Polytopal and structural aspects of matroids and related objects

by

Amanda Cameron

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Department of Mathematics
Queen Mary University of London
United Kingdom

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Declaration

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Details of collaboration and publications: parts of this work have been completed in collaboration with ALEX FINK and DILLON MAYHEW and are published in the following papers:

- *A lattice point counting generalisation of the Tutte polynomial*, joint with Alex Fink, accepted to FPSAC 2016. This is an extended abstract of the work in Chapters 4-7.

- *A splitter theorem for connected clutters*, joint with Dillon Mayhew, submitted. This appears as is in Chapter 9.

- *An excluded minor characterisation of split matroids*, joint with Dillon Mayhew, submitted. This appears in Chapter 8, with a truncated introduction and one additional proof, that of Theorem 8.9.

Abstract

This thesis consists of three self-contained but related parts. The first is focussed on polymatroids, these being a natural generalisation of matroids. The Tutte polynomial is one of the most important and well-known graph polynomials, and also features prominently in matroid theory. It is however not directly applicable to polymatroids. For instance, deletion-contraction properties do not hold. We construct a polynomial for polymatroids which behaves similarly to the Tutte polynomial of a matroid, and in fact contains the same information as the Tutte polynomial when we restrict to matroids.

The second section is concerned with split matroids, a class of matroids which arises by putting conditions on the system of split hyperplanes of the matroid base polytope. We describe these conditions in terms of structural properties of the matroid, and use this to give an excluded minor characterisation of the class.

In the final section, we investigate the structure of clutters. A clutter consists of a finite set and a collection of pairwise incomparable subsets. Clutters are natural generalisations of matroids, and they have similar operations of deletion and contraction. We introduce a notion of connectivity for clutters that generalises that of connectivity for matroids. We prove a splitter theorem for connected clutters that has the splitter theorem for connected matroids as a special case: if M and N are connected clutters, and N is a proper minor of M, then there is an element in $E(M)$ that can be deleted or contracted to produce a connected clutter with N as a minor.

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Chapter 1

Introduction

Matroids were first described in 1935 by Whitney [47] as a generalisation of linear independence in vector spaces. For a given matrix with columns representing vectors, take the set of sets of columns which are linearly independent. Whitney noted certain properties which must hold for such columns, then discovered that the same properties admit set-systems which cannot be represented as a matrix. A set-system is a collection of subsets of a given set: in this case, the relevant vector space. Whitney also drew allusions to graphs, these being a subset of matroids, thus explaining the migration of terminology from both graph theory and linear algebra. Graph theory is similar to linear algebra in that it is concerned with independence – here independence refers to cycle-free sets of edges. One way in which all three objects are similar is that they all have a notion of dual constructions. Every planar graph has a dual graph, and every matroid of a planar graph has a dual matroid which is the matroid of the dual graph. We also have a notion of duality for vector configurations: every full dimensional configuration has a Gale dual. A compelling reason to consider matroids to be the natural generalisation of graphs is that every matroid has a dual, not just those that arise from planar graphs.

The above suggests defining matroids as certain set systems, capturing which sets are independent. While this is one standard definition, many equivalent definitions are also used, and we choose one based on the rank function, with an eye to a generalisation in Chapter 3. The rank of a set is the size of its largest independent subset (defined below). Let $\mathcal{P}(E)$ be the power set of a set E.

Definition 1.1. *A matroid $M = (E, r)$ consists of a finite ground set E and a* rank function $r : \mathcal{P}(E) \to \mathbb{Z}^+ \cup \{0\}$ *such that, for $X, Y \in \mathcal{P}(E)$, the following conditions hold:*

R1. $r(X) \leq |X|$,

R2. *if* $Y \subseteq X$, *then* $r(Y) \leq r(X)$, *and*

R3. $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

Note that $r(M) = r(E)$. A set X is *independent* if and only if $r(X) = |X|$. If $r(X) = r(M)$ we call X a *basis* of M. If a set contains a basis, it is called *spanning.* Matroids can also be defined in terms of the bases, or their minimal dependent sets (known as *circuits*), or indeed in several other cryptomorphic ways. Some of these, and further terminology, will be discussed in Chapter 2. In Part III, Chapter 9, we will primarily be focussing on the circuit axioms.

A viewpoint of a matroid that we will take a lot is that of a polytope. For a set $U \subseteq E$, let $\mathbf{e}_U \in \mathbb{R}^E$ be the indicator vector of U, that is, $\mathbf{e}_U$ is the sum of the unit vectors $\mathbf{e}_i$ for all $i \in U$. Let the set of bases of a matroid M be $\mathcal{B}$, and let Conv denote the convex hull of a set of vectors.

Definition 1.2. *The* base polytope *of M is*

$$P(M) = \mathrm{Conv}\{\mathbf{e}_B \mid B \in \mathcal{B}\}.$$

Matroid polytopes were extensively studied by Edmonds a few decades after the discovery of matroids, and used by him to prove the famous matroid intersection theorem in 1971 [14]. This is in fact a generalisation of König's matching theorem [38, Theorem 16.2, and Section 41.1a]. The use of polytopes facilitates the application of matroids to optimisation problems. We may want to find the maximum cost basis in a given matroid, the minimum spanning set contained in both of two given matroids, or the maximum branching of a given digraph. The last two are solved by matroid intersection. Matroid polytopes are also useful in algebraic and tropical geometry, with objects such as the Grassmanian and the algebraic torus. For example, the polytope of any representable matroid is the moment polytope of a particular toric variety of a point in the Grassmanian [22]. Edmonds also considered a generalisation of matroids which arise by removing one rank condition, which can likewise be represented with a polytope. These are called *polymatroids*:

Definition 1.3. *A* polymatroid (E, r) *consists of a ground set* E *and a rank function* $r : \mathcal{P}(E) \to \mathbb{Z}^+ \cup \{0\}$ *such that, for* $X, Y \in \mathcal{P}(E)$*, the following conditions hold:*

P1. $r(\emptyset) = 0$*,*

P2. *if* $Y \subseteq X$*, then* $r(Y) \leq r(X)$*, and*

P3. $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$*.*

Edmonds was one of the first to discuss this object, and actually defined it in terms of the polymatroid polytope rather than with the rank axioms. This polytope is a similar construction to that of a matroid polytope, and the details are given in Chapter 3. This chapter also contains further background on polytopes.

In Part I of this thesis, we construct a polymatroidal generalisation of a famous matroid polynomial, the Tutte polynomial. This was originally formulated for graphs, in terms of the connectivity function.

Definition 1.4. *Let* $M = (E, r)$ *be a matroid with ground set* E *and rank function* $r : \mathcal{P}(E) \to \mathbb{Z}^+ \cup \{0\}$*. The* Tutte polynomial *of* M *is*

$$T_M(x, y) = \sum_{S \subseteq E} (x-1)^{r(M)-r(S)} (y-1)^{|S|-r(S)}.$$

This polynomial has a diverse range of applications. The most direct of these is classifying Tutte invariants, these being properties of matroids or graphs which can be enumerated by an evaluation of the Tutte polynomial. We can obtain the number of bases and the number of independent sets in a given matroid, and the number of acyclic orientations of a graph, amongst other information. These invariants, and other properties, are discussed in Chapter 2. The Tutte invariants also include other classical invariants, such as the chromatic polynomial, which gives the numbers of graph colourings, and the flow and reliability polynomials used in network theory. Multivariate versions of the Tutte polynomial also appear in other disciplines, containing as specialisations the Potts model [45] from statistical physics and the Jones polynomial [43] from knot theory. The Tutte polynomial also has practical applications in coding theory. The presentation of the Tutte polynomial in Definition 1.4 is given in terms of the *corank-nullity polynomial*: up to a change of variables, it is the generating function for subsets S of the ground set by their corank $r(M) - r(S)$ and nullity $|S| - r(S)$. The corank-nullity polynomial can be defined for polymatroids, but the resulting function gives far less easily accessible information akin to that of matroids, and the terms do not always have non-negative degrees. It is not even a Laurent polynomial in the variables x and y.

One difference between matroids and polymatroids is that matroids have a theory of *minors* analogous to graph minors: for each ground set element one can define a deletion and contraction, and knowing these two determines the matroid. The deletion-contraction recurrence for the Tutte polynomial reflects this structure. Polymatroids, however, lack satisfying properties of deletion and contraction such as this. Also, given any element in a matroid, every basis (or the basis minus the element) is in either the deletion or contraction of that element. This is not true in polymatroids, causing difficulty in arguments using basis-counting. It is however possible to salvage some features of the deletion-contraction recurrence in restricted cases: this is done by Oxley and Whittle [34] for polymatroids where singletons have rank at most 2 (a *2-polymatroid*), where the corank-nullity polynomial is still universal for a form of deletion-contraction recurrence. In their paper they point out that a polymatroid can be considered as a multiset of flats of some matroid, and that every polymatroid can be obtained in this way. They call this matroid a *representation* of the polymatroid. The definition of deletion in a 2-polymatroid given by Oxley and Whittle is a generalisation of matroid deletion, while their definition of contraction corresponds to contraction in the representing matroid. Therefore while these definitions do not give the properties of matroids we would like to have, they are natural choices which would be hard to improve on.

We create a polynomial which will be an evaluation of the Tutte polynomial when applied to a matroid and has analogous enumerative properties for polymatroids. This is done from a polytopal approach, inspired by the notion of *activity* of bases. The standard definition of activity provides for another description of the Tutte polynomial (Equation 1.5.1, below), and has recently been generalised by Kálmán [29] so as to apply to polymatroids. These generalisations are given in Section 3.5 of Chapter 3. In the case of a matroid, the definitions are as follows. Note that a *cocircuit* is a circuit in the dual matroid (a definition of this dual is given in Chapter 2).

Definition 1.5. *Take a matroid $M = (E, r)$, and give E some ordering. Let B be a basis of M.*

i. We say that $e \in E - B$ is externally active *with respect to B if e is the smallest element in the unique circuit contained in $B \cup e$, with respect to the ordering on E.*

ii. We say that $e \in B$ is internally active *with respect to B if e is the smallest element in the unique cocircuit in $(E - B) \cup e$.*

If an element is not active, in whichever sense, we say it is *inactive*. Let the number of internally active elements with respect to B be denoted with $\iota(B)$ and let the number of externally active elements be denoted by $\varepsilon(x)$. When M is a matroid, these numbers provide an alternative formulation of the Tutte polynomial,

$$T_M(x, y) = \sum_{B \in \mathcal{B}} x^{\iota(B)} y^{\varepsilon(B)}. \tag{1.5.1}$$

These definitions are the analogies of those originally formulated using spanning trees of graphs. The analogy between the internal and external polynomials of a polymatroid and the same polynomials under the graph definitions were what suggested that a two-variable polynomial similar to the Tutte polynomial could be found for polymatroids. In the case of a polymatroid, the definitions are as follows. Note that neither of these polynomials depend on the order on E that was used to define them. Here, $\bar{\iota}(B)$ is the number of internally *inactive* elements with respect to the basis B, and $\bar{\varepsilon}(B)$ is the number of externally inactive elements with respect to B.

Definition 1.6 ([29]). *Let $M = (E, r)$ be a polymatroid. Define the* internal polynomial *and* external polynomial *of M by*

$$I_M(\xi) = \sum_{x \in \mathcal{B}_M \cap \mathbb{Z}^E} \xi^{\bar{\iota}(x)} \quad \text{and} \quad X_M(\eta) = \sum_{x \in \mathcal{B}_M \cap \mathbb{Z}^E} \eta^{\bar{\varepsilon}(x)}.$$

In fact, by Theorem 4.3, the invariant we are constructing is the bivariate analogue of Kálmán's activity polynomials, which is something his paper sought. Kálmán's original interest in these objects related to enumerating spanning trees of bipartite graphs according to their vector of degrees at the vertices on one side. In this context Oh [32] has investigated a polyhedral construction similar to ours, as a way of proving Stanley's pure O-sequence conjecture for cotransversal matroids.

We will construct a two-variable polynomial which is equivalent to the Tutte polynomial for matroids and includes some activity information for polymatroids. We will form a polynomial which counts the lattice points in a particular polytope which we construct from $P(M)$ in a way which introduces the stylistically necessary two variables. This construction will be described in full detail in Chapter 4. In Chapters 5 through 7, we give properties and facts about this polynomial, including its relationship to the Tutte polynomial and a geometric interpretation of its coefficients.

Part III is also related to matroid polytopes. A matroid M is an *excluded minor* for a class of matroids if it is not in the class but all of its proper minors are. We give an excluded minor characterisation of a particular class of matroids, namely *split matroids*, which arise by considering particular subdivisions of the base polytope of the matroid.

Definition 1.7. *Let P be a polytope. A* subdivision *of P is a collection (complex) $\mathcal{C}$ of polytopes such that*

i. the empty polytope is in $\mathcal{C}$,
ii. if Q is in $\mathcal{C}$, so are all faces of Q,
iii. the intersection of any two polytopes $Q_1, Q_2 \in \mathcal{C}$ is a face of both Q_1 and Q_2, and
iv. $\bigcup_{C \in \mathcal{C}} C = P$.

The elements of $\mathcal{C}$ are called the cells*.*

We require that all vertices of $Q \in \mathcal{C}$ are also vertices of P.

Let a matroid M have some subdivision. The Tutte polynomial of M is equal to the alternating sum of the Tutte polynomials of the cells in its subdivision not on the boundary, through inclusion-exclusion based on dimensions of the cells [2], giving a further reason why matroid polytopes are useful.

In [28] Joswig and Schröter introduce a class of matroids called *split matroids*, defined in terms of split hyperplanes of the matroid base polytope. A *split* of a polytope is a polytopal subdivision with exactly two maximal cells, whose intersection is called a *split hyperplane*. Joswig and Schröter define a pair of splits of a polytope to be *compatible* if their split hyperplanes do not meet in a relative interior point (Definition 3.7) of P. Every flat of the matroid defines a face of the polytope. A *flacet* is a flat minimal under inclusion with respect to its hyperplane intersecting $P(M)$ in a facet. If the intersection spans a split hyperplane, we say the flacet is a *split flacet*. We can now present the class of split matroids:

Definition 1.8 ([28])**.** *A matroid is a split matroid if its split flacets form a compatible system of hypersimplex splits.*

Joswig and Schröter prove that the class of split matroids contains the class of sparse paving matroids, which is conjectured to dominate the class of matroids. While this class is new, the authors note that there have been occasions when split hyperplanes have been used in structural results, such as in [9]. As well as being interesting from a polytopal structural perspective, split matroids have strong ties to tropical geometry. Joswig and Schröter use them to solve some open questions about tropical Grassmanians and Dressians. Like representability in matroid theory, representability of tropical linear spaces is an important question in tropical geometry. Tropical linear spaces and split matroids are interrelated: a tropical linear space is a polytopal subdivision of a hypersimplex (or a regular subdivision of a matroid polytope) into matroid polytope cells, and is cryptomorphic to a valuated matroid. Representability of a tropical linear space is thus representability of valuated matroids [13], obtained in a natural way from standard matroid representability. Joswig and Schröter use split matroids and the Dressian to construct a number of nonrepresentable tropical linear spaces, and give a characterisation of matroid representability in terms of these spaces. Speyer [41] has also considered a similar problem, where the matroids in question are restricted to being series-parallel. Hyperplane splits also show up in a different context: Herrmann and Joswig note that the complex of splits of the regular octahedron is a particular graph (the link of the origin L_{n-1}, see [4]) in the space of phylogenetic trees, when $n = 4$. This space is in fact a tropical Grassmanian, $\mathrm{GR}(2, n)$. When $n > 4$, the link of the origin is the complex of splits of the hypersimplex $\Delta(2, n)$, which is the polytope of the uniform matroid $U_{2,n}$.

Joswig and Schröter end on a few open questions, one being that of an excluded minor characterisation of the class of split matroids. They conjecture this to be a set of five particular matroids, four of which are connected. A matroid is connected when, for every pair of elements in the matroid, there is a circuit which contains them both. We give a second definition for a split matroid, entirely in terms of structural matroid terminology. This is split into two cases, based on whether the matroid is connected or not. For details on why this definition is equivalent to that given above, see Chapter 8.

Definition 1.9. *Let M be a connected matroid, and let Z be a proper cyclic flat of M.*

If both $M|Z$ and M/Z are connected matroids, but at least one of them is a non-uniform matroid, then we say that Z is a certificate *for* non-splitting. *If M is connected, and has no such certificate, then M is a* split *matroid.*

Now consider disconnected matroids.

Definition 1.10. *Let M be a disconnected matroid, with connected components $C_1, \ldots, C_t$. Then M is a* split *matroid if and only if each connected matroid, $M|C_i$, is a split matroid, and at most one of these matroids is non-uniform.*

Using these definitions, we settle the conjecture positively:

Theorem 1.11. *The only disconnected excluded minor for the class of split matroids is S_0. The only connected excluded minors are S_1, S_2, S_3, and S_4.*

Figure 1.1 shows geometric representations of the four connected rank-3 matroids, each with six elements. It is easy to confirm that these matroids are indeed excluded minors for the class of split matroids. Note that $S_1^* \cong S_2$, whereas S_3 and S_4 are both self-dual matroids. The final excluded minor, S_0, is that constructed from the direct sum $U_{2,3} \oplus U_{2,3}$ by adding one parallel point to each of the two connected components.

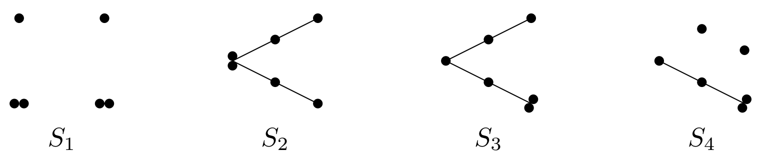


Figure 1.1: Connected excluded minors for split matroids.

Part IV is concerned with a different generalisation of matroids. A *clutter* is a pair $(E, \mathcal{A})$, where E is a finite set, and $\mathcal{A}$ is a collection of subsets of E, with the property that if A and A' are distinct members of $\mathcal{A}$, then $A \not\subseteq A'$. Clutters are also referred to as antichains and Sperner families. We will call the members of $\mathcal{A}$ the *rows* of the clutter. For an example of a clutter, we may take $\mathcal{A}$ to be the set of circuits of a matroid, or the set of bases, or indeed the sets minimal with respect to any given property. If the subsets A all have cardinality two, then they are the edges of a simple graph, and so clutters are naturally tied to both matroids and graphs. Clutters are natural generalisations of matroids and graphs in another sense: they admit the notion of a minor-relation, with contraction and deletion operations. The details of this are given in Chapter 9. Like with matroids and polymatroids, polytopes and algebraic geometry concepts arise with clutters quite naturally. Every clutter has an independence complex and thus a simplicial complex associated to it. The blocker of a clutter is a special case, using a Stanley-Reisner complex, of Alexander duality. When the clutter is from a chordal graph, there are combinatorial geometrical statements which can be made about the independence complex, such that it is shellable [44] and sequentially Cohen-Macauley [19]. Woodroofe [49] develops the idea of a chordal clutter and shows these are also both shellable and sequentially Cohen-Macauley. Clutters also have two polytopes associated to them, the independence system and covering polytopes. These polytopes, like matroid polytopes, are useful in combinatorial optimisation. One example of this is Fulkerson [20], [21], who uses these polytopes in his study of blocking and antiblocking polyhedra, which are related to the well-known problems of maximum-packing and minimum-cover respectively.

An important notion which arises in conjunction with matroid and graph minors is that of connectivity. We show that some connectivity behaviour in matroids is actually just a special case of a clutter phenomenon. To do so, we must develop a notion of connectivity for clutters.

Definition 1.12. *Let $M = (E, \mathcal{A})$ be a clutter. A* separation *of M is a partition of E into non-empty parts, X and Y, such that every row is contained in X or Y. If M admits no separation then it is* connected.

This is a natural way to define connectivity for clutters, since it generalises connectivity for graphs and for matroids. Brylawski [8] and Seymour [39] independently proved that if M is a connected matroid with a connected proper minor, N, then we can delete or contract an element from M in such a way to preserve connectivity, and the minor N. We prove that this is a special case of a clutter property:

Theorem 1.13. *Let M and N be connected clutters and assume that N is a proper* *minor of M. There exists an element, $v \in E(M)$, such that either $M\backslash v$ or M/v is connected and has N as a minor.*

This type of theorem is known as a *splitter theorem*, after Seymour's well-known splitter theorem for 3-connected matroids [40]. We obtain, as a corollary, a weaker type of statement, known as a *chain theorem*.

Corollary 1.14. *Let M be a non-empty connected clutter. Then there is an element, $v \in E(M)$, such that either $M\backslash v$ or M/v is a connected clutter.*

Seymour's splitter theorem was itself a generalisation of Tutte's Wheels and Whirls Theorem, which states that it is possible to contract or delete an element from a given 3-connected matroid M and obtain a 3-connected minor, unless M is a wheel or whirl. Oxley, Semple, and Whittle [35] give a version of this for 3-connected 2-polymatroids. Splitter theorems in general are useful as they are often important components of decomposition theorems. For instance, Seymour's splitter theorem can be used to show that every regular matroid can be constructed from a set of graphic and cographic matroids, and the matroid R_{10}, put together using 1-, 2-, and 3-sums. Given some conditions on a matroid, splitter theorems can also be used to get information on minors contained in that matroid, information highly useful for excluded minor characterisation of particular classes of matroids. Generalisations of the splitter theorem, such as only allowing contractions, lead to results in matroid representability. As examples, Whittle [48] uses this method when considering the number of inequivalent representations of a matroid over a given field, and Geelen, Gerards and Kapoor [23] use Seymour's splitter theorem as an ingredient in their proof of the set of excluded minors for GF(4)-representable matroids.

Part I

Preliminaries

Chapter 2

Matroid Fundamentals

To begin with, we will cover the basic concepts in matroid theory which will be used throughout this thesis. All of the following concepts and results can be found in [33].

Definition 2.1. *A* matroid $M = (E, \mathcal{I})$ *consists of a finite ground set* E *and a collection of subsets* $\mathcal{I} \subseteq E$ *such that:*

I1. $\varnothing \in \mathcal{I}$,

I2. *if* $I \in \mathcal{I}$ *and* $J \subseteq I$, *then* $J \in \mathcal{I}$, *and*

I3. *if* $I, J \in \mathcal{I}$ *and* $|I| < |J|$, *there exists* $x \in J - I$ *such that* $I \cup \{x\} \in \mathcal{I}$.

Any subset of E contained in $\mathcal{I}$ is referred to as an *independent set*, while any subset of E which is not contained in $\mathcal{I}$ is called *dependent*. A dependent set of cardinality one is called a *loop*. We may use $E(M)$ in the place of E at times, in order to make it clear which matroid is being referred to.

Definition 2.2. *Take a matroid* M *with ground set* E. *The* rank *of a subset* X *of* E, *denoted by* $r(X)$, *is the cardinality of the largest independent subset of* X.

Lemma 2.3. *A matroid* M *can be described by the ground set* E *and a* rank function $r : \mathcal{P}(E) \to \mathbb{Z}^+ \cup \{0\}$ *such that, for* $X, Y \in \mathcal{P}(E)$, *the following conditions hold:*

R1. $r(X) \leq |X|$,

R2. *if* $Y \subseteq X$, *then* $r(Y) \leq r(X)$, *and*

R3. $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

We say that $r(M) = r(E)$. A set X is independent if and only if $r(X) = |X|$. If $r(X) = r(M)$ we call X a *basis* of M. If a set contains a basis, it is called *spanning*. An element with rank zero is called a *loop*. A *parallel class* is a set of rank 1 which contains no loops.

A result that will prove important in a later proof is that of matroid partition. The following result is due to Edmonds.

Theorem 2.4 ([14]). *Let* $M_1, \ldots, M_k$ *be a set of matroids all with a ground set* E. *Let* $M_i = (E, \mathcal{I}_i)$. *Then* E *can be partitioned into a family* $I_1, \ldots, I_k$ *where* $I_i \in \mathcal{I}_i$, *if and only if there is no* $A \subseteq E$ *such that* $|A| > \sum_i r_i(A)$ *where* r_i *is the rank function of* M_i.

[14] gives an algorithm for finding such a partition.

2.1 Dependencies

Definition 2.5. *The* closure *of a set* X *is denoted by* $\mathrm{cl}(X)$, *where*

$$\mathrm{cl}(X) = X \cup \{e \in E - X \mid r(X \cup e) = r(X)\}.$$

Lemma 2.6. *The closure function of a matroid satisfies the following conditions:*

CL1. *If* $X \subseteq E$, *then* $X \subseteq cl(X)$.

CL2. *If* $X \subseteq Y$, *then* $cl(X) \subseteq cl(Y)$.

CL3. *If* $X \subseteq E$, *then* $cl(cl(X)) = cl(X)$.

CL4. *If* $X \subseteq E$ *and* $x \in E$, *and* $y \in cl(X \cup x) - cl(X)$, *then* $x \in cl(X \cup y)$.

The closure function corresponds to the notion of span of a vector space, and is sometimes referred to as such. A *flat* is a set whose closure is equal to the set itself, i.e. $\mathrm{cl}(X) = X$. If a flat has rank $r(M) - 1$, it is called a *hyperplane*. We say that a matroid is *connected* if and only if, for any two elements e, f of $E(M)$, there is a flat which contains them both.

A minimally dependent set – that is, a dependent set where every proper subset of that set is independent – is called a *circuit*. A matroid can be described entirely by its set of circuits $\mathcal{C}$.

Lemma 2.7. *$(E, \mathcal{C})$ describes a matroid when the following conditions hold.*

C1. $\varnothing \notin \mathcal{C}$.

C2. *If $C, D \in \mathcal{C}$ and $C \subseteq D$, then $C = D$.*

C3. *If C, D are distinct elements of $\mathcal{C}$ amd $e \in C \cup D$, then $(C \cup D) - e$ contains a circuit.*

A *circuit-hyperplane* is a set which is both a circuit and a hyperplane.

Definition 2.8. *Let M be a matroid and let H be a circuit-hyperplane of M. H has rank equal to $r(M) - 1$. We say that we* relax *H when we make it independent, i.e. we form a matroid M' whose set of bases is $\mathcal{B}(M) \cup H$. When we reverse this operation, we say that we* tighten *H.*

2.2 Representability

Definition 2.9. *If V is a set of vectors in a vector space, and for every subset X of V, we define $r(X)$ to be the linear rank of X, then (V, r) is a matroid, which we say is* representable.

If these vectors come from a finite field $\mathbb{K}$, we say that M is $\mathbb{K}$-*representable.*

2.3 Minors

Definition 2.10. *We can remove an element e of a matroid $M = (E, r)$ by* deleting *it. This yields a matroid $M\backslash e = (E - \{e\}, r_{M\backslash e})$, where $r_{M\backslash e}(X) = r_M(X)$ for all $X \subseteq E - \{e\}$.*

Definition 2.11. *We can also remove an element e of a matroid $M = (E, r)$ by* contracting *it. This gives a matroid $M/e = (E-\{e\}, r_{M/e})$ where $r_{M/e}(X) = r_M(X \cup \{e\}) - r(\{e\})$ for all $X \subseteq E - \{e\}$.*

The *restriction* of M to Z is denoted by $M|Z$, and is equal to $M\backslash(E - Z)$.

Any matroid produced by a sequence of deletions and contractions is called a *minor* of M.

We say that a class of matroids $\mathcal{M}$ is *minor-closed* if, for every matroid M in $\mathcal{M}$, each of its minors is also in $\mathcal{M}$.

A matroid M is an *excluded minor* for a minor-closed class of matroids $\mathcal{M}$ if it is a minimal minor not in $\mathcal{M}$. A matroid M is contained in $\mathcal{M}$ if and only if M does not contain an excluded minor for $\mathcal{M}$.

2.4 Duality

Definition 2.12. *From M we can construct the* dual matroid *M^*. This has ground set equal to the ground set E of M, and the rank of any subset X is found using the function $r^*(X) = |X| + r(E - X) - r(M)$.*

A basis of M^* is called a *cobasis* of M. Note that if B is a basis of M, then $E - B$ is a cobasis of M. Similarly, the rank function, circuits, loops and independent sets of M^* are called the *corank* function, *cocircuits*, *coloops* and *coindependent sets* of M. A parallel class in M^* is called a *series class* in M.

Lemma 2.13 ([33, Proposition 2.1.7]). *Let M be a matroid. Relax a circuit-hyperplane H of M to yield the matroid M'. Then $(M')^*$ is identical to the matroid yielded from M^* by relaxing the circuit-hyperplane $E - H$ of M^*.*

Lemma 2.14 ([33, Proposition 3.3.5]). *Let H be a circuit-hyperplane of a matroid M, and let M' be the matroid obtained from M by relaxing H.*

i. *When $e \in E(M) - H$, $M/e = M'/e$, and, unless e is a coloop of M, $M'\backslash e$ is obtained from $M\backslash e$ by relaxing the circuit-hyperplane H of $M\backslash e$.*

ii. *Dually, when $f \in H$, $M\backslash f = M'\backslash f$ and, unless f is a loop of M, M'/f is obtained from M/f by relaxing the circuit-hyperplane $X - \{f\}$ of M/f.*

2.5 Extensions

When we delete e from a matroid $M = (E, r)$ to get another matroid N, we say that M is an *extension* of N. As a subset of these, we can have *free extensions*:

Definition 2.15. *Take a matroid $M = (E, r)$. An element e is* freely placed *in a flat F of M if, for any set $Z \subseteq E$, $e \in cl(Z)$ implies $F \in cl(Z)$.*

Definition 2.16. *Let $M = (E, r)$ be a matroid. Add an element $e \notin E$ freely to E. This gives a matroid $(E \cup \{e\}, r)$, which we call a* free extension *of M.*

The other form of extension we will use is that of *parallel extensions*.

