ON THE VARIANCE OF SUMS OF DIVISOR FUNCTIONS IN SHORT INTERVALS

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Abstract. Given a positive integer n the k-fold divisor function $d_k(n)$ equals the number of ordered k-tuples of positive integers whose product equals n. In this article we study the variance of sums of $d_k(n)$ in short intervals and establish asymptotic formulas for the variance of sums of $d_k(n)$ in short intervals of certain lengths for $k = 3$ and for $k \geq 4$ under the assumption of the Lindelöf hypothesis.

1. Introduction and main results

Let $k \geq 2$ be an integer and $d_k(n)$ denote the number of ordered k-tuples of positive integers whose product is n. Also, write

$$\Delta_k(x) = \sum_{n \leq x} d_k(n) - \operatorname{Res}_{s=1}\Big(\zeta^k(s)\frac{x^s}{s}\Big),$$

where $\zeta(s)$ is the Riemann zeta-function and the residue on the right-hand side equals $xR(\log x)$ where $R(x)$ is a polynomial of degree $k-1$.

Asymptotic formulas for the mean square of $\Delta_k(x)$, which is the variance of sums of $d_k(n)$ with $1 \leq n \leq x$, have been given by Cramér [2], with $k = 2$, and Tong [15] for $k = 3$ as well as for $k \geq 4$ under assumption of the Lindelöf hypothesis. In this article we study the variance of sums of $d_k(n)$ in short intervals. Short intervals with $x < n < x + h$ and $h = o(x)$ capture the erratic nature of $d_k(n)$ better than long intervals do and the variance of sums of $d_k(n)$ over short intervals gives stronger information about its behavior. Additionally, it has long been understood that there is a connection between the variance of sums of $d_k(n)$ and the $2k$th moment of the Riemann zeta-function. This connection becomes more pronounced when looking at short intervals.

We first note that from the previously stated estimates for the variance of sums of $d_k(n)$ with $1 \leq n \leq x$ it follows that for $k = 2, 3$, and, for $k \geq 4$ assuming the Lindelöf hypothesis that

$$\frac{1}{X}\int_X^{2X} \Big(\Delta_k(x+H) - \Delta_k(x)\Big)^2 \, dx \ll X^{1-\frac{1}{k}}, \tag{1.1}$$

for $2 \leq H \leq X$. When H is small this bound is not very good and one expects that when $H = o(X^{1-\frac{1}{k}})$ this can be improved. Using a method of Selberg [12], Milinovich

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement n° 320755.

and Turnage-Butterbaugh [10] have given an elegant argument, which provides a better bound than (1.1) for $H = o(X^{1-\frac{1}{k}})$. Assuming the Riemann hypothesis and applying Harper's [4] sharp refinement of Soundararajan's [13] bound for the $2k$th moment of the Riemann zeta-function, their argument shows that

$$\frac{1}{X}\int_X^{2X} \Big(\Delta_k\Big(x+\frac{x\cdot X^{-\frac{1}{k}}}{L}\Big)-\Delta_k(x)\Big)^2\,dx \ll \frac{X^{1-\frac{1}{k}}}{L}(\log L)^{k^2} \tag{1.2}$$

where $2 \le L \ll X^{1-\frac{1}{k}-\varepsilon}$ for some $\varepsilon > 0$. However, this upper bound is most likely not sharp and the true order of magnitude is probably of size $(X^{1-\frac{1}{k}}/L)\cdot(\log L)^{k^2-1}$.

More precise estimates than (1.2) in the case that $k = 2$ are given by Jutila [8] and Ivić [7] (see also [1] and [9]). In particular, Ivić [7] derives an explicit asymptotic formula for the variance of sums of $d_2(n) = d(n)$ in short intervals with $x < n \le x + h$ and $x^\varepsilon \ll h \ll x^{\frac{1}{2}-\varepsilon}$. For $X^\varepsilon \ll L \ll X^{\frac{1}{2}-\varepsilon}$, for some $\varepsilon > 0$, Ivić proves that

$$\frac{1}{X}\int_X^{2X} \Big(\Delta_2\Big(x+\frac{X^{1/2}}{L}\Big)-\Delta_2(x)\Big)^2\,dx = \frac{8}{\pi^2}\frac{X^{1/2}}{L}(\log L)^3 + O\Big(\frac{X^{1/2}}{L}(\log L)^2\Big). \tag{1.3}$$

A main tool in Jutila's method for estimating the variance of $d(n)$ in short intervals is the Voronoi summation formula. This formula expresses $\Delta_2(x)$ in terms of a trigonometric polynomial and similar formulas can be established for $\Delta_k(x)$ (see for instance [3]). However, as k becomes larger these trigonometric polynomials become more complex and even when $k = 3$ this method seems to no longer work.

In this article we derive asymptotic formulas for the variance of sums of $d_k(n)$ in short intervals of certain lengths for $k = 3$ and under the assumption of the Lindelöf hypothesis for $k \ge 4$. Our main innovation is to (essentially) combine Jutila's approach with Selberg's method. This enables us to handle the large frequencies in the trigonometric polynomial approximation to $\Delta_k(x)$ that are a significant obstacle in this problem. Our formulas only hold for intervals of certain lengths and computing this variance in even shorter intervals than those in Theorem 1.2 seems difficult and would be very interesting.

Let

$$a_k = \prod_p \Big((1-p^{-1})^{k^2}\sum_{j=0}^{\infty}\Big(\frac{\Gamma(k+j)}{\Gamma(k)j!}\Big)^2\frac{1}{p^j}\Big) \quad \text{and} \quad C_k = \frac{2^{2-\frac{1}{k}}-1}{2-\frac{1}{k}}\cdot\frac{k^{k^2-1}}{\Gamma(k^2)}\cdot a_k. \tag{1.4}$$

Our first main result gives an estimate for the variance of sums of $d_3(n)$ in short intervals of certain lengths.

