Superstatistics of Blaschke products

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We consider a dynamics generated by families of maps whose invariant density depends on a parameter a and where a itself obeys a stochastic or periodic dynamics. For slowly varying a the long-term behavior of iterates is described by a suitable superposition of local invariant densities. We provide rigorous error estimates how good this approximation is. Our method generalizes the concept of superstatistics, a useful technique in nonequilibrium statistical mechanics, to maps. Our main example are Blaschke products, for which we provide rigorous error estimate on the difference between Birkhoff density and the superstatistical approximation.

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I. INTRODUCTION

Dynamics often takes place in a changing environment. This means, given some control parameter a and a local dynamics $x_{n+1} = f_a(x_n)$ generated by some mapping f_a, the control parameter a itself will also slowly change in time. In nonequilibrium statistical mechanics, these environmental fluctuations, if taking place on a large time scale, are modeled by a very useful concept, so-called superstatistics. The superstatistics concept was introduced in [1] and has since then provided a powerful tool to describe a large variety of complex systems for which there is change of environmental conditions [2–9]. The basic idea is to characterize the complex system under consideration by a superposition of several statistics, one corresponding to local equilibrium statistical mechanics (on a mesoscopic level modeled by a linear Langevin equation leading to locally Gaussian behavior) and the other one corresponding to a slowly varying parameter a of the system. Essential for this approach is the fact that there is sufficient time scale separation, i.e. the local relaxation time of the system must be much shorter than the typical time scale on which the parameter a changes.

In most applications in nonequilibrium statistical mechanics the varying control parameter a is the local inverse temperature β of the system, i.e. $a = \beta$. However, in some applications beyond the immediate scope of statistical mechanics the control parameter a can also have a different meaning. There are many interesting applications of the superstatistics concept to real-world problems, for example to train delay statistics[10], hydrodynamic turbulence [11] and cancer survival statistics [12]. Further applications are described in [13–18].

In this paper we want to extend the superstatistics concept to maps, which usually have invariant densities different from Gaussian distributions, and thus analyze this problem in a more general context. Our generalization assumes that the local dynamics is not anymore restricted to a linear Langevin dynamics (as in the nonequilibrium statistical mechanics applications), but given by an *a priori* arbitrary map with strong mixing properties. We will make the superstatistics concept mathematically rigorous by considering simple model examples of local maps where everything can be proved explicitly and by estimating the error terms. In our new approach described here, we allow for *a priori* arbitrary local invariant densities $\rho_a(x)$ and consider a dynamics given by long term iteration of a map f_a with slowly varying a. This leads to a mixing of various invariant densities $\rho_a(x)$ with different parameter a, in a way that we will analyze in detail in this paper.

If a changes on a long time scale, long as compared to the relaxation time of the local map f_a, the resulting long-term probability distribution of iterates is closely approximated by a superposition of local invariant densities $\rho_a(x)$. For particular examples, Blaschke products, we will indeed provide estimates of the error terms involved and prove how fast the Birkhoff density approaches the superstatistical approximation. On the other hand, if a changes rapidly, then a different dynamics arises which is not properly described by a mixing of the various local invariant densities. Rather, in this case one has to look at fixed points of the Perron-Frobenius operator of higher iterates of composed maps f_a with varying a. Depending on the time scale of the changes of a, there are transition scenarios between both cases.

This paper is organized as follows. In section 2, we will introduce the superstatistics concept for maps. We will study several examples in this section to illustrate the concept and to motivate our rigorous treatment in the later sections. In section 3 we state our main result, estimating the error terms for alternating block iteration of Blaschke products. In section 4 we present a rigorous theory for Blaschke products. We will prove the existence of invariant measures, determine the invariant measure explicitly as a function of the parameters involved and prove our main result, an error estimate on the difference between Birkhoff density and the superstatistical approximation.

II. SUPERSTATISTICAL DYNAMICS OF MAPS

Let us consider families of maps f_a depending on a control parameter a. These can be *a priori* arbitrary maps in arbitrary dimensions. Later we will restrict ourselves to mixing maps and assume that an absolutely continuous invariant density $\rho_a(x)$ exists for each value of the control parameter a. The local dynamics is

$$x_{n+1} = f_a(x_n). \tag{1}$$

We now allow for a time dependence of a and study the long-term behavior of iterates given by

$$x_n = f_{a_n} \circ f_{a_{n-1}} \circ \dots f_{a_1}(x_0). \tag{2}$$

Clearly, the problem now requires the specification of the sequence of control parameters $a_1, \dots, a_n$ as well, at least in a statistical sense. One possibility is a periodic orbit of control parameters of length L. Another possibility is to regard the a_j as random variables and to specify the properties of the corresponding stochastic process in parameter space.

In general, rapidly fluctuating parameters a_j will lead to a very complicated dynamics. However, there is a significant simplification if the parameters a_j change slowly. This is the analogue of the slowly varying temperature parameters in the superstatistical treatment of nonequilibrium statistical mechanics [1, 19]. The basic assumption of superstatistics is that an environmental control parameter a changes only very slowly, much slower than the local relaxation time of the dynamics. For maps this means that significant changes of a occur only over a large number T of iterations. In practice, one can model this superstatistical case as follows: One keeps a_1 constant for T iterations ($T >> 1$), then switches after T iterations to a new value a_2, after T iterations one switches to the next values a_3, and so on.