Definition 2.17. *Let M and N be matroids, and let $f \in E(M)$. If f is in a parallel pair in M and $M\backslash f = N$, then M is a* parallel extension *of N.*

We will refer to this as adding a parallel point to N. There is a similar notion involving the dual object, series pairs. A series pair in M is a parallel pair in M^*.

Definition 2.18. *Let M and N be matroids, and let $f \in E(M)$. If f is in a series pair in M and $M/f = N$, then M is a* series extension *of N.*

2.6 Tutte polynomial

The Tutte polynomial originated with graphs, and so we will explain it in this context before applying it to matroids.

Take a graph $G = (V, E)$ and order the elements of E arbitrarily. Let $\mathcal{T}$ be the set of spanning trees of G. In order to define the Tutte polynomial, we must first define the notion of activity.

Definition 2.19. *Let $\Gamma \in \mathcal{T}$ be a spanning tree.*

i. An edge $e \in \Gamma$ is internally active *with respect to Γ if, after removing e from Γ, connectedness of the subgraph cannot be restored by adding an edge to $\Gamma \backslash \{e\}$ that is smaller than e under the given order.*

ii. An edge $e \notin \Gamma$ is externally active *if, after adding e to Γ, cycle-freeness cannot be restored by removing an edge from $\Gamma \cup \{e\}$ that is smaller than e.*

Definition 2.20. *Let Γ be a spanning tree. Define $\iota(\Gamma)$ to be the number of internally active edges with respect to Γ, under the given order. Define $\varepsilon(\Gamma)$ to be the number of externally active edges with respect to Γ.*

Definition 2.21. *Let G be any graph. The* Tutte polynomial *of G is*

$$T_G(x, y) = \sum_{\Gamma \in \mathcal{T}} x^{\iota(\Gamma)} y^{\varepsilon(\Gamma)}. \tag{2.21.1}$$

The Tutte polynomial is independent of the chosen edge order. The definition above is only one of multiple methods of finding the polynomial. An alternative method is the iterated deletion and contraction of edges of G, using the following facts:

i. $T_G(x, y) = xT_{G/e}(x, y)$ if e is an isthmus.

ii. $T_G(x, y) = yT_{G\backslash e}(x, y)$ if e is a loop.

iii. $T_G(x, y) = T_{G\backslash e}(x, y) + T_{G/e}(x, y)$ if e is neither an isthmus nor a loop.

Every matroid also has an associated Tutte polynomial.

Definition 2.22. *Let $M = (E, r)$ be a matroid. The* Tutte polynomial *of M is*

$$T_M(x, y) = \sum_{S \subseteq E} (x-1)^{r(M)-r(S)}(y-1)^{|S|-r(S)}.$$

When $x-1$ is replaced with u and $y-1$ is replaced with v, this is known as the corank–nullity polynomial, due to the following definitions: the *corank* of M is $\text{cork}(M) := r(M) - r(S)$ and the *nullity* of M is $\text{null}(M) := |S| - r(S)$. Note that the corank is different from the dual rank, an unfortunate lapse from the previous terminology.

This definition of the Tutte polynomial can also be used for graphs: let $r_G(S) = V(G) - k_G(S)$, where $k_G(S)$ is the number of connected components of $G_2 = (V(G), S)$.

By substituting in values for x and y, the Tutte polynomial reveals various facts about the matroid or the graph being used. For instance, $T(1,1)$ gives the number of bases in a matroid or the number of spanning trees in a graph, while $T(2,1)$ gives the number of independent sets in a matroid or the number of forests in graphs. Invariants that can be obtained in this way are called *Tutte Invariants*.

Chapter 3

Polytope fundamentals

3.1 Polytopes

An alternative way to view matroids is as polytopes. This viewpoint is more commonly used for an extension of matroids, namely polymatroids. This chapter focuses on the background theory behind polymatroids and polytopes, incorporating the Tutte polynomial once again. As matroid polytopes are convex, we will mainly limit the discussion to these. We will begin with some general terminology, then relate this to matroid theory.

Recall that a subset P of $\mathbb{R}^n$ is *convex* if any two points of P are connected by a straight line which lies inside P. The *convex hull* of a set of points P is the intersection of all convex sets containing P.

Definition 3.1. *A* convex polytope *is the convex hull of a finite set of points in* $\mathbb{R}^n$*.*

Such a description of a convex polytope is often called a *vertex representation* (often shortened to v-representation). A vertex corresponds, intuitively, to a point of the polytope, either on the boundary or interior. For a technical definition, see directly below Definition 3.3. Note that the intersection of two polytopes is another polytope.

We can alternatively describe the polytope by a set of linear inequalities the vertices must satisfy: it can be defined as a set of vectors

$$\{\mathbf{x} \in \mathbb{R}^d \mid A\mathbf{x} \leq \mathbf{b}\}$$

where d is the dimension of the polytope, $\mathbf{x}$ is an integer vector, and $A\mathbf{x} \leq \mathbf{b}$ is a system of rank inequalities describing the polytope with $A \in \mathbb{Z}^{m \times d}, \mathbf{b} \in \mathbb{Z}^m$. Thus finding the number of lattice points inside the polytope is equivalent to finding the number of integer solutions to $A\mathbf{x} \leq \mathbf{b}$. This is called the *half-space*- or *hyperplane*-representation (h-representation). The complexity of going from one representation to the other is an ongoing problem.

Definition 3.2. *A* half-space *is either of the two parts into which a hyperplane divides an affine space.*

Any subspace connecting a point in one side of the partition to one point in the other must intersect the hyperplane. If the space is two-dimensional, then a half-space is called a half-plane. In one-dimensional space, a half-space is called a ray. A half-space is a convex set, and any convex set can be described as the intersection of half-spaces.

As mentioned, a half-space can be specified by a linear inequality, derived from the linear equality which specifies the hyperplane:

$$a_1x_1 + \cdots + a_nx_n \leq b.$$

A strict inequality specifies an open half-space, while a non-strict specifies a closed half-space. With this notation, we can regard a closed convex polytope as the set of solutions to such a system, or more compactly, as $A\mathbf{x} \leq \mathbf{b}$. With this terminology, we can now give the promised definition of a vertex, and more.

Definition 3.3. *Let a polytope P be defined by the set of solutions to the equalities $A\mathbf{x} \leq \mathbf{b}$. Let C be a submatrix of A found by removing some rows of A, and $\mathbf{d}$ be a vector created from $\mathbf{b}$ by removing the rows in corresponding positions. A* face *F of a polytope P consists of the set of solutions to this subset of the defining equalities of P,*

$F = \{x \in P \mid C\mathbf{x} \leq \mathbf{d}\}$.

Note the above definition includes both the empty set and the entire polytope P as faces. A *vertex* is a 0-dimensional face of the polytope, a 1-dimensional face is an *edge*, and a $(n-1)$-dimensional face is called a *facet*.

An important difference between the two representations is the notion of boundedness: a polytope is *bounded* if there is a ball of finite radius that contains it. Also, a polytope is *finite* if it is the convex hull of a finite set of points. When the convex polytope is given by an h-representation, it need not be either bounded or finite. All the polytopes we use will, however, be both bounded and finite.

Remark 3.4. A convex polytope is a *lattice* polytope if its vertices are lattice points. *Integral* is commonly used in the place of "lattice". All the polytopes we will be considering are lattice polytopes. Note that a convex lattice polytope is determined by its set of lattice points. Throughout this thesis, we will thus restrict our attention to these points, and statements made about the points of polytopes will be made at the lattice point level. For instance, when proving that two lattice polytopes are equal, we will use the technique that showing their sets of lattice points are the same.

A commonly used polytopal construction is that of the *cone*:

Definition 3.5. *A* convex cone *is the convex hull of a (finite) set of half-lines which originate from a single point.*

In other words, a cone is a set of points which are solutions to $\lambda(x) \geq 0$ where λ is linear. A *face of a polynomial cone* is a subset of the cone given by replacing some of the inequalities with $\lambda(x) = 0$. Every face F of a polytope has a unique *tangent* cone: this is the intersection of all closed half-spaces containing P whose boundary contains F. There is also a *normal* cone. Given a polytope (or any set) P and a vector x, the *maximiser set* of x over P is $\{f \in P \mid x \cdot f = \sup_{p \in S} x \cdot p\}$. Now, a *normal cone* of a face F is the set of vectors for which F is the maximiser set over P. This set of vectors is in the dual vector space to that P sits in, and hence determine a linear functional, $x \cdot f$, on

P.

A notion we will make use of is that of the *relative interior* of a polytope P. This relies on the notions of *affine subspaces* and *affine spans*.

Definition 3.6. *An* affine subspace $A \subseteq \mathbb{R}^n$ *of dimension* d *is a translate by some fixed* $y \in \mathbb{R}^n$ *of a* d*-dimensional linear subspace of* $\mathbb{R}^n$*. Given a subset* V *of* $\mathbb{R}^n$*, the* affine span *of* V *is the intersection of all affine subspaces of* $\mathbb{R}^n$ *containing* V*.*

Definition 3.7. *Let* P *be a polytope of dimension* d*. The* relative interior *of* P *is the interior of* P *with respect to the embedding of* P *into its affine span.*

When P is convex, the following definition is equivalent:

Definition 3.8. *Let* P *be a convex polytope. Then*

$$relint(P) = \{x \in P \mid \forall y \in P\ \exists \lambda > 1 \text{ such that } \lambda x + (1-\lambda)y \in P\}.$$

The interior of a polytope is the set of all points of the polytope except those on the boundary. The relative interior is the interior relative to the subset of the affine space it spans. For example, the interior of a point is empty, while the relative interior of the point is the point itself.

3.2 Minkowski sum

Definition 3.9. *The* Minkowski sum *of two polytopes* P *and* Q *in* $\mathbb{R}^n$ *is*

$$P + Q = \{p + q \mid p \in P,\ q \in Q\}.$$

It is easy to see that this will always be another polytope, and convex in the case that both P and Q are. This construction is crucial for the work in the first part of this thesis.

Lemma 3.10. *Let F be a face of $P+Q$. Then there are unique faces P_X, Q_Y of P and Q respectively such that $F = P_X + Q_Y$.*

Note that the converse does not always apply.

We also have a notion of Minkowski difference, which can explain what is means to scale a polytope by a negative amount. First, when $t \in \mathbb{Z}^+$, if we scale every coordinate of the points of P by t, we get the *dilated* polytope tP. When we have a Minkowski sum of a polytope with itself, we get the same polytope as in the dilation. In the following definition only, we will allow t to be negative.

Definition 3.11. *The* Minkowski difference *of two polytopes P and Q in $\mathbb{R}^n$ is*

$$P - Q = \{t \in \mathbb{R}^n \mid t + Q \subseteq P\}.$$

Given this, we could now define $P + (-Q)$ to be $P - Q$. Note that this causes us to lose associativity in $+$, and so in any result which makes use of this, we must do all additions first. Also, $(P - Q) + Q$ may not be equal to P, but instead only a subset of P. This subset would be the parts of P that can be covered with a translate of Q lying entirely inside P.

3.3 Subdivisions

Definition 3.12. *Let P be a polytope. A* subdivision *of P is a collection (complex) of polytopes $\mathcal{C}$ such that*

i.) *the empty polytope is in $\mathcal{C}$,*

ii.) *if Q is in $\mathcal{C}$, so are all faces of Q,*

iii.) *the intersection of any two polytopes $Q_1, Q_2 \in \mathcal{C}$ is a (possibly empty) face of both Q_1 and Q_2, and*

iv.) $\bigcup_{C \in \mathcal{C}} C = P$.

We require that any vertex of $Q \in \mathcal{C}$ is also a vertex of P. The maximal elements of $\mathcal{C}$ are called the *cells* of the subdivision. The set of faces of $\mathcal{C}$ consists of the cells and all their faces. A subdivision is called a *triangulation* when every cell is a simplex.

A *lower face* of P is a face "visible from below", that is, a maximiser set which is negative on the last coordinate. The standard projection map $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is such that, for $p \in P$, $\pi(p_1, \ldots, p_{d+1}) = (p_1, \ldots, p_d)$.

A subdivision of $P \in \mathbb{R}^d$ is called *regular* if it is constructed from a polytope $Q \in \mathbb{R}^{d+1}$ in a particular way:

i.) We have that $\pi(Q) = P$.
ii.) The cells of the subdivision are the *lower faces* of Q.

In other words, a subdivision of P in $\mathbb{R}^d$ is regular if there are heights α_i for every lattice point $p_i \in P$ such that the cells of the subdivision are given by projections of the lower faces of the polytope $Q := \mathrm{Conv}\{(p_i, \alpha_i) \in \mathbb{R}^{d+1} \mid p_i \in P\}$. We call Q the *lifted polytope* of P, and the vector $(\alpha_1, \ldots, \alpha_n)$ P's *height function*.

We will give an example of this type of subdivision in Example 3.15, but as it is also an example of other types of subdivisions we will define these first.

Taking the Minkowski sum of a pair of lifted polytopes induces *mixed* subdivisions of the projections. Take a family of polytopes $P_1, \ldots, P_n$. The *weighted* Minkowski sum of this family is $\lambda_1 P_1 + \cdots + \lambda_n P_n$, where $\sum\limits_i \lambda_i = 1$ and $0 < \lambda_i < 1$, for all i. A *Minkowski cell* is a cell which is equal to $\lambda_1 F_1 + \cdots + \lambda_n F_n$ for faces $F_i \subset P_i$. Given a linear functional ϕ, let m_ϕ be the function sending any polytope to its ϕ-maximising face.

Definition 3.13. *A subdivision of the weighted Minkowski sum $Q = \lambda_1 P_1 + \cdots + \lambda_n P_n$ is* mixed *if each face F of Q has a given Minkowski cell structure $F = \lambda_1 H_1(F) + \cdots + \lambda_n H_n(F)$ for faces $H_i \subset P_i$ such that, for each face $m_\phi(F)$ of F, we have $H_i(m_\phi(F)) = m_\phi(H_i(F))$.*

We will define one final type of subdivision, that of a fine mixed subdivision. Multiple definitions of this appear in the literature, not all of them consistent. Throughout this thesis, we will be using only the following one:

Definition 3.14. *Let $F = \lambda_1 H_1(F) + \cdots + \lambda_n H_n(F)$ and $G = \lambda_1 H_1(G) + \cdots + \lambda_n H_n(G)$ be two Minkowski cells of a weighted Minkowski sum. We say that $F \leq G$ if $H_i(F) \subseteq H_i(G)$ for all i. If Q and R are are two mixed subdivisions of the weighted Minkowski sum, we say that $Q \leq R$ if every cell of Q is less than or equal to some cell of R. The minimal elements of the poset of mied subdivisions are called* fine mixed subdivisions. *The cells of the subdivision are called* fine mixed cells.

The subdivisions we use (see Example 3.15 and Section 7.1) are mixed, but not typically fine. They are, however, finest of those which do not subdivide $P(M)$.

Example 3.15. Let Δ be the standard simplex in $\mathbb{R}^{|E|}$ of dimension equal to $|E| - 1$, that is

$$\Delta = \mathrm{Conv}\{\mathbf{e}_i \mid i \in E\},$$

and ∇ be its reflection through the origin, $\nabla = \{-x \mid x \in \Delta\}$. The faces of Δ are the polyhedra

$$\Delta_S = \mathrm{Conv}\{\mathbf{e}_i \mid i \in S\}$$

for all nonempty subsets S of E; similarly, the faces of ∇ are the polyhedra ∇_S given as the reflections of the Δ_S. There are thus $|\mathcal{P}(E)| - 1 = 2^{|E|} - 1$ faces in Δ and also in ∇, including the polytopes themselves. For a slightly more combinatorial argument, note that there are $|E|$ vertices, and so $\binom{|E|}{2}$ edges, and $\binom{|E|}{i+1}$ i-faces for any i. The total number of faces is thus $\sum_{i=0}^{|E|-1} \binom{|E|}{i+1} = 2^{|E|} - 1$.

We will lift $\Delta + \nabla$, giving $(\mathrm{Conv}\{(u\mathbf{e}_i, \alpha_i)\} + \mathrm{Conv}\{(-t\mathbf{e}_j, \beta_i)\}$, where $\alpha_1 < \cdots < \alpha_n$, $\beta_1 < \cdots < \beta_n$ are positive reals. The associated height function on the lattice points of $\Delta + P(M) + \nabla$ is

$$h(x) := \min\{\alpha_i + \beta_j \mid \mathbf{e}_i - \mathbf{e}_j \in \mathcal{B}_M\}.$$

By Lemma 3.10, the faces of the subdivision are of the form $\Delta_S + \nabla_T$, where these are faces of the respective summands, which obey certain conditions. This is a fine mixed subdivision, so by [37, Proposition 2.3] we have that the affine span of the faces must be independent and thus transverse, and so must have opposite dimension and meet in no more than one point. This ensures that the dimension of the Minkowski sum is the sum of the dimensions of summands. It requires that, for highest-order faces, $S \cup T = E$, and $|S| + |T| = |E| + 1$. For other faces, we need $|S| + |T| = d + 2$, where d is the dimension of the face.

Given the above height function, $\mathbf{e}_i - \mathbf{e}_i$ isn't the label on any vertex of the mixed subdivision for any $i \neq 1$, as $\mathbf{e}_1 - \mathbf{e}_1$ is always lower. That is, in $(\mathbf{e}_i, \alpha_i) + (-\mathbf{e}_i, \beta_i)$, the last coordinate is greater than the last coordinate of $(\mathbf{e}_1, \alpha_1) + (-\mathbf{e}_1, \beta_1)$. Therefore neither can we have a face $\Delta_S + \nabla_T$ where the single common element of S and T is some $i \neq 1$. If it did, by the definition of mixed subdivision we could choose a functional ϕ so that $\mathbf{e}_i - \mathbf{e}_i$ appears as a label of a vertex in the face, a contradiction. The functional ϕ is created by making the coefficient of x_i greatest amongst all the coefficients for elements of S, but least among all the coefficients for elements of T.

We will now count the number of faces. All elements other than 1 can be in S or T, or neither, but not both. The element 1 is additionally allowed to be in both. This gives an initial count of $4 \cdot 3^{n-1}$. We cannot have either S or T empty, which occurs $2 \cdot 2^n$ times. Finally, we've double-counted the case where both S and T are empty. So, the number of faces of the subdivision is thus $\frac{4}{3}3^n - 2 \cdot 2^n + 1$.

3.4 Matroid polytopes

Now we can return to matroids and their extensions, polymatroids. First we will present matroids in a different manner to the previous chapter. Let the set of bases of a matroid M be $\mathcal{B}$, and let E be its finite ground set. We work in the vector space $\mathbb{R}^E = \{(r_i \mid i \in E)\}$, where $r_i \in \mathbb{R}$. For a set $U \subseteq E$, $\mathbf{e}_U \in \mathbb{R}^E$ is the indicator vector of U, that is, $\mathbf{e}_U$ is the sum of the unit vectors $\mathbf{e}_i$ for all $i \in U$. When we write $(\mathbf{e}_U)_i$, we will be referring to the i-th coordinate of $\mathbf{e}_U$. We will abbreviate $\mathbf{e}_{\{i\}}$ to $\mathbf{e}_i$.

Definition 3.16. *The* base polytope *of M is*

$$P(M) = \mathrm{Conv}\{\mathbf{e}_B \mid B \in \mathcal{B}\}.$$

The base polytope is contained inside the *independent set polytope* of M, which is the convex hull of indicator vectors of the independent sets of M. These definitions in terms of convex hulls are the v-representations, as described in Section 3.1. Both of these polytopes can also be given by the h-representation. For the independent set polytope, we have

$$I(M) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for } i \in [n],\ x \cdot \mathbf{e}_A \leq r(A)\ \forall A \subseteq E\}.$$

For the base polytope, the h-representation is

$$P(M) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for } i \in [n],\ x \cdot \mathbf{e}_A = r(A)\ \forall A \subseteq E\}.$$

The edges of the polytope reflect the basis exchange property: two vertices, $\mathbf{e}_{B_1}$ and $\mathbf{e}_{B_2}$, are joined by an edge in $P(M)$ if and only if $\mathbf{e}_{B_1} - \mathbf{e}_{B_2} = \mathbf{e}_i - \mathbf{e}_j$ for some i, j. This is equivalent to saying that a zero-one polytope is a matroid base polytope if and only every edge is parallel to an edge of the simplex Δ, as first shown by Gelfand, Goresky, MacPherson and Serganova [24].

Example 3.17. Let $\Delta(r, n)$ be the $(n-1)$-dimensional hypersimplex, that is, the convex hull of the set of $0,1$-vectors with exactly r ones. Then $\Delta(r, n)$ is the polytope of the uniform matroid $U_{r,n}$.

Note that the polytope of any matroid on n elements is a subpolytope of the hypersimplex.

A third definition of the matroid base polytope can be given in a way which links the polytope back to the Tutte polynomial.

Definition 3.18. *Let $M = (E, r)$ be a matroid. The* beta invariant *of M is*

$$\beta(M) = (-1)^{r(M)} \sum_{X \subseteq E} (-1)^{|X|} r(X).$$

If we write the Tutte polynomial of M as

$$T_M(x, y) = \sum_{i,j} b_{ij} x^i y^j, \tag{3.18.1}$$

then $\beta(M) = b_{10} = b_{01}$ for $|E| \geq 2$. The following facts are well known:

Lemma 3.19 ([11])**.** *Let $M = (E, r)$ be a matroid and $T_M(x, y) = \sum_{i,j} b_{ij} x^i y^j$ its Tutte polynomial. Then M is connected if and only if $\beta(M) > 0$.*

Lemma 3.20 ([7, Theorem 1.6(vi)])**.** *Let $M = (E, r)$ be a matroid and $T_M(x, y) = \sum_{i,j} b_{ij} x^i y^j$ its Tutte polynomial. Then M is series-parallel if and only if $\beta(M) = 1$.*

We can give an alternative definition of the matroid polytope $P(M)$ using the *signed* beta invariant.

Definition 3.21. *The* signed beta invariant *of M is $\widetilde{\beta}(M) = (-1)^{r(M)+1}\beta(M)$.*

Lemma 3.22 ([1, Theorem 4.5])**.** *Let Δ_I be the convex hull of points $\mathbf{e}_i$ for $i \in I$. Then*

$$P(M) = \sum_{A \subseteq E,\ \widetilde{\beta} > 0} \widetilde{\beta}(M/A)\Delta_{E-A} + \sum_{A \subseteq E,\ \widetilde{\beta} < 0} \widetilde{\beta}(M/A)\Delta_{E-A}.$$

Note that, in the above lemma, we separated the terms based on sign due to the definition and non-associativity of Minkowski difference (recall Section 3.2).

3.5 Polymatroids

A natural extension of matroids are polymatroids, which are a class of objects formed by relaxing one matroid rank axiom.

Definition 3.23. *A* polymatroid (E, r) *consists of a finite ground set* E *and a rank function* $r : \mathcal{P}(E) \to \mathbb{Z}^+ \cup \{0\}$ *such that, for* $X, Y \in \mathcal{P}(E)$*, the following conditions hold:*

P1. $r(\emptyset) = 0$*,*

P2. *If* $Y \subseteq X$*, then* $r(Y) \leq r(X)$*, and*

P3. $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$*.*

If the rank of every singleton is bounded by some integer k, we can call the polymatroid a k-polymatroid.

Example 3.24. We can construct a polymatroid from any graph in a different way to that of the graphic matroid. The ground set E will still be the edge set of the graph. Let $X \subseteq E$. Then we define $r(X)$ to be the number of vertices incident with X. We will refer to this as a *graphic polymatroid*. When the graph is simple, this gives a 2-polymatroid.

Example 3.25. Given two matroids with rank functions r_1 and r_2, $r_1 + r_2$ is always the rank function of a 2-polymatroid.

As with matroids, we can represent polymatroids as polytopes. Let $r : \mathcal{P}(E) \to \mathbb{N}$ be a rank function, and $M = (E, r)$ the associated polymatroid. We again have the independent set polytope, which is also referred to as the *extended polymatroid* of M:

$$EP(M) = \{\mathbf{x} \in \mathbb{R}^E \mid x \cdot \mathbf{e}_U \leq r(U) \text{ for all } U \subseteq E\}.$$

Note that, unlike in the matroid case, this definition allows for points with negative coordinates. If we also require that $x \geq \mathbf{0}$, we have the polymatroid *base polytope* of M,

$$P(M) = EP(M) \cap \{x \in \mathbb{R}^E, x \geq \mathbf{0} \mid x \cdot \mathbf{e}_E = r(E)\}.$$

As before, the base polytope is a face of the extended polymatroid. Note the abuse in notation: we use $P(M)$ to refer to the polytope of both a matroid and a polymatroid. When we mean M to be strictly a matroid, we will make this clear. Note that every matroid is a polymatroid, but not the converse. When a polymatroid happens to be a matroid, then the extended polymatroid is equal to the independent set polytope, and the base polytope of the polymatroid is the same as the base polytope defined for matroids.

Both the base polytope and extended polymatroid contain all the information in the rank function. In [36], polymatroid base polytopes are dubbed "generalised permutohedra." A base polytope is always a lattice polytope, as is an extended polymatroid. The dimension of the polytope is equal to $|E|$ minus the number of connected components of the matroid.

Edmonds [15], amongst others (often those with backgrounds in optimisation rather than combinatorics), use $EP(M)$ as the definition of a polymatroid. We prefer to adopt the stance that, as with the various sets of axioms which can be used to define a matroid, $EP(M)$ and to Definition 3.23 are simply alternative ways to define the same object. One could give further combinatorial definitions for polymatroids, such as defining bases (Definition 3.27) as particular multisets rather than as vectors satisfying a certain inner product.

What the $P(M)$ definition means in effect is that we form the vertices of a polytope by using the Greedy Algorithm: Choose some ordering of the ground set E, and let S_i be the set of the i least elements according to the chosen ordering. Form a vector $x = (x_1, \ldots, x_{|E|})$ by:

$$x_1 = r(S_1), \text{ and}$$

$$x_i = r(S_i) - r(S_{i-1}), \text{ for all } i \in \{2, \ldots, |E|\}.$$

The vertices of a polymatroid are these vectors x formed for all possible orderings of the ground set.

We have a notion of polymatroid basis exchange:

Lemma 3.26 ([27, Theorem 4.1])**.** *Let P be a polymatroid polytope. Take any two lattice* *points $p, q \in P$. If $p_i > q_i$, then there exists a j such that $p_j < q_j$ and $p - \mathbf{e}_i + \mathbf{e}_j \in P$.*

3.6 Activity

Polymatroids have a notion of internal and external activity associated with them, akin to the commonly used notion for graphs. An important distinction between matroids and polymatroids is that in polymatroids, bases need not be equicardinal, a well-known and easily proven fact of matroids. Note that when the basis of a polymatroid is given by a vector of the polytope, we use cardinality to refer to the number of non-zero coordinates in the vector. We thus first have to define exactly what we mean by a basis of a polymatroid. The following three definitions for the polymatroid generalisation are from [29], and in fact apply to all polytopes, given a submodular function r. Let E be a finite set, which will serve as the ground set of our (poly)matroid. Take a polymatroid $M = (E, r)$.

Definition 3.27. *A vector $x \in \mathbb{Z}^E$ is called a* basis *if $x \cdot \mathbf{e}_E = r(E)$ and $x \cdot \mathbf{e}_S \leq r(S)$ for all subsets $S \subseteq E$.*

Let $\mathcal{B}_M$ be the set of all bases of M.

Definition 3.28. *A* transfer *is possible from $u_1 \in E$ to $u_2 \in E$ in the basis $x \in \mathcal{B}_M \cap \mathbb{Z}^E$ if by decreasing the u_1-component of x by 1 and increasing its u_2-component by 1 we get another basis.*

Definition 3.29 (Polymatroid activity)**.** *Order the elements of E arbitrarily.*

i. We say that $u \in E$ is internally active *with respect to the basis x if no transfer is possible in x* from *u to a smaller element of E.*

ii. We say that $u \in E$ is externally active *with respect to the basis x if no transfer is possible in x* to *u from a smaller element of E.*

For $x \in \mathcal{B}_M \cap \mathbb{Z}^E$, let the set of internally active elements with respect to x be denoted with $\mathrm{Int}(x)$, and let $\iota(x) = |\mathrm{Int}(x)|$; likewise, let the set of externally active elements be denoted with $\mathrm{Ext}(x)$ and $\varepsilon(x) = |\mathrm{Ext}(x)|$. Let $\bar{\iota}(x), \bar{\varepsilon}(x)$ denote the respective numbers of inactive elements. When M is a matroid, these numbers provide an alternative formulation of the Tutte polynomial,

$$T_M(x, y) = \sum_{B \in \mathcal{B}} x^{\iota(B)} y^{\varepsilon(B)}.$$

In this formula, we are actually using the following simplified versions of the definition of activity, which are more common for matroids:

Definition 3.30 (Matroid activity)**.** *Take a matroid $M = (E, r)$, and give E some ordering. Let B be a basis of M.*

i. We say that $e \in E - B$ is externally active *with respect to B if e is the smallest element in the unique circuit contained in $B \cup e$, with respect to the ordering on E.*

ii. We say that $e \in B$ is internally active *with respect to B if e is the smallest element in the unique cocircuit in $(E \setminus B) \cup e$.*

These definitions are the analogies of those originally formulated using spanning trees of graphs.