Theorem 1.1. *Suppose that $2 \le L \ll X^{\frac{1}{12}-\varepsilon}$ for some fixed $\varepsilon > 0$. Then*

$$\frac{1}{X}\int_X^{2X} \Big(\Delta_3\Big(x+\frac{x^{2/3}}{L}\Big)-\Delta_3(x)\Big)^2\,dx = C_3\cdot\frac{X^{2/3}}{L}(\log L)^8 + O\Big(\frac{X^{2/3}}{L}(\log L)^7\Big).$$

We also examine the variance of sums of $d_k(n)$ in short intervals for $k \ge 3$ under the assumption of the Lindelöf hypothesis.

Theorem 1.2. *Assume the Lindelöf hypothesis. Suppose* $2 \le L \ll X^{\frac{1}{k(k-1)}-\varepsilon}$ *for some fixed* $\varepsilon > 0$*. Then for each integer* $k \ge 3$ *we have*

$$\frac{1}{X}\int_X^{2X} \Big(\Delta_k\Big(x+\frac{x^{1-\frac{1}{k}}}{L}\Big)-\Delta_k(x)\Big)^2\,dx = C_k\cdot\frac{X^{1-\frac{1}{k}}}{L}(\log L)^{k^2-1} + O\Big(\frac{X^{1-\frac{1}{k}}}{L}(\log L)^{k^2-2}\Big).$$

Remark. There is a slight difference between C_2 and the leading coefficient in (1.3). This arises because the lengths of our intervals depend on the variable x. To clarify this discrepancy note that

$$C_k = \Big(\int_1^2 x^{1-\frac{1}{k}}\,dx\Big)\cdot\frac{k^{k^2-1}}{\Gamma(k^2)}\cdot a_k.$$

Ivić [7] gives a more precise formula for the left-hand side of (1.3) that includes lower order terms and an error term with a power savings in X. We can also prove more precise formulas than those stated in Theorems 1.1 and 1.2. In particular, assuming the Lindelöf hypothesis, we can show for each $k \ge 3$ and $L = X^\delta$ with $\varepsilon < \delta < \frac{1}{k(k-1)} - \varepsilon$, for some $\varepsilon > 0$, that

$$\frac{1}{X}\int_X^{2X} \Big(\Delta_k\Big(x+\frac{x^{1-\frac{1}{k}}}{L}\Big)-\Delta_k(x)\Big)^2\,dx = \sum_{j=0}^{k^2-1} c_j\cdot\frac{X^{1-\frac{1}{k}}}{L}(\log L)^j + O\Big(\frac{X^{\vartheta(\delta,k)}}{L}\Big)$$

where $\vartheta(\delta,k) \le 1-\frac{1}{k}-\eta$ for some fixed $\eta=\eta(\varepsilon)>0$. We have not computed the coefficients c_j for $0 \le j \le k^2-2$.

In concurrent work, Keating, Rodgers, Roditty-Gershon, and Rudnick (see [11]) have established strong results on an analog of this problem in the setting of function fields over a finite field $\mathbb{F}_q$ in the case $q\to\infty$ and degree of the polynomials, n, is fixed. In this setting they succeed in unconditionally computing the variance of sums of divisor functions in very short intervals. From their results we expect that the order of the left-hand side of (1.2) to be of size $(X^{1-\frac{1}{k}}/L)\cdot(\log L)^{k^2-1}$ for $X^\varepsilon \ll L \ll X^{1-\frac{1}{k}-\varepsilon}$. Additionally, their analysis suggests that leading order constant should have a very elaborate and interesting behavior. For instance, a transition appears there when the lengths of the intervals become smaller than those considered in Theorem 1.2. This is consistent with our analysis, since when the intervals become shorter than those considered in Theorem 1.2 our method fails for several reasons. Not only does the polynomial approximation to $\Delta_k(x+h)-\Delta_k(x)$ become too long to handle, but it also seems to no longer effectively approximate $\Delta_k(x+h)-\Delta_k(x)$ in mean square. These breaking points coincide precisely at this transition.

2. Main propositions

Our first main step approximates $\Delta_k(x)$ on average by short trigonometric polynomials. This may be compared to what can be proved for pointwise approximations (see [3]).

Proposition 2.1. *Let* $0 < \theta \leq \frac{1}{2}$ *and*

$$P_3(x;\theta) = \frac{x^{\frac{1}{3}}}{\pi\sqrt{3}} \sum_{n \leq X^\theta} \frac{d_3(n)}{n^{\frac{2}{3}}} \cos\left(6\pi\sqrt[3]{nx}\right).$$

Then for any $\varepsilon > 0$ *we have*

$$\frac{1}{X}\int_X^{2X} \Big(\Delta_3(x) - P_3(x;\theta)\Big)^2 \, dx \ll X^{\frac{2}{3}-\frac{\theta}{6}+\varepsilon}. \tag{2.1}$$

Assuming the Lindelöf hypothesis we are able to prove a stronger result for the ternary divisor function as well gives analogous result for $d_k(n)$ for each $k \geq 4$.

Proposition 2.2. *Assume the Lindelöf hypothesis. Let* $k \geq 3$ *be an integer,* $0 < \theta \leq \frac{1}{k-1}$*, and*

$$P_k(x;\theta) = \frac{x^{\frac{1}{2}-\frac{1}{2k}}}{\pi\sqrt{k}} \sum_{n \leq X^\theta} \frac{d_k(n)}{n^{\frac{1}{2}+\frac{1}{2k}}} \cos\left(2\pi k\sqrt[k]{nx} + \frac{k-3}{4}\pi\right).$$

Then for any $\varepsilon > 0$ *we have*

$$\frac{1}{X}\int_X^{2X} \Big(\Delta_k(x) - P_k(x;\theta)\Big)^2 \, dx \ll X^{1-\frac{1}{k}\cdot(1+\theta)+\varepsilon}. \tag{2.2}$$

For $k = 2$ and $0 < \theta \leq 1$ the inequality (2.2) is known to hold unconditionally (see equation (12.4.4) of Titchmarsh [14]). The strength of the upper bound is significant and better bounds in (2.1) correspond to being able to compute the variance of sums of $d_k(n)$ in shorter intervals. In Proposition 2.2 we obtain a better estimate in the case $k = 3$ than (2.1). This allows us to compute the variance of sums of $d_3(n)$ in even shorter intervals, assuming the Lindelöf hypothesis. Heath-Brown [6] has also obtained an estimate for the left-hand side of (2.1) by estimating the mean values of $\Delta_3(x)P_3(x;\theta)$ and $P_3(x;\theta)^2$ and then applying Tong's formula for the mean square of $\Delta_3(x)$. Our upper bound strengthens the estimate given by Heath-Brown. Additionally, our proof of (2.1) is significantly different. Particularly, it does not use Tong's results. In fact, our argument gives a new proof of Tong's formulas.

