UK Term Structure Decompositions at the Zero Lower Bound

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Abstract

This paper employs a Zero Lower Bound (ZLB) consistent shadow-rate model to decompose UK nominal yields into expectation and term premium components. Compared to a standard affine term structure model, it performs relatively better in a ZLB setting by capturing the stylized facts of the yield curve. The ZLB model is then exploited to estimate inflation expectations and risk premiums. This entails jointly pricing and decomposing nominal and real UK yields. We find evidence that medium- and long-term inflation expectations are contained within narrower bounds since the early 1990s, suggesting monetary policy credibility improved after the introduction of inflation targeting.

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Keywords: term structure, zero lower bound, risk premiums, inflation expectations.

1.   Introduction

In March 2009, the Monetary Policy Committee announced a cut of the policy rate to 0.5%, from a level of 4.5% six months earlier. This decision was accompanied by an economic stimulus via large-scale asset purchase programs. Since 2009, UK short yields stemming from nominal sovereign bonds reached historically low levels.

When nominal yields come close to the Zero Lower Bound (ZLB), further downward movements are considered unlikely (due to the non-negativity of nominal interest rates), thus resulting in an asymmetry in expectations (because paths involving negative yields are excluded) and a reduction in volatility (because only upward movements are possible).[1] In such a situation, the yield curve is anchored at the short end, agents' expectations reflect the belief that the policy rate would not be (substantially) further reduced, and the volatility of short-term rates falls.

These considerations lead to question the use of standard Gaussian affine term structure models, because they do not take into account the non-linearity existing in proximity of the lower bound. In periods of low nominal yields, conditional expectations and variances produced by these models might well be violating the inherent asymmetry in the time series evolution of nominal yields. As a result, these models can generate both implausible nominal risk premiums (see, Kim and Singleton, 2012) and imprecise expected inflation projections.

In addition to very low nominal yield levels, estimating inflation expectations has also preoccupied policymakers. Break-Even Inflation (BEI) rates (i.e., the difference between nominal and real yields) provide a proxy for market expectations of future inflation levels. However, even assuming nominal and real government bonds are equally liquid, this measure is an imperfect representation of inflation expectations as it is polluted by an inflation risk premium.

This paper considers two main issues. First, the paper aims at analyzing whether traditional models produce different results than ZLB-consistent models for the UK.[2] We

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[1]Negative nominal yields remain a possibility in periods of crisis (e.g., Denmark, the euro area, Switzerland), when bondholders require an insurance to safe-guard their investments, however it seems that an effective lower bound does exist and is a by-product of the level of the policy rate and the convenience yield.

[2]We provide results for the United Kingdom. Previous results have mainly focused on Japan and the United States (e.g., Kim and Singleton, 2012; Christensen and Rudebusch, 2013).

show that using a term structure model that incorporates the ZLB restriction provides a better representation of the nominal UK yield curve in terms of replicating the stylized facts of the term structure in low interest rate environments.

Second, the paper provides a decomposition of the UK term structure and estimates of inflation expectations and inflation risk premiums. To this end, the existing literature uses standard Gaussian affine term structure models for both the nominal and the real yield curve. In this paper we combine a ZLB-consistent term structure model for the nominal yield curve with a standard Gaussian affine term structure model for the real yields. This novel modeling choice offers the possibility to consistently estimate market-implied inflation expectations and associated risk premiums across sample periods that include nominal yield curve data constrained by the ZLB. Specifically, our proposed model preserves the empirical regularity of counter-cyclical inflation risk premiums.

We address both issues by building on the framework recently proposed by Christensen and Rudebusch (2015), which in turn is based on contributions by Black (1995) and Krippner (2012). Specifically, we use a Shadow-rate Lower Bound Nelson and Siegel (SLB-NS) term structure model that imposes the non-negativity of interest rates. Unlike Kim and Singleton (2012)'s model, this particular representation has the benefit of being capable of encompassing more than two factors (which will be important in fitting both nominal and real yields, jointly). Additionally, the factor loadings, borrowed from Nelson and Siegel (1987)'s model, facilitate the tractability of the model and offer an interpretation of the factors in terms of level, slope and curvature of the term structure.

Our methodological contribution is to extend the shadow-rate model to allow for the joint pricing of nominal and real sovereign bonds such that only nominal yields are bound to be non-negative. As far as future inflation projections are concerned, the benefits of using an asset pricing model come into play by enabling the disentanglement of inflation risk premiums from BEI rates, thus providing estimates of pure inflation expectations.

There is a considerable number of papers examining inflation expectations and risk premiums (see, Chen et al., 2005; Christensen et al., 2010; Chun, 2011; Chernov and Mueller, 2012; Grishchenko and Huang, 2012; D'Amico et al., 2014; Hordahl and Tristani, 2014). Only a limited literature is available for UK yields, despite the fact that the UK's inflation-linked bond market is one of the most liquid ones and the UK Debt Management

Office — an Executive Agency of HM Treasury — is committed to maintain this liquidity with regular issuance of inflation-linked bonds. A few exceptions include Joyce et al. (2010) which provide inflation projections up to 2009 (i.e., before unconventional monetary policies were put in place), and Abrahams et al. (2016) which use an affine term structure model for the joint pricing of nominal and real yields that accounts for potential lack of liquidity on US and UK data.

Importantly, all of the contributions cited above use standard term structure models to extract inflation expectations, which means no ZLB is imposed on the nominal yield curve. This would imply model misspecification if applied to UK data, since recent UK nominal yields are arguably constrained by the ZLB.

Our analysis of UK yield curves uses data ranging from January 1986 to August 2014. We find that Gaussian affine term structure models and ZLB-consistent models generate different results at the ZLB. Compared to a standard affine term structure model, a ZLB-consistent model performs relatively better in a ZLB setting and effectively captures the stylized facts of yield curves in a low interest rate environment. These stylized facts include: (i) the non-negativity of UK nominal yields and, (ii) the volatility compression of short- and medium-term yields.

The ZLB model is then exploited to estimate inflation expectations and risk premiums. We find evidence that medium- and long-term inflation expectations are contained within narrower bounds since the early 1990s, suggesting monetary policy credibility improved after the introduction of inflation targeting. In addition, we show that the sharp increase in inflation risk premiums in the late 2008 is likely to be partially driven by liquidity and pricing distortions in the inflation-linked bond market. Though inflation risk premiums dropped soon after March 2009, they have been steadily increasing since August 2013 as investors might have been placing more weight on future inflation uncertainty.

The paper is structured as follows. In Section 2 we estimate term structure models for nominal yields. In Section 3 we estimate a joint term structure model of nominal and real curves using an SLB-NS model that restricts solely nominal yields in a positive domain. Section 4 provides and analyzes the decomposition of BEI rates into two components, namely inflation expectations and risk premiums. We provide concluding remarks in Section 5. Appendix 6 encloses further technical details.

2. Modeling nominal UK yields at the zero lower bound

In this section we estimate term structure models for the nominal yield curve. In particular, we consider two alternative term structure models. The first is affine and Gaussian (and therefore does not take into account the ZLB restriction) while the second is a shadow-rate model (and therefore does take into account the ZLB restriction). We then compare the two models in terms of: (i) replicating yield curve stylized facts in a low interest rate environment, (ii) in-sample fit, (iii) estimated expectations, and (iv) estimated term premiums. Our empirical evidence points towards supporting the use of shadow-rate models in fitting UK nominal yields.

We start by describing the affine term structure model, and how it can be modified to implement the ZLB. We adopt the framework laid down Christensen et al. (2011) and Christensen and Rudebusch (2015) from which this section draws heavily on and to which the interested reader may refer to for a complete discussion. Here, to make the paper self-contained, we provide a brief outline of these models.

2.1. AFNS model for nominal yields

We consider the Arbitrage Free Nelson and Siegel dynamic term structure model (AFNS) of Christensen et al. (2011). The AFNS is an asset pricing model which relies on the existence of a physical measure $\mathbb{P}$ and a risk-neutral measure $\mathbb{Q}$ to disentangle expectation components from risk premium components. This model uses three latent factors (level, slope, and curvature, respectively denoted by L_t^N, S_t^N, C_t^N) to describe the yield curve.[3] The factor loadings feature the empirically popular Nelson and Siegel (1987) functional form and are consistent with the absence of arbitrage, while yields are affine in the latent factors. Specifically, the latent state vector $X_t^N = \left(L_t^N, S_t^N, C_t^N \right)'$ solves the following system of stochastic differential equations under the risk-neutral measure $\mathbb{Q}$:

$$dX_t^N = \kappa^{N,\mathbb{Q}} \left[\theta^{N,\mathbb{Q}} - X_t^N \right] dt + \sigma^N dW_t^{X^N,\mathbb{Q}}, \tag{1}$$

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[3]It is widely accepted in the literature that three pricing factors are typically considered sufficient (see Litterman and Scheinkman, 1991; Ang and Piazzesi, 2003). This is also confirmed via a principal component analysis. Further details are available in Subsection 2.3.

where $\theta^{N,\mathbb{Q}}$ is the unconditional mean of the process, $\kappa^{N,\mathbb{Q}}$ is the mean-reversion matrix and $W_t^{\mathbb{Q}}$ denotes a three-dimensional Wiener process.

Since the pricing factors are latent, a set of normalization restrictions is used to identify the model. Specifically, $\theta^{N,\mathbb{Q}} = [0,0,0]'$, the diffusion σ^N is a diagonal matrix with entries $(\sigma_{11,N}, \sigma_{22,N}, \sigma_{33,N})'$, and $\kappa^{N,\mathbb{Q}}$ is defined as:

$$
\kappa^{N,\mathbb{Q}} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \lambda^N & -\lambda^N \\ 0 & 0 & \lambda^N \end{pmatrix},
\tag{2}
$$

where λ^N is a mean-reversion parameter and $\varepsilon = 10^{-6}$ to obtain a near unit root behavior for the level factor.