In graphs and matroids, if u is internally active in B then $u \in B$. Applying the polymatroid definition of internally activity to a matroid, however, says that every element not in B is internally active, and only elements in B can be externally active. This is the only difference between the two notions of external activity. Compared to the matroid activity definitions, this definition gives $|E| - r(M)$ extra internally active elements, and $r(M)$ extra externally active elements. This means we have a second formula for the Tutte polynomial using activity, when we use the polymatroid definitions, as follows:

Definition 3.31. *Let $M = (E, r)$ be a matroid, and give E some ordering. Using the polymatroid definitions of activity, Definition 3.29, we have that*

$$T_M(x, y) = \frac{1}{x^{|E|-r(M)} y^{r(M)}} \cdot \sum_{B \in \mathcal{B}_M \cap \mathbb{Z}^E} x^{\iota(B)} y^{\varepsilon(B)}.$$

This analogy between the internal and external polynomials of a polymatroid and the same polynomials under the graph definitions suggested that a two-variable polynomial similar to the Tutte polynomial could be found for polymatroids. In the case of a polymatroid, the definitions are as follows. Note that these polynomials do not depend on the order on E that was used to define them, by [29, Theorem 5.4].

Definition 3.32 ([29])**.** *Let M be a polymatroid. Define the* internal polynomial *and* external polynomial *of M by*

$$I_M(\xi) = \sum_{x \in \mathcal{B}_M \cap \mathbb{Z}^E} \xi^{\bar{\iota}(x)} \quad \text{and} \quad X_M(\eta) = \sum_{x \in \mathcal{B}_M \cap \mathbb{Z}^E} \eta^{\bar{\varepsilon}(x)}.$$

Our work generalises formula (3.6), creating a two-variable polynomial which is equivalent to the Tutte polynomial for matroids (Definition 4.0.1) and specialises to the two activity polynomials above for polymatroids (Theorem 4.3). That is, the invariant we construct is the bivariate analogue of Kálmán's activity polynomials, which is something his paper [29] sought.

3.7 Classical and mixed Ehrhart theory

Before we describe our construction, we will give one more section of background theory, which will be implicitly used in our work.

We denote the number of lattice points in a polytope P by $\#(P \cap \mathbb{Z}^n)$. This is called the *discrete volume* of P. Recall that if we scale every coordinate of the points of P by $t \in \mathbb{Z}^+$, we get the *dilated* polytope tP. Let P have dimension n, and recall that P is a lattice polytope. The number of lattice points in the dilation has long been known to be a polynomial of degree n in t:

Definition 3.33. *Let $t \in \mathbb{Z}^+ \cup \{0\}$. The* Ehrhart polynomial *of P is*

$$L(P,t) := \#(tP \cap \mathbb{Z}^n) = a_0 + a_1 t \cdots + a_n t^n.$$

We have that $a_0 = 1$, and a_n is the volume of P. There is also a geometric interpretation of a_{n-1}. Given a facet F of P, the *relative volume* of F is

$$\mathrm{relvol}(F) := \lim_{t\to\infty} \frac{1}{t^{n-1}} \#(tF \cap \mathbb{Z}^n).$$

Then $a_{n-1} = \frac{1}{2} \sum\limits_{F} \mathrm{relvol}(F)$.

It is not known if the remaining coefficients have such interpretations.

Example. Let P be the three-dimensional cube. Then $L(P,t) = 1 + 3t + 3t^2 + t^3$, and is in fact the cube of the Ehrhart polynomial of the line segment of length one, $t+1$.

Note that $tP \cap \mathbb{Z}^n$ is not always equal to $t(P \cap \mathbb{Z}^n)$. When this does occur for all t, P is said to be *normal.* Any lattice polytope, such as the matroid base polytope, is normal for $t \geq n-1$. Moreover, all matroid polytopes are normal [46, Theorem 1].

We can also count the number of lattice points in the interior of a convex polytope, using the Ehrhart–Macdonald reciprocity theorem:

$$L(\mathrm{Int}(P), t) = (-1)^n L(P, -t).$$

The *Ehrhart series* is found by forming a generating function with $L(P,t)$:

$$\mathrm{Ehr}_P(z) := \sum_{t \geq 0} L(P,t) z^t.$$

This can be written as a rational function, using the well-known h^*-vectors:

$$\mathrm{Ehr}_P(z) = \frac{h^*(t)}{(1-t)^n},$$

where $h^*(t)$ is a polynomial in t with degree $\leq n$. This is called the h^*-polynomial, and the vector of coefficients in the polynomial the h^*-vector. Like in the Ehrhart polynomial, all the coefficients are non-negative ([42]).

Now take a set of n-dimensional polytopes $P_1, \dots P_k$. The Minkowski sum of dilated polytopes $t_1P_1 + \cdots + t_kP_k$ is a multivariate polynomial in $t_1, \dots, t_k$. This is known as the multivariate Ehrhart polynomial, and we will denote it by $\mathrm{Ehr}_{\mathbf{P}}(t_1, \dots, t_k)$. The degree of t_i is dim P_i, and the total degree is $\dim(P_1 + \cdots + P_k)$. Bihan [3] developed the idea of a *mixed* Ehrhart theory, by defining a *discrete mixed volume* of a collection of polytopes $P_1, \dots, P_k$:

$$DMV(P_1, \dots, P_k) = \sum_{J \subseteq [k]} (-1)^{k-|J|} \#(P_J \cap \mathbb{Z}^d)$$

where

$$P_J := \sum_{j \in J} P_j \text{ for } \emptyset \neq J \subseteq [k] \text{ and } P_\emptyset = \{0\}.$$

The *mixed Ehrhart polynomial* is then

$$\begin{aligned} ME_{P_1,\dots,P_k}(n) &:= DMV(nP_1, \dots, nP_k) \\ &= \sum_{J \subseteq [k]} (-1)^{k-|J|} \mathrm{Ehr}_{P_J}(n) \end{aligned}$$

where $Ehr_{P_J}(n)$ is the multivariate Ehrhart polynomial with all $t_i = n$. Bihan showed that the discrete mixed volume, and therefore the coefficients of the mixed Ehrhart polynomial, are non-negative for lattice polytopes.

Let $\mathbf{P} = (P_1, \dots, P_k)$. In the case that $P_1, \dots P_k$ are all the same d-dimensional polytope P, we get expressions in terms of the h^*-vector:

$$DMV(\mathbf{P}) = \sum_{j=0}^{d} \binom{d-j}{d-k} h_j^*(P), \text{ and}$$

$$ME_{\mathbf{P}}(n) = \sum_{j=0}^{d} \left(\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \binom{in+d-j}{a} \right) h_j^*(P).$$

We also have the notion of a *mixed h^*-vector* ([25]). If we write

$$ME_{\mathbf{P}}(n) = h_0^*(\mathbf{P})\binom{n+d}{d} + \cdots + h_d^*(P)\binom{n}{d}$$

where $d = \dim(P_1 + \cdots + P_K)$, then the mixed h^*-vector is $(h_0^*, \ldots, h_d^*)$. Moreover, for all $i \in [d]$,

$$h_i^*(\mathbf{P}) = \sum_{\emptyset \neq J \subseteq [k]} (-1)^{j-|J|} h_i^*(P_J) + (-1)^{k+i}\binom{d}{i}.$$

Note that $h_o^*(\mathbf{P}) = 0$ and if $k = 2$, then $h_1^*(P_1, P_2) = DMV(P_1, P_2)$. Unlike in the case of the non-mixed h^*-vectors, the mixed ones are not always non-negative. Haase et al [25, Corollary 4.6] show that if each polytope P_i is dilated by a large enough t, non-negativity can be recovered.

Finally, we could ask if there are interpretations of the coefficients of the mixed Ehrhart polynomial akin to those of the standard polynomial. This turns out to be a similarly hard problem, with only certain coefficients having geometric meanings. Let me_i be the coefficient of n^i in $ME_{\mathbf{P}}(n)$, and let $P[\alpha]$ mean we take α copies of P. Given a linear functional, a, let P_i^a be the face of P_i where a is maximised. Then we have

$$me_d = \sum_{\substack{\alpha \in \mathbb{Z}^k_{\geq 1} \\ |\alpha| = d}} \binom{d}{\alpha_1, \ldots, \alpha_k} MV_d(P_1[\alpha_1], \ldots, P_k[\alpha_k]), \text{ and}$$

$$me_{d-1} = \frac{1}{2}\sum_a \sum_{\substack{\alpha \in \mathbb{Z}^k_{\geq 1} \\ |\alpha| = d-1}} \binom{d-1}{\alpha_1, \ldots, \alpha_k} MV_a(P_1^a[\alpha_1], \ldots, P_k^a[\alpha_k])$$

where a ranges over primitive facet normals of $P_1 + \cdots + P_k$. Here, *primitive* means that there is a unique way to scale the normal vector to the relevant facet in such a way that the entries are relatively prime integers.

Part II

An Ehrhart-theory generalisation of the Tutte poynomial

Chapter 4

Construction

As the base polytope of a matroid is the convex hull of a set of zero-one vectors, the number of lattice points in it is equal to the number of its vertices, that is, the number of bases in its underlying matroid. Recall that one important property of the Tutte polynomial is that $T_M(1,1)$ counts the number of bases in the matroid. With these facts in mind, we will form a polynomial which counts the lattice points of a particular Minkowski sum of polyhedra, with the aim of it having such an enumerative property. Note that, by [38, Corollary 46.2c], the set of lattice points in a sum of polymatroid polytopes is the set of sums of lattice points from the summands.

Let Δ be the standard simplex in $\mathbb{R}^E$ of dimension equal to $|E|-1$, that is

$$\Delta = \operatorname{conv}\{\mathbf{e}_i \mid i \in E\},$$

and ∇ be its reflection through the origin, $\nabla = \{-x \mid x \in \Delta\}$. The faces of Δ are the polyhedra

$$\Delta_S = \operatorname{conv}\{\mathbf{e}_i \mid i \in S\}$$

for all nonempty subsets S of E; similarly, the faces of ∇ are the polyhedra ∇_S given as the reflections of the Δ_S.

We consider the polytope given by the Minkowski sum $P(M) + u\Delta + t\nabla$ where $M = (E, r)$ is any polymatroid and $u, t \in \mathbb{N} = \mathbb{Z}^+ \cup \{0\}$. By Theorem 7 of [30], the number of lattice points inside the polytope is a polynomial in t and u, of degree $\dim(P(M) + u\Delta + t\nabla) = |E| - 1$. This polynomial we write in the form

$$Q_M(t, u) := \#(P(M) + u\Delta + t\nabla) = \sum_{i,j} c_{ij} \binom{u}{j} \binom{t}{i}. \tag{4.0.1}$$

Changing the basis of the vector space of rational polynomials gives the polynomial

$$Q'_M(x, y) = \sum_{ij} c_{ij} (x-1)^i (y-1)^j \tag{4.0.2}$$

where the c_{ij} are equal to those in the previous equation. Remark 4.4 explains the reasoning behind this choice of basis change.

One motivation for this particular Minkowski sum is that it provides a polyhedral translation of Kálmán's construction of activities in a polymatroid.

Lemma 4.1. *Let P be a polymatroid polytope, and choose $t \in \mathbb{Z}^+ \cup \{0\}$. Give the natural ordering to the elements of the polymatroid. At every point $f \in P$, attach the scaled simplex*

$$f + t\,\mathrm{Conv}(\{-\mathbf{e}_i \mid i \textit{ is internally active in } f \textit{ or } i \notin f\}) =: t\Delta_f.$$

This operation partitions the set of lattice points of $P + t\nabla$ into a collection of translates of faces of $t\nabla$, with the simplex attached at f having codimension $\bar{\iota}(f)$ within P.

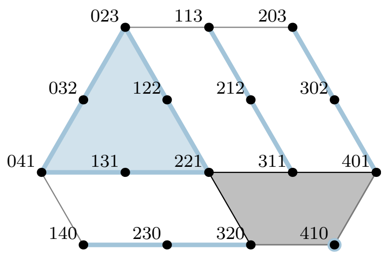


Figure 4.1: A polytope (grey) with Δ_f attached at lattice points f (blue)

Figure 4.1 shows a case of this operation, to illustrate what is meant by "attaching" a simplex. Our polymatroid $P(M)$ is coloured grey, and the polytope drawn is $P(M) + 2\Delta + \nabla$. Coordinate labels are written without parentheses and commas. The blue areas are faces of the scaled simplices $2\Delta_f$, and can be seen to be a partition of the lattice points.

Proof. We will first show that this operation covers all of $P + t\nabla$. Let $g \in P + t\nabla$ be a lattice point. We will find f such that $g \in t\Delta_f$.

Let $g_t = g$. For $i \in \{0, \ldots, t-1\}$, define

$$g_i = \begin{cases} g_{i+1} + \mathbf{e}_1 & \text{if } g_{i+1} + \mathbf{e}_1 \in P + (t-i)\nabla \\ g_{i+1} + \mathbf{e}_2 & \text{if } g_{i+1} + \mathbf{e}_1 \notin P + (t-i)\nabla, g_{i+1} + \mathbf{e}_2 \in P + (t-i)\nabla \\ \quad\vdots & \\ g_{i+1} + \mathbf{e}_n & \text{if } g_{i+1} + \mathbf{e}_h \notin P + (t-i)\nabla,\ \forall j \in [n],\ g_{i+1} + \mathbf{e}_n \in P + (t-i)\nabla \end{cases}$$

In other words, at each iteration i, we are adding an element $\mathbf{e}_j$ which is internally active with respect to g_{i+1}. We cannot replace $\mathbf{e}_j$ with $\mathbf{e}_i$ where $i < j$ and remain inside $P + t\nabla$. Let $\mathbf{e}_{j_i}$ be the element added in iteration t. We get that

$$g = g_t = g_0 - \mathbf{e}_{j_1} - \ldots - \mathbf{e}_{j_t} \in P + t\nabla.$$

Note that if we added $\mathbf{e}_i$ at some stage g_s of the iteration, and $\mathbf{e}_j$ at stage g_{s-1}, then $j \geq i$. Thus if we take a tuple $\mathbf{e}_k$ such that $(k_1, \ldots, k_t) < (j_1, \ldots, j_t)$ with respect to the lexicographic ordering, then $g_0 - \sum_t \mathbf{e}_{j_t} + \sum_t \mathbf{e}_{k_t} \notin P$, so each $\mathbf{e}_j$ is internally active. Thus $g_0 = f$ and the $\mathbf{e}_j$ found define a simplex Δ_f such that $g \in t\Delta_f$.

Now we will show that this operation gives disjoint sets. We have that $\{t\Delta_f\}$ covers $P + t\nabla$, and that $\{(t-1)\Delta_f\}$ partitions $P + (t-1)\nabla$. Thus in order to show that $\{t\Delta_f\}$ is in fact a partition of the lattice points of P, it suffices to prove that if $g_t \in t\Delta_f$, then $g_{t-1} \in (t-1)\Delta_f$. Say that $f = g_t + \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_t}$. This means that each element $\mathbf{e}_{i_k}$ is internally active at f for all $k \in \{0, \ldots, t\}$. Now, for a contradiction, let $g_0 = f' \neq f$, so that $g_{t-1} = g_t + \mathbf{e}_i \in (t-1)\Delta_{f'}$. Apply the same iterative process as before to get

$$\begin{aligned} f' = g_0 &= g_{t-1} + \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_{t-1}} \\ &= g_t + \mathbf{e}_i + \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_{t-1}} \\ &= f - \mathbf{e}_{i_t} + \mathbf{e}_i. \end{aligned}$$

Thus $\mathbf{e}_{i_t}$ was internally inactive at f, contradicting our construction of $t\Delta_f$. □

The exterior analogue can be proven in an identical manner:

Lemma 4.2. *Let P be a lattice polytope, and choose $t \in \mathbb{Z}^+ \cup \{0\}$. At every lattice point $p \in P$, attach the scaled simplex*

$$p + t\,\mathrm{Conv}(\{-\mathbf{e}_i \mid i \textit{ is externally active in } p \textit{ or } i \in f\}).$$

This operation partitions the set of lattice points of $P + t\Delta$ into a collection of translates of faces of $t\Delta$, with the simplex attached at p having dimension $\overline{\varepsilon}(p)$ within P.

The following is a direct consequence of applying Lemma 5.9 to Definitions 3.31 and 3.32. We state it here for the motivation it gives.

Theorem 4.3. *Let M be a polymatroid with rank function r. Then $I_M(\xi) = \xi \cdot Q'_M(\xi, 1)$ and $X_M(\eta) = \eta \cdot Q'_M(1, \eta)$.*

Remark 4.4. It is this result which first motivated the particular change of basis made from Q_M to Q'_M, since an i-dimensional face of $t\Delta$ has $\binom{t+i}{i} = \sum_{k=0}^{i} \binom{i}{k}\binom{t}{i}$ lattice points. The coefficients $\binom{i}{k}$ are exactly absorbed by the binomial theorem when passing from powers of $x - 1$ to powers of x, so the overall outcome is that the dimensions of the simplices in a decomposition are, notionally, just the coefficients of Q'_M.

The bivariate enumerator of internal and external activities for polymatroids is not order-independent, and so we do not have that $Q'_M = \sum_{x \in \mathcal{B}_M \cap \mathbb{Z}^{|E|}} \xi^{\iota(x)} \eta^{\varepsilon(x)}$. For example, take the polymatroid with bases $\{(2,1,1), (1,2,1), (1,1,2), (2,0,2)\}$. Using the natural ordering on [3], the only externally active element in $(2, 1, 1)$ is 1. If we instead use the ordering $2 < 3 < 1$, the same bases has two externally active elements: 1 and 3.

In the following chapters, we will provide further information about this polynomial and its invariants, and provide links to the Tutte polynomial.

Chapter 5

Relationship to the Tutte polynomial

When we restrict M to be a matroid, $Q'_M(x,y)$ is an evaluation of the Tutte polynomial, and in fact one that contains precisely the same information:

Theorem 5.1. *Let $M = (E, r)$ be a matroid. Then we have that*

$$Q'_M(x,y) = \frac{x^{|E|-r(M)}y^{r(M)}}{x+y-1} \cdot T_M\left(\frac{x+y-1}{y}, \frac{x+y-1}{x}\right) \tag{5.1.1}$$

as an identification of rational functions.

The proof of this is given in the second section of this chapter. We can also invert Equation 5.1.1, giving the Tutte polynomial as an evaluation of Q'_M:

Theorem 5.2. *Let $M = (E, r)$ be a matroid. Then*

$$T_M(x,y) = \frac{(xy-x-y)^{|E|-1}}{(-y)^{r(M)-1}(-x)^{|E|-r(M)-1}} \cdot Q'_M\left(\frac{-x}{xy-x-y}, \frac{-y}{xy-x-y}\right)$$

as an identification of rational functions.

As such, the Tutte polynomial can be directly evaluated by lattice point counting methods, which is a novel approach:

Theorem 5.3. *Let $M = (E, r)$ be a matroid and let $u, t \in \mathbb{Z}$. Then*

$$T_M(x,y) = (xy - x - y)^{|E|}(-x)^{r(M)}(-y)^{|E|-r(M)}.$$
$$\times \sum_{u,t\geq 0} Q_M(t,u)\left(\frac{y-xy}{xy-x-y}\right)^t \left(\frac{x-xy}{xy-x-y}\right)^u.$$

This can likewise be inverted, a direct proof of which will be given below. We will first work through an example showing these results in practice.

Example 5.4. Let M be the matroid on ground set $[3] = \{1, 2, 3\}$ with $\mathcal{B}_M = \{\{1\}, \{2\}\}$. When $u = 2$ and $t = 1$, the sum $P(M) + u\Delta + t\nabla$ is the polytope of Figure 5.1, with 16 lattice points.

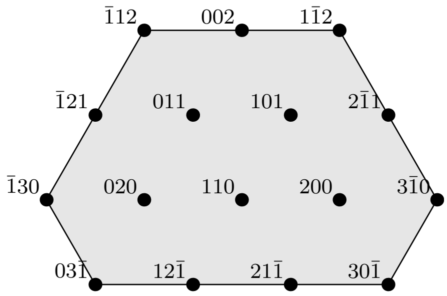


Figure 5.1: The polytope $P(M) + u\Delta + t\nabla$ of Example 5.4. The coordinates are written without parentheses or commas, and $\bar{1}$ means -1.

To compute $Q_M(x, y)$, it is enough to count the lattice points in $P(M) + u\Delta + t\nabla$ for a range of non-negative integers u and t. Since Q_M is a polynomial of degree 2, in order to form it from a set of its values, it is sufficient to take t and u as non-negative integers with sum at most 2. These are the bold entries in the table below:

$t \setminus u$	0	1	2
0	**2**	**5**	**9**
1	**5**	**10**	16
2	**9**	16	24

We can then fit a polynomial to this data, and find

$$Q_M(t,u) = \binom{t}{2} + 2tu + \binom{u}{2} + 3t + 3u + 2,$$

so

$$\begin{aligned} Q'_M(x,y) &= (x-1)^2 + 2(x-1)(y-1) + (y-1)^2 + 3(x-1) + 3(y-1) + 2 \\ &= x^2 + 2xy + y^2 - x - y. \end{aligned}$$

Finally, by Theorem 5.2,

$$\begin{aligned} T_M(x,y) &= -\frac{(xy-x-y)^2}{(-y)^0(-x)^1} \cdot \left(\frac{y^2+2xy+x^2}{(xy-x-y)^2} + \frac{y+x}{xy-x-y} \right) \\ &= xy + y^2, \end{aligned}$$

which is indeed the Tutte polynomial of M.

As a second example of how Q'_M looks compared to T_M, we can consider uniform matroids. A uniform matroid $M = U_{m,n}$ is the matroid of ground set $|E| = n$ whose set of bases consist of all subsets of E of size m. The Tutte polynomial can be written as the sum of a polynomial in y and a polynomial in x, but we conjecture the same is not true for the Q' polynomial.

Lemma 5.5 ([31, Equation 18])**.**

$$T_{U_{m,n}}(x,y) = \sum_{i=1}^{n} \binom{n-i-1}{n-m-1} x^i + \sum_{j=1}^{n-m} \binom{n-j-1}{m-1} y^j$$

Conjecture 5.6.

$$\begin{aligned} Q'_{U_{m,n}}(x,y) = \sum_{i,j} (-1)^{n-1-i-j} x^i y^j \Bigg[&\binom{n-1}{i}\binom{n-m-1-i}{j-m} \\ &+ \binom{n-1}{j}\binom{m-1-j}{i-n+m} \Bigg] \end{aligned}$$

We also have two conjectures on the form Q'_M takes for graphic polymatroids, namely those from paths and cycles:

Conjecture 5.7. *Let M be the graphic polymatroid of the path with k edges. Then*

$$Q'_M(x, y) = (x + y)^{k-1}.$$

Conjecture 5.8. *Let M be the graphic polymatroid of the k-cycle. Then*

$$Q'_M(x, y) = \frac{(x + y)^k - 1}{x + y - 1}.$$

Some calculations of Q' for both matroids and polymatroids are given in the Appendix.

5.1 Proofs of results

We first present the proof of Theorem 5.1: Q' is an evaluation of the Tutte polynomial, when we restrict to matroids.

Theorem 5.1. *Let $M = (E, r)$ be a matroid. Then we have that*

$$Q'_M(x, y) = \frac{x^{|E|-r(M)}y^{r(M)}}{x + y - 1} \cdot T_M\left(\frac{x + y - 1}{y}, \frac{x + y - 1}{x}\right) \qquad (5.0.1)$$

as an identification of rational functions.

Proof. Let $q = e_B + \mathbf{e}_{x_1} + \cdots + \mathbf{e}_{x_i} - \mathbf{e}_{y_1} - \cdots - \mathbf{e}_{y_j}$ be a point in $P(M) + i\Delta + j\nabla$, where $\mathbf{e}_B \in P(M)$. We say that the expression for q has a *cancellation* if $x_k = y_l$ for some k, l. Let k be the number of cancellations in the expression for q, allowing a summand to appear in only one cancellation. For instance, if $x_k = y_l = y_m$ is the complete set of equalities, there is one cancellation, while $x_k = x_n = y_l = y_m$ would give two. We will partition the set of lattice points of $P(M) + i\Delta + j\nabla$ according to how many coordinates are non-negative, and then construct Q by counting the lattice points in each part of the partition. Given a lattice point q, let $S = \{1 \leq i \leq n \mid q_i > 0\}$. In order to construct $\mathbf{e}_B$ from $\mathbf{e}_S$ we need to use $|S| - r(S)$ of the Δ summands: as B is spanning, we must have used a Δ-summand every time $|S|$ rises above $r(S)$. Similarly, we must use $r(M) - r(S)$ ∇-summands to account for any fall in rank. Now, we will set the remaining Δ-summands equal to those coordinates already positive, that is, set them equal to indicator vectors of elements of S. The remaining ∇-summands we will set equal to indicator vectors of elements not in S.

We will ensure, through choice of B and k, that this is the largest q (in terms of sum of coordinates) we can find given i and j. There are two ways the expression can fail to be maximal in this sense:

- when we decrease k, we could construct q using less Δ and ∇ summands, and
- if we write q using B' where we have summands $\mathbf{e}_a - \mathbf{e}_b$ such that $(B' \cup a) - b$ is a valid basis exchange, we would again be able to construct q using less summands.

We will choose B in the expression for q and the maximal k so that describing all lattice points q can be done uniquely in the way described.

Now we have that $|S|$ is the number of non-negative coordinates in at least one point of $P(M) + i\Delta + j\nabla$, and all our positive summands of such a point are assigned to such coordinates. The sum of these summands must be $r(M) + i - k - |S| = i - k - \mathrm{null}(S)$. If we ensure that $|E - S|$ is the number of negative integers in the respective points of $P(M) + i\Delta + j\nabla$, summing over these sets S will give a count of all lattice points. By the reasoning above, we must have that the $|E - S|$ non-negative integers sum to $j - k - r(M) + r(S) = j - k - \mathrm{cork}(S)$. Thus,

$$\begin{aligned}\#(P(M) + i\Delta + j\nabla) = \sum_S \sum_k [&\#(|S| \text{ non-negative integers summing to } i - k - \mathrm{null}(S)) \\ &\times \#(|E - S| \text{ non-negative integers summing to } j - k - \mathrm{cork}(S))]\end{aligned}$$

which is

$$\#(P(M) + i\Delta + j\nabla) = \sum_S \sum_k \binom{i - k + |S| - \mathrm{null}(S) - 1}{|S| - 1} \times \binom{j - k + |E - S| - \mathrm{cork}(S) - 1}{|E - S| - 1}. \quad (5.8.1)$$

Now form the generating function

$$\sum_{i,j} \#(P(M) + i\Delta + j\nabla) v^i w^j = \sum_i \sum_j \#(P(M) + i\Delta + j\nabla) v^i w^j.$$

Substituting Equation 5.8.1 into the generating function gives

$$\sum_i \sum_j \sum_S \sum_{k \geq 0} \binom{i - k + |S| - \mathrm{null}(S) - 1}{|S| - 1} v^{i-k} \times \binom{j - k + |E - S| - \mathrm{cork}(S) - 1}{|E - S| - 1} w^{j-k} (vw)^k.$$

Using the identity $\sum_i \binom{i+a}{b} x^i = \dfrac{x^{b+a}}{(1-x)^{b+1}}$ simplifies this to

$$\sum_S \sum_k \frac{v^{\mathrm{null}(S)}}{(1-v)^{|S|}} \cdot \frac{w^{\mathrm{cork}(S)}}{(1-w)^{|E-S|}} \cdot (vw)^k$$

which we can write as

$$\sum_S \frac{v^{\mathrm{null}(S)}}{(1-v)^{\mathrm{null}(S) - \mathrm{cork}(S) + r(M)}} \cdot \frac{w^{\mathrm{cork}(S)}}{(1-w)^{\mathrm{cork}(S) - \mathrm{null}(S) + |E| - r(M)}} \cdot \sum_k (vw)^k.$$

Collecting like exponents, we end up with

$$\begin{aligned}\sum_{i,j} \#(P(M) + i\Delta + j\nabla)v^i w^j &= \frac{1}{1-vw} \cdot \frac{1}{(1-v)^{r(M)}(1-w)^{|E|-r(M)}} \\ &\qquad \times \sum_S \left(\frac{v(1-w)}{1-v}\right)^{\mathrm{null}(S)} \left(\frac{w(1-v)}{1-w}\right)^{\mathrm{cork}(S)} \\ &= \frac{1}{1-vw} \cdot \frac{1}{(1-v)^{r(M)}(1-w)^{|E|-r(M)}} \\ &\qquad \times T\left(\frac{w(1-v)}{1-w} + 1, \frac{v(1-w)}{1-v} + 1\right) \\ &= \frac{1}{1-vw} \cdot \frac{1}{(1-v)^{r(M)}(1-w)^{|E|-r(M)}} \\ &\qquad \times T_M\left(\frac{1-vw}{1-w}, \frac{1-vw)}{1-v}\right) \qquad (5.8.2)\end{aligned}$$

where T_M is the Tutte polynomial of M. Now it remains to be shown that the left-hand side contains an evaluation of our polynomial Q'_M. Using our original definition of Q_M, Equation 4.0.1, we have that

$$\begin{aligned}\sum_{i,j} \#(P(M) + i\Delta + j\nabla)v^i w^j &= \sum_{i,j,k,l} c_{kl} \binom{i}{l}\binom{j}{k} v^i w^j \\ &= \sum_{k,l} c_{kl} \cdot \frac{v^l}{(1-v)^{l+1}} \cdot \frac{w^k}{(1-w)^{k+1}}.\end{aligned}$$

If we let $\dfrac{w}{1-w} = x-1$ and $\dfrac{v}{1-v} = y-1$, then

$$\begin{aligned}\sum_{i,j} \#(P(M) + i\Delta + j\nabla)v^i w^j &= \sum_{k,l} c_{kl} \cdot \frac{v^l}{(1-v)^{l+1}} \cdot \frac{w^k}{(1-w)^{k+1}} \\ &= (1-v)(1-w)\sum_{k,l} c_{kl}(x-1)^k(y-1)^l \\ &= (1-v)(1-w)Q'_M(x,y).\end{aligned}$$

So, from Equation (5.8.2), we have that

$$(1-v)(1-w)Q'_M(x,y) = \frac{1}{1-vw} \cdot \frac{1}{(1-v)^{r(M)}(1-w)^{|E|-r(M)}} \cdot T_M\left(\frac{1-vw}{1-w}, \frac{1-vw}{1-v}\right).$$

Solving for w and v in terms of x and y gives that $w = \dfrac{x-1}{x}, v = \dfrac{y-1}{y}$. Substitute these into the above equation to get

$$Q'_M(x,y) = \frac{x^{|E|-r(M)}y^{r(M)}}{x+y-1} \cdot T_M\left(\frac{x+y-1}{y}, \frac{x+y-1}{x}\right). \tag{5.8.3}$$

□

As mentioned at the start of this chapter, we can invert this formula. This is simply done by setting $x' = \dfrac{x+y-1}{x}, y' = \dfrac{x+y-1}{y}$, rearranging, and then relabelling.

