ON THE TOPOLOGICAL COMPLEXITY OF ASPHERICAL SPACES

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ABSTRACT. The well-known theorem of Eilenberg and Ganea [12] expresses the Lusternik-Schnirelmann category of an aspherical space $K(\pi, 1)$ as the cohomological dimension of the group π. In this paper we study a similar problem of determining algebraically the topological complexity of the Eilenberg-MacLane spaces $K(\pi, 1)$. One of our main results states that in the case when the group π is hyperbolic in the sense of Gromov the topological complexity $\mathsf{TC}(K(\pi, 1))$ either equals or is by one larger than the cohomological dimension of $\pi \times \pi$. We approach the problem by studying *essential cohomology classes*, i.e. classes which can be obtained from the powers of the canonical class (defined in [7]) via coefficient homomorphisms. We describe a spectral sequence which allows to specify a full set of obstructions for a cohomology class to be essential. In the case of a hyperbolic group we establish a vanishing property of this spectral sequence which leads to the main result.

MSC: 55M99

1. Introduction

In this paper we study a numerical topological invariant $\mathsf{TC}(X)$ of a topological space X, originally introduced in [14], see also [15], [17]. The concept of $\mathsf{TC}(X)$ is related to the motion planning problem of robotics where a system (robot) has to be programmed to be able to move autonomously from any initial state to any final state. In this situation a motion of the system is represented by a continuous path in the configuration space X and a motion planning algorithm is a section of the path fibration

$$p : PX \to X \times X, \quad p(\gamma) = (\gamma(0), \gamma(1)). \tag{1}$$

Here PX denotes the space of all continuous paths $\gamma : [0, 1] \to X$ equipped with the compact-open topology. The topological complexity $\mathsf{TC}(X)$ is an integer reflecting the complexity of this fibration, it has several different characterisations, see [15]. Intuitively, $\mathsf{TC}(X)$ is a measure of the navigational complexity of X viewed as the configuration space of a system. $\mathsf{TC}(X)$ is similar in spirit to the classical Lusternik - Schnirelmann category $\mathsf{cat(X)}$. The invariants $\mathsf{TC}(X)$ and $\mathsf{cat(X)}$ are special cases of a more general notion of the genus of a fibration introduced by A. Schwarz [30]. A recent survey of the concept $\mathsf{TC}(X)$ and robot motion planning algorithms in interesting configuration spaces can be found in [18].

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Definition 1.1. Given a path-connected topological space X, the topological complexity of X is defined as the minimal number $\mathsf{TC}(X) = k$ such that the Cartesian product $X \times X$ can be covered by k open subsets $X \times X = U_1 \cup U_2 \cup \dots U_k$ with the property that for any $i = 1, 2, ..., k$ there exists a continuous section $s_i : U_i \to PX$, $\pi \circ s_i = \mathrm{id}$, over U_i. If no such k exists we will set $\mathsf{TC}(X) = \infty$.

Note that in the mathematical literature there is also a *reduced* version of the topological complexity which is one less compared to the one we are dealing with in this paper.

One of the main properties of $\mathsf{TC}(X)$ is its *homotopy invariance* [14], i.e. $\mathsf{TC}(X)$ depends only on the homotopy type of X. This property is helpful for the task of computing $\mathsf{TC}(X)$ in various examples since cohomological tools can be employed. In the case when the configuration space X is aspherical, i.e. $\pi_i(X) = 0$ for all $i > 1$, the number $\mathsf{TC}(X)$ depends only on the fundamental group $\pi = \pi_1(X)$ and it was observed in [15] that one has to be able to express $\mathsf{TC}(X)$ in terms of algebraic properties of the group π alone.

A similar question for the Lusternik - Schnirelmann category $\mathsf{cat(X)}$ was solved by S. Eilenberg and T. Ganea in 1957 in the seminal paper [12]. Their theorem relates $\mathsf{cat(X)}$ and the cohomological dimension of the fundamental group π of X.

The problem of computing $\mathsf{TC}(K(\pi, 1))$ as an algebraic invariant of the group π attracted attention of many mathematicians. Although no general answer is presently known, many interesting results were obtained.

The initial papers [14], [15] contained computations of $\mathsf{TC}(X)$ for graphs, closed orientable surfaces and tori. In [19] the number $\mathsf{TC}(X)$ was computed for the case when X is the configuration space of many particles moving on the plane without collisions. D. Cohen and G. Pruidze [5] calculated the topological complexity of complements of general position arrangements and Eilenberg – MacLane spaces associated to certain right-angled Artin groups.

As a recent breakthrough, the topological complexity of closed non-orientable surfaces of genus $g \geq 2$ has only recently been computed by A. Dranishnikov for $g \geq 4$ in [11] and by D. Cohen and L. Vandembroucq for $g = 2, 3$ in [6]. In both these articles it is shown that $\mathsf{TC}(K(\pi, 1))$ attains its maximum, i.e. coincides with $\mathrm{cd}(\pi \times \pi) + 1$.

The estimates of M. Grant [20] give good upper bounds for $\mathsf{TC}(K(\pi, 1))$ for nilpotent fundamental groups π. In [21], M. Grant, G. Lupton and J. Oprea proved that $\mathsf{TC}(K(\pi, 1))$ is bounded below by the cohomological dimension of $A \times B$ where A and B are subgroups of π whose conjugates intersect trivially. Using these estimates, M. Grant and D. Recio-Mitter [22] have computed $\mathsf{TC}(K(\pi, 1))$ for certain subgroups of Artin's braid groups.

Yuli Rudyak [29] showed that for any pair of positive integers k, ℓ satisfying $k \leq \ell \leq 2k$ there exists a finitely presented group π such that $\mathrm{cd}(\pi) = k$ and $\mathsf{TC}(K(\pi, 1)) = \ell + 1$.

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2. Statements of the main results

In this section we state the main results obtained in the present paper.

Theorem 1. *Let X be a connected aspherical finite cell complex with hyperbolic fundamental group $\pi = \pi_1(X)$. Then the topological complexity $\mathsf{TC}(X)$ equals either $\mathrm{cd}(\pi \times \pi)$ or $\mathrm{cd}(\pi \times \pi) + 1$.*

The symbol $\mathrm{cd}(\pi \times \pi)$ stands for the cohomological dimension of $\pi \times \pi$ and similarly for $\mathrm{cd}(\pi)$. The Eilenberg-Ganea theorem [4] states that $\mathrm{cd}(\pi \times \pi) = \mathsf{gd}(\pi \times \pi)$ (with one possible exception) and hence the general dimensional upper bound for $\mathsf{TC}(X)$ (see [14], [17]) gives

$$\mathsf{TC}(X) \leq \mathsf{gd}(\pi \times \pi) + 1 = \mathrm{cd}(\pi \times \pi) + 1. \tag{2}$$

Here $\mathsf{gd}(\pi \times \pi)$ denotes *the geometric dimension* of $\pi \times \pi$, i.e. the minimal dimension of a cell complex with fundamental group $\pi \times \pi$. Thus Theorem 1 essentially states that the topological complexity $\mathsf{TC}(K(\pi, 1))$, where π is hyperbolic, is either maximal (as allowed by the dimensional upper bound) or is by one smaller than the maximum.

The notion of a hyperbolic group was introduced by M. Gromov in [23]; we also refer the reader to the monograph [3]. Hyperbolic groups are "typical", i.e. they appear with probability tending to 1, in many models of random groups including Gromov's well-known density model [24], [28].

As an example, consider the case of the fundamental group $\pi = \pi_1(\Sigma_g)$ of a closed orientable surface of genus $g \geq 2$. It is torsion-free hyperbolic and $\mathsf{TC}(\Sigma_g) = \mathrm{cd}(\pi \times \pi) + 1 = 5$, see [17], in accordance with the maximal option of Theorem 1. Similarly, if $\pi = F_\mu$ is a free group on μ generators then, according to [17], Proposition 4.42, $\mathsf{TC}(K(F_\mu, 1)) = 3$ (for $\mu > 1$) and $\mathrm{cd}(F_\mu \times F_\mu) = 2$; here again Theorem 1 is satisfied in the maximal version.

The only known to us example of an aspherical space $X = K(\pi, 1)$ with π hyperbolic where $\mathsf{TC}(X) = \mathrm{cd}(\pi \times \pi)$ is the case of the circle $X = S^1$. It would be interesting to learn if some other examples of this type exist.

Theorem 1 follows from the following statement:

Theorem 2. *Let X be a connected aspherical finite cell complex with fundamental group $\pi = \pi_1(X)$. Suppose that (1) the centraliser of any nontrivial element $g \in \pi$ is cyclic and (2) $\mathrm{cd}(\pi \times \pi) > \mathrm{cd}(\pi)$. Then the topological complexity $\mathsf{TC}(X)$ equals either $\mathrm{cd}(\pi \times \pi)$ or $\mathrm{cd}(\pi \times \pi) + 1$.*

We do not know examples of finitely presented groups π such that $\mathrm{cd}(\pi \times \pi) = \mathrm{cd}(\pi)$, i.e. such that the assumption (2) of Theorem 2 is violated. A. Dranishnikov [9] constructed examples with $\mathrm{cd}(\pi_1 \times \pi_2) < \mathrm{cd}(\pi_1) + \mathrm{cd}(\pi_2)$; see also [8], page 157. In [9] he also proved that $\mathrm{cd}(\pi \times \pi) = 2\mathrm{cd}(\pi)$ for any Coxeter group π.

To state another main result of this paper we need to recall the notion of TC-*weight* of cohomology classes as introduced in [16]; this notion is similar but not identical to the concept of TC-weight introduced in [17], §4.5; both these notions were inspired by the notion of category weight of cohomology classes initiated by E. Fadell and S. Husseini [13].

Definition 2.1. *Let $\alpha \in H^*(X\times X, A)$ be a cohomology class, where A is a local coefficient system on $X\times X$. We say that α has weight $k\geq 0$ (notation $\mathrm{wgt}(\alpha)=k$) if k is the largest integer with the property that for any continuous map $f: Y \to X\times X$ (where Y is a topological space) one has $f^*(\alpha)=0\in H^*(Y, f^*(A))$ provided the space Y admits an open cover $U_1\cup U_2\cup\dots\cup U_k = Y$ such that each restriction map $f|U_j : U_j\to X\times X$ admits a continuous lift $U_j\to PX$ into the path-space fibration (1).*

A cohomology class $\alpha\in H^*(X\times X, A)$ has a positive weight $\mathrm{wgt}(\alpha)\geq 1$ if and only if α is a *zero-divisor*, i.e. if its restriction to the diagonal $\Delta_X\subset X\times X$ vanishes,

$$0=\alpha\,|\,\Delta_X\in H^*(X,\tilde{A}),$$

see [16], page 3341. Here $\tilde{A}$ denotes the restriction local system $A\,|\,\Delta_X$. Note that in [16] the authors considered untwisted coefficients but all the arguments automatically extend to general local coefficient systems. In particular, by Proposition 2 from [16] we have

$$\mathrm{wgt}(\alpha_1\cup\alpha_2)\geq \mathrm{wgt}(\alpha_1)+\mathrm{wgt}(\alpha_2) \tag{3}$$

for cohomology classes $\alpha_i\in H^{d_i}(X\times X, A_i)$, $i=1,2$, where the cup-product $\alpha_1\cup\alpha_2$ lies in $H^{d_1+d_2}(X\times X, A_1\otimes_{\mathbf{Z}} A_2)$.