As another application of the above propositions we will establish asymptotic formulas for the variance of sums of $d_k(n)$ in intervals with $x < n \leq x+h$ with $x^{1-\frac{1}{k}+\varepsilon} \ll h \ll x^{1-\varepsilon}$. In this regime $\Delta_k(x+h)$ and $\Delta_k(x)$ interact as if they are uncorrelated.

Theorem 2.3. *Suppose that* $X^{1-\frac{1}{k}+\varepsilon} \ll H \ll X^{1-\varepsilon}$ *for some* $\varepsilon > 0$*. Then for* $k = 2, 3$ *and for* $k \geq 4$ *under the assumption of the Lindelöf hypothesis we have as* $X \to \infty$

$$\frac{1}{X}\int_X^{2X} \Big(\Delta_k(x+H) - \Delta_k(x)\Big)^2 \, dx \sim B_k \cdot X^{1-\frac{1}{k}}$$

where

$$B_k = \frac{2^{2-\frac{1}{k}} - 1}{2-\frac{1}{k}} \cdot \frac{1}{\pi^2 k} \sum_{n=1}^{\infty} \frac{d_k(n)^2}{n^{1+\frac{1}{k}}}.$$

The leading order constant here is essentially twice the one that appears in Tong's formula [15] for the mean square of $\Delta_k(x)$. As we will see, this reflects that the covariance between $\Delta_k(x+h)$ and $\Delta_k(x)$ tends to zero as $X \to \infty$ in this regime.

Before proving Propositions 2.1 and 2.2 we first require several preliminary lemmas. The first of these lemmas cites a stationary phase estimate. Here and throughout $\chi(s) = \pi^{s-\frac{1}{2}}\Gamma(\frac{1-s}{2})/\Gamma(\frac{s}{2})$ is the functional equation factor for $\zeta(s)$, that is $\zeta(s) = \chi(s)\zeta(1-s)$.

Lemma 2.4. *Suppose $k \geq 2$. For sufficiently large $Y < x$ we have*

$$\frac{1}{2\pi i}\int_{-\epsilon-iY}^{-\epsilon+iY} \chi^k(s)\zeta^k(1-s)x^s\frac{ds}{s} = \frac{x^{\frac{1}{2}-\frac{1}{2k}}}{\pi\sqrt{k}}\sum_{n\leq N}\frac{d_k(n)}{n^{\frac{1}{2}+\frac{1}{2k}}}\cos\Big(2\pi k\sqrt[k]{nx} + \frac{k-3}{4}\pi\Big) + O\Big(Y^{\frac{k}{2}-1}x^{\epsilon} + x^{1+\epsilon}Y^{-\frac{1}{2}-\frac{k}{2}}\Big)$$

where $N = (\frac{Y}{2\pi})^k x^{-1}$.

Proof. This estimate is due to Friedlander and Iwaniec. See pages 497-499 of [3]. □

Let $\lambda \geq 0$ be a number such that $|\zeta(\frac{1}{2}+it)| \ll t^{\lambda+\varepsilon}$ for every $\varepsilon > 0$. It is well-known that by the Phragmen-Lindelöf principle (or otherwise) one has for $\frac{1}{2} \leq \sigma \leq 1$ and every $\varepsilon > 0$ that

$$|\zeta(\sigma+it)| \ll t^{2\lambda(1-\sigma)+\varepsilon}. \tag{2.3}$$

Lemma 2.5. *Let $k \geq 2$ and $0 \leq \delta \leq \frac{1}{k}$. Also, let*

$$I_k(x;\theta,\delta) = \mathrm{Re}\,\frac{1}{\pi i}\int_{\frac{1}{2}-\delta+iY}^{\frac{1}{2}-\delta+iX} \zeta^k(s)x^s\frac{ds}{s},$$

where $Y = 2\pi X^{\frac{1+\theta}{k}}$ and $0 \leq \theta \leq \frac{1}{2}\cdot(k-1)$. Then uniformly for $X \leq x \leq 2X$ we have

$$\Delta_k(x) = \frac{x^{\frac{1}{2}-\frac{1}{2k}}}{\pi\sqrt{k}}\sum_{nx\leq X^{1+\theta}}\frac{d_k(n)}{n^{\frac{1}{2}+\frac{1}{2k}}}\cos\Big(2\pi k\sqrt[k]{nx} + \frac{k-3}{4}\pi\Big) + I_k(x;\theta,\delta) + E_k(\theta,\delta,X),$$

where, for any $\varepsilon > 0$,

$$E_k(\theta,\delta,X) \ll X^{\varepsilon}\Big(X^{(\frac{1}{2}-\frac{1}{k})\cdot(1+\theta)} + X^{\frac{1}{2}-\frac{(1+\theta+k\theta)}{2k}} + X^{\frac{1}{2}-\delta}(X^{\frac{(1+\theta)}{k}} + X)^{k(\lambda+\delta-2\lambda\delta)-1}\Big).$$

Remark. In the bound for E_k the term $X^{(\frac{1}{2}-\frac{1}{k})(1+\theta)}$ is significant and is smaller than $X^{\frac{1}{2}-\frac{(1+\theta)}{2k}}$ only for $\theta < \frac{1}{k-1}$. This accounts for the limitation in the range of θ in Proposition (2.2).