The instantaneous risk-free rate is an affine function of the state variables and is specifically defined as the sum of the level and slope factors:

$$
r_t^N = L_t^N + S_t^N.
\tag{3}
$$

Note that the curvature factor does not feature in the instantaneous risk-free rate as the latter is a driver of yields with medium-term maturities.[4] As shown in e.g. Ang and Piazzesi (2003), nominal zero-coupon bond prices are exponentially affine functions of the state variables. As an immediate consequence, the representation of nominal zero-coupon yields with maturity T at time t is given by an affine function of the state variables, as shown below:

$$
\begin{aligned}
y^N(t,T) &= -\frac{A^N(t,T)}{T-t} - \frac{B^N(t,T)'}{T-t} X_t^N \\
&= L_t^N + \left(\frac{1-e^{-\lambda^N(T-t)}}{\lambda^N(T-t)} \right) S_t^N + \left(\frac{1-e^{-\lambda^N(T-t)}}{\lambda^N(T-t)} - e^{-\lambda^N(T-t)} \right) C_t^N - \frac{A^N(t,T)}{T-t},
\end{aligned}
\tag{4}
$$

where $A^N(t,T)$ and $B^N(t,T)$ are the unique solutions to a system of Riccati equations. $A^N(t,T)$ is known as the adjustment term (see, Christensen et al., 2011, for the derivation)

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[4]The level factor drives all maturities (long and short), while the slope and curvature factors predominantly drive short-term and medium-term maturities, respectively.

and $B^N(t, T)$ matches the Nelson-Siegel factor loadings. The AFNS model is formulated in continuous time and Girsanov's theorem ensures the change from the physical to the risk-neutral measure, $dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \Gamma_t^N dt$, where Γ_t^N is the market price of risk. Under essentially affine risk premium specifications (see, Duffee, 2002; Cheridito et al., 2007), it takes the form below, with γ_0^N being a three-dimensional vector and γ_1^N a 3x3 matrix:

$$\Gamma_t^N = \gamma_0^N + \gamma_1^N X_t^N. \tag{5}$$

We can now extract the latent state variables $X_t^N = \left(L_t^N, S_t^N, C_t^N\right)'$ under the physical measure. The dynamics are given by the following stochastic differential equation:

$$dX_t^N = \kappa^{N,\mathbb{P}} \left[\theta^{N,\mathbb{P}} - X_t^N\right] dt + \sigma^N dW_t^{X^N,\mathbb{P}}. \tag{6}$$

The key parameters are $\kappa^{N,\mathbb{P}}$ (on which we impose the restrictions stemming from the general-to-specific method detailed in Subsection 2.3), $\theta^{N,\mathbb{P}}$ (which is unrestricted) and σ^N (which has a diagonal structure).

2.2. SLB-NS model for nominal yields

We now consider how the model described above is modified in order to implement the ZLB restriction. The adjustments described in this Subsection imply that a non-linearity is introduced in the framework, and therefore the model implementing the ZLB is no longer Gaussian nor affine, but still features loadings in line with Nelson and Siegel (1987).[5] This model is introduced in Christensen and Rudebusch (2015), and relies on contributions by Black (1995) and Krippner (2012).[6]

The introduction of the ZLB hinges on the definition of an unobservable variable — the shadow-rate — which can be thought of as the policy rate that would generate the observed yield curve had the ZLB not been binding. The shadow-rate will have the same dynamics as the instantaneous risk-free rate under the AFNS, while the new dynamics for

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[5]Specifically, the loadings of the shadow yield curve are as for the Nelson-Siegel model, while the loadings for the lower-bound yield curve, which are those relevant to the observed data, are attenuated versions of the Nelson-Siegel model. For further details refer to Krippner (2015).

[6]Note that the model is not arbitrage-free, but is a very close and tractable approximation to the framework by Black (1995). These points are discussed further in Christensen and Rudebusch (2015) and Krippner (2015).

the instantaneous rate will consist of the maximum between the shadow-rate and zero.[7] The latent shadow-rates and instantaneous rates are respectively defined as:

$$s_t^N = L_t^N + S_t^N, \tag{7}$$

$$\underline{r}_t^N = \max \left\{ 0, s_t^N \right\}. \tag{8}$$

As in the AFNS, the state dynamics under the risk-neutral $\mathbb{Q}$ measure and the physical $\mathbb{P}$ measure are given by Equations (1) and (6), respectively. We now use a few important concepts borrowed from the bond option price literature.[8] Recently, Krippner (2012) and Krippner (2013) develop a shadow-rate framework in which a representation for the ZLB instantaneous forward rate is provided. This representation is valid for all Gaussian term structure models, including the AFNS, and depends on the instantaneous forward shadow-rates as well as an additional component, which is a function of the conditional variance of a European call. In the case of the SLB-NS model, analytical solutions for the instantaneous forward shadow-rates and the conditional variance are provided by Christensen and Rudebusch (2015), and we report them in Appendix 6 for reference.[9] The ZLB zero-coupon bond yields, denoted by $\underline{y}^N(t,T)$, are given by:

$$\underline{y}^N(t,T) = \frac{1}{T-t} \int_t^T \left[f(t,s) \Phi \left(\frac{f(t,s)}{\omega(t,s)} \right) + \omega(t,s) \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} \left[\frac{f(t,s)}{\omega(t,s)} \right]^2 \right) \right] ds. \tag{9}$$

It is important to note at this stage that $\underline{y}^N(t,T)$ is no longer a linear function of the state variables, unlike in the AFNS model. This non-linearity is translated in the estimation procedure, whereby a conventional Kalman filter cannot be used and is replaced

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[7]The same analysis can be conducted with a different threshold. Recent developments in Denmark and Switzerland have shown that despite the existence of physical cash, interest rates can go negative; nonetheless, rates seem to be bound below by a threshold known as the convenience yield. In the case of the UK, we opt for zero, as we want to reflect an "effective" lower bound for the UK that accounts for the convenience yield as well as the possibility of future downward revisions of the policy rate. Indeed, outside the sample considered in this paper, the bank rate has been further reduced to 25 basis points in response to the Brexit referendum. Similar lower bound thresholds of zero for the UK have been supported by the literature (see, Andreasen and Meldrum, 2015).

[8]As formulated in Krippner (2012), in the presence of currency, the investor has the option but not the obligation to hold cash. Therefore, if a zero-coupon bond trades below par (implying a positive yield), the investor will choose to hold the bond; however if the bond trades above par (implying a negative yield), the investor's return can be maximized by holding cash, with the bond trading at par. Hence, when the optionality of cash exists, the price of a bond can be expressed as the price of a shadow-rate zero-coupon bond (which can trade above par) minus a call option whose underlying is the shadow-bond price.

[9]This is done by setting the vector $(X_1, X_2, X_3)'$, and variables $\sigma_{11}, \sigma_{22}, \sigma_{33}$, found in Appendix 6, equal to $(L_t^N, S_t^N, C_t^N)'$ and $\sigma_{11,N}, \sigma_{22,N}, \sigma_{33,N}$, respectively.

by an extended Kalman filter.[10]

2.3. Data and empirical results

We estimate both models using nominal zero-coupon UK yields. The data set consists of continuously-compounded monthly nominal yields spanning from October 1986 to August 2014 and includes a set of seven maturities, namely 6, 12, 24, 36, 60, 84 and 120 months.[11] Interestingly, the time period under analysis incorporates three main changes in monetary policy practices in the UK: the introduction of inflation targeting in September 1992, the Bank of England's independence in May 1997, and the introduction of 'Quantitative Easing' in March 2009.

We first conduct a principal component analysis (PCA) to determine the number of pricing factors required to explain the cross-sectional variation of nominal yields. Second, for each of our two models (AFNS and SLB-NS), we use a general-to-specific method in order to impose the relevant restrictions on the mean-reversion matrix $\kappa^{N,\mathbb{P}}$.[12]

Table 1 displays the loadings from the PCA for the set of maturities and the percentage variation of yields that is being captured by each component. Note that the first component is characteristic of a level factor due to its homogeneity, the second component incorporates a sign switch between shorter and longer maturities hence displaying a slope feature and finally the third component, being U-shaped, has the behavior of a curvature factor. Additionally, the first three components explain 99.99% of the cross-sectional yield variation. The PCA results validate the literature's widespread use of three factors bearing the interpretation of level, slope and curvature.

It is at this point that the general-to-specific strategy comes into play. We implement it to find the best specification for the mean-reversion matrix $\kappa^{N,\mathbb{P}}$. The procedure goes as follows. First, we estimate the model without setting any restrictions on $\kappa^{N,\mathbb{P}}$. Subsequently, we run a second estimation, this time setting the least significant element of $\kappa^{N,\mathbb{P}}$

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[10]Additional information regarding the extended Kalman filter is provided in Online Appendix A. Note that the use of the extended Kalman filter is conventional in this literature, while, alternatives to this procedure are the iterated extended Kalman filter and the unscented Kalman filter.

[11]The UK DMO issues bonds that have maturities of up to around 55 years. The aim of this study is to only analyze rate dynamics up to 10 years.

[12]Note that using the so-called preferred specification is of importance due to the sensitivity of results to different specifications (see, Joslin et al., 2011, 2014; Christensen and Rudebusch, 2015). The issue of sensitivity is particularly relevant when considering the estimation of risk premiums, given they rely heavily on the estimation of $\kappa^{N,\mathbb{P}}$.