Theorem 3. *Let X be a connected aspherical finite cell complex. Suppose that the fundamental group $\pi=\pi_1(X)$ is such that the centraliser of any nontrivial element $g\in\pi$ is infinite cyclic. Then any degree n zero-divisor $\alpha\in H^n(X\times X, A)$, where $n\geq 1$, has weight $\mathrm{wgt}(\alpha)\geq n-1$.*

For obvious reasons this theorem is automatically true for $n=1,2$; it becomes meaningful only for $n>2$.

Here is a useful corollary of Theorem 3:

Theorem 4. *Under the assumptions of Theorem 2, one has*

$$\mathfrak{v}^{n-1}\neq 0\in H^{n-1}(\pi\times\pi, I^{n-1})$$

where $n=\mathrm{cd}(\pi\times\pi)$. Here $\mathfrak{v}\in H^1(\pi\times\pi, I)$ denotes the canonical class, see §3 below.

The statement of Theorem 4 becomes false if we remove the assumptions on the fundamental group. For example in the case of an abelian group $\pi=\mathbf{Z}^k$ (see §6) we have $n=\mathrm{cd}(\pi\times\pi)=2k$ and $\mathfrak{v}^k$ is the highest nontrivial power of the canonical class.

Question: Let π be a noncommutative hyperbolic group and let n denote $\mathrm{cd}(\pi\times\pi)$. Is it true that the n-th power of the canonical class $\mathfrak{v}^n\in H^n(\pi\times\pi, I^n)$ is nonzero, $\mathfrak{v}^n\neq 0$?

A positive answer to this question would imply that for any noncommutative hyperbolic group π one has $\mathsf{TC}(K(\pi,1))=\mathrm{cd}(\pi\times\pi)+1$.

The proofs of Theorems 1, 2, 3 and 4 are given in §9.

In §10, we present an application of Theorem 3 to the topological complexity of symplectically aspherical manifolds.

3. The canonical class

First we fix notations which will be used in this paper. We shall consider a discrete torsion-free group π with unit element $e \in \pi$ and left modules M over the group ring $\mathbf{Z}[\pi \times \pi]$. Any such module M can be equivalently viewed as a $\pi - \pi$-bimodule using the convention

$$(g,h) \cdot m = gmh^{-1}$$

for $g, h \in \pi$ and $m \in M$. Recall that for two left $\mathbf{Z}[\pi \times \pi]$-modules A and B the module $\mathrm{Hom}_{\mathbf{Z}}(A, B)$ has a canonical $\mathbf{Z}[\pi \times \pi]$-module structure given by $((g,h) \cdot f)(a) = gf(g^{-1}ah)h^{-1}$ where $g, h \in \pi$, $a \in A$ and $f : A \to B$ is a group homomorphism.

Besides, the tensor product $A \otimes_{\mathbf{Z}} B$ has a left $\mathbf{Z}[\pi \times \pi]$-module structure given by $(g,h) \cdot (a \otimes b) = (gah^{-1}) \otimes (gbh^{-1})$ where $g, h \in \pi$ and $a \in A$, $b \in B$; we shall refer to this action as *the diagonal action.*

For a left $\mathbf{Z}[\pi \times \pi]$-module A we shall denote by $\tilde{A}$ the same abelian group viewed as a $\mathbf{Z}[\pi]$-module via *the conjugation action*, i.e. $g \cdot a = gag^{-1}$ for $g \in \pi$ and $a \in A$.

The group ring $\mathbf{Z}[\pi]$ is a $\mathbf{Z}[\pi \times \pi]$-module with respect to the action

$$(g,h) \cdot a = gah^{-1}, \quad \text{where} \quad g, h, a \in \pi.$$

The augmentation homomorphism $\epsilon : \mathbf{Z}[\pi] \to \mathbf{Z}$ is a $\mathbf{Z}[\pi \times \pi]$-homomorphism where we consider the trivial $\mathbf{Z}[\pi \times \pi]$-module structure on $\mathbf{Z}$. The *augmentation ideal* $I = \ker \epsilon$ is hence a $\mathbf{Z}[\pi \times \pi]$-module and we have a short exact sequence of $\mathbf{Z}[\pi \times \pi]$-modules

$$0 \to I \to \mathbf{Z}[\pi] \xrightarrow{\epsilon} \mathbf{Z} \to 0. \tag{4}$$

In this paper we shall use the the formalism (described in [25], Chapter IV, §9) which associates a well defined class

$$\theta \in \mathrm{Ext}^n_R(M, N)$$

with any exact sequence

$$0 \to N \to L_n \to L_{n-1} \to \cdots \to L_1 \to M \to 0$$

of left R-modules and R-homomorphisms, where R is a ring. This construction can be briefly summarised as follows. If

$$\cdots \to C_2 \to C_1 \to C_0 \to M \to 0$$

is a projective resolution of M over R, one obtains a commutative diagram

$$\begin{array}{ccccccccccccccc}
C_{n+1} & \to & C_n & \to & C_{n-1} & \to & \dots & C_1 & \to & C_0 & \to & M & \to & 0 \\
\downarrow & & \downarrow f & & \downarrow & & \dots & \downarrow & & \downarrow & & \downarrow = & & \\
0 & \to & N & \to & L_n & \to & \dots & L_2 & \to & L_1 & \to & M & \to & 0.
\end{array}$$

The homomorphism $f : C_n \to N$ is a cocycle of the complex $\mathrm{Hom}_R(C_*, N)$, which is defined uniquely up to chain homotopy. The class θ is the cohomology class of this cocycle

$$\theta \;=\; \{f\} \in H^n(\mathrm{Hom}_R(C_*, N)) = \mathrm{Ext}^n_R(M, N).$$

Note that the definition of Bourbaki (see [2] §7, n. 3) is slightly different since [2] introduces additionally a sign factor $(-1)^{n(n+1)/2}$.

An important role plays the class

$$\mathfrak{v} \in \mathrm{Ext}^1_{\mathbf{Z}[\pi\times\pi]}(\mathbf{Z}, I) = H^1(\pi\times\pi, I)$$

associated with the exact sequence (4). It was introduced in [7] under the name of *the canonical class*.

To describe the cocycle representing the canonical class $\mathfrak{v}$ consider the bar resolution C_* of $\mathbf{Z}$ over $\mathbf{Z}[\pi\times\pi]$, see [4], page 19. Here $\dots \to C_1 \xrightarrow{d} C_0 \to \mathbf{Z} \to 0$ where C_0 is a free $\mathbf{Z}[\pi\times\pi]$-module generated by the symbol $[\,]$ and C_1 is the free $\mathbf{Z}[\pi\times\pi]$-module generated by the symbols $[(g,h)]$ for all $(g,h)\in\pi\times\pi$. The boundary operator d acts by

$$d[(g,h)] = ((g,h)-1)\,[\,].$$

We obtain the chain map

$$\begin{array}{ccccccccc}
C_2 & \to & C_1 & \xrightarrow{d} & C_0 & \to & \mathbf{Z} & \to & 0\\
\downarrow & & \downarrow f & & \downarrow \mu & & \downarrow= & & \\
0 & \to & I & \to & \mathbf{Z}[\pi] & \xrightarrow{\epsilon} & \mathbf{Z} & \to & 0
\end{array}$$

where $\mu([\,]) = 1$ and

$$f([(g,h)]) = gh^{-1} - 1 \in I. \tag{5}$$

Thus, the cocycle $f : C_1 \to I$ is given by the crossed homomorphism (5). Comparing with [7], we see that the definition of the canonical class given above coincides with the definition given in [7], page 110.

We shall also describe the cocycle representing the canonical class in *the homogeneous standard resolution* of $\pi\times\pi$, see [4], page 18:

$$\dots \to C'_2 \xrightarrow{d} C'_1 \xrightarrow{d} C'_0 \to \mathbf{Z} \to 0. \tag{6}$$

Here C'_i is a free $\mathbf{Z}$-module generated by the $(i+1)$-tuples $((g_0,h_0),\dots,(g_i,h_i))$ with $g_j, h_j \in \pi$ for any $j = 0,1,\dots,i$. Using (5) we obtain that the cocycle $f' : C'_1 \to I$ representing the canonical class $\mathfrak{v}$ is given by the formula

$$f'((g_0,h_0),(g_1,h_1)) = g_1h_1^{-1} - g_0h_0^{-1}, \quad g_j, h_j \in \pi. \tag{7}$$

The canonical class $\mathfrak{v}$ is closely related to *the Berstein-Schwarz class* (see [1], [10]); the latter is crucial for the study of the Lusternik-Schnirelmann category cat. The Berstein-Schwarz class can be defined as the class

$$\mathfrak{b} \in \mathrm{Ext}^1_{\mathbf{Z}[\pi]}(\mathbf{Z}, I) = H^1(\pi, I)$$

which corresponds to the exact sequence (4) viewed as a sequence of left $\mathbf{Z}[\pi]$-modules via the left action of $\pi = \pi\times 1 \subset \pi\times\pi$. For future reference we state

$$\mathfrak{v}\,|\,\pi\times 1 \;=\; \mathfrak{b}; \tag{8}$$

here $\pi\times 1\subset\pi\times\pi$ denotes the left factor viewed as a subgroup.

The main properties of the canonical class $\mathfrak{v}$ are as follows.

Let X be a finite connected cell complex with fundamental group $\pi_1(X) = \pi$. We may view I as a local coefficient system over $X \times X$ and form the cup-product $\mathfrak{v} \cup \mathfrak{v} \cup \cdots \cup \mathfrak{v} = \mathfrak{v}^k$ (k times) which lies in the cohomology group

$$\mathfrak{v}^k \in H^k(X \times X, I^k),$$

where I^k denotes the tensor product $I \otimes_{\mathbf{Z}} I \otimes_{\mathbf{Z}} \cdots \otimes_{\mathbf{Z}} I$ of k copies of I viewed as a left $\mathbf{Z}[\pi \times \pi]$-module via the diagonal action as explained above. Let n denote the dimension of X. It is known that in general the topological complexity satisfies $\mathsf{TC}(X) \leq 2n+1$ and the equality

$$\mathsf{TC}(X) = 2\dim(X) + 1 = 2n + 1$$

happens if and only if $\mathfrak{v}^{2n} \neq 0$; see [7], Theorem 7.

Another important property of $\mathfrak{v}$ is that it is *a zero-divisor*, i.e.

$$\mathfrak{v} \,|\, \Delta_\pi = 0 \in H^1(\pi, \tilde{I}) \tag{9}$$

where $\Delta_\pi \subset \pi \times \pi$ is the diagonal subgroup, $\Delta_\pi = \{(g, g); g \in \pi\}$. This immediately follows from the observation that the cocycle f representing $\mathfrak{v}$ (see (5)) vanishes on the diagonal Δ_π.

