Algebraic Semantics and Model Completeness for Intuitionistic Public Announcement Logic

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Abstract

In the present paper, we start studying epistemic updates using the standard toolkit of duality theory. We focus on public announcements, which are the simplest epistemic actions, and hence on Public Announcement Logic (PAL) without the common knowledge operator. As is well known, the epistemic action of publicly announcing a given proposition is semantically represented as a transformation of the model encoding the current epistemic setup of the given agents; the given current model being replaced with its submodel relativized to the announced proposition. We dually characterize the associated submodel-injection map as a certain pseudo-quotient map between the complex algebras respectively associated with the given model and with its relativized submodel. As is well known, these complex algebras are complete atomic BAOs (Boolean algebras with operators). The dual characterization we provide naturally generalizes to much wider classes of algebras, which include, but are not limited to, arbitrary BAOs and arbitrary modal expansions of Heyting algebras (HAOs). Thanks to this construction, the benefits and the wider scope of applications given by a point-free, intuitionistic theory of epistemic updates are made available. As an application of this dual characterization, we axiomatize the intuitionistic analogue of PAL, which we refer to as IPAL, prove soundness and completeness of IPAL w.r.t. both algebraic and relational models, and show that the well known Muddy Children Puzzle can be formalized in IPAL.

2010 AMS Subject Classification 03B42, 06D20, 06D50, 06E15.

Keywords and phrases Public Announcement Logic, duality, intuitionistic modal logic, algebraic models, pointfree semantics, Intuitionistic Public Announcement Logic.

1 Introduction

Dynamic logics (DLs) are perhaps the logical formalisms with the widest span of applications. They are designed to describe and reason about *change* brought about by *actions* of diverse nature: updates on the memory state of a computer, displacements of a moving robot in a closed environment, measurements in a model of quantum physics, interactions between cognitive agents performing given communication protocols, belief-revisions changing the common ground between different agents, actions which change the contextually available referents in a conversation, knowledge update, etc (the latter ones are examples of *epistemic actions*). In each of these areas, DL-formulas express the properties of the model encoding the given state of affairs, as well as the pre- and post-conditions of a given action. Actions are semantically represented as *transformations* of the current model into another one, which encodes the state of affairs after the given action has taken place. DL-languages are *expansions* of classical (propositional or modal) logic (their *static* underlying logic) with *dynamic operators*, parametrized with actions; the semantic interpretation of a dynamic operator is given in terms of the transformation of models corresponding to its action-parameter.

Similarly to [2], the line of research started with the present paper is motivated by the observations that 'dynamic phenomena' are independent of their underlying static logic being classical, and that this assumption is unrealistic in many important contexts; for instance, in all those contexts (such as scientific experiments, acquisition of legal evidence, verification of programs, etc.) where the notion of truth is *procedural*. In these contexts, affirming ϕ means demonstrating that *some* appropriate *instance* of the procedure applies to ϕ; refuting ϕ means demonstrating that *some* appropriate *instance* of the procedure applies to $\neg\phi$; however, neither instance might be available in some cases. Hence, the law of excluded middle fails in these contexts, and so classical logic is not viable as their underlying reasoning formalism. A more appropriate alternative is e.g. *intuitionistic logic*.

Desirable and conceptually important as it is, the more general problem of identifying the *right* intuitionistic counterparts of (modal-like) expansions of classical logic (such as modal logics themselves, or hybrid logics, etc.) has proven to be difficult, and for most of these logics, this question is still open. Indeed, different axiomatizations which—in the presence of classical tautologies—define the same logic become nonequivalent against an intuitionistic background. Hence, each classical axiomatization might have infinitely many nonequivalent intuitionistic potential counterparts. The most widely accepted proposals of intuitionistic counterparts of given (modal-like) expansions of classical logic have been defined by means of *syntactic* approaches (cf. for instance, the extensive discussion in [22], or more recently [8]), which consist in either weakening the proof systems for the classical versions of these expanded logics so as to make them compatible with the principles of intuitionistic logic, or by defining translations into intuitionistic first-order, or classical propositional languages. However, to the knowledge of the authors, these approaches do not take the *performances* of the given candidate intuitionistic counterpart as the main desideratum, but are rather aimed at establishing *a priori* what the given intuitionistic counterpart should *be*; the performances of the given candidate are then tested, to verify its adequacy.

The main contribution of the present paper is the introduction of a uniform methodology for defining the intuitionistic counterparts of dynamic logics; this methodology is grounded on *semantics* rather than on syntax, and takes *performances* as its main design criterion. As to the first feature, this methodology is based on the dual characterizations of the transformations of models which interpret the actions. For the sake of simplicity, we address one concrete case study, and restrict our attention to the Logic of Public Announcements (PAL), which is one of the simplest yet best known logical framework within the family of Dynamic Epistemic Logics (DELs). PAL was introduced by Plaza in [16] and subsequently intensively studied, both specifically and as part of the DEL-family, viz. [1, 10, 4] and references therein. As epistemic actions, public announcements correspond to transformations of models which are called *relativizations*. Namely, publicly announcing the formula α corresponds to shifting the given model M to its *submodel* M^α, based on the subset $[\![\alpha]\!]_M$ of the states of M on which α is satisfied.

As mentioned early on, in the present paper, relativization—which is encoded in the injection map $i_\alpha : M^\alpha \hookrightarrow M$—is characterized on algebras via classical Stone duality. Unsurprisingly, this injection map is dually characterized as a certain pseudo-quotient between the complex algebras of the underlying frames of M and of M^α (which, as is well known, are—up to isomorphism—complete atomic BAOs). The advantage brought about by this pseudo-quotient construction is that its definition naturally holds in much more general contexts than the one given by the algebras which are dually equivalent to Kripke frames. These more general contexts include—but are not limited to—arbitrary BAOs, and arbitrary modal expansions of Heyting algebras (HAOs).

Therefore, each of these wider classes of algebras lends itself to the role of generalized semantic environment for *some* logic of public announcements (cf. Definition 14). For instance, based on Definition 14, it is easy to see that the class of algebraic models based on *arbitrary* BAOs (which properly extends the class of complete and atomic BAOs) provides sound and complete *pointfree* semantics for PAL.

Likewise, the set of axioms describing the behaviour of the intuitionistic dynamic connectives (cf. Subsection 4.1) naturally arises from the class of algebraic models based on *Heyting algebras with operators* (HAOs) (which, for the sake of the present paper, are understood as Heyting algebras expanded with one normal $\Box$ operator and one normal $\Diamond$ operator). The axiomatization of HAOs does not imply the existence of any interaction between the static box and diamond operations, and of course, for the purposes of describing the epistemic setup of each agent, it is desirable to have at least as strong an axiomatization as one which forces the pairs of box and diamond operators associated with each agent to be interpreted by *the same* relation. The intuitionistic basic modal logic IK [11, 22] is the weakest axiomatization which implies the desired connection between the modal operations; its canonically associated class of algebras is a subclass of HAO which we refer to as Fischer Servi algebras, or *FS-algebras* (cf. Definition 3). The logic IPAL introduced in the present paper arises as the logic of public announcements associated with the class of algebraic models based on FS-algebras. In fact, in parallel to the mentioned definition, a second way to define IPAL is proposed, which reflects the idea that the epistemic set-up of agents might be encoded by *equivalence* relations. To account for this possibility, Prior's MIPC [20] can be taken as the underlying static logic, and *monadic Heyting algebras* can be taken in place of the more general FS-algebras; however, the results presented in what follows proceed modularly w.r.t. these two options. For the sake of the axiomatic definition of IPAL, the second feature of our methodology becomes relevant: indeed, the crucial performance aspect of classical PAL is that its set of axioms is designed in such a way (cf. Proposition 1) that the completeness of PAL w.r.t. relational models follows from the completeness of its static fragment w.r.t. the same class of models. The axiomatic definition of IPAL takes this performance aspect as its main desideratum.

The structure of the paper goes as follows: Section 2 collects the needed preliminaries on classical PAL and intuitionistic modal logic. In Section 3, the dual, algebraic characterization of public announcements is introduced. In Section 4, the intuitionistic public announcement logic IPAL is axiomatically defined, as well as its interpretation on models based on Heyting algebras. Moreover, the relational semantics for intuitionistic modal logic/IPAL is described in detail. Finally, the soundness of IPAL w.r.t. algebraic (hence relational) models, and the completeness of IPAL w.r.t. relational (hence algebraic) models are proven. In Section 5, it is shown how IPAL can be used to describe and reason about the well known epistemic scenario of the Muddy Children. Details of all the proofs in the previous sections are collected in Section 6, the appendix.

2 Preliminaries

2.1 The logic of public announcements

Let AtProp be a countable set of proposition letters. The formulas of (single-agent) public announcement logic PAL are built by the following inductive rule:

$$\phi ::= p \in \mathsf{AtProp} \mid \neg\phi \mid \phi \vee \phi \mid \Diamond\phi \mid \langle\phi\rangle\phi.$$

The standard stipulations hold for the defined connectives $\top$, $\bot$, $\wedge$, $\rightarrow$ and $\leftrightarrow$. Models of PAL are Kripke models $M = (W, R, V)$ such that R is an equivalence relation. The evaluation of the static fragment of the language is standard. For every Kripke frame $\mathcal{F} = (W, R)$ and every $a \subseteq W$, the *subframe* of $\mathcal{F}$ *relativized to* a is the Kripke frame $\mathcal{F}^a = (W^a, R^a)$ defined as follows: $W^a := a$ and

$R^a := R \cap (a \times a)$. Given this preliminary definition, formulas of form $\langle\alpha\rangle\phi$ are evaluated as follows:

$$M, w \Vdash \langle\alpha\rangle\phi \quad \text{iff} \quad M, w \Vdash \alpha \text{ and } M^\alpha, w \Vdash \phi,$$

where $M^\alpha = (W^\alpha, R^\alpha, V^\alpha)$ is defined as follows: the underlying frame of M^α is the underlying frame of M relativized to $[\![\alpha]\!]_M$, i.e. $W^\alpha := [\![\alpha]\!]_M$, and $R^\alpha := R \cap (W^\alpha \times W^\alpha)$; for every $p \in \mathsf{AtProp}$, $V^\alpha(p) = V(p) \cap W^\alpha$.

▶ Proposition 1 ([3, Theorem 27]). PAL is axiomatized completely by the axioms and rules for the modal logic S5 plus the following axioms:

1. $\langle\alpha\rangle p \leftrightarrow (\alpha \wedge p)$;
2. $\langle\alpha\rangle\neg\phi \leftrightarrow (\alpha \wedge \neg\langle\alpha\rangle\phi)$;
3. $\langle\alpha\rangle(\phi \vee \psi) \leftrightarrow (\langle\alpha\rangle\phi \vee \langle\alpha\rangle\psi)$;
4. $\langle\alpha\rangle\Diamond\phi \leftrightarrow (\alpha \wedge \Diamond(\alpha \wedge \langle\alpha\rangle\phi))$.

2.2 The intuitionistic modal logics MIPC and IK

Respectively introduced by Prior with the name MIPQ [20], and by Fischer Servi [11], the two intuitionistic modal logics the present subsection focuses on are largely considered the intuitionistic analogues of S5 and of K, respectively. These logics have been studied by many authors, viz. [5, 6, 22] and the references therein. In the present subsection, the notions and facts needed for the purposes of the present paper will be briefly reviewed. The reader is referred to [5, 6, 22] for their attributions. The set of formulas $\mathcal{L}$ for both logics are built by the following inductive rule:

$$\phi ::= \bot \mid p \in \mathsf{AtProp} \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \to \psi \mid \Diamond\phi \mid \Box\phi.$$

Let $\top$ be defined as $\bot \to \bot$ and, for all formulas ϕ and ψ, let $\neg\phi$ be defined as $\phi \to \bot$ and $\phi \leftrightarrow \psi$ be defined as $(\phi \to \psi) \wedge (\psi \to \phi)$. The logic IK is the smallest set of formulas in the language above which contains all the axioms of intuitionistic propositional logic, the following modal axioms:

$\Box(p \to q) \to (\Box p \to \Box q)$,
$\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$, $\neg\Diamond\bot$,
FS1. $\Diamond(p \to q) \to (\Box p \to \Diamond q)$,
FS2. $(\Diamond p \to \Box q) \to \Box(p \to q)$,

and is closed under uniform substitution, modus ponens and necessitation ($\vdash \varphi / \vdash \Box\varphi$). The logic MIPC is the smallest set of formulas in the language above which contains all the axioms of intuitionistic propositional logic, the following modal axioms:

$\Box p \to p$, $p \to \Diamond p$,
$\Box(p \to q) \to (\Box p \to \Box q)$, $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$,
$\Diamond p \to \Box\Diamond p$, $\Diamond\Box p \to \Box p$,
$\Box(p \to q) \to (\Diamond p \to \Diamond q)$,

and is closed under uniform substitution, modus ponens and necessitation ($\vdash \varphi / \vdash \Box\varphi$).

The relational structures for IK (resp. MIPC), called *IK-frames* (resp. *MIPC-frames*), are triples $\mathcal{F} = (W, \leq, R)$ such that $(W, \leq)$ is a nonempty poset and R is a binary (equivalence) relation such that

$$(R \circ \geq) \subseteq (\geq \circ R), \qquad (\leq \circ R) \subseteq (R \circ \leq), \qquad R = (\geq \circ R) \cap (R \circ \leq).$$

Notice that, in the case of MIPC-frames, R being symmetric implies that the second condition is equivalent to the first one, and the third condition is equivalent to R being the equivalence relation induced by the preorder $(R \circ \leq)$. *IK-models* (resp. *MIPC-models*) are structures $M = (\mathcal{F}, V)$ such that $\mathcal{F}$ is an IK-frame (resp. an MIPC-frame) and $V : \mathsf{AtProp} \longrightarrow \mathcal{P}^{\downarrow}(W)$ is a function mapping proposition letters to downward-closed subsets of W.[1] For any such model, its associated extension

[1] For every poset $(W, \leq)$, a subset Y of W is *downward-closed* if for every $x, y \in W$, if $x \leq y$ and $y \in Y$ then $x \in Y$.

map $[\![\cdot]\!]_M : \mathcal{L} \longrightarrow \mathcal{P}^{\downarrow}(W)$ is defined recursively as follows:

$$\begin{array}{rcl}
[\![p]\!]_M & = & V(p) \\
[\![\bot]\!]_M & = & \varnothing \\
[\![\phi \vee \psi]\!]_M & = & [\![\phi]\!]_M \cup [\![\psi]\!]_M \\
[\![\phi \wedge \psi]\!]_M & = & [\![\phi]\!]_M \cap [\![\psi]\!]_M \\
[\![\phi \to \psi]\!]_M & = & ([\![\phi]\!]_M \cap [\![\psi]\!]_M^c){\uparrow}^c \\
[\![\Diamond\phi]\!]_M & = & R^{-1}[[\![\phi]\!]_M] \\
[\![\Box\phi]\!]_M & = & ((\geq \circ R)^{-1}[[\![\phi]\!]_M^c])^c
\end{array}$$

where, for every $Y \subseteq W$, we let $Y^c := \{x \in W \mid x \notin Y\}$, and $Y{\uparrow} := \{x \in W \mid y \leq x \text{ for some } y \in Y\}$. For any model M and any formula ϕ, we write:

$M, w \Vdash \phi$ if $w \in [\![\phi]\!]_M$;
$M \Vdash \phi$ if $[\![\phi]\!]_M = W$;
$\mathcal{F} \Vdash \phi$ if $[\![\phi]\!]_M = W$ for any model M based on $\mathcal{F}$.

► Proposition 2. IK (resp. MIPC) is sound and complete with respect to the class of IK-frames (resp. MIPC-frames).

The algebraic semantics for IK (MIPC) is given by Fischer Servi algebras (monadic Heyting algebras), the definitions of which are reported below:

► **Definition 3.** The algebra $\mathbb{A} = (A, \wedge, \vee, \to, \bot, \Diamond, \Box)$ is a *Fischer Servi algebra* (FSA) if $(A, \wedge, \vee, \to, \bot)$ is a Heyting algebra and the following inequalities hold:
$\Box(x \to y) \leq \Box x \to \Box y$,
$\Diamond(x \vee y) \leq (\Diamond x \vee \Diamond y)$, $\Diamond\bot \leq \bot$,
$\Diamond(x \to y) \leq \Box x \to \Diamond y$,
$\Diamond x \to \Box y \leq \Box(x \to y)$.
The algebra $\mathbb{A}$ is a *monadic Heyting algebra* (MHA) if $(A, \wedge, \vee, \to, \bot)$ is a Heyting algebra and the following inequalities hold:
$\Box x \leq x$, $x \leq \Diamond x$;
$\Box(x \to y) \leq \Box x \to \Box y$, $\Diamond(x \vee y) \leq (\Diamond x \vee \Diamond y)$;
$\Diamond x \leq \Box\Diamond x$, $\Diamond\Box x \leq \Box x$;
$\Box(x \to y) \leq \Diamond x \to \Diamond y$.
It is well known and can be readily verified that every monadic Heyting algebra is an FS-algebra. The inequalities above can be equivalently written as equalities, thanks to the fact that, in any Heyting algebra, $x \leq y$ iff $x \to y = \top$. Clearly, any formula in the language $\mathcal{L}$ of IK (MIPC) can be regarded as a term in the algebraic language of FSAs (MHAs). Therefore, given an algebra $\mathbb{A}$ and an interpretation V : AtProp $\longrightarrow \mathbb{A}$, an $\mathcal{L}$-formula ϕ is *true* in $\mathbb{A}$ under the interpretation V (notation: $(\mathbb{A}, V) \models \phi$) if the unique homomorphic extension of V, denoted by $[\![\cdot]\!]_V : \mathcal{L} \longrightarrow \mathbb{A}$, maps ϕ to $\top^{\mathbb{A}}$. An $\mathcal{L}$-formula is *valid* in $\mathbb{A}$ (notation: $\mathbb{A} \models \phi$), if $(\mathbb{A}, V) \models \phi$ for every interpretation V.

IK-frames give rise to complex algebras, just as Kripke frames do: for any IK-frame $\mathcal{F}$, the *complex algebra* of $\mathcal{F}$ is

$$\mathcal{F}^+ = (\mathcal{P}^{\downarrow}(W), \cap, \cup, \Rightarrow, \varnothing, \langle R\rangle, [\geq \circ R]),$$

where for all $X, Y \in \mathcal{P}^{\downarrow}(W)$,

$$\langle R\rangle X = R^{-1}[X], \quad [\geq \circ R]X = ((\geq \circ R)^{-1}[X^c])^c, \quad X \Rightarrow Y = (X \cap Y^c){\uparrow}^c.$$

Clearly, given a model $M = (\mathcal{F}, V)$, the extension map $[\![\cdot]\!]_M : \mathcal{L} \longrightarrow \mathcal{F}^+$ is the unique homomorphic extension of V : AtProp $\longrightarrow \mathcal{F}^+$.

► Proposition 4. For every IK-model $(\mathcal{F}, V)$ and every $\mathcal{L}$-formula ϕ,

1. $(\mathcal{F}, V) \Vdash \phi$ iff $(\mathcal{F}^+, V) \models \phi$.
2. $\mathcal{F}^+$ is an FS-algebra.
3. If R is an equivalence relation, then $\mathcal{F}^+$ is a monadic Heyting algebra.