One of the simplest examples is a period-2 orbit in the parameter space. That is, we have an alternating sequence a_1, a_2 that repeats itself, with switching between the two possible values taking place after T iterations. We are interested in the long-term behavior of iterates obtained for $n \to \infty$. Possible sequences of parameters $a_1, a_2, \dots, a_L$ of period length L could be studied equally well, with a switching to the new parameter value always taking place after $T >> 1$ iterations. Another possibility are stochastic parameter changes on the long time scale T.

To illustrate and motivate the superstatistics concept for maps, we will now deal with three important examples of families of maps f_a.

Example 1 We take for f_a the asymmetric tent map on $[0, 1]$ given by

$$f_a(x) = \begin{cases} \frac{1}{a}x & x \leq a \\ 1 - \frac{x-a}{1-a} & x > a \end{cases} \tag{3}$$

with $a \in (0, 1)$. This example is somewhat trivial, because the invariant density $\rho_a(x)$ is independent of a and given by the uniform distribution for any value of a. Hence, whatever the statistics of the varying parameter sequence $a_1, a_2, \dots$ is, we get for the long-term distribution of iterates given by (2), (3) the uniform distribution

$$p(x) = 1 \tag{4}$$

Example 2 We take for f_a a map of linear Langevin type [20, 21]. This means f_a is a 2-dimensional map given by a skew product of the form

$$x_{n+1} = g(x_n) \tag{5}$$
$$y_{n+1} = e^{-a\tau} y_n + \tau^{1/2}(x_n - \bar{g}) \tag{6}$$

Here $\bar{g}$ denote the average of iterates of g. It has been shown in [20] that for $\tau \to 0$, $t = n\tau$ finite this deterministic chaotic map generates a dynamics equivalent to a linear Langevin equation [22], provided the map g has the so-called φ-mixing property [30], and regarding the initial values $x_0 \in [0, 1]$ as a smoothly distributed random variable. Consequently, in this limit the variable y_n converges to the Ornstein-Uhlenbeck process [22] and its stationary density is given by

$$\rho_\beta(y) = \sqrt{\frac{\beta}{2\pi}} e^{-\frac{1}{2}\beta y^2} \tag{7}$$

The variance parameter β of this Gaussian depends on the map g and the damping constant a. If the parameter a changes on a very large time scale, much larger than the local relaxation time to equilibrium, one expects for the long-term distribution of iterates a mixture of Gaussian distributions with different variances β^{-1}. For example, a period 2 orbit of parameter changes yields a mixture of two Gaussians

$$p(y) = \frac{1}{2}\left(\sqrt{\frac{\beta_1}{2\pi}}e^{-\frac{1}{2}\beta_1 y^2} + \sqrt{\frac{\beta_2}{2\pi}}e^{-\frac{1}{2}\beta_2 y^2}\right). \tag{8}$$

Generally, for more complicated parameter changes on the long time scale T, the long-term distribution of iterates y_n will be mixture of Gaussians with a suitable weight function $h(\beta)$ for $\tau \to 0$:

$$p(y) \sim \int d\beta\; h(\beta)e^{-\frac{1}{2}\beta y^2} \tag{9}$$

This is just the usual form of superstatistics used in statistical mechanics, based on a mixture of Gaussians with fluctuating variance with a given weight function [1]. Thus for this example of skew products the superstatistics of the map f_a reproduces the concept of superstatistics in nonequilibrium statitistical mechanics, based on the Langevin equation. In fact, the map f_a can be regarded as a possible microscopic dynamics underlying the Langevin equation. The random forces pushing the particle left and right are in this case generated by deterministic chaotic map g governing the dynamics of the variable x_n. Generally it is possible to consider any φ-mixing map here [20]. Based on functional limit theorems, one can prove equivalence with the Langevin equation in the limit $\tau \to 0$.

Example 3: Blaschke products We now want to consider further examples beyond the immediate scope of statistical mechanics where the invariant density of the local map is non-Gaussian but still a full analytic treatment is possible. Consider mappings of a complex variable z given by

$$f(z) = b_0 \prod_{j=1}^{d} \frac{z - b_j}{1 - \bar{b}_j z}, \tag{10}$$

where $|b_0| = 1$ and $|b_j| < 1$ for $j = 1, 2, \ldots, d$ and $d \geq 2$. We are interested in a dynamics restricted to the unit circle S^1 and write $u \in S^1$ as

$$u = e^{i2\pi\varphi}, \tag{11}$$

so that $\varphi \in [0, 1)$. According to Martin [27], and as established in the following sections in much more detail, the invariant density of f with respect to the variable u is given by

$$\rho^u(u) = \frac{1}{2\pi}\frac{1 - |z_0|^2}{|u - z_0|^2}. \tag{12}$$

Here $z_0 = f(z_0)$ is a fixed point of f. Blaschke products usually exhibit very strong chaotic behaviour and can be used as the map g in equation (5) if an extension to a Langevin dynamics is wished for for physical reasons. The remarkable property of Blaschke products is that the knowledge of a fixed point z_0 of the map uniquely fixes the shape of the invariant density ρ^u, making an analytic treatment very convenient.

Transformation of variables $u \to \varphi$ yields the invariant density ρ^φ with respect to the variable φ as

$$\rho^\varphi(\varphi) = \rho^u(u)\left|\frac{du}{d\varphi}\right| \tag{13}$$

$$= \frac{1 - |z_0|^2}{|e^{i2\pi\varphi} - z_0|^2} \tag{14}$$

$$= \frac{1 - a^2 - b^2}{1 + a^2 + b^2 - 2a\cos 2\pi\varphi - 2b\sin 2\pi\varphi}. \tag{15}$$

Here a denotes the real part and b the imaginary part of the fixed point z_0, i.e. $z_0 = a + ib$.