Our next goal is to describe an exact sequence representing the power $\mathfrak{v}^n$ of the canonical class. The exact sequence (4) splits over $\mathbf{Z}$ and hence for any left $\mathbf{Z}[\pi \times \pi]$-module M tensoring over $\mathbf{Z}$ we obtain an exact sequence

$$0 \to I \otimes_{\mathbf{Z}} M \to \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} M \stackrel{\epsilon}{\to} M \to 0. \tag{10}$$

In (10) we consider the diagonal action of $\pi \times \pi$ on the tensor products. Taking here $M = I^s$ where $I^s = I \otimes_{\mathbf{Z}} I \otimes_{\mathbf{Z}} I \cdots \otimes_{\mathbf{Z}} I$ we obtain a short exact sequence

$$0 \to I^{s+1} \stackrel{i \otimes 1}{\to} \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^s \stackrel{\epsilon \otimes 1}{\to} I^s \to 0. \tag{11}$$

Here $i : I \to \mathbf{Z}[\pi]$ is the inclusion and $\epsilon : \mathbf{Z}[\pi] \to \mathbf{Z}$ is the augmentation. Splicing exact sequences (11) for $s = 0, 1, \ldots, n-1$ we obtain an exact sequence

$$0 \to I^n \to \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^{n-1} \to \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^{n-2} \to \ldots \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I \to \mathbf{Z}[\pi] \to \mathbf{Z} \to 0. \tag{12}$$

Lemma 3.1. *The cohomology class*

$$\mathfrak{v}^n \in H^n(\pi \times \pi, I^n) = \mathrm{Ext}^n_{\mathbf{Z}[\pi \times \pi]}(\mathbf{Z}, I^n)$$

is represented by the exact sequence (12).

Proof. Consider again the homogeneous standard resolution (6) of $\pi \times \pi$. Define $\mathbf{Z}[\pi \times \pi]$-homomorphisms

$$\kappa_j : C'_j \to \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^j, \quad \text{where} \quad j = 0, 1, \ldots, n-1, \tag{13}$$

by the formula

$$\kappa_j((g_0, h_0), (g_1, h_1), \ldots, (g_j, h_j)) = x_0 \otimes (x_1 - x_0) \otimes \cdots \otimes (x_j - x_{j-1}), \tag{14}$$

where the symbol x_i denotes $g_i h_i^{-1} \in \pi$ for $i = 0, \ldots, j$.

We claim that the homomorphisms κ_j, for $j = 0, \dots, n$, determine a chain map from the homogeneous standard resolution (6) into the exact sequence (12). In other words, we want to show that

$$\kappa_{j-1}(d((g_0,h_0),(g_1,h_1),\dots,(g_j,h_j))) = (x_1 - x_0) \otimes \dots \otimes (x_j - x_{j-1}). \tag{15}$$

This statement is obvious for $j = 1$. To prove it for $j > 1$ we apply induction on j. Denoting

$$\Pi_j(x_0, x_1, \dots, x_{j-1}) = x_0 \otimes (x_1 - x_0) \otimes \dots \otimes (x_{j-1} - x_{j-2})$$

we may write

$$\kappa_{j-1}(d((g_0,h_0),(g_1,h_1),\dots,(g_j,h_j))) = \sum_{i=0}^{j} (-1)^i \Pi_j(x_0, \dots, \hat{x}_i, \dots, x_j).$$

The last two terms in this sum (for $i = j-1$ and $i = j$) sum up to

$$(-1)^{j-1} x_0 \otimes (x_1 - x_0) \otimes \dots (x_{j-2} - x_{j-3}) \otimes (x_j - x_{j-1}).$$

Thus we see that the LHS of (15) can be written as

$$\left[\sum_{i=0}^{j-1} (-1)^i \Pi_{j-1}(x_0, \dots, \hat{x}_i, \dots, x_{j-1})\right] \otimes (x_j - x_{j-1})$$

and our statement follows by induction.

The homomorphism $f_n : C'_n \to I^n$ which appears in the commutative diagram

$$\begin{array}{ccccccccc}
C'_{n+1} & \to & C'_n & \xrightarrow{d} & C'_{n-1} & \xrightarrow{d} & C'_{n-2} & \to & \dots \\
 & & \downarrow f_n & & \downarrow \kappa_{n-1} & & \downarrow \kappa_{n-2} & & \\
0 & \to & I^n & \to & \mathbf{Z}[\pi] \otimes I^{n-1} & \to & \mathbf{Z}[\pi] \otimes I^{n-2} & \to & \dots
\end{array}$$

is given by the formula

$$f_n((g_0,h_0),(g_1,h_1),\dots,(g_n,h_n)) = (x_1 - x_0) \otimes (x_2 - x_1) \otimes \dots (x_n - x_{n-1}) \tag{16}$$

where $x_i = g_i h_i^{-1}$. Since the cocycle representing $\mathfrak{v}$ is given by $x_1 - x_0$ (see (7)), using the diagonal approximation in the standard complex (see [4], page 108) we find that f_n represents $\mathfrak{v}^n$. □

Remark 3.2. Lemma 3.1 also follows by applying Theorems 4.2 and 9.2 from [27], Chapter VIII.

The canonical class $\mathfrak{v}$ allows to describe the connecting homomorphisms in cohomology as we shall exploit several times in this paper. Let M be a left $\mathbf{Z}[\pi \times \pi]$-module. The Bockstein homomorphism

$$\beta : H^i(\pi \times \pi, M) \to H^{i+1}(\pi \times \pi, I \otimes M)$$

of the exact sequence (10) acts as follows

$$\beta(u) = \mathfrak{v} \cup u, \quad \text{for} \quad u \in H^i(\pi \times \pi, M). \tag{17}$$

This follows from Lemma 5 from [7] and from [4], chapter V, (3.3).

4. Universality of the Berstein - Schwarz class

In the theory of Lusternik - Schnirelmann category an important role plays the following result which was originally stated (without proof) by A.S. Schwarz [30], Proposition 34. A recent proof can be found in [10].

Theorem 5. *For any left $\mathbf{Z}[\pi]$-module A and for any cohomology class $\alpha \in H^n(\pi, A)$ one may find a $\mathbf{Z}[\pi]$-homomorphism $\mu : I^n \to A$ such that $\alpha = \mu_*(\mathfrak{b}^n)$.*

Recall that we view the tensor power $I^n = I \otimes_{\mathbf{Z}} I \otimes_{\mathbf{Z}} \cdots \otimes_{\mathbf{Z}} I$ as a left $\mathbf{Z}[\pi]$-module using the diagonal action of π from the left, i.e. $g \cdot (\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n) = g\alpha_1 \otimes g\alpha_2 \otimes \cdots \otimes g\alpha_n$ where $g \in \pi$ and $\alpha_i \in I$ for $i = 1, \cdots, n$.

In other words, Theorem 5 states that the powers of the Berstein - Schwarz class $\mathfrak{b}^n$ are universal in the sense that any other degree n cohomology class can be obtained from $\mathfrak{b}^n$ by a coefficient homomorphism. This result implies that the Lusternik - Schnirelmann category of an aspherical space is at least $\mathrm{cd}(\pi) + 1$.

We include below a short proof of Theorem 5 (following essentially [10]) for completeness.

Proof. First one observes that I^s is a free abelian group and hence $\mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^s$ is free as a left $\mathbf{Z}[\pi]$-module; here we apply Corollary 5.7 from chapter III of [4]. Hence the exact sequence

$$\cdots \to \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^n \to \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^{n-1} \to \cdots \to \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I \to \mathbf{Z}[\pi] \to \mathbf{Z} \to 0 \tag{18}$$

is a free resolution of $\mathbf{Z}$ over $\mathbf{Z}[\pi]$. The differential of this complex is given by

$$\mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^n = \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I \otimes_{\mathbf{Z}} I^{n-1} \xrightarrow{\epsilon \otimes i \otimes 1} \mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^{n-1} = \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^{n-1};$$

here ϵ is the augmentation and $i : I \to \mathbf{Z}[\pi]$ is the inclusion.

Using resolution (18), any degree n cohomology class $\alpha \in H^n(\pi, A)$ can be represented by an n-cocycle $f : \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^n \to A$ which is a $\mathbf{Z}[\pi]$-homomorphism vanishing on the image I^{n+1} of the boundary homomorphism $\mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^{n+1} \to \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^n$. In view of the short exact sequence

$$0 \to I^{n+1} \to \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^n \xrightarrow{\epsilon \otimes 1} I^n \to 0$$

we see that *there is a 1-1 correspondence between cocycles $f : \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^n \to A$ and homomorphisms $\mu : I^n \to A$.*

Let $\mu : I^n \to A$ be the $\mathbf{Z}[\pi]$-homomorphism corresponding to a cocycle representing the class α.

Using the definition of a class associated to an exact sequence (see beginning of §3), Lemma 3.1 and formula (8) we see that the identity map $I^n \to I^n$ corresponds to the n-th power of the Berstein - Schwarz class $\mathfrak{b}^n$. Combining all these mentioned results we obtain $\mu_*(\mathfrak{b}^n) = \alpha$. $\square$

5. Essential cohomology classes

It is easy to see that the analogue of Theorem 5 fails when we consider cohomology classes $\alpha \in H^n(\pi \times \pi, A)$ and ask whether such classes can be obtained from powers of the canonical class $\mathfrak{v} \in H^1(\pi \times \pi, I)$ by coefficient homomorphisms. The arguments of the proof of Theorem 5 are not applicable since the $\mathbf{Z}[\pi \times \pi]$-modules $\mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^s$ are neither free nor projective over the ring $\mathbf{Z}[\pi \times \pi]$.

Definition 5.1. *We shall say that a cohomology class $\alpha \in H^n(\pi \times \pi, A)$ is essential if there exists a homomorphism of $\mathbf{Z}[\pi \times \pi]$-modules $\mu : I^n \to A$ such that $\mu_*(\mathfrak{v}^n) = \alpha$.*

One wants to have verifiable criteria which guarantee that a given cohomology class $\alpha \in H^n(\pi \times \pi, A)$ is essential. Since $\mathfrak{v}$ and all its powers are zero-divisors, it is obvious that any essential class must also be a zero-divisor, i.e. satisfy

$$\alpha \,|\, \Delta_\pi \,=\, 0 \in H^n(\pi, \tilde{A}),$$

see above. For degree one cohomology classes this condition is sufficient, see Lemma 5.2. However, as we shall see, a degree $n \geq 2$ zero-divisor does not need to be essential.

Clearly, the set of all essential classes in $H^n(\pi \times \pi, A)$ forms a subgroup.

Moreover, the cup-product of two essential classes $\alpha_i \in H^{n_i}(\pi \times \pi, A_i)$, where $i = 1, 2$, is an essential class

$$\alpha_1 \cup \alpha_2 \in H^{n_1+n_2}(\pi \times \pi, A_1 \otimes_{\mathbf{Z}} A_2).$$

Indeed, suppose $\mu_i : I^{n_i} \to A_i$ are $\mathbf{Z}[\pi \times \pi]$-homomorphisms such that $\mu_{i*}(\mathfrak{v}^{n_i}) = \alpha_i$, where $i = 1, 2$. Then $\mu = \mu_1 \otimes \mu_2 : I^{n_1} \otimes_{\mathbf{Z}} I^{n_2} \to A_1 \otimes_{\mathbf{Z}} A_2$ satisfies

$$\mu_*(\mathfrak{v}^{n_1+n_2}) = \mu_*(\mathfrak{v}^{n_1} \cup \mathfrak{v}^{n_2}) = \mu_{1*}(\mathfrak{v}^{n_1}) \cup \mu_{2*}(\mathfrak{v}^{n_2}) = \alpha_1 \cup \alpha_2.$$

Lemma 5.2. *A degree one cohomology class $\alpha \in H^1(\pi \times \pi, A)$ is essential if and only if it is a zero-divisor.*

The proof will be postponed until we have prepared the necessary algebraic techniques.