(stemming from the previous estimation) to zero. We repeat this procedure until we are left with a diagonal $\kappa^{N,\mathbb{P}}$, and at each iteration, we compute the Akaike Information Criterion (AIC) and Bayes Information Criterion (BIC). We rule our decision by minimizing the AIC (when the AIC and BIC decision rules do not coincide).[13]

The results of the general-to-specific analysis on the AFNS model are displayed in Table 2, and the preferred specification is given by specification (6), which is consistent with Christensen and Rudebusch (2012)'s findings for the UK. Table 3 indicates the parameter estimates of the AFNS specification. The results of the general-to-specific method applied to the SLB-NS model are found on Table 4 and indicate that the preferred specification is given by specification (5). The estimated parameters are found on Table 5. The in-sample fit of the AFNS and SLB-NS models is comparable, with the average root mean squared errors amounting to roughly 4 basis points. Rather than the in-sample fit, the main benefit of our model is instead the ability to replicate key stylized facts such as volatility attenuation and a zero probability of having short-term negative rates, which are both not attainable using a Gaussian model. We devote the next Subsection 2.4 to the importance of accounting for the ZLB in term structure models.

2.4.   The importance of modeling the ZLB

In this Subsection, we explain why including the ZLB in a term structure model is important for the UK, and how it affects the results.

Figure 1 displays the state variables, namely the level, slope and curvature, estimated using the AFNS and SLB-NS models, respectively. The Figure shows that, prior to the ZLB period, state variables estimated from the two models roughly coincide and have a correlation of approximately 99%. During the ZLB, this feature persists for both the level and the slope; however the curvature factor exhibits a significant change in behavior from one model to another, with the correlation now dropping to roughly 84%. This could be explained by the fact that the ZLB imposes a non-linear restriction (see Eq. (9)), which potentially is best translated into effects on the non-linear curvature state variable.

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[13]Since in any given step of the procedure the removed coefficient is (generally) an insignificant one, the overall effect of the removal of such a coefficient on the likelihood is small, when one moves from one estimated model to the one immediately following it. However, when more restrictions get imposed, the difference between the likelihoods becomes significant. Note that a similar pattern can be found in other papers such as Christensen et al. (2010) and Christensen and Rudebusch (2012).

Nominal yields can be further decomposed into two components: the so called risk-neutral yields and the term premiums. The latter can be computed through numerical methods and is given by:

$$TP^N(t,T) = \underline{y}^N(t,T) - \frac{1}{T-t} \int_t^T \mathbb{E}_t^{\mathbb{P}} \left[\underline{r}_s^N \right] ds. \tag{10}$$

In order to assess the effect of accounting for the ZLB on expectations, Panel (a) of Figure 2 depicts the expectation components of the 10-year nominal yield obtained using the preferred AFNS and SLB-NS models, respectively. We observe that neglecting the ZLB restriction leads to an overestimation of the fitted expectation term of the ten-year yield by up to 1%. This is consistent with Christensen and Rudebusch (2012)'s result which states that declines in US treasury yields mainly reflect lower expectations. In Panel (b) of Figure 2, we provide the estimates of the 10-year fitted term premiums of nominal yields, with and without the ZLB assumption. At first glance, we notice the two series do not coincide even prior to the ZLB period. This finding is consistent with a similar comparison conducted by Ichiue and Ueno (2013). This difference can be justified by the highly sensitive nature of term premiums to different preferred specifications used by each of these models. More importantly, prior to the ZLB, both term premiums track each other and move in the same direction. Conversely, in recent years, models neglecting the ZLB restriction tend to generate implausibly large negative term premiums (see, Kim and Singleton, 2012). Moreover, our findings corroborate Malik and Meldrum (2016)'s result whereby UK bond term premiums are positively related to uncertainty about future inflation. Indeed, in line with their findings, our ZLB-consistent 5-year term premium displays a correlation of 89% with 3-year survey-based inflation uncertainty measures, unlike its Gaussian counterpart which displays a correlation of -21%.[14]

Moreover, it is worth mentioning that the expectation component under the Gaussian model is typically higher than under the shadow-rate model due to the fact that Gaussian models have a tendency to revert back to the mean relatively fast. In contrast, shadow-rate models are designed to maintain model-implied yields and their expectation terms relatively low for prolonged periods of time (Christensen and Rudebusch, 2013).

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[14]3-year survey-based inflation uncertainty measures are constructed using the conditional variance implied by the Bank of England's Survey of External Forecasters' aggregate distribution function.

Finally, another key fact that a non-ZLB model generates is a positive probability of negative rates. Following the analysis by Christensen and Rudebusch (2013), in Panel (a) of Figure 3 we show that — in the absence of the ZLB restriction — the probability of negative rates implied by the Gaussian AFNS model jumps to about 50% after the financial crisis. Similarly, Panel (b) of Figure 3 shows that while the Gaussian AFNS model imposes homoskedasticity throughout the sample, the ZLB-consistent SLB-NS model replicates the stylized fact of volatility attenuation in yields, during the period 2008-2014 (see, Filipovic et al., 2016). Note that such volatility measures are defined as the square root of the model-implied conditional yield variances, and those are obtained using the conditional covariance matrix of the state variables. In the case of the AFNS model the volatility is available in closed-form, while in the case of the SLB-NS model, standard Monte Carlo simulations are used.[15]

Therefore, compared to a standard affine Gaussian term structure model, a ZLB-consistent model performs relatively better by capturing the stylized facts of the yield curve at the ZLB.[16]

3. Joint modeling of nominal and real UK yields at the zero lower bound

In this Section, we propose and estimate a new model that combines a ZLB-consistent term structure model for the nominal yield curve with a standard Gaussian affine term structure model for real yields. This novel modeling choice offers the possibility to consistently estimate market-implied inflation expectations and associated risk premiums across sample periods that include nominal yield curve data constrained by the ZLB, which will be discussed in Section 4.

3.1. The model

Our methodological contribution is to extend the shadow-rate model to allow for the joint pricing of nominal and real sovereign bonds such that only nominal yields are bound to

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[15]Note that the volatility can also be computed using numerical integration.

[16]Note that the shadow-rate series we obtain is similar to that of Wu and Xia (2016) for the UK.

be non-negative. As far as future inflation projections are concerned, the benefits of using an asset-pricing model come into play by enabling the disentanglement of inflation risk premiums from BEI rates, thus providing estimates of pure inflation expectations.

We first consider the structure of our joint SLB-NS model. The joint latent state vector is given by $X_t^J = \left(L_t^N, S_t^N, C_t, L_t^R, S_t^R \right)'$.[17] The state vector X_t^J solves the following stochastic differential equations under the risk-neutral measure $\mathbb{Q}$:

$$dX_t^J = \kappa^{J,\mathbb{Q}} \left[\theta^{J,\mathbb{Q}} - X_t^J \right] dt + \sigma^J dW_t^{X^J,\mathbb{Q}}, \tag{11}$$

where $dW_t^{\mathbb{Q}}$ is a five-dimensional Wiener process.

For identification purposes, the following restrictions are imposed: $\theta^{J,\mathbb{Q}} = [0,0,0,0,0]'$, the diffusion σ^J is a diagonal matrix whose elements are $(\sigma_{11,J}, \sigma_{22,J}, \sigma_{33,J}, \sigma_{44,J}, \sigma_{55,J})'$, and $\kappa^{J,\mathbb{Q}}$ is defined as:

$$\kappa^{J,\mathbb{Q}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 & 0 \\ 0 & \lambda^N & -\lambda^N & 0 & 0 \\ 0 & 0 & \lambda^N & 0 & 0 \\ 0 & 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \lambda^R & -\lambda^R \end{pmatrix}, \tag{12}$$

where λ^N and λ^R are scalars that represent the speed of mean-reversion for nominal and real yields respectively, and $\varepsilon = 10^{-6}$.

The joint SLB-NS model restricts nominal yields in the positive domain while simultaneously keeping real yields unrestricted. The instantaneous risk-free nominal and real rates are thus given respectively by:

$$\underline{r}_t^N = \max \left\{ 0, L_t^N + S_t^N \right\}, \tag{13}$$

$$r_t^R = L_t^R + S_t^R. \tag{14}$$

We note that the nominal instantaneous risk-free rate is the maximum between zero and

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[17]We thank an anonymous referee for pointing out this specification. This specification matches the empirical characteristics of the data as determined by the PCA in Table 1. That is, the nominal yield curve data is well-explained by three factors (level, slope, and curvature), and the real yield curve data is well-explained by two factors (level and slope).

the nominal shadow-rate, while the real instantaneous risk-free rate is linear in the state variables. Let us denote by $\underline{y}^N(t,T)$ and $y^R(t,T)$ the ZLB nominal zero-coupon bond yields and the real zero coupon bond yields, respectively. In Appendix 6, we provide further details on $\underline{y}^N(t,T)$.[18] Their representations are given as follows:

$$\underline{y}^N(t,T) = \frac{1}{T-t} \int_t^T \left[f^N(t,s)\Phi\left(\frac{f^N(t,s)}{\omega^N(t,s)}\right) + \omega^N(t,s)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left[\frac{f^N(t,s)}{\omega_N(t,s)}\right]^2\right)\right]ds,$$
(15)

$$y^R(t,T) = L_t^R + \left(\frac{1-e^{-\lambda^R\tau}}{\lambda^R\tau}\right)S_t^R + \alpha^R\left(\frac{1-e^{-\lambda^R\tau}}{\lambda^R\tau} - e^{-\lambda^R\tau}\right)C_t - \frac{A^R(\tau)}{\tau},$$
(16)

where we denote by α^R the weight of real yields on the curvature of nominal yields.

This model can be written in a state-space representation and estimated via quasi-maximum likelihood. Note that nominal yields and real yields are non-linear and affine functions of the state vector, respectively. As a consequence, to accommodate for the non-linearity, the computation of the likelihood requires the use of an extended Kalman filter.