For $z_0 = 0$ we get $a = b = 0$ and hence $\rho^\varphi(\varphi) = 1$. This is just the invariant density of a d-ary shift map on [0,1], noting that for $b_j = 0$ the Blaschke product becomes

$$z \to z^d \Longleftrightarrow \varphi \to d \cdot \varphi \; mod\; 1 \tag{16}$$

Another example would be a Blaschke product with a real fixed point $z_0 = a$. In this case the invariant density is

$$\rho^\varphi(\varphi) = \frac{1 - a^2}{1 + a^2 - 2a\cos 2\pi\varphi}. \tag{17}$$

A particular example, taken from [27], is $b_1 = b_2 = \frac{1}{2}$, i.e.

$$f(z) = \frac{(z-\frac{1}{2})^2}{(1-\frac{1}{2}z)^2} \tag{18}$$

The fixed point condition $z_0 = f(z_0)$ is solved by

$$z_0 = \frac{1}{2}(7-\sqrt{45}) \approx 0.145898... = a \tag{19}$$

This is the unique fixed point in the unit disk $D = \{z| \, |z| < 1\}$.

We are now in a position to explicitly do superstatistics for Blaschke products, since the invariant densities are known explicitly as a function of the parameters of the map. While a rigorous treatment will be worked out in detail in the following sections, we here just consider a simple example to illustrate the general idea. Let us consider two different Blaschke products, and a periodic orbit of length 2 of the parameters. For example, we may consider an alternating dynamics of the two maps

$$f_1(z) \;=\; z^2 \tag{20}$$
$$f_2(z) \;=\; \frac{(z-\frac{1}{2})^2}{(1-\frac{1}{2}z)^2}. \tag{21}$$

If we iterate f_2 for a long time T, then iterate f_1 for the same long time T, then switch back to f_2, then to f_1, and so on, the Birkhoff density will be a mixture of both invariant densities. In the φ variable we expect to get for $T \to \infty$ the superstatistical result

$$\rho_\infty(\varphi) \;=\; \frac{1}{2}\rho_1(\varphi) + \frac{1}{2}\rho_2(\varphi) \tag{22}$$
$$\;=\; \frac{1}{2}\left(1 + \frac{1-a^2}{1+a^2-2a\cos 2\pi\varphi}\right) \tag{23}$$

with $a \approx 0.145898...$. This will be confirmed by our rigorous treatment in the following section. If, on the other hand, we switch maps after each iteration step, the result will be different. In this case we need to determine the invariant density of the composed map $f_1 \circ f_2(z)$ (or $f_2 \circ f_1(z)$, depending on which map is iterated first). The composed map is again a Blaschke product, now with $d = 4$:

$$f_{12}(z) := f_1 \circ f_2(z) = \frac{(z-\frac{1}{2})^4}{(1-\frac{1}{2}z)^4} \tag{24}$$

The fixed point condition

$$z_0 = f_{12}(z_0) = \frac{(z_0-\frac{1}{2})^4}{(1-\frac{1}{2}z_0)^4} \tag{25}$$

yields $z_0 \approx 0.0464774... =: c$ and hence the invariant density of $f_{12} = f_1 \circ f_2$ is given by

$$\rho_{12}(\varphi) = \frac{1-c^2}{1+c^2-2c\cos 2\pi\varphi}. \tag{26}$$

Similarly the other composed map is also a Blaschke product with $d = 4$:

$$f_{21}(z) := f_2 \circ f_1(z) = \frac{(z^2-\frac{1}{2})^2}{(1-\frac{1}{2}z^2)^2} \tag{27}$$

The fixed point condition

$$z_0 = f_{21}(z_0) = \frac{(z_0^2-\frac{1}{2})^2}{(1-\frac{1}{2}z_0^2)^2} \tag{28}$$

yields $z_0 = c^{\frac{1}{2}} \approx 0.0464774...^{\frac{1}{2}} \approx 0.215586...$ and hence the invariant density of $f_{21} = f_2 \circ f_1$ is given by

$$\rho_{21}(\varphi) = \frac{1-c}{1+c-2\sqrt{c}\cos 2\pi\varphi}. \tag{29}$$

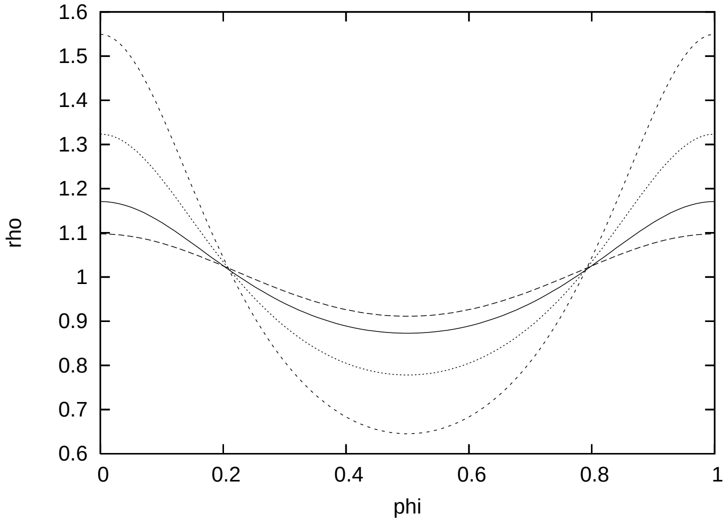

FIG. 1: The density $\rho_{12}(\varphi)$ (lower dashed line at $\varphi = 0$) and the density $\rho_{21}(\varphi)$ (upper dashed line at $\varphi = 0$), obtained by switching between f_1 and f_2 on the short time scale $T = 1$. The Birkhoff density $\frac{1}{2}\rho_{12}(\varphi) + \frac{1}{2}\rho_{21}(\varphi)$ corresponds to the dotted line. In the limit $T \to \infty$, the Birkhoff density will converge to the superstatistical composition $\rho_\infty(\varphi)$ (solid line).