3 Epistemic updates on algebras

In the present section, the operation of epistemic update via public announcement on Kripke models is dualized: for every algebra $\mathbb{A}$ and every $a \in \mathbb{A}$, a quotient-like algebra $\mathbb{A}^a$ is defined in such a way that, whenever $\mathbb{A}$ is the algebraic dual of some frame $\mathcal{F}$ (and hence a is a subset of the universe of $\mathcal{F}$), the algebra $\mathbb{A}^a$ is isomorphic to the algebraic dual of the relativized subframe $\mathcal{F}^a$. We will proceed in two stages: first (cf. Subsection 3.2), for every $a \in \mathbb{A}$, the equivalence relation will be introduced which will be used to quotient out the algebra $\mathbb{A}$. This equivalence relation is a congruence on several classes of algebras which include Boolean and Heyting algebras, but it is not in general compatible with the modal operators, hence it is not in general a congruence on BAOs or on HAOs. This is of course unsurprising, since the corresponding operation of epistemic update on relational models produces submodels which are not in general *generated submodels* of the original models. The second stage (cf. Subsection 3.3) focuses on the definition of the modal operators on the algebra $\mathbb{A}^a$, the proof of their being normal modal operations, and that indeed, whenever $\mathbb{A}$ is the algebraic dual of some frame $\mathcal{F}$, the algebra $\mathbb{A}^a$ is (isomorphic to) the algebraic dual of the subframe $\mathcal{F}^a$. This construction and set of facts hold uniformly in the setting of Heyting algebra.

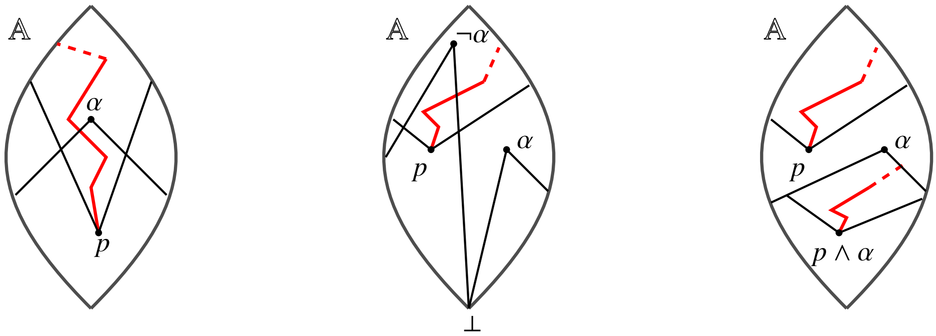
3.1 Dual characterizations, informally

At various points early on, we stated that the contributions of the present paper are grounded on a *dual characterization*. Namely, the *relativization* construction on relational models is dually characterized as a certain *quotient* construction on their dual algebras. In the present subsection, we aim at motivating this dual characterization via an informal discussion, starting by positioning dual characterizations in the more general context of dualities in logic.

Stone-type dualities serve to establish systematic connections between two different types of semantics for a given logical system: the *state-based*, geometric semantics, incarnated by Kripke models, coalgebras, descriptive general frames and similar structures, and the *algebraic* semantics. These two types of semantics work rather differently, the most basic difference being that formulas are interpreted in algebras as *elements*, and in state-based models as *collections of states*. For the sake of highlighting this difference even more dramatically, let us depart from the standard view on semantics, and think of the interpretation of a given formula not as a primitive notion, but rather as an approximation process, the limit of which converges to the standard interpretation of that formula. Then the difference between the algebraic and the state-based semantics translates into a *different approximation process* in each type of semantics. Namely, in state-based models, the approximation process is cumulative and consists in progressively adding states of the model satisfying the given formula; in its limit, this process delivers the interpretation of the given formula *extensionally*. On the other hand, in algebras, the approximation process consists in specifying which elements of the algebra are part of the "logical space" of that formula, the logical space being encoded either as the logical filter or the logical ideal associated with the formula[2]; in its limit, this process would deliver the interpretation of the given formula *intensionally*.

[2] Principal ideals and filters are respectively up-directed and down-directed sets; therefore, in each case, their generator can be thought of as the limit of approximating sequences of elements in the filter (or ideal).

The discussion so far can be summarized in the following slogan: dualities systematically connect the extensional and the intensional descriptions of mathematical notions and facts.[3] These considerations generally apply to logical systems, independently of their specific features. Moving on to the case of PAL, we are interested in capturing the change brought about by the public announcement of a formula α. In Kripke models, this change is encoded *extensionally*, and the encoding consists in the well-known relativization construction, which drops the states of the original model which do not satisfy α. How would the change be encoded *intensionally*? Answering this question requires understanding how the logical space of a given formula, e.g., of a proposition p, changes as a consequence of the public announcement of α. Concretely, given an algebra and an interpretation map, let an approximation path of the interpretation of p be any down-directed sequence of elements in the principal filter associated with p, converging to the interpretation of p. If p entails α, then no significant change takes place in the logical space of p after the announcement of α. Indeed, any approximation path will eventually enter and never leave the logical interval between p and α (that is, the intersection between the principal ideal of α and the principal filter of p), as represented in the leftmost picture below; this means that every approximation path will eventually become more accurate than α.



The opposite extreme is the case in which p is inconsistent with α. This inconsistence will be witnessed at some stage of any approximation path converging to p; indeed, p entails $\neg\alpha$ (since $\neg\alpha$ is by definition the weakest formula which is inconsistent with α), and so, as in the previous case, any approximation path converging to p will eventually enter and never leave the interval between p and $\neg\alpha$, as represented in the center picture above. So, after the announcement of α, the logical space of p is identified with the logical space of $\perp$. When α is neither implied by p nor inconsistent with p, any approximation path converging to p can be pointwise transformed into some approximation path converging to $p \wedge \alpha$, as represented in the rightmost picture. In this case, after the announcement of α, the logical space of p will coincide with the logical space of $p \wedge \alpha$ (which, as discussed in the first case, is not changed by the announcement). A moment of reflection will convince the reader that the latter case in fact incorporates the previous two. As a consequence of this fact, for all propositions p and q, if $p \wedge \alpha$ and $q \wedge \alpha$ have the same interpretation in the algebra, then p and q will have the same logical space (i.e., the same intensional description) after the announcement of α.

Summing up: the change brought about by the public announcement of α is encoded extensionally by dropping the states (of given models) which do not satisfy α, and intensionally by identifying formulas (elements of the algebra) which are logically equivalent (i.e., coincide in the algebra) when in conjunction with α.

So far in our discussion, the intensional and the extensional description of epistemic updates have been accounted for in isolation. However, and most importantly, they are related: to see this,

[3] For a differently motivated, but conceptually related perspective on this, the reader is referred to Vaughn Pratt's line of research on concurrency via Stone duality, e.g. [17], and on Chu spaces [18, 19].

notice that the natural map recording the change from the original model to the updated model is the *submodel injection*, mapping every point in the new model to its original copy in the old model, and hence pointing in the direction *opposite* to the "arrow of time" along which the change takes place. Since in the new model there are less states, there are less potential witnesses for telling formulas apart. So, in the change from the old to the new state of affairs, formulas might be identified; hence the natural map recording the change on the intensional side must be a *quotient*, pointing in *the same* direction as the arrow of time.

Using the insight of the informal discussion above, we can move on to the formal construction. As mentioned in Subsection 2.1, for any model $M = (W, R, V)$ and any PAL-formula α, the updated model M^α is based on the subset $a = [\![\alpha]\!]_V \subseteq W$. The natural map encoding this transformation is the inclusion $i : M^\alpha \hookrightarrow M$ pointing in the direction opposite to the arrow of time. The (Boolean) algebras dually associated with M and M^α are $\mathcal{P}(W)$ and $\mathcal{P}(a)$, respectively. The map dually representing the injection $i : M^\alpha \hookrightarrow M$ is the surjection $\pi : \mathcal{P}(W) \twoheadrightarrow \mathcal{P}(a)$, defined by mapping every subset $X \subseteq W$ to its inverse image $i^{-1}[X] = X \cap a$. By the isomorphism theorem (in Boolean algebras), the algebra $\mathcal{P}(a)$ is (BA-)isomorphic to—hence can be identified with—the quotient algebra $\mathcal{P}(W)/Ker(\pi)$, with $Ker(\pi)$ being the BA-congruence of $\mathcal{P}(W)$ defined by the stipulation

$$(X, Y) \in Ker(\pi) \quad \text{iff} \quad X \cap a = \pi(X) = \pi(Y) = Y \cap a,$$

and the isomorphisms $\mu : \mathcal{P}(W)/Ker(\pi) \longrightarrow \mathcal{P}(a)$ and $\nu : \mathcal{P}(a) \longrightarrow \mathcal{P}(W)/Ker(\pi)$ being defined by the assignments $[X] \mapsto X \cap a = \pi(X)$ and $Y \mapsto [i[Y]]$, for any $X \subseteq W$ and $Y \subseteq a$, respectively.

Since the map $i : M^\alpha \hookrightarrow M$ is not in general a p-morphism (cf. next subsection for more details on this), the map $\pi : \mathcal{P}(W) \twoheadrightarrow \mathcal{P}(a)$ is not in general a BAO-homomorphism; in other words, the quotient algebra $\mathcal{P}(W)/Ker(\pi)$ does not canonically inherit the modal operations from $\mathcal{P}(W)$. This is of course unsurprising: indeed, $\mathcal{P}(W)/Ker(\pi)$ has been identified with $\mathcal{P}(a)$ up to BA-isomorphism, and $\mathcal{P}(a)$ is endowed with a natural structure of BAO by its being the complex algebra of M^α. Hence, it is the modal operators of $\mathcal{P}(a)$—not of $\mathcal{P}(W)$—which need to be translated into $\mathcal{P}(W)/Ker(\pi)$, and this should be done along the isomorphisms μ and ν.

In other words, we aim at defining a modal operator $\Diamond^\alpha$ on $\mathcal{P}(W)/Ker(\pi)$ in such a way that the BA-isomorphisms μ and ν become BAO-isomorphisms. There is exactly one way to achieve this, which is to define $\Diamond^\alpha$ on $\mathcal{P}(W)/Ker(\pi)$ as the composition $\nu \circ \Diamond^{\mathcal{P}(a)} \circ \mu$, as represented in the following diagram:

$$\begin{tikzcd}
\mathcal{P}(a) \arrow[r, "\Diamond^{\mathcal{P}(a)}"] & \mathcal{P}(a) \arrow[d, "\nu"] \\
\mathcal{P}(W)/Ker(\pi) \arrow[u, "\mu"] \arrow[r, "\Diamond^\alpha"] & \mathcal{P}(W)/Ker(\pi).
\end{tikzcd}$$

Since $\Diamond^{\mathcal{P}(a)}$ is a normal modal operator and μ and ν are BA-isomorphisms, $\Diamond^\alpha$ is normal, which implies that $(\mathcal{P}(W)/Ker(\pi), \Diamond^\alpha)$ is a BAO. However, the definition above has been given in terms of the complex algebra of M^α, whereas we need that $\Diamond^\alpha$ be defined purely in terms of the resources of the quotient $\mathcal{P}(W)/Ker(\pi)$. To provide such a definition, let us recall that the accessibility relation on M^α is $R^\alpha \subseteq a \times a$, defined as follows:

$$wR^\alpha v \text{ iff } w \in a \text{ and } v \in a \text{ and } wRv.$$

The operation $\Diamond^{\mathcal{P}(a)}$ is defined as usual by taking R^α-inverse images; that is, $\Diamond^{\mathcal{P}(a)} Y = R^{\alpha-1}[Y] = R^{-1}[Y] \cap a$ for any $Y \subseteq a$. Hence, for every $X \subseteq W$,

$$\begin{array}{rclclcl}
\Diamond^\alpha[X] & = & \nu\Diamond^{\mathcal{P}(a)}\mu[X] & = & \nu\Diamond^{\mathcal{P}(a)}(X \cap a) & = & \nu R^{\alpha-1}[X \cap a] \\
& = & \nu(R^{-1}[X \cap a] \cap a) & = & [R^{-1}[X \cap a] \cap a]_{Ker(\pi)} & = & [R^{-1}[X \cap a]]_{Ker(\pi)} \\
& = & [\Diamond^{\mathcal{P}(W)}(X \cap a)]_{Ker(\pi)}. & & & &
\end{array}$$

Thus, we have proved the following:

▶ Proposition 5. For M, α, a and M^α as above, let $\mathbb{A}$ be the complex algebra of M. The complex algebra $\mathbb{A}^\alpha$ of M^α can be identified up to BAO-isomorphism with $(\mathbb{A}/Ker(\pi), \Diamond^\alpha)$, where $\pi : \mathbb{A} \longrightarrow \mathbb{A}^\alpha$ is defined by the assignment $b \mapsto b \wedge a$, and $\Diamond^\alpha$ is defined by the assignment $[b]_{Ker(\pi)} \mapsto [\Diamond^{\mathbb{A}}(b \wedge a)]_{Ker(\pi)}$.

Two remarks are in order about the proposition above. Firstly, the (pseudo-)quotient algebra $(\mathbb{A}/Ker(\pi), \Diamond^\alpha)$ has been defined in *purely algebraic* terms: indeed, the interpretation $a \in \mathbb{A}$ of α and the operations $\wedge$ and $\Diamond$ of $\mathbb{A}$ are the only ingredients of this definition, which, in particular, makes no reference to M and M^α. Secondly, the proposition states that $(\mathbb{A}/Ker(\pi), \Diamond^\alpha)$ is BAO-isomorphic to the complex algebra of M^α, which is defined in *set-theoretic* terms. Hence, Proposition 5 yields an *equivalent* encoding of public announcement-type epistemic updates, formulated *independently* of the way in which these epistemic updates are represented on Kripke models; this encoding is the *dual characterization* of these epistemic updates on algebras. The advantage brought about by the dual characterization is that the quotient construction applies to a much more general setting than complex algebras: namely, it can be performed on any meet-semilattice $\mathbb{A}$ expanded with a unary operation $\Diamond^{\mathbb{A}}$. In the following subsection, we will take this vastly generalized setting as our starting point.

3.2 Updates as pseudo-quotients

Throughout the present subsection, and unless specified otherwise, let $\mathbb{A}$ be a $\wedge$-semilattice and let $a \in \mathbb{A}$. Define the following equivalence relation $\equiv_a$ on $\mathbb{A}$: for every $b, c \in \mathbb{A}$,

$$b \equiv_a c \ \text{ iff } \ b \wedge a = c \wedge a.$$

Let $[b]_a$ be the equivalence class of $b \in \mathbb{A}$. Usually, the subscript will be dropped when there is no risk of confusion. Let the quotient set $\mathbb{A}/\equiv_a$ be denoted by $\mathbb{A}^a$. Clearly, $\mathbb{A}^a$ is an ordered set by putting $[b] \leq [c]$ iff $b' \leq_{\mathbb{A}} c'$ for some $b' \in [b]$ and some $c' \in [c]$. Let $\pi_a : \mathbb{A} \longrightarrow \mathbb{A}^a$ be the canonical projection, given by $b \mapsto [b]$.

The relation $\equiv_a$ and its properties are well known[4]. Firstly, $\equiv_a$ is a congruence[5] if $\mathbb{A}$ is a Boolean algebra, a Heyting algebra, a bounded distributive lattice or a frame (as shown in Fact 8 below). Hence, $\mathbb{A}^a$ inherits the same algebraic structure of $\mathbb{A}$ in each of these cases. The following properties of $\equiv_a$ are as crucial to our purposes as they are straightforward:

▶ Fact 6. Let $\mathbb{A}$ be a $\wedge$-semilattice and let $a \in \mathbb{A}$.

1. $[b \wedge a] = [b]$ for every $b \in \mathbb{A}$. Hence, for every $b \in \mathbb{A}$, there exists a unique $c \in \mathbb{A}$ such that $c \in [b]_a$ and $c \leq a$.
2. For all $b, c \in \mathbb{A}$, we have that $[b] \leq [c]$ iff $b \wedge a \leq c \wedge a$.
3. If $\mathbb{A}$ is a Heyting algebra, then $[a \to b] = [b]$ for every $b \in \mathbb{A}$.

Proof. 1. Since $\wedge$ is idempotent, $(b \wedge a) \wedge a = b \wedge a$; this proves the first part of the statement and the existence claim of the second part. As to uniqueness, if $c \in [b]_a$ and $c \leq a$, then $c = c \wedge a = b \wedge a$. 2. The right-to-left direction follows immediately from the definitions involved. Conversely, if $b' \leq c'$

[4] Cf. e.g. [14, Chapter 2], [15, Chapter 3, Section 4, pp 22-23], where it is used to define the local operators on locales which will be used to define the open sublocales (ibid. Chapter 5, Section 2). On Boolean algebras, cf. [21, Ch. II, sec. 6], and in modal logic, similar constructions are used in [23], and more recently in [7].

[5] Recall that a congruence on a a given algebra $\mathbb{A}$ is an equivalence relation $\cong$ on the domain of $\mathbb{A}$ such that for any $n \in \mathbb{N}$ and any n-ary operation f of $\mathbb{A}$, if $a_i \cong b_i$ for $1 \leq i \leq n$, then $f(a_1, \ldots, a_n) \cong f(b_1, \ldots, b_n)$. Therefore, the notion of congruence depends on each specific algebraic signature, and hence on the given algebraic setting.

for some $b' \in [b]$ and some $c' \in [c]$, then $b \wedge a = b' \wedge a \leq c' \wedge a = c \wedge a$.
3. From $b \leq a \to b$ we get $[b] \leq [a \to b]$. From $a \wedge (a \to b) \leq b$ we get $[a \to b] \leq [b]$. ◀

Item 1 of the fact above implies that each $\equiv_a$-equivalence class has a canonical representant, namely the only element in the given class which is less than or equal to a. Hence, the (injective) map $i' = i'_a : \mathbb{A}^a \longrightarrow \mathbb{A}$ given by $[b] \mapsto b \wedge a$ is well defined, and $\pi \circ i'$ is the identity map[6] on $\mathbb{A}^a$. The map i' will be a critical ingredient for the definition of the interpretation of IPAL-formulas on algebraic models (cf. Definition 14), motivated by the fact, shown in the following Proposition 7, that whenever $\mathbb{A} = \mathcal{F}^+$ for some (classical) Kripke frame $\mathcal{F}$, the map i' can be identified with the direct-image map associated with the map $i : \mathcal{F}^a \hookrightarrow \mathcal{F}$ modulo the BAO-isomorphism $\mathbb{A}^a \cong \mathcal{F}^{a+}$ (cf. Proposition 5).