Lemma 5.3. *Consider two left $\mathbf{Z}[\pi \times \pi]$-modules M and N. Let $\tilde{M}$ and $\tilde{N}$ denote the left $\mathbf{Z}[\pi]$-module structures on M and N correspondingly via conjugation, i.e. $g \cdot m = gmg^{-1}$ and $g \cdot n = gng^{-1}$ for $g \in \pi$ and $m \in \tilde{M}$, $n \in \tilde{N}$. Let*

$$\Phi : \mathrm{Hom}_{\mathbf{Z}[\pi\times\pi]}(\mathbf{Z}[\pi] \otimes_{\mathbf{Z}} M, N) \to \mathrm{Hom}_{\mathbf{Z}[\pi]}(\tilde{M}, \tilde{N}) \tag{19}$$

be the map which associates with any $\mathbf{Z}[\pi \times \pi]$-homomorphism $f : \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} M \to N$ its restriction $f \,|\, e \otimes M$ onto

$$M = e \otimes_{\mathbf{Z}} M \subset \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} M$$

where $e \in \pi$ is the unit element. Then Φ is an isomorphism.

Proof. The inverse map

$$\Psi : \mathrm{Hom}_{\mathbf{Z}[\pi]}(\tilde{M}, \tilde{N}) \to \mathrm{Hom}_{\mathbf{Z}[\pi\times\pi]}(\mathbf{Z}[\pi] \otimes_{\mathbf{Z}} M, N) \tag{20}$$

can be defined as follows. Given $\phi \in \mathrm{Hom}_{\mathbf{Z}[\pi]}(\tilde{M}, \tilde{N})$ let $\hat{\phi} : \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} M \to N$ be defined by

$$\hat{\phi}(g \otimes m) = g\phi(g^{-1}m) = \phi(mg^{-1})g$$

for $g \in \pi$ and $m \in M$. For $a, b \in \pi$ we have

$$\begin{aligned}\hat{\phi}(agb^{-1} \otimes amb^{-1}) &= agb^{-1}\phi(bg^{-1}a^{-1} \cdot amb^{-1}) \\ &= a\hat{\phi}(g \otimes m)b^{-1}\end{aligned}$$

which shows that $\hat{\phi}$ is a $\mathbf{Z}[\pi \times \pi]$-homomorphism. We set $\Psi(\phi) = \hat{\phi}$. One checks directly that Ψ and Φ are mutually inverse. ☐

As the next step we prove the following generalisation of the previous lemma.

Lemma 5.4. *For two left $\mathbf{Z}[\pi \times \pi]$-modules M and N and any $i \geq 0$ the map*

$$\Phi : \mathrm{Ext}^i_{\mathbf{Z}[\pi\times\pi]}(\mathbf{Z}[\pi] \otimes_{\mathbf{Z}} M, N) \to \mathrm{Ext}^i_{\mathbf{Z}[\pi]}(\tilde{M}, \tilde{N}) \tag{21}$$

is an isomorphism. The map Φ acts by first restricting the $\mathbf{Z}[\pi \times \pi]$-module structure to the conjugate action of $\mathbf{Z}[\pi]$ (where $\pi = \Delta_\pi \subset \pi \times \pi$ is the diagonal subgroup) and secondly by taking the restriction on the $\mathbf{Z}[\pi]$-submodule $\tilde{M} = e \otimes_{\mathbf{Z}} M \subset \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} M$.

Proof. Consider an injective resolution $0 \to N \to J_0 \to J_1 \to \dots$ of N over $\mathbf{Z}[\pi \times \pi]$; we may use it to compute $\mathrm{Ext}^i_{\mathbf{Z}[\pi\times\pi]}(\mathbf{Z}[\pi] \otimes_{\mathbf{Z}} M, N)$. By Lemma 5.3, we have an isomorphism

$$\Phi : \mathrm{Hom}_{\mathbf{Z}[\pi\times\pi]}(\mathbf{Z}[\pi] \otimes_{\mathbf{Z}} M, J_i) \to \mathrm{Hom}_{\mathbf{Z}[\pi]}(\tilde{M}, \tilde{J}_i), \quad i = 0, 1, \dots$$

The statement of Lemma 5.4 follows once we show that each module $\tilde{J}_i$ is injective with respect to the conjugate action of $\mathbf{Z}[\pi]$.

Consider an injective $\mathbf{Z}[\pi \times \pi]$-module J, two $\mathbf{Z}[\pi]$-modules $X \subset Y$ and a $\mathbf{Z}[\pi]$-homomorphism $f : X \to \tilde{J}$ which needs to be extended onto Y. Note that $\mathbf{Z}[\pi \times \pi]$ is free when viewed as a right $\mathbf{Z}[\pi]$-module where the right action is given by $(g, h) \cdot k = (gk, hk)$ for $(g, h) \in \pi \times \pi$ and $k \in \pi$. Hence we obtain the $\mathbf{Z}[\pi \times \pi]$-modules

$$\mathbf{Z}[\pi \times \pi] \otimes_{\mathbf{Z}[\pi]} X \hookrightarrow \mathbf{Z}[\pi \times \pi] \otimes_{\mathbf{Z}[\pi]} Y$$

and the homomorphism $f : X \to \tilde{J}$ determines

$$f' : \mathbf{Z}[\pi \times \pi] \otimes_{\mathbf{Z}[\pi]} X \to J$$

by the formula: $f'((g, h) \otimes x) = gf(x)h^{-1}$, where $g, h \in \pi$ and $x \in X$. It is obvious that f' is well-defined and is a $\mathbf{Z}[\pi \times \pi]$-homomorphism. Since J is $\mathbf{Z}[\pi \times \pi]$-injective, there is a $\mathbf{Z}[\pi \times \pi]$-extension

$$f'' : \mathbf{Z}[\pi \times \pi] \otimes_{\mathbf{Z}[\pi]} Y \to J.$$

The restriction of f'' onto $Y = (e, e) \otimes Y$ is a $\mathbf{Z}[\pi]$-homomorphism $Y \to \tilde{J}$ extending f. Hence $\tilde{J}$ is injective. This completes the proof. ☐

Proof of Lemma 5.2. Consider the short exact sequence of $\pi \times \pi$-modules

$$0 \to I \to \mathbf{Z}[\pi] \xrightarrow{\epsilon} \mathbf{Z} \to 0$$

and the associated long exact sequence

$$\dots \to \mathrm{Hom}_{\mathbf{Z}[\pi\times\pi]}(I, A) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathbf{Z}[\pi\times\pi]}(\mathbf{Z}, A) \xrightarrow{\epsilon^*} \mathrm{Ext}^1_{\mathbf{Z}[\pi\times\pi]}(\mathbf{Z}[\pi], A) \to ...$$

The condition that $\alpha \in H^1(\pi\times\pi, A) = \mathrm{Ext}^1_{\mathbf{Z}[\pi\times\pi]}(\mathbf{Z}, A)$ is essential is equivalent to the requirement that α lies in the image of δ. By exactness, it is equivalent to $\epsilon^*(\alpha) = 0 \in \mathrm{Ext}^1_{\mathbf{Z}[\pi\times\pi]}(\mathbf{Z}[\pi], A)$. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathbf{Z}[\pi\times\pi]}(\mathbf{Z}, A) & \stackrel{\epsilon^*}{\to} & \mathrm{Ext}^1_{\mathbf{Z}[\pi\times\pi]}(\mathbf{Z}[\pi], A) \\ \downarrow = & & \simeq\downarrow \Phi \\ H^1(\pi\times\pi, A) & \stackrel{\Delta^*}{\to} & \mathrm{Ext}^1_{\mathbf{Z}[\pi]}(\mathbf{Z}, \tilde{A}) = H^1(\pi, \tilde{A}). \end{array} \tag{22}$$

Here $\Delta : \pi \to \pi\times\pi$ is the diagonal. The isomorphism Φ is given by Lemma 5.4. The commutativity of the diagram follows from the explicit description of Φ. Thus we see that a cohomology class $\alpha \in H^1(\pi\times\pi, A)$ is essential if and only if $\Delta^*(\alpha) = 0 \in H^1(\pi, \tilde{A})$, i.e. if α is a zero-divisor. □

Corollary 5.5. *For any $\pi\times\pi$ module A one has an isomorphism*

$$\Gamma : H^i(\pi\times\pi, \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\pi], A)) \to H^i(\pi, \tilde{A}).$$

This isomorphism acts as follows:

$$v \mapsto \omega_*(v\,|\,\pi), \quad v \in H^i(\pi\times\pi, \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\pi], A))$$

where $\pi \subset \pi\times\pi$ is the diagonal subgroup and $\omega : \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\pi], A) \to A$ is the homomorphism $\omega(f) = f(e) \in A$. The symbol e denotes the unit element $e \in \pi$.

Proof. Consider a free resolution P_* of $\mathbf{Z}$ over $\mathbf{Z}[\pi\times\pi]$. Then

$$\mathrm{Hom}_{\mathbf{Z}[\pi\times\pi]}(P_*, \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\pi], A)) \simeq \mathrm{Hom}_{\mathbf{Z}[\pi\times\pi]}(P_* \otimes_{\mathbf{Z}} \mathbf{Z}[\pi], A) \simeq \mathrm{Hom}_{\mathbf{Z}[\pi]}(\tilde{P}_*, \tilde{A})$$

according to Lemma 5.3. Our statement now follows since $\tilde{P}_*$ is a free resolution of $\mathbf{Z}$ over $\mathbf{Z}[\pi]$. □

6. The case of an abelian group

Throughout this section we shall assume that the group π is abelian. We shall fully describe the essential cohomology classes in $H^n(\pi\times\pi, A)$.

First, we note that it makes sense to impose an additional condition on the $\mathbf{Z}[\pi\times\pi]$-module A.

For a $\mathbf{Z}[\pi\times\pi]$-module B let $B' \subset B$ denote the submodule $B' = \{b \in B; gb = bg \text{ for any } g \in \pi\}$. Any $\mathbf{Z}[\pi\times\pi]$-homomorphism $\mu : A \to B$ restricts to a homomorphism $\mu : A' \to B'$.

Clearly, $I' = I$ and similarly $(I^n)' = I^n$. Hence any $\mathbf{Z}[\pi\times\pi]$-homomorphism $\mu : I^n \to A$ takes values in the submodule $A' \subset A$. Hence discussing essential cohomology classes $\alpha \in H^n(\pi\times\pi, A)$ we may assume that $A' = A$.

Consider the map

$$\phi : \pi\times\pi \to \pi, \quad \text{where} \quad \phi(x, y) = xy^{-1}. \tag{23}$$

It is a group homomorphism (since π is abelian). Besides, let A be a $\mathbf{Z}[\pi\times\pi]$-module with $A' = A$. Then there exists a unique $\mathbf{Z}[\pi]$-module B such that $A = \phi^*(B)$.

Theorem 6. *Assume that the group π is abelian. Let B be a $\mathbf{Z}[\pi]$-module and let $\alpha \in H^n(\pi\times\pi, \phi^*(B))$ be a cohomology class. Then α is essential if and only if $\alpha = \phi^*(\beta)$ for some $\beta \in H^n(\pi, B)$.*

It follows from Theorem 6 that there are no nonzero essential cohomology classes $\alpha \in H^n(\pi\times\pi, A)$ with $n > \mathrm{cd}(\pi)$. Moreover, we see that if $\mathrm{cd}(\pi) < n \leq \mathrm{cd}(\pi\times\pi)$ then any cohomology class $\alpha \in H^n(\pi\times\pi, A)$ is a zero-divisor which is not essential.