The Birkhoff density, which describes the long-term distribution of iterates independent of the phase of the periodic orbit, is then given by $\frac{1}{2}\rho_{12}(\varphi) + \frac{1}{2}\rho_{21}(\varphi)$.

Fig. 1 shows the densities ρ_{12} and ρ_{21} (dashed lines), the Birkhoff density $\frac{1}{2}(\rho_{12}+\rho_{21})$ (dotted line) and the superstatistical composition ρ_∞ (solid line) as a function of φ. Apparently, all curves are different. But the difference between Birkhoff density $\frac{1}{2}\rho_{12} + \frac{1}{2}\rho_{21}$ and the superstatistical composition ρ_∞ will decrease if the time scale T of switching between parameters is increased. There is a transition scenario from the Birkhoff density to the superstatistical result ρ_∞ for $T \to \infty$.

III. STATEMENT OF MAIN RESULT

Physicists use superstatistical techniques in many applications [2, 5, 7–10, 13, 14, 18], but a rigorous proof and estimate for the error terms involved in the superstatistical approximation are missing. It would be highly desirable to have a very general theorem delivering this. Unfortunately, in full generality such a theorem is out of reach presently. Hence in this paper we restrict ourselves to a first step in this direction, a rigorous proof for particular dynamics (Blaschke products) and a particular orbit structure in parameter space, namely blocks of two alternating Blaschke products.

Let us prepare the mathematical background for our main theorem. Let λ_0 be the normalized Lebesgue measure on the unit circle $\mathbf{S}^1$. For any z in the unit disk, let λ_z denote the harmonic measure on the unit circle whose density ρ_z with respect to the normalized Lebesgue measure λ_0 on $\mathbf{S}^1$ is given by the Poisson kernel $\frac{1-|z|^2}{|u-z|^2}$, $u \in \mathbf{S}^1$.

Suppose we are given two Blaschke products A and B which expand Lebesgue measure λ_0 on the unit circle. Given an arbitrary composition $C = C_l \circ \ldots \circ C_1$ (or word) with $C_i \in \{A, B\}$ define $\lambda_C = \lambda_\gamma$ where γ is the attracting fixed point of the composition C. Thus $\lambda_A = \lambda_\alpha$ where α is the attracting fixed point of A and $\lambda_B = \lambda_\beta$ where β is the attracting fixed point of B. We are interested in alternate block iteration of A of length m and of B of length n. When iterated cyclically the two maps A and B induce a Birkhoff measure

$$\frac{1}{m+n}\{\sum_{i=1}^{m} \lambda_{A^i \circ B^n \circ A^{m-i}} + \sum_{j=1}^{n} \lambda_{B^j \circ A^m \circ B^{n-j}}\} \ .$$

The sum over i and j takes care of the fact that we average over all possible phases of iteration of A^m and B^n in a cyclic way, analogous to eq. (24) and (27) for $m = n = 1$. Our main result is that as m and n tend to infinity with fixed ratio $p : q$ (satisfying $p + q = 1$) this Birkhoff measure tends to the super-statistical limit $p\lambda_A + q\lambda_B$. One thus gets in this limit a significant simplification — just the superstatistical approximation used by physicists.

To formulate our main theorem, it is useful to proceed to densities. Writing $\rho_C(= \rho_\gamma)$ for the (Poisson) density of

λ_C we obtain the quantity

$$\frac{1}{m+n}\{\sum_{i=1}^{m} \rho_{A^i \circ B^n \circ A^{m-i}} + \sum_{j=1}^{n} \rho_{B^j \circ A^m \circ B^{n-j}}\}$$

as the Birkhoff density for the cycle of maps A and B.

Given a Blaschke product B and a point α in the unit disk, define the error $\varepsilon_B(\alpha)$ by

$$\varepsilon_B(\alpha) = \sum_{j=1}^{\infty} \varepsilon_{B^j,\alpha} \ .$$

Similarly, given a Blaschke product A and a point β in the unit disk, define the error $\varepsilon_A(\beta)$ by

$$\varepsilon_A(\beta) = \sum_{i=1}^{\infty} \varepsilon_{A^i,\beta} \ .$$

We are interested in the case α is the attracting fixed point of A and β is the attracting fixed point of B.

The individual terms $\varepsilon_{B^j,\alpha}$ are given by the difference in Poisson densities $\rho_{B^j(\alpha)} - \rho_\beta$ and the individual terms $\varepsilon_{A^i,\beta}$ are given by the difference in Poisson densities $\rho_{A^i(\beta)} - \rho_\alpha$. These Poisson differences converge to zero, in the supremum norm on densities, exponentially fast as i and j tend to infinity. Hence the above errors are well-defined.

Given $|b|$ with $0 < |b| < 1/\sqrt{3}$ put

$$r_{|b|} = \frac{1 - |b|^2 - \sqrt{(1+|b|^2)(1-3|b|^2)}}{2|b|^2}$$

which satisfies $0 < r_{|b|} < 1$.