Theorem 5.2. *Let $M = (E, r)$ be a matroid. Then*

$$T_M(x,y) = \frac{(xy-x-y)^{|E|-1}}{(-y)^{r(M)-1}(-x)^{|E|-r(M)-1}} \cdot Q'_M\left(\frac{-x}{xy-x-y}, \frac{-y}{xy-x-y}\right)$$

as an identification of rational functions.

If we restrict the construction to the cases where $u = 0$ or $t = 0$, we get the following formulae. These can easily be found by simplifying the above formula.

Lemma 5.9. *Let $M = (E, r)$ be a matroid. Then*

$$\frac{Q'_M(x,1)}{x^{|E|-r(M)-1}} = T_M(x,1), \text{ and}$$

$$\frac{Q'_M(1,y)}{y^{r(M)-1}} = T_M(1,y).$$

We can also give a combinatorial-geometric intepretation of these results, using Lemma 4.2:

Remark 5.10. Note that $Q'_M(x,1) = \#(P(M) + t\nabla)$ and $Q'_M(1,y) = \#(P(M) + u\Delta)$. By Lemma 4.2, these polyhedra are partitioned by the the Minkowski sum of each point with faces of simplices formed by elements internally and externally active to that point. If we divide through by the number of lattice points in such a face, we gain the original matroid. Using the calculations used above, we get the necessary scale factors.

We can now prove the formula for the Tutte polynomial directly in terms of lattice point counting, restated here:

Theorem 5.11. *Let $M = (E, r)$ be a matroid and let $u, t \in \mathbb{Z}$. Then*

$$\sum_{u,t \geq 0} Q_M(t,u) v^t w^u = \frac{1}{(1-w)^{r(M)}(1-v)^{|E|-r(M)}(1-vw)} \cdot T_M\left(\frac{1-vw}{1-v}, \frac{1-vw}{1-w}\right).$$

Proof. Consider the power series $\sum_{u,t\geq 0} Q_M(t,u) a^t b^u$. Note that

$$\sum_{u,t\geq 0} \binom{t}{i}\binom{u}{j} a^t b^u = \frac{1}{ab} \cdot \left(\frac{a}{1-a}\right)^{i+1} \cdot \left(\frac{b}{1-b}\right)^{j+1}.$$

We can thus write our power series as

$$\frac{1}{ab} \sum_{i,j} c_{ij} \left(\frac{a}{1-a}\right)^{i+1} \left(\frac{b}{1-b}\right)^{j+1}.$$

Substituting $a = \frac{x-1}{x}$ and $b = \frac{y-1}{y}$ turns this into

$$\frac{xy}{(x-1)(y-1)} \sum_{i,j} c_{ij} (x-1)^{i+1} (y-1)^{j+1}$$

which is a scaled version of $Q'_M(x,y)$. We can now apply Theorem 5.2:

$$\begin{aligned}
\sum_{u,t\geq 0} Q_M(t,u) \left(\frac{x-1}{x}\right)^t \left(\frac{y-1}{y}\right)^u &= Q'_M(x,y) \\
&= \frac{x^{|E|-r(M)+1} y^{r(M)+1}}{x+y-1} \cdot T_M\left(\frac{x+y-1}{y}, \frac{x+y-1}{x}\right).
\end{aligned}$$

Substitute $v = \frac{x-1}{x}$ and $w = \frac{y-1}{y}$ to get the stated result. $\square$

A further substitution gives the following corollary (stated as a theorem at the beginning of this chapter):

Corollary 5.12. *Let $M = (E, r)$ be a matroid and let $u, t \in \mathbb{Z}$. Then*

$$T_M(x,y) = (xy - x - y)^{|E|}(-x)^{r(M)}(-y)^{|E|-r(M)} \times \sum_{u,t \geq 0} Q_M(t,u) \cdot \left(\frac{y - xy}{xy - x - y}\right)^t \left(\frac{x - xy}{xy - x - y}\right)^u.$$

Our formula for the Tutte polynomial can be viewed as a close relative of the algebro-geometric formula for the Tutte polynomial in [18]. This is due to the computations on the Grassmannian in that work being done in terms of $P(M)$, the moment polytope of a certain torus orbit closure. We have that Δ and ∇ are the moment polytopes of the two dual copies of $\mathbb{P}^{n-1}$, the K-theory ring of whose product $\mathbb{Z}[x,y]/(x^n, y^n)$ is identified with the ambient ring of the Tutte polynomial.

Chapter 6

Properties

From the definition of Q'_M, it is not difficult to describe its behaviour under the polymatroid generalisation of many standard matroid operations; we see that it retains versions of formulae true of the Tutte polynomial in many cases. For instance, there is a polymatroid analogue of the direct sum of matroids: given two polymatroids $M_1 = (E_1, r_1), M_2 = (E_2, r_2)$ with disjoint ground sets, their *direct sum* $M = (E, r)$ has ground set $E = E_1 \sqcup E_2$ and rank function $r(S) = r_1(S \cap E_1) + r_2(S \cap E_2)$.

Recall the following property of the Tutte polynomial:

Proposition 6.1. *Let $M_1 \oplus M_2$ be the direct sum of two matroids $M_1 = (E_1, r_1), M_2 = (E_2, r_2)$ with disjoint ground sets. Then $T_{M_1 \oplus M_2}(x, y) = T_{M_1}(x, y) T_{M_2}(x, y)$.*

Using this, we gain the following property of Q'_M:

Proposition 6.2. *Let $M_1 \oplus M_2$ be the direct sum of two matroids $M_1 = (E_1, r_1), M_2 = (E_2, r_2)$ with disjoint ground sets. Then $Q'_{M_1 \oplus M_2}(x, y) = \dfrac{Q'_{M_1}(x, y) Q'_{M_2}(x, y)}{x + y - 1}$.*

Proof.

$$
\begin{aligned}
Q'_{M_1 \oplus M_2}(x,y) &= \frac{x^{r_1(M_1)+r_2(M_2)} y^{|E_1|+|E_2|-r_1(M_1)-r_2(M_2)}}{x+y-1} \cdot T_{M_1 \oplus M_2}\left(\frac{x+y-1}{x}, \frac{x+y-1}{y}\right) \\
&= \left(\frac{x^{r_1(M_1)} y^{|E_1|-r_1(M_1)}}{x+y-1}\right) x^{r_2(M_2)} y^{|E_2|-r_2(M_2)} \cdot T_{M_1}\left(\frac{x+y-1}{x}, \frac{x+y-1}{y}\right) \\
&\qquad \times T_{M_2}\left(\frac{x+y-1}{x}, \frac{x+y-1}{y}\right) \\
&= Q'_{M_1}(x,y) x^{r_2} y^{|E_2|-r_2(M_2)} \cdot T_{M_2}\left(\frac{x+y-1}{x}, \frac{x+y-1}{y}\right) \\
&= \frac{Q'_{M_1}(x,y) Q'_{M_2}(x,y)}{x+y-1}
\end{aligned}
$$

□

In particular, in the matroid setting where one of the summands is a loop or a coloop, we obtain:

Corollary 6.3. *Take a matroid $M = (E, r)$. Let $M' = M \cup \{e\}$ where e is either a loop or a coloop. Then $Q'_{M'}(x,y) = (x+y-1)Q'_M(x,y)$.*

Proof. This can also be proven using the Tutte polynomial deletion-contraction equations given in Section 2.6, as follows.

First let e be a loop of M. We have $r(M) = r(M')$. So,

$$
\begin{aligned}
Q'_{M'}(x,y) &= \frac{x^{r(M)} y^{|E(M')|-r(M)}}{x+y-1} \cdot T_{M'}\left(\frac{x+y-1}{x}, \frac{x+y-1}{y}\right) \\
&= \frac{x^{r(M)} y^{|E(M)|+1-r(M)}}{x+y-1} \cdot \frac{x+y-1}{y} \cdot T_{M'/e}\left(\frac{x+y-1}{x}, \frac{x+y-1}{y}\right) \\
&= (x+y-1) Q'_M(x,y).
\end{aligned}
$$

Now let e be a coloop of M. We have that $r(M') = r(M) + 1$. We have that:

$$\begin{aligned} Q'_{M'}(x,y) &= \frac{x^{r(M')}y^{|E(M')|-r(M')}}{x+y-1} \cdot T_{M'}\left(\frac{x+y-1}{x}, \frac{x+y-1}{y}\right) \\ &= \frac{x^{r(M)+1}y^{|E(M)|+1-r(M)-1}}{x+y-1} \cdot \frac{x+y-1}{x} \cdot T_{M'/e}\left(\frac{x+y-1}{x}, \frac{x+y-1}{y}\right) \\ &= (x+y-1)Q'_M(x,y). \end{aligned}$$

□

Next we see Q' exchanges its two variables under duality, as does the Tutte polynomial. The best analogue of duality for polymatroids requires a parameter s greater than or equal to the rank of any singleton; then if $M = (E, r)$ is a polymatroid, its *s-dual* is the polymatroid $M^* = (E, r^*)$ with

$$r^*(S) = r(E) + s|E - S| - r(E - S).$$

Proposition 6.4. *For any matroid $M = (E, r)$, $Q'_{M^*}(x, y) = Q'_M(y, x)$.*

This can easily be seen from the equation

$$Q'_M(x,y) = \frac{x^{r(M)}y^{|E|-r(M)}}{x+y-1} \cdot T_M\left(\frac{x+y-1}{x}, \frac{x+y-1}{y}\right),$$

but we also provide a geometric proof.

Proof. In order to get an inequality description for $P(M^*) + u\Delta + t(-\Delta)$, we use

$$\phi(P) = \{\phi(p) \mid Ap \leq b\}$$

where ϕ is a bijection which takes elements of $P(M)$ to elements of $P(M^*)$. Such a bijection clearly exists as both matroids have the same ground set and thus the same number of bases. This gives that

$$\begin{aligned}\#(P(M) + u\Delta + t(-\Delta)) &= \#(\phi(P(M)) + u\phi(\Delta) + t\phi(-\Delta)) \\ &= \#(P(M^*) + u(\nabla + 1_E) + t(\Delta + 1_E) \\ &= \#(P(M^*) + (t-u)1_E + u\nabla + t\Delta) \\ &= \#(P(M^*) + u\nabla + t\Delta)\end{aligned}$$

where the last line is true due to the polytope being a translation of the one in the line above. The statement follows. □

Given a subdivision $P_1, \dots, P_n$ of a polytope P, a *valuation* is a function f such that $f(P) = \sum_{P_i} f(P_i) - \sum_{P_i, P_j} f(P_i \cap P_j) + \dots + (-1)^{n-1} f(P_1 \cap \dots \cap P_n)$. The number of lattice points in a polytope, and thus the invariant Q'_M, is a polytope valuation of polymatroids. That is:

Proposition 6.5. *Let $\mathcal{F}$ be a polyhedral complex whose total space is a polymatroid base polytope $P(M)$, and each of whose faces F is a polymatroid base polytope $P(M(F))$. Then*

$$Q'_M(x,y) = \sum_{F \text{ a face of } \mathcal{F}} (-1)^{\dim(P(M)) - \dim F} Q'_{M(F)}(x,y).$$

For example, if M is a matroid and we relax a circuit-hyperplane, we get the following result:

Corollary 6.6. *Take a matroid $M = (E, r)$ and let $C \subseteq E$ be a circuit-hyperplane of M. Let M' be the matroid formed by relaxing C. Then $Q'_M(x,y) = Q'_{M'}(x,y) - x^{n-r(M)-1} y^{r(M)-1}$.*

Proof. Note that $r(M') = r(M)$ and that $|E(M)| = |E(M')| = |E|$. Using Lemma 6.7, we have that

$$\begin{aligned}
Q'_M(x,y) &= \frac{x^{r(M)}y^{|E(M)|-r(M)}}{x+y-1} \cdot T_M\left(\frac{x+y-1}{x}, \frac{x+y-1}{y}\right) \\
&= \frac{x^{r(M)}y^{|E(M)|-r(M)}}{x+y-1} \cdot \left[T_{M'}\left(\frac{x+y-1}{x}, \frac{x+y-1}{y}\right) - \frac{x+y-1}{x}\right. \\
&\qquad\qquad \left. - \frac{x+y-1}{y} + \frac{(x+y-1)^2}{xy}\right] \\
&= Q'_{M'}(x,y) - \frac{x^{r(M)}y^{|E|-r(M)}}{x+y-1} \cdot \left[\frac{(x+y-1)^2}{xy} - \frac{x+y-1}{x} - \frac{x+y-1}{y}\right] \\
&= Q'_{M'}(x,y) - x^{r(M)-1}y^{|E|-r(M)-1}.
\end{aligned}$$

□

Lemma 6.7 ([31]). *Take a matroid $M = (E, r)$ and let $C \subseteq E$ be a circuit-hyperplane of M. Let M' be the matroid formed by relaxing C. Then $T_M(x,y) = T_{M'}(x,y) - x - y + xy$.*

Most importantly, when M is a matroid, we have a deletion-contraction recurrence for Q', akin to that of the Tutte polynomial:

Proposition 6.8. *Let $M = (E, r)$ be a matroid with $|E| = n$. Then, for $e \in E$,*

i. $Q'_M(x,y) = xQ_{M\backslash e}(x,y) + yQ'_{M/e}(x,y)$ *when e is not a loop or coloop, and*

ii. $Q'_M(x,y) = (x+y-1)Q'_{M/e}(x,y) = (x+y-1)Q'_{M\backslash e}(x,y)$ *otherwise.*

Proof. Part *ii* is Corollary 6.3, as when e is a (co)loop, $M\backslash e = M/e$. For part *i*, recall that if e is neither a loop nor a coloop, then $E(M\backslash e) = E - e = E(M/e)$, $r(M\backslash e) = r(M)$, and $r(M/e) = r(M) - 1$. Take the equation $T_M(x,y) = T_{M\backslash e}(x,y) + T_{M/e}(x,y)$ and rewrite it in terms of Q', as per Corollary 5.2:

$$\begin{aligned}
-\frac{(xy-x-y)^{n-1}}{(-y)^{r(M)-1}(-x)^{n-r(M)-1}} \cdot Q'_M(x,y) = &-\frac{(xy-x-y)^{n-2}}{(-y)^{r(M)-1}(-x)^{n-r(M)-2}} \cdot Q'_{M\backslash e}(x,y) \\
&-\frac{(xy-x-y)^{n-2}}{(-y)^{r(M)-2}(-x)^{n-r(M)-1}} \cdot Q'_{M/e}(x,y)
\end{aligned}$$

Multiplying through by $-\dfrac{(-y)^{r(M)-1}(-x)^{n-r(M)-1}}{(xy-x-y)^{n-1}}$ gives the result. $\square$

We have that $P(M\backslash a) = \{p \in P(M) \mid p_a = \underline{k}\}$, where $\underline{k}$ is the minimum value p_a takes (this will be 0 unless a is a coloop), and that $P(M/a) = \{p \in P(M) \mid p_a = \overline{k}\}$, where $\overline{k}$ is the maximum value p_a takes. When M is a matroid, these two sets partition $P(M)$. However, when M is a polymatroid, we can have points in $P(M)$ where $\underline{k} < p_a < \overline{k}$. Let $N_k := \{p \in P(M) \mid p_a = k\}$, and let $P(N_k)$ be the polytope consisting of the convex hull of such points. Now we have that $P(M\backslash a)$, $P(M/a)$, and the collection of $P(N_k)$ for $k \in \{\underline{k}+1, \ldots, \overline{k}-1\}$ partition $P(M)$. We will refer to each of these parts, when they exist, as an *a-slice* of $P(M)$. When we do not include the deletion and contraction slices, we can talk about *(strictly) interior* slices.

Theorem 6.9. *Let $M = (E, r)$ be a polymatroid and take $a \in E(M)$. Let N be an a-slice of $P(M)$. Then*

$$Q'_M(x,y) = (x-1)Q'_{M\backslash a}(x,y) + (y-1)Q'_{M/a}(x,y) + \sum_{N} Q'_N(x,y).$$

Note that when M is a matroid, the statement simplifies to the formulae given in Lemma 6.8: if a is neither a loop nor coloop, then the a-slices are $P(M\backslash a)$ and $P(M/a)$, so

$$\begin{aligned}
Q'_M(x,y) &= (x-1)Q'_{M\backslash a}(x,y) + (y-1)Q'_{M/a}(x,y) + \sum_{N_k} Q'_{N_k}(x,y) \\
&= (x-1)Q'_{M\backslash a}(x,y) + (y-1)Q'_{M/a}(x,y) + Q'_{M\backslash a}(x,y) + Q'_{M/a}(x,y) \\
&= xQ'_{M\backslash a}(x,y) + yQ'_{M/a}(x,y).
\end{aligned}$$

When a is a loop or coloop, $M\backslash a = M/a$, and we have only one a-slice: $P(M\backslash a) =$

$P(M/a)$. So we get that

$$\begin{aligned} Q'_M(x,y) &= (x-1)Q'_{M\backslash a}(x,y) + (y-1)Q'_{M/a}(x,y) + \sum_{N_k} Q'_{N_k}(x,y) \\ &= (x+y-1)Q'_{M/a}(x,y) \end{aligned}$$

as in Lemma 6.8.

Also note that this result gives another proof of Theorem 5.1 as a corollary.

Proof. In this proof, all statements made about points of polytopes will be made on the lattice point level.

Let M be a polymatroid. If the rank function of M is a matroid rank function summed with a function of the form $S \mapsto \sum_{i \in S} c_i$, then $P(M)$ will be a translate of a matroid polytope, and the same argument as above will hold. Assume now that this is not the case. This means that for any $a \in E(M)$, there will be at least one a-slice of $P(M)$, $P(N_k)$, which is not equal to $P(M/a)$ or $P(M\backslash a)$.

Claim 6.10. *Define R to be the polytope $\{q \in P(M) + u\Delta_E + t\nabla_E \mid q_a = k\}$, and define S to be $P(N_k) + u\Delta_{E-a} + t\nabla_{E-a}$. If R intersects the set of lattice points of $P(M)$, then $R = S$.*

Proof of Claim 6.10. It is clear that the lattice points of S are contained in R. Take a point in R, $q_1 = p_1 + \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_u} - \mathbf{e}_{j_1} - \cdots - \mathbf{e}_{j_t}$. We will show that we can write this as a point contained in S, $q_2 = p_2 + \mathbf{e}_{m_1} + \cdots + \mathbf{e}_{m_u} - \mathbf{e}_{n_1} - \cdots - \mathbf{e}_{n_t}$, where no m_i or n_j can be equal to a.

If $(p_1)_a = k$, then we simply choose p_2 to be p_1 and choose $m_k = i_k$, $n_k = j_k$ for all $k \in \{1, \ldots, t\}$, with one possible change: if we have $i_k = j_l = a$ in q_1, in q_2 replace m_k and n_l with b, where b is any other element in $E(M)$. Note $\mathbf{e}_a$ must always appear paired in this way, such that $(q_1)_a = k$, and so this change does not affect the coordinate values of q_2.

If $(p_1)_a \neq k$, we first must rewrite the expression for q_1. Recall that by Lemma 3.26, for p_1 and any point $p_3 \in P(M)$, if $(p_1)_i > (p_3)_i$ there exists l such that $(p_1)_l < (p_3)_l$ and $p_1 - \mathbf{e}_i + \mathbf{e}_l \in P(M)$. Let $(p_1)_a = k + \lambda$, where $\lambda > 0$. Then, by repeatedly applying the exchange property, we get that $p_1 - \lambda\mathbf{e}_a + \mathbf{e}_{l_1} + \cdots + \mathbf{e}_{l_\lambda} \in P(M)$. Then we can find q_2 by setting $p_2 = p_1 - \lambda\mathbf{e}_a + \mathbf{e}_{l_1} + \cdots + \mathbf{e}_{l_\lambda}$, so

$$q_2 = p_2 + \lambda\mathbf{e}_a - \mathbf{e}_{l_1} - \cdots - \mathbf{e}_{l_\lambda} + \mathbf{e}_{i_1} + \ldots + \mathbf{e}_{i_u} - \mathbf{e}_{j_1} - \cdots - \mathbf{e}_{j_t} = q_1.$$

Note that as $(q_1)_a = k$ and $(p_2)_a = k$, there must be λ $-\mathbf{e}_{j_k}$ terms equal to $-\mathbf{e}_a$, so

$$q_2 = p_2 - \mathbf{e}_{l_1} - \cdots - \mathbf{e}_{l_\lambda} + \mathbf{e}_{i_1} + \ldots + \mathbf{e}_{i_u} - \mathbf{e}_{j_1} - \cdots - \mathbf{e}_{j_{t-\lambda}}$$

which is of the correct form, completing the proof of Claim 6.10. ∎

Claim 6.11. *Let N_i be a strictly interior slice of $P(M)$. Then* $P(M) + t\Delta_E + t\nabla_E = (P(M/a) + u\Delta_E + t\nabla_{E-a}) \sqcup \bigsqcup_i (P(N_i) + t\Delta_{E-a} + t\nabla_{E-a}) \sqcup (P(M\backslash a) + t\Delta_{E-a} + t\nabla_E)$.

Proof of Claim 6.11. Take $P(M) + u\Delta_E + t\nabla_E$ and split it into a collection of polytopes according to the value of q_a for all points $q \in P(M) + u\Delta_E + t\nabla_E$. The disjoint union of the lattice points of these parts clearly will give back those of the original polytope. By the previous result, if one of these parts intersects $P(M)$ we can write it as $P(N_k) + u\Delta_{E-a} + t\nabla_{E-a}$. Otherwise, we must be able to write the part as $P(M/a) + (u - \lambda)\Delta_{E-a} + \lambda\mathbf{e}_a + t\nabla_{E-a}$, where $\lambda > \overline{k}$, or as $P(M\backslash a) + t\Delta_{E-a} - \mu\mathbf{e}_a + (t-\mu)\nabla_{E-a}$, where $\mu > \underline{k}$.

We will show that

$$\bigsqcup_\lambda P(M/a) + (u - \lambda)\Delta_{E-a} + \lambda\mathbf{e}_a + t\nabla_{E-a} = P(M/a) + u\Delta_E + t\nabla_{E-a}. \tag{6.11.1}$$

It is clear that the sets of lattice points of the summands are pairwise disjoint as the a-coordinates in each set must be different. It is also clear that the lattice points contained in the polytope on the left hand side are contained in that of the right hand side. Take a point $q_1 = p_1 + \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_u} - \mathbf{e}_{j_1} - \cdots - \mathbf{e}_{j_t}$ contained in $P(M/a) + u\Delta_E + t\nabla_{E-a}$. Let $(q_1)_a = \overline{k} + \mu$, where $\mu > 0$. We need to write q_1 as $p_2 + \mathbf{e}_{m_1} + \cdots + \mathbf{e}_{m_{u-\lambda}} + \lambda\mathbf{e}_a - \mathbf{e}_{n_1} - \cdots - \mathbf{e}_{n_t}$, a lattice point contained in one of the summands on the left hand side. Choose $\mu = \lambda$, $p_2 = p_1$, $\{j_\alpha\} = \{n_\alpha\}$, and $\{i_\beta \mid i_\beta \neq a\} = \{m_\beta\}$ and the equality follows.

The same arguments show that

$$\bigsqcup_{\mu} P(M\backslash a) + t\Delta_{E-a} - \mu\mathbf{e}_a + (t-\mu)\nabla_{E-a} = P(M\backslash a) + u\Delta_{E-a} + t\nabla_E \tag{6.11.2}$$

and the claim follows. ■

Claim 6.12. *We have that*

$$\#(P(M/a) + u\Delta_E + t\nabla_{E-a}) = \sum_{j=0}^{u} \#(P(M/a) + j\Delta_{E-a} + t\nabla_{E-a})$$

and

$$\#(P(M\backslash a) + u\Delta_{E-a} + t\nabla_E) = \sum_{j=0}^{u} \#(P(M\backslash a) + u\Delta_{E-a} + i\nabla_{E-a}).$$

Proof of Claim 6.12. Take the cardinalities of both sides of Equations 6.11.1 and 6.11.2. ■

Continuing the proof of the theorem, we now that have

$$Q_M(t,u) = \sum_{N_k} Q_{N_k}(t,u) + \sum_{j=0}^{u} Q_{M/a}(t,j) + \sum_{i=0}^{t} Q_{M\backslash a}(i,u) \tag{6.12.1}$$

where $k \in \{\underline{k}+1, \ldots, \overline{k}-1\}$, that is, N_k is always a strictly interior slice of $P(M)$.

We now work out how the change of basis from Q to Q' transforms the sums in

Equation 6.12.1. Take a term in $Q_{M/a}$, $c_{ik}\binom{j}{k}\binom{t}{i}$. We have that

$$\begin{aligned}\sum_{j=0}^{u} c_{ik}\binom{j}{k}\binom{t}{i} &= c_{ik}\binom{u+1}{k+1}\binom{t}{i} \\ &= c_{ik}\binom{t}{i}\left(\binom{u}{k}+\binom{u}{k+1}\right).\end{aligned}$$

Now apply the change of basis to get

$$\begin{aligned}c_{ik}(x-1)^i((y-1)^k+(y-1)^{k+1}) &= c_{ik}(x-1)^i\left((y-1)^k(1+y-1)\right) \\ &= c_{ik}(x_1)^i(y-1)^k y.\end{aligned}$$

Thus

$$\sum_{j=0}^{u} Q_{M/a}(t,j) = yQ'_{M/a}(t,u)$$

and similarly,

$$\sum_{i=0}^{t} Q_{M\backslash a}(i,u) = xQ'_{M\backslash a}(t,u).$$

Finally, putting this together with Claim 6.11 and Equations 6.11.1 6.11.2 gives:

$$\begin{aligned}Q'_M(t,u) &= xQ'_{M/a}(t,u) + yQ'_{M/a}(t,u) + \sum_{\text{interior } N_k} Q'_{N_k}(t,u) \\ &= (x-1)Q'_{M/a}(t,u) + (y-1)Q'_{M/a}(t,u) + \sum_{N_k} Q'_{N_k}(t,u).\end{aligned}$$

This completes the proof of Theorem 6.9.

□

Chapter 7

Coefficients and Terms

We will restate the two polynomials here for convenience:

$$Q_M(x,y) := \#(P(M) + u\Delta + t(-\Delta)) = \sum_{i,j} c_{ij} \binom{u}{j}\binom{t}{i}$$
$$Q'_M(x,y) = \sum_{ij} c_{ij}(x-1)^i(y-1)^j$$

In this chaper, we will only allow M to be a matroid.

7.1 Coefficients of $Q'(x,y)$

Some coefficients of the Tutte polynomial provide structural information about the matroid in question. Let $b_{i,j}$ be the coefficient of $x^i y^j$ in $T_M(x,y)$. It is well known that M is connected only if $b_{1,0}$, known as the *beta invariant*, is non-zero; moreover, $b_{1,0} = b_{0,1}$ when $|E| \geq 2$. Not every coefficient yields such an appealing result, though of course they do count the bases with internal and external activity of fixed sizes. We are able to provide a geometric interpretation of the coefficients of $Q'_M(x,y)$ when M is a matroid, which is the focus of this section.

