $\epsilon(n,\omega)$ satisfying $|\epsilon(n,\omega)| \le \varepsilon$ and such that

$$\begin{aligned} &||A_n(\omega)\dots A_2(\omega)A_1(\omega)v_0(\omega)|| = \bar\lambda(\omega)\bar\lambda(f\omega)\dots\bar\lambda(f^{n-1}\omega) = \\ &\exp\left(\sum_{i=0}^{n-1}\ln\bar\lambda(f^j\omega)\right) = \exp(n(\boldsymbol{\lambda}+\epsilon(n,\omega))) \end{aligned} \tag{B.3}$$

which implies the exponential decay in (B.1).

Remark B.1. One more immediate corollary of (B.3) is the following inequality which holds uniformly in $\omega \in \Omega$ for all $n \ge 1$:

$$\begin{aligned} &||A_n(\omega)\dots A_2(\omega)A_1(\omega)|| = ||A_n(\omega)\dots A_2(\omega)A_1(\omega)\mathbf{1}|| \\ &\le \mathrm{Const}||A_n(\omega)\dots A_2(\omega)A_1(\omega)v_0|| \le \mathrm{Const}e^{n\boldsymbol{\lambda}/2}. \end{aligned} \tag{B.4}$$

This follows from the property $v_0 \ge m\varepsilon^2\mathbf{1}$ explained in Remark 2.4.

The CLT holds for any initial distribution of the walk. In order, to simplify several formulae below we choose the initial distribution as follows:

$$P_{\omega,(0,\cdot)}\{\xi_\omega(0) = (0,i)\} = \sigma_0(i), \quad 1 \le i \le m, \tag{B.5}$$

where σ_0 is defined by (2.9). We would like to emphasize that the proof presented below works, with minor modifications (see e.g. the comment following formula (B.6)), for arbitrary initial distribution.

Let us list some properties of the vectors σ_n which will be used below. It follows directly from (2.10) that $\sigma_n = \sigma_k\zeta_k\dots\zeta_{n-1}$ for any $k < n$ (here we also use the relation $\zeta_n(\omega) = \zeta(f^n\omega)$). Next, $\sigma_0(\omega)$ is a continuous function of ω. This fact follows form Proposition A.1 applied to $\Omega \times U$, where U is the set of probability vectors of dimension m. The corresponding skew product transformation is given by $(\omega, y) \to (f\omega, y\zeta(\omega))$. The related fiber contraction property is the standard property of stochastic matrices ζ with $\zeta(i,j) \ge \varepsilon$, where ε is the same as in (2.5) (because $(I-R-Q\zeta)^{-1}P \ge (I-R)^{-1}P$; see proof of (2.15) (b)).

In what follows, we use the following notations and conventions. $\xi_t = \xi_\omega(t) = (X(t), Y(t))$ is the walk in RE ω starting from a random point in layer 0 which is distributed according to (B.5). More precise notations, such as e. g. $\xi_{\omega,(0,\cdot)}(t)$ will also be used where appropriate. The same convention applies to P_ω and E_ω.

As in Remark 2.8, denote $\mathfrak{t}_n$ the hitting time, by the RW, of layer n, $\mathfrak{t}_n = \min\{t : X_t = n\}$. Recall that if a RW is recurrent or transient to the right then the entries of the matrix ζ_n have the following probabilistic meaning:

$$\zeta_n(i,j) = P_\omega\left(\text{RW starting from } (n,i) \text{ hits } L_{n+1} \text{ at } (n+1,j)\right).$$

Since ξ is a Markov chain, it follows for $n \ge 1$ that

$$P_{\omega,(0,\cdot)}\left(\xi(\mathfrak{t}_n) = (n,i)\right) = \sigma_n(i), \quad 1 \le i \le m. \tag{B.6}$$

(If the initial distribution of the walker is different from σ_0 then σ_n in (B.6) has to be replaced by some $\tilde{\sigma}_n$ which is exponentially close to σ_n.)