Proof. Applying Perron's formula we get that

$$\sum_{n\leq x} d_k(n) = \int_{1+\epsilon-iX}^{1+\epsilon+iX} \zeta^k(s)x^s\frac{ds}{s} + O(X^{\epsilon}).$$

Next, pull the contour to the line $\mathrm{Re}(s) = \frac{1}{2} - \delta$ picking up the residue at $s = 1$. To estimate the horizontal contours, apply (2.3) and use the functional equation for $\zeta(s)$ along with Stirling's formula to see that they are $\ll X^{\varepsilon}(1 + X^{\frac{1}{2}-\delta}X^{k(\lambda+\delta-2\lambda\delta)-1})$. Thus,

$$\Delta_k(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}-\delta-iY}^{\frac{1}{2}-\delta+iY} \zeta^k(s)x^s\frac{ds}{s} + I_k(x;\theta,\delta) + O\Big(X^{\varepsilon}\Big(1 + X^{\frac{1}{2}-\delta}X^{k(\lambda+\delta-2\lambda\delta)-1}\Big)\Big).$$

Now pull the first integral on the right-hand side to the line $\mathrm{Re}(s) = -\epsilon$ and note that the residue at $s = 0$ contributes $O(1)$. Arguing as before, the horizontal contours are $\ll X^{\varepsilon}(Y^{\frac{k}{2}-1} + X^{\frac{1}{2}-\delta}Y^{k(\lambda+\delta-2\lambda\delta)-1})$. Finally, applying the functional equation we see that

$$\Delta_k(x) = \frac{1}{2\pi i}\int_{-\epsilon-iY}^{-\epsilon+iY} \chi^k(s)\zeta^k(1-s)x^s\frac{ds}{s} + I_k(x;\theta,\delta) + O\Big(X^{\varepsilon}\Big(Y^{\frac{k}{2}-1} + X^{\frac{1}{2}-\delta}(X+Y)^{k(\lambda+\delta-2\lambda\delta)-1}\Big)\Big).$$

To complete the proof apply Lemma 2.4. □

We now show that the mean square of $I_k(x;\theta,\delta)$ can be estimated in terms of the $2k$th moment of the Riemann zeta-function. This is essentially Plancherel's theorem and we will give a direct proof.

Lemma 2.6. *Let w be a smooth function that is compactly supported in the positive real numbers. Suppose that $0 \le \delta < \frac{1}{2k}$ and for every $\epsilon > 0$ that*

$$\int_0^T |\zeta(\tfrac{1}{2} + \delta + it)|^{2k}\,dt \ll T^{1+\epsilon}. \tag{2.4}$$

Then we have for any $\varepsilon > 0$ that

$$\frac{1}{X}\int_{\mathbb{R}}\Big|I_k(x;\theta,\delta)\Big|^2 w\Big(\frac{x}{X}\Big)\,dx \ll X^{1+2\delta\theta-\frac{(1+\theta)}{k}+\varepsilon}.$$

Proof. Changing the order of integration and making a change of variables we get that

$$\frac{1}{X}\int_{\mathbb{R}}\Big|I_k(x;\theta_k,\delta)\Big|^2 w\Big(\frac{x}{X}\Big)\,dx \le \frac{X^{1-2\delta}}{\pi^2}\int_Y^X\int_Y^X \frac{\zeta^k(\frac{1}{2}-\delta+it)\zeta^k(\frac{1}{2}-\delta-iv)}{(\frac{1}{2}-\delta+it)(\frac{1}{2}-\delta-iv)}X^{i(t-v)}\mathcal{I}(t-v)\,dv\,dt,$$

where $\mathcal{I}(y) := \int_{\mathbb{R}} u^{1-2\delta+iy}w(u)\,du$. Observe that by repeatedly integrating by parts $\mathcal{I}(y) \ll_{w,A} \min(1, |y|^{-A})$. Hence, by this, Lemma 2.3, and the functional equation for $\zeta(s)$ along with Stirling's formula we get for $U = X^{\eta}$ with $0 < \eta \le \frac{1}{k}$ fixed that the portion of the above integral on the right-hand side with $|t - v| \ge U$ is $\ll U^{-A}X^{3+k/3}$ for any $A \ge 1$. Thus, taking $A = (4 + k/3)/\eta$ the portion of the above integral with

$|t-v| \geq U$ is $\ll X^{-1}$. To bound the remaining portion of the integral apply the functional equation to see that it is

$$\ll X^{1-2\delta} \iint_{|v-t|\leq U} |\zeta^k(\tfrac{1}{2}+\delta-it)\zeta^k(\tfrac{1}{2}+\delta+iv)|(tv)^{\delta k-1} dv\, dt$$
$$\ll UX^{1-2\delta} \int_{Y-U}^{X+U} |\zeta(\tfrac{1}{2}+\delta+it)|^{2k} \frac{dt}{t^{2-2\delta k}} \ll X^{1-2\delta+\eta} Y^{2\delta k-1+\epsilon},$$

where in the last step we have used (2.4). □

Proof of Propositions 2.1 and 2.2. To prove Proposition 2.1 first apply Lemmas 2.5 and 2.6, where the smooth function w is taken so that it majorizes the indicator function of the interval $[1,2]$. A result of Heath-Brown [5] allows us to take $\delta = \frac{1}{12}$. Also, Weyl's bound gives $\lambda = \frac{1}{6}$. It follows for $0 < \theta \leq \frac{1}{2}$ that

$$\frac{1}{X}\int_X^{2X} \left(\Delta_3(x) - \frac{x^{\frac{1}{3}}}{\pi\sqrt{3}} \sum_{n \leq \frac{1}{x}\cdot X^{1+\theta}} \frac{d_3(n)}{n^{\frac{2}{3}}} \cos\left(6\pi\sqrt[3]{nx}\right)\right)^2 dx \ll X^{\frac{2}{3}-\frac{\theta}{6}+\varepsilon}. \tag{2.5}$$

If we assume the Lindelöf hypothesis we may take $\delta = \lambda = 0$. Arguing in the same way as before we get for $k \geq 3$ and $0 < \theta \leq \frac{1}{k-1}$ that