Proof. Assume that $\alpha \in H^n(\pi\times\pi, \phi^*(B))$ is such that $\alpha = \phi^*(\beta)$ where $\beta \in H^n(\pi, B)$. We want to show that α is essential. By Theorem 5 there exists a $\mathbf{Z}[\pi]$-homomorphism $\mu : I^n \to B$ such that $\mu_*(\mathfrak{b}^n) = \beta$ where $\mathfrak{b} \in H^1(\pi, I)$ is the Berstein - Schwarz class. Note that $\phi^*(I) = I$ and

$$\phi^*(\mathfrak{b}) = \mathfrak{v} \tag{24}$$

where $\mathfrak{v} \in H^1(\pi\times\pi, I)$ is the canonical class. To prove (24) we consider two subgroups $G_1 = \pi\times 1 \subset \pi\times\pi$ and $G_2 = \Delta_\pi \subset \pi\times\pi$ and since $\pi\times\pi \simeq G_1\times G_2$ we can view the Eilenberg-MacLane space $K(\pi\times\pi, 1)$ as the product $K(G_1, 1)\times K(G_2, 1)$. The restriction of the classes $\phi^*(\mathfrak{b})$ and $\mathfrak{v}$ onto G_1 coincide (as follows from (8) and from the definition of ϕ). On the other hand, the restriction of the classes $\phi^*(\mathfrak{b})$ and $\mathfrak{v}$ onto G_2 are trivial (as follows from (9) and from the definition of ϕ). Now the equality (24) follows from the fact that the inclusion $K(G_1, 1)\vee K(G_2, 1) \to K(G_1, 1)\times K(G_2, 1)$ induces a monomorphism on 1-dimensional cohomology with any coefficients.

Consider the commutative diagram

$$\begin{array}{ccc} H^n(\pi, I^n) & \xrightarrow{\mu_*} & H^n(\pi, B) \\ \phi^*\downarrow & & \downarrow\phi^* \\ H^n(\pi\times\pi, I^n) & \xrightarrow{\mu_*} & H^n(\pi\times\pi, \phi^*(B)). \end{array} \tag{25}$$

The upper left group contains the power $\mathfrak{b}^n$ of the Berstein - Schwarz class which is mapped onto $\beta = \mu_*(\mathfrak{b}^n)$ and $\phi^*(\beta) = \alpha$. Moving in the other direction we find $\alpha = \mu_*(\phi^*(\mathfrak{b}^n)) = \mu_*(\mathfrak{v}^n)$, i.e. α is essential.

To prove the inverse statement, assume that a cohomology class

$$\alpha \in H^n(\pi\times\pi, \phi^*(B))$$

is essential, i.e. $\alpha = \mu_*(\mathfrak{v}^n)$ for a $\mathbf{Z}[\pi\times\pi]$-homomorphism $\mu : I^n \to \phi^*(B)$. We may also view μ as a $\mathbf{Z}[\pi]$-homomorphism $I^n \to B$ which leads to the commutative diagram (25). Using (24) we find that $\phi^*(\mu_*(\mathfrak{b}^n)) = \mu_*(\phi^*(\mathfrak{b}^n)) = \mu_*(\mathfrak{v}^n) = \alpha$. Hence we see that $\alpha = \phi^*(\beta)$ where $\beta = \mu_*(\mathfrak{b}^n)$. ∎

If we wish to be specific, let $\pi = \mathbf{Z}^N$ and consider the trivial coefficient system $A = \mathbf{Z}$. Then N is the highest dimension in which essential cohomology classes

$$\alpha \in H^N(\mathbf{Z}^N \times \mathbf{Z}^N; \mathbf{Z})$$

exist. Due to Theorem 6, all N-dimensional essential cohomology classes are integral multiples of a single class which we are going to describe.

Denote by $x_1, \dots, x_N \in H^1(\mathbf{Z}^N, \mathbf{Z})$ a set of generators. Then each class

$$\alpha_i = x_i \otimes 1 - 1 \otimes x_i \in H^1(\mathbf{Z}^N \times \mathbf{Z}^N; \mathbf{Z}), \quad i = 1, \dots, N,$$

is a zero-divisor and hence is essential by Lemma 5.2. Their product

$$\alpha = \alpha_1 \cup \alpha_2 \cup \dots \cup \alpha_N \in H^N(\mathbf{Z}^N \times \mathbf{Z}^N; \mathbf{Z})$$

is essential as a product of essential classes. We may write α as the sum of 2^N terms

$$\alpha = (-1)^N \cdot \sum_K (-1)^{|K|} \cdot x_K \otimes x_{K^c}$$

where $K \subset \{1, 2, \dots, N\}$ runs over all subsets of the index set and K^c denotes the complement of K. For $K = \{i_1, i_2, \dots, i_k\}$ with $i_1 < i_2 < \dots < i_k$ the symbol x_K stands for the product $x_{i_1} x_{i_2} \dots x_{i_k}$.

7. The spectral sequence

In this and in the subsequent sections we abandon the assumption that π is abelian and return to the general case, i.e. we consider an arbitrary discrete group π.

Let A be a left $\mathbf{Z}[\pi \times \pi]$-module. We shall describe an exact couple and a spectral sequence which will allow us to find a sequence of obstructions for a cohomology class $\alpha \in H^*(\pi \times \pi, A)$ to be essential.

We introduce the following notations:

$$E_0^{rs} = \mathrm{Ext}^r_{\mathbf{Z}[\pi \times \pi]}(\mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^s, A) \quad \text{and} \quad D_0^{rs} = \mathrm{Ext}^r_{\mathbf{Z}[\pi \times \pi]}(I^s, A).$$

The long exact sequence associated to the short exact sequence (11) can be written in the form

$$\dots \to E_0^{rs} \xrightarrow{k_0} D_0^{r,s+1} \xrightarrow{i_0} D_0^{r+1,s} \xrightarrow{j_0} E_0^{r+1,s} \to \dots \tag{26}$$

Here $i_0 : D_0^{rs} \to D_0^{r+1,s-1}$ is the connecting homomorphism

$$\mathrm{Ext}^r_{\mathbf{Z}[\pi \times \pi]}(I^s, A) \to \mathrm{Ext}^{r+1}_{\mathbf{Z}[\pi \times \pi]}(I^{s-1}, A) \tag{27}$$

corresponding to the exact sequence (11). Note that

$$D_0^{n,0} = H^n(\pi \times \pi, A) \quad \text{and} \quad D_0^{0,n} = \mathrm{Hom}_{\mathbf{Z}[\pi \times \pi]}(I^n, A). \tag{28}$$

Lemma 7.1. *The set of essential cohomology classes in $H^n(\pi \times \pi, A)$ coincides with the image of the composition of n maps i_0:*

$$\mathrm{Hom}_{\mathbf{Z}[\pi \times \pi]}(I^n, A) = D_0^{0,n} \xrightarrow{i_0} D_0^{1,n-1} \xrightarrow{i_0} \dots \xrightarrow{i_0} D_0^{n,0} = H^n(\pi \times \pi, A). \tag{29}$$

Proof. Applying the technique described in [25], chapter IV, §9, we obtain that the image of a homomorphism $f \in \mathrm{Hom}_{\mathbf{Z}[\pi\times\pi]}(I^n, A)$ under the composition i_0^n is an element of $\mathrm{Ext}^n_{\mathbf{Z}[\pi\times\pi]}(\mathbf{Z}, A)$ represented by the exact sequence

$$0 \to A \to X_f \to \mathbf{Z}[\pi]\otimes_{\mathbf{Z}} I^{n-2} \to \mathbf{Z}[\pi]\otimes_{\mathbf{Z}} I^{n-3} \to \cdots \to \mathbf{Z}[\pi] \to \mathbf{Z} \to 0$$

where X_f appears in the push-out diagram

$$\begin{array}{ccc} I^n & \to & \mathbf{Z}[\pi]\otimes_{\mathbf{Z}} I^{n-1} \\ \downarrow f & & \downarrow \\ A & \to & X_f. \end{array}$$

Using Lemma 3.1 we see that the same exact sequence represents the element $f_*(\mathfrak{v}^n)$. $\square$

A different proof of Lemma 7.1 will be given later in this section.

The exact sequences (26) can be organised into a bigraded exact couple as follows. Denote

$$E_0 = \bigoplus_{r,s\geq 0} E_0^{rs} = \bigoplus_{r,s\geq 0} \mathrm{Ext}^r_{\pi\times\pi}(\mathbf{Z}[\pi]\otimes_{\mathbf{Z}} I^s, A), \tag{30}$$

and

$$D_0 = \bigoplus_{r,s\geq 0} D_0^{rs} = \bigoplus_{r,s\geq 0} \mathrm{Ext}^r_{\pi\times\pi}(I^s, A). \tag{31}$$

The exact sequence (26) becomes an exact couple

$$\begin{array}{ccccc} D_0 & & \xrightarrow{i_0} & & D_0 \\ & k_0 \nwarrow & & \swarrow j_0 & \\ & & E_0 & & \end{array}$$

Here the homomorphism i_0 has bidegree $(1, -1)$, the homomorphism k_0 has bidegree (0,1), and the homomorphism j_0 has bidegree $(0, 0)$. Applying the general formalism of exact couples, we may construct the p-th derived couple

$$\begin{array}{ccccc} D_p & & \xrightarrow{i_p} & & D_p \\ & k_p \nwarrow & & \swarrow j_p & \\ & & E_p & & \end{array}$$

where $p = 0, 1, \ldots$. The module D_p^{rs} is defined as

$$D_p^{rs} = \mathrm{Im}[i_{p-1} : D_{p-1}^{r-1,s+1} \to D_{p-1}^{rs}] = \mathrm{Im}[i_0 \circ \cdots \circ i_0 : D_0^{r-p,s+p} \to D_0^{r,s}].$$

and

$$E_p^{*,*} = H(E_{p-1}^{*,*}, d_{p-1})$$

is the homology of the previous term with respect to the differential $d_{p-1} = j_{p-1} \circ k_{p-1}$. The degrees are as follows:

$$\begin{aligned} \deg j_p &= (-p, p), \\ \deg i_p &= (1, -1), \\ \deg k_p &= (0, 1), \\ \deg d_p &= (-p, p+1). \end{aligned}$$

Using this spectral sequence we can express the set of essential classes as follows:

Corollary 7.2. *The group $D_n^{n,0} \subset H^n(\pi \times \pi, A) = D_0^{n,0}$ coincides with the set of all essential cohomology classes in $H^n(\pi \times \pi, A)$.*

Proof. This is equivalent to Lemma 7.1. $\square$

We want to express the homomorphism $i_0 : D_0^{r,s+1} \to D_0^{r+1,s}$ with $s \geq 0$ through the canonical class $\mathfrak{v} \in H^1(\pi \times \pi, I)$. This will be used to give a different proof of Lemma 26 and will have some other interesting applications. According to the definition, i_0 is the connecting homomorphism

$$i_0 : \operatorname{Ext}^r_{\mathbf{Z}[\pi\times\pi]}(I^{s+1}, A) \to \operatorname{Ext}^{r+1}_{\mathbf{Z}[\pi\times\pi]}(I^s, A)$$

corresponding to the short exact sequence (11). Note that

$$D_0^{r,s} = \operatorname{Ext}^r_{\mathbf{Z}[\pi\times\pi]}(I^s, A) = H^r(\pi \times \pi, \operatorname{Hom}_{\mathbf{Z}}(I^s, A)),$$

see [4], chapter III, Proposition 2.2. Under this identification i_0 turns into the Bockstein homomorphism

$$\beta : H^r(\pi \times \pi; \operatorname{Hom}_{\mathbf{Z}}(I^{s+1}, A)) \to H^{r+1}(\pi \times \pi; \operatorname{Hom}_{\mathbf{Z}}(I^s, A)) \tag{32}$$

corresponding to the short exact sequence of $\mathbf{Z}[\pi \times \pi]$-modules

$$0 \to \operatorname{Hom}_{\mathbf{Z}}(I^s, A) \to \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^s, A) \to \operatorname{Hom}_{\mathbf{Z}}(I^{s+1}, A) \to 0. \tag{33}$$

The sequence (33) is obtained by applying the functor $\operatorname{Hom}_{\mathbf{Z}}(\,\cdot\,, A)$ to the exact sequence (11) (note that (11) splits over $\mathbf{Z}$).