The market price of risk under the essentially affine risk premium specifications takes the form:

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \Gamma_t^J dt,$$
(17)

$$\Gamma_t^J = \gamma_0^J + \gamma_1^J X_t^J.$$
(18)

By applying the change of measure, we extract the latent state variable vector $X_t^J = \left(L_t^N, S_t^N, C_t, L_t^R, S_t^R\right)'$ which solves the stochastic differential equations below:

$$dX_t^J = \kappa^{J,\mathbb{P}}\left[\theta^{J,\mathbb{P}} - X_t^J\right]dt + \sigma^J dW_t^{X^J,\mathbb{P}},$$
(19)

where $\mathbb{P}$ denotes the physical measure.

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[18]This is done by setting the vector $(X_1, X_2, X_3)'$, and variables $\sigma_{11}, \sigma_{22}, \sigma_{33}$, found in Appendix 6, equal to $(L_t^N, S_t^N, C_t)'$ and $\sigma_{11,J}, \sigma_{22,J}, \sigma_{33,J}$, respectively.

3.2. Data and empirical results

We consider a data set combining nominal and real zero-coupon bond yields. Specifically, the data consists of continuously-compounded monthly nominal and real yields spanning from October 1986 to August 2014 and includes a set of seven maturities for nominal yields, namely, 6, 12, 24, 36, 60, 84 and 120 months, and an additional set of six maturities for real yields: 60, 72, 84, 96, 108 and 120 months. Note that we have chosen longer maturities for real yields, in comparison to nominal yields, due to a reduced liquidity of real sovereign bonds in the short-end.[19]

As in the nominal case, before enforcing the zero lower-bound on nominal yields, we need to first find the preferred specification of our mean reversion matrix $\kappa^{J,\mathbb{P}}$. We hence proceed in conducting such a strategy on a joint SLB-NS model, which imposes the non-negativity assumption solely on nominal yields.

The results of the general-to-specific method applied to the joint SLB-NS model are found on Table 6 and indicate that the preferred specification is given by specification (20). The estimated parameters stemming from this preferred specification are found on Table 7. The fit of both nominal and real yields is very satisfactory (i.e., the average root mean squared errors is roughly equal to 5 basis points) and further allows us to explore inflation expectations and risk premiums, which we discuss in the next Section.

4. Inflation expectations and risk premiums

There is a considerable number of papers examining inflation expectations and risk premiums using standard term structure models. Prominent examples include Chen et al. (2005), Christensen et al. (2010), Chun (2011), Chernov and Mueller (2012), Grishchenko and Huang (2012), D'Amico et al. (2014) and Hordahl and Tristani (2014), while studies on the UK, in particular, include Joyce et al. (2010) and Abrahams et al. (2016). Importantly, all of these contributions do not make allowances for the ZLB on the nominal

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[19]The BEI data are provided by the DMO and are not publicly available. However, they are virtually equivalent to the data provided by the Bank of England. Specifically, the models of the DMO and the Bank of England use Variable Roughness Penalty (VRP) estimates of nominal and real spot rates, which are computed following Anderson and Sleath (2001). The only difference is that, unlike the Bank of England, the DMO does not use General Collateral repo rates but only gilt data with maturity greater than 3 months. In practice, this difference has an extremely limited impact —of no more than two basis point —on the estimated BEI rates.

yield curve. This implies model misspecification if applied to UK data, since recent UK nominal yields are arguably constrained by the ZLB (see Subsection 2.4).

In this Section we address the decomposition of BEI rates into inflation risk premiums and expectations. The existence of risk-neutral and physical measures provides us this decomposition. We denote by $\frac{dM_t^N}{M_t^N}$ and $\frac{dM_t^R}{M_t^R}$, the nominal and real pricing kernel dynamics, respectively, and provide their expressions below:

$$\frac{dM_t^N}{M_t^N} = -\underline{r}_t^N dt - \Gamma_t^{J'} dW_t^{J,\mathbb{P}},\tag{20}$$

$$\frac{dM_t^R}{M_t^R} = -r_t^R dt - \Gamma_t^{J'} dW_t^{J,\mathbb{P}}.\tag{21}$$

By using the two stochastic discount factors above, one can extract the following system of equations (see, Christensen et al., 2010, for further details):

$$\begin{aligned}
BEI(t,T) &= \underline{y}^N(t,T) - y^R(t,T) \\
&= y^N(t,T) + Z(t,T) - y^R(t,T) \\
&= y^R(t,T) + \pi^e(t,T) + \phi(t,T) + Z(t,T) - y^R(t,T) \\
&= \pi^e(t,T) + Z(t,T) + \phi(t,T) \\
&= \underline{\pi}^e(t,T) + \phi(t,T),
\end{aligned}\tag{22}$$

$$\underline{\pi}^e(t,T) = -\frac{1}{T-t} ln \left\{ \mathbb{E}_t^{\mathbb{P}} \left[\exp\left(-\int_t^T (\underline{r}_u^N - r_u^R) du \right) \right] \right\},\tag{23}$$

where $\pi^e(t,T)$, $\underline{\pi}^e(t,T)$ and $\phi(t,T)$ denote respectively the inflation expectation obtained using the shadow-rate, the ZLB-consistent inflation expectation (obtained using the short-rate) and inflation risk premium for maturity T (at time t). The term $Z(t,T)$ represents the "option effect", which captures the value of the option of holding cash at the ZLB. This effect is such that $\underline{y}^N(t,T) = y^N(t,T) + Z(t,T)$ and $\underline{\pi}^e(t,T) = \pi^e(t,T) + Z(t,T)$. Figure A in Online Appendix B shows that the magnitude of $Z(t,T)$ can reach up to approximately 80 basis points during the ZLB.[20] The solution to the expression in curly brackets in Eq. (23) is obtained through numerical procedures.[21]

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[20]Online Appendix B provides further details on the decomposition of breakeven inflation rates in a ZLB environment. The matter of how option effects should be treated in such a decomposition is interesting in its own rights and would benefit from further research.

[21]Note that $\underline{\pi}^e(t,T)$ is a continuous process as it is implicitly a function of latent factors, hence it is

In Panel (a) of Figure 4, we display the 5- and 10-year inflation expectations. We identify a handful of key monetary policy events over the sample, including the adoption of inflation targeting in September 1992 (sparked by the withdrawal of the pound sterling from the European Exchange Rate Mechanism), the independence of the Bank of England in setting monetary policy in May 1997, the cut of the bank rate to 0.5% and launch of the asset purchase program in March 2009, the asset purchase program reaching a running total of £375bn in July 2012 (thus amounting to roughly 30% of debt at the time), and finally the adoption of forward guidance in August 2013 and February 2014. We note that since 1992 inflation expectations have decreased, possibly as a result of investors' confidence in the new monetary policy framework that was reinforced in the Bank of England Act 1998; similar results are found in Joyce et al. (2010) and Andreasen (2012). Since the mid-2000s, there is a tendency for the 5- and 10-year spot inflation projections to be below the current inflation level, while at a 10-year horizon inflation projections systematically undershoot target inflation after 2008.[22]

Panel (b) of Figure 4 depicts 5- and 10-year inflation risk premiums. We observe that the compensation for inflation risk significantly dropped after the independence of the Bank of England, suggesting a gained credibility in inflation targeting practices and conveying a period of lower uncertainty. Moreover, there are indications that the fall in term premiums observed in Figure 2 might very well be driven by lower inflation risk premiums during that period, while the sharp increase in inflation risk premiums in the late 2008 is likely to be partially driven by liquidity and pricing distortions in the inflation-linked market (refer to Panel (a) of Figure 5, for liquidity premium estimates). Though inflation risk premiums dropped soon after March 2009, they have been steadily increasing since August 2013 as investors might have been placing more weight on future inflation uncertainty.[23]

In 2008, inflation expectations decreased abruptly, reaching a trough of -3% in January 2009, stayed around 0 immediately after this period, and then eventually slowly reverted

not directly comparable to observed inflation.

[22]We take into account that inflation expectations are RPI based since real sovereign bonds differ from nominal ones in that payments are adjusted in line with movements in RPI. Note that in December 2003, the Bank of England changed its inflation target from a 2.5% level of RPIX to a 2% level of CPI.

[23]It is worth noting that, towards the end of our sample, UK breakeven rates are relatively higher than in the US. Specifically, during the period 2013-2014, 5-year UK breakeven rates hovered around 3% while their US counterparts fluctuated around 2%.

back to up to 2%. The large downward swing observed between November 2008 and the end of 2012 is decisively far relative to survey-based inflation expectations, which instead hovered around 2% during this period.

This swing occurred in conjunction with a large volatility in the inflation-linked bond's market, which suffered reduced liquidity. At that time, the spread between BEI and inflation-linked swap rates sharply widened to historical highs. As a result, it is likely that part of the large blip observed in January 2009 has been affected by this event and is therefore liquidity-related. To verify this conjecture we consider adjusting our estimates for this liquidity effect.

Rather than embedding a liquidity risk premium in the model, we use a very simple proxy measure of liquidity, given by the difference between inflation-linked swaps and BEI rates (refer to Panel (a) of Figure 5). While this measure cannot be considered a pure measure of liquidity premiums, it serves our goal of confirming the conjecture that the large downward swing in expectations of January 2009 is at least partially due to liquidity issues. Moreover, this proxy measure, proposed by Christensen and Gillan (2011), tracks well, and actually is more conservative than those obtained using more sophisticated model-based approaches (e.g. D'Amico et al., 2014; Andreasen et al., 2017).

Panel (b) of Figure 5 plots 5-year inflation expectations implied by our joint SLB-NS estimation and their 5-year liquidity-adjusted counterparts. Liquidity-adjusted inflation expectations are obtained by adding the estimated liquidity premium to model-implied inflation expectations. Moreover, Figure 5 contains the time series of 3-year survey-based inflation expectations (taken from the Bank of Englands Survey of External Forecasters).