It is proven in [16] that if (B.1) holds then there are positive constants $\mathbf{v}$ and σ such that with probability 1

$$\frac{E_\omega(\mathfrak{t}_n)}{n} \to \frac{1}{\mathbf{v}} \tag{B.7}$$

and

$$\frac{\mathfrak{t}_n - E_\omega(\mathfrak{t}_n)}{\sigma\sqrt{n}} \text{ converges in distribution to a standard normal distribution.} \tag{B.8}$$

Define $\mathfrak{b}_n$ by the condition

$$E_\omega(\mathfrak{t}_{\mathfrak{b}_n - 1}) < n \leq E_\omega(\mathfrak{t}_{\mathfrak{b}_n}).$$

We are now in a position to prove the precise version of Theorem 3.12.

Theorem B.2. $\frac{X_n - \mathfrak{b}_n}{\sigma \mathbf{v}^{3/2}\sqrt{n}}$ *converges to a standard normal distribution almost surely.*

Proof. We need the following estimate of the probability of return from layer L_n to L_a which is uniform in ω: there is a $\theta > 0$ such that for all a, n, $a < n$, $n \geq 0$ and all $\omega \in \Omega$

$$P_\omega\,(\text{RW visits } L_a \text{ after visiting } L_n) \leq \text{Const}e^{-\theta(n-a)} \tag{B.9}$$

This estimate is a strengthening of Lemma 3.2 from [10]. Its proof relies strongly on the unique ergodicity property of the environment and is different from that of Lemma 3.2 in [10].

We also need two strengthenings of (B.7) for uniquely ergodic environments. First, as $k \to \infty$

$$\frac{E_\omega(\mathfrak{t}_k)}{k} \to \frac{1}{\mathbf{v}} \text{ uniformly in } \omega \tag{B.10}$$

and hence

$$\frac{E_\omega(\mathfrak{t}_{\mathfrak{i}+k} - \mathfrak{t}_{\mathfrak{i}})}{k} = \frac{E_{f^{\mathfrak{i}}\omega}(\mathfrak{t}_k)}{k} \to \frac{1}{\mathbf{v}} \text{ uniformly in } \mathfrak{i} \text{ and } \omega. \tag{B.11}$$

Second

$$E_\omega(\mathfrak{t}_1) \text{ is bounded} \tag{B.12}$$

and hence

$$E_\omega(\mathfrak{t}_{\mathfrak{b}_n}) = n + O(1). \tag{B.13}$$

The proofs of (B.9), (B.10) and (B.12) will be given later. Let us first see how these facts imply the theorem. Given x we have

$$P_\omega(\mathfrak{t}_{\mathfrak{b}_n + \sqrt{n}x + \ln n} \leq n) - P_\omega(\mathcal{A}_n) \leq P_\omega\left(X_n - \mathfrak{b}_n \geq \sqrt{n}x\right) \leq P_\omega(\mathfrak{t}_{\mathfrak{b}_n + \sqrt{n}x} \leq n)$$

where $\mathcal{A}_n$ is the event that X returns to level $\mathfrak{b}_n + \sqrt{n}x$ after visiting level $\mathfrak{b}_n + x\sqrt{n} + \ln n$. By (B.9)

$$P_\omega(\mathcal{A}_n) \le \text{Const}\, e^{-\theta \ln n} = \frac{\text{Const}}{n^\theta}.$$

Therefore to complete the proof of the CLT for X it suffices to obtain the asymptotic behaviour of $P_\omega(\mathfrak{t}_{\mathfrak{b}_n+k_n} \le n)$ under the assumption that $k_n/\sqrt{n} \to x$ as $n \to \infty$. Next,

$$P_\omega(\mathfrak{t}_{\mathfrak{b}_n+k_n} \le n) = P_\omega\left(\frac{\mathfrak{t}_{\mathfrak{b}_n+k_n} - E_\omega(\mathfrak{t}_{\mathfrak{b}_n+k_n})}{\sqrt{\mathfrak{b}_n + k_n}} \le \frac{n - E_\omega(\mathfrak{t}_{\mathfrak{b}_n+k_n})}{\sqrt{\mathfrak{b}_n + k_n}}\right).$$

By (B.11) and (B.13), we have

$$E_\omega(\mathfrak{t}_{\mathfrak{b}_n+k_n}) = E_\omega(\mathfrak{t}_{\mathfrak{b}_n}) + E_{f^{\mathfrak{t}_{\mathfrak{b}_n}}\omega}(\mathfrak{t}_{k_n}) = n + O(1) + E_{f^{\mathfrak{t}_{\mathfrak{b}_n}}\omega}(\mathfrak{t}_{k_n}).$$