Let

$$\mathbf{ev} : I \otimes_{\mathbf{Z}} \operatorname{Hom}_{\mathbf{Z}}(I^{s+1}, A) \to \operatorname{Hom}_{\mathbf{Z}}(I^s, A)$$

denote the homomorphism given by

$$x_0 \otimes f \mapsto (x_1 \otimes \cdots \otimes x_s \mapsto f(x_0 \otimes x_1 \otimes \cdots \otimes x_s))$$

for $f \in \operatorname{Hom}_{\mathbf{Z}}(I^s, A)$ and $x_i \in I$ for $i = 0, 1, \ldots, s$.

Proposition 7.3. *For any cohomology class $u \in H^r(\pi \times \pi, \operatorname{Hom}_{\mathbf{Z}}(I^{s+1}, A))$ one has*

$$\beta(u) = -\mathbf{ev}_*(\mathfrak{v} \cup u), \tag{34}$$

where $\mathfrak{v} \in H^1(\pi \times \pi, I)$ denotes the canonical class.

Proof. Using [4], chapter V, property (3.3) and Lemma 5 from [7], we obtain $\delta(u) = \mathfrak{v} \cup u$, where

$$\delta : H^r(\pi \times \pi; \mathrm{Hom}_{\mathbf{Z}}(I^{s+1}, A)) \to H^{r+1}(\pi \times \pi; I \otimes \mathrm{Hom}_{\mathbf{Z}}(I^s, A))$$

is the Bockstein homomorphism associated with the short exact coefficient sequence

$$0 \to I \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(I^{s+1}, A) \to \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(I^{s+1}, A) \overset{\epsilon \otimes \mathrm{id}}{\to} \mathrm{Hom}_{\mathbf{Z}}(I^{s+1}, A) \to 0\ .$$

The latter sequence is obtained by tensoring (4) with $\mathrm{Hom}_{\mathbf{Z}}(I^{s+1}, A)$ over $\mathbf{Z}$. To prove Proposition 7.3 it is enough to show that $\beta = -\,\mathbf{ev}_* \circ \delta$. Having this goal in mind, we denote by

$$F : \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(I^{s+1}, A) \to \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^s, A)$$

the homomorphism which extends $\mathbf{Z}$-linearly the following map

$$F(x \otimes f)(z \otimes y) = f((z - x) \otimes y)$$

for $x, z \in \pi,\ y \in I^s$ and $f \in \mathrm{Hom}_{\mathbf{Z}}(I^{s+1}, A)$. We compute:

$$\begin{aligned} & F((g,h) \cdot (x \otimes f))(z \otimes y) = F(gxh^{-1} \otimes (g,h)f)(z \otimes y) \\ = \quad & ((g,h)f)((z - gxh^{-1}) \otimes y) = gf((g^{-1}zh - x) \otimes g^{-1}yh)h^{-1} \\ = \quad & gF(x \otimes f)(g^{-1}zh \otimes g^{-1}yh)h^{-1} = ((g,h) \cdot F(x \otimes f))(z \otimes y)\ . \end{aligned}$$

Hence F is a $\mathbf{Z}[\pi \times \pi]$-homomorphism. Next we claim that the following diagram with exact rows commutes:

$$\begin{array}{ccccccccc} 0 & \to & I \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(I^{s+1}, A) & \overset{i \otimes \mathrm{id}}{\longrightarrow} & \mathbf{Z}[\pi] \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(I^{s+1}, A) & \overset{\epsilon \otimes \mathrm{id}}{\longrightarrow} & \mathrm{Hom}_{\mathbf{Z}}(I^{s+1}, A) & \to & 0 \\ & & \downarrow \mathbf{ev} & & \downarrow F & & \downarrow \mathrm{id} & & \\ 0 & \to & \mathrm{Hom}_{\mathbf{Z}}(I^s, A) & \overset{-\epsilon^*}{\longrightarrow} & \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[\pi] \otimes_{\mathbf{Z}} I^s, A) & \overset{i^*}{\longrightarrow} & \mathrm{Hom}_{\mathbf{Z}}(I^{s+1}, A) & \to & 0 \end{array}$$

Indeed, we compute for $g, h \in \pi$, $y \in I^s$ and $f \in \mathrm{Hom}_{\mathbf{Z}}(I^{s+1}, A)$,

$$(-\epsilon^* \circ \mathbf{ev})((g-1) \otimes f)(h \otimes y) = -\epsilon(h) f((g-1) \otimes y) = -f((g-1) \otimes y)$$

and

$$\begin{aligned} & (F \circ (i \otimes \mathrm{id})((g-1) \otimes f)(h \otimes y) \\ = \quad & (F(g \otimes f))(h \otimes y) - (F(1 \otimes f))(h \otimes y) \\ = \quad & f((h-g) \otimes y) - f((h-1) \otimes y) \\ = \quad & -f((g-1) \otimes y)\ . \end{aligned}$$

Hence, we see that the left square in the above diagram commutes. We further observe that for all $g, h \in \pi$, $y \in I^s$ and $f \in \mathrm{Hom}_{\mathbf{Z}}(I^{s+1}, A)$ one has

$$\begin{aligned} & ((i^* \circ F)(g \otimes f))((h-1) \otimes y) = F(g \otimes f)(h \otimes y) - F(g \otimes f)(1 \otimes y) \\ = \quad & f((h-g) \otimes y) - f((1-g) \otimes y) = f((h-1) \otimes y) \\ = \quad & \epsilon(g) f((h-1) \otimes y) = ((\epsilon \otimes \mathrm{id})(g \otimes f))((h-1) \otimes y), \end{aligned}$$

and hence the right square of the diagram commutes as well.

The commutativity of the above diagram implies that the Bockstein homomorphisms satisfy

$$\mathbf{ev}_* \circ \delta = \beta' \circ \mathrm{id} = \beta' \ ,$$

where β' denotes the Bockstein homomorphism of the bottom row exact sequence. Since this sequence coincides with the sequence associated with β up to a sign change in the first map, one derives from the snake lemma that $\beta' = -\beta$. This completes the proof. $\square$

Corollary 7.4. *Let $\alpha \in H^n(\pi \times \pi, A)$ be a cohomology class and let $k = 1, 2, \ldots, n-1$ be an integer. Then the following conditions are equivalent:*

(1) α lies in $D_k^{n,0}$.

(2) $\alpha = \psi_*(\mathfrak{v}^k \cup u)$ for a cohomology class $u \in H^{n-k}(\pi \times \pi, \mathrm{Hom}_{\mathbf{Z}}(I^k, A))$ where

$$\psi : I^k \otimes \mathrm{Hom}_{\mathbf{Z}}(I^k, A) \to A$$

is the coefficient pairing

$$\psi(x_1 \otimes \cdots \otimes x_k \otimes f) = f(x_k \otimes x_{k-1} \otimes \cdots \otimes x_1). \tag{35}$$

Proof. The condition $\alpha \in D_k^{n,0}$ means that $\alpha = i_0^k(u)$ for some $u \in D_0^{n-k,k}$. We know that $D_0^{n-k,k} = \mathrm{Ext}^{n-k}_{\mathbf{Z}[\pi\times\pi]}(I^k, A) = H^{n-k}(\pi \times \pi, \mathrm{Hom}_{\mathbf{Z}}(I^k, A))$. Our statement follows by applying iteratively Proposition 7.3. $\square$

We may use Proposition 7.3 to give another proof of Lemma 7.1. By Corollary 7.4, classes $\alpha \in D_n^{n,0}$ are characterised by the property $\alpha = \mathfrak{v}^n \cup u$ where the cup product is given with respect to the pairing $I^n \otimes \mathrm{Hom}_{\mathbf{Z}}(I^n, A) \to A$ given by the formula (35) for some $u \in H^0(\pi \times \pi, \mathrm{Hom}_{\mathbf{Z}}(I^n, A)) = \mathrm{Hom}_{\mathbf{Z}[\pi\times\pi]}(I^n, A)$. Thus u is a $\mathbf{Z}[\pi \times \pi]$-homomorphism $I^n \to A$ and applying the definition of the cup product (see [4], chapter V, §3) we see that $\alpha \in D_n^{n,0}$ if and only if $\alpha = \phi_*(\mathfrak{v}^n)$ for a $\mathbf{Z}[\pi \times \pi]$-homomorphism $\phi : I^n \to A$.

Corollary 7.5. *If for some $\mathbf{Z}[\pi \times \pi]$-module A and for an integer k the module $D_k^{n,0}$ is nonzero then $\mathsf{TC}(K(\pi, 1)) \geq k + 1$.*

Proof. Using Corollary 7.4 we see that $D_k^{n,0} \neq 0$ then for $\alpha \in D_k^{n,0}$, $\alpha \neq 0$, we have $\alpha = \psi_*(\mathfrak{v}^k \cup u)$ and hence $\mathfrak{v}^k \neq 0$. Since $\mathfrak{v}$ is a zero-divisor, our statement follows from [17], Corollary 4.40. $\square$

Using the spectral sequence we may describe a complete set of n obstructions for a cohomology class $\alpha \in H^n(\pi \times \pi, A) = D_0^{n,0}$ to be essential. We shall apply Lemma 7.1 and act inductively. The class α is essential if it lies in the image of the composition of n maps i_0. For this to happen we first need to guarantee that α lies in the image of the last map $i_0 : D_0^{n-1,1} \to D_0^{n,0}$. Because of the exact sequence

$$\cdots \to D_0^{n-1,1} \xrightarrow{i_0} D_0^{n,0} \xrightarrow{j_0} E_0^{n,0} \xrightarrow{k_0} \ldots$$

we see that α lies in the image of i_0 if and only if

$$j_0(\alpha) = 0 \in E_0^{n,0}. \tag{36}$$

We have the commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^n_{\mathbf{Z}[\pi\times\pi]}(\mathbf{Z}, A) & \stackrel{j_0}{\to} & \mathrm{Ext}^n_{\mathbf{Z}[\pi\times\pi]}(\mathbf{Z}[\pi], A) \\
\downarrow = & & \Phi \downarrow \simeq \\
H^n(\pi\times\pi, A) & \stackrel{r^*}{\to} & H^n(\pi, \tilde{A}).
\end{array}$$

Here Φ is the isomorphism of Lemma 5.4 and $j_0 = \epsilon^*$ is the homomorphism induced by the augmentation; the homomorphism r^* is induced by the inclusion $r : \pi \to \pi\times\pi$ of the diagonal subgroup. We see that the class α lies in the image of i_0 if and only if it is a zero divisor, i.e. $r^*(\alpha) = 0$.