THEOREM: Given two Blaschke products A and B with opposite zeroes: $A(z) = a_0 \frac{z^2-a^2}{1-\bar{a}^2 z^2}$ and $B(z) = b_0 \frac{z^2-b^2}{1-\bar{b}^2 z^2}$ (with $|a_0| = |b_0| = 1$) and satisfying $|a|, |b| < 1/\sqrt{3}$, and given r with $\max\{r_{|a|}, r_{|b|}\} \leq r < 1$ then, putting $K = \frac{2r}{1+r^2}$ (< 1), we have, for all $m, n \in \mathbf{N}$:

$$\left| \sum_{i=1}^{m} \rho_{A^i \circ B^n \circ A^{m-i}} - m\rho_\alpha - \varepsilon_A(\beta) \right| < \frac{4r}{(1-r)^2} \left(\frac{K^{n+1} + K^{m+1}}{1-K} \right) \ ,$$

$$\left| \sum_{j=1}^{n} \rho_{B^j \circ A^m \circ B^{n-j}} - n\rho_\beta - \varepsilon_B(\alpha) \right| < \frac{4r}{(1-r)^2} \left(\frac{K^{m+1} + K^{n+1}}{1-K} \right) \ .$$

COROLLARY:

$$\sum_{i=1}^{m} \rho_{A^i \circ B^n \circ A^{m-i}} + \sum_{j=1}^{n} \rho_{B^j \circ A^m \circ B^{n-j}} - (m\rho_\alpha + n\rho_\beta) \ \to \ \varepsilon_A(\beta) + \varepsilon_B(\alpha)$$

exponentially fast as m and n tend to infinity.

COROLLARY: The Birkhoff density

$$\frac{1}{m+n}\{\sum_{i=1}^{m} \rho_{A^i \circ B^n \circ A^{m-i}} + \sum_{j=1}^{n} \rho_{B^j \circ A^m \circ B^{n-j}}\}$$

tends to the super-statistical limit $p\rho_A + q\rho_B$ as m and n tend to infinity with fixed ratio $p : q$ (satisfying $p + q = 1$).

Note that the linear combination $p\rho_A + q\rho_B$ occuring above is just the analogue of the superstatistical approximation used by physicists, with the invariant densities of the two different Blaschke products replacing the two different stationary Gaussian distributions that occur in equation (8). The limit $m \to \infty$ and $n \to \infty$ corresponds to the assumption of time scale separation made by physicists: The system has enough time to relax to the stationary state of A before the next parameter change takes place, changing the dynamics to B. The fact that p and q occur simply means that it is relevant how long the system stays in state A as compared to state B. All this makes physical sense.

In the following sections we show how to arrive at this rigorous result, by proving statements for the existence and uniqueness of invariant measures of Blaschke products and estimating the relevant error terms of the superstatistical approximation.

IV. DETAILED CALCULATIONS FOR BLASCHKE PRODUCTS

Degree d maps of the circle

Let λ_0 be normalized Lebesgue measure on the unit circle $\mathbf{S}^1$. Suppose $\tau : \mathbf{S}^1 \to \mathbf{S}^1$ is C^1 and has degree $d > 1$. Then its pushforward action $\tau^* : \mu \mapsto \mu \circ \tau^{-1}$ on probability measures μ which are absolutely continuous with respect to Lebesgue measure is given by the transfer operator $\mathcal{L}_\tau$ on densities. Thus $\tau^* : \rho\lambda_0 \mapsto \mathcal{L}_\tau(\rho)\lambda_0$ where

$$(\mathcal{L}_\tau(\rho))(e^{i\theta}) = \sum_{e^{i\zeta} \in \tau^{-1}(e^{i\theta})} \frac{\rho(e^{i\zeta})}{|\tau'(e^{i\zeta})|} \ .$$

The main example which we are concerned with is when τ is the restriction to $\mathbf{S}^1$ of a Blaschke product

$$B(z) = b_0 \prod_{j=1}^{d} \frac{z - b_j}{1 - \bar{b}_j z}$$

(where $|b_0| = 1$ and $|b_j| < 1$ for $j = 1, 2, \ldots, d$).

Action of Blaschke products on Poisson measures

For any z in the unit disk, let λ_z denote the harmonic measure on the unit circle whose density ρ_z with respect to the normalized Lebesgue measure λ_0 on $\mathbf{S}^1$ is given by the Poisson kernel $\frac{1-|z|^2}{|u-z|^2}$, $u \in \mathbf{S}^1$.

As observed in [27], a Blaschke product pushes forward Poisson measures to Poisson measures.

PROPOSITION 1: If $f : \mathbf{D} \to \mathbf{D}$ is an analytic function on the unit disk $\mathbf{D}$ whose extension to $\bar{\mathbf{D}}$ is continuous and where the restriction τ to $\mathbf{S}^1$ takes values in $\mathbf{S}^1$. Then, for all $z \in \mathbf{D}$, we have

$$\tau^* \lambda_z = \lambda_{f(z)} \ .$$

Proof. The proof is given in [27], but we repeat it here in order to make the paper self-consistent. Given a continuous function $\psi : \mathbf{S}^1 \to \mathbf{R}$ the unique extension $\bar{\psi}$ of ψ which is continuous on $\bar{\mathbf{D}}$ and harmonic in $\mathbf{D}$ is given by

$$\bar{\psi}(z) = \int_{\mathbf{S}^1} \psi d\lambda_z$$

where λ_z is the harmonic measure on $\mathbf{S}^1$ determined by $z \in \mathbf{D}$. (See [24] Chapter 10.)

Given $f : \mathbf{D} \to \mathbf{D}$ an analytic function on the unit disk $\mathbf{D}$ whose boundary value τ to $\mathbf{S}^1$ is continuous, mapping $\mathbf{S}^1$ to $\mathbf{S}^1$, then for any continuous $\psi : \mathbf{S}^1 \to \mathbf{R}$ we have that $\bar{\psi} \circ f$ is a harmonic function with boundary value $\psi \circ \tau$ whence equals $\overline{\psi \circ \tau}$ (by the uniqueness theorem for harmonic extensions).