▶ Proposition 7. If $\mathbb{A} = \mathcal{F}^+$ and $a \in \mathbb{A}$, then $i'(c) = i[\mu(c)]$ for every $c \in \mathbb{A}^a$, where $\mu : \mathbb{A}^a \longrightarrow \mathcal{F}^{a+}$ is the BAO-isomorphism identifying the two algebras. Diagrammatically:

$$\begin{tikzcd}
\mathcal{F}^{+a} \arrow[rr, "\mu"] \arrow[dr, "i'"'] & & \mathcal{F}^{a+} \arrow[dl, "{i[\cdot]}"] \\
& \mathcal{F}^{+} &
\end{tikzcd}$$

It immediately follows that $i[c] = i'(\nu(c))$ for every $c \in \mathcal{F}^{a+}$, where $\nu : \mathcal{F}^{a+} \longrightarrow \mathbb{A}^a$ is the inverse of μ.

Proof. The map μ identifies $\mathbb{A}^a$ and $\mathcal{F}^{a+}$ by sending any element $c \in \mathbb{A}^a$ to its canonical representant $\mu(c) \in \mathcal{F}^{a+}$. Hence, $c = [\mu(c)]$ and $\mu(c) \subseteq a$, and so $i'(c) = i'([\mu(c)]) = \mu(c) \cap a = \mu(c) = i[\mu(c)]$. ◀

In the remainder of this subsection, the mentioned compatibility properties of $\equiv_a$ are established, and it is shown that $\equiv_a$ is not in general compatible with the modal operators.

▶ Fact 8. For every $\wedge$-semilattice $\mathbb{A}$ and every $a \in \mathbb{A}$,
1. the relation $\equiv_a$ is a congruence of $\mathbb{A}$.
2. If $\mathbb{A}$ is a distributive lattice, then $\equiv_a$ is a congruence of $\mathbb{A}$.
3. If $\mathbb{A}$ is a frame, then $\equiv_a$ is a congruence of $\mathbb{A}$.
4. If $\mathbb{A}$ is a Boolean algebra, then $\equiv_a$ is a congruence of $\mathbb{A}$.
5. If $\mathbb{A}$ is a Heyting algebra, then $\equiv_a$ is a congruence of $\mathbb{A}$.

Proof. 1. Let $b_i \equiv_a c_i$, $i = 1, 2$. Now we have $(b_1 \wedge b_2) \wedge a = b_1 \wedge (b_2 \wedge a) = b_1 \wedge (c_2 \wedge a) = (b_1 \wedge a) \wedge c_2 = (c_1 \wedge a) \wedge c_2 = (c_1 \wedge c_2) \wedge a$.

2. This item is a special case of 3.

3. Let us show that for every $S \subseteq \mathbb{A}$, $\bigvee\{[s] \mid s \in S\} = [\bigvee S]$: clearly $s \leq \bigvee S$ for every $s \in S$ implies the '$\leq$' direction; as to the '$\geq$', it is enough to show that if $[c]$ is such that $[s] \leq [c]$ for every $s \in S$ then $[\bigvee S] \leq [c]$, i.e. $\bigvee S \leq c'$ for some c' such that $c' \wedge a = c \wedge a$. By assumption, for every $s \in S$ there exist some s' and some c_s such that $c \wedge a = c_s \wedge a$, $s' \wedge a = s \wedge a$ and $s' \leq c_s$. Hence, for every $s \in S$ there exists some $c_s \in [c]$ such that

$$s \wedge a = s' \wedge a \leq c_s \wedge a.$$

Let $c' := \bigvee\{c_s \mid s \in S\}$; then by frame distributivity $c' \wedge a = \bigvee\{c_s \wedge a \mid s \in S\} = \bigvee\{c \wedge a \mid s \in S\} = \bigvee\{c \mid s \in S\} \wedge a = c \wedge a$; moreover

$$\bigvee S \wedge a = \bigvee\{s \wedge a \mid s \in S\} \leq \bigvee\{c_s \wedge a \mid s \in S\} \leq c'.$$

[6] The fact that i' and π form a section/retraction pair of maps is the reason why the construction which is here accounted for as a pseudo-quotient can be alternatively accounted for—as is done elsewhere in the literature (cf. [21, 7])—as a pseudo-subalgebra construction, dual to a pseudo-quotient (e.g. filtrations on Kripke models).

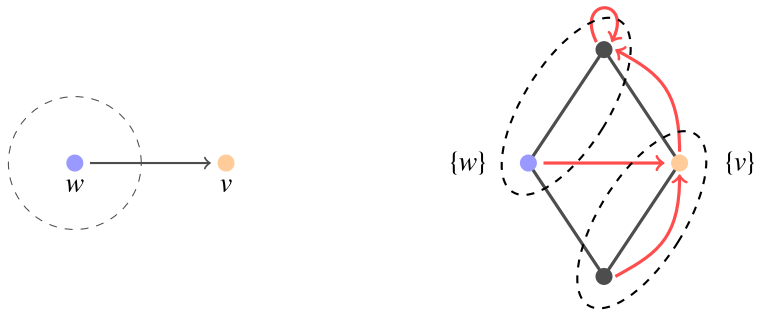


Figure 1 Frame and complex algebra of Example 9.

4. Assume that $b \equiv_a c$; then $b \wedge a = c \wedge a$ hence, by distributivity and de Morgan law,

$$\neg b \wedge a = (\neg b \vee \neg a) \wedge a = (\neg c \vee \neg a) \wedge a = \neg c \wedge a.$$

5. Recall that for every $b, c \in \mathbb{A}$, $b \to c = \bigvee\{x \mid b \wedge x \leq c\}$. Define $[b] \to [c] = \bigvee\{[x] \mid b \wedge x \leq c\}$. By item 3, $[b] \to [c] = [b \to c]$. Let us verify that for every $b, c, x \in \mathbb{A}$,

$$[b] \wedge [x] \leq [c] \ \text{ iff } \ [x] \leq [b] \to [c]:$$

$[b] \wedge [x] \leq [c]$ iff $(b \wedge a) \wedge (x \wedge a) \leq c \wedge a$ iff $b \wedge a \leq (x \wedge a) \to (c \wedge a) = a \to (x \to c)$ iff $[b] \leq [a \to (x \to c)] = [x \to c] = [x] \to [c]$. As to the last 'iff', from left to right it is clear; from right to left, $[b] \leq [a \to (x \to c)]$ implies that the canonical representant of $[b]$, which is $b \wedge a$, is less than or equal to the canonical representant of $[a \to (x \to c)]$, which is the minimum of $[a \to (x \to c)]$ and so in particular is less than or equal to $a \to (x \to c)$. ◀

▶ **Example 9.** Let us consider the model $M = (\{w, v\}, \{(w, v)\}, V(p) = \{w\})$. The submodel $M^a = M^p = (\{w\}, \varnothing, V^p)$, represented in Figure 1 on the left by a dashed circle, is not a generated submodel of M. If $\mathcal{F}$ is the underlying Kripke frame of M, the complex algebra $\mathcal{F}^+$ is depicted in above figure, on the right side, the arrows of which represent the modal operator $\Box$. The dashed ellipses represent the equivalence classes of the relation $\equiv_a$, for $a = \{w\} \in \mathcal{F}^+$. Then clearly, $a \equiv_a \top$, but $\Box a = \{v\} \not\equiv_a \top = \Box\top$.

3.3 Modal operations on the pseudo-quotient algebra

Since $\equiv_a$ is not in general compatible with the modal operators, $\mathbb{A}^a$ does not canonically inherit the structure of modal expansion from $\mathbb{A}$. In the present subsection, the modalities will be defined on the algebra $\mathbb{A}^a$ in such a way that, when $\mathbb{A} = \mathcal{F}^+$ for some Kripke frame $\mathcal{F}$, we get $\mathbb{A}^a \cong_{BAO} \mathcal{F}^{a+}$. Throughout the present subsection, $\mathbb{A}$ will be a Heyting algebra. In order to enable a separate treatment for each modal operator, just in the present subsection we will consider Heyting algebras expanded with possibly one normal modal operator at a time. In these situations, we will use symbols such as $(\mathbb{A}, \Diamond)$ and $(\mathbb{A}, \Box)$.

3.3.1 The diamond operation

Let $(\mathbb{A}, \Diamond)$ be a HAO. Define for every $b \in \mathbb{A}$,

$$\Diamond^a[b] := [\Diamond(b \wedge a) \wedge a] = [\Diamond(b \wedge a)].$$

The stipulation above is of course justified by the discussion in Subsection 3.1, and by Proposition 5 in particular. Facts 10.3 and 11.3 have exactly the same content of Proposition 5; however, while in that subsection the definition of the quotient algebra has been obtained from the desideratum that the statement of the Proposition 5 be true, in the present subsection the definition of the quotient algebra, given in its full generality, is taken as the starting point, so the proofs below reflect this change in perspective.

▶ **Fact 10.** For every HAO $(\mathbb{A}, \Diamond)$ and every $a \in \mathbb{A}$,

1. $\Diamond^a$ is a normal modal operator. Hence $(\mathbb{A}^a, \Diamond^a)$ is a HAO.
2. For every Kripke frame $\mathcal{F} = (W, R)$ and all $X, a \subseteq W$, $R^{a-1}[X \cap a] = R^{-1}[X \cap a] \cap a$.
3. If $\mathbb{A} = \mathcal{F}^+$ for some Kripke frame $\mathcal{F}$, then $\mathbb{A}^a \cong_{BAO} \mathcal{F}^{a+}$.

Proof. 1. $\Diamond^a[\bot] := [\Diamond(\bot \wedge a) \wedge a] = [\Diamond(\bot) \wedge a] = [\bot \wedge a] = [\bot]$. By distributivity, and the fact that π commutes with $\vee$,

$$\begin{array}{lcl}
\Diamond^a[b \vee c] & := & [\Diamond((b \vee c) \wedge a) \wedge a] \\
& = & [\Diamond((b \wedge a) \vee (c \wedge a)) \wedge a] \\
& = & [(\Diamond(b \wedge a) \vee \Diamond(c \wedge a)) \wedge a] \\
& = & [(\Diamond(b \wedge a) \wedge a) \vee (\Diamond(c \wedge a) \wedge a)] \\
& = & [\Diamond(b \wedge a) \wedge a] \vee [\Diamond(c \wedge a) \wedge a] \\
& = & \Diamond^a[b] \vee \Diamond^a[c].
\end{array}$$

2. Immediate.

3. By definition, $\mathcal{F}^{a+} = (\mathcal{P}(W^a), \langle R^a \rangle)$ and $\mathcal{F}^{+a} = \mathcal{F}^+/\equiv_a$. By Fact 6, for every $[X] \in \mathcal{F}^{+a}$ there exists a unique subset $Y = X \cap a$ of W that is a member of $[X]$ and is also a subset of a. So the assignment $[X] \mapsto X \cap a$ defines a map $\mu : \mathcal{F}^{+a} \longrightarrow \mathcal{F}^{a+}$. If $[X] \neq [X']$ then $X \cap a \neq X' \cap a$, which proves that μ is injective. If $X \in \mathcal{F}^{a+}$, then $X \subseteq a \subseteq W$, hence $X \in \mathcal{F}^+$; so $[X] \in \mathcal{F}^{+a}$ and moreover $\mu([X]) = X$, which shows that μ is surjective. Let us show that μ is a BAO homomorphism:

$$\mu(\neg^{\mathcal{F}^{+a}}[X]) = \neg^{\mathcal{F}^{a+}}(\mu([X]))$$

Since $\equiv_a$ is compatible with Boolean negation, $\neg^{\mathcal{F}^{+a}}[X] = [\neg^{\mathcal{F}^+}X] = [W \setminus X]$; hence $\mu(\neg^{\mathcal{F}^{+a}}[X]) = (W \setminus X) \cap a = a \setminus X$. On the other hand, $\neg^{\mathcal{F}^{a+}}(\mu([X])) = a \setminus \mu([X]) = a \setminus (X \cap a) = a \setminus X$. Let us show that

$$\mu(\Diamond^a[X]) = \langle R^a \rangle \mu([X]).$$

By definition, $\Diamond^a[X] = [\Diamond^{\mathcal{F}^+}(X \cap a)] = [\langle R \rangle (X \cap a)] = [R^{-1}[X \cap a]]$, hence $\mu(\Diamond^a[X]) = R^{-1}[X \cap a] \cap a$. On the other hand, $\langle R^a \rangle \mu([X]) = R^{a-1}[\mu([X])] = R^{a-1}[X \cap a]$. Then the claim immediately follows from the item 2 above. The remaining cases are left to the reader. ◀

3.3.2 The box operation

Let $(\mathbb{A}, \Box)$ be a HAO. Define for every $b \in \mathbb{A}$,

$$\Box^a[b] := [a \to \Box(a \to b)] = [\Box(a \to b)].$$

The second equality holds since, by Fact 25.1, $a \wedge (a \to \Box(a \to b)) \leq \Box(a \to b)$, and by Fact 25.3, $a \wedge \Box(a \to b) \leq a \to \Box(a \to b)$.

▶ **Fact 11.** For every HAO $(\mathbb{A}, \Box)$ and every $a \in \mathbb{A}$,

1. $\Box^a$ is a normal modal operator.
2. If $(\mathbb{A}, \Box)$ is a BAO and $\Box = \neg\Diamond\neg$, then $\Box^a = \neg\Diamond^a\neg$.
3. If $\mathbb{A} = \mathcal{F}^+$ for some Kripke frame $\mathcal{F}$, then $\Box^a = [R^a]$, hence $\mathbb{A}^a \cong_{BAO} \mathcal{F}^{a+}$.

Proof. 1. Since $\top \leq a \to \top$, $\Box^a[\top] := [a \to \Box(a \to \top)] = [a \to \Box(\top)] = [a \to \top] = [\top]$.

$$
\begin{array}{rcll}
\Box^a[b \wedge c] & := & [a \to \Box(a \to (b \wedge c))] & \\
& = & [a \to (\Box((a \to b) \wedge (a \to c)))] & \text{(Fact 25.6)} \\
& = & [a \to (\Box(a \to b) \wedge \Box(a \to c))] & \\
& = & [(a \to \Box(a \to b)) \wedge (a \to \Box(a \to c))] & \text{(Fact 25.6)} \\
& = & [(a \to \Box(a \to b))] \wedge [(a \to \Box(a \to c))] & \\
& = & \Box^a[b] \wedge \Box^a[c]. &
\end{array}
$$

2.

$$
\begin{array}{rclcl}
\Box^a[b] & = & [a \to \Box(a \to b)] & = & [\neg a \vee \Box(\neg a \vee b)] \\
& = & [\neg\neg(\neg a \vee \Box(\neg a \vee b))] & = & \neg[\neg(\neg a \vee \Box(\neg a \vee b))] \\
& = & \neg[a \wedge \neg\Box(\neg a \vee b))] & = & \neg[a \wedge \neg\Box(\neg\neg(\neg a \vee b)))] \\
& = & \neg[a \wedge \neg\Box\neg(\neg(\neg a \vee b))] & = & \neg[a \wedge \Diamond(a \wedge \neg b)] \\
& = & \neg\Diamond^a[\neg b] & = & \neg\Diamond^a\neg[b].
\end{array}
$$

3. By Fact 10.4, $(\mathbb{A}^a, \Diamond^a) \cong (\mathcal{P}(W^a), \langle R^a \rangle)$. So the statement immediately follows from this and item 2 above. ◀

3.3.3 Pseudo-quotients of FSAs and of MHAs

The following fact shows that the pseudo-quotient construction preserves the axioms defining the algebras (MHAs and FSAs) that are immediately relevant to the present paper; in the item 3 of the fact below, we observe that the adjunction relations $\Diamond \dashv \blacksquare$ and $\blacklozenge \dashv \Box$, well known from tense modal logic, are also preserved. A *tense HAO* is a Heyting algebra expansion $(\mathbb{A}, \Diamond, \Box, \blacklozenge, \blacksquare)$ such that the adjunction relations above hold (more details can be found in the proof of the item 3).

▶ Fact 12. **1.** For every MHA $(\mathbb{A}, \Diamond, \Box)$ and every $a \in \mathbb{A}$, the algebra $(\mathbb{A}^a, \Diamond^a, \Box^a)$ is a MHA.
2. For every FSA $(\mathbb{A}, \Diamond, \Box)$ and every $a \in \mathbb{A}$, the algebra $(\mathbb{A}^a, \Diamond^a, \Box^a)$ is a FSA.
3. For every tense HAO $(\mathbb{A}, \Diamond, \Box, \blacklozenge, \blacksquare)$ and every $a \in \mathbb{A}$, the algebra $(\mathbb{A}^a, \Diamond^a, \Box^a, \blacklozenge^a, \blacksquare^a)$ is a tense HAO.

Proof. 1. By Fact 8.5, $\mathbb{A}^a$ is a HA, so we only need to show the validity of the modal axioms. Since $\mathbb{A}$ is a MHA, in particular, for every $b \in \mathbb{A}$, the following inequalities hold: $b \wedge a \leq \Diamond(b \wedge a)$ and $\Box(a \to b) \leq a \to b$, which imply that $[b] \leq \Diamond^a[b]$ and $\Box^a[b] \leq [b]$. Likewise, for all $b, c \in \mathbb{A}$, the following Heyting inequality holds: $(a \to (b \to c)) \leq (a \to b) \to (a \to c)$, which yields, by monotonicity, $\Box(a \to (b \to c)) \leq \Box((a \to b) \to (a \to c)) \leq \Box(a \to b) \to \Box(a \to c)$; this implies that $\Box^a([b] \to [c]) \leq \Box^a[b] \to \Box^a[c]$. For all $b, c \in \mathbb{A}$, we have:

$$(a \wedge b) \to (a \wedge c) = ((a \wedge b) \to a) \wedge ((a \wedge b) \to c) = \top \wedge (a \to (b \to c)) = (a \to (b \to c));$$

hence, $\Box(a \to (b \to c)) = \Box((a \wedge b) \to (a \wedge c)) \leq \Diamond(a \wedge b) \to \Diamond(a \wedge c)$, which yields $\Box^a([b] \to [c]) \leq \Diamond^a[b] \to \Diamond^a[c]$. By monotonicity, from $a \wedge \Box(a \to b) \leq \Box(a \to b)$ we get:

$$\Diamond(a \wedge \Box(a \to b)) \leq \Diamond\Box(a \to b) \leq \Box(a \to b),$$

which yields $\Diamond^a\Box^a[b] \leq \Box^a[b]$. Likewise, by monotonicity, from $\Diamond(a \wedge b) \leq a \to \Diamond(a \wedge b)$ we get:

$$\Diamond(a \wedge b) \leq \Box\Diamond(a \wedge b) \leq \Box(a \to \Diamond(a \wedge b)),$$

which yields $\Diamond^a[b] \leq \Box^a\Diamond^a[b]$.
2. Since $a \wedge (b \to c) \leq (a \to b) \to (a \wedge c)$, by monotonicity we have:

$$\Diamond(a \wedge (b \to c)) \leq \Diamond((a \to b) \to (a \wedge c)) \leq \Box(a \to b) \to \Diamond(a \wedge c),$$

which yields $\Diamond^a([b] \to [c]) \leq \Box^a[b] \to \Diamond^a[c]$. Since $(a \wedge b) \to (a \to c) \leq (a \to (b \to c))$, by monotonicity we have:

$$\Diamond(a \wedge b) \to \Box(a \to c) \leq \Box((a \wedge b) \to (a \to c)) \leq \Box(a \to (b \to c)),$$

which yields $\Diamond^a[b] \to \Box^a[c] \leq \Box^a([b] \to [c])$.
3. By assumption we have that for all $x, y \in \mathbb{A}$,

$$\Diamond x \leq y \quad \text{iff} \quad x \leq \blacksquare y \qquad \text{and} \qquad \blacklozenge x \leq y \quad \text{iff} \quad x \leq \Box y.$$