In order to do this, we will make use of a particular regular mixed subdivision of $u\Delta + P(M) + t\nabla$. This will be the regular subdivision determined by the lifted polytope $(P(M)\times\{0\})+\mathrm{Conv}\{(u\mathbf{e}_i,\alpha_i)\}+\mathrm{Conv}\{(-t\mathbf{e}_i,\beta_i)\}$ lying in $\mathbb{R}^E\times\mathbb{R}$, where $\alpha_1 < \cdots < \alpha_n$, $\beta_1 < \cdots < \beta_n$ are positive reals. When $t = u = 1$, the associated height function on the lattice points of $\Delta + P(M) + \nabla$ is

$$h(x) := \min\{\alpha_i + \beta_j \mid x - \mathbf{e}_i + \mathbf{e}_j \in \mathcal{B}_M\};$$

in general, one subtracts t standard basis vectors and adds u of them.

Let $\mathfrak{F}$ be the set of lower faces of the lifted polytope. For each face $F \in \mathfrak{F}$, let $\pi(F)$ be its projection back to $\mathbb{R}^n$. Now $\mathcal{F} := \{\pi(F) \mid F \in \mathfrak{F}\}$ is a regular subdivision of $u\Delta + P(M) + t\nabla$.

The structure of the face poset of this polyhedral subdivision does not depend on t and u as long as these are positive. By Lemma 3.10, the faces of our Minkowski sum consist of sums of faces of the summands. The face lattice of $u\Delta$ does not depend on the value of u: the face lattice of a $(n-1)$-dimensional simplex is a $(n-1)$-dimensional cube, and the addition of u just scales the coordinates of the cube. Likewise, the face lattice of $t\nabla$ is independent of the value of t. When we name a face of $\mathcal{F}$ as a sum of three polytopes, we mean this to be the given Minkowski cell structure from Definition 3.13.

Definition 7.1. *A cell $F+G+H$ of $\mathcal{F}$ is a* top degree face *when G is a vertex of $P(M)$ and there exists no cell $F + G' + H$ of $\mathcal{F}$ where $G' \subsetneq G$.*

The result we will be working towards in this chapter is the following:

Theorem 7.2. *Let M be a matroid. Take the regular mixed subdivision $\mathcal{F}$ of $u\Delta + P(M) + t\nabla$. We have that $|[x^iy^j]Q'_M(x,y)|$ counts the cells $F + G + H$ of the mixed subdivision where G is a vertex of $P(M)$ and there exists no cell $F + G' + H$ where $G' \subsetneq G$ and $i = dim(F), j = dim(H)$.*

The key fact in the proof is the following.

Proposition 7.3. *In the subdivision $\mathcal{F}$, each of the lattice points of $u\Delta + P(M) + t\nabla$ lies in a top degree face.*

To expose the combinatorial content of this proposition, we need to describe the top degree faces more carefully. All top degree faces are of dimension $|E| - 1$, and therefore have the form $u\Delta_X + \mathbf{e}_B + t\nabla_Y$. Recall that a mixed subdivision of $\Delta + \nabla$ was described in Example 3.15. The addition of $\mathbf{e}_B$ does not change the combinatorics of the choice of X and Y as it is a zero-dimensional polytope and thus has only one face. Thus, by prior work, we know that X and Y are subsets of E so that the affine spans of Δ_X and ∇_Y are transverse (and thus only meet at the origin) and of dimensions summing to $|E| - 1$, which implies that $X \cup Y = E$ and $|X \cap Y| = 1$. In fact the conditions on the α and β imply that $X \cap Y = \{1\}$. We thus have $2^{|E|-1}$ top degree faces, one for each valid choice of X and Y – each element (except 1) is either in X, or it is in Y.

Lemma 7.4. *Take subsets X and Y of E with $X \cup Y = E$ and $X \cap Y = \{1\}$. There is a unique basis B such that $u\Delta_{1\cup X} + \mathbf{e}_B + t\nabla_{1\cup Y}$ is a top-degree face. It is the unique basis B such that no elements of X are externally inactive and no elements of Y are internally inactive with respect to B, where activity is defined with respect to reversed natural order on E.*

Note in the service of readability we write 1 instead of $\{1\}$.

The basis B can be found using the simplex algorithm for linear programming on $P(M)$, which relies on the fact that if an objective function of a linear program has a maximum, it occurs at an extremal point of the feasible set. The algorithm describes how to move from extremal point to extremal point so that, in each direction moved, there is an increase in the objective function. In terms of a polytope, this corresponds to moving from vertex to vertex via the edges. We apply this algorithm to a linear functional constructed from the α and β encoding the activity conditions. This procedure can be completely combinatorialised, giving a way to start from a randomly chosen initial basis and make a sequence of exchanges which yields a unique output B regardless of the input choice. The proof is as follows.

Proof. Choose any basis, B_0, and order the elements $b_1, \dots, b_r$ lexicographically. Perform the following algorithm to find the basis B:

- Take b_1.

  (a) Test whether b_1 can be replaced with a larger element of X to give another basis. If so, replace b_1 with the largest such element. Call this a check of type (a).

  (a) If b_1 was not replaced, and $b_1 \in Y$, see if b_1 can be replaced by a smaller element of E to give another basis. If so, replace b_1 with the smallest such element. Call this a check of type (b).

- Repeat the above steps with the successive elements b_{i+1}, with $b_{r+1} = b_1$.

- Terminate when, after a full run of the algorithm on the elements of the basis, the basis has remained unchanged at each iteration.

Now let $\gamma_1, \dots, \gamma_n \in \mathbb{R}$ be such that $0 = |\gamma_1| \ll \cdots \ll |\gamma_n|$, and $\gamma_a > 0$ if $a \in X$, $\gamma_a < 0$ is $a \in Y$.

Claim 7.5. *Let B_i and B_{i+1} be two bases of M found consecutively by the algorithm. Then $\sum\limits_{a \in B_i} \gamma_a < \sum\limits_{a \in B_{i+1}} \gamma_a$ for all i. That is, the sum $\sum\limits_{a \in B} \gamma_a$ is increasing with the algorithm.*

Proof of Claim 7.5. Moves of type (a) replace an element b of a basis with a larger element c in X, so regardless of whether b was in X or Y, this must increase the sum as $\gamma_b > 0$. Moves of type (b) replace an element $y \in Y$ in the basis with a smaller element d. If $d \in Y$, we are replacing γ_y with a smaller negative, as $|\gamma_d| \ll |\gamma_y|$. If $d \in X$, we are replacing a negative γ_y with a positive γ_d. So $\sum\limits_{a \in B} \gamma_a$ is increasing in every case. ■

We will write the symmetric difference of two sets A and B as $A \triangle B$. The next result follows as a corollary of the previous claim.

Claim 7.6. *$B \triangle Y$ written with the largest elements first is lexicographically increasing with the algorithm.*

Proof of Claim 7.6. A move of type (a) replaces an element of B with a larger element in X. This puts a larger element into $B \triangle Y$ than was in B originally, and so must cause a lexicographic increase. A move of type (b) removes a Y element in B (and so, an element not in $B \triangle Y$), and puts a smaller element z into B. Removing the Y element from B adds it to $B \triangle Y$, and adding an element to a set cannot decrease the lexicographic order. If the smaller element z is in Y, then this move removes z from $B \triangle Y$. As we have replaced it with a larger element, the lexicographic order of $B \triangle Y$ is increased. If z is in X, the move adds z to $B \triangle Y$, increasing the lexicographic order of $B \triangle Y$. ■

Claim 7.7. *The algorithm described above terminates and gives an output independent of B_0.*

Proof of Claim 7.7. Order all bases of the matroid based on the increasing lexicographic order of $B \triangle Y$. As we chose the elements δ_a to be much greater than the previous element, only the largest element of $B \triangle Y$ determines the total ordering. We have shown in the previous corollary that this sequence is increasing with the algorithm. As there is a finite number of bases, there must be a greatest element, and thus the algorithm terminates.

We now need to show that there is a unique basis for which the algorithm can terminate.

The structure $(E, \{B_i \triangle Y \mid B_i \in \mathcal{B}_M\})$ is what is known as a *delta-matroid*, a generalisation of a matroid allowing bases to have different sizes. This delta-matroid is a *twist* of M by the set Y [5]. The subsets $B_i \triangle Y$ are called the feasible sets. It is a result of Bouchet ([5]) that feasible sets of largest size form the bases of a matroid.

In order to show uniqueness of termination bases, we will first show that if B is a termination basis and $B \triangle Y$ is not of largest size, then any basis B' with $|B' \triangle Y| > |B \triangle Y|$ is terminal. Suppose this is not the case. As $|B \triangle Y| = |B| + |Y| - 2|B \cap Y|$, this requires that $|B' \cap Y| < |B \cap Y|$.

As B' is not a termination basis, there is either an element $b \in B'$ such that $(B' - b) \cup c \in \mathcal{B}$ for some element $c \in X$, where $c > b$, or there is an element $b \in Y \cap B$ such that $(B' - b) \cup a \in \mathcal{B}$ for some $a < b$. If we have $b, c \in X$, $|((B' - b) \cup c) \cap Y| = |B' \cap Y|$. If $b \in Y$, $|((B' - b) \cup c) \cap Y| < |B' \cap Y|$. If $a, b \in Y$, then $|((B' - b) \cup a) \cap Y| = |B' \cap Y|$. Finally, if $b \in Y$ and $a \in X$, then $|((B' - b) \cup a) \cap Y| < |B' \cap Y|$. In every case we have a contradiction.

As the algorithm terminates, we know that after a finite number of such exchanges, we produce B from B'. Let the bases constructed in each step form a chain

$$B', B_1, B_2, \ldots, B_n, B.$$

From above, we have that $|B' \cap Y| \geq |B_1 \cap Y| \geq \cdots \geq |B_n \cap Y| \geq |B \cap Y|$. This contradicts the initial assumption that $|B' \cap Y| < |B \cap Y|$.

Now assume the algorithm can terminate with two bases B_1, B_2. Take $B_1 \triangle Y$ and $B_2 \Delta Y$, and choose the earliest element $b \in B_1 \triangle Y - B_2 \triangle Y$ (assuming this comes lexicographically first in $B_1 \triangle Y$). If $b \in X$, then $b \in B_1 - B_2$. If $b \in Y$, then $b \in B_2 - B_1$. Similarly, if $c \in B_2 \triangle Y - B_1 \triangle Y$, if $c \in X$ then $c \in B_2 - B_1$, or if $c \in Y$ then $c \in B_1 - B_2$.

Apply the delta-matroid exchange algorithm to $B_1 \triangle Y$ and $B_2 \triangle Y$ to get that $(B_1 \triangle Y) \triangle \{b, c\}$ is a feasible set, for some element $c \in (B_1 \triangle Y) \triangle (B_2 \triangle Y)$. Given we have a twist of a matroid, we must have that $(B_1 \triangle Y) \triangle \{b, c\} = B_3 \triangle Y$ for some basis B_3, and so $|(B_1 \triangle \{b, c\}| = |B_3| = |B_1|$ as $\triangle$ is associative. This means we must have that exactly one of $\{b, c\}$ is in B_1. If $b \in X$, $(B_1 \triangle Y - b) \cup c = ((B_1 - b) \cup c) \triangle Y$, so $(B_1 - b) \cup c \in \mathcal{B}$ and we must have $c \in X$ by the above paragraph. As b was the earliest element different in either basis, we must have $c > b$, and so B_1 was not a termination basis of the original algorithm. If $b \in Y$, $(B_1 \triangle Y - b) \cup c = B_1 \triangle((Y - b) \cup c)$. But we cannot change Y, so must have $[(B_1 - c) \cup b] \triangle Y$ and $c \in Y$. This means that again B_1 was not a termination basis, as we are replacing an element of Y with a smaller one. This completes the proof of Claim 7.7. ■

This claim completes the proof of Lemma 7.4. □

Before we can get to the proof of Theorem 7.2, we first need two results on how these top degree cells interact.

Lemma 7.8. *Take two distinct partitions* (X_1, Y_1),(X_2, Y_2) *of* $[n]\backslash\{1\}$. *The algorithm finds two bases* B_1, B_2 *such that we have two top degree cells* $T_i = \mathbf{e}_{B_i} + \Delta_{1\cup X_i} + \nabla_{1\cup Y_i}$, $i \in \{1, 2\}$. *If* $T_1 \cap T_2 \neq \emptyset$, *then* $B_1 = B_2$.

Proof. We will show that a top degree cell cannot contain more than one basis. Suppose that $T = \mathbf{e}_B + \Delta_{1\cup X} + \nabla_{1\cup Y}$ contains a basis $B_2 \neq B$. Any elements of $B_2 - B$ must be in X, and as X, Y partition $[n]\backslash\{1\}$, there can be no elements of $B_2 - B$ in Y, other than 1. If we take $a \in X - B$, where $a \in B_2$, there is some element $b \in B - B_2$ such that $(B - b) \cup a \in \mathcal{B}$. Then a must be smaller than b – otherwise this would contradict B being the termination basis of the algorithm. Note this implies b cannot be 1. Moreover, as B is the termination basis, move (a) of the algorithm tells us that $b \in X$. Now, we have that $\mathbf{e}_B - \mathbf{e}_b + \mathbf{e}_a \in \mathbf{e}_B + \Delta_{1\cup X} + \nabla_{1\cup Y}$. This implies that b is in Y – a contradiction to X, Y being a partition.

Define $\overline{A} := E - A$. Let $\overline{X_1} \neq \overline{X_2}$. We must have that there is a set $D \in (B_2 \cap \overline{X_1}) - B_1$ and $E \in (B_1 \cap \overline{X_2}) - B_2$ such that $(B_1 - E) \cup D = B_2$ – not by basis exchange, by equality. So both T_1 and T_2 contain B_1 and B_2. By the above paragraph, we must have $B_1 = B_2$. If $\overline{X_1} = \overline{X_2}$, we must have $(B_1 - A) \cup B = B_2$, where $B \in \overline{Y_2}, A \in \overline{Y_1}$. □

Lemma 7.9. *Take two distinct partitions* $P_1 = (X_1, Y_1)$,$P_2 = (X_2, Y_2)$ *of* $[n]\backslash\{1\}$ *such that their corresponding top degree cells contain a common point* p. *Now let* $P_3 = (X_3, Y_3)$

be a partition of $[n]\backslash\{1\}$ *such that if* $x \in X_1 \cap X_2$ *then* $x \in X_3$*, and if* $y \in Y_1 \cap Y_2$ *then* $y \in Y_3$*. Then* $p \in T_3$*, and* $B_3 = B_1 = B_2 := B^*$*.*

Proof. By Lemma 7.8, $T_1 = \mathbf{e}_{B^*} + \Delta_{1\cup X_1} + \nabla_{1\cup Y_1}$ and $T_2 = \mathbf{e}_{B^*} + \Delta_{1\cup X_2} + \nabla_{1\cup Y_2}$ where B^* is the basis found by the algorithm. The lexicographically earliest $B \triangle Y_1$ is that with $B = B^*$, likewise for $B \triangle Y_2$. Clearly if $Y_3 = Y_1$ or $Y_3 = Y_2$ then B^* will again give the lexicographically earliest $B \triangle Y_3$, so let $Y_1, Y_2 \neq Y_3$. Take $e \in Y_3$ such that $e \in Y_1 - Y_2$. Then if $e \in B^*$, we have that $e \in B^* \triangle Y_2 - B^* \triangle Y_1$. Recall that the lexicographically earliest $B \triangle Y$ is determined by the largest element of this set. This means that e must not be the largest element of $B^* \triangle Y_2$, or B^* would not be best for Y_1. Morever, this means that the largest element of $B^* \triangle Y_2$ must be in $Y_1 \cap Y_2$, and thus in Y_3. Let this element be e^*. Suppose that the greatest element in the lexicographically earliest $B \triangle Y_3$, $B' \triangle Y_3$, is f, and let $f > e^*$. First suppose $f \in Y_3 - B'$. As every element of Y_3 must be in Y_1 or Y_2, we can assume without loss of generalisation that $f \in Y_2$. As the largest element of $B^* \triangle Y_2$ was e^*, we must have that $f \in B^*$. Changing from B^* to B' will give a better $B \triangle Y_2$, contradiction. So there is no such f. Now let $f \in B' - Y_3$. In order for B' to not have given the lexicographically earliest $B' \triangle Y_1$ and $B' \triangle Y_2$, we must have that $f \in Y_1 \cap Y_2$. But then $f \in Y_3$, contradiction. If $e \notin B^*$, a symmetric argument holds. Thus the greatest element in $B' \triangle Y_3$ is e^*.

Let e_2' be the second greatest element in $B' \triangle Y_3$ and e_2 be the second greatest element in $B^* \triangle Y_i$ for $i \in \{1, 2\}$. Suppose $e_2' > e_2$ and $e_2' \notin B^* \Delta Y_i$. If $e_2' \in Y_3 - B'$, then $e_2' \in Y_i$ for at least one i. This means that, as $e_2' \notin B^* \triangle Y_i$, we have $e_2' \in B^*$. So we must have $e_2' \in Y_1 \cap Y_2$ and $e_2' \in B^* - B'$. Without loss of generality, suppose that $Y_1 \nsubseteq Y_2$. Now, by basis exchange, there exists an element $e \in B' - B^*$ such that $(B^* - e_2') \cup e \in \mathcal{B}$. As the two bases agree on largest elements, we must have that $e < e_2$. Recall $e_2 \in Y_1 \cap Y_2$. This is a contradiction to the algorithm terminating with B^* for both Y_1 and Y_2. So we must have that $e_2' \in B' - Y_3$. This requires that $e_2' \in X_i$ for at least one i, forcing $e_2' \notin B^*$. We must also have $e_2' \in X_2$. Now, by dual basis exchange, there exists an element $e \in B^* - B'$ such that $(B^* - e) \cup e_2 \in \mathcal{B}$. We must again have that $e < e_2$, and since we have $e_2 \in X$, we again have a contradiction to the termination basis of the algorithm. Thus B', B^* must agree on the second greatest element.

Repeating this argument for each successive element (in reverse order) of B' will give that $B' = B^*$. $\square$

The following result is an immediate corollary of Lemma 7.9:

Corollary 7.10. *Define $T_Y = u\Delta_X + \mathbf{e}_B + t\nabla_Y$. For every face F of the mixed subdivision, if F is contained in any top degree face, then the set of Y such that F is contained in T_Y is an interval in the boolean lattice.*

We now have all the ingredients we need to prove the main result of this section, restated here:

Theorem 7.2. *Let M be a matroid. Take the regular mixed subdivision $\mathcal{F}$ of $u\Delta + P(M) + t\nabla$. We have that $|[x^i y^j]Q'_M(x,y)|$ counts the cells $F + G + H$ of the mixed subdivision where G is a vertex of $P(M)$ and there exists no cell $F + G' + H$ where $G' \subsetneq G$ and $i = dim(F), j = dim(H)$.*

Proof. First, we must show that all the lattice points of $P(M) + u\Delta + t\nabla$ lie in a top degree face:

Claim 7.11. *Any $x \in (t\nabla + P(M) + u\Delta) \cap \mathbb{Z}^n$ is of the form $-\mathbf{e}_{i_1} - \cdots - \mathbf{e}_{i_t} + \mathbf{e}_B + \mathbf{e}_{j_1} + \cdots + \mathbf{e}_{j_u}$.*

Proof of Claim 7.11. We will make use of the matroid partition theorem stated earlier as Theorem 2.4. Moreover, we will employ the algorithm of Edmonds [14] to find such a partition. In order to do so, we need to adjust the expression for x above so that it no longer allows for repetition of elements and so that our vector sum can be treated as a set union. First write $t\nabla + P(M) + u\Delta$ as $P(M) + \Delta + \cdots + \Delta + \nabla + \cdots + \nabla$. Note that Δ is the matroid polytope of $U_{1,n}$, and ∇ is the matroid polytope of $U_{n-1,n}$. We will relabel each of these so their ground sets are disjoint.

To each of these summands, except $P(M)$, in turn, create bijective functions from $[n]$ to $\{n+1,\dots,2n\}$, $\{2n+1,\dots,3n\}$,...,$\{(u+t)n,\dots,(u+t+1)n\}$. Write out the set of bases of each matroid – recall this is the set of vertices – using each applicable ground set. Now let $E=[(u+t+1)n]$ and run the matroid partition algorithm. This will give a basis for $t\nabla+P(M)+u\Delta$ – which corresponds to a lattice point in $(t\nabla+P(M)+u\Delta)\cap\mathbb{Z}^n$ – as a union of bases from each summand. For each such union, apply the inverse of the bijection to the elements within. Finally, the vector sum of the elements of the resulting (multi)set gives an expression for x as in the statement of the claim, when the ground sets of the Δ and ∇ are reidentified. ■

Recall that π is the projection map from $\mathbb{R}^{n+1}\to\mathbb{R}^n$.

Claim 7.12. *Any $\pi(x)\in(t\nabla+P(M)+u\Delta)\cap\mathbb{Z}^n$ on $\mathfrak{F}$ is of the form $(-\mathbf{e}_{i_1},\beta_1)+\cdots+(-\mathbf{e}_{i_t},\beta_t)+(\mathbf{e}_B,0)+(\mathbf{e}_{j_1},\alpha_1)+\cdots+(\mathbf{e}_{j_u},\alpha_u)$.*

Proof of Claim 7.12. Claim 7.11 provides an expression for x of the requisite form except that the sum may be incorrect in the last coordinate. Take some x in the form described in Claim 7.11 and add an extra coordinate as stated above. To obtain an expression where the last coordinate is correct, we will now rewrite this to show that x in fact lies on $\mathfrak{F}$: this is equivalent to showing that there exists a partition $X\sqcup Y=[n]\backslash 1$ such that every i is in $1\cup Y$, every j is in $1\cup X$, and the algorithm of Theorem 7.4, given X and Y, yields B. This is because, as we know the height function used to lift the top-degree faces, finding this X and Y will give a top-degree face containing $\pi(x)$, and we would then have the correct last coordinate.

By the algorithm of Theorem 7.4, we require that

1. if there exists an element $d\notin B$ such that $d<e\in B$ and $(B-e)\cup d\in\mathcal{B}$ then $e\in X$, and
2. if there exists an element $e\in B$ such that $e<f\notin B$ and $(B-e)\cup f\in\mathcal{B}$ then $f\in Y$.

Choose any X and any Y. Given these, we will construct X' and Y' such that $X' \sqcup Y' = [n] \backslash 1$.

- Suppose $e \in B$ and $e = i_k$, and there exists $d < e$ with $d \in B$ such that $(B-e) \cup d \in \mathcal{B}$. In the expression for x, replace $-\mathbf{e}_e + \mathbf{e}_B$ with $-\mathbf{e}_d + \mathbf{e}_{(B-e)\cup d}$. Add d to Y'.
- Suppose $f \in B$ and $f = j_l$, and there exists $e < f$ with $e \in B$ such that $(B-e) \cup f \in \mathcal{B}$. In the expression for x, replace $\mathbf{e}_B + \mathbf{e}_f$ with $\mathbf{e}_{(B-e)\cup f} + \mathbf{e}_e$. Add e to X'.
- If we have an element $i \in X \cap Y$, in the expression for x replace $-\mathbf{e}_i + \mathbf{e}_i$ with $-\mathbf{e}_1 + \mathbf{e}_1$. Remove i from both X' and Y'.

The above three operations always replace a term $\pm\mathbf{e}_a$ with a smaller term. As we have a finite ground set, there is a finite amount of such operations, and so this construction must terminate with a X', Y' which fits the criteria. At this point, the expression we have for x will be that required by the claim. ∎

Continuing the proof of the theorem, we now form a poset P where the elements are the top degree faces and all nonempty intersections of sets of these, ordered by containment. This poset is a subposet of the face lattice of the $(|E|-1)$-dimensional cube whose vertices correspond to the top degree faces. Proposition 7.3 shows that every lattice point of $u\Delta + P(M) + t\nabla$ lies in at least one face in P. The total number of lattice points is given by inclusion-exclusion on the function on P assigning to each element of P the number of lattice points in that face. Let $[\cdot]$ denote the number of lattice points of the corresponding face. So we have that

$$Q_M(t,u) = \sum_{i,j} c_{ij} \binom{u}{j}\binom{t}{i} = \sum_{k\geq 1} (-1)^k \sum_{\substack{S \subseteq \text{atoms}(P) \\ |S|=k}} \left[\bigwedge S\right] \tag{7.12.1}$$

$$= \sum_{x \in P} \mu(0,x)[x]$$

where μ is the Möbius function. Now, as the face poset of the cubical complex $\mathcal{C}$ is

Eulerian, we have that $\mu(x,y) = (-1)^{r(y)-r(x)}$. This means that

$$\sum_{k\geq 1}(-1)^k \sum_{\substack{S\subseteq \text{atoms}(P)\\ |S|=k}} \left[\bigwedge S\right] = \sum_{E \text{ face of } \mathcal{C}} (-1)^{\text{codim}E}\binom{t+i}{i}\binom{u+j}{j} \tag{7.12.2}$$

where E is the product of an i-dimensional face of Δ with a j-dimensional face of ∇, that is, E corresponds to a face of type $tF + G + uH$, where G is a basis of $P(M)$.

Now take Q'_M and expand it:

$$Q'_M(x,y) = \sum_{i,j} c_{ij}(x-1)^i(y-1)^j = \sum_{i,j,k,l} c_{ij}\binom{i}{k}x^k(-1)^{i-k}\binom{j}{l}y^l(-1)^{j-l}.$$

The coefficient of x^ky^l is $\sum_{i,j} c_{ij}\binom{i}{k}(-1)^{i-k}\binom{j}{l}(-1)^{j-l}$. To compare this to the count in the lattice, we need to expand $\binom{t}{i}$ (and $\binom{u}{j}$) in the basis of $\binom{t+i}{i}$ (and $\binom{u+j}{j}$). This gives that

$$\binom{t}{i} = \sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\binom{t+k}{k},$$

as proven below:

Claim 7.13. *For any positive integers i, t,*

$$\binom{t}{i} = \sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\binom{t+k}{k}.$$

Proof of Claim 7.13. The Vandermonde identity gives that

$$\binom{t+i}{i} = \sum_{k=0}^{i}\binom{t}{k}\binom{i}{i-k} = \sum_{k=0}^{i}\binom{t}{k}\binom{i}{k}.$$

We will use the binomial inversion theorem to get $\binom{t}{i}$. Rewriting the above identity in this language, we have that

$$f_i = \sum_{k=0}^{i} g_k\binom{i}{k},$$

where $f_i = \binom{t+i}{i}$ and $g_k = \binom{t}{k}$. Then,

$$g_i = \sum_{k=0}^{i} (-1)^{i+k} f_k \binom{i}{k}.$$

Substituting in values again gives that

$$\binom{t}{i} = \sum_{k=0}^{i} (-1)^{i+k} \binom{t+k}{k} \binom{i}{k},$$

as in the statement of the claim. ■

Substitute this into Equation 7.12.1 to get

$$\sum_{k\geq 1} (-1)^k \sum_{\substack{S\subseteq \text{atoms}(P)\\ |S|=k}} \left[\bigwedge S\right] = \sum_{i,j,k,l} c_{ij} (-1)^{i-k} \binom{i}{k} \binom{t+k}{k} (-1)^{j-l} \binom{j}{l} \binom{u+l}{l}$$

$$= \sum_{k,l} [x^k y^l] Q'_M(x,y) \binom{t+k}{k} \binom{u+l}{l}.$$

Comparing this to Equation 7.12.2 proves Theorem 7.2. □

Furthermore, the above proof immediately yields the following result:

Corollary 7.14. *The signs of the coefficients of $Q'_M(x,y)$ are alternating.*

This is parallel (if opposite) to the Tutte polynomial, where the coefficients are all positive. The coefficients of Q'_M, up to sign, have the combinatorial interpretation of counting elements of P of form $u\Delta_X + \mathbf{e}_B + t\nabla_Y$ by the cardinalities of $X \setminus \{1\}$ and $Y \setminus \{1\}$. In particular the top degree faces are counted by the collection of coefficients of Q'_M of top degree (hence the name), and the degree $|E|-1$ terms of Q'_M are always $(x+y)^{|E|-1}$. This proof is given in the final section of this chapter.

The appearance of basis activities in Lemma 7.4 reveals that P is intimately related to a familiar object in matroid theory, the *Dawson partition* [12]. Give the lexicographic order to the power set $\mathcal{P}(E)$. A partition of $\mathcal{P}(E)$ into intervals $[S_1, T_1], \ldots, [S_p, T_p]$ with

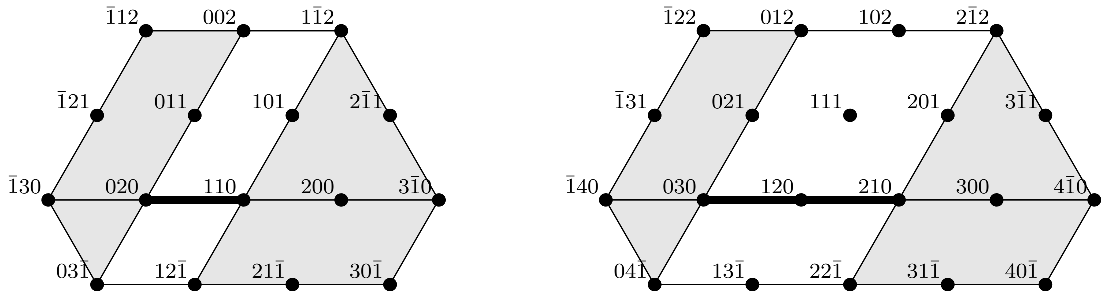


Figure 7.1: At left, the regular subdivision $\mathcal{F}$ associated to the Minkowski sum of Example 5.4, with $P(M)$ bolded and the top degree faces shaded in grey. At right, the regular subdivision for a related polymatroid, still with $(t, u) = (2, 1)$.

indices such that $S_1 < \ldots < S_p$ is a Dawson partition if and only if $T_1 < \ldots < T_p$. In particular, every matroid gives rise to a Dawson partition in which these intervals are $[B \setminus \mathrm{Int}(B), B \cup \mathrm{Ext}(B)]$ for all $B \in \mathcal{B}_M$ [6, Example 1.1].