The Figure lends support to the conjecture that the large downward swing observed between November 2008 and the end of 2012 can be partially explained by liquidity issues causing a distortion in market prices, however, even after adjusting for this effect, model-based expectations appear to be well below survey expectations at the peak of the crisis. This might reflect a difficulty of our parametric model to fully capture such an extreme event, which included for example the sharp and sudden fall in RPI as shown in Panel (a) of Figure 4.[24]. Subsequently to this sharp drop, model-implied inflation expectations

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[24]It is also worth noting that survey-based inflation expectations tend to be more persistent than their market-based counterparts, and are available at a lower frequency

picked up and realigned with survey-based expectations by the beginning of 2011.

4.1. Discussion

Previous studies on the UK term structure include Joyce et al. (2010) and Abrahams et al. (2016). The former only considers pre-2008 data in a Gaussian framework (limiting thus comparability with our sample period), while the latter includes data on the financial crisis without accounting for the inherent non-linearity in the presence of the ZLB. Our nominal yield and BEI rate decompositions plotted in Figures 2 and 4 are in line with the decompositions provided by Abrahams et al. (2016). Specifically, nominal term premiums are of similar magnitudes, and tend to be, on average, negative during the period 1997-2008, and starting 2012. Inflation risk premiums are embedded in nominal term premiums and hence, negative term premiums can be suggestive of negative inflation risk premiums. Indeed, both our inflation risk premiums (see Figure 4) and Abrahams et al. (2016)'s are negative and of similar magnitude during the period 1997-2000 and tend to oscillate around zero thereafter, until the outbreak of the Great Recession. However, unlike Abrahams et al. (2016)'s estimates, our inflation risk premiums increase and turn significantly positive during the crisis. Importantly, the imposition of the ZLB constraint increases the persistence of the model, hence leading to a slower pace of policy normalization and larger nominal term premiums for a given level of yields, leading to a larger inflation risk premium.

Notably, our ZLB-consistent 5- and 10-year inflation risk premiums display a correlation of 68% and 72% with survey-based inflation uncertainty measures and of -26% and -21% with year-over-year annual GDP growth, respectively. Instead, if one were to use the 5- and 10-year inflation risk premium Gaussian counterparts that do not account for the ZLB, these correlations would have the opposite signs: -47% and -4% (with survey-based inflation uncertainty) and 17% and 3% (with year-over-year annual GDP growth), respectively. These results suggest that accounting for the ZLB can generate counter-cyclical inflation risk premiums (given the negative correlations obtained between model-implied inflation risk premiums and GDP growth).[25]

---

[25] We further regress our model-implied 5- and 10-year inflation risk premiums on an inflation uncertainty proxy and find that the respective slope coefficients are positive and statistically significant at a 1% significance level, while the R-squared coefficients are equal to 47% and 51%, respectively. We replicate

5. Conclusions

This paper first examines whether traditional models produce different results than ZLB-consistent models for the UK, using data ranging from January 1986 to August 2014. We find that, compared to a standard affine term structure model, a ZLB-consistent model performs relatively better in a ZLB setting, in terms of replicating the stylized facts of the yield curve in low interest-rate environments. We find that imposing the ZLB in the model specification allows correcting for the unreasonably low term premium projections stemming from a standard model not featuring a ZLB restriction after 2009.

Based on this superior performance of the shadow-rate model at the ZLB (vis-à-vis standard Gaussian affine term structure models) for the nominal yield curve, we subsequently build a ZLB-consistent model that jointly prices nominal and real yields in the UK. We specify and estimate a joint SLB-NS model that is able to impose the ZLB restriction on nominal yields while allowing real yields to fall below zero. The model proposed is consistent with the behavior of UK data in the last decade, since it takes into account the existence of a ZLB. Previous studies on the UK term structure only consider pre-2008 data (in the case of Joyce et al., 2010) and do not allow for the non-linearity inherent in the presence of the lower bound (in the case of Abrahams et al., 2016).

The model we propose is used to estimate inflation expectations and risk premiums in the presence of the ZLB. Our decompositions provide evidence supporting the conclusion that the Bank of England Act 1998 established credibility in inflation targeting. Finally, we find that inflation premiums have been steadily increasing since August 2013, suggesting that investors might be placing more weight on future inflation uncertainty.

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this analysis on the Gaussian counterparts and conversely find that the slope coefficients are negative and statistically significant in the case of the 5-year inflation risk premium and insignificant in the case of the 10-year inflation risk premium.

6.   Appendix: SLB-NS model à la Krippner

The instantaneous shadow forward rates are obtained by deriving the logarithmic bond prices $P(t, T)$ with respect to the maturity T, as follows:

$$
\begin{aligned}
f(t, T) &= -\frac{\partial}{\partial T} \ln P(t, T) \\
&= X_1 + e^{-\lambda(T-t)} X_2 + \lambda(T-t) e^{-\lambda(T-t)} X_3 + A^f(t, T),
\end{aligned}
\tag{24}
$$

where $A^f(t, T)$ is obtained below:

$$
\begin{aligned}
A^f(t, T) &= -\frac{\partial A(t, T)}{\partial T} \\
&= -\frac{1}{2}\sigma_{11}^2 (T-t)^2 - \frac{1}{2}\sigma_{22}^2 \left(\frac{1 - e^{-\lambda(T-t)}}{\lambda}\right)^2 - \frac{1}{2}\sigma_{33}^2 \left((T-t)e^{-\lambda(T-t)} - \frac{1 - e^{-\lambda(T-t)}}{\lambda}\right)^2.
\end{aligned}
\tag{25}
$$

Transforming the conditional variance of a European call option maturing at time T, contingent on the zero-coupon bond with maturity $T + \epsilon$, we can obtain a representation of $\omega(t, T)^2$:

$$
\begin{aligned}
\omega(t, T)^2 &= \sigma_{11}^2 (T-t) + \sigma_{22}^2 \left(\frac{1 - e^{-2\lambda(T-t)}}{2\lambda}\right) \\
&+ \sigma_{33}^2 \left[\frac{1 - e^{-2\lambda(T-t)}}{4\lambda} - \frac{1}{2}(T-t)e^{-2\lambda(T-t)} - \frac{1}{2}\lambda(T-t)^2 e^{-2\lambda(T-t)}\right].
\end{aligned}
\tag{26}
$$

Note that Eqs. (25) and (26) hold for the case of a diagonal covariance matrix. Christensen and Rudebusch (2015) and Krippner (2015) provide expressions allowing for correlations.

Let us now denote by $\underline{f}(t, T)$, the ZLB instantaneous forward rate. Setting $\Phi(.)$ to be the standard normal cumulative probability, we obtain a representation for $\underline{f}(t, T)$:

$$
\underline{f}(t, T) = f(t, T)\Phi\left(\frac{f(t, T)}{\omega(t, T)}\right) + \omega(t, T)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left[\frac{f(t, T)}{\omega(t, T)}\right]^2\right).
\tag{27}
$$

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Table 1: Principal components in nominal and real yields

Maturity	First PC	Second PC	Third PC
NOMINAL YIELDS			
6 months	0.4212	-0.4861	0.5232
12 months	0.4120	-0.3699	0.0981
24 months	0.3971	-0.1723	-0.3303
36 months	0.3841	-0.0029	-0.4839
60 months	0.3622	0.2596	-0.3315
84 months	0.3428	0.4339	0.0451
120 months	0.3146	0.5844	0.5113
% explained	97.90	1.95	0.14
REAL YIELDS			
60 months	0.4321	0.6563	0.5152
72 months	0.4199	0.3210	-0.2525
84 months	0.4099	0.0396	-0.4941
96 months	0.4017	-0.1922	-0.3526
108 months	0.3949	-0.3805	0.0350
120 months	0.3893	-0.5320	0.5488
% explained	98.96	1.03	0.01

We provide the loadings on the three first principal components of nominal yields at maturities of 6, 12, 24, 36, 60, 84 and 120 months and the three first principal components of real yields at maturities of 60, 72, 84, 96, 108 and 120 months. The percentage cross-sectional variation accounted for by each component is displayed on the final row. The data comprises of monthly nominal and real zero coupon bond yields from October 1986 to August 2014.

Table 2: Evaluation of alternative specifications of the three factor AFNS model for nominal rates

Alternative specifications	logL	m	p-value	AIC	BIC
(1) Unrestricted $\kappa^{N,\mathbb{P}}$	13324.1389	23		-26602.2779	-26514.5529
(2) $\kappa^{N,\mathbb{P}}_{31} = 0$	13324.1386	22	0.9803	-26604.2773	-26520.3664
(3) $\kappa^{N,\mathbb{P}}_{31} = \kappa^{N,\mathbb{P}}_{32} = 0$	13324.1379	21	0.9993	-26606.2759	-26526.1791
(4) $\kappa^{N,\mathbb{P}}_{31} = \kappa^{N,\mathbb{P}}_{32} = \kappa^{N,\mathbb{P}}_{21} = 0$	13324.1174	20	0.9978	-26608.2347	-26531.9521
(5) $\kappa^{N,\mathbb{P}}_{31} = ... = \kappa^{N,\mathbb{P}}_{12} = 0$	13324.0991	19	0.9998	-26610.1982	-26537.7297
(6) $\kappa^{N,\mathbb{P}}_{31} = ... = \kappa^{N,\mathbb{P}}_{13} = 0$	13323.8107	18	0.9890	**-26611.6215**	-26542.9671
(7) $\kappa^{N,\mathbb{P}}_{31} = ... = \kappa^{N,\mathbb{P}}_{23} = 0$	13321.4142	17	0.5706	-26608.8284	**-26543.9882**

We estimate and evaluate seven alternative specifications of the individual standard AFNS model on nominal yields. The restrictions imposed on $\kappa^{N,\mathbb{P}}$ for each alternative specification are reported on the first column. For the estimation of each specification, we record its log-likelihood ($LogL$), number of parameters (m) and the p-value of a likelihood ratio test of the hypothesis that a specification with $(m-i)$ parameters is different from the one with $(m-i+1)$ parameters. The information criteria (AIC and BIC) are reported and we display their minimum in bold.