$\Diamond^a[x] \leq [y]$ iff $a \wedge \Diamond(a \wedge x) \leq y$, iff $\Diamond(a \wedge x) \leq a \to y$, iff $a \wedge x \leq \blacksquare(a \to y)$, iff $[x] \leq \blacksquare^a[y]$. The second equivalence is proved analogously. ◀

4 Intuitionistic PAL

4.1 Axiomatization

Let AtProp be a countable set of proposition letters. The formulas of the (single-agent) *intuitionistic public announcement logic* IPAL are built up by the following inductive rule:

$$\phi ::= p \in \mathsf{AtProp} \mid \bot \mid \phi \vee \phi \mid \phi \wedge \phi \mid \phi \to \phi \mid \Diamond\phi \mid \Box\phi \mid \langle\phi\rangle\phi \mid [\phi]\phi.$$

The same stipulations hold for the defined connectives $\top$, $\neg$ and $\leftrightarrow$ as introduced early on. IPAL is axiomatically defined by the axioms and rules of IK (MIPC) plus the following axioms:

Interaction with logical constants	**Preservation of facts**
$\langle\alpha\rangle\bot \leftrightarrow \bot$, $\langle\alpha\rangle\top \leftrightarrow \alpha$	$\langle\alpha\rangle p \leftrightarrow \alpha \wedge p$
$[\alpha]\top \leftrightarrow \top$, $[\alpha]\bot \leftrightarrow \neg\alpha$	$[\alpha]p \leftrightarrow \alpha \to p$
Interaction with disjunction	**Interaction with conjunction**
$\langle\alpha\rangle(\phi \vee \psi) \leftrightarrow \langle\alpha\rangle\phi \vee \langle\alpha\rangle\psi$	$\langle\alpha\rangle(\phi \wedge \psi) \leftrightarrow \langle\alpha\rangle\phi \wedge \langle\alpha\rangle\psi$
$[\alpha](\phi \vee \psi) \leftrightarrow \alpha \to (\langle\alpha\rangle\phi \vee \langle\alpha\rangle\psi)$	$[\alpha](\phi \wedge \psi) \leftrightarrow [\alpha]\phi \wedge [\alpha]\psi$
Interaction with implication	
$\langle\alpha\rangle(\phi \to \psi) \leftrightarrow \alpha \wedge (\langle\alpha\rangle\phi \to \langle\alpha\rangle\psi)$	
$[\alpha](\phi \to \psi) \leftrightarrow \langle\alpha\rangle\phi \to \langle\alpha\rangle\psi$	
Interaction with diamond	**Interaction with box**
$\langle\alpha\rangle\Diamond\phi \leftrightarrow \alpha \wedge \Diamond\langle\alpha\rangle\phi$	$\langle\alpha\rangle\Box\phi \leftrightarrow \alpha \wedge \Box[\alpha]\phi$
$[\alpha]\Diamond\phi \leftrightarrow \alpha \to \Diamond\langle\alpha\rangle\phi$	$[\alpha]\Box\phi \leftrightarrow \alpha \to \Box[\alpha]\phi$

4.2 Models

▶ **Definition 13.** An *algebraic model* is a tuple $M = (\mathbb{A}, V)$ such that $\mathbb{A}$ is an FSA (resp. an MHA) (cf. Definition 3) and $V : \mathsf{AtProp} \to \mathbb{A}$.

Given such a model, we want to define its associated extension map $[\![\cdot]\!]_M : Fm \to \mathbb{A}$ so that, when $\mathbb{A} = \mathcal{F}^+$ for some Kripke frame $\mathcal{F}$, we recover the familiar extension map associated with the model $M = (\mathcal{F}, V)$. Notice that the satisfaction condition for $\langle\alpha\rangle$-formulas

$$M, w \Vdash \langle\alpha\rangle\phi \quad \text{iff} \quad M, w \Vdash \alpha \text{ and } M^\alpha, w \Vdash \phi$$

can be equivalently written as follows:

$$w \in [\![\langle\alpha\rangle\phi]\!]_M \quad \text{iff} \quad \exists w' \in W^\alpha \text{ such that } i(w') = w \in [\![\alpha]\!]_M \text{ and } w' \in [\![\phi]\!]_{M^\alpha}.$$

Because the map $i : M^\alpha \hookrightarrow M$ is injective, we get that $w' \in [\![\phi]\!]_{M^\alpha}$ iff $w = i(w') \in i[[\![\phi]\!]_{M^\alpha}]$. Hence,

$$w \in [\![\langle\alpha\rangle\phi]\!]_M \quad \text{iff} \quad w \in [\![\alpha]\!]_M \cap i[[\![\phi]\!]_{M^\alpha}],$$

from which we get that

$$[\![\langle\alpha\rangle\phi]\!]_M = [\![\alpha]\!]_M \cap i[[\![\phi]\!]_{M^\alpha}]. \tag{1}$$

Likewise, equivalently rewriting the following satisfaction condition for $[\alpha]$-formulas

$$M, w \Vdash [\alpha]\phi \quad \text{iff} \quad M, w \Vdash \alpha \text{ implies } M^\alpha, w \Vdash \phi$$

yields:

$$[\![[\alpha]\phi]\!]_M = [\![\alpha]\!]_M \Rightarrow i[[\![\phi]\!]_{M^\alpha}], \tag{2}$$

where $X \Rightarrow Y = (W \setminus X) \cup Y$ for every $X, Y \subseteq W$.

Finally, by Proposition 7, the direct image map $i[\cdot] : \mathcal{F}^{\alpha+} \to \mathcal{F}^+$ can be identified with the map $i' : \mathcal{F}^{+\alpha} \to \mathcal{F}^+$ under the identification $\mathcal{F}^{\alpha+} \cong \mathcal{F}^{+\alpha}$.

So we can adopt equations (1) and (2), modified by replacing $i[\cdot]$ with i', as the definitions of the extensions of $\langle\alpha\rangle\phi$ and $[\alpha]\phi$ respectively in any algebraic model $(\mathbb{A}, V)$:

▶ **Definition 14.** For every algebraic model $M = (\mathbb{A}, V)$, the *extension map* $[\![\cdot]\!]_M : \mathcal{L} \to \mathbb{A}$ is defined recursively as follows:

$$\begin{array}{rcl}
[\![p]\!]_M & = & V(p) \\
[\![\bot]\!]_M & = & \bot^{\mathbb{A}} \\
[\![\phi \vee \psi]\!]_M & = & [\![\phi]\!]_M \vee^{\mathbb{A}} [\![\psi]\!]_M \\
[\![\phi \wedge \psi]\!]_M & = & [\![\phi]\!]_M \wedge^{\mathbb{A}} [\![\psi]\!]_M \\
[\![\phi \to \psi]\!]_M & = & [\![\phi]\!]_M \to^{\mathbb{A}} [\![\psi]\!]_M \\
[\![\Diamond\phi]\!]_M & = & \Diamond^{\mathbb{A}}[\![\phi]\!]_M \\
[\![\Box\phi]\!]_M & = & \Box^{\mathbb{A}}[\![\phi]\!]_M \\
[\![\langle\alpha\rangle\phi]\!]_M & = & [\![\alpha]\!]_M \wedge^{\mathbb{A}} i'([\![\phi]\!]_{M^\alpha}) \\
[\![[\alpha]\phi]\!]_M & = & [\![\alpha]\!]_M \to^{\mathbb{A}} i'([\![\phi]\!]_{M^\alpha})
\end{array}$$

Here, $M^\alpha = (\mathbb{A}^\alpha, V^\alpha)$ such that $\mathbb{A}^\alpha = \mathbb{A}^{[\![\alpha]\!]_M}$ and $V^\alpha : \mathsf{AtProp} \to \mathbb{A}^\alpha$ is $\pi \circ V$, i.e. for every $p \in \mathsf{AtProp}$,

$$[\![p]\!]_{M^\alpha} = V^\alpha(p) = \pi(V(p)) = \pi([\![p]\!]_M).$$

Notice that, by Proposition 4, the above definition specializes to those algebraic models $(\mathbb{A}, V)$ such that $\mathbb{A} = \mathcal{F}^+$ is the complex algebra of some IK-frame (MIPC-frame) $\mathcal{F}$, and from those, to their relational counterparts $(\mathcal{F}, V)$. Hence, as a special case of the definition above we get an interpretation of IPAL on relational IK-models (MIPC-models). In the next subsection, we are going to expand on this.

4.3 Relational semantics for IPAL

In order to be able to see what the semantics of Definition 14 amounts to over IK-frames (MIPC-frames), we need to dualize back the FSAs (MHAs) and the pseudo quotient map $\mathbb{A} \to \mathbb{A}^a$. As is well known (cf. [6]), dualizing the FSAs (MHAs) is possible in full generality, and the resulting construction involves the intuitionistic counterparts of descriptive general frames in classical modal logic, i.e. relational structures endowed with topologies. However, obtaining the purely relational IK-frames (MIPC-frames) is possible for certain special FSAs (MHAs), which we call *perfect* FSAs

(MHAs), in order to define which, we need some preliminary definitions: A poset P is a *complete lattice* if the joins and meets of arbitrary subsets of P exist, in which case, P is *completely distributive* if arbitrary meets distribute over arbitrary joins. For every complete lattice $P = (X, \leq)$, a non-bottom element $x \in X$ is *completely join-prime* if, for every $S \subseteq X$ such that $x \leq \bigvee S$, there exists some $s \in S$ such that $x \leq s$; a non-top element $y \in X$ is *completely meet-prime* if, for every $S \subseteq X$ such that $\bigwedge S \leq y$, there exists some $s \in S$ such that $s \leq y$. Let $J^\infty(P)$ and $M^\infty(P)$ respectively denote the set of the completely join-prime elements and the set of the completely meet-prime elements in P. A poset P is *completely join-generated* (resp. *completely meet-generated*) by a given $S \subseteq P$ if for every $x \in P$, $x = \bigvee S'$ (resp. $x = \bigwedge S'$) for some $S' \subseteq S$.

► **Definition 15.** An HA $\mathbb{A}$ is *perfect* if it is a complete and completely distributive lattice w.r.t. its natural ordering, and is also completely join-generated by $J^\infty(\mathbb{A})$ (or equivalently, completely meet-generated by $M^\infty(\mathbb{A})$). An HAO $(\mathbb{A}, \Diamond, \Box)$ is *perfect* if $\mathbb{A}$ is a perfect HA, and moreover, $\Diamond$ distributes over arbitrary joins and $\Box$ distributes over arbitrary meets. A *perfect* FSA (MHA) is an FSA (MHA) which is also a perfect HAO.

Clearly, any finite HA(O) is perfect. It is well known that a Heyting algebra $\mathbb{A}$ is perfect iff it is isomorphic to $\mathcal{P}^{\downarrow}(P)$, where $P = (J^\infty(\mathbb{A}), \leq)$ and $\leq$ is the restriction of the natural ordering of $\mathbb{A}$ to $J^\infty(\mathbb{A})$. The Boolean self-duality $u \mapsto \neg u$ generalizes, in the HA setting, to the maps $\kappa : \mathbb{A} \to \mathbb{A}$, given by $x \mapsto \bigvee\{x' \mid x' \nleq x\}$, and $\lambda : \mathbb{A} \to \mathbb{A}$, given by $y \mapsto \bigwedge\{y' \mid y \nleq y'\}$. These maps induce order isomorphisms $\kappa : J^\infty(\mathbb{A}) \to M^\infty(\mathbb{A})$ and $\lambda : M^\infty(\mathbb{A}) \to J^\infty(\mathbb{A})$ (seen as subposets of $\mathbb{A}$). Clearly, $x \nleq \kappa(x)$ (resp. $\lambda(y) \nleq y$) for every $x \in J^\infty(\mathbb{A})$ (resp. $y \in M^\infty(\mathbb{A})$); moreover, for every $u \in \mathbb{A}$ and every $x \in J^\infty(\mathbb{A})$,

$$j \leq u \quad \text{iff} \quad u \nleq \kappa(j).$$

By the theory of adjunction on posets, it is well known that, in a perfect HAO $\mathbb{A}$, the properties of complete distributivity enjoyed by the modal operations imply that they are parts of adjoint pairs: unary operations $\blacklozenge$ and $\blacksquare$ are defined on $\mathbb{A}$ so that for all $x, y \in \mathbb{A}$,

$$\Diamond x \leq y \quad \text{iff} \quad x \leq \blacksquare y \qquad \text{and} \qquad \blacklozenge x \leq y \quad \text{iff} \quad x \leq \Box y.$$

As is done in Subsection **??**, these adjunction relations are denoted by writing $\Diamond \dashv \blacksquare$ and $\blacklozenge \dashv \Box$. One member of the adjunction relation completely determines the other. The choice of notation is a reminder of the fact that, by the general theory, $\blacklozenge$ distributes over arbitrary joins (i.e., it enjoys exactly the characterizing property of a 'diamond' operator on perfect algebras), and $\blacksquare$ distributes over arbitrary meets (i.e., it enjoys the characterizing property of a 'box' operator on perfect algebras). In particular, they are both order-preserving. Well known pairs of adjoint modal operators occur in temporal logic: its axiomatization essentially states that, when interpreted on algebras, the forward-looking diamond is *left adjoint* to the backward-looking box, and the backward-looking diamond is left adjoint to the forward-looking box. This is actually an essential feature: indeed R is the accessibility relation for one operation iff R^{-1} is the accessibility relation for the other.

Let us now introduce the intuitionistic counterpart of the atom structures for complete atomic BAOs:

► **Definition 16.** For every perfect FSA (MHA) $\mathbb{A}$, let us define $R \subseteq J^\infty(\mathbb{A}) \times J^\infty(\mathbb{A})$ by setting

$$xRy \quad \text{iff} \quad x \leq \Diamond y \text{ and } y \leq \blacklozenge x.$$

The *prime structure* associated with $\mathbb{A}$ is the relational structure $\mathbb{A}_+ := (J^\infty(\mathbb{A}), \leq, R)$.

Notice that $y \leq \blacklozenge x \quad \text{iff} \quad \blacklozenge x \nleq \kappa(y) \quad \text{iff} \quad x \nleq \Box\kappa(y)$.

► Fact 17. For every perfect HAO $\mathbb{A}$,

1. if $\mathbb{A}$ is an FSA, then $\mathbb{A}_+$ is an IK-frame;
2. if $\mathbb{A}$ is an MHA, then $\mathbb{A}_+$ is an MIPC-frame.

Proof. 1. In order to show that $(R \circ \geq) \subseteq (\geq \circ R)$, let $x, y', y \in J^\infty(\mathbb{A})$ such that xRy' and $y' \geq y$, i.e., $x \leq \Diamond y'$, and $x \nleq \Box\kappa(y')$, and $y' \geq y$. We need to show that $x' \leq x$ and $x' \leq \Diamond y$ and $x' \nleq \Box\kappa(y)$ for some $x' \in J^\infty(\mathbb{A})$. Because $\mathbb{A}$ is completely join-generated by $J^\infty(\mathbb{A})$, it is enough to show that $x \wedge \Diamond y \nleq \Box\kappa(y)$. Suppose for contradiction that $x \wedge \Diamond y \leq \Box\kappa(y)$. Then, by residuation and axiom FS2, we get $x \leq \Diamond y \to \Box\kappa(y) \leq \Box(y \to \kappa(y))$; since $\blacklozenge \dashv \Box$, we get that $\blacklozenge x \leq y \to \kappa(y)$, which, again by residuation, is equivalent to

$$\blacklozenge x \wedge y \leq \kappa(y).$$

Notice that the assumptions $y \leq y'$ and $x \nleq \Box\kappa(y')$ (the latter one can be equivalently rewritten as $y' \leq \blacklozenge x$) imply that $y \leq \blacklozenge x$. Hence, the centered inequality implies that $y \leq \kappa(y)$, contradiction.

In order to show that $(\leq \circ R) \subseteq (R \circ \leq)$, let $x, x', y \in J^\infty(\mathbb{A})$ such that $x \leq x'$ and $x'Ry$, i.e., $x \leq x'$, and $x' \leq \Diamond y$, and $x' \nleq \Box\kappa(y)$. We need to show that some $y' \in J^\infty(\mathbb{A})$ exists such that $x \leq \Diamond y'$ and $x \nleq \Box\kappa(y')$ and $y' \leq y$, i.e., such that $y' \nleq \blacksquare\kappa(x)$, and $y' \leq \blacklozenge x$, and $y' \leq y$. Because $\mathbb{A}$ is completely join-generated by $J^\infty(\mathbb{A})$, it is enough to show that $y \wedge \blacklozenge x \nleq \blacksquare\kappa(x)$. Suppose for contradiction that $y \wedge \blacklozenge x \leq \blacksquare\kappa(x)$. Then, by residuation, FS1, and Lemma 27, we get $y \leq \blacklozenge x \to \blacksquare\kappa(x) \leq \blacksquare(x \to \kappa(x))$; since $\Diamond \dashv \blacksquare$, we get that $\Diamond y \leq x \to \kappa(x)$, which, again by residuation, is equivalent to

$$\Diamond y \wedge x \leq \kappa(x).$$

The assumptions $x \leq x'$ and $x' \leq \Diamond y$ imply that $x \leq \Diamond y$. Hence, the centered inequality implies that $x \leq \kappa(x)$, contradiction.
Finally, the identity $R = (\geq \circ R) \cap (R \circ \leq)$ easily follows from the definition and the monotonicity of $\Diamond$ and $\blacklozenge$.
2. Because every MHA is a FSA, it only remains to be shown that the accessibility relation R in $\mathbb{A}_+$ is an equivalence relation. This can be readily seen by correspondence theory (cf. [9]); for instance, the axioms $p \to \Diamond p$ and $\Box p \to p$ respectively correspond to $x \leq \Diamond x$ and $x \leq \blacklozenge x$ for every state x of $\mathbb{A}_+$, which together express the reflexivity of R; in the presence of reflexivity, the correspondents of the axioms $\Diamond\Box p \to \Box p$ and of $\Diamond p \to \Box\Diamond p$ express a condition implying that R is the total relation on $J^\infty(\mathbb{A})$. ◀

▶ Proposition 18. For every perfect FSA $\mathbb{A}$, and every IK-frame $\mathcal{F}$,

$$\mathbb{A} \cong_{HAO} (\mathbb{A}_+)^+ \quad \text{and} \quad \mathcal{F} \cong (\mathcal{F}^+)_+.$$