Proposition 7.15. *Let $[S_1, T_1], \ldots, [S_p, T_p]$ be the Dawson partition of M. The poset P is a disjoint union of face posets of cubes $C_1, \ldots, C_p$ where the vertices of C_i are the top-degree faces $u\Delta_X + \mathbf{e}_B + t\nabla_Y$ such that $X \in [S_i, T_i]$.*

The description of the cubes comes from Lemma 7.9. Note that the element 1 is both internally and externally active with respect to every basis, due to it being the smallest element in the ordering. So, even though 1 is in both X and Y, it is in $T_i - S_i$ for all i.

Here is an example to illustrate this construction of Q'_M and show that Theorem 7.2 fails for polymatroids.

Example 7.16. The left of Figure 7.1 displays the subdivision $\mathcal{F}$ for the sum of Example 5.4. We see that the four grey top degree faces contain all the lattice points between them, and the poset P contains two other faces which are pairwise intersections thereof, the horizontal segment on the left with $(X, Y) = (1, 12)$ and the one on the right with $(X, Y) = (12, 1)$. These are indeed enumerated, up to the alternation of sign, by the polynomial $Q'_M(x, y) = x^2 + 2xy + y^2 - x - y$ found earlier.

By contrast, the right of the figure displays $\mathcal{F}$ for the polymatroid M_2 obtained by doubling the rank function of M. The corresponding polynomial is $Q'_{M_2}(x, y) =$

$x^2 + 2xy + y^2 - 1$, in which the signs are not alternating, dashing hopes of a similar enumerative interpretation. In the figure we see that there are lattice points not on any grey face.

7.2 Terms of $Q'(x, y)$

Lemma 7.17. *Let $M = (E, r)$ be any matroid, and let $n = |E|$. The leading terms of $Q'_M(x, y)$ are x^{n-1} and y^{n-1}.*

Proof. By Theorem 7.2, we have that $|[x^i y^j]|Q'_M(x, y)$ counts cells of the form $F+G+H$, where F is an i-dimensional face of Δ, H is a j-dimensional face of ∇, and G is a vertex of $P(M)$. As $\dim(P(M) + u\Delta + t\nabla = n - 1$, we must have that $i + j \leq n - 1$. This means we can have no terms in Q' of the form $x^{n-1}y^k$ where $k \geq 0$, or terms of the form $x^k y^{-1}$ where $k \geq 0$. It is clear that there is precisely one cell of the type described in Theorem 7.2 when $i = n - 1$ and $j = 0$ – namely $u\Delta + G + H$, where H is a vertex of ∇. There is also one such cell when $i = 0$ and $j = n - 1$, and so the result follows. $\square$

Lemma 7.18. *Let $M = (E, r)$ be any matroid. The constant term of $Q'_M(x, y)$ is zero.*

Proof. We have that

$$\begin{aligned}
Q'_M(x, y) &= \frac{x^{|E|-r(M)} y^{r(M)}}{x + y - 1} \cdot T_M\left(\frac{x+y-1}{y}, \frac{x+y-1}{x}\right) \\
&= \frac{x^{|E|-r(M)} y^{r(M)}}{x + y - 1} \cdot \sum_S \left(\frac{x+y-1}{y} - 1\right)^{r(M)-r(S)} \left(\frac{x+y-1}{x} - 1\right)^{|S|-r(S)} \\
&= \frac{x^{|E|-r(M)} y^{r(M)}}{x + y - 1} \cdot \sum_S \left(\frac{x-1}{y}\right)^{r(M)-r(S)} \left(\frac{y-1}{x}\right)^{|S|-r(S)}.
\end{aligned}$$

Take a single term from this sum:

$$\begin{aligned} &\frac{(x-1)^{r(M)-r(S)}(y-1)^{|S|-r(S)}}{x+y-1} \cdot \frac{x^{|E|-r(M)}y^{r(M)}}{x^{|S|-r(S)}y^{r(M)-r(S)}} \\ = &\frac{(x-1)^{r(M)-r(S)}(y-1)^{|S|-r(S)}}{x+y-1} \cdot \frac{x^{|E|-r(M)}}{x^{|S|-r(S)}} \cdot y^{r(S)}. \end{aligned} \tag{7.18.1}$$

Note that $|E| - r(M) \geq |S| - r(S)$ since removing an element from the set E can reduce the rank by at most 1. Thus, as $x, y \to 0$, each term is 0 and so

$$\lim_{x,y \to 0} Q'_M(x, y) = 0.$$

□

Lemma 7.19. *Let $M = (E, r)$ be any matroid. The lowest order terms of $Q'_M(x, y)$ have coefficients ± 1. Moreover, if M has c loops and k coloops, they are $\pm x^{|E|-r(M)-c}$ and $\pm y^{r(M)-k}$.*

Proof. From Equation 7.18.1, it can be seen that, when $y = 0$, the terms of Q'_M are non-zero exactly when $S = \emptyset$ or when S is a set of loops. First suppose M has no loops. In this case, the terms of Q'_M are

$$\frac{(x-1)^{r(M)}x^{|E|-r(M)}}{x-1} = (x-1)^{r(M)-1}x^{|E|-r(M)}$$

and so the lowest order term in x is $x^{|E|-r}$ and has coefficient $(-1)^{r-1}$. Now suppose M has loops. The terms of Q'_M are now

$$(x-1)^{r(M)-1}x^{|E|-r(M)} + \frac{(x-1)^{r(M)}(-1)^{|S|}}{x-1} \cdot \frac{x^{|E|-r(M)}}{x^{|S|}}$$

$$= (x-1)^{r(M)-1}(x^{|E|-r(M)} + (-1)^{|S|}x^{|E|-|S|-r(M)}).$$

The lowest order term is when S is as large as possible, so when S contains all loops of M. The term is $(-1)^{|S|+r(M)-1}x^{|E|-|S|-r(M)}$. Thus if M has $c \geq 0$ loops, the lowest order term of Q'_M is $\pm x^{|E|-r(M)-c}$.

Now let $x = 0$. The non-zero terms of Q'_M are when $|E| - r(M) = |S| - r(S)$. This occurs when $S = E$, or S is E minus any of the coloops of M. Our terms are then

$$(-1)^{r(M)-r(S)}(y-1)^{|E|-r(M)-1}y^{r(S)}.$$

When $S = E$, this becomes $(-1)^{r(M)-r(S)}(y-1)^{|E|-r(M)-1}y^r$, and so, if there are no coloops, the lowest order term is $\pm y^{r(M)}$. Suppose now there are k coloops in M. S could be formed by removing any number of these from E, but the lowest order term will arise when we remove all k elements. This set S has $r(S) = r(M) - k$, and so the terms it contributes to Q'_M are $(-1)^k(y-1)^{|E|-r(M)-1}y^{r(M)-k}$, with lowest order term $\pm y^{r(M)-k}$.

$\square$

Part III

An excluded minor characterisation of split matroids

Chapter 8

Split Matroids

This chapter is submitted for publication under the same name, and was joint work with Dillon Mayhew. The introduction below is truncated, due to introductory material already appearing in Chapter 3. The proof of Theorem 8.9 does not appear in the paper, due to a different proof of the result already appearing in the literature. Note that in the introduction we give an alternative definition of faces and facets to those in Chapter 3. This is purely to exclude the case that the entire polytope is a face of itself.

8.1 Introduction

Our aim is to give an excluded-minor characterisation of the class of split matroids, defined by Joswig and Schröter, and motivated by natural considerations from the polyhedral view of matroids. Roughly speaking, a split of a polytope is a division into two polytopes by a hyperplane, called a split hyperplane. If all pairs of split hyperplanes in a matroid polytope satisfy a certain compatibility condition, then the matroid is split. We provide more details below.

Let X be a finite set of points in $\mathbb{R}^n$. Let P be the convex hull of the finite set X. Let A be the affine subspace (Definition 3.6) of $\mathbb{R}^n$ spanned by P, and let H be a hyperplane of A. Thus $A - H$ is partitioned into two open half-spaces (Definition 3.2) of A. If one of these has an empty intersection with P, and yet $H \cap P$ is non-empty, then $H \cap P$ is a *face* of P. A *facet* is a maximal face, and a *vertex* is a minimal face. A point in P that is in no face is an *interior* point of P. Every vertex of P is a point in X, but the converse is false unless P consists of a single point.

The following definition comes from [26]. We let P be a polytope. A *split* of P is a subdivision (Definition 3.12) of P with exactly two maximal cells. The affine subspace spanned by the intersection of the two maximal cells is called a *split hyperplane*.

The base polytope of a matroid can be written in terms of flats, as well as in terms of bases as described earlier (Definition 3.16). Let M be a rank-r matroid with $E(M) = \{1, \ldots, n\}$. If x is in $\mathbb{R}^n$, then x_i stands for the entry of x indexed by $i \in E(M)$. Recall from Example 3.17 that $\Delta(r, n)$ is the rank-r hypersimplex, the matroid polytope of $U_{r,n}$. Edmonds [15] proved that

$$P(M) = \left\{ x \in \Delta(r, n) \mid \sum_{i \in F} x_i \leq r(F) \text{ for all flats } F \text{ of } M \right\}.$$

Let F be a flat of M. Then the *F-hyperplane*, $H(F)$, is the set

$$\left\{ x \in \mathbb{R}^n \mid \sum_{i \in F} x_i = r(F) \right\}.$$

Let A be the affine subspace spanned by $P(M)$. Then $H(F) \cap A$ is a hyperplane of A. If F is minimal under inclusion with respect to $H(F)$ intersecting $P(M)$ in a given facet of $P(M)$, then we say that F is a *flacet* of M. If, in addition, $H(F) \cap \Delta(r, n)$ spans a split hyperplane of the hypersimplex $\Delta(r, n)$, then we say that F is a *split flacet* of M.

Say that two elements in the matroid M are equivalent if they are equal, or if they are contained in a common circuit. Then this is an equivalence relation on $E(M)$, and the equivalence classes are called *connected components*. A matroid is *connected* if it has only one connected component. A matroid is connected if and only if its dual (Definition 2.12) is. The next result is [17, Proposition 2.6].

Proposition 8.1. *Let F be a flat of the connected matroid M. Then F is a flacet of M if and only if both $M|F$ and M/F are connected.*

Definition 8.2 ([28])**.** *Assume that M is a rank-r matroid with $E(M) = \{1, \ldots, n\}$. Let A be the affine subspace spanned by $P(M)$. We use $[0,1]^n$ to denote the closed unit cube. Assume that the following holds for any distinct split flacets, F_1 and F_2, of M: no point in $F_1 \cap F_2$ is in the interior of $A \cap [0,1]^n$. Then we say that M is a* split matroid.

Joswig and Schröter prove that every sparse paving matroid is a split matroid, so it is possible that asymptotically every matroid is split. The following property is proved in [28, Proposition 44]. For the sake of completeness of this thesis, we give an alternative proof of this later (Proposition 8.9).

Proposition 8.3. *The class of split matroids is closed under duality and under taking minors.*

Therefore we can reasonably ask what the excluded minors are for the class of split matroids. Joswig and Schröter identify five excluded minors. The main result of this paper shows that that their list of excluded minors is complete. Figure 8.1 shows geometric representations of four connected rank-3 matroids, each with six elements. Note that $S_1^* \cong S_2$, whereas S_3 and S_4 are both self-dual matroids. In addition, S_0 is constructed from the direct sum $U_{2,3} \oplus U_{2,3}$ by adding one parallel point to each of the two connected components.

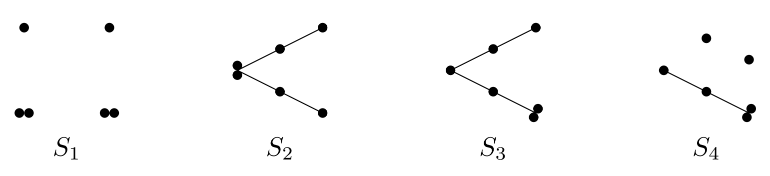


Figure 8.1: Connected excluded minors for split matroids.

Theorem 8.4. *The excluded minors for the class of split matroids are S_0, S_1, S_2, S_3, and S_4.*

In order to prove this theorem, we rely on Joswig and Schröter's equivalent formulation of Definition 8.2 that relies entirely on matroidal structural concepts.

We say that a flat, F, of the matroid, M, is *proper* if $0 < r(F) < r(M)$. A set is *cyclic* if it is a union of circuits or it is the empty set. We first note that the following result (which combines Lemma 10 and Proposition 15 of [28]) means that we need only concern ourselves with characterising connected split matroids.

Proposition 8.5. *Let M be a disconnected matroid, with connected components $C_1, \ldots, C_t$. Then M is a split matroid if and only if each connected matroid, $M|C_i$, is a split matroid, and at most one of these matroids is non-uniform.*

Definition 8.6. *Let M be a matroid, and let Z be a proper cyclic flat of M. If both $M|Z$ and M/Z are connected matroids, but at least one of them is a non-uniform matroid, then we say that Z is a* certificate *for* non-splitting.

Lemma 8.7. *A connected matroid is a split matroid if and only if it has no certificate for non-splitting.*

Proof. This will follow immediately from Theorem 11 in [28] provided that we can demonstrate that the flat Z is a split flacet if and only if it is a proper cyclic flat such that $M|Z$ and M/Z are connected.

Assume that Z is a proper cyclic flat of M such that $M|Z$ and M/Z are connected. Then Z is a flacet by Proposition 8.1. Furthermore $0 < r(Z) < |Z|$, since Z is a proper flat and is not independent. As Z and $E(M) - Z$ are non-empty, we can find an element in $E(M) - Z$ that is not a coloop, since M is connected. Now Lemma 6 of [28] implies that Z is a split flacet.

For the converse, we let Z be a split flacet. Then $M|Z$ and M/Z are connected by Proposition 8.1. Assume $|Z| \leq 1$. Now Proposition 4 of [28] asserts that there must be a positive integer, μ, which satisfies both $\mu < r(M)$ and $\mu > r(M) - |Z| \geq r(M) - 1$. Since this is impossible, $|Z| \geq 2$, so Z is a cyclic flat by Proposition 13 in [28]. It remains only to show that Z is a proper flat. If not, then $Z = E(M)$, as $|Z| \geq 2$. But every point, x, in $P(M)$ satisfies $\sum x_i \leq r(M)$, which means that $P(M)$ is contained in the hyperplane

$H(E(M))$, so this hyperplane does not intersect $P(M)$ in a facet. This completes the proof. $\square$

8.2 Proof of the main theorem

We discuss some preliminaries: Let M be a matroid on the ground set E, and let U be a subset of E. Recall that $\lambda(U)$ is defined to be $r(U) + r(E - U) - r(M)$. This is equal to $r(U)+r^*(U)-|U|$. A *k-separation* is a partition, (U, V), of E, such that $|U|, |V| \geq k$, and $\lambda(U) < k$. A matroid is *n-connected* if it has no k-separation with $k < n$. A matroid is *connected* if it is 2-connected (equivalently, if every pair of distinct elements is contained in a circuit). We refer to a 1-separation as a *separation*. We make use of the fact that if M is a connected matroid, and $e \in E(M)$, then either $M\backslash e$ or M/e is connected [33, Theorem 4.3.1]. In addition, if a single-element extension of a connected matroid by the element e is not connected, then e is a loop or a coloop in the extension [33, Proposition 8.2.7].

Lemma 8.8. *Let Z be a proper cyclic flat of the connected matroid M. Then $E(M) - Z$ is a proper cyclic flat of M^*.*

Proof. Let E be the ground set of M. The fact that $E - Z$ is a cyclic flat of M^* is well-known and easy to verify. Suppose it is not proper, that is $r^*(E - Z) = r(M^*)$ or $r^*(E - Z) = 0$. First, consider the case where $r^*(E - Z) = r(M^*) = |E| - r(M)$. Then the corank function gives

$$|E| - r(M) = r(Z) + |E - Z| - r(M),$$

meaning $r(Z) = |Z|$, so Z is an independent set. The only set that is cyclic and independent is the empty set, and this is impossible, as Z is a proper flat. Now suppose $r^*(E - Z) = 0$. As Z is a proper flat it cannot be equal to E. Therefore $E - Z$ contains an element, and this element is a coloop. The only connected matroid with a coloop has a ground set of size one, but this is impossible since Z and $E - Z$ are both non-empty. □

In order for an excluded-minor characterisation to make sense, we require the class to be closed under taking minors. This is proved in an alternative way by Joswig and Schröter in [28].

Proposition 8.9. *The class of split matroids is closed under minors and duality.*

Proof. First we will consider duality. Note F is a cyclic flat of M if and only if $E - F$ is a cyclic flat of M^*, and that M is connected if and only if M^* is connected. First suppose M is connected. Let Z be a proper cyclic flat of M. Then $M|Z$ and M/Z are both connected and uniform. We have that $(M|Z)^* = (M\backslash(E - Z))^* = M^*/(E - Z)$, which must be connected, as must be $(M/Z)^* = M^*\backslash Z = M^*|(E - Z)$. Furthermore, both $M^*/(E - Z)$ and $M^*|(E - Z)$ are uniform as being uniform is a dual-closed property. As proper cyclic flats of M^* are in one-to-one correspondence with those of M, this proves that M^* has no certificate for non-splitting if and only if M does not. Now let M be disconnected. If $M = N_1 \oplus \cdots \oplus N_t$, then $M^* = N_1^* \oplus \cdots \oplus N_t^*$. As M is split, each N_i is split, and we have just shown that duals of connected split matroids are themselves split. We also have that uniform matroids are closed under duality, and so M^* is split.

We will now show that the class of split matroids is minor-closed. Let M be a split matroid. Since we have proved that the class is closed under duality, we need only prove that $M\backslash e$ is a split matroid, for every element $e \in E(M)$. In the first case, we will assume that M is connected.

Assume that $M\backslash e$ is connected. Assume also that $M\backslash e$ has a certificate Z, so $(M\backslash e)|Z = (M|Z)\backslash e$ and $(M\backslash e)/Z = (M/Z)\backslash e$ are connected, and at least one of these two matroids is non-uniform. Then either $M|Z$ or M/Z is non-uniform as being uniform is a minor-closed property. As M is connected and Z is a cyclic flat, $M|Z$ must be connected and loop-free, and so $(M|Z)\backslash e$ is also connected. Suppose M/Z contains a loop, x. Then the rank function of M/Z implies that $r_M(x \cup Z) = r_M(Z)$, so $x \in \mathrm{cl}(Z)$, contradicting Z being a flat. Therefore $(M/Z)\backslash e$ is also connected. Thus Z is a certificate for M, contradicting M being a split matroid, so long as Z is a flat of M. That is, we need that $e \notin \mathrm{cl}_M(Z)$. If not, then $Z \cup e$ is a flat of M, and a certificate for M: we have that $(M\backslash e)|(Z \cup e) = (M\backslash e)|Z$ is connected, and so $M|(Z \cup e)$ is connected as this has no loops. We also have that $(M\backslash e)/(Z \cup e) = M/Z$. which must be connected as $(M/Z)\backslash e$ was, and (M/Z) has no loops. Again, one of $(M\backslash e)|Z$, and $(M\backslash e)/Z$ and thus $M|(Z \cup e)$ must be non-uniform, and so M is not split, a contradiction.

Now we will assume that $M\backslash e$ is not connected. Assume also that $M\backslash e$ is not split. Then there is a connected component, Z, of $M\backslash e$, such that $M|Z$ is not uniform, for otherwise $M\backslash e$ is split. By definition, $M|Z$ is connected. Thus it is a union of circuits in M. Assume that Z is not a flat, so that there is an element, $x \notin Z$, in the closure of Z. Let C be a circuit contained in $Z \cup x$ that contains x. Since Z is a component of $M\backslash e$, it follows that $x = e$. However, this implies that $(Z \cup e, E(M) - (Z \cup e))$ is a separation in M, a contradiction as M is connected. Therefore we can conclude that Z is a cyclic flat in M. Since $M|Z$ is non-uniform, all we need now do is prove that M/Z is connected. This will establish that Z is a certificate for non-splitting in M, a contradiction.

Assume that (U, V) is a separation of M/Z. We can assume that e is in U. Let v be an element of V, and let Y be the connected component of $M\backslash e$ that contains v. Any pair of elements in Y are contained in a common circuit of $M\backslash e$. As Z is a component of $M\backslash e$, it follows that any pair of elements in Y also share a common circuit in $(M\backslash e)/Z$, and hence in M/Z. This implies that no separation of M/Z can partition Y. Hence $Y \subseteq V$. Note that $(E(M) - (Y \cup e), Y)$ is a separation of $M\backslash e$. If e were in the closure of $E(M) - (Y \cup e)$, then $(E(M) - Y, Y)$ would be a separation of M, which is impossible. Therefore e is not in the closure of $E(M) - (Y \cup e)$, which means that it is in the coclosure of Y. Let C^* be a cocircuit of M contained in $Y \cup e$ such that e is in C^*. Then C^* is also a cocircuit in M/Z, and now C^* is a cocircuit of M/Z that contains elements of both U and V. This contradicts the fact that (U, V) is a separation of M/Z, so we must conclude that M/Z is connected, and hence M is not a split matroid. Since this is contradictory, we have shown that $M\backslash e$ is a split matroid.

Now we can assume that M is not connected. Let $M = M|X_1 \oplus \cdots \oplus M|X_t$, where $t \geq 2$ and each component is a split matroid. Take $e \in X_1$. Then $M\backslash e = (M|X_1)\backslash e \oplus M|X_2 \oplus \cdots \oplus M|X_t$. Suppose this is indeed disconnected. As $M|X_1$ is connected and split, our previous arguments give us that $(M|X_1)\backslash e$ is also split. Now the only way $M\backslash e$ can fail to be split is if it has two non-uniform components. Suppose $M|X_2$ is non-uniform. Then $M|X_1$ must be uniform, as M is split, implying $(M|X_1)\backslash e$ is uniform as well. Now suppose $M|X_1$ is the non-uniform component in M. Then all the other components are uniform. If deleting e from X_1 divides X_1 into new components, then at most one of these is non-uniform, or else $(M|X_1)\backslash e$ is not split, and this contradicts the previous paragraph. Thus $M\backslash e$ has at most one non-uniform component, and so is split. $\square$

First we note that it is easy to confirm that S_0 is not split, by Proposition 8.5, and in fact is an excluded minor for the class of split matroids. Moreover, the connected matroids S_1, S_2, S_3, and S_4 all contain certificates for non-splitting, and are indeed excluded minors.

We now show that there is only one disconnected excluded minor for split matroids. Recall that S_0 is the matroid constructed from the direct sum, $U_{2,3} \oplus U_{2,3}$, by adding a parallel point to each of the two connected components.

Proposition 8.10. *The only disconnected excluded minor for the class of split matroids is S_0.*

Proof. Suppose M is a disconnected excluded minor. This means M is not a split matroid, but every proper minor of M is. Let the connected components of M be $X_1, \ldots, X_t$, where $t > 1$. Suppose that $M|X_i$ is not split for some i. Choose an element $e \notin X_i$. Then, as deletion distributes over direct sums, $M|X_i$ is a component of $M\backslash e$. Thus $M\backslash e$ has a non-split component, and is therefore itself not split. This contradiction shows that every component of M is split. If at most one component of M is non-uniform, then M will be split, which is a contradiction. So let $M|X_i$ and $M|X_j$ be non-uniform, where $1 \leq i < j \leq t$. If there is an element $e \notin X_i \cup X_j$, then $M|X_i$ and $M|X_j$ are both non-uniform components of $M\backslash e$, which is a contradiction as $M\backslash e$ is split. So we must have $i = 1$ and $j = t = 2$.

Let e be an arbitrary element in X_1. Adding a loop to a split matroid produces another split matroid. It follows that e is not a loop in M, so $|X_1| > 1$. Since $M\backslash e$ is split and $M|X_2$ is not uniform, $(M|X_1)\backslash e$ must be either connected and uniform, or disconnected with all components uniform. Either deleting or contracting e from $M|X_1$ produces a connected matroid, and by duality, we can assume that $(M|X_1)\backslash e$ is connected, and therefore uniform. Note $r(X_1) > 1$, for otherwise $r(X_1) = 1$, and $M|X_1$ is a rank-one uniform matroid. Let C be a smallest circuit of $M|X_1$ that contains e, and note that C is not spanning, since $M|X_1$ is not uniform but $(M|X_1)\backslash e$ is. Take $c \in C - e$. If $|C| > 2$, then e is not a loop in $(M|X_1)/c$, and hence this matroid is a connected extension of the uniform matroid $((M|X_1)\backslash e)/c$. It is also non-uniform, since $C - c$ is a non-spanning circuit. Thus M/c contains two non-uniform components: $(M/c)|(X_1 - c)$ and $(M/c)|X_2$. Therefore M/c is not split and we have a contradiction. Hence $|C| = 2$. Let x be an element in $X_1 - C$. Then $(M|X_1)/x$ is a parallel extension of a uniform matroid with rank $r(X_1) - 1 \geq 1$. Since this matroid must be uniform, we conclude it actually has rank one. Thus $r(X_1) = 2$, so $M|X_1$ is a parallel extension of a rank-2 uniform matroid. If $|X_1| > 4$, then we can let x be an element not in the parallel pair, and $(M|X_1)\backslash x$ is connected and non-uniform. Thus $M|X_1$ is a parallel extension of $U_{2,3}$. Note that $M|X_1$ is self-dual. Now symmetric arguments show that $M|X_2$ is also $U_{2,3}$ plus a parallel point, and so M is isomorphic to S_0. □

Lemma 8.11. *Let (U, V) be a 2-separation in the connected matroid, M, and assume that there is no parallel pair contained in U. Then there is an element, $u \in U$, such that M/u is connected.*

Proof. Assume that the lemma fails, and that we have chosen a counterexample with $|U|$ as small as possible. We first note that if U contains a series pair, $\{u, v\}$, then $M\backslash u$

is not connected, as v is a coloop in this matroid. This implies that M/u is connected, contrary to hypothesis. Thus U does not contain a series pair. As $|U| \geq 2$, and U contains no parallel pairs, we see that $r(U) > 1$. Therefore $r(V) < r(M)$.

Assume that $|U| \leq 3$, which implies that $|U| = 2$ or $|U| = 3$, as (U, V) is a 2-separation. Note that $\lambda(U) = 1$ implies $r(U) + r^*(U) \leq 4$, which is possible only if $r(U) = r^*(U) = 2$, as U contains no parallel pair and no series pair. If $|U| = 2 = r^*(U)$, then U is coindependent, so V contains a basis, contradicting the earlier conclusion that V is not spanning. Hence $|U| = 3$, and U is both a circuit and a cocircuit. Let u be an element of U. Then M/u has a separation, (X, Y), by hypothesis. Since $U - u$ is a circuit in M/u, we can assume that $U - u \subseteq X$. But since U is a cocircuit of M, it follows that $r^*(U) = r^*(U \cup u)$, and now it is easy to verify that $(X \cup u, Y)$ is a separation of M, a contradiction. Therefore $|U| \geq 4$.

As V does not span M, we can chose an arbitrary element, u, in $U - \mathrm{cl}(V)$. If u is in a parallel pair with the element z, then by hypothesis, z is in V, implying u is in the span of V, contrary to our choice. Hence M/u contains no loops. By assumption, M/u is not connected. Let (X, Y) be a separation of M/u. Since M/u has no loops, it follows that $|X| \geq 2$ and $|Y| \geq 2$. Now standard rank calculations show that both $(X \cup u, Y)$ and $(X, Y \cup u)$ are 2-separations of M. Since $|U - u| \geq 3$, we can assume without loss of generality that $|U \cap X| \geq 2$.

The submodularity of the connectivity function [33, Lemma 8.2.9] implies that $\lambda(U \cap X) + \lambda(U \cup X) \leq \lambda(U) + \lambda(X) = 2$. Assume that $\lambda(U \cap X) \leq 1$. Then it is clear that $(U \cap X, V \cup Y \cup u)$ is a 2-separation of M, and as $U \cap X$ contains no parallel pairs, we contradict the minimality of U, since $U \cap X \subseteq U - u$. It follows that $\lambda(U \cup X) = 0$. As M is connected, this means that $V \cap Y = \emptyset$. But then Y is a proper subset of U, and as (X, Y) is a 2-separation, we again reach a contradiction to our assumption on $|U|$. $\square$

We can now prove the rest of the characterisation.