This, together with $\mathfrak{b}_n = n\mathbf{v} + o(n)$ and $E_{f^{\mathfrak{t}_{\mathfrak{b}_n}}\omega}(\mathfrak{t}_{k_n}) = k_n/\mathbf{v} + o(k_n)$, shows that

$$\lim_{n\to\infty} \frac{n - E_\omega(\mathfrak{t}_{\mathfrak{b}_n+k_n})}{\sqrt{\mathfrak{b}_n + k_n}} = \frac{-x}{\mathbf{v}^{3/2}}.$$

So (B.8) gives

$$\lim_{n\to\infty} P_\omega(\mathfrak{t}_{\mathfrak{b}_n+k_n} \le n) = \int_{-\infty}^{-x/(\sigma \mathbf{v}^{3/2})} \frac{e^{-s^2/2}}{\sqrt{2\pi}} ds$$

proving the CLT for X.

It remains to establish (B.9), (B.10) and (B.12). We start with (B.10) and (B.12).

Denote by $\mathfrak{e}_n$ the column vector whose i^{th} coordinate $\mathfrak{e}_n(i)$ is $E_{\omega,(0,i)}(\mathfrak{t}_n)$ (the expectation of $\mathfrak{t}_n$ conditioned on $\xi(0) = (0,i)$). By [16, equation (4.27)] for $n \ge 1$

$$\mathfrak{e}_n = \sum_{j=0}^{n-1} \zeta_0 \dots \zeta_j \sum_{i=0}^{\infty} A_j \dots A_{j-i+1}(I - Q_{j-i}\zeta_{j-i-1} - R_{j-i})^{-1}\mathbf{1},$$

where we use the following conventions: for any k $\zeta_k \dots \zeta_k = I$, $A_j \dots A_{j+i-1} = I$ if $i = 0$, and $A_j \dots A_{j-i+1} = A_j$ if $i = 1$. Since $E_\omega(\mathfrak{t}_n) = \sum_{i=1}^m \sigma_0(i)\mathfrak{e}_n(i)$, we have

$$E_\omega(\mathfrak{t}_n) = \sigma_0 \mathfrak{e}_n = \sum_{j=0}^{n-1} \sigma_j \sum_{i=0}^{\infty} A_j \dots A_{j-i+1}(I - Q_{j-i}\zeta_{j-i-1} - R_{j-i})^{-1}\mathbf{1}$$

and in particular

$$E_\omega(\mathfrak{t}_1) = \sigma_0 \sum_{i=0}^{\infty} A_0 \dots A_{-i+1}(I - Q_{-i}\zeta_{-i-1} - R_{-i})^{-1}\mathbf{1}.$$

Estimate (2.15) (a) means that $\|(I - Q_{-i}\zeta_{-i-1} - R_{-i})^{-1}\| \le \text{Const}$ and hence

$$E_\omega(\mathfrak{t}_1) \le \text{Const} \sum_{i=0}^{\infty} ||A_0 \dots A_{-i+1}||.$$

Due to (B.2) and (B.3)

$$||A_i \dots A_0|| \le \text{Const} \prod_{k=0}^{i-1} \bar{\lambda}(f^k\omega) \tag{B.14}$$

proving (B.12). Next, denote $\mathfrak{u}(\omega) = E_\omega(\mathfrak{t}_1)$. Obviously $E_\omega(\mathfrak{t}_n) = \sum_{j=0}^{n-1} \mathfrak{u}(f^j\omega)$ and since $\mathfrak{u}$ is continuous the unique ergodicity implies that $\frac{1}{n}\sum_{j=0}^{n-1} \mathfrak{u}(f^j\omega)$ converges uniformly in ω which proves (B.10).