Write τ^* for the action on probability measures on $\mathbf{S}^1$ induced by τ. Then for all $\psi : \mathbf{S}^1 \to \mathbf{R}$ continuous we have

$$\int_{\mathbf{S}^1} \psi d(\tau^* \lambda_z) = \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{i\theta})(\mathcal{L}_\tau(\rho_z))(e^{i\theta}) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{i\theta}) \sum_{e^{i\zeta} \in \tau^{-1}(e^{i\theta})} \frac{\rho_z(e^{i\zeta})}{|\tau'(e^{i\zeta})|} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \psi(\tau(e^{i\zeta}))\rho_z(e^{i\zeta}) d\zeta$$

$$= \int_0^{2\pi} (\psi \circ \tau)(e^{i\zeta}) d\lambda_z(\zeta)$$

$$= \int_{\mathbf{S}^1} (\psi \circ \tau) d\lambda_z = \overline{\psi \circ \tau}(z)$$

$$= (\bar{\psi} \circ f)(z) = \bar{\psi}(f(z)) = \int_{\mathbf{S}^1} \psi d\lambda_{f(z)} \ .$$

It follows that $\tau^* \lambda_z = \lambda_{f(z)}$ by uniqueness of the probability measure. □

Consequently, if B is a Blaschke product with a fixed point z in the unit disk then λ_z is an invariant measure for the action (τ) on $\mathbf{S}^1$. With respect to such smooth invariant measure $\tilde{\lambda}$ the transfer equation $\mathcal{L}_{\tilde{\tau}}(\tilde{\rho}) = \tilde{\rho}$ (with $\tilde{\rho} = 1$ and $\tilde{\tau}$ the straightened circle map) together with an upper bound on $|\tilde{\tau}'|$ shows (via $|\tilde{\tau}'| > 1$) that τ expands such measure. By Walters [29] (Theorem 18) there is only one invariant probability measure absolutely continuous with respect to Lebesgue and so this must be λ_z. The measure is mixing whence ergodic.

If a Blaschke product of degree d expands some measure on the unit circle then there must be exactly $d-1$ fixed points on $\mathbf{S}^1$, but since there are $d+1$ fixed points on the Riemann sphere, there must be two further fixed points, one in each component of the complement of $\mathbf{S}^1$, and both attracting (by the Schwarz lemma).

Blaschke products which expand Lebesgue measure

Martin [27] gives a sufficient condition for a Blaschke product

$$B(z) = b_0 \prod_{j=1}^{d} \frac{z - b_j}{1 - \bar{b}_j z}$$

(where $|b_0| = 1$ and $|b_j| < 1$ for $j = 1, 2, \ldots, d$ and $d \geq 2$) to expand Lebesgue measure on the unit circle. The sufficient condition is

$$\sum_{j=1}^{d} \frac{1 - |b_j|}{1 + |b_j|} > 1 \ .$$

(In the special case $d = 2$ and $b_1 + b_2 = 0$ this can be improved to

$$\sum_{j=1}^{2} \frac{1 - |b_j|^2}{1 + |b_j|^2} > 1 \ ,$$

which reduces to $|b_j| < 1/\sqrt{3}$.)

Dynamics on the unit disk

Any holomorphic self map of unit disk does not increase the *pseudo hyperbolic distance*

$$d(z, w) = \frac{|z - w|}{|1 - \bar{w} z|}$$

(a metric which is invariant under Möbius transformations preserving the disk). In the case the self map is a Blaschke product of degree at least two we get contraction uniform on compact subsets of the disk.

LEMMA 2: For z and w in the unit disk we have

$$\frac{|z - w|}{|1 - \bar{w} z|} \leq \frac{|z| + |w|}{1 + |z||w|} \ .$$

Proof. We consider w fixed and z varying around a circle centre the origin and radius $r < 1$. Then the pseudo hyperbolic distance between z and w is the absolute value of

$$v = \frac{z - w}{1 - \bar{w} z} \ .$$

Inverting this yields

$$z = \frac{v + w}{1 + \bar{w}v}$$

and the absolute value of this equals r. Hence

$$|v + w|^2 = r^2|1 + \bar{w}v|^2$$

This can be written as

$$\big|(1 - r^2 w\bar{w})v + (1 - r^2)w\big| = r(1 - w\bar{w})$$

which is the equation of a circle in v. The centre is

$$-\frac{(1 - r^2)w}{1 - r^2 w\bar{w}}$$

and the radius is

$$\frac{r(1 - w\bar{w})}{1 - r^2 w\bar{w}}$$

whence the maximum of $|v|$ on the circle is

$$\frac{(1 - r^2)|w| + r(1 - |w|^2)}{1 - r^2|w|^2}$$

$$= \frac{|w| - r^2|w| + r - r|w|^2}{(1 - r|w|)(1 + r|w|)}$$

$$= \frac{(1 - r|w|)(r + |w|)}{(1 - r|w|)(1 + r|w|)} = \frac{r + |w|}{1 + r|w|} \ . \ \square$$

PROPOSITION 3: In the case B is a Blaschke product of degree two with opposite zeros we have

$$\frac{d(B(z), B(w))}{d(z, w)} = \left| \frac{z + w}{1 + \bar{w}z} \right| \ .$$