Table 3: Three factor AFNS estimates for nominal rates

$\kappa_t^{N,\mathbb{P}}$	$\kappa_{.,1}^{N,\mathbb{P}}$	$\kappa_{.,2}^{N,\mathbb{P}}$	$\kappa_{.,3}^{N,\mathbb{P}}$	$\theta^{N,\mathbb{P}}$	$\sigma_{i,i}^{N}$
$\kappa_{1,.}^{N,\mathbb{P}}$	0.0848	-	-	0.0824	0.0118
	(0.031624)			(0.031623)	(0.033686)
$\kappa_{2,.}^{N,\mathbb{P}}$	-	0.3706	-0.2413	-0.0214	0.0174
		(0.031623)	(0.031623)	(0.031631)	(0.033907)
$\kappa_{3,.}^{N,\mathbb{P}}$	-	-	0.4538	-0.0103	0.0304
			(0.031623)	(0.031628)	(0.031768)

The estimated parameters of the $\kappa^{N,\mathbb{P}}$ matrix, $\theta^{N,\mathbb{P}}$ vector, and diagonal diffusion matrix $\sigma_{i,i}^{N}$ are given for our preferred individual three-factor standard AFNS model for nominal yields. The estimated value of λ^N is 0.4321 with standard deviation of 0.031623. The numbers in parentheses are the standard deviations of the estimated parameters.

Table 4: Evaluation of alternative specifications of the three factor SLB-NS model for nominal rates

Alternative specifications	logL	m	p-value	AIC	BIC
(1) Unrestricted $\kappa^{N,\mathbb{P}}$	13591.3729	23		-27136.7458	-27049.0208
(2) $\kappa_{31}^{N,\mathbb{P}} = 0$	13591.3717	22	0.9614	-27138.7434	-27054.8326
(3) $\kappa_{31}^{N,\mathbb{P}} = \kappa_{12}^{N,\mathbb{P}} = 0$	13591.2280	21	0.8661	-27140.4559	-27060.3592
(4) $\kappa_{31}^{N,\mathbb{P}} = \kappa_{12}^{N,\mathbb{P}} = \kappa_{32}^{N,\mathbb{P}} = 0$	13591.1876	20	0.9940	-27142.3752	-27066.0926
(5) $\kappa_{31}^{N,\mathbb{P}} = ... = \kappa_{13}^{N,\mathbb{P}} = 0$	13590.6782	19	0.9069	**-27143.3564**	-27070.8879
(6) $\kappa_{31}^{N,\mathbb{P}} = ... = \kappa_{21}^{N,\mathbb{P}} = 0$	13589.1025	18	0.6767	-27142.2050	**-27073.5507**
(7) $\kappa_{31}^{N,\mathbb{P}} = ... = \kappa_{23}^{N,\mathbb{P}} = 0$	13586.1000	17	0.4226	-27138.1999	-27073.3597

We estimate and evaluate seven alternative specifications of the individual SLB-NS model on nominal yields. The restrictions imposed on $\kappa^{N,\mathbb{P}}$ for each alternative specification are reported on the first column. For the estimation of each specification, we record its log-likelihood ($LogL$), number of parameters (m) and the p-value of a likelihood ratio test of the hypothesis that a specification with $(m-i)$ parameters is different from the one with $(m-i+1)$ parameters. The information criteria (AIC and BIC) are reported and we display their minimum in bold.

Table 5: Three factor SLB-NS estimates for nominal rates

$\kappa_t^{N,\mathbb{P}}$	$\kappa_{.,1}^{N,\mathbb{P}}$	$\kappa_{.,2}^{N,\mathbb{P}}$	$\kappa_{.,3}^{N,\mathbb{P}}$	$\theta^{N,\mathbb{P}}$	$\sigma_{i,i}^{N}$
$\kappa_{1,.}^{N,\mathbb{P}}$	0.0362	-	-	0.0513	0.0157
	(0.034041)			(0.007385)	(0.000488)
$\kappa_{2,.}^{N,\mathbb{P}}$	0.1103	0.3359	-0.2286	-0.0005	0.0206
	(0.072374)	(0.047283)	(0.032776)	(0.012462)	(0.000833)
$\kappa_{3,.}^{N,\mathbb{P}}$	-	-	0.4507	-0.0164	0.0324
			(0.031654)	(0.006607)	(0.001453)

The estimated parameters of the $\kappa^{N,\mathbb{P}}$ matrix, $\theta^{N,\mathbb{P}}$ vector, and diagonal diffusion matrix $\sigma_{i,i}^{N}$ are given for our preferred individual three-factor SLB-NS model for nominal yields. The estimated value of λ^{N} is 0.4622 with standard deviation of 0.009396. The numbers in parentheses are the standard deviations of the estimated parameters.

Table 6: Evaluation of alternative specifications of the five factor joint SLB-NS model

| Alternative specifications	logL	m	p-value	AIC	BIC
(1) Unrestricted $\kappa^{J,\mathbb{P}}$	26608.7127	51		-53115.4253	-52920.9047
(2) $\kappa_{45}^{J,\mathbb{P}} = 0$	26608.4935	50	0.5080	-53116.9870	-52926.2805
(3) $\kappa_{45}^{J,\mathbb{P}} = \kappa_{24}^{J,\mathbb{P}} = 0$	26608.3524	49	0.8684	-53118.7048	-52931.8124
(4) $\kappa_{45}^{J,\mathbb{P}} = \kappa_{24}^{J,\mathbb{P}} = \kappa_{52}^{J,\mathbb{P}} = 0$	26608.2849	48	0.9873	-53120.5698	-52937.4915
(5) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{32}^{J,\mathbb{P}} = 0$	26608.2836	47	1.0000	-53122.5673	-52943.3031
(6) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{35}^{J,\mathbb{P}} = 0$	26608.2781	46	1.0000	-53124.5563	-52949.1063
(7) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{15}^{J,\mathbb{P}} = 0$	26608.2621	45	1.0000	-53126.5242	-52954.8883
(8) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{12}^{J,\mathbb{P}} = 0$	26608.2057	44	1.0000	-53128.4113	-52960.5896
(9) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{13}^{J,\mathbb{P}} = 0$	26608.2011	43	1.0000	-53130.4022	-52966.3946
(10) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{25}^{J,\mathbb{P}} = 0$	26607.8972	42	0.9999	-53131.7944	-52971.6009
(11) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{31}^{J,\mathbb{P}} = 0$	26607.8868	41	1.0000	-53133.7737	-52977.3943
(12) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{43}^{J,\mathbb{P}} = 0$	26607.8501	40	1.0000	-53135.7003	-52983.1350
(13) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{34}^{J,\mathbb{P}} = 0$	26607.5563	39	1.0000	-53137.1126	-52988.3615
(14) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{51}^{J,\mathbb{P}} = 0$	26607.3570	38	1.0000	-53138.7139	-52993.7770
(15) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{42}^{J,\mathbb{P}} = 0$	26607.1214	37	1.0000	-53140.2428	-52999.1200
(16) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{54}^{J,\mathbb{P}} = 0$	26606.9659	36	1.0000	-53141.9318	-53004.6231
(17) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{21}^{J,\mathbb{P}} = 0$	26606.9071	35	1.0000	-53143.8141	-53010.3196
(18) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{23}^{J,\mathbb{P}} = 0$	26606.8994	34	1.0000	-53145.7988	-53016.1183
(19) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{14}^{J,\mathbb{P}} = 0$	26606.4894	33	1.0000	-53146.9789	-53021.1126
(20) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{53}^{J,\mathbb{P}} = 0$	26606.0090	32	1.0000	**-53148.0180**	**-53025.9658**
(21) $\kappa_{45}^{J,\mathbb{P}} = ... = \kappa_{41}^{J,\mathbb{P}} = 0$	26590.4770	31	0.0544	-53118.9540	-53000.7159

We estimate and evaluate thirteen alternative specifications of the joint SLB-NS model on nominal and real yields. The restrictions imposed on $\kappa^{J,\mathbb{P}}$ for each alternative specification are reported on the first column. For the estimation of each specification, we record its log-likelihood ($LogL$), number of parameters (m) and the p-value of a likelihood ratio test of the hypothesis that a specification with $(m-i)$ parameters is different from the one with $(m-i+1)$ parameters. The information criteria (AIC and BIC) are reported and we display their minimum in bold.