We next prove (B.9). Define events $\mathcal{A}_{a,n} \stackrel{\text{def}}{=} \{\text{RW visits } L_a \text{ after visiting } L_n\}$ and $\mathcal{B}_{a,n,b} \stackrel{\text{def}}{=} \{\text{RW visits } L_b \text{ before } L_a \text{ after visiting } L_n\}$, where $n < b$. Due to the choice of the initial distribution σ_0 of the walk (see (B.5)) and the fact that the walk is transient to the right we have

$$P_\omega(\mathcal{A}_{a,n}) \equiv P_{\omega,(0,\cdot)}(\mathcal{A}_{a,n}) = P_{\omega,(n,\cdot)}(\mathcal{A}_{a,n}), \tag{B.15}$$

where the distribution of the starting point in L_n is now given by σ_n (see (B.6)). So, from now on we shall be proving (B.9) for the walk starting from n.

Observe next that

$$P_{\omega,(n,\cdot)}(\mathcal{A}_{a,n}) = 1 - \lim_{b\to\infty} P_{\omega,(n,\cdot)}(\mathcal{B}_{a,n,b}). \tag{B.16}$$

We shall now compute $P_{\omega,(n,\cdot)}(\mathcal{B}_{a,n,b})$ in terms of of products of matrices φ defined in (7.2). Let $\mathbf{h}_{a,n,b} = (h_{n,a,b}(j))_{1\le j\le m}$ be a column vector with components $h_{a,n,b}(j) = P_{\omega,(n,j)}(\mathcal{B}_{a,n,b})$ (in words, they are the probabilities of reaching b before a starting from (n,j)). It is routine (see [2]) that vectors $\mathbf{h}_{a,n,b}$ solve equation (7.1) with $\mathbf{h}_{a,a,b} = 0$ and $\mathbf{h}_{a,b,b} = \mathbf{1}$ and therefore, by (7.3),

$$\mathbf{h}_{a,n,b} = \varphi_n\varphi_{n+1}\dots\varphi_{b-1}\mathbf{1} = (\zeta_n - \Delta_n)(\zeta_{n+1} - \Delta_{n+1})\dots(\zeta_{b-1} - \Delta_{b-1})\mathbf{1},$$

where as before $\Delta_j = \zeta_j - \varphi_j$. Since $(\zeta_j - \Delta_j)\mathbf{1} = \mathbf{1} - \Delta_j\mathbf{1} \ge (1 - \|\Delta_j\|)\mathbf{1}$ and $\|\Delta_j\| \le 1$, we obtain by induction (on b) that

$$\mathbf{h}_{a,n,b} \ge (1 - \|\Delta_n\|)(1 - \|\Delta_{n+1}\|)\dots(1 - \|\Delta_{b-1}\|)\mathbf{1} \ge \left(1 - \sum_{j=n}^{\infty} \|\Delta_j\|\right)\mathbf{1}.$$

But then $P_{\omega,(n,\cdot)}(\mathcal{B}_{a,n,b}) = \sigma_n\mathbf{h}_{a,n,b} \ge 1 - \sum_{j=n}^{\infty} \|\Delta_j\|$ and therefore (B.16) gives

$$P_{\omega,(n,\cdot)}(\mathcal{A}_{a,n}) \le \sum_{j=n}^{\infty} \|\Delta_j\|. \tag{B.17}$$

From (7.11) (with $-b$ replaced by a) we have for any $n > a$

$$\|\Delta_n(\omega)\| = \|A_n(\omega)\dots A_{a+1}(\omega)\Delta_a\varphi_{a+1}\dots\varphi_n\| \le \|A_n((\omega)\dots A_{a+1}(\omega)\|. \tag{B.18}$$

Due to (B.4) and since (B.18) holds uniformly in ω, we obtain that

$$\|\Delta_n(\omega)\| \le \text{Const}e^{-(n-a)\theta},$$

where $\theta = -\boldsymbol{\lambda}/2$. Finally this together with (B.17) implies

$$P_{\omega,(n,\cdot)}(\mathcal{A}_{a,n}) \le \text{Const} e^{-(n-a)\theta}$$

and this finishes the proof of (B.9). □

Note that Theorem B.2 requires a random centering by $\mathfrak{b}_n(\omega)$. On the other hand if f is a translation on $\mathbb{T}^d$, $(\bar{P}, \bar{Q}, \bar{R})$ are C^r, and (3.7) and (3.8) hold then σ_0 and hence $\mathfrak{u}$ are C^r. We now set $\bar{\mathfrak{u}} = \int_{\mathbb{T}^d} \mathfrak{u}(\omega) d\omega$ and apply (12.2) to $\mathfrak{u} - \bar{\mathfrak{u}}$. This gives $\mathfrak{u}(\omega) = \bar{\mathfrak{u}} + \hat{\Phi}(\omega + \gamma) - \hat{\Phi}(\omega)$, where $\hat{\Phi}$ is continuous and hence