(2.6)
$$\frac{1}{X}\int_X^{2X} \left(\Delta_k(x) - \frac{x^{\frac{1}{2}-\frac{1}{2k}}}{\pi\sqrt{k}} \sum_{n\leq \frac{1}{x}\cdot X^{1+\theta}} \frac{d_k(n)}{n^{\frac{1}{2}+\frac{1}{2k}}} \cos\left(2\pi k\sqrt[k]{nx} + \frac{k-3}{4}\pi\right)\right)^2 dx \ll X^{1-\frac{(1+\theta)}{k}+\varepsilon}.$$

To complete the proof, we will now remove the dependence on the variable x from the length of the sum. Let $a_n = d_k(n)n^{-\frac{1}{2}-\frac{1}{2k}}e\left(\frac{k-3}{8}\right)$, where $e(x) = e^{2\pi i x}$, and integrate term-by-term to see

$$\int_X^{2X} \Bigg| \sum_{\frac{1}{x}\cdot X^{1+\theta}\leq n\leq X^\theta} a_n e(k\sqrt[k]{nx}) \Bigg|^2 dx = \sum_{\frac{1}{2}\cdot X^\theta \leq m,n\leq X^\theta} a_n \overline{a}_m \int_Z^{2X} e(k\sqrt[k]{x}(\sqrt[k]{n}-\sqrt[k]{m}))\, dx,$$

where $Z = \max(m^{-1}, n^{-1})\cdot X^{1+\theta}$. The diagonal terms with $m = n$ are $\ll X^{1-\frac{\theta}{k}+\varepsilon}$. To bound the off-diagonal terms with $m \neq n$ we integrate by parts to see that the above integral is $\ll X^{1-\frac{1}{k}}/|\sqrt[k]{m}-\sqrt[k]{n}|$. Also, for $m > n$ we use the bound $\sqrt[k]{m}-\sqrt[k]{n} \gg (m-n)m^{\frac{1}{k}-1}$. Hence, by symmetry, the off-diagonal terms are bounded by

$$\ll X^{1-\frac{1}{k}} \sum_{\substack{\frac{1}{2}\cdot X^\theta\leq m,n\leq X^\theta \\ m>n}} \frac{|a_m a_n|}{|\sqrt[k]{n}-\sqrt[k]{m}|} \ll X^{1-\frac{1}{k}} \sum_{\substack{\frac{1}{2}\cdot X^\theta\leq m,n\leq X^\theta \\ m>n}} \frac{|a_m a_n| m^{1-\frac{1}{k}}}{|m-n|}$$
$$\ll X^{1-\frac{1}{k}} X^{\theta\cdot\frac{(k-1)}{k}} \log X \sum_{\frac{1}{2}\cdot X^\theta \leq n\leq X^\theta} |a_n|^2.$$

Since $\theta \leq \frac{1}{k-1}$ this is $\ll X^{1-\frac{1}{k}+\theta\cdot\frac{(k-1)}{k}-\frac{\theta}{k}+\varepsilon} \ll X^{1-\frac{\theta}{k}+\varepsilon}$. Thus, Proposition 2.1 follows from this and (2.5). Proposition 2.2 follows from this and (2.6). □

3. The proofs of Theorems 1.1, 1.2 and 2.3

Lemma 3.1. *Suppose $0 \le \alpha \le 1$ and write $e(x) = e^{2\pi i x}$. For any complex numbers a_n we have*

$$\int_X^{2X} x^\alpha \Big| \sum_{1 \le n \le N} a_n e(k\sqrt[k]{nx}) \Big|^2 \, dx = \sum_{1 \le n \le N} |a_n|^2 \cdot \Big(\frac{2^{1+\alpha}-1}{1+\alpha} X^{1+\alpha} + O\Big(X^{1+\alpha-\frac{1}{k}} N^{1-\frac{1}{k}} \log N \Big) \Big).$$

Proof. Integrating term-by-term we see that the diagonal terms give the main term. To bound the off-diagonal terms we argue as in the previous proof. Integrate by parts and then use the estimate $|\sqrt[k]{m} - \sqrt[k]{n}| \gg |m-n| (\max(m,n))^{\frac{1}{k}-1}$ to get that

$$\begin{aligned} \sum_{\substack{1 \le m,n \le N \\ m > n}} |a_m \overline{a_n}| \Big| \int_X^{2X} x^\alpha e(k\sqrt[k]{nx}) \, dx \Big| &\ll X^{1+\alpha-\frac{1}{k}} \sum_{\substack{1 \le m,n \le N \\ m > n}} \frac{|a_m a_n|}{|\sqrt[k]{n} - \sqrt[k]{m}|} \\ &\ll X^{1+\alpha-\frac{1}{k}} \sum_{\substack{1 \le m,n \le N \\ m > n}} \frac{|a_m a_n| m^{1-\frac{1}{k}}}{|m-n|} \\ &\ll X^{1+\alpha-\frac{1}{k}} N^{1-\frac{1}{k}} \log N \sum_{1 \le n \le N} |a_n|^2. \end{aligned}$$

□

Write

$$\mathcal{M}_k(N, L) = X^{1-\frac{1}{k}} \cdot \frac{2\,(2^{2-\frac{1}{k}}-1)}{\pi^2 k (2 - \frac{1}{k})} \sum_{n \le N} \frac{d_k^2(n)}{n^{1+\frac{1}{k}}} \sin^2\Big(\pi \frac{\sqrt[k]{n}}{L} \Big).$$

Lemma 3.2. *Let $\varepsilon_1 > 0$ and suppose that $0 < \theta \le \frac{1}{k-1} - \varepsilon_1$ and $L \ge 2$. Then there exists $\varepsilon_2 > 0$ such that*

$$\frac{1}{X} \int_X^{2X} \Big(P_k\Big(x + \frac{x^{1-\frac{1}{k}}}{L}; \theta \Big) - P_k(x;\theta) \Big)^2 dx = \mathcal{M}_k(X^\theta, L) \Big(1 + O(X^{-\varepsilon_2}) \Big) + O\Big(\frac{X^{1-\frac{1}{k}-\varepsilon_2}}{L} \Big).$$