Proof. Introduce $[z, w] = \frac{z-w}{1-\bar{w}z}$. Then a degree two Blaschke product can be written $B(z) = b_0[z, b_1][z, b_2]$. When B is a Blaschke product of degree d in z then the derived map

$$B^{\triangle}(z, w) = \frac{[B(z), B(w)]}{[z, w]}$$

turns out to be a Blaschke product of degree $d-1$ in z [23]. We prove this in the case B has degree two with opposite zeros b_1 and b_2 (written $\pm b$) obtaining the precise formula.

$$[B(z), B(w)] = \frac{B(z) - B(w)}{1 - \overline{B(w)}B(z)}$$

$$= \frac{b_0[z, b_1][z, b_2] - b_0[w, b_1][w, b_2]}{1 - \bar{b}_0[\bar{w}, \bar{b}_1][\bar{w}, \bar{b}_2]b_0[z, b_1][z, b_2]}$$

$$= \frac{b_0\left[\frac{z^2 - b^2}{1 - \bar{b}^2 z^z} - \frac{w^2 - b^2}{1 - \bar{b}^2 w^2}\right]}{1 - b_0\bar{b}_0\left(\frac{\bar{w}^2 - \bar{b}^2}{1 - b^2\bar{w}^2}\right)\left(\frac{z^2 - b^2}{1 - \bar{b}^2 z^2}\right)}$$

After a short calculation one obtains

$$[B(z), B(w)] = \frac{b_0(z-w)(z+w)}{(1-\bar{w}z)(1+\bar{w}z)} \left(\frac{1-b^2\bar{w}^2}{1-\bar{b}^2w^2} \right) \ .$$

Hence

$$B^{\triangle}(z,w) = \frac{[B(z), B(w)]}{[z,w]}$$

$$= \frac{b_0(z+w)}{(1+\bar{w}z)} \left(\frac{1-b^2\bar{w}^2}{1-\bar{b}^2w^2} \right)$$

is a degree one Blaschke product in z whose absolute value is

$$\left| \frac{z+w}{1+\bar{w}z} \right| \ . \square$$

COROLLARY 4: We have a bound on the uniform contraction on compact subsets of the disk, which is given by

$$\frac{d(B(z), B(w))}{d(z,w)} \le \frac{|z|+|w|}{1+|z||w|} \ .\square$$

Write $|b|$ for the common absolute value of b_1 and b_2. When $|b| < 1/\sqrt{3}$ put

$$r_{|b|} = \frac{1-|b|^2 - \sqrt{(1+|b|^2)(1-3|b|^2)}}{2|b|^2} \ .$$

PROPOSITION 5: Given a Blaschke product B with zeros $\pm b$ satisfying $|b| < 1/\sqrt{3}$ and an r satisfying $r_{|b|} \le r < 1$ then the closed disk D_r centre 0 radius r is mapped inside itself by B.

Proof. We first show that, for $|b| < 1$ and $r < 1$, a Blaschke product B with zeros $\pm b$ maps the disk D_r inside the disk D_s centre 0 radius s where

$$s = \frac{r^2 + |b|^2}{1+|b|^2 r^2} \ .$$

Treating

$$B(z) = b_0 \frac{z^2 - b^2}{1-\bar{b}^2 z^2}$$

as a function of z^2 (and b^2) we can apply the lemma and obtain

$$\frac{|z^2 - b^2|}{|1-\bar{b}^2 z^2|} \le \frac{|z^2| + |b^2|}{1+|z^2||b^2|}$$

whence the inclusion follows since s is monotone increasing in r.

Finally the hypothesis $r \ge r_{|b|}$ then guarantees that $s \le r$. $\square$

If r satisfies the hypotheses for two Blaschke products A and B, each with opposite zeros, then an iterated function system on D_r with uniform contraction $K = \frac{2r}{1+r^2}$ (with respect to the pseudo-hyperbolic distance on $\mathbf{D}$) is created. This is the situation considered by Hutchinson [26].

When iterated cyclically the two maps A and B induce a Birkhoff measure

$$\frac{1}{m+n} \{ \sum_{i=1}^{m} \lambda_{A^i \circ B^n \circ A^{m-i}} + \sum_{j=1}^{n} \lambda_{B^j \circ A^m \circ B^{n-j}} \} \ .$$

Superstatistics for Blaschke products

The map $z \mapsto \lambda_z$ (from the unit disk) is continuous relative to the supremum norm on densities, as is seen in the following

PROPOSITION 6: For z and w in the unit disk and u in the unit circle

$$\left| \frac{1-|w|^2}{|u-w|^2} - \frac{1-|z|^2}{|u-z|^2} \right| \leq \frac{2|z-w|}{(1-|z|)(1-|w|)} \ .$$

Proof. For $|z| < 1$ and $|u| = 1$ we have

$$\frac{1-|z|^2}{|u-z|^2} = \frac{1-|z|^2}{|(u-z)(1-\bar{z}u)|}$$

$$= \left| \frac{1}{u-z} + \frac{\bar{z}}{1-\bar{z}u} \right| \ .$$

Now

$$\frac{1}{u-w} - \frac{1}{u-z} = \frac{w-z}{(u-w)(u-z)}$$

and

$$\frac{\bar{w}}{1-\bar{w}u} - \frac{\bar{z}}{1-\bar{z}u} = \frac{\bar{w}-\bar{z}}{(1-\bar{w}u)(1-\bar{z}u)} \ .$$