Proof. It is well known that the canonical map $\eta : \mathbb{A} \to (\mathbb{A}_+)^+$, given by the assignment $u \mapsto \{x \mid x \in J^\infty(\mathbb{A}) \text{ and } x \leq u\}$, is a HA isomorphism. So we need to show that (a) $\eta(\Diamond u) = \langle R\rangle\eta(u)$ and (b) $\eta(\Box u) = [\geq \circ R]\eta(u)$.
(a): for the left-to-right inclusion, let $x \in J^\infty(\mathbb{A})$ such that $x \leq \Diamond u = \bigvee\{\Diamond y' \mid y' \in J^\infty(\mathbb{A}) \text{ and } y' \leq u\}$; hence, $x \leq \Diamond x'$ for some $x' \in J^\infty(\mathbb{A})$ such that $x' \leq u$. We need to show that some $y' \in J^\infty(\mathbb{A})$ exists such that $y' \leq u$, and $x \leq \Diamond y'$, and $y' \leq \blacklozenge x$, i.e., such that $y' \leq u$, and $y' \nleq \blacksquare\kappa(x)$, and $y' \leq \blacklozenge x$. Because $\mathbb{A}$ is completely join-generated by $J^\infty(\mathbb{A})$, it is enough to show that $x' \wedge \blacklozenge x \nleq \blacksquare\kappa(x)$. Suppose for contradiction that $x' \wedge \blacklozenge x \leq \blacksquare\kappa(x)$. Then, by residuation, FS1, and Lemma 27, we get $x' \leq \blacklozenge x \to \blacksquare\kappa(x) \leq \blacksquare(x \to \kappa(x))$; since $\Diamond \dashv \blacksquare$, we get that $\Diamond x' \leq x \to \kappa(x)$, which, again by residuation, is equivalent to

$$\Diamond x' \wedge x \leq \kappa(x).$$

But since $x \leq \Diamond x'$, the centered inequality implies that $x \leq \kappa(x)$, contradiction. The converse inclusion is immediate.
(b): for the left-to-right inclusion, let $x \in J^\infty(\mathbb{A})$ such that $x \leq \Box u = \bigvee\{y' \mid y' \in J^\infty(\mathbb{A}) \text{ and } y' \leq \Box u\}$;

hence, $x \leq x'$ for some $x' \in J^\infty(\mathbb{A})$ such that $x' \leq \Box u$, i.e., $\blacklozenge x' \leq u$. We need to show that, for all $y', z \in J^\infty(\mathbb{A})$, if $x \geq z$ and zRy', then $y' \leq u$. Indeed, the assumptions, and the monotonicity of $\blacklozenge$, imply that the following chain of inequalities holds: $y' \leq \blacklozenge z \leq \blacklozenge(x \wedge \Diamond y') \leq \blacklozenge x \leq \blacklozenge x' \leq u$. Conversely, let $x \in J^\infty(\mathbb{A})$ such that, for all $y', z \in J^\infty(\mathbb{A})$, if $x \geq z$ and $z \leq \Diamond y'$, and $y' \leq \blacklozenge z$, then $y' \leq u$; we need to show that $x \leq \Box u$, i.e., that $\blacklozenge x \leq u$. Because $\mathbb{A}$ is completely join-generated by $J^\infty(\mathbb{A})$, it is enough to show that, if $y' \in J^\infty(\mathbb{A})$ and $y' \leq \blacklozenge x$, then $y' \leq u$. By the assumptions on x (for $z = x$), it is enough to show that $x \leq \Diamond y'$, i.e., that $y' \nleq \blacksquare\kappa(x)$. Suppose for contradiction that $y' \nleq \blacksquare\kappa(x)$. Then, by Proposition 28 and Fact 26, $0 \neq y' \leq \blacksquare\kappa(x) \wedge \blacklozenge x \leq \blacklozenge(\kappa(x) \wedge x) = \blacklozenge 0 = 0$, contradiction.

Let $(\mathcal{F}^+)_+ = (J^\infty(\mathcal{P}^\downarrow(W)), \subseteq, R^*)$. Clearly, the canonical map $\epsilon : \mathcal{F} \to (\mathcal{F}^+)_+$, given by the assignment $x \mapsto x{\downarrow}$, is an order-isomorphism between $(W, \leq)$ and $(J^\infty(\mathcal{P}^\downarrow(W)), \subseteq)$. For every $x, y \in W$,

$$\begin{array}{rll}
x{\downarrow}R^*y{\downarrow} & \text{iff} & x{\downarrow} \subseteq \langle R\rangle y{\downarrow} \quad \text{and} \quad y{\downarrow} \subseteq \langle(\geq \circ R)^{-1}\rangle x{\downarrow} \\
& \text{iff} & x{\downarrow} \subseteq R^{-1}[y{\downarrow}] \quad \text{and} \quad y{\downarrow} \subseteq (\geq \circ R)[x{\downarrow}] \\
& \text{iff} & x{\downarrow} \subseteq (R \circ \leq)^{-1}[y] \quad \text{and} \quad y{\downarrow} \subseteq (\geq \circ R)[x] \\
(**) & \text{iff} & x \in (R \circ \leq)^{-1}[y] \quad \text{and} \quad y \in (\geq \circ R)[x] \\
(*) & \text{iff} & xRy.
\end{array}$$

The condition $(R \circ \leq) \subseteq (\leq \circ R)$ implies that $x{\downarrow} \subseteq (R \circ \leq)^{-1}[y]$ iff $x \in (R \circ \leq)^{-1}[y]$; the condition $(\geq \circ R) \subseteq (R \circ \geq)$ implies that $y{\downarrow} \subseteq (\geq \circ R)[x]$ iff $y \in (\geq \circ R)[x]$; the equivalence marked with $(*)$ holds because $R = (R \circ \leq) \cap (\geq \circ R)$. ◀

The bijective correspondence above, between perfect FSAs and IK-frames, specializes to MHAs and MIPC-frames, and also extends to homomorphisms and p-morphisms; in short, it is a duality, but treating it in detail is out of the aims of the present paper.

▶ **Definition 19.** For every IK-frame $\mathcal{F} = (W, \leq, R)$ and every down-set $a \subseteq W$, let $\mathcal{F}^a = (W^a, \leq^a, R^a)$ be defined in the usual way, i.e., $W^a := a$, and $\leq^a := \leq \cap (W^a \times W^a)$ and $R^a := R \cap (W^a \times W^a)$.

Because a is a down-set, it is easy to see that $\mathcal{F}$ being an IK-frame implies that $\mathcal{F}^a$ is an IK-frame. The remainder of the present subsection focuses on showing that, for every perfect FSA $\mathbb{A}$ and every $a \in \mathbb{A}$,

$$(\mathbb{A}^a)_+ \cong (\mathbb{A}_+)^a.$$

▶ Fact 20. For every HA $\mathbb{A}$ and every $a \in \mathbb{A} \setminus \{\bot\}$, we have $[b] \in J^\infty(\mathbb{A}^a)$ iff $(b \wedge a) \in J^\infty(\mathbb{A})$.

Proof. Clearly, $[b] \neq \bot^{\mathbb{A}^a}$ iff $(b \wedge a) \neq \bot^{\mathbb{A}^a}$. From left to right, let $b \wedge a \leq \bigvee S$ for some $S \subseteq \mathbb{A}$. Then $[b] \leq [\bigvee S] = \bigvee\{[x] \mid x \in S\}$. By assumption, $[b] \leq [s]$ for some $s \in S$, i.e. by Fact 6.2, $b \wedge a \leq s \wedge a \leq s$. Conversely, let $[b] \leq \bigvee\{[x] \mid x \in S\} = [\bigvee S]$. Then $b \wedge a \leq \bigvee S \wedge a = \bigvee\{x \wedge a \mid x \in S\}$. By assumption, $b \wedge a \leq s \wedge a$ for some $s \in S$, hence $[b] \leq [s]$, which proves the statement. ◀

The fact above implies that the assignment $[b] \mapsto b \wedge a$ defines a bijective correspondence between the states in $(\mathbb{A}^a)_+$ and the states in $(\mathbb{A}_+)^a$. If $[x], [y] \in J^\infty(\mathbb{A}^a)$, then $[x] \leq [y]$ iff $x \wedge a \leq y \wedge a$; moreover $[x]R[y]$ iff $[x] \leq \Diamond^a[y]$ and $[y] \leq \blacklozenge^a[x]$, iff $x \wedge a \leq \Diamond(y \wedge a)$ and $y \wedge a \leq \Diamond(x \wedge a)$. This finishes the proof that $(\mathbb{A}^a)_+ \cong (\mathbb{A}_+)^a$.

The identification between these two relational structures implies that the mechanism of epistemic update for public announcements remains largely unchanged when generalizing from the Boolean to the intuitionistic setting.

4.4 Soundness and completeness for IPAL

▶ Proposition 21. IPAL is sound with respect to algebraic IK-models (MIPC-models), hence with respect to relational IK- models (MIPC-models).

Proof. The soundness of the preservation of facts and logical constants follows from Lemma 30. The soundness of the remaining axioms is proved in Lemmas 31, 32, 33, 34, 35 of the appendix. ◀

▶ **Theorem 22.** *IPAL is complete with respect to relational IK-models (MIPC-models).*

Proof. The proof is analogous to the proof of completeness of classical PAL [3, Theorem 27], and follows from the reducibility of IPAL to IK (MIPC) via the reduction axioms. Let ϕ be a valid IPAL formula. Let us consider some innermost occurrence of a dynamic modality in ϕ. Hence, the subformula ψ having that occurrence labelling the root of its generation tree is either of the form $[\alpha]\psi'$ or of the form $\langle\alpha\rangle\psi'$, for some formula ψ' in the static language. The distribution axioms make it possible to equivalently transform ψ by pushing the dynamic modality down the generation tree, through the static connectives, until it attaches to a proposition letter or to a constant symbol. Here, the dynamic modality disappears, thanks to an application of the appropriate 'preservation of facts' or 'interaction with logical constant' axiom. This process is repeated for all the dynamic modalities of ϕ, so as to obtain a formula ϕ' which is provably equivalent to ϕ. Since ϕ is valid by assumption, and since the process preserves provable equivalence, by soundness we can conclude that ϕ' is valid. By Proposition 2, we can conclude that ϕ' is provable in IK (MIPC), hence in IPAL. This, together with the provable equivalence of ϕ and ϕ', concludes the proof. ◀

5 The Muddy Children Puzzle, intuitionistically

> After having played outside, $k \geq 1$ of n children have got mud on their foreheads. They can only see the others, so they do not know their own status. Now their Father comes along and says: "At least one of you is dirty". He then asks: "Do you know whether your own forehead is dirty?" Children answer truthfully, and this is repeated round by round. As questions and answers repeat, what will happen?
> There is a straightforward proof by induction that the first $k - 1$ times he asks the question, they will all answer "No," but then, at the kth time, the children with muddy foreheads will all answer "Yes."

If $k = 1$, then the only dirty child knows that all the other children are clean, so his/her uncertainty is about whether the total number of dirty children is 0 or 1 (in the latter case, he/she will be dirty). *Learning* from Father that there is at least one dirty child among them takes away the uncertainty, and enables the conclusion that he/she is dirty[7]. As to the inductive step, suppose the statement is true for k dirty children, and let us show it for $k + 1$ dirty children. In this case, each dirty child sees k dirty children, so his/her uncertainty is about whether the total number of dirty children is k or $k + 1$ (in the latter case, he/she will be dirty). If there were only k dirty children, then, by induction hypothesis, each of them would know at round $k - 1$. However, at round $k - 1$, each dirty child *learns* that none of the others knows. Again, this takes away the uncertainty, and enables the conclusion that he/she is dirty[8].

7 Notice that the difference with the clean children is that each clean child sees one dirty child, so each clean child's uncertainty is about whether the total number of dirty children is 1 or 2 (in the latter case, he/she will be dirty). Father's public announcement is uninformative for the clean children; the only point at which they learn what they need to conclude something about their own status is when the dirty child announces that he /she knows.

8 Again, the same holds for clean children, but because they see more dirty children than each dirty child, the clean children are one step behind.

The epistemic scenario of the Muddy Children Puzzle described above has become perhaps the main test case in the literature on dynamic epistemic logic, starting with [12, 13], which formalized the argument above in the object language and entailment of classical PAL with common knowledge. Since [12, 13], various generalizations, refinements and extensions of this original scenario have been treated in the literature. The present section is aimed at showing that the argument above can be formalized in the object language and by the entailment of IPAL. While it is well known that, this being a finite scenario, the common knowledge operator can be readily replaced by a suitable finitary approximation of it (see more on this below), the present analysis highlights the non-obvious fact that the crucial step of this reasoning ("If there were only k dirty children, then each of them would know at round $k-1$") is *not* an argument by contradiction, and hence, its nature is *not* inherently classical. This grounds our conviction that the phenomenon of dynamic updates can and is worth being studied in nonclassical contexts.

In what follows, we will need the n-agent version of IPAL, which we denote IPAL_n, whose language, if the set of agents is taken to be $\{1,\ldots,n\}$, is defined as one expects by considering indexed epistemic modalities $\Box_i$ and $\Diamond_i$ for $1 \le i \le n$, and whose axiomatization is given by correspondingly indexed copies of the IPAL axioms[9]. Derived modalities can be defined in the language of IPAL_n, which will act as finitary approximations of common knowledge: for every IPAL_n-formula ϕ, let $E\phi := \bigwedge_{i=1}^{n} \Box_i\phi$. The intended meaning of the defined connective E is 'Everybody knows'.[10] It is easy to see that $E\top \dashv\vdash_{IK_n} \top$ and $E(\phi \wedge \psi) \dashv\vdash_{IK_n} E\phi \wedge E\psi$. So E is a box-type normal modality. The following fact will be important:

▶ Fact 23. If $\mathcal{L}$ is an extension of IPAL_n with the axiom scheme $\Box_1 p \to p$, then $\vdash_{\mathcal{L}} Ep \to p$.

Proof. By induction on n. The base case is trivially true. Assume that the statement is true for $n-1$ agents, i.e., that $\vdash_{\mathcal{L}} (\bigwedge_{i=1}^{n-1} \Box_i p) \to p$; since $y \to (x \to y)$ and $[(x \wedge y) \to z] \leftrightarrow [x \to (y \to z)]$ are intuitionistic axiom schemes, the assumption implies that $\vdash_{\mathcal{L}} \Box_n p \to [(\bigwedge_{i=1}^{n-1} \Box_i p) \to p] \dashv\vdash_{\mathcal{L}} [(\bigwedge_{i=1}^{n} \Box_i p) \to p]$. ◀

For the sake of this example, the set of atomic propositions can be restricted to $At = \{D_i, C_i \mid 1 \le i \le n\}$, where D_i is the proposition saying 'child i is dirty', and C_i is the proposition saying 'child i is clean'. Let us introduce the following abbreviations:

- father $:= \bigvee_{i=1}^{n} D_i$ expresses the proposition publicly announced by Father;
- vision $:= \bigwedge\{(D_i \to \Box_j D_i) \wedge (C_i \to \Box_j C_i) \mid 1 \le i, j \le n \text{ and } i \ne j\}$ expresses the fact that every child knows whether each other child is clean or dirty;
- aut $:= \bigwedge_{i=1}^{n}[(C_i \to \bot) \leftrightarrow D_i]$ expresses the fact that being clean or dirty are not only *mutually incompatible* conditions, but they are also *exhaustive*;
- no $:= \bigwedge_{i=1}^{n}(\Diamond_i D_i \wedge \Diamond_i C_i)$ expresses the ignorance of the children about their own status;
- dirty$(J) := (\bigwedge_{j\in J} D_j) \wedge (\bigwedge_{h\notin J} C_h)$, for each $J \subseteq \{1,\ldots,n\}$, expresses that all and only the children in J have dirty foreheads.

The aim of this section is proving the following

▶ Proposition 24. Let $\mathcal{L}$ be an extension of IPAL_n with aut and the axiom scheme $\Box_1 p \to p$. For every $\varnothing \ne J \subseteq \{1,\ldots,n\}$ such that $|J| = k$,

$$\mathsf{dirty}(J), E^k(\mathsf{vision}) \vdash_{\mathcal{L}} [\mathsf{father}][\mathsf{no}]^{k-1}\Box_j D_j$$

for each $j \in J$.

[9] For the remainder of this section, if $\mathcal{L}$ is one of the logics introduced so far, $\mathcal{L}_n$ will denote its n-agent version. For any logic $\mathcal{L}$, the relation of provable equivalence relative to $\mathcal{L}$ will be denoted by $\dashv\vdash_{\mathcal{L}}$.

[10] The connective E appears in the literature also under the name *distributed knowledge*.

Proof. By induction on k. If $J = \{1\}$, the statement becomes

$$\mathsf{dirty}(\{1\}), E(\mathsf{vision}) \vdash_{\mathcal{L}} [\mathsf{father}]\Box_1 D_1.$$

By the reduction axioms, we equivalently need to show that

$$\mathsf{dirty}(\{1\}), E(\mathsf{vision}) \vdash_{\mathcal{L}} \mathsf{father} \to \Box_1(\mathsf{father} \to D_1).$$

By the Deduction Detachment Theorem (DDT), this is equivalent to showing that

$$\mathsf{dirty}(\{1\}), E(\mathsf{vision}), \mathsf{father} \vdash_{\mathcal{L}} \Box_1(\mathsf{father} \to D_1).$$

Since $\mathsf{father} \to D_1 = (\bigvee_{i=1}^{n} D_i) \to D_1 \dashv\vdash_{\mathcal{L}} \bigwedge_{i=1}^{n}(D_i \to D_1) \dashv\vdash_{\mathcal{L}} \bigwedge_{h\neq 1}(D_h \to D_1)$, we also have that $\Box_1(\mathsf{father} \to D_1) \dashv\vdash_{\mathcal{L}} \bigwedge_{h\neq 1} \Box_1(D_h \to D_1)$. Hence, it is enough to show that, for every $h \neq 1$,

$$\mathsf{dirty}(\{1\}), E(\mathsf{vision}), \mathsf{father} \vdash_{\mathcal{L}} \Box_1(D_h \to D_1).$$

Fix $1 < h \leq n$. By FS2, it is enough to show that

$$\mathsf{dirty}(\{1\}), E(\mathsf{vision}), \mathsf{father} \vdash_{\mathcal{L}} \Diamond_1 D_h \to \Box_1 D_1,$$

and by DDT, this is equivalent to showing that

$$\mathsf{dirty}(\{1\}), E(\mathsf{vision}), \mathsf{father}, \Diamond_1 D_h \vdash_{\mathcal{L}} \Box_1 D_1.$$

Observe that, because of the direction $D_h \to (C_h \to \bot)$ of aut, we have $C_h \wedge D_h \vdash_{\mathcal{L}} \bot$. Hence, by the monotonicity of $\Diamond_1$, we get

$$\Diamond_1(C_h \wedge D_h) \vdash_{\mathcal{L}} \Diamond_1 \bot \vdash_{\mathcal{L}} \bot \vdash_{\mathcal{L}} \Box_1 D_1.$$

Hence, to finish the proof of the base case, it is enough to show that

$$\mathsf{dirty}(\{1\}), E(\mathsf{vision}), \mathsf{father}, \Diamond_1 D_h \vdash_{\mathcal{L}} \Diamond_1(C_h \wedge D_h).$$

By Fact 23, $E(\mathsf{vision}) \vdash_{\mathcal{L}} \mathsf{vision}$; moreover,

$$\mathsf{dirty}(\{1\}), \mathsf{vision} \vdash_{\mathcal{L}} \bigwedge_{h\neq 1} \Box_1 C_h \vdash_{\mathcal{L}} \Box_1 C_h.$$