Proof. Write

$$M(x) = \frac{-2 \cdot x^{\frac{1}{2}-\frac{1}{2k}}}{\pi\sqrt{k}} \sum_{n \le X^\theta} \frac{d_k(n)}{n^{\frac{1}{2}+\frac{1}{2k}}} \sin\Big(\pi \frac{\sqrt[k]{n}}{L} \Big) \sin\Big(2\pi k \sqrt[k]{n} \Big(\sqrt[k]{x} + \frac{1}{2kL} \Big) + \frac{k-3}{4}\pi \Big).$$

It follows from some basic manipulations that

$$P_k\Big(\Big(\sqrt[k]{x} + \frac{1}{kL} \Big)^k ; \theta \Big) - P_k(x;\theta) = M(x) + \mathcal{R}(x) \tag{3.1}$$

where, for $X < x \le 2X$,

$$\mathcal{R}(x) = O\bigg(\frac{x^{\frac{1}{2}-\frac{3}{2k}}}{L}\bigg|\sum_{n\le X^\theta}\frac{d_k(n)}{n^{\frac{1}{2}+\frac{1}{2k}}}\cos\Big(2\pi k\sqrt[k]{n}\Big(\sqrt[k]{x}+\frac{1}{kL}\Big)+\frac{k-3}{4}\pi\Big)\bigg|\bigg).$$

Now write $a_n = d_k(n)n^{-\frac{1}{2}-\frac{1}{2k}}\sin\Big(\pi\frac{\sqrt[k]{n}}{L}\Big)e\Big(\frac{\sqrt[k]{n}}{2L}+\frac{k-3}{8}\Big)$, so that

$$M(x) = \frac{-x^{\frac{1}{2}-\frac{1}{2k}}}{\pi i\sqrt{k}}\bigg(\sum_{n\le X^\theta}a_n e(k\sqrt[k]{nx}) - \overline{\sum_{n\le X^\theta}a_n e(k\sqrt[k]{nx})}\bigg).$$

Using Lemma 3.1 it is not hard to see that

$$\frac{1}{X}\int_X^{2X} M(x)^2\,dx = \mathcal{M}_k(X^\theta, L)\cdot\Big(1+O(X^{\theta(1-\frac{1}{k})-\frac{1}{k}}\log X)\Big). \tag{3.2}$$

Also, by Lemma 3.1 it follows that

$$\frac{1}{X}\int_X^{2X}\mathcal{R}(x)^2\,dx \ll \frac{X^{1-\frac{3}{k}}}{L^2}\sum_{n\le X^\theta}\frac{d_k^2(n)}{n^{1+\frac{1}{k}}} \ll \frac{X^{1-\frac{3}{k}}}{L^2}. \tag{3.3}$$

Next, to shorten notation write

$$\Sigma(x) = \sum_{n\le X^\theta}\frac{d_k(n)}{n^{\frac{1}{2}+\frac{1}{2k}}}\cos\Big(2\pi kx\sqrt[k]{n}+\frac{k-3}{4}\cdot\pi\Big).$$

Also, let $x_1 = \sqrt[k]{x}+\frac{1}{kL}$ and $x_2 = \Big(x+\frac{x^{1-\frac{1}{k}}}{L}\Big)^{\frac{1}{k}}$. We have for $X \le x \le 2X$ that

$$\begin{aligned}|x_1^{\frac{k-1}{2}}\Sigma(x_1) - x_2^{\frac{k-1}{2}}\Sigma(x_2)| \ll& x_2^{\frac{k-1}{2}}|\Sigma(x_1)-\Sigma(x_2)| + |x_2^{\frac{k-1}{2}} - x_1^{\frac{k-1}{2}}||\Sigma(x_1)| \\ \ll& |x_2-x_1|\cdot\Big(X^{\frac{1}{2}-\frac{1}{2k}}\sum_{n\le X^\theta}\frac{d_k(n)}{n^{\frac{1}{2}-\frac{1}{2k}}} + X^{\frac{k-3}{2k}}|\Sigma(x_1)|\Big) \\ \ll& \frac{1}{L^2X^{\frac{1}{k}}}\cdot\Big(X^{\frac{1}{2}-\frac{1}{2k}+\theta(\frac{1}{2}+\frac{1}{2k})+\frac{\varepsilon_1}{2k}} + X^{\frac{k-3}{2k}}|\Sigma(x_1)|\Big).\end{aligned}$$

Thus, applying Lemma 3.1 and using that $\theta \le \frac{1}{k-1}-\varepsilon_1$ we have

$$\begin{aligned}\frac{1}{X}\int_X^{2X}\Big(P_k\Big(\Big(\sqrt[k]{x}+\frac{1}{kL}\Big)^k;\theta\Big) - P_k\Big(x+\frac{x^{1-\frac{1}{k}}}{L};\theta\Big)\Big)^2\,dx \ll& \frac{X^{1-\frac{3}{k}+\theta(1+\frac{1}{k})+\frac{\varepsilon_1}{k}}}{L^4} + \frac{X^{1-\frac{5}{k}}}{L^4} \\ \ll& \frac{X^{1-\frac{1}{k}+\frac{3-k}{k(k-1)}-\varepsilon_1}}{L^4} + \frac{X^{1-\frac{5}{k}}}{L^4}.\end{aligned}$$

Combining this with (3.1), (3.2), and (3.3) and then applying Cauchy-Schwarz we complete the proof. □

Lemma 3.3. *Let $\theta > 0$ and $k \geq 2$. Suppose that $2 \leq L = o(X^{\theta/k})$ as $X \to \infty$. We have*

$$\sum_{n \leq X^\theta} \frac{d_k^2(n)}{n^{1+\frac{1}{k}}} \sin^2\Big(\pi \frac{\sqrt[k]{n}}{L}\Big) = \frac{k^{k^2}\pi^2}{2\,\Gamma(k^2)}\, a_k \cdot \frac{(\log L)^{k^2-1}}{L} + O\bigg(\frac{(\log L)^{k^2-2}}{L} + \frac{(\log L)^{k^2-1}}{X^{\theta/k}}\bigg),$$

where a_k is as given in (1.4).