To describe the second obstruction let us assume that $\alpha \in H^n(\pi\times\pi, A)$ is a zero-divisor, i.e. (36) is satisfied. Then $\alpha \in D_1^{n,0}$. One has $\alpha \in D_2^{n,0}$ if and only if

$$j_1(\alpha) = 0 \in E_1^{n-1,1}. \tag{37}$$

This follows from the exact sequence

$$\cdots \to D_1^{n-1,1} \stackrel{i_1}{\to} D_1^{n,0} \stackrel{j_1}{\to} E_1^{n-1,1} \stackrel{k_1}{\to} D_1^{n-1,2} \to \dots$$

where i_1 is the restriction of i_0 onto $D_1 \subset D_0$.

Continuing these arguments and using the exact sequences

$$\cdots \to D_p^{n-1,1} \stackrel{i_p}{\to} D_p^{n,0} \stackrel{j_p}{\to} E_p^{n-p,p} \stackrel{k_p}{\to} D_p^{n-p,p+1} \to \dots$$

we arrive at the following conclusion:

Corollary 7.6. *Let k and n be integers with $0 < k \le n$.*

(1) A cohomology class $\alpha \in H^n(\pi\times\pi, A)$ lies in the group $D_k^{n,0} = \mathrm{Im}[i_0^k : D_0^{n-k,k} \to D_0^{n,0}]$ if and only if the following k obstructions

$$j_s(\alpha) \in E_s^{n-s,s}, \quad \textit{where} \quad s = 0, 1, \dots, k-1, \tag{38}$$

vanish.

(2) The condition $j_0(\alpha) = 0$ is equivalent for α to be a zero-divisor.

(3) Each obstruction $j_s(\alpha)$ is defined once the previous obstruction $j_{s-1}(\alpha)$ vanishes.

(4) The triviality of all obstructions $j_0(\alpha), j_1(\alpha), \dots, j_{n-1}(\alpha)$ is necessary and sufficient for the cohomology class α to be essential.

Figure 1 shows the locations of the obstructions $j_k(\alpha) \in E_k^{n-k,k}$.

8. Computing the term $E_0^{r,s}$ for $s \ge 1$

In this section we compute the initial term $E_0^{r,s}$ of the spectral sequence. Using Lemma 21 we find

$$\begin{aligned} E_0^{r,s} &= \mathrm{Ext}^r_{\mathbf{Z}[\pi\times\pi]}(\mathbf{Z}[\pi]\otimes_{\mathbf{Z}} I^s, A) \\ &\simeq \mathrm{Ext}^r_{\mathbf{Z}[\pi]}(\tilde{I}^s, \tilde{A}) \end{aligned} \tag{39}$$

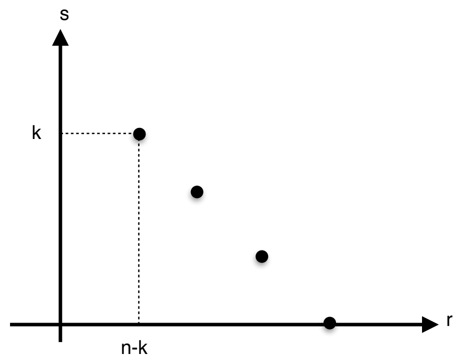

FIGURE 1. The groups in the E-term of the spectral sequence containing the obstructions $j_k(\alpha)$.

where in the second line the tilde $\tilde{}$ above the corresponding modules means that we consider these modules with respect to the conjugation action, i.e. $g \cdot a = gag^{-1}$ for $a \in A$ and $g \in \pi$. We exploit below the simple structure of the module $\tilde{I}^s$ where $s \geq 1$ and compute explicitly the E_0-term.

For $s \geq 1$ consider the action of π on the Cartesian power $\pi^s = \pi \times \pi \times \cdots \times \pi$ (s times) via conjugation, i.e. $g \cdot (g_1, \dots, g_s) = (gg_1g^{-1}, \dots, gg_sg^{-1})$ for $g, g_1, \dots g_s \in \pi$. The orbits of this action are joint conjugacy classes of s-tuples of elements of π. We denote by $\mathcal{C}_{\pi^s}$ the set of orbits and let $\mathcal{C}'_{\pi^s} \subset \mathcal{C}_{\pi^s}$ denote the set of orbits of nontrivial elements, i.e. such that $g_i \neq 1$ for all $i = 1, \dots, s$.

Let $C \in \mathcal{C}'_{\pi^s}$ be an orbit. The isotropy subgroup $N_C \subset \pi$ of an s-tuple $(g_1, \dots, g_s) \in C \subset \pi^s$ is the intersection of the centralisers of the elements $g_1, \dots, g_s$. The subgroup N_C, viewed up to conjugation, depends only on the orbit C.

Theorem 7. *For any left $\mathbf{Z}[\pi \times \pi]$-module A and for integers $r \geq 0$ and $s \geq 1$ one has*

$$E_0^{r,s} \simeq \prod_{C \in \mathcal{C}'_{\pi^s}} H^r(N_C, A \,|\, N_C). \tag{40}$$

Here $A \,|\, N_C$ denotes A viewed as $\mathbf{Z}[N_C]$-module with $N_C \subset \pi = \Delta_\pi \subset \pi \times \pi$.

Proof. For any $C \in \mathcal{C}'_{\pi^s}$ consider the set $J_C \subset \tilde{I}^s$ generated over $\mathbf{Z}$ by the tensors of the form

$$(g_1 - 1) \otimes \cdots \otimes (g_s - 1) \tag{41}$$

for all $(g_1, \dots, g_s) \in C$. It is clear that J_C is a $\mathbf{Z}[\pi]$-submodule of $\tilde{I}^s$ (since we consider the conjugation action). Moreover, we observe that

$$\tilde{I}^s = \bigoplus_{C \in \mathcal{C}'_{\pi^s}} J_C. \tag{42}$$

Indeed, the elements $g-1$ with various $g \in \pi^*$ (where we denote $\pi^* = \pi - \{1\}$) form a free $\mathbf{Z}$-basis of I; therefore elements of the form $(g_1 - 1)\otimes \cdots \otimes (g_s - 1)$ with all possible $g_1, \dots, g_s \in \pi^*$ form a free $\mathbf{Z}$-basis of I^s. The formula (42) is now obvious.

For $C \in \mathcal{C}'_{\pi^s}$ let $\mathbf{Z}[C]$ denote the free abelian group generated by C. Since π acts on C, the group $\mathbf{Z}[C]$ is naturally a left $\mathbf{Z}[\pi]$-module which is isomorphic to J_C via the isomorphism

$$(g_1, \dots, g_s) \mapsto (g_1 - 1)\otimes \cdots \otimes (g_s - 1),$$

where $(g_1, \dots, g_s) \in C$. For a left $\mathbf{Z}[\pi]$-module B we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Z}[\pi]}(J_C, B) &= \mathrm{Hom}_{\mathbf{Z}[\pi]}(\mathbf{Z}[C], B|_{N_C}) \\ &= \mathrm{Hom}_{N_C}(\mathbf{Z}, B) \\ &= H^0(N_C, B|_{N_C}). \end{aligned}$$

Here we used the fact that the action of π on C is transitive and hence a $\mathbf{Z}[\pi]$-homomorphism $f : \mathbf{Z}[C] \to B$ is uniquely determined by one of its values $f(c)$ where $c \in C$.

Consider a free resolution

$$P_* : \quad \cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbf{Z} \to 0$$

of $\mathbf{Z}$ over $\mathbf{Z}[\pi]$. Since $\mathbf{Z}[C]$ is free as an abelian group we have the exact sequence

$$\cdots \to \mathbf{Z}[C] \otimes_{\mathbf{Z}} P_n \to \mathbf{Z}[C] \otimes_{\mathbf{Z}} P_{n-1} \to \cdots \to \mathbf{Z}[C] \otimes_{\mathbf{Z}} P_0 \to \mathbf{Z}[C] \to 0 \tag{43}$$

of $\mathbf{Z}[\pi]$-modules. It is easy to see that each module $\mathbf{Z}[C] \otimes_{\mathbf{Z}} P_n$ (equipped with the diagonal action) is free as a $\mathbf{Z}[\pi]$-module (see [4], chapter III, Corollary 5.7). Thus we see that (43) is a free $\mathbf{Z}[\pi]$-resolution of $\mathbf{Z}[C]$ and we may use it to compute $\mathrm{Ext}^r_{\mathbf{Z}[\pi]}(\mathbf{Z}[C], B)$. We have

$$\mathrm{Hom}_{\mathbf{Z}[\pi]}(\mathbf{Z}[C] \otimes_{\mathbf{Z}} P_n, B) = \mathrm{Hom}_{\mathbf{Z}[\pi]}(\mathbf{Z}[C], \mathrm{Hom}_{\mathbf{Z}}(P_n, B)) = \mathrm{Hom}_{\mathbf{Z}[N_C]}(P_n, B|N_C).$$

Thus we see that the complex $\mathrm{Hom}_{\mathbf{Z}[\pi]}(\mathbf{Z}[C] \otimes P_*, B)$ which computes $\mathrm{Ext}^r_{\mathbf{Z}[\pi]}(J_C, B)$, coincides with the complex

$$\mathrm{Hom}_{\mathbf{Z}[N_C]}(P_*|_{N_C}, B|N_C). \tag{44}$$

Since P_n is free as a $\mathbf{Z}[N_C]$-module, we see that the cohomology of the complex (44) equals $H^r(N_C, B \,|\, N_C)$. Thus we obtain isomorphisms

$$\mathrm{Ext}^r_{\mathbf{Z}[\pi]}(J_C, B) \simeq \mathrm{Ext}^r_{\mathbf{Z}[\pi]}(\mathbf{Z}[C], B) \simeq H^r(N_C; B|N_C). \tag{45}$$

Combining the isomorphisms (39), (42) and (45) we obtain the isomorphism (40). □

Corollary 8.1. *Let π be a discrete torsion-free group such that the centraliser of any nontrivial element $g \in \pi$, $g \neq 1$ is infinite cyclic. Then*

$$E_0^{r,s} = 0$$

for all $r > 1$ and $s \geq 1$.

Proof. Applying Theorem 7 we see that each group N_C, where $C \in \mathcal{C}'_{\pi^s}$, is a subgroup of $\mathbf{Z}$ and hence it is either $\mathbf{Z}$ or trivial. The result now follows from (40) since we assume that $r > 1$. □

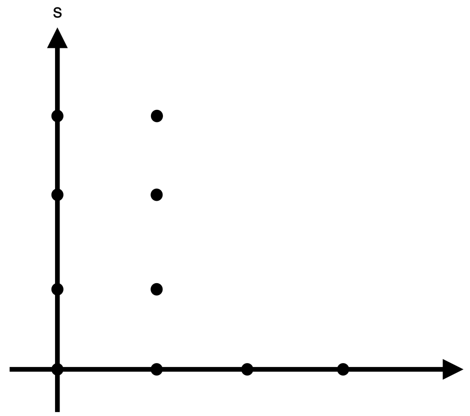

FIGURE 2. The nontrivial groups in the E_0-term of the spectral sequence.

Figure 2 shows potentially nontrivial groups in the E_0-term in the case when all centralisers of nontrivial elements are cyclic.