Hence,

$$\begin{array}{rcl} \mathsf{dirty}(\{1\}), E(\mathsf{vision}), \Diamond_1 D_h & \vdash_{\mathcal{L}} & \mathsf{dirty}(\{1\}), \mathsf{vision}, \Diamond_1 D_h \\ & \vdash_{\mathcal{L}} & \Box_1 C_h \wedge \Diamond_1 D_h \\ \text{(FS1, Fact 26)} & \vdash_{\mathcal{L}} & \Diamond_1(C_h \wedge D_h), \end{array}$$

which finishes the proof of the base case. As to the induction step, the statement for $J = \{1, \ldots, k+1\}$ becomes:

$$\mathsf{dirty}(J), E^{k+1}\mathsf{vision} \vdash_{\mathcal{L}} [\mathsf{father}][\mathsf{no}]^k \Box_1 D_1.$$

It is enough to show that the following chain of entailments holds in IPAL[11]:

$$\begin{array}{rcl} & & \mathsf{dirty}(J), E^{k+1}\mathsf{vision} \\ \text{(Claim 1)} & \vdash_{\mathcal{L}} & \Box_1((C_1 \wedge \mathsf{father}) \to [\mathsf{father}][\mathsf{no}]^{k-1}\Box_2 D_2) \\ \text{(Claim 2)} & \vdash_{\mathcal{L}} & \Box_1(\mathsf{father} \to [\mathsf{father}][\mathsf{no}]^k D_1) \\ \text{(Claim 3)} & \vdash_{\mathcal{L}} & [\mathsf{father}]\Box_1[\mathsf{no}]^k D_1 \\ \text{(Claim 4)} & \vdash_{\mathcal{L}} & [\mathsf{father}][\mathsf{no}]^k \Box_1 D_1. \end{array}$$

[11] Modulo small differences in notation, this is the same chain of entailments in the proof of [12, Proposition 7.2]

As to Claim 1, let $J' = \{2, \ldots, k+1\}$; by the induction hypothesis,

$$\mathsf{dirty}(J'), E^k\mathsf{vision} \vdash_{\mathcal{L}} [\mathsf{father}][\mathsf{no}]^{k-1}\Box_2 D_2.$$

By Fact 23 we also have that

$$E^{k+1}\mathsf{vision} \vdash_{\mathcal{L}} \mathsf{vision} \wedge \Box_1 E^k\mathsf{vision};$$

moreover, observe that

$$\mathsf{dirty}(J), \mathsf{vision} \vdash_{\mathcal{L}} \Box_1(C_1 \to \mathsf{dirty}(J')) :$$

indeed, $\mathsf{dirty}(J') \vdash_{\mathcal{L}} \mathsf{dirty}(J')$ implies that

$$\bigwedge_{h \notin J} C_h \wedge \bigwedge_{j \in J'} D_j \vdash_{\mathcal{L}} C_1 \to \mathsf{dirty}(J'),$$

hence, by the monotonicity of $\Box_1$ we get

$$\mathsf{dirty}(J), \mathsf{vision} \vdash_{\mathcal{L}} \Box_1(\bigwedge_{h \notin J} C_h \wedge \bigwedge_{j \in J'} D_j) \vdash_{\mathcal{L}} \Box_1(C_1 \to \mathsf{dirty}(J')).$$

Therefore:

$$\begin{array}{ll} & \mathsf{dirty}(J), E^{k+1}\mathsf{vision} \\ \vdash_{\mathcal{L}} & \mathsf{dirty}(J), \mathsf{vision} \wedge \Box_1 E^k\mathsf{vision} \\ \vdash_{\mathcal{L}} & \Box_1(C_1 \to \mathsf{dirty}(J')) \wedge \Box_1 E^k\mathsf{vision} \\ \vdash_{\mathcal{L}} & \Box_1[(C_1 \to \mathsf{dirty}(J')) \wedge E^k\mathsf{vision}]. \end{array}$$

By the monotonicity of $\Box_1$, the following chain of entailments is then enough to finish the proof of Claim 1:

$$\begin{array}{lll} & & (C_1 \to \mathsf{dirty}(J')) \wedge E^k\mathsf{vision} \\ & \vdash_{\mathcal{L}} & (C_1 \to \mathsf{dirty}(J')) \wedge (C_1 \to E^k\mathsf{vision}) \\ & \vdash_{\mathcal{L}} & C_1 \to (\mathsf{dirty}(J') \wedge E^k\mathsf{vision}) \\ \text{(IH)} & \vdash_{\mathcal{L}} & C_1 \to [\mathsf{father}][\mathsf{no}]^{k-1}\Box_2 D_2 \\ & \vdash_{\mathcal{L}} & (C_1 \wedge \mathsf{father}) \to [\mathsf{father}][\mathsf{no}]^{k-1}\Box_2 D_2. \end{array}$$

As to Claim 2, by the monotonicity of $\Box_1$, it is enough to show that

$$(C_1 \wedge \mathsf{father}) \to [\mathsf{father}][\mathsf{no}]^{k-1}\Box_2 D_2 \vdash_{\mathcal{L}} \mathsf{father} \to [\mathsf{father}][\mathsf{no}]^k D_1,$$

and since the premise of the entailment above is intuitionistically equivalent to $\mathsf{father} \to (C_1 \to [\mathsf{father}][\mathsf{no}]^{k-1}\Box_2 D_2)$, by the monotonicity of $\to$ in its second argument, it is enough to show that

$$C_1 \to [\mathsf{father}][\mathsf{no}]^{k-1}\Box_2 D_2 \vdash_{\mathcal{L}} [\mathsf{father}][\mathsf{no}]^k D_1.$$

By the reduction axioms we have:

$$\begin{array}{lll} [\mathsf{father}][\mathsf{no}]^k D_1 & \dashv\vdash_{\mathcal{L}} & [\mathsf{father}][\mathsf{no}]^{k-1}(\mathsf{no} \to D_1) \\ & \dashv\vdash_{\mathcal{L}} & \langle\mathsf{father}\rangle\langle\mathsf{no}\rangle^{k-1}\mathsf{no} \to \langle\mathsf{father}\rangle\langle\mathsf{no}\rangle^{k-1} D_1; \end{array}$$

hence, by DDT, proving the entailment above is equivalent to proving that

$$C_1 \to [\mathsf{father}][\mathsf{no}]^{k-1}\Box_2 D_2, \langle\mathsf{father}\rangle\langle\mathsf{no}\rangle^{k-1}\mathsf{no} \vdash_{\mathcal{L}} \langle\mathsf{father}\rangle\langle\mathsf{no}\rangle^{k-1} D_1.$$

Indeed,

$$\begin{array}{rll} & & C_1 \to [\mathsf{father}][\mathsf{no}]^{k-1}\Box_2 D_2, \langle\mathsf{father}\rangle\langle\mathsf{no}\rangle^{k-1}\mathsf{no} \\ & \vdash_{\mathcal{L}} & C_1 \to [\mathsf{father}][\mathsf{no}]^{k-1}\Box_2 D_2, C_1 \to \langle\mathsf{father}\rangle\langle\mathsf{no}\rangle^{k-1}\mathsf{no} \\ & \vdash_{\mathcal{L}} & C_1 \to ([\mathsf{father}][\mathsf{no}]^{k-1}\Box_2 D_2 \wedge \langle\mathsf{father}\rangle\langle\mathsf{no}\rangle^{k-1}\mathsf{no}) \\ & \vdash_{\mathcal{L}} & C_1 \to ([\mathsf{father}][\mathsf{no}]^{k-1}\Box_2 D_2 \wedge \langle\mathsf{father}\rangle\langle\mathsf{no}\rangle^{k-1}\Diamond_2 C_2) \\ \text{(Lemma 36.3)} & \vdash_{\mathcal{L}} & C_1 \to \bot. \end{array}$$

By the direction $(C_1 \to \bot) \to D_1$ of aut, we have $C_1 \to \bot \vdash_{\mathcal{L}} D_1$. Hence, the following chain of entailments holds:

$$\begin{array}{rll} & & C_1 \to [\mathsf{father}][\mathsf{no}]^{k-1}\Box_2 D_2, \langle\mathsf{father}\rangle\langle\mathsf{no}\rangle^{k-1}\mathsf{no} \\ & \vdash_{\mathcal{L}} & \langle\mathsf{father}\rangle\langle\mathsf{no}\rangle^{k-1}\mathsf{no}, C_1 \to \bot \\ & \vdash_{\mathcal{L}} & \langle\mathsf{father}\rangle\langle\mathsf{no}\rangle^{k-1}\mathsf{no}, D_1 \\ \text{(Lemma 36.6)} & \vdash_{\mathcal{L}} & \langle\mathsf{father}\rangle\langle\mathsf{no}\rangle^{k-1} D_1. \end{array}$$

Claim 3 immediately follows from the fact that, by the reduction axioms,

$$[\mathsf{father}]\Box_1[\mathsf{no}]^k D_1 \quad \dashv\vdash_{\mathcal{L}} \quad \mathsf{father} \to \Box_1(\mathsf{father} \to [\mathsf{father}][\mathsf{no}]^k D_1),$$

and the fact that $p \vdash q \to p$ in intuitionistic logic.
Finally, Claim 4 follows immediately by Lemma 36.5 and the monotonicity of [father]. ◀

Acknowledgements The research of the first author is supported by the grant 12CZX054 of the China national fund of social sciences. The research of the second author was supported by the Vidi grant 016.138.314 of the Netherlands Organization for Scientific Research (NWO). The research of the third author was supported by the EPSRC grant EP/F042728/1.

6 Appendix

6.1 Identities and inequalities valid on Heyting algebras

In a Heyting algebra $\wedge$ and $\to$ are *residuated*, namely, for all $x, y, z \in \mathbb{A}$,

$$x \wedge y \leq z \quad \text{iff} \quad x \leq y \to z. \tag{3}$$

Hence, by the general theory of residuation,

$$y \to z = \bigvee \{x \mid x \wedge y \leq z\}. \tag{4}$$

Using (3) and (4) above, it is not difficult to prove the following

► Fact 25. For every Heyting algebra $\mathbb{A}$ and all $x, y, z \in \mathbb{A}$,

1. $x \wedge (x \to y) \leq y$.
2. $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.
3. $x \wedge y \leq x \to y$.
4. $x \to y = x \to (x \wedge y)$.
5. $(x \wedge y) \to z = x \to (y \to z)$.
6. $x \wedge (y \to z) = x \wedge ((x \wedge y) \to z)$.

► Fact 26. The following are provably equivalent in IK:

1. $\Diamond(p \to q) \leq \Box p \to \Diamond q$;
2. $\Box p \wedge \Diamond q \leq \Diamond(p \wedge q)$;
3. $\Box(p \to q) \leq \Diamond p \to \Diamond q$.

Proof. 1 implies 2: by residuation, it is enough to show that $\Diamond q \leq \Box p \to \Diamond(p \wedge q)$. Indeed,

$$\Diamond q \leq \Diamond(p \to q) = \Diamond(p \to (p \wedge q)) \leq \Box p \to \Diamond(p \wedge q).$$

2 implies 1: by residuation, it is enough to show that $\Box p \wedge \Diamond(p \to q) \leq \Diamond q$. Indeed,

$$\Box p \wedge \Diamond(p \to q) \leq \Diamond(p \wedge (p \to q)) \leq \Diamond q.$$

2 implies 3: by residuation, it is enough to show that $\Diamond p \wedge \Box(p \to q) \leq \Diamond q$. Indeed, by 2,

$$\Diamond p \wedge \Box(p \to q) \leq \Diamond(p \wedge (p \to q)) \leq \Diamond q.$$

3 implies 2: by residuation, it is enough to show that $\Box p \leq \Diamond p \to \Diamond(p \wedge q)$. Indeed,

$$\Box q \leq \Box(p \to q) \leq \Box(p \to (p \wedge q)) \leq \Diamond p \to \Diamond(p \wedge q).$$

◀

▶ **Lemma 27.** *For every perfect HAO $\mathbb{A}$,*

1. *the following are equivalent:*
   a. $\mathbb{A} \models \Diamond(p \to q) \leq \Box p \to \Diamond q$;
   b. $\mathbb{A} \models \blacklozenge j \to \blacksquare n \leq \blacksquare(j \to n)$, *where j ranges over $J^\infty(\mathbb{A})$ and n ranges over $M^\infty(\mathbb{A})$.*
2. *The following are equivalent:*
   a. $\mathbb{A} \models \Diamond p \to \Box q \leq \Box(p \to q)$;
   b. $\mathbb{A} \models \Diamond j \to \Box n \leq \Box(j \to n)$, *where j ranges over $J^\infty(\mathbb{A})$ and n ranges over $M^\infty(\mathbb{A})$.*

Proof. The following proof is an instance of the algorithmic correspondence for DML developed in [9]. The reader is referred to that paper for the detailed justification of the following chain of logical equivalences. As to notation, the variables i, j range over $J^\infty(\mathbb{A})$, and the variables m, n range over $M^\infty(\mathbb{A})$.

$$
\begin{array}{ll}
 & \forall p \forall q[\Diamond(p \to q) \leq \Box p \to \Diamond q] \\
\text{iff} & \forall p \forall q \forall i \forall m[(i \leq \Diamond(p \to q) \;\&\; \Box p \to \Diamond q \leq m) \Rightarrow i \leq m] \\
\text{iff} & \forall p \forall q \forall i \forall m \forall j[(i \leq \Diamond(p \to q) \;\&\; j \to \Diamond q \leq m \;\&\; j \leq \Box p) \Rightarrow i \leq m] \\
\text{iff} & \forall p \forall q \forall i \forall m \forall j[(i \leq \Diamond(p \to q) \;\&\; j \to \Diamond q \leq m \;\&\; \blacklozenge j \leq p) \Rightarrow i \leq m] \\
\text{iff} & \forall q \forall i \forall m \forall j[(i \leq \Diamond(\blacklozenge j \to q) \;\&\; j \to \Diamond q \leq m) \Rightarrow i \leq m] \\
\text{iff} & \forall q \forall i \forall m \forall j \forall n[(i \leq \Diamond(\blacklozenge j \to q) \;\&\; j \to n \leq m \;\&\; \Diamond q \leq n) \Rightarrow i \leq m] \\
\text{iff} & \forall q \forall i \forall m \forall j \forall n[(i \leq \Diamond(\blacklozenge j \to q) \;\&\; j \to n \leq m \;\&\; q \leq \blacksquare n) \Rightarrow i \leq m] \\
\text{iff} & \forall i \forall m \forall j \forall n[(i \leq \Diamond(\blacklozenge j \to \blacksquare n) \;\&\; j \to n \leq m) \Rightarrow i \leq m] \\
\text{iff} & \forall i \forall j \forall n[i \leq \Diamond(\blacklozenge j \to \blacksquare n) \Rightarrow \forall m[j \to n \leq m \Rightarrow i \leq m]] \\
\text{iff} & \forall i \forall j \forall n[i \leq \Diamond(\blacklozenge j \to \blacksquare n) \Rightarrow i \leq j \to n] \\
\text{iff} & \forall j \forall n[\Diamond(\blacklozenge j \to \blacksquare n) \leq j \to n] \\
\text{iff} & \forall j \forall n[\blacklozenge j \to \blacksquare n \leq \blacksquare(j \to n)].
\end{array}
$$

$$
\begin{array}{ll}
 & \forall p \forall q[\Diamond p \to \Box q \leq \Box(p \to q)] \\
\text{iff} & \forall p \forall q \forall i \forall m[(i \leq \Diamond p \to \Box q \;\&\; \Box(p \to q) \leq m) \Rightarrow i \leq m] \\
\text{iff} & \forall p \forall q \forall i \forall m \forall n[(i \leq \Diamond p \to \Box q \;\&\; \Box(p \to n) \leq m \;\&\; q \leq n) \Rightarrow i \leq m] \\
\text{iff} & \forall p \forall i \forall m \forall n[(i \leq \Diamond p \to \Box n \;\&\; \Box(p \to n) \leq m) \Rightarrow i \leq m] \\
\text{iff} & \forall p \forall i \forall m \forall n \forall j[(i \leq \Diamond p \to \Box n \;\&\; \Box(j \to n) \leq m \;\&\; j \leq p) \Rightarrow i \leq m] \\
\text{iff} & \forall i \forall m \forall n \forall j[(i \leq \Diamond j \to \Box n \;\&\; \Box(j \to n) \leq m) \Rightarrow i \leq m] \\
\text{iff} & \forall i \forall n \forall j[i \leq \Diamond j \to \Box n \Rightarrow \forall m[\Box(j \to n) \leq m \Rightarrow i \leq m]] \\
\text{iff} & \forall i \forall n \forall j[i \leq \Diamond j \to \Box n \Rightarrow i \leq \Box(j \to n)] \\
\text{iff} & \forall n \forall j[\Diamond j \to \Box n \leq \Box(j \to n)].
\end{array}
$$

◄

► Proposition 28. For every perfect HA $\mathbb{A}$, its modal expansion $(\mathbb{A}, \Diamond, \Box)$ is a perfect FSA iff $(\mathbb{A}, \blacklozenge, \blacksquare)$ is.

Proof. By Lemma 27.1, FS1 is valid on $(\mathbb{A}, \Diamond, \Box)$ iff the restricted version of FS2 is valid on $(\mathbb{A}, \blacklozenge, \blacksquare)$; by Lemma 27.2 applied to $(\mathbb{A}, \blacklozenge, \blacksquare)$, this is equivalent to FS2 being valid on $(\mathbb{A}, \blacklozenge, \blacksquare)$. By Lemma 27.2, FS2 is valid on $(\mathbb{A}, \Diamond, \Box)$ iff its restricted version is valid on $(\mathbb{A}, \Diamond, \Box)$; by Lemma 27.1 applied to $(\mathbb{A}, \blacklozenge, \blacksquare)$, this is equivalent to FS1 being valid on $(\mathbb{A}, \blacklozenge, \blacksquare)$. ◄

6.2 Properties of the map i'

► Fact 29. Let $\mathbb{A}$ be an MIPC-algebra, $a \in \mathbb{A}$, and let $i' : \mathbb{A}^a \to \mathbb{A}$ given by $[b] \mapsto b \wedge a$. Then, for every $b, c \in \mathbb{A}^a$,

1. $i'(b \vee c) = i'(b) \vee i'(c)$;
2. $i'(b \wedge c) = i'(b) \wedge i'(c)$;
3. $i'(b \to c) = a \wedge (i'(b) \to i'(c))$;
4. $i'(\Diamond^a b) = \Diamond(i'(b) \wedge a) \wedge a$;
5. $i'(\Box^a b) = a \wedge \Box(a \to i'(b))$.

Proof. 1. Let $b', c' \in \mathbb{A}$ such that $b = [b']$ and $c = [c']$. Then $i'(b) = b' \wedge a$ and $i'(c) = c' \wedge a$, and because π commutes with $\vee$, we have $b \vee c = [b'] \vee [c'] = [b' \vee c']$. By distributivity, $i'(b \vee c) = i'([b' \vee c']) = (b' \vee c') \wedge a = (b' \wedge a) \vee (c' \wedge a) = i'(b) \vee i'(c)$.