Table 7: Five factor joint SLB-NS estimates

$\kappa_t^{J,\mathbb{P}}$	$\kappa_{.,1}^{J,\mathbb{P}}$	$\kappa_{.,2}^{J,\mathbb{P}}$	$\kappa_{.,3}^{J,\mathbb{P}}$	$\kappa_{.,4}^{J,\mathbb{P}}$	$\kappa_{.,5}^{J,\mathbb{P}}$	$\theta^{J,\mathbb{P}}$	$\sigma_{i,i}^{J}$
$\kappa_{1,.}^{J,\mathbb{PP}}$	0.0890	-	-	-	-	0.0551	0.0161
	(0.031623)					(0.036095)	(0.031627)
$\kappa_{2,.}^{J,\mathbb{PP}}$	-	0.1327	-	-	-	-0.0523	0.0205
		(0.031623)				(0.036508)	(0.031666)
$\kappa_{3,.}^{J,\mathbb{PP}}$	-	-	0.2461	-	-	0.0181	0.0321
			(0.031623)			(0.031633)	(0.031631)
$\kappa_{4,.}^{J,\mathbb{PP}}$	-0.2432	-	-	0.4420	-	0.0166	0.0061
	(0.031623)			(0.031623)		(0.031644)	(0.031770)
$\kappa_{5,.}^{J,\mathbb{PP}}$	-	-	-	-	0.4000	-0.0186	0.0271
					(0.031623)	(0.031623)	(0.031625)

The estimated parameters of the $\kappa^{J,\mathbb{P}}$ matrix, $\theta^{J,\mathbb{P}}$ vector, and diagonal diffusion matrix $\sigma_{i,i}^{J}$ are given for our preferred joint five-factor SLB-NS model for nominal and real yields. The estimated value of λ^N is 0.4606 with standard deviation of 0.031623 and the estimated value of λ^R is 0.4558 with standard deviation of 0.031623. The estimated value of α^R is 0.2782 with standard deviation of 0.031623. The numbers in parentheses are the standard deviations of the estimated parameters.
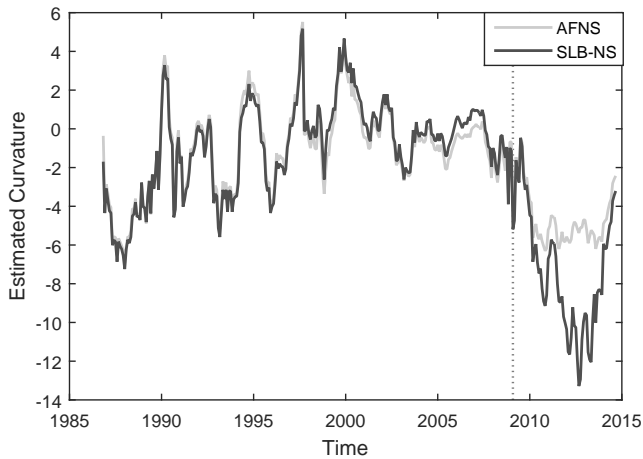
Figure 1: Estimated state variables
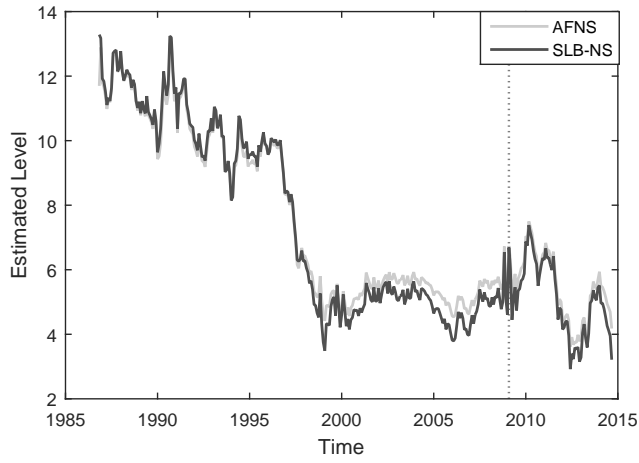
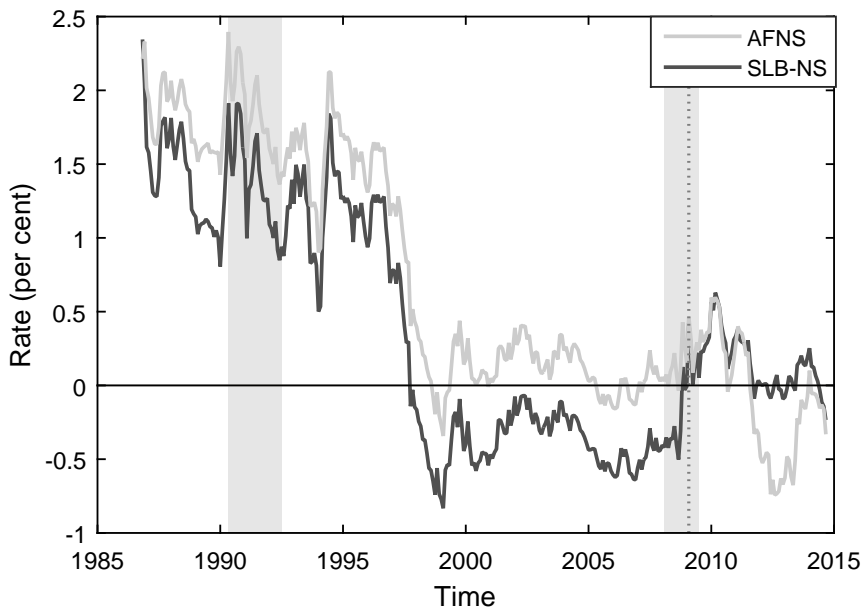(a) Level factor
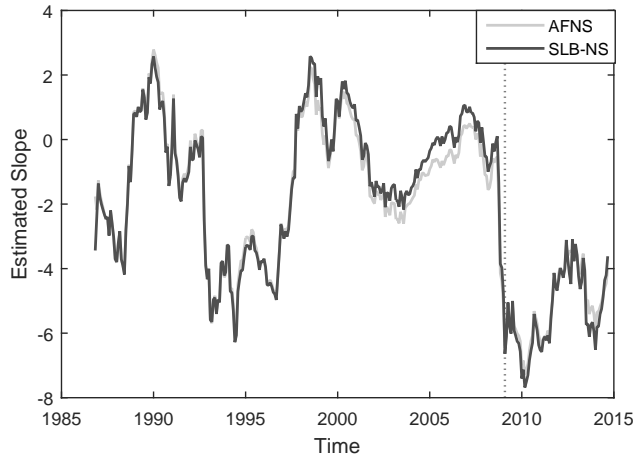
(b) Slope factor

(c) Curvature factor

State variables (measured in per cent), estimated using the preferred AFNS and SLB-NS models. The vertical dotted line represents the date (March 2009) when the Monetary Policy Committee announced a cut of the policy rate to 0.5% and an economic stimulus via large-scale asset purchase programs.

Figure 2: Fitted 10-year expectation component and term premium
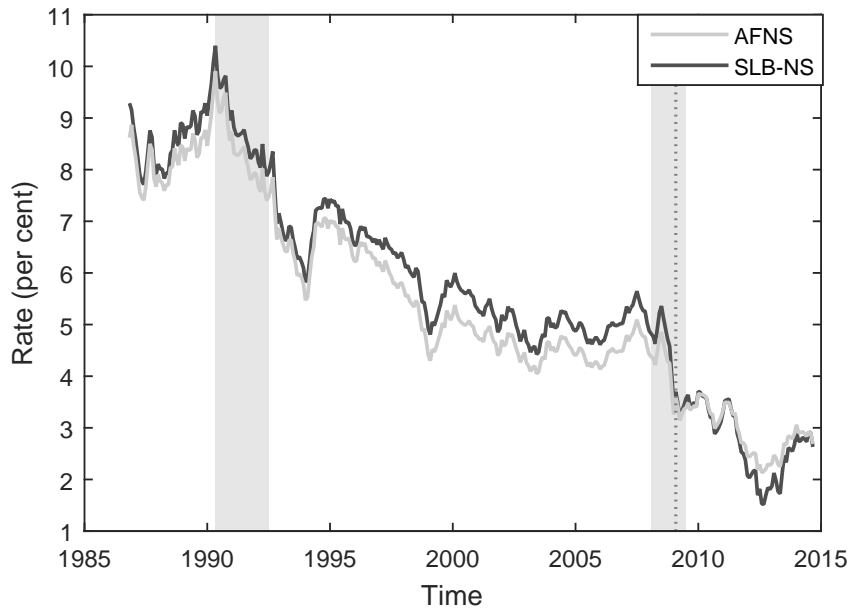
(a) Fitted expectation term of the 10-year nominal yield

(b) Fitted 10-year nominal term premium

Fitted expectation term of the 10-year nominal yield — Panel (a) — and 10-year fitted term premiums of nominal yields — Panel (b) — estimated using the preferred AFNS and SLB-NS models. The light-gray bars indicate UK recessions. The vertical dotted line represents the date (March 2009) when the Monetary Policy Committee announced a cut of the policy rate to 0.5% and an economic stimulus via large-scale asset purchase programs.

Figure 3: The importance of the ZLB
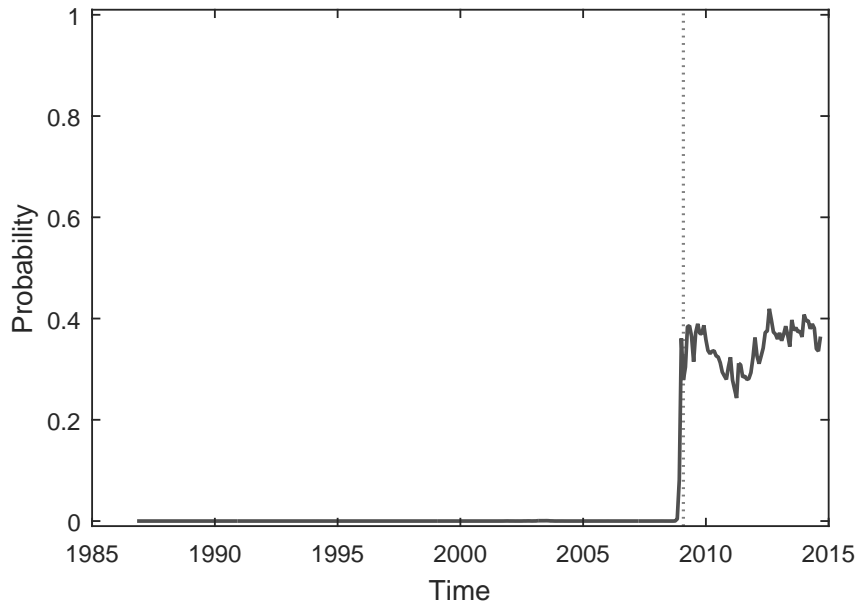
(a) Probability of the 6-month yield being negative using the AFNS
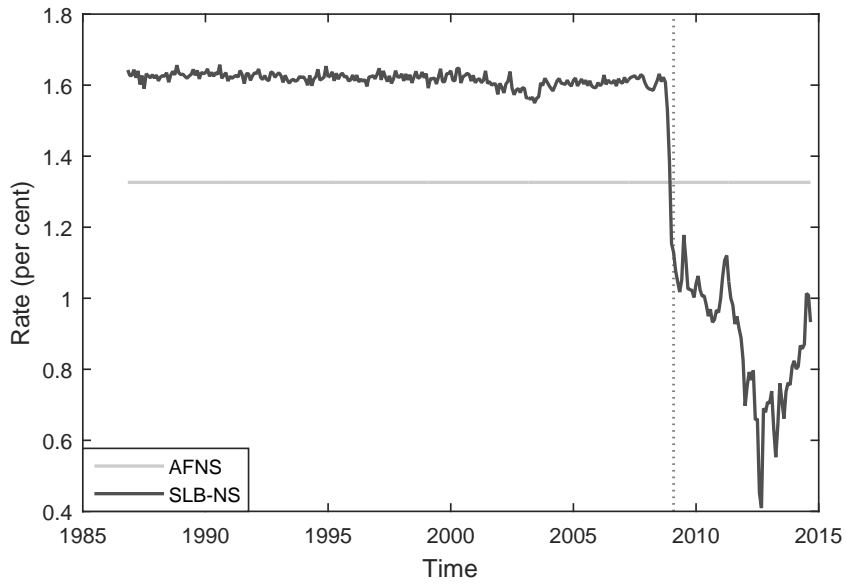