Proof. We first require an estimate for the summatory function of $d_k(n)^2$, which follows from a standard argument that we will briefly sketch. Start with the generating series

$$G(s) = \sum_{n=1}^{\infty} \frac{d_k(n)^2}{n^s} = \prod_p \Big(\sum_{j=0}^{\infty} \frac{d_k(p^j)^2}{p^{js}} \Big) = \prod_p \Big(\sum_{j=0}^{\infty} \Big(\frac{\Gamma(k+j)}{\Gamma(k) j!} \Big)^2 \frac{1}{p^{js}} \Big) = \zeta^{k^2}(s) g(s).$$

Here the function $g(s)$ is analytic for $\mathrm{Re}(s) > \frac{1}{2}$ and is given by

$$g(s) = \prod_p \Big((1-p^{-s})^{k^2} \sum_{j=0}^{\infty} \Big(\frac{\Gamma(k+j)}{\Gamma(k) j!} \Big)^2 \frac{1}{p^{js}} \Big).$$

Also, note that $g(s)$ is bounded for $\mathrm{Re}(s) > \frac{1}{2} + \varepsilon$ (see pages 173-174 of Titchmarsh [14]). Applying Perron's formula, shifting contours of integration, and using Theorem 7.7 of Titchmarsh [14] one can show that

$$\sum_{n \leq N} d_k(n)^2 = N\, Q_{k^2-1}(\log N) + O(N^{1-\frac{1}{k^2}+\varepsilon}), \tag{3.4}$$

where $Q_{k^2-1}(x) = \sum_{j=0}^{k^2-1} b_j x^j$ and $b_{k^2-1} = a_k/\Gamma(k^2)$, where a_k is the arithmetic factor in (1.4).

Using (3.4) we have that

$$\sum_{n \leq X^\theta} \frac{d_k^2(n)}{n^{1+\frac{1}{k}}} \sin^2\Big(\pi \frac{\sqrt[k]{n}}{L}\Big) = \int_1^{X^\theta} \frac{Q_{k^2-1}(\log x) + Q'_{k^2-1}(\log x)}{x^{1+\frac{1}{k}}} \sin^2\Big(\pi \frac{\sqrt[k]{x}}{L}\Big)\, dx + O\Big(\frac{1}{L}\Big).$$

Make the change of variables $u = \sqrt[k]{x}/L$ and assume that $X^{\theta/k}/L \to \infty$. The integral on the right-hand side equals

$$\begin{aligned}
&\frac{k}{L} \int_{1/L}^{X^{\theta/k}/L} \Big(Q_{k^2-1}(k \log(Lu))) + Q'_{k^2-1}(k\log(Lu)) \Big) \frac{\sin^2(\pi u)}{u^2}\, du \\
&\quad = \frac{b_{k^2-1} \cdot k^{k^2} (\log L)^{k^2-1}}{L} \int_{1/L}^{X^{\theta/k}/L} \frac{\sin^2(\pi u)}{u^2}\, du + O\bigg(\frac{(\log L)^{k^2-2}}{L}\bigg) \\
&\quad = \frac{b_{k^2-1} \cdot k^{k^2} (\log L)^{k^2-1}}{L} \int_0^{\infty} \frac{\sin^2(\pi u)}{u^2}\, du + O\bigg(\frac{(\log L)^{k^2-2}}{L} + \frac{(\log L)^{k^2-1}}{X^{\theta/k}}\bigg).
\end{aligned}$$

Note that $\int_0^\infty \frac{\sin^2(\pi u)}{u^2}\, du = \frac{\pi^2}{2}$. □

Proof of Theorems 1.1 and 1.2. Let

$$S_k(x) = \Delta_k\Big(x + \frac{x^{1-\frac{1}{k}}}{L}\Big) - \Delta_k(x) \quad \text{and} \quad \mathcal{P}_k(x) = P_k\Big(x + \frac{x^{1-\frac{1}{k}}}{L}; \theta\Big) - P_k(x;\theta).$$

From Lemmas 3.2 and 3.3 we deduce that for $0 < \theta \le \frac{1}{k-1} - \varepsilon$ and $L = o(X^{\theta/k})$ that

$$\frac{1}{X}\int_X^{2X} \mathcal{P}_k(x;\theta)^2\,dx = C_k \frac{X^{1-\frac{1}{k}}}{L}(\log L)^{k^2-1}\Big(1 + O\Big(\frac{1}{\log L}\Big)\Big) + O\Big(X^{1-\frac{(1+\theta)}{k}+\varepsilon}\Big), \tag{3.5}$$

where C_k is as given in (1.4). Proposition 2.1 states for $0 < \theta \le \frac{1}{2} - \varepsilon$ that

$$\frac{1}{X}\int_X^{2X} \Big(S_3(x) - \mathcal{P}_3(x;\theta)\Big)^2\,dx \ll X^{\frac{2}{3}-\frac{\theta}{6}+\varepsilon}.$$

If $\theta = \frac{1}{2} - \varepsilon$ and $L \ll X^{\frac{1}{12}-2\varepsilon}$ this is smaller than (3.5), with $k = 3$, with a power savings in X. Now apply Cauchy-Schwarz to see for $L \ll X^{\frac{1}{12}-2\varepsilon}$ that

$$\frac{1}{X}\int_X^{2X} S_3(x)^2\,dx = C_3 \frac{X^{2/3}}{L}(\log L)^{k^2-1} + O\Big(\frac{X^{2/3}}{L}(\log L)^{k^2-2}\Big).$$

This proves Theorem 1.1. Theorem 1.2 follows from a similar argument, except that here we use Proposition 2.2 with $\theta = \frac{1}{k-1} - \varepsilon$ in place of Proposition 2.1, so that we may take $2 \le L \ll X^{\frac{1}{k(k-1)}-2\varepsilon}$. □

Proof of Theorem 2.3. Write $a_n = d_k(n) n^{-\frac{1}{2}-\frac{1}{2k}} e\Big(\frac{k-3}{8}\Big)$ so that

$$P_k(x;\theta) = \frac{x^{\frac{1}{2}-\frac{1}{2k}}}{2\pi\sqrt{k}}\Bigg(\sum_{n \le X^\theta} a_n e(k\sqrt[k]{nx}) + \overline{\sum_{n \le X^\theta} a_n e(k\sqrt[k]{nx})}\Bigg).$$