$$E_\omega(\mathfrak{t}_n) = n\bar{\mathfrak{u}} + \hat{\Phi}(\omega + n\gamma) - \hat{\Phi}(\omega) = \frac{n}{\mathbf{v}} + O(1), \text{ where } \frac{1}{\mathbf{v}} = \bar{\mathfrak{u}}$$

Accordingly $\mathfrak{b}_n = \mathbf{v}n + O(1)$ and we obtain

Corollary B.3. *In the quasiperiodic environment satisfying* (3.7), (3.8) *and* $\mathbf{v} \neq 0$

$$\frac{(X_n - n\mathbf{v})}{\sqrt{n}\mathbf{v}^{3/2}\sigma}$$

converges to a standard normal distribution where σ is the constant from (B.8).

Appendix C. Bounded ergodic sums.

The following lemma is a variation of the Gottschalk-Hedlund Theorem [17, Theorem 14.11]. We include the proof of this lemma for the sake of completeness and because it is very short.

Lemma C.1. *Let T be an ergodic transformation and Φ be a measurable function. Then there exists a constant K such that for almost all ω and all $n \in \mathbb{N}$*

$$\left| \sum_{j=0}^{n-1} \Phi(T^j\omega) \right| \le K \tag{C.1}$$

if and only if there exists a bounded function $\tilde{\Phi}$ such that

$$\Phi(\omega) = \tilde{\Phi}(T\omega) - \tilde{\Phi}(\omega) \tag{C.2}$$

Proof. (C.2) implies (C.1) since in that case $\sum_{j=0}^{n-1} \Phi(T^j\omega) = \tilde{\Phi}(T^n\omega) - \tilde{\Phi}(\omega)$. Conversely, if (C.1) holds then one can set $\tilde{\Phi}(\omega) = -\liminf_{n\to\infty} \sum_{j=0}^{n-1} \Phi(T^j\omega)$. □

References

[1] S. Alili: Asymptotic behaviour for random walks in random environments. J. Appl. Prob. **36** (1999) 334–349.

[2] E. Bolthausen, I. Goldsheid: Recurrence and transience of random walks in random environments on a strip. Commun. Math. Phys. **214**, 429–447 (2000).

[3] E. Bolthausen, I. Goldsheid: Lingering random walks in random environment on a strip. Comm. Math. Phys. **278**, no. 1 (2008) 253–288.