Theorem 8.4. *The excluded minors for the class of split matroids are S_0, S_1, S_2, S_3,* *and S_4.*

Proof. Let M be a connected excluded minor for the class of split matroids. If M contains a loop, then it is isomorphic to the uniform matroid $U_{0,1}$, and is therefore a split matroid. Hence M is loopless. As M is connected and not split, it contains a certificate, Z, for non-splitting. Both $M|Z$ and M/Z are connected matroids and either $M|Z$ or M/Z is non-uniform. We would like to assume that $M|Z$ is non-uniform, so consider the case when this fails. Then M/Z is non-uniform. We will apply duality. Note that $E(M) - Z$ is a proper cyclic flat of M^* by Lemma 8.8, and that $M^*|(E(M) - Z) = (M/Z)^*$ while $M^*/(E(M) - Z) = (M|Z)^*$. Both of these matroids are connected, and $M^*|(E(M)-Z) = (M/Z)^*$ is non-uniform. Therefore we relabel M^* as M, and $E(M)-Z$ as Z. Now we can assume without loss of generality that $M|Z$ is not uniform. In the following analysis, we should expect to encounter S_2, S_3, and S_4, but not S_1, since it does not possess a proper cyclic flat of this type. Instead, we will encounter its dual, S_2.

Claim 8.12. *Let z be an element in Z such that $(M|Z)/z$ is connected and non-uniform. Then $(Z, E(M) - Z)$ is a 2-separation of M, and $z \in \mathrm{cl}_M(E(M) - Z)$.*

Proof of Claim 8.12. Note that since M is loopless and $M|Z$ is non-uniform, it follows that $r(Z) > 1$. Now it is very easy to confirm that $Z - z$ is a proper cyclic flat of M/z. Moreover, $(M/z)/(Z - z) = M/Z$ is connected, since Z is a certificate of non-splitting in M. We have assumed that $(M/z)|(Z - z) = (M|Z)/z$ is connected. Furthermore, $(M/z)|(Z-z)$ is not uniform by assumption. Thus $Z-z$ is a certificate for non-splitting in M/z. If M/z is connected, then this implies that M/z is not a split matroid, which is impossible as M is an excluded minor for the class of split matroids. Therefore we let (U, V) be a separation in M/z.

If both U and V contain elements of Z, then $(U \cap Z, V \cap Z)$ is a separation of $(M/z)|(Z - z) = (M|Z)/z$, and we have assumed this matroid is connected. Therefore we can assume without loss of generality that $Z - z \subseteq U$. If both U and V contain elements of $E(M) - Z$, then $(U - Z, V - Z)$ is a separation of the connected matroid

$(M/z)/(Z-z) = M/Z$. Therefore we must have $V = E(M) - Z$, and $U = Z - z$. As M is connected, (U, V) is not a separation of M, and standard rank calculations show that $(U \cup z, V)$ is a 2-separation in M satisfying $z \in \mathrm{cl}_M(V)$. This is exactly what we set out to prove. ■

Claim 8.13. *Let z_1 and z_2 be distinct elements in Z such that $(M|Z)/z_1$ and $(M|Z)/z_2$ are both non-uniform. Then at most one of $(M|Z)/z_1$ and $(M|Z)/z_2$ is connected.*

Proof of Claim 8.13. Assume that both $(M|Z)/z_1$ and $(M|Z)/z_2$ are connected. Then Claim 8.12 implies that $(Z, E(M) - Z)$ is a 2-separation of M, and both z_1 and z_2 are in $\mathrm{cl}_M(E(M) - Z)$. This means that $r((E(M) - Z) \cup \{z_1, z_2\}) = r(E(M) - Z)$, and as $(Z, E(M) - Z)$ is a 2-separation, we can use the submodularity of the rank function to establish that

$$\begin{aligned} r(\{z_1, z_2\}) \leq r(Z) + r((E(M) - Z) \cup \{z_1, z_2\}) - r(E(M)) \\ = r(Z) + r(E(M) - Z) - r(M) = 1. \end{aligned}$$

Because M has no loops, this implies that $\{z_1, z_2\}$ is a parallel pair of M. Thus z_2 is a loop in $(M|Z)/z_1$. Since this matroid is connected, it must consist of the single loop, z_2. Therefore $Z = \{z_1, z_2\}$. This implies that $M|Z$ is isomorphic to the uniform matroid $U_{1,2}$, which is impossible as we have assumed that $M|Z$ is non-uniform. ■

Claim 8.14. *Z contains a parallel pair.*

Proof of Claim 8.14. We assume otherwise. Since $M|Z$ is not uniform, it contains a non-spanning circuit C. Let z be an arbitrary element in C. Then $C - z$ is a non-spanning circuit in $(M/z)|(Z - z) = (M|Z)/z$, so this matroid is non-uniform. Choose distinct elements z and z' from C. From Claim 8.13 we see that at most one of $(M|Z)/z$ and $(M|Z)/z'$ is connected. Without loss of generality, we assume that $(M|Z)/z$ has a separation, (U, V). By assumption, z is not in a parallel pair in $M|Z$. Therefore $(M|Z)/z$ contains no loops. This implies that $|U| \geq 2$ and $|V| \geq 2$. Since $M|Z$ is connected, we deduce that both $(U, V \cup z)$ and $(U \cup z, V)$ are 2-separations of $M|Z$. As neither U nor V contains a parallel pair, we can apply Lemma 8.11, and deduce that there are elements $z_1 \in U$ and $z_2 \in V$ such that $(M|Z)/z_1$ and $(M|Z)/z_2$ are connected. If both $(M|Z)/z_1$ and $(M|Z)/z_2$ are non-uniform, then we have a violation of Claim 8.13. Therefore we can assume without loss of generality that $(M|Z)/z_1$ is uniform.

It cannot be the case that z_1 is a coloop of $(M|Z)/z$, for then it would be a coloop in the connected matroid $M|Z$. Therefore we let C' be a circuit of $(M|Z)/z$ that contains z_1. Since (U, V) is a separation, it follows that $C' \subseteq U$. There is a circuit, C, of $M|Z$ such that C is equal to either C' or $C' \cup z$. Then $C - z_1$ is a circuit of the uniform matroid $(M|Z)/z_1$, so $C - z_1$ spans $(M|Z)/z_1$. Thus C is a spanning circuit of $M|Z$ that is contained in $U \cup z$. Therefore $(U \cup z, V)$ is a 2-separation of $M|Z$ satisfying $r(U \cup z) = r(M|Z)$. This implies that $r(V) = 1$. But $|V| \geq 2$, so $V \subseteq Z$ contains a parallel pair and we have a contradiction. ■

Henceforth we let $\{x, y\}$ be a parallel pair contained in Z.

Claim 8.15. *$\{x, y\}$ is the only parallel pair in M.*

Proof of Claim 8.15. Assume that $\{a, b\}$ is a parallel pair not equal to $\{x, y\}$. Without loss of generality, we can assume that $a \notin \{x, y\}$. It is an easy exercise to show that deleting an element from a parallel pair does not disconnect a connected matroid. Therefore $M\backslash a$ is connected.

In the first case, assume that a, and hence b, is in Z. Then $Z - a$ is a proper cyclic flat of $M\backslash a$, and $(M\backslash a)|(Z - a)$ is connected. As a is in the span of $Z - a$, it is a loop in $M/(Z - a)$. Therefore

$$M/Z = M/(Z - a)/a = M/(Z - a)\backslash a = (M\backslash a)/(Z - a)$$

so $(M\backslash a)/(Z - a)$ is also connected. Furthermore $(M\backslash a)|(Z - a)$ is non-uniform, since $r(Z - a) = r(Z) > r(\{x, y\})$, so $\{x, y\}$ is a non-spanning circuit in $(M\backslash a)|(Z - a)$.

Therefore $Z - a$ is a certificate for non-splitting in the connected matroid $M\backslash a$, and we have a contradiction as $M\backslash a$ is a proper minor of M.

For the second case, we assume that a is not in Z. Therefore Z is a proper cyclic flat of $M\backslash a$, and $(M\backslash a)|Z = M|Z$ is connected and non-uniform. We know that M/Z is connected, and $\{a, b\}$ is a parallel pair in this matroid, since neither a nor b is in the span of Z. Therefore $(M\backslash a)/Z = (M/Z)\backslash a$ is obtained from a connected matroid by deleting an element from a parallel pair, and is hence connected. Thus Z is a certificate for non-splitting in the connected matroid $M\backslash a$, and we again have a contradiction. ■

Claim 8.16. $r(Z) = 2$

Proof of Claim 8.16. Assume that $r(Z) > 2$. Let z be an arbitrary element in $Z - \{x, y\}$. Then $(M|Z)/z$ is non-uniform, because it has rank at least two, but it also contains a parallel pair. From Claim 8.13 we deduce that if z and z' are distinct elements in $Z - \{x, y\}$ then at most one of $(M|Z)/z$ and $(M|Z)/z'$ is connected. Thus we choose distinct elements z and z' in $Z - \{x, y\}$, and without loss of generality, we assume that $(M|Z)/z$ has a separation (U, V). Since z is not in a parallel pair, it follows that $(M|Z)/z$ has no loops, so $|U| \geq 2$ and $|V| \geq 2$. We deduce that $(U, V \cup z)$ and $(U \cup z, V)$ are 2-separations in $(M|Z)/z$. Since $\{x, y\}$ is a circuit in $(M|Z)/z$, and (U, V) is a separation in this matroid, we relabel as necessary and assume that $x, y \in V$. Therefore U contains no parallel pair of $M|Z$, so we can apply Lemma 8.11 to $(U, V \cup z)$ and deduce that there is an element $z_1 \in U$ such that $(M|Z)/z_1$ is connected. The earlier conclusion shows that if w is an element in $Z - \{x, y, z_1\}$, then $(M|Z)/w$ is not connected.

Let w be an arbitrary element in $Z - \{x, y, z_1\}$ and let (U_w, V_w) be a separation of $(M|Z)/w$. As $(M|Z)/w$ has no loops, we deduce that $|U_w| \geq 2$ and $|V_w| \geq 2$, and both $(U_w, V_w \cup w)$ and $(U_w \cup w, V_w)$ are 2-separations of $M|Z$. Without loss of generality, we assume that z_1 is in V_w. We claim that $x, y \in U_w$. If this is not the case, then U_w contains no parallel pair of $M|Z$. Therefore we can apply Lemma 8.11 to $(U_w, V_w \cup w)$ and deduce that there is an element $u \in U_w \subseteq Z - \{x, y, z_1\}$ such that $(M|Z)/u$ is connected. This contradicts an earlier conclusion, so $x, y \in U_w$, as claimed. Assume that we have chosen w from $Z - \{x, y, z_1\}$ in such a way that $|U_w|$ is as small as possible.

If w is not in the closure of U_w in $M|Z$, then it is in the coclosure of V_w. This implies that $(U_w, V_w \cup w)$ is a separation in the connected matroid $M|Z$. Therefore w is in the closure of U_w. The same argument shows that w is in the closure of V_w. Let $C \subseteq U_w \cup w$ be a circuit of $M|Z$ that contains w. Then $C \nsubseteq \{x, y, w\}$, as $\{x, y\}$ is a circuit in $M|Z$, and w is not parallel to x or y. Let t be an element in $C - \{x, y, w\}$, so that t belongs to U_w, and also to $Z - \{x, y, z_1\}$. Therefore $(M|Z)/t$ has a separation (U_t, V_t) where $z_1 \in V_t$ and $x, y \in U_t$. As before, we see that $|U_t| \geq 2$ and $|V_t| \geq 2$, and both $(U_t, V_t \cup t)$ and $(U_t \cup t, V_t)$ are 2-separations of $M|Z$, where t is in the closure of both U_t and V_t.

Let λ be the connectivity function of $M|Z$. Then λ is submodular, so

$$\lambda(U_w \cap U_t) + \lambda(U_w \cup U_t) \leq \lambda(U_w) + \lambda(U_t) = 2.$$

Neither $U_w \cap U_t$ nor $E(M) - (U_w \cup U_t)$ is empty (the former contains x and y, the latter contains z_1). Therefore neither $\lambda(U_w \cap U_t)$ nor $\lambda(U_w \cup U_t)$ is zero. We deduce that $\lambda(U_w \cap U_t) = 1$. We can apply the same argument to $\lambda(U_w \cap (U_t \cup t)) + \lambda(U_w \cup (U_t \cup t))$ and deduce that $\lambda(U_w \cap (U_t \cup t)) = 1$. Since t is in the closure of V_t it follows that

$$r(Z - (U_w \cap U_t)) = r(Z - (U_w \cap (U_t \cup t))).$$

From $\lambda(U_w \cap U_t) = \lambda(U_w \cap (U_t \cup t))$ we can deduce that $r(U_w \cap U_t) = r(U_w \cap (U_t \cup t))$ and therefore t is in the closure of $U_w \cap U_t$. We have now shown that $(U_w \cap U_t, Z - (U_w \cap U_t))$ is a 2-separation of $M|Z$, and that t is in the closure of both sides. Standard rank calculations now show that $(U_w \cap U_t, Z - (U_w \cap (U_t \cup t)))$ is a separation of $(M|Z)/t$. Note that $U_w \cap U_t$ contains $\{x, y\}$. But $U_w \cap U_t$ does not contain $t \in U_w$, so $|U_w \cap U_t| < |U_w|$, and we have a contradiction to our choice of w. This contradiction completes the proof of Claim 8.16. ■

Claim 8.17. $|Z| = 4$

Proof of Claim 8.17. We now know that Z is a rank-two cyclic flat containing a parallel pair, $\{x, y\}$. Therefore $|Z| > 2$. If $|Z| = 3$ then the element in $Z - \{x, y\}$ would be a coloop of $M|Z$, and thus Z would not be cyclic.

Suppose that $|Z| > 4$, and let a, b, and c be elements of $Z-\{x, y\}$. Then $M|\{a, b, c, x\}$ is isomorphic to $U_{2,4}$. Assume that $M\backslash a$ has a separation, (U, V). Without loss of generality, U contains two elements of $\{b, c, x\}$. Then a is in the closure of U, so $(U \cup a, V)$ is a separation of M, a contradiction. Therefore $M\backslash a$ is connected. Exactly the same argument shows that $(M|Z)\backslash a = (M\backslash a)|(Z - a)$ is connected. We also know that $(M\backslash a)/(Z-a) = M/Z$ is connected. It is clear that $Z-a$ is a proper cyclic flat of $M\backslash a$. Furthermore $(M\backslash a)|(Z - a)$ is not uniform, as it has rank two and contains a parallel pair. Therefore $Z - a$ is a certificate for non-splitting in the connected matroid $M\backslash a$. This is a contradiction, so the proof of the claim is complete. ■

Claim 8.18. *If $r(M) = 3$ then M is isomorphic to S_2, S_3, or S_4.*

Proof of Claim 8.18. Assume that $r(M) = 3$. Let w be an arbitrary element not in Z. Note that $(M\backslash w)|Z = M|Z$ is connected and non-uniform. Also, $(M\backslash w)/Z$ is a rank-1 matroid. It is loopless because Z is a flat. Hence it is connected. Since Z is a proper cyclic flat of $M\backslash w$, it is a certificate for $M\backslash w$. Since $M\backslash w$ is a split matroid, $M\backslash w$ cannot be connected. Note that $M\backslash w$ contains no parallel pair other than $\{x, y\}$, by Claim 8.15. Since $M\backslash w$ has rank three and is not connected, it now follows that it is equal to the direct sum of $M|Z$ with a coloop, w'. Hence $\{w, w'\}$ is a series pair in M, and M contains six elements. When we contract w, the element w' is projected onto the line spanned by Z. Thus, in M/w, the element w' is in a parallel class of size three, two, or one. These cases correspond to M being isomorphic to S_2, S_3, or S_4, respectively. ■

Henceforth we assume that $r(M) > 3$.

Claim 8.19. *If $w \notin Z$, then $(M/Z)/w$ is not connected.*

Proof of Claim 8.19. Assume otherwise, so that $(M/Z)/w$ is connected. Assume that Z is not a flat in M/w, and let x be an element in $\mathrm{cl}_{M/w}(Z) - Z$. Then x is a loop of $(M/Z)/w$, and since this matroid is connected, it follows that it contains a single element, x. Thus x and w are the only elements in $E(M) - Z$. As $r(M) \geq r(Z) + 2$, both x and w are coloops, a contradiction. Therefore Z is a proper cyclic flat of M/w. Moreover, $(M/w)|Z = M|Z$ is connected and non-uniform and $(M/w)/Z$ is connected by assumption. Hence Z is a certificate for M/w. As M/w is a split matroid, it cannot be connected. Let (U, V) be a separation of M/w.

As $(M/w)|Z$ is connected, we can assume that $Z \subseteq U$. In fact, $U = Z$, for otherwise $(U - Z, V)$ is a separation of $(M/w)/Z$, and this matroid is connected. Let C be a circuit of M that contains an element of Z and an element of V. Then $w \notin C$, for otherwise $C - w$ is a circuit in M/w that contains elements in U and V. As C is not a circuit of M/w, it is a union of at least two circuits. Let z be an element in $Z \cap C$, and let $D \subseteq C$ be a circuit of M/w that contains z. Thus D is contained in $U = Z$. Note that D is not a circuit of M, for it is properly contained in C. Therefore $D \cup w$ is a circuit of M. This implies that $w \in \mathrm{cl}(Z)$, which is impossible as Z is a flat. ■

Claim 8.20. *If $w \notin Z$, then $M \backslash w$ is not connected.*

Proof of Claim 8.20. Because M/Z is connected, but $(M/Z)/w$ is not by Claim 8.19, it follows that $(M/Z)\backslash w$ is connected. Certainly Z is a proper cyclic flat in $M \backslash w$. Also $(M\backslash w)|Z = M|Z$ is connected and non-uniform, and $(M\backslash w)/Z = (M/Z)\backslash w$ is connected. Thus Z is a certificate for non-splitting in $M\backslash w$. As $M\backslash w$ is a split matroid, it cannot be connected. ■

Now, continuing the proof of Theorem 8.4 we choose an element $w \notin Z$, and let $X_1, \ldots, X_t$ be the connected components of $M\backslash w$. Since $(M|Z)\backslash w = M|Z$ is connected we can assume that $Z \subseteq X_1$. Because M/Z is connected, but $(M/Z)/w$ is not, by Claim 8.19, we see that $(M/Z)\backslash w = (M\backslash w)/Z$ is connected. The only way that this can occur is if $X_1 = Z$, and $t = 2$. Since $M\backslash w$ is a split matroid, and $M|Z = M|X_1$ is not uniform, we deduce that $M|X_2$ is uniform. As $r(Z) = 2$ and $r(M) > 3$, the rank of $M|X_2$ is at least two. Now M/Z is connected and is an extension of the uniform matroid $M|X_2$ by the element w. The rank of M/Z is at least two. Thus we can choose $w' \in M/Z$ that is not equal or parallel to w. Therefore $(M/Z)/w'$ is an extension of a uniform matroid by the element w and $r((M/Z)/w') \geq 1$. As w is not a loop or coloop in $(M/Z)/w'$, it follows that $(M/Z)/w'$ is connected, contradicting Claim 8.19. Thus we have a final contradiction that completes the proof of Theorem 8.4. $\square$

Part IV

A splitter theorem for connected clutters

Chapter 9

Clutters

This chapter has been submitted for publication under the same name, and was joint work with Dillon Mayhew.

9.1 Introduction

A *clutter* is a pair $(E, \mathcal{A})$, where E is a finite set, and $\mathcal{A}$ is a collection of subsets of E, with the property that if A and A' are distinct members of $\mathcal{A}$, then $A \nsubseteq A'$. We refer to E as the *ground set* of the clutter, and we call members of $\mathcal{A}$ *rows* of the clutter. In the literature, elements of the ground set are often referred to as vertices, while rows are called edges. Since we will later represent rows of a clutter by vertices in a graph, we prefer to avoid this terminology. If M is a clutter, then $E(M)$ denotes its ground set. Clutters are also referred to as *antichains* and *Sperner families*.

For an example of a clutter, we may take the rows to be the circuits of a matroid, or the set of bases of a matroid. Thus clutters are natural generalisations of matroids: they lie somewhere on the spectrum between matroids, and completely general hypergraphs. It may seem as though clutters are significantly more general objects than matroids, but there are some reasons to view them as being closer to the matroid end of the spectrum.

In particular, there are notions of deletion and contraction for clutters. If $M = (E, \mathcal{A})$ is a clutter, and v is an element of E, we define $M\backslash v$ to be the clutter

$$(E - v, \{A \in \mathcal{A} \colon v \notin A\})$$

and we let M/v be the clutter on the set $E - v$ whose rows are the sets in $\{A - v \colon A \in \mathcal{A}\}$ that are minimal under subset-inclusion. Note here we include the sets A which do not contain v. We say that $M\backslash v$ and M/v are produced by *deleting* and *contracting* v respectively. These clutter operations extend the matroidal operations: if M is the clutter of circuits in the matroid N, then $M\backslash v$ and M/v are the clutters of circuits in the matroids $N\backslash v$ and N/v. Any clutter produced from M by a (possibly empty) sequence of deletions and contractions is a *minor* of M. A minor produced by a non-empty sequence of deletions and contractions is a *proper minor*.

The following result is in [10], and shows that the order of deletion and contraction is immaterial.

Proposition 9.1. *Let $M = (E, \mathcal{A})$ be a clutter, and let v and v' be elements of E. Then (i) $(M\backslash v)\backslash v' = (M\backslash v')\backslash v$, (ii) $(M/v)/v' = (M/v')/v$, and (iii) $(M\backslash v)/v' = (M/v')\backslash v$.*

Clutters, moreover, have a duality involution that is analogous to matroid duality. If $M = (E, \mathcal{A})$ is a clutter, then the *blocker* of M, written $b(M)$, has E as its ground set, and its rows are the minimal subsets of E that have non-empty intersection with each row of M. Edmonds and Fulkerson [16] proved that $b(b(M)) = M$. This involution swaps deletion and contraction, just as matroid duality does. Thus $b(M\backslash v) = b(M)/v$ and $b(M/v) = b(M)\backslash v$.

In this chapter we present evidence that pushes clutters further in the matroid direction along the matroid-hypergraph continuum. We show that some connectivity behaviour in matroids is actually just a special case of a clutter phenomenon. To do so, we must develop a notion of connectivity for clutters.

Definition 9.2. *Let $M = (E, \mathcal{A})$ be a clutter. A* separation *of M is a partition of E into* *non-empty parts, X and Y, such that every row is contained in X or Y. If M admits no separation then it is* connected.

This is a natural way to define connectivity for clutters, since it generalises connectivity for graphs and for matroids. If $M = (E, \mathcal{A})$ is a clutter and each row has cardinality two, then M can be identified with a simple graph G, with vertex set E, whose edges are the rows of M. In this case, M is connected if and only if G is. Similarly, if the rows of M are the circuits of a matroid, N, then separations of M and N exactly coincide. Therefore M is connected if and only if N is.

We would like to know which inductive properties of matroid connectivity extend to connected clutters. Our first observation is a negative one. If N is a connected matroid, and e is an element of its ground set, then either $N \backslash e$ or N/e is a connected matroid [33, Theorem 4.3.1]. This phenomenon does not extend to clutters. To see this, consider a clutter, M, whose edges all have cardinality two, and therefore correspond to the edges of a graph, G. Assume v is a cut-vertex in G. Then $M \backslash v$ corresponds to the graph produced from G by deleting v and all edges incident with it. This is certainly not a connected clutter. On the other hand, M/v is produced by removing v, all rows containing v, all rows containing a neighbour of v in G, and then adding all such neighbours as singleton rows. It is clear that this clutter will also fail to be connected.

On the other hand, our main theorem is positive. Brylawski [8] and Seymour [39] independently proved that if M is a connected matroid with a connected proper minor, N, then we can delete or contract an element from M in such a way to preserve connectivity, and the minor N. We prove that this is a special case of a clutter phenomenon.

Theorem 9.3. *Let M and N be connected clutters and assume that N is a proper minor of M. There exists an element, $v \in E(M)$, such that either $M \backslash v$ or M/v is connected and has N as a minor.*

This type of theorem is known as a *splitter theorem*, after Seymour's well-known splitter theorem for 3-connected matroids [40]. We obtain, as a corollary, a weaker type of statement, known as a *chain theorem*.

Corollary 9.4. *Let M be a non-empty connected clutter. Then there is an element, $v \in E(M)$, such that either $M\backslash v$ or M/v is a connected clutter.*

Proof. Since every clutter has the empty clutter as a minor, we simply apply Theorem 9.3 with N equal to the empty clutter. $\square$

We note that our notion of connectivity is not invariant under taking blockers. To see this, let M be a clutter whose rows are the circuits of a matroid, N. Assume that N admits a separation, (X, Y), but that the dual matroid, N^*, has no circuits of size less than three. By an earlier observation, (X, Y) is also a separation of M. The rows of $b(M)$ are the bases of N^* [21]. Assume that (X', Y') is a separation of $b(M)$, and let x and y be elements from X' and Y', respectively. Then no basis of N^* contains both x and y, so N^* contains a circuit of size at most two, contrary to hypothesis. Thus $b(M)$ is a connected clutter, even though M is not.

The main tool we use to prove Theorem 9.3 is the *incidence graph* of a clutter. Let $M = (E, \mathcal{A})$ be a clutter. We use $G(M)$ to denote the incidence graph of M. The vertex set of $G(M)$ is $E \cup \mathcal{A}$. We say that vertices in E are *black* and vertices in $\mathcal{A}$ are *white*. Every edge of $G(M)$ joins a black vertex to a white vertex, so $G(M)$ is bipartite. The vertex $v \in E$ is adjacent to $A \in \mathcal{A}$ in $G(M)$ if and only if v is contained in A.

The incidence graph allows us to study clutter connectivity in graph theoretical terms.

Proposition 9.5. *Let M be a clutter. If $G(M)$ is connected, then M is connected. If M is connected, and is not the clutter with a single element and one, empty, row, then $G(M)$ is connected.*

Proof. Assume that $G(M)$ is not connected. We will prove that either M is not connected, or M is equal to the special clutter described in the statement of the proposition. Let (A, B) be a partition of the vertices of $G(M)$ into non-empty parts, such that no edge joins a vertex in A to a vertex in B. Assume that both A and B contain elements of $E(M)$. Then $(A \cap E(M), B \cap E(M))$ is clearly a separation of M, and M is not connected. Therefore we will assume that A contains no element of $E(M)$. Thus every vertex in A is white. It follows that $G(M)$ has a white vertex that is connected to no black vertex, and that therefore M has an empty row. Since M is a clutter, it follows that M has exactly one, empty, row, and that therefore $G(M)$ contains a single white vertex, and no edges. Note that $E(M)$ is non-empty, for otherwise $G(M)$ contains a single vertex, and moreover is connected. If $E(M)$ contains at least two elements, then we can find a separation of M. Thus we assume that $E(M)$ contains exactly one element, and deduce that M is the clutter described in the proposition.

Now suppose M is not connected and has a separation (A, B). White vertices corresponding to rows in A are incident only with elements of A; white vertices corresponding to rows in B are incident only with elements of B. There are no other white vertices in $G(M)$, so this means that there are no paths between vertices in A and vertices in B. Thus $G(M)$ must be disconnected. □

9.2 Proof of the main theorem

If v is a vertex of a graph, then $\mathrm{Neigh}(v)$ represents the set of neighbours of v (this set excludes v). We say $\mathrm{Neigh}(v)$ is the *open neighbourhood* of v. We write $\overline{\mathrm{Neigh}}(v)$ for the *closed neighbourhood* of v. That is, $\overline{\mathrm{Neigh}}(v) = \mathrm{Neigh}(v) \cup \{v\}$. In order for a bipartite graph with black and white vertices to be the incidence graph of a clutter, it is necessary and sufficient that, if u and v are distinct white vertices of G, $\mathrm{Neigh}(u)$ cannot be a subset of $\mathrm{Neigh}(v)$.

The next result follows immediately from the definition of deletion in clutters.

Proposition 9.6. *If M is a clutter, and v is in $E(M)$, then $G(M\backslash v) = G(M)\backslash\overline{\mathrm{Neigh}}(v)$.*

Clutter contraction is somewhat more complicated to observe in the incidence graph. We will use only one special case of contraction. We say that the black vertices, u and

v, are *twins* if $\mathrm{Neigh}(u) = \mathrm{Neigh}(v)$.

Proposition 9.7. *Let M be a connected clutter. If v and v' are twin black vertices, then $G(M/v) = G(M)\backslash v$ and is therefore connected.*

Proof. We form M/v by removing the element v from $E(M)$ and taking the rows, with v deleted, which are minimal under subset-inclusion. Suppose that $G(M)\backslash v$ has two distinct white vertices, u and w, such that every neighbour of u is a neighbour of w. This property does not hold in $G(M)$, so v must have been adjacent to u but not w. As v' is a twin of v, then v' is also adjacent to u and not w. These adjacencies remain in $G(M)\backslash v$, and so we have $\mathrm{Neigh}(u) \nsubseteq \mathrm{Neigh}(w)$. This shows that in $G(M)\backslash v$, there is no pair of distinct white vertices, one of whose neighbourhood is contained in the other. It follows that $G(M)\backslash v$ is the incidence graph of M/v.

Finally, it is clear that $G(M)\backslash v$ is connected as for every path using v, replacing v with v' gives a second path, and so deleting v cannot increase the number of components in the graph. □

Note this result says that if $M = (E, \mathcal{A})$ is a connected clutter with twins v, v', then $(E, \{A - v \mid A \in \mathcal{A}\}$ is a clutter, with no need for the usual minimality condition.

With this setup, we can immediately begin the proof of the main result.