2. Let $b', c' \in \mathbb{A}$ such that $b = [b']$ and $c = [c']$. Then $i'(b) = b' \wedge a$ and $i'(c) = c' \wedge a$, and because π commutes with $\wedge$, we have $b \wedge c = [b'] \wedge [c'] = [b' \wedge c']$. By idempotence and commutativity of $\wedge$, $i'(b \wedge c) = i'([b' \wedge c']) = (b' \wedge c') \wedge a = (b' \wedge a) \wedge (c' \wedge a) = i'(b) \wedge i'(c)$.

3. Let $b', c' \in \mathbb{A}$ such that $b = [b']$ and $c = [c']$. Then $i'(b) = b' \wedge a$ and $i'(c) = c' \wedge a$, and because π commutes with $\to$, we have

$$\begin{array}{rclcl} i'(b \to c) & = & i'([b'] \to [c']) & = & i'([b' \wedge a] \to [c' \wedge a]) \\ & = & i'([(b' \wedge a) \to (c' \wedge a)]) & = & a \wedge ((b' \wedge a) \to (c' \wedge a)) \\ & = & a \wedge (i'(b) \to i'(c)). & & \end{array}$$

4. Let $b' \in \mathbb{A}$ such that $b = [b']$. Then $i'(b) = b' \wedge a$, hence

$$b' \wedge a = (b' \wedge a) \wedge a = i'(b) \wedge a. \tag{5}$$

Moreover, $\Diamond^a b = [\Diamond(b' \wedge a)]$, hence $i'(\Diamond^a b) = \Diamond(b' \wedge a) \wedge a = \Diamond(i'(b) \wedge a) \wedge a$.

5. Let $b' \in \mathbb{A}$ with $b = [b']$. Then $i'(b) = a \wedge b'$ and $\Box^a b = [\Box(a \to b')]$. By Fact 25.4,

$$\begin{array}{rclcl} i'(\Box^a b) & = & i'([\Box(a \to b')]) & = & i'([\Box(a \to (a \wedge b'))]) \\ & = & i'([\Box(a \to i'(b))]) & = & a \wedge \Box(a \to i'(b)). \end{array}$$

◄

6.3 Soundness Lemmas

In this subsection, the lemmas are collected which serve to prove Proposition 21.

► **Lemma 30.** *Let $M = (\mathbb{A}, V)$ be an algebraic model. Let ϕ be a formula such that $[\![\phi]\!]_{M^\alpha} = \pi([\![\phi]\!]_M)$ for every formula α and model M. Then for every formula α,*

1. $[\![\langle\alpha\rangle\phi]\!]_M = [\![\alpha]\!]_M \wedge [\![\phi]\!]_M$.
2. $[\![[\alpha]\phi]\!]_M = [\![\alpha]\!]_M \to [\![\phi]\!]_M$.

Proof. 1.

$$\begin{array}{rcll}
[\![\langle\alpha\rangle\phi]\!]_M & = & [\![\alpha]\!]_M \wedge i'([\![\phi]\!]_{M^\alpha}) & \\
& = & [\![\alpha]\!]_M \wedge i'(\pi([\![\phi]\!]_M)) & \\
& = & [\![\alpha]\!]_M \wedge ([\![\phi]\!]_M \wedge [\![\alpha]\!]_M) & \\
& = & [\![\alpha]\!]_M \wedge [\![\phi]\!]_M. &
\end{array}$$

2.

$$\begin{array}{rcll}
[\![[\alpha]\phi]\!]_M & = & [\![\alpha]\!]_M \to i'([\![\phi]\!]_{M^\alpha}) & \\
& = & [\![\alpha]\!]_M \to i'(\pi([\![\phi]\!]_M)) & \\
& = & [\![\alpha]\!]_M \to ([\![\phi]\!]_M \wedge [\![\alpha]\!]_M) & \\
& = & [\![\alpha]\!]_M \to [\![\phi]\!]_M. & \text{(Fact 25.4)}
\end{array}$$

◀

▶ **Lemma 31.** *Let $M = (\mathbb{A}, V)$ be an algebraic model. For every formula α, ϕ and ψ,*

1. $[\![\langle\alpha\rangle(\phi \vee \psi)]\!]_M = [\![\langle\alpha\rangle\phi]\!]_M \vee [\![\langle\alpha\rangle\psi]\!]_M$.

2. $[\![[\alpha](\phi \vee \psi)]\!]_M = [\![\alpha]\!]_M \to ([\![\langle\alpha\rangle\phi]\!]_M \vee [\![\langle\alpha\rangle\psi]\!]_M)$.

Proof. 1.

$$\begin{array}{rcll}
[\![\langle\alpha\rangle(\phi \vee \psi)]\!]_M & = & [\![\alpha]\!]_M \wedge i'([\![\phi \vee \psi]\!]_{M^\alpha}) & \\
& = & [\![\alpha]\!]_M \wedge i'([\![\phi]\!]_{M^\alpha} \vee [\![\psi]\!]_{M^\alpha}) & \\
& = & [\![\alpha]\!]_M \wedge (i'([\![\phi]\!]_{M^\alpha}) \vee i'([\![\psi]\!]_{M^\alpha})) & \text{(Fact 29.1)} \\
& = & ([\![\alpha]\!]_M \wedge i'([\![\phi]\!]_{M^\alpha})) \vee ([\![\alpha]\!]_M \wedge i'([\![\psi]\!]_{M^\alpha}))) & \\
& = & [\![\langle\alpha\rangle\phi]\!]_M \vee [\![\langle\alpha\rangle\psi]\!]_M. &
\end{array}$$

2.

$$\begin{array}{rcll}
\llbracket [\alpha](\phi \vee \psi)\rrbracket_M & = & [\![\alpha]\!] \to i'([\![\phi \vee \psi]\!]_{M^\alpha}) & \\
& = & [\![\alpha]\!]_M \to (i'([\![\phi]\!]_{M^\alpha}) \vee i'([\![\psi]\!]_{M^\alpha})) & \text{(Fact 29.1)} \\
& = & [\![\alpha]\!]_M \to ([\![\alpha]\!]_M \wedge (i'([\![\phi]\!]_{M^\alpha}) \vee i'([\![\psi]\!]_{M^\alpha}))) & \text{(Fact 25.4)} \\
& = & [\![\alpha]\!]_M \to (([\![\alpha]\!]_M \wedge i'([\![\phi]\!]_{M^\alpha})) \vee ([\![\alpha]\!]_M \wedge i'([\![\psi]\!]_{M^\alpha}))) & \\
& = & [\![\alpha]\!]_M \to ([\![\langle\alpha\rangle\phi]\!]_M \vee [\![\langle\alpha\rangle\psi]\!]_M). &
\end{array}$$

◀

▶ **Lemma 32.** *Let $M = (\mathbb{A}, V)$ be an algebraic model. For every formula α, ϕ and ψ,*

1. $[\![\langle\alpha\rangle(\phi \wedge \psi)]\!]_M = [\![\langle\alpha\rangle\phi]\!]_M \wedge [\![\langle\alpha\rangle\psi]\!]_M$.

2. $[\![[\alpha](\phi \wedge \psi)]\!]_M = [\![[\alpha]\phi]\!]_M \wedge [\![[\alpha]\psi]\!]_M$.

Proof. 1.

$$\begin{array}{rcll}
[\![\langle\alpha\rangle(\phi \wedge \psi)]\!]_M & = & [\![\alpha]\!]_M \wedge i'([\![\phi \wedge \psi]\!]_{M^\alpha}) & \\
& = & [\![\alpha]\!]_M \wedge (i'([\![\phi]\!]_{M^\alpha}) \wedge i'([\![\psi]\!]_{M^\alpha})) & \text{(Fact 29.2)} \\
& = & ([\![\alpha]\!]_{M^\alpha} \wedge i'([\![\phi]\!]_{M^\alpha})) \wedge ([\![\alpha]\!]_M \wedge i'([\![\psi]\!]_{M^\alpha})) & \\
& = & [\![\langle\alpha\rangle\phi]\!]_M \wedge [\![\langle\alpha\rangle\psi]\!]_M. &
\end{array}$$

2.

$$\begin{array}{rcll}
\llbracket [\alpha](\phi \wedge \psi)\rrbracket_M & = & [\![\alpha]\!]_M \to i'([\![\phi \wedge \psi]\!]_{M^\alpha}) & \\
& = & [\![\alpha]\!]_M \to i'([\![\phi]\!]_{M^\alpha} \wedge [\![\psi]\!]_{M^\alpha}) & \\
& = & [\![\alpha]\!]_M \to (i'([\![\phi]\!]_{M^\alpha}) \wedge i'([\![\psi]\!]_{M^\alpha})) & \text{(Fact 29.2)} \\
& = & ([\![\alpha]\!]_M \to i'([\![\phi]\!]_{M^\alpha})) \wedge ([\![\alpha]\!]_M \to i'([\![\psi]\!]_{M^\alpha})) & \text{(Fact 25.2)} \\
& = & [\![[\alpha]\phi]\!]_M \wedge [\![[\alpha]\psi]\!]_M. &
\end{array}$$

◀

▶ **Lemma 33.** *Let $M = (\mathbb{A}, V)$ be an algebraic model. For every formula α, ϕ and ψ,*

1. $[\![[\alpha](\phi \to \psi)]\!]_M = [\![\langle\alpha\rangle\phi]\!]_M \to [\![\langle\alpha\rangle\psi]\!]_M$.
2. $[\![\langle\alpha\rangle(\phi \to \psi)]\!]_M = [\![\alpha]\!]_M \wedge ([\![\langle\alpha\rangle\phi]\!]_M \to [\![\langle\alpha\rangle\psi]\!]_M)$.

Proof. We preliminarily observe that

$$
\begin{array}{cll}
 & ([\![\alpha]\!]_M \wedge i'([\![\phi]\!]_{M^\alpha})) \to i'([\![\psi]\!]_{M^\alpha}) & \\
= & ([\![\alpha]\!]_M \wedge i'([\![\phi]\!]_{M^\alpha})) \to (([\![\alpha]\!]_M \wedge i'([\![\phi]\!]_{M^\alpha})) \wedge i'([\![\psi]\!]_{M^\alpha})) & \text{(Fact 25.4)} \\
= & [\![\langle\alpha\rangle\phi]\!]_M \to ([\![\langle\alpha\rangle\phi]\!]_M \wedge [\![\langle\alpha\rangle\psi]\!]_M) & \\
= & [\![\langle\alpha\rangle\phi]\!]_M \to [\![\langle\alpha\rangle\psi]\!]_M. & \text{(Fact 25.4)}
\end{array}
$$

Hence: 1.

$$
\begin{array}{rcll}
[\![[\alpha](\phi \to \psi)]\!]_M & = & [\![\alpha]\!]_M \to i'([\![\phi \to \psi]\!]_{M^\alpha}) & \\
 & = & [\![\alpha]\!]_M \to ([\![\alpha]\!]_M \wedge (i'([\![\phi]\!]_{M^\alpha}) \to i'([\![\psi]\!]_{M^\alpha}))) & \text{(Fact 29.3)} \\
 & = & [\![\alpha]\!]_M \to (i'([\![\phi]\!]_{M^\alpha}) \to i'([\![\psi]\!]_{M^\alpha})) & \text{(Fact 25.4)} \\
 & = & ([\![\alpha]\!]_M \wedge i'([\![\phi]\!]_{M^\alpha})) \to i'([\![\psi]\!]_{M^\alpha}) & \text{(Fact 25.5)} \\
 & = & [\![\langle\alpha\rangle\phi]\!]_M \to [\![\langle\alpha\rangle\psi]\!]_M. &
\end{array}
$$

2.

$$
\begin{array}{rcll}
[\![\langle\alpha\rangle(\phi \to \psi)]\!]_M & = & [\![\alpha]\!]_M \wedge i'([\![\phi \to \psi]\!]_{M^\alpha}) & \\
 & = & [\![\alpha]\!]_M \wedge ([\![\alpha]\!]_M \wedge (i'([\![\phi]\!]_{M^\alpha}) \to i'([\![\psi]\!]_{M^\alpha}))) & \text{(Fact 29.3)} \\
 & = & [\![\alpha]\!]_M \wedge (i'([\![\phi]\!]_{M^\alpha}) \to i'([\![\psi]\!]_{M^\alpha})) & \text{(Fact 25.4)} \\
 & = & [\![\alpha]\!]_M \wedge (([\![\alpha]\!]_M \wedge i'([\![\phi]\!]_{M^\alpha})) \to i'([\![\psi]\!]_{M^\alpha})) & \text{(Fact 25.6)} \\
 & = & [\![\alpha]\!]_M \wedge ([\![\langle\alpha\rangle\phi]\!]_M \to [\![\langle\alpha\rangle\psi]\!]_M). &
\end{array}
$$

◀

▶ **Lemma 34.** *Let $M = (\mathbb{A}, V)$ be an algebraic model. For every formula α and ϕ,*

1. $[\![\langle\alpha\rangle\Diamond\phi]\!]_M = [\![\alpha]\!]_M \wedge \Diamond^{\mathbb{A}}[\![\langle\alpha\rangle\phi]\!]_M$.
2. $[\![[\alpha]\Diamond\phi]\!]_M = [\![\alpha]\!]_M \to \Diamond^{\mathbb{A}}[\![\langle\alpha\rangle\phi]\!]_M$.

Proof. We preliminarily observe that

$$
\begin{array}{rcll}
i'([\![\Diamond\phi]\!]_{M^\alpha}) & = & [\![\alpha]\!]_M \wedge \Diamond^{\mathbb{A}}([\![\alpha]\!]_M \wedge i'([\![\phi]\!]_{M^\alpha})) & \text{(Fact 29.4)} \\
 & = & [\![\alpha]\!]_M \wedge \Diamond^{\mathbb{A}}[\![\langle\alpha\rangle\phi]\!]_M. &
\end{array}
$$

Hence: 1.

$$
\begin{array}{rcl}
[\![\langle\alpha\rangle\Diamond\phi]\!]_M & = & [\![\alpha]\!]_M \wedge i'([\![\Diamond\phi]\!]_{M^\alpha}) \\
 & = & [\![\alpha]\!]_M \wedge ([\![\alpha]\!]_M \wedge \Diamond^{\mathbb{A}}[\![\langle\alpha\rangle\phi]\!]_M) \\
 & = & [\![\alpha]\!]_M \wedge \Diamond^{\mathbb{A}}[\![\langle\alpha\rangle\phi]\!]_M.
\end{array}
$$

2.

$$
\begin{array}{rcll}
[\![[\alpha]\Diamond\phi]\!]_M & = & [\![\alpha]\!]_M \to i'([\![\Diamond\phi]\!]_{M^\alpha}) & \\
 & = & [\![\alpha]\!]_M \to ([\![\alpha]\!]_M \wedge \Diamond^{\mathbb{A}}[\![\langle\alpha\rangle\phi]\!]_M) & \\
 & = & [\![\alpha]\!]_M \to \Diamond^{\mathbb{A}}[\![\langle\alpha\rangle\phi]\!]_M. & \text{(Fact 25.4)}
\end{array}
$$

◀

▶ **Lemma 35.** *Let $M = (\mathbb{A}, V)$ be an algebraic model. For every formula α and ϕ,*

1. $[\![\langle\alpha\rangle\Box\phi]\!]_M = [\![\alpha]\!]_M \wedge \Box^{\mathbb{A}}[\![[\alpha]\phi]\!]_M$.
2. $[\![[\alpha]\Box\phi]\!]_M = [\![\alpha]\!]_M \to \Box^{\mathbb{A}}[\![[\alpha]\phi]\!]_M$.

Proof. We preliminarily observe that

$$
\begin{array}{rcll}
i'([\![\Box\phi]\!]_{M^\alpha}) & = & [\![\alpha]\!]_M \wedge \Box^{\mathbb{A}}([\![\alpha]\!]_M \to i'([\![\phi]\!]_{M^\alpha})) & \text{(Fact 29.5)} \\
 & = & [\![\alpha]\!]_M \wedge \Box^{\mathbb{A}}[\![[\alpha]\phi]\!]_M. &
\end{array}
$$

Hence: 1.

$$\begin{aligned}
[\![\langle\alpha\rangle\Box\phi]\!]_M &= [\![\alpha]\!]_M \wedge i'([\![\Box\phi]\!]_{M^\alpha}) \\
&= [\![\alpha]\!]_M \wedge ([\![\alpha]\!]_M \wedge \Box^{\mathbb{A}}[\![[\alpha]\phi]\!]_M) \\
&= [\![\alpha]\!]_M \wedge \Box^{\mathbb{A}}[\![[\alpha]\phi]\!]_M.
\end{aligned}$$

2.

$$\begin{aligned}
[\![[\alpha]\Box\phi]\!]_M &= [\![\alpha]\!]_M \rightarrow i'([\![\Box\phi]\!]_{M^\alpha}) \\
&= [\![\alpha]\!]_M \rightarrow ([\![\alpha]\!]_M \wedge \Box^{\mathbb{A}}([\![[\alpha]\phi]\!]_M)) \\
&= [\![\alpha]\!]_M \rightarrow \Box^{\mathbb{A}}[\![[\alpha]\phi]\!]_M. && \text{(Fact 25.4)}
\end{aligned}$$

◀

6.4 Muddy children reduction lemma

▶ **Lemma 36.** *For all formulas α, β, ϕ and ψ, for all proposition letters p and q, and for every $k \in \mathbb{N}$,*

1. $\vdash_{IPAL} [\alpha][\beta]^k\Box p \leftrightarrow (\langle\alpha\rangle\langle\beta\rangle^{k-1}\beta \rightarrow \langle\alpha\rangle\langle\beta\rangle^k\Box p)$.
2. $\vdash_{IPAL} \langle\alpha\rangle\langle\beta\rangle^k\Box p \leftrightarrow (\langle\alpha\rangle\langle\beta\rangle^{k-1}\beta \wedge \Box(\langle\alpha\rangle\langle\beta\rangle^{k-1}\beta \rightarrow \langle\alpha\rangle\langle\beta\rangle^{k-1}p))$.
3. *If* $p \wedge q \vdash \bot$, *then* $[\alpha][\beta]^k\Box p \wedge \langle\alpha\rangle\langle\beta\rangle^k\Diamond q \vdash \bot$.
4. $\vdash_{IPAL} [\alpha]^k(\phi \rightarrow \psi) \leftrightarrow (\langle\alpha\rangle^k\phi \rightarrow \langle\alpha\rangle^k\psi)$.
5. $\Box[\alpha]^k p \vdash_{IPAL} [\alpha]^k\Box p$.
6. $\langle\alpha\rangle\langle\beta\rangle^k\beta, p \vdash_{IPAL} \langle\alpha\rangle\langle\beta\rangle^k p$.