[4] E. Bolthausen, A.-S. Sznitman: Ten lectures on random media. DMV Seminar, **32** Birkhauser, Basel (2002).
[5] J. Brémont: On some random walks on $\mathbb{Z}$ in random medium, Ann. Prob. **30** (2002) 1266–1312.
[6] J. Brémont: One-dimensional finite range random walk in random medium and invariant measure equation. Ann. Inst. H. Poincare Prob/Stat, **45** (2009) 70–103.
[7] J. Brémont: Random walk in quasi-periodic random environment. Stochastics and Dynamics, **9** (1) (2009) 47–70.
[8] I. P. Cornfeld, S. V. Fomin, Ya. G.Sinai: Ergodic theory. Grundlehren der Mathematischen Wissenschaften **245** Springer, New York, (1982) x+486 pp.
[9] D. Dolgopyat and I. Goldsheid: Quenched limit theorems for nearest neighbour random walks in 1D random environment. Comm. Math. Phys. **315** (2012) 241–277.
[10] D. Dolgopyat, I. Goldsheid: Limit theorems for random walks on a strip in subdiffusive regimes. Nonlinearity **26** (2013) 1741–1782.
[11] D. Dolgopyat, I. Goldsheid: Invariant measure for random walks on ergodic environments on a strip. submitted.
[12] R. Durrett: Probability: theory and examples. 4th ed. Cambridge Univ. Press, (2010).
[13] Enriquez N., C. Sabot, O. Zindy: Limit laws for transient random walks in random environment on $\mathbb{Z}$. Ann. Institut Fourier **59**, no. 6 (2009) 2469–2508.
[14] N. Enriquez, C. Sabot, L. Tournier, O. Zindy: Quenched limits for the fluctuations of transient random walks in random environment on $\mathbb{Z}$. Ann. Appl. Prob. **23**, no. 3 (2013) 1148–1187.
[15] I. Goldsheid: Simple transient random walks in one-dimensional random environment: the central limit theorem. Prob. Th. Rel. Fields **139** (2007) 41–64.
[16] I. Goldsheid: Linear and Sub-linear Growth and the CLT for Hitting Times of a Random Walk in Random Environment on a Strip. Prob. Th. Rel. Fields **141** (2008) 471–511.
[17] W. H. Gottschalk, G. A. Hedlund: Topological dynamics. AMS Colloquium Publ. **36** AMS, Providence, R. I. (1955).
[18] P. Hall, and C. Heyde: Martingale limit theory and its application. Academic Press, New York–London, (1980).
[19] M. W. Hirsch, C. C. Pugh, M. Shub: Invariant manifolds. Lect. Notes in Math. **583** Springer-Verlag, Berlin-New York (1977).
[20] H. Kesten, M. V. Kozlov, F. Spitzer: Limit law for random walk in a random environment. Composito Mathematica **30** (1975) 145–168.
[21] A. N. Kolmogorov: On dynamical systems with an integral invariant on the torus. Doklady Akad. Nauk SSSR, **93** (1953) 763–766.
[22] S. Kozlov: Averaging Random Structers. Soviet Math. Doklady, **19**(4) (1978) 950–954.
[23] S. Kozlov: The method of averaging and walks in inhomogeneous environments. Russian Math. Surveys **40**(2) (1985) 73–145.
[24] E. Key: Recurrence and transience criteria for random walk in a random environment. Ann. Prob. **12**, (1984) 529–560.
[25] A.V. Letchikov: Localization of one-dimensional random walks in random environment. Soviet Sci. Reviews-C: Math. Phys. Rev. Harwood Academic Publishers, (1989) 173–220.
[26] C. Liverani: Invariant measures and their properties. A functional analytic point of view. In: Dynamical systems. Part II, 185–237, Pubbl. Cent. Ric. Mat. Ennio Giorgi, Scuola Norm. Sup., Pisa (2003).
[27] A. N. Livsic: Cohomology of dynamical systems. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972) 1296–1320.

[28] A. N. Livsic, Ya. G. Sinai: Invariant measures that are compatible with smoothness for transitive C-systems. (Russian) Dokl. Akad. Nauk SSSR **207** (1972) 103–1041.
[29] E. Mayer-Wolf, A. Roitershtein, O. Zeitouni: Limit theorems for one-dimensional transient random walks in Markov environments. Ann. Inst. H. Poincare, Prob. Stat. **40**, no 5, (2004) 635–659.
[30] G. Papanicolau, S. R. S. Varadhan: Diffusion with random coefficients. Statistics and Probability: essays in honor of C. R. Rao. G. Kallianpur, P.R. Krishnajah,, J.K. Gosh, eds., North Holland, Amsterdam (1982) 547–552.
[31] J. Peterson, G. Samorodnitsky: Weak quenched limiting distributions for transient one-dimensional random walk in a random environment. Ann. Inst. H. Poincar Probab. Statist.,**49**, no 3 (2013) 722–752.
[32] J. Peterson: Large deviations for random walks in a random environment on a strip. ALEA, Lat. Am. J. Prob. Math. Stat. **11**, no 1 (2014) 1–41.
[33] A. Roitershtein: Transient random walks on a strip in a random environment. Ann. Prob. **36**, no 6 (2008) 2354–2387.
[34] Ya. G. Sinai: The limiting behaviour of a one-dimensional random walk in a random medium. Theory Prob. Appl. **27** (1982) 256–268.
[35] Ya. G. Sinai: Simple random walks on tori. J. Statist. Phys. **94** (1999) 695–708.
[36] F. Solomon: Random walks in a random environment. Ann. Prob. **3** (1975) 1–31.
[37] O. Zeitouni: Random walks in random environment, XXXI Summer school in probability, St Flour–2001 Lecture notes in Math. **1837** (2004) 193–312.

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