9. Proofs of Theorems 1, 2, 3 and 4

Proof of Theorem 3. Let $X = K(\pi, 1)$ where the group π satisfies our assumption that the centraliser of any nonzero element is infinite cyclic. Let $\alpha \in H^n(X\times X, A)$ be a zero-divisor. Here A is a local coefficient system over $X \times X$. By Corollary 7.6, statement (2), we have $j_0(\alpha) = 0$. Besides, applying Corollary 8.1 we see that the obstructions $j_s(\alpha) \in E_s^{n-s,s}$ vanish for $s = 1, 2, \dots, n-2$ since they lie in the trivial groups. Thus we obtain

$$\alpha \in D^{n,0}_{n-1}.$$

Next we apply Corollary 7.4 which gives

$$\alpha = \psi_*(\mathfrak{v}^{n-1} \cup u) \tag{46}$$

for some $u \in H^1(\pi \times \pi, \mathrm{Hom}_{\mathbf{Z}}(I^{n-1}, A))$ where ψ is given by (35).

To prove that $\mathrm{wgt}(\alpha) \geq n-1$ we observe that the canonical class $\mathfrak{v}$ has positive weight, $\mathrm{wgt}(\mathfrak{v}) \geq 1$, since it is a zero-divisor, see (9). Hence using (3) we obtain $\mathrm{wgt}(\mathfrak{v}^{n-1}) \geq n-1$. Let $f : Y \to X \times X$ be a continuous map as in Definition 2.1. Then

$$f^*(\alpha) = \psi_*(f^*(\mathfrak{v}^{n-1}) \cup f^*(u)) = 0$$

since $f^*(\mathfrak{v}^{n-1}) = 0$. This completes the proof. □

Proof of Theorem 2. Suppose that we are in the situation of Theorem 2, i.e. let X be an aspherical finite cell complex such whose fundamental group $\pi = \pi_1(X)$ has the properties (1) and (2). Denote $n = \mathrm{cd}(\pi \times \pi)$. We may find a local coefficient system A over $X \times X$ and a nonzero cohomology class $\alpha \in H^n(X \times X, A)$. Since $n > \mathrm{cd}(\pi)$ we obtain that α

is a zero-divisor. Next we apply Theorem 3 which implies that the weight of α satisfies $\mathrm{wgt}(\alpha) \geq n-1$. Thus we obtain that $\mathsf{TC}(X) \geq n$.

The inequality $\mathsf{TC}(X) \leq n+1$ follows from the Eilenberg - Ganea theorem and general dimensional upper bound for the topological complexity $\mathsf{TC}(X) \leq \dim(X \times X) + 1$.

Hence, $\mathsf{TC}(X)$ is either n or $n+1$. □

Proof of Theorem 4. As in the proof of Theorem 2 we may find a nonzero cohomology class $\alpha \in H^n(X \times X, A)$, where $n = \mathrm{cd}(\pi \times \pi)$, which is automatically a zero-divisor, since $n > \mathrm{cd}(\pi)$. Applying the arguments used in the proof of Theorem 3 we find that $\alpha = \psi_*(\mathfrak{v}^{n-1} \cup u)$, see (46), implying that $\mathfrak{v}^{n-1} \neq 0$.

□

Proof of Theorem 1. Let X be an aspherical finite cell complex with $\pi = \pi_1(X)$ hyperbolic. Then π is torsion-free and we may assume that $\pi \neq 1$ since in the simply connected case our statement is obvious. The centraliser

$$Z(g) = \{h \in \pi; hgh^{-1} = g\}$$

of any nontrivial element $g \in \pi$ is virtually cyclic, see [3], Corollary 3.10 in chapter III. It is well known that any torsion-free virtually cyclic group is cyclic. Thus, we see that the assumption (1) of Theorem 2 is satisfied.

Next we show that the assumption (2) of Theorem 2 is satisfied as well, i.e. $\mathrm{cd}(\pi \times \pi) > \mathrm{cd}(\pi)$.

We know that π has a finite $K(\pi, 1)$ and hence there exists a finite free resolution P_* of $\mathbf{Z}$ over $\mathbf{Z}[\pi]$. Here each $\mathbf{Z}[\pi]$-module P_i is finitely generated and free and P_i is nonzero only for finitely many i. Note that $P_* \otimes_{\mathbf{Z}} P_*$ is a free resolution of $\mathbf{Z}$ over $\mathbf{Z}[\pi \times \pi]$ (see [4], Proposition 1.1 in chapter V). For any two left $\mathbf{Z}[\pi]$-modules A_1, A_2 we have the natural isomorphism of chain complexes

$$\mathrm{Hom}_{\mathbf{Z}[\pi]}(P_*, A_1) \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}[\pi]}(P_*, A_2) \to \mathrm{Hom}_{\mathbf{Z}[\pi \times \pi]}(P_* \otimes_{\mathbf{Z}} P_*, A_1 \otimes_{\mathbf{Z}} A_2).$$

If at least one of these chain complexes is flat over $\mathbf{Z}$, the Künneth theorem is applicable and we obtain the monomorphism

$$H^j(\pi, A_1) \otimes_{\mathbf{Z}} H^{j'}(\pi, A_2) \to H^{j+j'}(\pi \times \pi, A \otimes_{\mathbf{Z}} A_2)$$

given by the cross-product $\alpha \otimes \alpha' \mapsto \alpha \times \alpha'$. We see that a certain cross-product $\alpha \times \alpha'$ is nonzero provided that the tensor product of the abelian groups $H^j(\pi, A_1) \otimes_{\mathbf{Z}} H^{j'}(\pi, A_2)$ is nonzero.

Let A_1, A_2 be left $\mathbf{Z}[\pi]$-modules chosen such that $H^n(\pi, A_1) \neq 0$, where $n = \mathrm{cd}(\pi)$, and $H^1(\pi, A_2) \simeq \mathbf{Z}$. We may take $A_1 = \mathbf{Z}[\pi]$, see [4], chapter VIII, Proposition 6.7. Hence $\mathrm{Hom}_{\mathbf{Z}[\pi]}(P_*, A_1)$ is a chain complex of free abelian groups. For some nonzero elements $\alpha \in H^n(\pi, A_1)$ and $\alpha' \in H^1(\pi, A_2)$ we shall have $\alpha \times \alpha' \neq 0$ since $H^n(\pi, A_1) \otimes_{\mathbf{Z}} H^1(\pi, A_2) \simeq H^n(\pi, A_1) \neq 0$. Thus, we obtain $\mathrm{cd}(\pi \times \pi) \geq n+1 = \mathrm{cd}(\pi) + 1$ as required.

We explain below how to construct the module A_2. Fix a non-unit element $g \in \pi$ and let $C = C_g \subset \pi$ denote the conjugacy class of g. Let $A_2 = \mathbf{Z}^C = \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[C], \mathbf{Z})$ denote the set of all functions $f : C \to \mathbf{Z}$; here $\mathbf{Z}[C]$ denotes the free abelian group generated by

C. Since π acts on C (via conjugation) we obtain the induced action of π on $\mathbf{Z}^C$, where $(g \cdot f)(h) = f(g^{-1}hg)$.

Let P_* denote a free resolution of $\mathbf{Z}$ over $\mathbf{Z}[\pi]$. We have the isomorphisms of chain complexes (which are similar to those used in the proof of Theorem 7)

$$\begin{aligned}\operatorname{Hom}_{\mathbf{Z}[\pi]}(P_*, A_2) &\simeq \operatorname{Hom}_{\mathbf{Z}[\pi]}(P_* \otimes_{\mathbf{Z}} \mathbf{Z}[C], \mathbf{Z}) \\ &\simeq \operatorname{Hom}_{\mathbf{Z}[\pi]}(\mathbf{Z}[C], \operatorname{Hom}_{\mathbf{Z}}(P_*, \mathbf{Z})) \\ &\simeq \operatorname{Hom}_{\mathbf{Z}[Z(g)]}(P_*, \mathbf{Z}).\end{aligned}$$

The complex P_* is a free resolution of $\mathbf{Z}$ over $\mathbf{Z}[Z(g)]$ by restriction. Thus, $H^i(\pi, A_2) \simeq H^i(Z(g), \mathbf{Z})$ for any i. Since we know that the centraliser $Z(g)$ is isomorphic to $\mathbf{Z}$ we have $H^1(\pi, A_2) \simeq \mathbf{Z}$.

This completes the proof. $\square$

10. An application

A symplectic manifolds (M, ω) is said to be *symplectically aspherical* [26] if

$$\int_{S^2} f^*\omega \;=\; 0 \tag{47}$$

for any continuous map $f : S^2 \to M$. It follows from the Stokes' theorem that a symplectic manifold which is aspherical in the usual sense is also symplectically aspherical.

Theorem 8. *Let (M, ω) be a closed $2n$-dimensional symplectically aspherical manifold. Suppose that the fundamental group $\pi = \pi_1(M)$ is of type FL and the centraliser of any nontrivial element of π is cyclic. Then the topological complexity $\mathsf{TC}(M)$ is either $4n$ or $4n + 1$.*

Proof. Our assumption on π being of type FL implies that there exists a finite cell complex $K = K(\pi, 1)$ (see [4], chapter VIII, Theorem 7.1). Let $g : M \to K$ be a map inducing an isomorphism of the fundamental groups. Let $u \in H^2(M, \mathbf{R})$ denote the class of the symplectic form. Then $u^n \neq 0$ since ω^n is a volume form on M.

Next we show that there exists a unique class $v \in H^2(K, \mathbf{R})$ with $g^*(v) = u$. With this goal in mind we consider the Hopf exact sequence (see [4], chapter II, Proposition 5.2)

$$\pi_2(M) \xrightarrow{h} H_2(M) \xrightarrow{g_*} H_2(K) \to 0$$

where h denotes the Hurewicz homomorphism and all homology groups are with integer coefficients. We may view u as the homomorphism $u_* : H_2(M) \to \mathbf{R}$; it vanishes on the image of h due to (47). Hence there exists a unique $v_* : H_2(K) \to \mathbf{R}$ with $u_* = v_* \circ g_*$ and this v_* is the desired cohomology class $v \in H^2(K, \mathbf{R})$.

Denote $\bar{u} = u \times 1 - 1 \times u \in H^2(M \times M, \mathbf{R})$ and also $\bar{v} = v \times 1 - 1 \times v \in H^2(K \times K, \mathbf{R})$. These classes are zero-divisors and $(g \times g)^*(\bar{v}) = \bar{u}$. Note that

$$\bar{u}^{2n} = \pm \binom{2n}{n} u^n \times u^n \neq 0.$$

Thus we see that $\bar{v}^{2n} \neq 0$ since $\bar{u}^{2n} = (g \times g)^*(\bar{v}^{2n})$.

Applying Theorem 3 to the class $\alpha = \bar{v}^{2n} \in H^{4n}(K, \mathbf{R})$ we obtain $\mathrm{wgt}(\alpha) \geq 4n - 1$.

We claim that $\mathrm{wgt}(\bar{u}^{2n}) = \mathrm{wgt}((g \times g)^*(\alpha)) \geq 4n - 1$. Indeed, let $f : Y \to M \times M$ be a map satisfying the properties of the Definition 2.1 with $k = 4n - 1$. Then

$$f^*(\bar{u}^{2n}) = f^*((g \times g)^*(\alpha)) = [(g \times g) \circ f]^* (\alpha) = 0.$$

Since $\bar{u}^{2n} \neq 0$, the inequality $\mathrm{wgt}(\bar{u}^{2n}) \geq 4n - 1$ implies that $\mathsf{TC}(M) \geq 4n$. The upper bound $\mathsf{TC}(M) \leq 4n + 1$ is standard (see [14], Theorem 4). Thus, $\mathsf{TC}(M) \in \{4n, 4n + 1\}$ as claimed. This completes the proof.

□

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