Proof. 1. By completeness, it is enough to show that

$$[\![[\alpha][\beta]^k\Box p]\!]_M = [\![\langle\alpha\rangle\langle\beta\rangle^{k-1}\beta]\!]_M \rightarrow [\![\langle\alpha\rangle\langle\beta\rangle^k\Box p]\!]_M$$

for any algebraic model $M = (\mathbb{A}, V)$. By induction on k: for $k = 1$, let $i' = i'_\alpha : \mathbb{A}^\alpha \rightarrow \mathbb{A}$, $i'' = i'_{\alpha\beta} : \mathbb{A}^{\alpha\beta} \rightarrow \mathbb{A}^\alpha$.

$$\begin{aligned}
[\![[\alpha][\beta]\Box p]\!]_M &= [\![\alpha]\!]_M \rightarrow i'([\![[\beta]\Box p]\!]_{M^\alpha}) \\
&= [\![\alpha]\!]_M \rightarrow i'([\![\beta]\!]_{M^\alpha} \rightarrow i''([\![\Box p]\!]_{M^{\alpha\beta}})) \\
&= [\![\alpha]\!]_M \rightarrow ([\![\alpha]\!]_M \wedge (i'([\![\beta]\!]_{M^\alpha}) \rightarrow i'(i''([\![\Box p]\!]_{M^{\alpha\beta}})))) && \text{(Fact 29.3)} \\
&= [\![\alpha]\!]_M \rightarrow (i'([\![\beta]\!]_{M^\alpha}) \rightarrow i'(i''([\![\Box p]\!]_{M^{\alpha\beta}}))) && \text{(Fact 25.4)} \\
&= [\![\alpha]\!]_M \rightarrow (i'([\![\beta]\!]_{M^\alpha}) \rightarrow i'(i''(\Box^{\mathbb{A}^{\alpha\beta}}[\![p]\!]_{M^{\alpha\beta}})) \\
&= [\![\alpha]\!]_M \rightarrow (i'([\![\beta]\!]_{M^\alpha}) \rightarrow i'([\![\beta]\!]_{M^\alpha} \wedge \Box^{\mathbb{A}^\alpha}([\![\beta]\!]_{M^\alpha} \rightarrow i''([\![p]\!]_{M^{\alpha\beta}})))) && \text{(Fact 29.5)} \\
&= [\![\alpha]\!]_M \rightarrow (i'([\![\beta]\!]_{M^\alpha}) \rightarrow i'([\![\beta]\!]_{M^\alpha} \wedge \Box^{\mathbb{A}^\alpha}([\![\beta]\!]_{M^\alpha} \rightarrow ([\![\beta]\!]_{M^\alpha} \wedge [\![p]\!]_{M^\alpha})))) && (*) \\
&= [\![\alpha]\!]_M \rightarrow (i'([\![\beta]\!]_{M^\alpha}) \rightarrow i'([\![\beta]\!]_{M^\alpha} \wedge \Box^{\mathbb{A}^\alpha}([\![\beta]\!]_{M^\alpha} \rightarrow [\![p]\!]_{M^\alpha}))) && \text{(Fact 25.4)} \\
&= [\![\alpha]\!]_M \rightarrow (i'([\![\beta]\!]_{M^\alpha}) \rightarrow i'([\![\beta]\!]_{M^\alpha} \wedge \Box^{\mathbb{A}^\alpha}[\![[\beta]p]\!]_{M^\alpha})) \\
&= [\![\alpha]\!]_M \rightarrow (i'([\![\beta]\!]_{M^\alpha}) \rightarrow i'([\![\langle\beta\rangle\Box p]\!]_{M^\alpha})) \\
&= [\![\langle\alpha\rangle\beta]\!]_M \rightarrow [\![\langle\alpha\rangle\langle\beta\rangle\Box p]\!]_M. && (**)
\end{aligned}$$

The equality marked $(*)$ holds because $[\![p]\!]_{M^{\alpha\beta}} = [[\![p]\!]_{M^\alpha}]_\beta$ and by the way i'' is defined; the equality marked $(**)$ holds by the preliminary observation in the proof of Lemma 33. For the induction step,

$$\begin{aligned}
[\![[\alpha][\beta]^{k+1}\Box p]\!]_M &= [\![\alpha]\!]_M \rightarrow i'([\![[\beta][\beta]^k\Box p]\!]_{M^\alpha}) \\
&= [\![\alpha]\!]_M \rightarrow i'([\![\langle\beta\rangle\langle\beta\rangle^{k-1}\beta]\!]_{M^\alpha} \rightarrow [\![\langle\beta\rangle\langle\beta\rangle^k\Box p]\!]_{M^\alpha}) && \text{(IH)} \\
&= [\![\langle\alpha\rangle\langle\beta\rangle^k\beta]\!]_M \rightarrow [\![\langle\alpha\rangle\langle\beta\rangle^{k+1}\Box p]\!]_M.
\end{aligned}$$

2. By completeness, it is enough to show that

$$[\![\langle\alpha\rangle\langle\beta\rangle^k\Box p]\!]_M = [\![\langle\alpha\rangle\langle\beta\rangle^{k-1}\beta]\!]_M \wedge \Box^{\mathbb{A}}([\![\langle\alpha\rangle\langle\beta\rangle^{k-1}\beta]\!]_M \rightarrow [\![\langle\alpha\rangle\langle\beta\rangle^{k-1}p]\!]_M)$$

for any algebraic model $M = (\mathbb{A}, V)$. By induction on k: for $k = 1$,

$$\begin{array}{rcll}
[\![\langle\alpha\rangle\langle\beta\rangle\Box p]\!]_M & = & [\![\alpha]\!]_M \wedge i'([\![\langle\beta\rangle\Box p]\!]_{M^\alpha}) & \\
& = & [\![\alpha]\!]_M \wedge i'([\![\beta]\!]_{M^\alpha} \wedge i''([\![\Box p]\!]_{M^{\alpha\beta}})) & \\
& = & [\![\alpha]\!]_M \wedge i'([\![\beta]\!]_{M^\alpha} \wedge i''(\Box^{\mathbb{A}^{\alpha\beta}}[\![p]\!]_{M^{\alpha\beta}})) & \\
& = & [\![\alpha]\!]_M \wedge i'([\![\beta]\!]_{M^\alpha} \wedge ([\![\beta]\!]_{M^\alpha} \wedge \Box^{\mathbb{A}^\alpha}([\![\beta]\!]_{M^\alpha} \to i''([\![p]\!]_{M^{\alpha\beta}})))) & \text{(Fact 29.5)} \\
& = & [\![\alpha]\!]_M \wedge i'([\![\beta]\!]_{M^\alpha} \wedge \Box^{\mathbb{A}^\alpha}([\![\beta]\!]_{M^\alpha} \to ([\![\beta]\!]_{M^\alpha} \wedge [\![p]\!]_{M^\alpha}))) & (*) \\
& = & [\![\alpha]\!]_M \wedge i'([\![\beta]\!]_{M^\alpha} \wedge \Box^{\mathbb{A}^\alpha}([\![\beta]\!]_{M^\alpha} \to [\![p]\!]_{M^\alpha})) & \text{(Fact 25.4)} \\
& = & ([\![\alpha]\!]_M \wedge i'([\![\beta]\!]_{M^\alpha})) \wedge i'(\Box^{\mathbb{A}^\alpha}([\![\beta]\!]_{M^\alpha} \to [\![p]\!]_{M^\alpha})) & \text{(Fact 29.2)} \\
& = & [\![\langle\alpha\rangle\beta]\!]_M \wedge ([\![\alpha]\!]_M \wedge \Box^{\mathbb{A}}([\![\alpha]\!]_M \to (i'([\![\beta]\!]_{M^\alpha}) \to i'([\![p]\!]_{M^\alpha})))) & \text{(Fact 29.5, 25.4)} \\
& = & [\![\langle\alpha\rangle\beta]\!]_M \wedge ([\![\alpha]\!]_M \wedge \Box^{\mathbb{A}}([\![\langle\alpha\rangle\beta]\!]_M \to [\![\langle\alpha\rangle p]\!]_M)) & (**) \\
& = & [\![\langle\alpha\rangle\beta]\!]_M \wedge \Box^{\mathbb{A}}([\![\langle\alpha\rangle\beta]\!]_M \to [\![\langle\alpha\rangle p]\!]_M). &
\end{array}$$

For the induction step,

$$\begin{array}{rcll}
[\![\langle\alpha\rangle\langle\beta\rangle^{k+1}\Box p]\!]_M & = & [\![\alpha]\!]_M \wedge i'([\![\langle\beta\rangle\langle\beta\rangle^k\Box p]\!]_{M^\alpha}) & \\
& = & [\![\alpha]\!]_M \wedge i'([\![\langle\beta\rangle\langle\beta\rangle^{k-1}\beta]\!]_{M^\alpha} \wedge \Box^{\mathbb{A}^\alpha}([\![\langle\beta\rangle\langle\beta\rangle^{k-1}\beta]\!]_{M^\alpha} \to [\![\langle\beta\rangle\langle\beta\rangle^{k-1}p]\!]_{M^\alpha})) & \text{(IH)} \\
& = & [\![\langle\alpha\rangle\langle\beta\rangle^k\beta]\!]_M \wedge i'(\Box^{\mathbb{A}^\alpha}([\![\langle\beta\rangle^k\beta]\!]_{M^\alpha} \to [\![\langle\beta\rangle^k p]\!]_{M^\alpha})) & \text{(Fact 29.2)} \\
& = & [\![\langle\alpha\rangle\langle\beta\rangle^k\beta]\!]_M \wedge \Box^{\mathbb{A}}([\![\langle\alpha\rangle\langle\beta\rangle^k\beta]\!]_{M^\alpha} \to [\![\langle\alpha\rangle\langle\beta\rangle^k p]\!]_{M^\alpha}). & \text{(Fact 29.5, }**)
\end{array}$$

3. By completeness, it is enough to show that, if $[\![p \wedge q]\!]_M = [\![\bot]\!]_M$, then

$$[\![[\alpha][\beta]^k\Box p]\!]_M \wedge [\![\langle\alpha\rangle\langle\beta\rangle^k\Diamond q]\!]_M = [\![\bot]\!]_M$$

for any algebraic model $M = (\mathbb{A}, V)$. By induction on k: for $k = 0$,

$$\begin{array}{rcl}
[\![[\alpha]\Box p]\!]_M \wedge [\![\langle\alpha\rangle\Diamond q]\!]_M & = & ([\![\alpha]\!]_M \to \Box([\![\alpha]\!]_M \to [\![p]\!]_M)) \wedge ([\![\alpha]\!]_M \wedge \Diamond([\![\alpha]\!]_M \wedge [\![q]\!]_M)) \\
& \leq & \Box([\![\alpha]\!]_M \to [\![p]\!]_M)) \wedge \Diamond([\![\alpha]\!]_M \wedge [\![q]\!]) \\
\text{(FS1, Fact 26)} & \leq & \Diamond(([\![\alpha]\!]_M \to [\![p]\!]_M) \wedge ([\![\alpha]\!]_M \wedge [\![q]\!]_M)) \\
& \leq & \Diamond([\![p \wedge q]\!]_M) \\
& = & \Diamond\bot \\
& = & \bot.
\end{array}$$

For the induction step,

$$\begin{array}{rcll}
[\![[\alpha][\beta]^{k+1}\Box p]\!]_M \wedge [\![\langle\alpha\rangle\langle\beta\rangle^{k+1}\Diamond q]\!]_M & = & ([\![\alpha]\!]_M \to i'([\![[\beta]^{k+1}\Box p]\!]_{M^\alpha})) \wedge ([\![\alpha]\!]_M \wedge i'([\![\langle\beta\rangle^{k+1}\Diamond q]\!]_{M^\alpha})) & \\
& \leq & i'([\![[\beta]^{k+1}\Box p]\!]_{M^\alpha}) \wedge i'([\![\langle\beta\rangle^{k+1}\Diamond q]\!]_{M^\alpha}) & \\
& = & i'([\![[\beta]^{k+1}\Box p \wedge \langle\beta\rangle^{k+1}\Diamond q]\!]_{M^\alpha}) & \\
& = & i'([\![[\beta][\beta]^k\Box p \wedge \langle\beta\rangle\langle\beta\rangle^k\Diamond q]\!]_{M^\alpha}) & \\
& = & i'([\![\bot]\!]_{M^\alpha}) & \text{(IH)} \\
& = & [\![\alpha]\!]_M \wedge [\![\bot]\!]_M & \\
& = & [\![\bot]\!]_M. &
\end{array}$$

4. By induction on k. If $k = 1$ the statement is one of the interaction axioms. Let $k \geq 1$ and assume that the statement is true for k. Then,

$$\begin{array}{rcl}
[\alpha]^{k+1}(\phi \to \psi) & \leftrightarrow & [\alpha][\alpha]^k(\phi \to \psi) \\
\text{(IH)} & \leftrightarrow & [\alpha](\langle\alpha\rangle^k\phi \to \langle\alpha\rangle^k\psi) \\
\text{(Int. axiom)} & \leftrightarrow & \langle\alpha\rangle\langle\alpha\rangle^k\phi \to \langle\alpha\rangle\langle\alpha\rangle^k\psi \\
& \leftrightarrow & \langle\alpha\rangle^{k+1}\phi \to \langle\alpha\rangle^{k+1}\psi.
\end{array}$$

5. If $k = 0$, then the statement reduces to $\Box[\alpha]p \vdash_{\mathcal{L}} [\alpha]\Box p$, which can be equivalently rewritten as

$$\Box(\alpha \to p) \vdash_{\mathcal{L}} \alpha \to \Box(\alpha \to p),$$

which is true, since $q \to (r \to q)$ is an intuitionistic axiom scheme. If $k \geq 1$, let us show that

$$\Box[\alpha][\alpha]^k p \vdash_{\mathcal{L}} [\alpha][\alpha]^k \Box p.$$

By items 1 and 2 of the present lemma,

$$[\alpha][\alpha]^k \Box p \leftrightarrow (\langle\alpha\rangle\langle\alpha\rangle^{k-1}\alpha \to \Box(\langle\alpha\rangle\langle\alpha\rangle^{k-1}\alpha \to \langle\alpha\rangle\langle\alpha\rangle^{k-1}p));$$

hence, again because $q \to (r \to q)$ is an intuitionistic axiom scheme, it is enough to show that

$$\Box[\alpha][\alpha]^k p \vdash_{\mathcal{L}} \Box(\langle\alpha\rangle^k\alpha \to \langle\alpha\rangle^k p).$$

By item 4 of the present lemma, this is equivalent to showing that

$$\Box[\alpha][\alpha]^k p \vdash_{\mathcal{L}} \Box[\alpha]^k(\alpha \to p),$$

which is true, since by the reduction axioms, $(\alpha \to p) \leftrightarrow [\alpha]p$.
6. Notice preliminarily that, from the defining clause of the interpretation of $\langle\beta\rangle\beta$ in any algebraic model and by completeness, it immediately follows that $\langle\beta\rangle\beta \vdash_{\mathcal{L}} \beta$. Hence, by repeated applications of monotonicity of dynamic diamonds,

$$\langle\alpha\rangle\langle\beta\rangle^k\beta \vdash_{\mathcal{L}} \langle\alpha\rangle\langle\beta\rangle^{k-1}\beta.$$

Let us prove the statement by induction on k: for $k = 0$, the statement reduces to

$$\langle\alpha\rangle\beta, p \vdash_{\mathcal{L}} \langle\alpha\rangle p,$$

which is true, since $\langle\alpha\rangle p$ is equivalent to $\alpha \wedge p$, and $\langle\alpha\rangle\beta \vdash_{\mathcal{L}} \alpha$. For the induction step, assume that

$$\langle\alpha\rangle\langle\beta\rangle^{k-1}\beta, p \vdash_{IPAL} \langle\alpha\rangle\langle\beta\rangle^{k-1}p$$

and let us show that

$$\langle\alpha\rangle\langle\beta\rangle^k\beta, p \vdash_{IPAL} \langle\alpha\rangle\langle\beta\rangle^k p.$$

Indeed, by the reduction axioms,

$$\langle\alpha\rangle\langle\beta\rangle^k p \dashv\vdash_{\mathcal{L}} \langle\alpha\rangle\langle\beta\rangle^{k-1}(\beta \wedge p) \dashv\vdash_{\mathcal{L}} \langle\alpha\rangle\langle\beta\rangle^{k-1}\beta \wedge \langle\alpha\rangle\langle\beta\rangle^{k-1}p.$$

The statement then immediately follows from this, the preliminary remark, and the induction hypothesis. ◀

References

1 A. Baltag, L. Moss, S. Solecki, The logic of public announcements, common knowledge, and private suspicions, CWI technical report SEN-R9922, 1999.

2 A. Baltag, B. Coecke and M. Sadrzadeh. Epistemic Actions as Resources. *J. Logic and Computation*, 17: 555-585, 2007.

3 J. van Benthem. *Modal Logic for Open Minds*. CSLI publications, 2010.

4 J. van Benthem. *Logical Dynamics of Information and Interaction*. Cambridge University Press, 2011.

5 G. Bezhanishvili. Varieties of Monadic Heyting Algebras. Part I. *Studia Logica* 61: 367-402, 1998.

6 G. Bezhanishvili. Varieties of Monadic Heyting Algebras Part II: Duality theory. *Studia Logica* 62: 21-48, 1999.

7 G. Bezhanishvili, S. Ghilardi and M. Jibladze. An Algebraic Approach to Subframe Logics. Modal Case. *Notre Dame Journal of Formal Logic* 52(2): 187-202, 2010.

8 T. Braüner, V. de Paiva, Intuitionistic hybrid logic. *J. Appl. Logic* 4(3): 231-255, 2006.

9 W. Conradie, A. Palmigiano. Algorithmic correspondence and canonicity for distributive modal logic. *Annals of Pure and Applied Logic*, 163: 338-376, 2012.

10 H. van Ditmarsch, W. van der Hoek and B. Kooi. *Dynamic Epistemic Logic*. Springer, 2007.

11 G. Fischer Servi. Axiomatizations for Some Intuitionistic Modal Logics. *Rend. Sem. Mat. Polit. di Torino* 42: 179-194, 1984.

12 J. Gerbrandy, *Bisimulations on Planet Kripke*, PhD dissertation, University of Amsterdam, 1999.

13 J. Gerbrandy and W. Groeneveld. Reasoning about Information Change. *J. Logic, Language, and Information* 6: 147-169, 1997.

14 P. Johnstone. *Stone Spaces*. Cambridge University Press, 1982.

15 A. Joyal and M. Tierney. An extension of the Galois theory of Grothendieck. *Memoires Amer. Math. Soc.* 51, no. 309, 1984.

16 J. Plaza. Logics of Public Communications. In: *Proceedings 4th International Symposium on Methodologies for Intelligent Systems*, 201-216, 1989.

17 V. Pratt. The Duality of Time and Information. In: *Proceedings CONCUR'92*, LNCS 630, 237-253, 1992.

18 V. Pratt. Rational Mechanics and Natural Mathematics. In *Proceedings TAPSOFT'95*, LNCS 915, 108-122, 1995.

19 V. Pratt. Chu Spaces. Course notes for the School in Category Theory and Applications, Coimbra, Portugal, 1999.

20 A. Prior. *Time and Modality*. Clarendon Press, 1957.

21 H. Rasiowa and R. Sikorski. *The mathematics of metamathematics*, Panstwowe Wydawnictwo. Naukowe: Warszawa, 1963.

22 A. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic*, PhD dissertation, University of Edinburgh, 1994.

23 F. Wolters. The structure of lattices of subframe logics. *Ann. Pure Appl. Logic* 86(1): 47-100, 1997.