Theorem 9.8. *Let M and N be connected clutters and assume that N is a proper minor of M. There exists an element, $v \in E(M)$, such that either $M\backslash v$ or M/v is connected and has N as a minor.*

Proof. Assume that M and N form a counterexample to the theorem. We will let G stand for $G(M)$.

Claim 9.9. *There is an element $v \in E(M)$ such that N is a minor of $M\backslash v$.*

Proof of Claim 9.9. Assume that this is not the case. Then N is a minor of M/u for some $u \in E(M)$. Now M/u is not connected, or else M and N would not give us a counterexample. Therefore $G(M/u)$ is disconnected by Proposition 9.5. Since N is connected, it follows easily that $E(N)$ is contained in a connected component of $G(M/u)$. Choose C, a component of $G(M/u)$ such that C does not contain $E(N)$. If C consists of a single white vertex, then M/u has an empty row, and this means that it has exactly one row. Hence $G(M/u)$ contains a single white vertex (and possibly some black vertices) and no edges. This means that N contains at most one element, or else it is not connected. Since M/u is not connected we have that $M/u \neq N$, so $E(M/u)$ must contain an element that is not in $E(N)$. Let u' be such an element. Then u' is an isolated black vertex in $G(M/u)$, so N is a minor of $M/u\backslash u'$ and hence of $M\backslash u'$, contrary to assumption. We must now assume that C contains a black vertex, u'. As C is a component of $G(M/u)$ and C does not contain any element of $E(N)$, we see that N is a minor of $M/u\backslash u'$ and hence of $M\backslash u'$, which is a contradiction. ■

Note that if u and v are black vertices, then $\mathrm{Neigh}(u)$ may be a subset of $\mathrm{Neigh}(v)$. Say that v is a *minimal* black vertex if there is no black vertex, u, such that $\mathrm{Neigh}(v)$ properly contains $\mathrm{Neigh}(u)$.

Claim 9.10. *There is a minimal black vertex, v, of G, such that N is a minor of $M\backslash v$.*

Proof of Claim 9.10. Assume the statement is false. By Claim 9.9, we can choose a non-minimal black vertex v' so that N is a minor of $M\backslash v'$. Let v' have the smallest possible degree. First assume $|E(N)| > 1$, so $G(N)$ is connected by Proposition 9.5. Say $G\backslash\overline{\mathrm{Neigh}}(v')$ has components $C_1, \ldots, C_t$, where $E(N) \subseteq C_1$. As v' is not minimal, we will choose a minimal black vertex v with $\mathrm{Neigh}(v) \subset \mathrm{Neigh}(v')$. Note this implies v is an isolated vertex in $G\backslash\overline{\mathrm{Neigh}}(v')$, and so is one of the components $C_1, \ldots, C_t$. Then v is clearly not in C_1, so N is a minor of $M\backslash v'\backslash v$, and hence of $M\backslash v$, as desired.

Now we consider the case that $|E(N)|$ is at most 1. We will still assume v' is not minimal, so $\mathrm{Neigh}(v)$ is properly contained in $\mathrm{Neigh}(v')$, for some minimal black vertex v. If $v \notin C_1$, then M is a minor of $M\backslash v'\backslash v$, and hence of $M\backslash v$, for the same reasons as in the previous case. Thus $v \in C_1$, implying C_1 is a single vertex as v is isolated

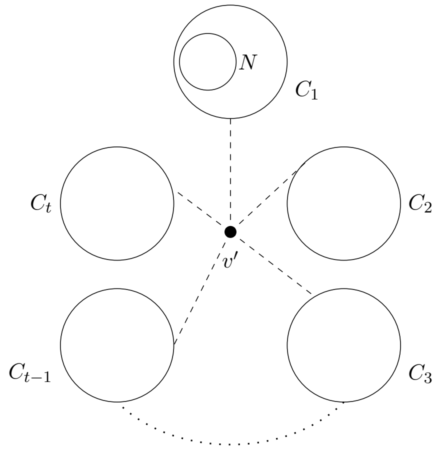


Figure 9.1: Structure of $G(M)$

in $G\backslash\overline{\text{Neigh}}(v')$. If $|E(N)| = 0$, then v can be any minimal black vertex and the result follows as N will be a minor of $M\backslash v$. Thus we assume that $E(N) = \{v\}$. Note that v is isolated after deleting $\overline{\text{Neigh}}(v')$. This means N is the clutter with $E(N) = \{v\}$ and no rows. Choose a black vertex $u' \in C_2$. Then N is a minor of $M\backslash u'$. If u' is a minimal black vertex, the result follows. So let u be a minimal black vertex with $\text{Neigh}(u) \subset \text{Neigh}(u')$. If $u \in C_2$, then N is a minor of $M\backslash u$ and the result also follows. If $u \notin C_2$, then no neighbour of u is in C_2, but any such neighbour is also a neighbour of u'. It follows that all the neighbours of u are also neighbours of v'. This means u is an isolated vertex after deleting $\overline{\text{Neigh}}(v')$, so N is a minor of $M\backslash u$, completing the proof of Claim 9.10. ■

Now fix v to be a minimal black vertex such that N is a minor of $M\backslash v$. Let $C_1, \ldots, C_t$ be the connected components of $G(M\backslash v)$, where $t \geq 2$. Since N is connected, we can assume that $E(N)$ is contained in C_1.

Claim 9.11. *If u is a black vertex which is not in C_1, then u has no twin vertex.*

Proof of Claim 9.11. First suppose that $u = v$, and let u' be a twin of u. We know that N is a minor of $M\backslash u$, and we have that u' is isolated in $G(M\backslash u)$. It follows easily from Propositions 9.5 and 9.7 that M/u' is connected. Moreover, as $G(M/u') = G(M)\backslash u'$, the only way M/u' cannot contain N as a minor is if $E(N) = \{u'\}$. But this would imply that M/v does contain N as a minor, and is connected, and the theorem would follow. Now let $u \neq v$. Assume that u is a black vertex in C_i where $i \neq 1$, and assume that u' is a twin of u. Since u is contained in C_i, a component of $G(M \backslash v)$, and $E(N)$ is contained in C_1, it follows that N is a minor of $M \backslash v/u$, and hence of M/u. But Propositions 9.5 and 9.7 imply that M/u is connected, a contradiction to our counterexample. ■

If u is any minimal black vertex, and C is any connected component of $G \backslash \overline{\mathrm{Neigh}}(u)$, then we define C to be a *good component*. Therefore $C_1, \ldots, C_t$ are good components. The next claim shows that good components contain minimal black vertices.

Claim 9.12. *Let u be a minimal black vertex, and assume that u is not in C_1. Let C be a component of $G \backslash \overline{\mathrm{Neigh}}(u)$. Then C contains a minimal black vertex.*

Proof of Claim 9.12. First, we prove that C contains a black vertex. If not, then C is a single white vertex, w. Since $G(M)$ is connected, w is adjacent with a black vertex, x, in $G(M)$. Neither w nor x belongs to $\overline{\mathrm{Neigh}}(u)$, so w and x are adjacent in $G \backslash \overline{\mathrm{Neigh}}(u)$. Thus C contains the black vertex, x, contrary to hypothesis. Therefore C contains at least one black vertex. Let x' be an arbitrary black vertex in C. If x' is minimal, the result follows. Hence assume that there is a black vertex x with $\mathrm{Neigh}(x) \subset \mathrm{Neigh}(x')$. Choose x so that its degree is as small as possible, implying that x is minimal. If $x \in C$, the result follows, so say $x \in D$ where D is some connected component of $G \backslash \overline{\mathrm{Neigh}}(u)$ other than C. If x is adjacent to a white vertex in D, we will clearly not have $\mathrm{Neigh}(x) \subset \mathrm{Neigh}(x')$. So $D = \{x\}$, and x is adjacent only to neighbours of u. As u is a minimal black vertex, we deduce that x and u are twin vertices, which contradicts Claim 9.11. ■

Note that it is possible for one good component to be contained in another. Let u be a minimal black vertex. We say that a component C in $G \backslash \overline{\mathrm{Neigh}}(u)$ is *minimal* if the vertex set of C does not properly contain the vertex set of a good component.

Claim 9.13. *Let u be a minimal black vertex, and let C be a component in $G \backslash \overline{\mathrm{Neigh}}(u)$*

that is disjoint from C_1. Assume that C is a minimal good component. If w is a black vertex in C, then w has a common neighbour with u.

Proof of Claim 9.13. Assume there is a black vertex in C that has no common neighbour with u. We will prove that there is a minimal black vertex with this property. Let w' be an arbitrary black vertex in C that has no white neighbour in common with u. If w' is not a minimal black vertex, then we can assume that w is a minimal black vertex and that $\mathrm{Neigh}(w) \subset \mathrm{Neigh}(w')$. Then w is joined to w' by a path of length 2, and this path does not contain a white vertex adjacent with u, since w' does not have a neighbour in common with u. This means that w and w' are joined by a path in $G\backslash\overline{\mathrm{Neigh}}(u)$, so both are in C. In fact, w cannot have a neighbour in common with u, because any such neighbour would also be a neighbour of w'. Thus w is a minimal black vertex in C having no common neighbours with u.

Any white vertex not in C that is adjacent to a black vertex in C must also be adjacent to u. It immediately follows that any white vertex not in C is not adjacent to w. Therefore every vertex not in C is also a vertex in $G\backslash\overline{\mathrm{Neigh}}(w)$. This implies that the vertices not in C are contained in a connected component of $G\backslash\overline{\mathrm{Neigh}}(w)$. Since $w \in C$, and C is disjoint from C_1, it follows that N is a minor of $M\backslash w$. Therefore $M\backslash w$ is not connected, since M and N form a counterexample to the theorem. It follows that $G\backslash\overline{\mathrm{Neigh}}(w)$ is not connected. Let D be a connected component in $G\backslash\overline{\mathrm{Neigh}}(u)$ that is different from the one containing the vertices not in C. Every vertex in D is also in C. But the vertex set of D is a proper subset of the vertex set of C, since it doesn't contain w. Recall that w is a minimal black vertex, and thus D is a good component. This contradicts the minimality of C, finishing the proof of Claim 9.13. ∎

Claim 9.14. *Let u be a minimal black vertex, where $u \notin C_1$, and let C be a component in $G\backslash\overline{\mathrm{Neigh}}(u)$. Assume that C is a minimal good component. Then C contains at least two black vertices.*

Proof of Claim 9.14. By Claim 9.12, we see that C contains a minimal black vertex, w.

Assume that w is the only black vertex in C. We also assume that C contains a white vertex, x. Then w is the only neighbour of x. By Claim 9.13, we can let y be a common neighbour of w and u. Then $\{w\} = \text{Neigh}(x) \subseteq \text{Neigh}(y)$, a contradiction, since G is the incidence graph of a clutter. Therefore $C = \{w\}$. Thus $\text{Neigh}(w) \subseteq \text{Neigh}(u)$. Since u is a minimal black vertex, this means that $\text{Neigh}(w) = \text{Neigh}(u)$, so u has a twin vertex. As $u \notin C_1$, this is a contradiction to Claim 9.11. ■

Claim 9.15. *Let u be a minimal black vertex that is not in C_1. Let C be a component of $G\backslash\overline{\text{Neigh}}(u)$, and assume that C is a minimal good component. If w is a minimal black vertex in C, then the component of $G\backslash\overline{\text{Neigh}}(w)$ that contains u also contains every vertex in $C\backslash\overline{\text{Neigh}}(w)$.*

Proof of Claim 9.15. Note that C contains at least two black vertices by Claim 9.14. Therefore $C\backslash\overline{\text{Neigh}}(w)$ contains at least one vertex. Let x be an arbitrary vertex in $C\backslash\overline{\text{Neigh}}(w)$, and let C' be the component of $G\backslash\overline{\text{Neigh}}(w)$ containing x. There must be a vertex in C' that is not in C, for otherwise the vertex set of C' is a proper subset of the vertex set of C, which contradicts the minimality of C. It follows that in $G\backslash\overline{\text{Neigh}}(w)$, there is a path from x to a vertex not in C. Any such path must contain a neighbour of u. It now follows that in $G\backslash\overline{\text{Neigh}}(w)$ there is a path from x to u. As x was chosen arbitrarily from $C\backslash\overline{\text{Neigh}}(w)$, we see that the component of $G\backslash\overline{\text{Neigh}}(w)$ that contains u also contains every vertex in $C\backslash\overline{\text{Neigh}}(w)$, exactly as desired. ■

Claim 9.16. *At least one of the components $C_2, \ldots, C_t$ is not minimal.*

Proof of Claim 9.16. Assume that $C_2, \ldots, C_t$ are all minimal good components. Say a black vertex, u, in one of $C_2, \ldots, C_t$ is *interesting* if v is in the same component as C_1 in $G\backslash\overline{\text{Neigh}}(u)$. Assume u is interesting, and chosen so that $|\text{Neigh}(u) \cap \text{Neigh}(v)|$ is smallest possible. We assume that u is in C_i, where $i \geq 2$. Note that N is a minor of $M\backslash u$, and hence $M\backslash u$ is not connected. Thus $G\backslash\overline{\text{Neigh}}(u)$ is not connected. Let D be a component in $G\backslash\overline{\text{Neigh}}(u)$ not containing v. Then D has no vertex in common with $C_i\backslash\overline{\text{Neigh}}(u)$, by Claim 9.15. It also has no vertex in common with C_1, since v is in the same component as C_1 in $G\backslash\overline{\text{Neigh}}(u)$. Therefore any vertex that is not in D, but is adjacent to a vertex in D, must be a neighbour of w that is not in C_i. Any such vertex is also a neighbour of v. Hence D is a connected component of $G\backslash\overline{\text{Neigh}}(v)$. Now we can assume that $D = C_j$, where $i \neq j$ and $j \geq 2$. Choose w, a black vertex in C_j. In order for C_j to be disconnected from the component containing v in $G\backslash\overline{\text{Neigh}}(u)$, we must have that $\text{Neigh}(w) \cap \text{Neigh}(v) \subseteq \text{Neigh}(u) \cap \text{Neigh}(v)$, implying that w is interesting. Now $\text{Neigh}(w)\cap\text{Neigh}(v) = \text{Neigh}(u)\cap\text{Neigh}(v)$ by the choice of u. Since w was arbitrary, this means $C_j\backslash\overline{\text{Neigh}}(w)$ is disconnected from v in $G\backslash\overline{\text{Neigh}}(w)$, contradicting Claim 9.15.

Now assume that there is no interesting vertex. Let y be a vertex in $\text{Neigh}(v)$ that is adjacent to a vertex in C_1. Let w be an arbitrary black vertex in C_2. Then C_1 is disconnected from v in $G\backslash\overline{\text{Neigh}}(w)$, as w is not interesting by assumption, which implies $y \in \text{Neigh}(w)$. As w was arbitrary, y is adjacent to every black vertex in C_2, as well as v. Thus if x is a white vertex in C_2, then $\text{Neigh}(x) \subset \text{Neigh}(y)$, a contradiction as G is the incidence graph of a clutter. ∎

From now on, we assume that C_2 is a good component, but not minimal. Let C be a minimal good component, and assume that the vertex set of C is properly contained in the vertex set of C_2. Let u be a minimal black vertex such that C is a component of $G\backslash\overline{\text{Neigh}}(u)$.

Claim 9.17. $u \in C_2$.

Proof of Claim 9.17. If this is not the case, then any common neighbour of u and a vertex in C must be a common neighbour of v and a vertex in C_2. This implies that C is a connected component of $G\backslash\overline{\text{Neigh}}(v)$, which is impossible because the veretx set of C is properly contained in the vertex set of a connected component of $G\backslash\overline{\text{Neigh}}(v)$. ∎

By Claim 9.12, we can choose a minimal black vertex, w, in C. Let H be the component of $G\backslash\overline{\text{Neigh}}(w)$ that contains u. By Claim 9.15, H also contains $C\backslash\overline{\text{Neigh}}(w)$. Assume that we have chosen C, u, and w, so that H is as large as possible.

Claim 9.18. *Let D be a component of $G\backslash\overline{\text{Neigh}}(w)$ not equal to H. Then D is a minimal good component.*

Proof of Claim 9.18. Note that D is a good component, since it is disconnected when we delete w and its neighbours. Assume D is not minimal. Let D' be a minimal good component such that the vertex set of D' is properly contained in the vertex set of D. Let u' be a minimal black vertex such that D' is a component in $G\backslash\overline{\text{Neigh}}(u')$. Suppose $u' \notin D$. Then any neighbour of u' that is adjacent to a vertex in D' is not in D, but is adjacent to a vertex in D. The only such vertices are in $\overline{\text{Neigh}}(w)$. This means that D' is a component of $G\backslash\overline{\text{Neigh}}(w)$, but this is impossible, since the vertex set of D' is properly contained in a the vertex set of a component of $G\backslash\overline{\text{Neigh}}(w)$. Therefore u' is in D.

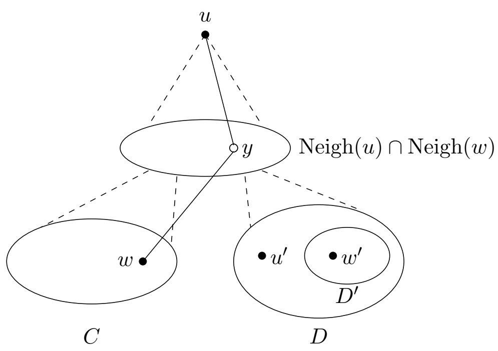


Figure 9.2: Further structure in $G(M)$

Assume that there is vertex, y, in $\text{Neigh}(w) \cap \text{Neigh}(u)$ that is adjacent to u'. Let w' be an arbitrary minimal black vertex contained in D', and assume that y is not a neighbour of w'. Let x be an arbitrary vertex in H. Then x and u are joined by a path, P_x, in $G\backslash\overline{\text{Neigh}}(w)$. By concatenating P_x with the two edges uy and yu', we obtain a path joining x to u'. Assume that this is not a path in $G\backslash\overline{\text{Neigh}}(w')$, so that some vertex in the path is adjacent to w'. Such a vertex can only be a white vertex, so it is not u or u'. Moreover, we have assumed that y is not adjacent to w'. Therefore some vertex in P_x is adjacent to w'. But P_x is a path in $G\backslash\overline{\text{Neigh}}(w)$ that contains u, and u is not in D, a connected component of $G\backslash\overline{\text{Neigh}}(w)$. Therefore there can be no edge from a vertex in D, such as w', to a vertex in P_x. Now we see that the component of $G\backslash\overline{\mathrm{Neigh}}(w')$ that contains u' also contains x. As x was arbitrary, this component contains every vertex in H, as well as u'. This violates our choice of C, u, and w, since we could have chosen D', u', and w' instead. We conclude that y is adjacent to w'. Since w' was an arbitrary minimal black vertex in D', we conclude that every minimal black vertex in D' is adjacent to y. Next we will show that every black vertex in D' is adjacent to y.

Let x' be a black vertex in D'. If x' is minimal, then we are done, so assume otherwise. Then there is a black vertex, x, such that $\mathrm{Neigh}(x) \subset \mathrm{Neigh}(x')$. We may as well assume that x is a minimal black vertex. If x is in D', then x is adjacent to y, so x' is adjacent to y, as desired. Therefore we assume that x is not in D'. Then $\mathrm{Neigh}(x) \subseteq \mathrm{Neigh}(u')$. As u' is a minimal black vertex, we deduce that $\mathrm{Neigh}(x) = \mathrm{Neigh}(u')$. Since u' is not contained in C_1, it cannot be the case that u' has a twin vertex, by Claim 9.11. Therefore x and u' are the same vertex. But y is adjacent to u', and now we again conclude that x, and hence x', is adjacent to y, as desired. Therefore every black vertex in D' is adjacent to y. This means that if z is an arbitrary white vertex in D', then every neighbour of z is a neighbour of y, so $\mathrm{Neigh}(z) \subseteq \mathrm{Neigh}(y)$ which is a contradiction to the fact that G is the incidence graph of a clutter. We must conclude that u' is not adjacent to any vertex in $\mathrm{Neigh}(w) \cap \mathrm{Neigh}(u)$. This also means that no black vertex in D' can be adjacent to a vertex in $\mathrm{Neigh}(w) \cap \mathrm{Neigh}(u)$, since any vertex that is not in D', but is adjacent to a black vertex in D', is adjacent to u'.

Let P be a shortest-possible path between u and u' in G. First assume that there is no vertex in P that is adjacent to a vertex in D'. Let w' be an arbitrary minimal black vertex in D'. Then P is a path from u to u' in $G\backslash\overline{\mathrm{Neigh}}(w')$. If x is an arbitrary vertex in H, then x is joined by a path, P_x, to u in $G\backslash\overline{\mathrm{Neigh}}(w)$. Since w' is not adjacent to any vertex in $\mathrm{Neigh}(u) \cap \mathrm{Neigh}(w)$, we see that P_x is also a path in $G\backslash\overline{\mathrm{Neigh}}(w')$. By concatenating P_x and P, we obtain a path from x to u' in $G\backslash\overline{\mathrm{Neigh}}(w')$. Therefore the component of $G\backslash\overline{\mathrm{Neigh}}(w')$ that contains u' also contains every vertex in H, and we again have a contradiction to our choice of C, u, and w. Thus there is a vertex in P that is a neighbour of a vertex in D'.

Note that any vertex not in D' that is a neighbour of a vertex in D' is in $\text{Neigh}(u')$, as D' is a connected component of $G \backslash \overline{\text{Neigh}}(u')$. Since P is a shortest path from u to u', we see that P contains exactly one vertex, y, that is adjacent to a vertex in D'. Let w' be an arbitrary minimal black vertex in D'. If y is not adjacent to w', then P is a path from u to u' in $G \backslash \overline{\text{Neigh}}(w')$. We get a contradiction to our choice of C, u, and w, exactly as before. Therefore y is adjacent to every minimal black vertex in D'.

We show that y is adjacent to every black vertex in D'. Let x' be a black vertex in D', and assume that x' is not adjacent to y. Then x' is not a minimal black vertex, so let x be a minimal black vertex such that $\text{Neigh}(x) \subset \text{Neigh}(x')$. If x is in D', then $y \in \text{Neigh}(x)$, and we have a contradiction, so $x \notin D'$. This means that $\text{Neigh}(x) \subseteq \text{Neigh}(u')$. Because u' is a minimal black vertex, and does not have a twin by Claim 9.11, this implies that $x = u'$. But y is in $\text{Neigh}(u')$, so we again see that y is adjacent to x. Thus y is adjacent to every black vertex in D'. If z is a white vertex in D', then every neighbour of z is a neighbour of y, which is impossible. From this final contradiction we see that D must be a minimal good component. This completes the proof of Claim 9.17. ∎

Now we can finish the proof of the main theorem. Let D be a component of $G \backslash \overline{\text{Neigh}}(w)$ that is not equal to H. By Claim 9.18, we see that D is a minimal good component. Any vertex not in D, but adjacent to a vertex in D, must be in $\text{Neigh}(u) \cap \text{Neigh}(w)$. It therefore follows that D is a component of $G \backslash \overline{\text{Neigh}}(u)$. By Claim 9.12 we see that D contains a minimal black vertex, w'. Let x be an arbitrary vertex in H, and let P_x be a path from x to u in $G \backslash \overline{\text{Neigh}}(w)$. Assume that P_x is not a path in $G \backslash \overline{\text{Neigh}}(w')$. Then some vertex of P_x is adjacent to w'. No vertex in P_x is in D, since P_x is a path in $G \backslash \overline{\text{Neigh}}(w)$ containing u, and D is a component of $G \backslash \overline{\text{Neigh}}(w)$ that does not contain u. Therefore there is a vertex in P_x that is not in D, but is adjacent to a vertex in D (namely w'). Any such vertex must be in $\text{Neigh}(u) \cap \text{Neigh}(w)$. But this is impossible, because no vertex of P_x is in $\overline{\text{Neigh}}(w)$. Let H' be the component of

$G\backslash\overline{\text{Neigh}}(w')$ that contains u. We have just shown that H' contains all the vertices of H. By Claim 9.15, we see that H' also contains $D - \overline{\text{Neigh}}(w')$, and Claim 9.14 implies that this set is not empty. Thus we have contradicted our choice of C, u, and w, because we could have chosen D, u, and w' instead. Thus there is no possible counterexample, and the result follows. $\square$

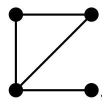
Appendix A

Tables of evaluations

The second table gives $Q'_M(x, y)$ and $T_M(x, y)$ for certain matroids. The table on the next page gives Q' for the following polymatroids:

- $P(M_1) = \text{Conv}\{(2, 1, 0, 1), (2, 0, 1, 1), (2, 0, 0, 2), (1, 0, 2, 1), (0, 1, 2, 1), (1, 2, 0, 1),$ $(0, 2, 1, 1), (1, 1, 0, 2), (0, 1, 1, 2), (1, 0, 1, 2)\}.$

  This is the graphic polymatroid of the graph

  .

- $P(M_2) = \text{Conv}\{(2, 1, 1, 1), (2, 1, 0, 2), (2, 0, 2, 1), (2, 0, 1, 2), (1, 2, 1, 1), (1, 2, 0, 2)\}$, which is the graphic polymatroid of the path P_4.

- $P(M_3)$ is the graphic polymatroid of the 4-cycle.

- $P(M_4) = \text{Conv}\{(2, 1, 1, 1, 1), (2, 1, 0, 1, 2), (2, 1, 0, 2, 1), (2, 0, 2, 1, 1), (2, 0, 1, 2, 1),$ $(1, 2, 1, 1, 1), (1, 2, 1, 0, 2), (1, 2, 0, 2, 1), (1, 2, 0, 1, 2), (1, 1, 1, 1, 2),$ $(1, 1, 2, 0, 2)\}.$

  This is the graphic polynomial of the path P_5.

- $P(M_5)$ is the graphic polynomial of the 5-cycle.

- $P(M_6) = \mathrm{Conv}\{(2,1,1,1),(1,2,1,1),(1,1,2,1),(1,1,1,2)\}$
- $P(M_7) = \mathrm{Conv}\{(2,1,2,1),(2,1,1,2),(1,2,2,1),(1,2,1,2)\}$
- $P(M_8) = \mathrm{Conv}\{(2,1,0,2),(2,0,1,2),(1,2,0,2),(1,0,2,2)\}$

Note that the last three polymatroids cannot be graphic: $P(M_6)$ and $P(M_8)$ would have four edges and five vertices, but neither is the polymatroid of P_5 above, while $P(M_7)$ would have four edges and six vertices, and it can be easily checked no graph with the correct structure can be found.

Polymatroid	$Q'(x,y)$
$P(M_1)$	$x^3+3x^2y+3xy^2+y^3+x^2+xy+x$
$P(M_2)$	$x^3+3x^2y+3xy^2+y^3=(x+y)^3$
$P(M_3)$	$x^3+3x^2y+3xy^2+y^3+x^2+2xy+y^2+x+y+1$
	$=\frac{(x+y)^4-1}{x+y-1}$
$P(M_4)$	$x^4+4x^3y+6x^2y^2+4xy^3+y^4=(x+y)^4$
$P(M_5)$	$\frac{(x+y)^5-1}{x+y-1}$
$P(M_6)$	$x^3+3x^2y+3xy^2+y^3-3xy-2y^2+y$
$P(M_7)$	$x^3+3x^2y+3xy^2+y^3-x^2-2xy-y^2$
$P(M_8)$	$x^3+3x^2y+3xy^2+y^3-xy-y^2-x$

Matroid	$Q'(x,y)$	$T(x,y)$
$U_{2,3}$	$x^2+2xy+y^2-x$	$x^2-x+2xy+y^2$
$U_{1,4}$	$x^3+3x^2y+3xy^2+y^3-3xy-2y^2+y$	$x+y+y^2+y^3$
$U_{2,4}$	$x^3+3x^2y+3xy^2+y^3-x^2-y^2$	$x^2+2x+2y+y^2$
$U_{2,5}$	$x^4+4x^3y+6x^2y^2+4xy^3+y^4-x^3-4xy^2-2y^3+y^2$	$x^2+3x+3y+2y^2+y^3$
$U_{2,4}$+ coloop	$x^4+4x^3y+6x^2y^2+4xy^3+y^4-2x^3-4x^2y-4xy^2-2y^3+x^2+y^2$	$x^3+2x^2+2xy+xy^2$
$U_{1,2}\oplus U_{1,2}$	$x^3-x^2+3x^2y-2xy+3xy^2-y^2+y^3$	$x^2+2xy+y^2$
$M(K_3)$	$x^2+2xy+y^2+x+y+1$	x^2+x+y
K_4	$x^4+4x^3y+6x^2y^2+4xy^3+y^4-2x^3-4x^2y-xy^2-y^3+x^2$	$x^3+3x^2+33xy+3y+3y^2+y^3$
$K_4\backslash e$	$x^4+4x^3y+6x^2y^2+4xy^3+y^4-2x^3-4x^2y-2xy^2-y^3+x^2$	$x^3+2x^2+x+2xy+y+y^2$
$(K_4\backslash e)^*$	$x^4+4x^3y+6x^2y^2+4xy^3+y^4-x^3-2x^2y-4xy^2-2y^3+y^2$	$x^2+x+2xy+2y^2+y^3$
K_4/e	$x^3+3x^2y+3xy^2+y^3-2x^2-3xy-y^2+x$	$x^2+x+2xy+2y+2y^2+y^3$

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