It follows, by the triangle inequality, that

$$\left| \frac{1-|w|^2}{|u-w|^2} - \frac{1-|z|^2}{|u-z|^2} \right| \leq \left| \frac{w-z}{(u-w)(u-z)} \right| + \left| \frac{\bar{w}-\bar{z}}{(1-\bar{w}u)(1-\bar{z}u)} \right|$$

$$= \frac{|w-z|}{|(u-w)(u-z)|} + \frac{|\bar{w}-\bar{z}|}{|(\bar{u}-\bar{w})(\bar{u}-\bar{z})|}$$

$$= \frac{2|w-z|}{|u-w| \cdot |u-z|} \leq \frac{2|w-z|}{(1-|w|)(1-|z|)} \ . \ \square$$

DEFINITION 7: As mentioned before in section 3, given a Blaschke product B and a point α in the unit disk, define the error $\varepsilon_B(\alpha)$ by

$$\varepsilon_B(\alpha) = \sum_{j=1}^{\infty} \varepsilon_{B^j,\alpha} \ .$$

Similarly, given a Blaschke product A and a point β in the unit disk, define the error $\varepsilon_A(\beta)$ by

$$\varepsilon_A(\beta) = \sum_{i=1}^{\infty} \varepsilon_{A^i,\beta} \ .$$

We are interested in the case α is the attracting fixed point of A and β is the attracting fixed point of B.

The individual terms $\varepsilon_{B^j,\alpha}$ are given by the difference in Poisson densities $\rho_{B^j(\alpha)} - \rho_\beta$ and the individual terms $\varepsilon_{A^i,\beta}$ are given by the difference in Poisson densities $\rho_{A^i(\beta)} - \rho_\alpha$. These Poisson differences converge to zero, in the supremum norm on densities, exponentially fast as i and j tend to infinity, by Proposition 6. Hence the above errors are well-defined.

Given an arbitrary composition $C = C_l \circ \ldots \circ C_1$ (or word) with $C_i \in \{A, B\}$ define $\varepsilon_C := \rho_C - \rho_{C_l}$ where $\rho_C = \rho_\gamma$ where γ is the attracting fixed point of the composition C. (Thus $\rho_{C_l} = \rho_\alpha$ if $C_l = A$ and $\rho_{C_l} = \rho_\beta$ if $C_l = B$.)

THEOREM 8: Given two Blaschke products A and B with opposite zeroes: $A(z) = a_0 \frac{z^2-a^2}{1-\bar{a}^2 z^2}$ and $B(z) = b_0 \frac{z^2-b^2}{1-\bar{b}^2 z^2}$ (with $|a_0| = |b_0| = 1$) and satisfying $|a|, |b| < 1/\sqrt{3}$, and given r with $\max\{r_{|a|}, r_{|b|}\} \leq r < 1$ then, putting $K = \frac{2r}{1+r^2}$ (< 1), we have, for all $m, n \in \mathbf{N}$:

$$\left| \sum_{i=1}^{m} \varepsilon_{A^i \circ B^n \circ A^{m-i}} - \varepsilon_A(\beta) \right| < \frac{4r}{(1-r)^2} \left(\frac{K^{n+1} + K^{m+1}}{1-K} \right) ,$$

$$\left| \sum_{j=1}^{n} \varepsilon_{B^j \circ A^m \circ B^{n-j}} - \varepsilon_B(\alpha) \right| < \frac{4r}{(1-r)^2} \left(\frac{K^{m+1} + K^{n+1}}{1-K} \right) .$$

Proof. For all i with $1 \leq i \leq m$ we have

$$\varepsilon_{A^i \circ B^n \circ A^{m-i}} = \rho_{A^i \circ B^n \circ A^{m-i}} - \rho_\alpha$$

and $\varepsilon_{A^i,\beta} = \rho_{A^i(\beta)} - \rho_\alpha$. Hence

$$\varepsilon_{A^i \circ B^n \circ A^{m-i}} - \varepsilon_{A^i,\beta} = \rho_{A^i \circ B^n \circ A^{m-i}} - \rho_{A^i(\beta)}$$

Furthermore

$$\left|\rho_{A^i \circ B^n \circ A^{m-i}} - \rho_{A^i(\beta)}\right| \leq \frac{2}{(1-r)^2} |A^i(B^n(\gamma)) - A^i(\beta)| \leq \frac{2}{(1-r)^2} \cdot 2rK^{i+n}$$

(since $d(\gamma, \beta) \leq \frac{2r}{1+r^2}$) where γ is the fixed point of $A^m B^n$. Then, for $i > m$, we have

$$\left|\varepsilon_{A^i,\beta}\right| \leq \frac{2}{(1-r)^2} |A^i(\beta) - \alpha| \leq \frac{2}{(1-r)^2} \cdot 2rK^i$$

(since $d(\beta, \alpha) \leq \frac{2r}{1+r^2}$). Summing over all $i \geq 1$ gives the first conclusion. A similar argument gives the second conclusion. □

COROLLARY 9:

$$\sum_{i=1}^{m} \rho_{A^i \circ B^n \circ A^{m-i}} + \sum_{j=1}^{n} \rho_{B^j \circ A^m \circ B^{n-j}} - (m\rho_\alpha + n\rho_\beta) \ \rightarrow \ \varepsilon_A(\beta) + \varepsilon_B(\alpha)$$

exponentially fast as m and n tend to infinity.

COROLLARY 10: The Birkhoff density

$$\frac{1}{m+n} \{ \sum_{i=1}^{m} \rho_{A^i \circ B^n \circ A^{m-i}} + \sum_{j=1}^{n} \rho_{B^j \circ A^m \circ B^{n-j}} \}$$

tends to the super-statistical limit $p\rho_A + q\rho_B$ as m and n tend to infinity with fixed ratio $p : q$ (satisfying $p + q = 1$).

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