(b) Conditional volatility of the 6-month yield

Panel (a) depicts the conditional probability of the UK's 6-month nominal interest rate being negative, implied by the estimated preferred AFNS model. Panel (b) depicts the conditional volatility of the UK's 6-month nominal interest rate, implied by the estimated preferred AFNS and SLB-NS models, respectively. The vertical dotted line represents the date (March 2009) when the Monetary Policy Committee announced a cut of the policy rate to 0.5% and an economic stimulus via large-scale asset purchase programs.

Figure 4: 5- and 10-year BEI rate decompositions
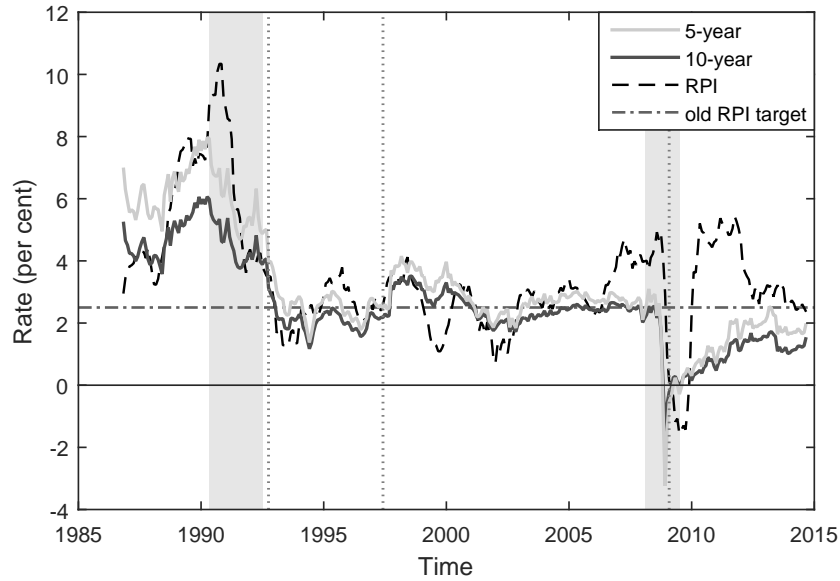
(a) Inflation expectations by maturity, RPI inflation and old RPI inflation target level
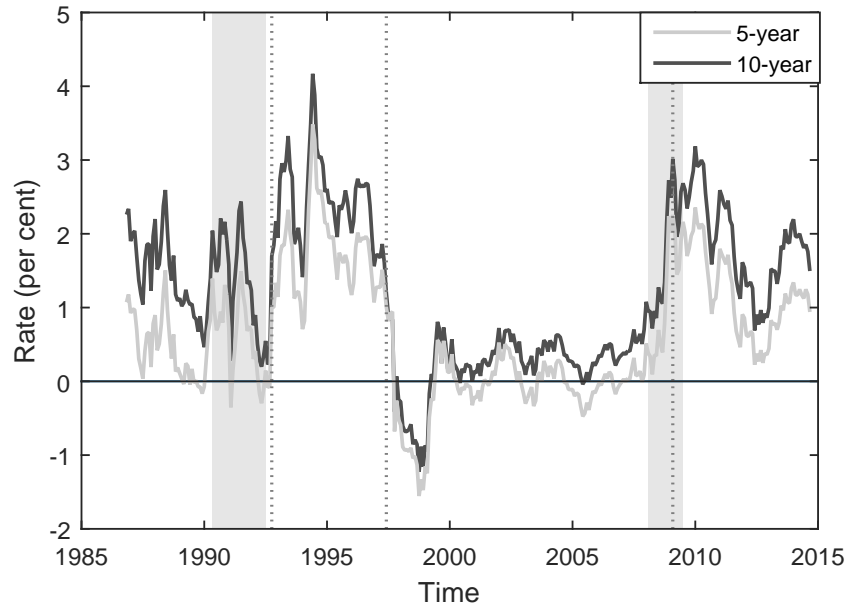

(b) Inflation premiums by maturity

The 5- and 10- year expected inflation rates — Panel (a) — and inflation risk premiums — Panel (b) — implied by the preferred joint SLB-NS model. Panel (a) includes the series for historical RPI inflation and the old RPI inflation target of 2.5%, which is approximately consistent with the new 2% CPI target. The data span from October 1986 to August 2014. The light-gray bars indicate UK recessions. The vertical dotted lines represent respectively: (i) September 1992, when the Bank of England adopted inflation targeting, (ii) May 1997, when the Bank of England gained independence in setting monetary policy and (iii) March 2009, when the Monetary Policy Committee announced a cut of the policy rate to 0.5% and an economic stimulus via large-scale asset purchase programs.

Figure 5: Liquidity proxy and comparison of market-based and survey-based inflation expectations
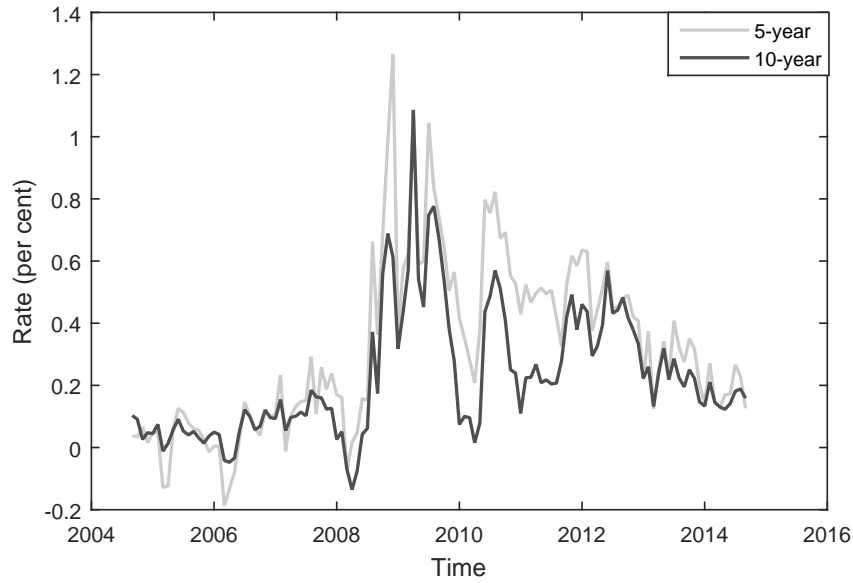
(a) 5-year and 10-year liquidity premium estimates
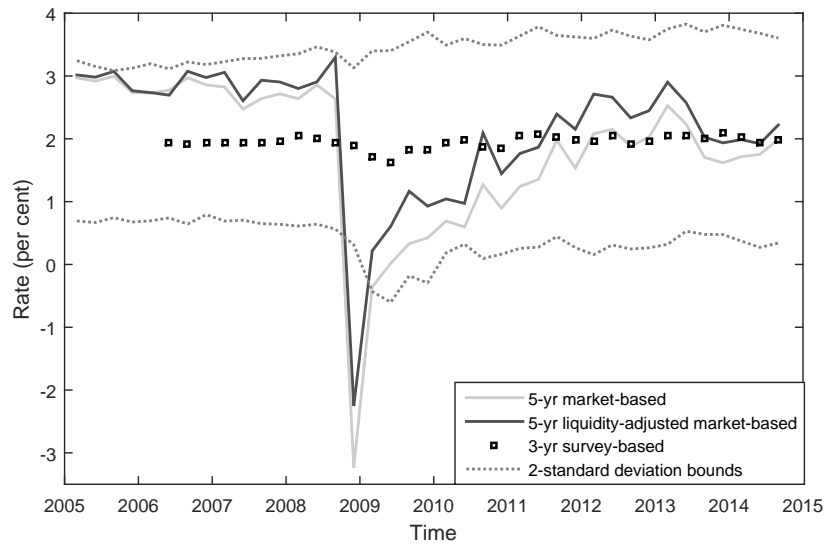
(b) Market-based and survey-based inflation expectations

Panel (a) reports estimates of 5- and 10-year liquidity premiums, obtained as the difference between inflation-linked swaps and BEI rates, as described in Christensen and Gillan (2011). Panel (b) reports 5-year inflation expectations implied by our joint SLB-NS estimation and their 5-year liquidity-adjusted counterparts. 3-year survey-based inflation expectations stemming from the Bank of England's Survey of External Forecasters are provided. 2-standard deviation bounds around these 3-year survey-based measures are constructed using the conditional variance implied by the survey's aggregate distribution function.