Applying Lemma 3.1 it is not difficult to see that for $\varepsilon \le \theta \le \frac{1}{k-1} - \varepsilon$ that as $X \to \infty$

$$\frac{1}{X}\int_X^{2X} \Big(P_k(x;\theta)\Big)^2 dx \sim \frac{B_k}{2}\cdot X^{1-\frac{1}{k}}. \tag{3.6}$$

Next note that $(X - H)^{1-\frac{1}{k}} \sim X^{1-\frac{1}{k}}$ for $H = o(X)$ as $X \to \infty$. Using this estimate, making the change of variables $u = x + H$ and applying Lemma 3.1 one has for $\varepsilon \le \theta \le \frac{1}{k-1} - \varepsilon$ that as $X \to \infty$

$$\frac{1}{X}\int_X^{2X} \Big(P_k(x+H;\theta)\Big)^2 dx \sim \frac{B_k}{2}\cdot X^{1-\frac{1}{k}}. \tag{3.7}$$

We next estimate the covariance term. Let

$$I = \frac{1}{X}\int_X^{2X} \frac{x^{1-\frac{1}{k}}}{4\pi^2 k}\sum_{m,n \le X^\theta} a_m \overline{a_n} e(k(\sqrt[k]{m(x+H)} - \sqrt[k]{nx})\,dx$$

and

$$J = \frac{1}{X}\int_X^{2X} \frac{x^{1-\frac{1}{k}}}{4\pi^2 k}\sum_{m,n \le X^\theta} a_m a_n e(k(\sqrt[k]{m(x+H)} + \sqrt[k]{nx})\,dx.$$

It follows that

$$\frac{1}{X}\int_X^{2X} P_k(x+H;\theta)P_k(x;\theta)\,dx = 2\,\mathrm{Re}\,\big(I+J\big). \tag{3.8}$$

We will now bound I and assume that $H = o(X^{1-\theta})$. Note that for real numbers $x, y > 0$ we have $|\sqrt[k]{x} - \sqrt[k]{y}| \gg |x-y|(\max(x,y))^{\frac{1}{k}-1}$. So for $m, n \le X^\theta$ with $m \ne n$ we have uniformly for $X \le x \le 2X$ that

$$\left| x^{\frac{1}{k}-1}\Big(\sqrt[k]{m}\Big(1+\frac{H}{x}\Big)^{\frac{1}{k}-1} - \sqrt[k]{n}\Big)\right| \gg X^{\frac{1}{k}-1}|m-n+o(1)|(\max(m,n))^{\frac{1}{k}-1} > 0.$$

Using this bound along with Lemma 4.3 of Titchmarsh [14], or alternatively integrating by parts, we have

$$\frac{1}{X}\left|\int_X^{2X} x^{1-\frac{1}{k}}\, e(k(\sqrt[k]{m(x+H)} - \sqrt[k]{nx})\,dx\right| \ll \begin{cases} \dfrac{X^{1-\frac{2}{k}}(\max(m,n))^{1-\frac{1}{k}}}{|m-n|} & \text{if } m \ne n, \\ \dfrac{X^{2-\frac{2}{k}}}{H\sqrt[k]{n}} & \text{if } m = n. \end{cases}$$

Thus, for $\theta < \frac{1}{k-1} - 3\varepsilon$ the contribution of the terms with $m \ne n$ to I is

$$\ll X^{1-\frac{2}{k}} \sum_{\substack{m,n\le X^\theta \\ m\ne n}} \frac{|a_m a_n|(\max(m,n))^{1-\frac{1}{k}}}{|m-n|} \ll X^{1-\frac{2}{k}+\theta(1-\frac{1}{k})+\varepsilon} \sum_{n\le X^\theta} |a_n|^2 \ll X^{1-\frac{1}{k}-\varepsilon}.$$

The terms with $m = n$ contribute

$$\ll \frac{X^{2-\frac{2}{k}}}{H} \sum_{n\le X^\theta} \frac{|a_n|^2}{\sqrt[k]{n}} \ll \frac{X^{2-\frac{2}{k}}}{H}.$$

It follows for $\theta < \frac{1}{k} - \varepsilon$ and $X^{1-\frac{1}{k}+\varepsilon} \ll H = o(X^{1-\theta})$ that $I = O(X^{1-\frac{1}{k}-\varepsilon})$. The proof of the analogous bound for J follows from a similar, but easier argument that we will omit. Using these bounds in (3.8) we get for $\theta < \frac{1}{k} - \varepsilon$ and $X^{1-\frac{1}{k}+\varepsilon} \ll H = o(X^{1-\theta})$ that

$$\frac{1}{X}\int_X^{2X} P_k(x+H;\theta)P_k(x;\theta)\,dx = O(X^{1-\frac{1}{k}-\varepsilon}).$$

Therefore, combining this with (3.6) and (3.7) we obtain for $\theta < \frac{1}{k} - \varepsilon$ and $X^{1-\frac{1}{k}+\varepsilon} \ll H = o(X^{1-\theta})$ that as $X \to \infty$

$$\frac{1}{X}\int_X^{2X} \Big(P_k\Big(x+H;\theta\Big) - P_k(x;\theta)\Big)^2 dx \sim B_k \cdot X^{1-\frac{1}{k}}. \tag{3.9}$$

To complete the proof for $k = 3$ we use (3.9), Proposition 2.1 with $\theta = 12\varepsilon$, and then Cauchy-Schwarz. For $k \ge 4$ one argues in the same way only now use Proposition 2.2 with $\theta = 2k\varepsilon$ in place of Proposition 2.1. For $k = 2$ we use a classical estimate of Titchmarsh (see [14] (12.4.4)) which unconditionally implies (2.2) in the case $k = 2$, so that this case now follows as well. □

Acknowledgments. I would like to thank Zeev Rudnick for his encouragement and for many helpful and interesting discussions.

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